Beauty in Proofs

Kant on Aesthetics in Mathematics

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Abstract

It is a common thought that mathematics can be not only true but also beautiful, and many of the greatest mathematicians have attached central importance to the aesthetic merit of their theorems, proofs and theories. But how, exactly, should we conceive of the character of beauty in mathematics? In this paper I suggest that Kant's philosophy provides the resources for a compelling answer to this question. Focusing on §62 of the 'Critique of Aesthetic Judgment', I argue against the common view that Kant's aesthetics leaves no room for beauty in mathematics. More specifically, I show that on the Kantian account beauty in mathematics is a non-conceptual response felt in light of our own creative activities involved in the process of mathematical reasoning. The Kantian proposal I thus develop provides a promising alternative to Platonist accounts of beauty widespread among mathematicians. While on the Platonist conception the experience of mathematical beauty consists in an intellectual insight into the fundamental structures of the universe, according to the Kantian proposal the experience of beauty in mathematics is grounded in our felt awareness of the imaginative processes that lead to mathematical knowledge. The Kantian account I develop thus offers to elucidate the connection between aesthetic reflection, creative imagination and mathematical cognition.

1. Introduction

It is a common thought that mathematics can be not only true but also beautiful, and some of the greatest mathematicians have attached central importance to the aesthetic merit of their work. Many have derived aesthetic pleasure from mathematical research, pointing out the incomparable beauty and elegance of particular theorems, proofs and theories. As the French mathematician and theoretical physicist Henri Poincaré put it, mathematical beauty is a 'real aesthetic feeling that all true mathematicians recognize.'[1] While talk of beauty in mathematics may be commonplace, however, the notion of mathematical beauty raises serious questions. How, exactly, should we conceive of the character of such beauty? In what sense can we meaningfully speak of the objects of mathematical investigation as appealing to us aesthetically? Is beauty a property of mathematical objects or statements, or is it a subjective response particular to the perspective of the mathematician? How, in sum, are we to make sense of judgments concerning the beauty of mathematics?[2]

My aim in this paper is to show that Kant's philosophy provides the resources for a compelling answer to these questions. More specifically, I argue that on the Kantian account I develop beauty in mathematics is a non-conceptual response felt in light of our own creative activities involved in the process of mathematical reasoning. This proposal may come as a surprise. For, according to the standard conception, Kant's aesthetics leaves no room for beauty in mathematics. This is because Kant claims that judgments of beauty are essentially concerned with feeling and non-conceptual reflection rather than rational cognition, while such cognition is, on the face of it, the aim of mathematics. Indeed, Kant himself draws some strikingly negative conclusions concerning the possibility of beauty in the mathematical and physical sciences, claiming in the *Critique of Judgment* that there is 'no beautiful science, but only beautiful art'.[3] In light of this, many conclude that the Kantian has little of interest to say about beauty in mathematics and the sciences more broadly.[4] As a recent commentator summarises, Kant 'famously breaks with tradition in declaring that there can be no "beautiful science", nor a "science of the beautiful", because the pleasure that is characteristic of the free play of the faculties in reflective judgment [of beauty] is not to be found in the exercise of determinative judgment in science'.[5]

In the following, I aim to show that this standard conception overlooks an important insight offered by the Kantian position. In particular, I suggest that the Kantian proposal I develop provides a promising alternative to widespread Platonist convictions that can be found among many mathematicians.[6] According to the Platonist approach, the experience of mathematical beauty is a particular intellectual insight into fundamental structures of the universe. The distinctive Kantian proposal I advance in this paper, by contrast, consists in the view that the experience of beauty in mathematics is grounded not in an intellectual insight into particular properties of mathematical objects but in our felt awareness of the imaginative processes that lead to mathematical knowledge. The Kantian account I propose thus offers to elucidate, as the Platonist approach does not, the connection between aesthetic reflection, creative imagination and mathematical cognition.

In order to argue for these claims, I focus on a surprisingly under-studied section of Kant's *Critique of Judgment*. In §62 Kant makes three key claims concerning the possibility of beauty in mathematics:

(i) It is because of their unexpected purposiveness for cognition that the properties of mathematical objects are customarily regarded as beautiful.

(ii) Contrary to this customary conception of the beauty of mathematical properties, the purposiveness of mathematical properties does not indicate beauty but a form of perfection.

(iii) While mathematical properties themselves are not beautiful, it is the demonstration of such properties that can be the object of aesthetic appreciation.

In the first two of these statements, Kant comes back to the reasons, spelt out in detail in the body of the 'Critique of Aesthetic Judgment', for arguing against a conception of mathematical objects and their properties as beautiful. In the third claim, however, Kant seems to retract from an outright rejection of the possibility of aesthetic experience in mathematics. Kant's position, this suggests, is more complex than one might expect upon first consideration. But how should we understand the qualification made in (iii)? In what sense can mathematical demonstrations, but not the properties of mathematical objects, be regarded as beautiful?

To answer these questions, I focus on Kant's three claims in order. In Section 1, I examine the type of purposiveness displayed by mathematical objects that, according to (i), is commonly regarded as the ground of beauty. I show that, according to Kant, mathematical objects display a formal and objective purposiveness insofar as they are conducive to solving mathematical problems without having been designed for the solution of those problems. Section 2 discusses the reasons why mathematical objects that are purposive in this way cannot, according to (ii), be regarded as beautiful but must rather be considered as displaying a kind of perfection. As I spell out in this section, reflection on the purposiveness of mathematical objects can only ever lead to cognitive claims about those objects, but not to non-conceptual judgments that are the grounds of aesthetic pleasure. In Sections 3 and 4, I then tackle the difficult question of how, according to (iii), we may nevertheless conceive of beauty in mathematical demonstrations. Thus, in Section 3, I show that what Kant allows as a 'rightful' (*CJ*, V 364) feeling of admiration in mathematics is a response to the surprising fit of our intellectual capacities involved in mathematical proofs. In particular, I suggest that on Kant's account we feel a sense of admiration in light of the harmony we experience between our imaginative combination of the sensory manifold and our conceptual understanding of mathematical objects and their properties. Moreover, I argue in Section 4 that this feeling of admiration can adequately be identified as an aesthetic response. For, as I show, in providing non-conceptual sensory unities, the imagination makes a distinctive contribution to the process of demonstrating mathematical claims. The experience of beauty in mathematics, I thus claim, consists in an emotional response to this free and non-conceptual imaginative activity involved in mathematical proofs. I conclude, in the final section, by raising two questions about the scope of the Kantian account developed in this paper and by offering a suggestion for how this account may be extended beyond certain limitations of Kant's theory of mathematics. I propose that understanding the experience of beauty in mathematics as a non-conceptual response generated by our own intellectual processes sheds new light on the phenomenon of mathematical beauty.

2. The formal purposiveness of mathematical objects

In §62 of the *Critique of Judgment*, Kant distinguishes the *formal objective* purposiveness of mathematical objects, such as geometrical figures and numbers, from the *formal subjective* purposiveness with which he is concerned in the 'Critique of Aesthetic Judgment', on the one hand, and, on the other, from the *material objective* purposiveness that is the main concern of the 'Critique of Teleological Judgment'.

The purposiveness of mathematical objects, Kant argues, consists in their conduciveness to solving 'a host of problems' in a way that appears surprisingly simple (*CJ*, V 362). Consider the geometrical figure of a circle, for instance. It is by means of this basic figure that we can solve the apparently difficult task of 'construct[ing] a triangle from a given baseline and the [right] angle opposite to it' (ibid.). While an infinite number of different solutions could be given to the problem, the circle that takes the baseline of the triangle as diameter provides one unified solution. It presents precisely that geometrical shape that contains all the missing points required for constructing a right-angled triangle.[7] Thus, by means of the simple figure of a circle we can encompass all possible solutions to the problem. It is in this sense that mathematical objects, as Kant argues, display 'a manifold and often admired [...] purposiveness' (ibid.).

Kant characterises the purposiveness thus attributed to mathematical objects as 'an objective purposiveness which is merely formal' (ibid.). The purposiveness is objective because it is a relation between objects, for example between the geometrical figures of circles and right-angled triangles. This contrasts with the subjective purposiveness of the objects of aesthetic appreciation, which stand in a purposive relation to the subject. As Kant explains in the 'Analytic of Aesthetic Judgment', the objects of aesthetic appreciation are subjectively purposive insofar as our cognitive faculties, by reflecting on the form of these objects, are 'unintentionally brought into accord' (*CJ*, V 190) with one another.[8] Moreover, the purposiveness of mathematical objects is 'formal' (ibid.) since, although the objects are conducive to the solution of various problems and thus have the form of purposiveness, they have not in fact been created for the solution of those problems. This contrasts with the 'material' (*CJ*, V 362) purposiveness of objects that can only be conceived as the realisation of an intended purpose, which is 'the cause of the [object] (the real ground of its possibility)' (*CJ*, V 220). The purposiveness of the properties of a circle for constructing right-angled triangles thus differs from that of the properties of an artefact, such as a garden, for example, whose parts (e.g. trees, flower beds, and paths) have intentionally been designed and arranged so as to constitute a garden of a particular shape.

It is because mathematical objects display the form of purposiveness, even though they have not been designed for the realisation of any particular purpose, that, as Kant argues, their purposiveness is 'not expected from the simplicity of their construction' (*CJ*, V 366) and may, for that reason, surprise us. Moreover, it is because of this unexpected purposiveness that, as Kant claims further, the properties of mathematical objects are commonly described in aesthetic terms:

(i) It is customary to call the properties of geometrical shapes as well as of numbers […] *beauty*, on account of a certain a priori purposiveness, not expected from the simplicity of their construction, for all sorts of cognitive use, and to speak of this or that *beautiful* property of, e.g., a circle, which is discovered in this way or that (*CJ*, V 365 f.).

Kant explicitly attributes this view to Plato, but also seems to have in mind the perfectionist conception, put forward by such rationalist thinkers as Christian Wolff, Alexander Gottlieb Baumgarten and Georg Friedrich Meier, and going back to Leibniz, which identifies beauty with the perception of some form of objective purposiveness in the object (or, alternatively, in the artistic representation of the object).[9] According to this conception, our experience of beauty consists in the sensory recognition that the perceived object displays a certain internal unity and harmony and a conduciveness to purposes. While proponents of this conception may have held different views about the precise nature of such recognition of perfection, they agree on the key claim that the experience of beauty is, at least in part, based on the experience of a property of purposiveness in the object itself.[10] Insofar as this objective purposiveness is something that can be known in the object, Kant also calls it an 'intellectual' purposiveness, that is, a purposiveness that 'is cognised through reason' (*CJ*, V 362). As he critically points out, it is the recognition of such – often unexpected and hence surprising – objective and intellectual purposiveness that is commonly associated with the experience of beauty in mathematical objects.

3. Relative perfection rather than intellectual beauty

Kant crucially disagrees with the 'customary' view presented in (i). On his account, there is an important difference between the objective formal purposiveness we find in mathematics, on the one hand, and the subjective formal purposiveness that can be the ground of an experience of beauty on the other:

(ii) it is not an aesthetic judging by means of which we find them [i.e., the properties of mathematical objects] purposive, not a judging without a concept, which makes noticeable a merely *subjective* purposiveness in the free play of our cognitive faculties, but an intellectual judging in accordance with concepts, which gives us distinct cognition of an objective purposiveness, i.e., serviceability for all sorts of (infinitely manifold) purposes. One would have to call it a *relative perfection* rather than a beauty of the mathematical figure (*CJ*, V 366).

Kant agrees with the rationalists that the objective purposiveness of mathematical objects may be called a kind of perfection, that is, a perfection relative to the instantiation of a type, or the realisation of a purpose. He importantly disagrees with the further claim, however, that the perception of such relative perfection constitutes an experience of beauty.[11] Rather than consisting in the perception of some purposiveness in the object, the experience of beauty, for Kant, is the awareness of a purely subjective purposiveness that is independent of any conceptual cognition of the object. In contrast with our experience of the objective purposiveness of mathematical objects, the experience of beauty is not identifiable with any conceptual representation of purposiveness but can be perceived only through feeling.

In order to understand this claim, it is necessary to consider Kant's general views about aesthetic pleasure. As Kant explains in the 'Analytic of Aesthetic Judgment', the objects of aesthetic appreciation can be regarded as subjectively purposive insofar as, in reflecting on such objects, our cognitive faculties are brought into harmony with one another, a harmony that is experienced as aesthetically pleasing. What is important about this harmonious interaction of the faculties is that it conforms to the general conditions of cognitive judgment without, however, leading to cognition. Successful cognitive judgments, on Kant's account, depend on the work of the imagination, the capacity to combine different sensible states and to be aware of such combination as a unity, and the understanding, the conceptual capacity by means of which we make sense of such unity as a unity of a specific sort.[12] Cognition is thus made possible, Kant argues, by an imaginative synthesis of the sensory manifold, which is subsumed under concepts by the understanding. Aesthetic judgments, by contrast, differ from such cognitive judgments insofar as they do not lead to any conceptual subsumption of the synthesis brought about by the imagination. While the imagination unifies the sensory manifold in a way that could in principle be brought under concepts by the understanding, no concept is in fact applied. Kant describes the interaction of imagination and understanding in such judgments as harmonious insofar as it is in accordance with the conditions of cognition in general. It can nevertheless be regarded as free and spontaneous insofar as the creative activity of the imagination, which 'gathers' and 'unites' elements of the sensory given, is unconstrained by the conceptual guidance and determination of the understanding (*CPR*, A 77 f./B 103). The resulting judgment remains purely reflective, that is, it consists in the free and non-conceptual consideration of the object without, however, leading to any determinate conceptual claim about the object. It is this attunement of our cognitive faculties in the face of beautiful objects, independent of any specific judgment about the object itself that, Kant argues, we experience as aesthetically pleasing. In regarding an object as beautiful, he claims, 'we are conscious' of the harmony of our intellectual capacities 'with the sensation of satisfaction' (*CJ*, V 204).

In contrast with the rationalists, Kant thus conceives of aesthetic judgments as only indirectly concerned with the object regarded as beautiful. Rather than consisting in determinate claims about some purposive property in the object, aesthetic judgments express the awareness of the free and creative activity of our own mental capacities, triggered by consideration of the object. Aesthetic judgments may thus be regarded as expressions of our feeling that something makes sense to us, where this feeling of making sense is only indirectly related to the objects thus experienced and directly connected with the harmony of own intellectual activities that we experience in the face of such objects. It is in this sense that beautiful objects may be regarded as subjectively and formally purposive – purposive, that is, for our intellectual capacities without, however, fulfilling any particular purpose that can be ascribed to the object.

On Kant's account, the subjective purposiveness appreciated aesthetically through feeling thus differs significantly from the objective purposiveness grasped cognitively in mathematical objects. It is for this reason that Kant rejects the rationalist view that identifies the objective purposiveness of mathematical objects with the ground of an aesthetic appreciation. As we have seen, Kant argues that the objective purposiveness of mathematical objects and their properties must be conceived as a 'relative perfection' (*CJ*, V 366). This relative or, as Kant also calls it, 'external' (*CJ*, V 226) perfection of an object consists in its 'utility' (ibid.) or 'serviceability' (*CJ*, V 366) for some external purpose, that is, for the solution of mathematical problems. A judgment on the perfection of mathematical objects, rather than consisting in free and non-conceptual reflection, ascribes concepts to objects in a determinate way. Thales' Theorem, for instance, gives a conceptual account of the relation between the circle and the infinite number of right-angled triangles that have the diameter of the circle as their base. When, in mathematics, we feel pleased at the sight of a simple geometrical figure that aids the solution of a complex problem, the pleasure thus generated cannot, therefore, be a type of aesthetic appreciation. It is brought about not by the 'free and indeterminately purposive entertainment of the mental powers with that which we call beautiful' but by 'the approval of the solution that answers a problem' (*CJ*, V 242). Hence, judgments about the relative perfection of mathematical objects, Kant concludes, are not identifiable with aesthetic judgments but are concerned with a type of intellectual pleasure. Indeed, Kant leaves no room for ambiguity in distinguishing the two types of pleasure when he argues that '[t]he designation of an *intellectual beauty* cannot legitimately be allowed at all, for otherwise the word "beauty" would have to lose all determinate meaning' (*CJ*, V 366). The appreciation of beauty, in short, has nothing to do with the satisfaction associated with the solution of intellectual problems. The rationalists fall into error, Kant concludes, when they mistake intellectual for aesthetic pleasure in their analysis of our experience of mathematical objects.

4. Admiration for mathematical demonstrations

After rejecting the rationalist, or 'customary', view that ascribes aesthetic qualities to the purposive properties of mathematical objects, Kant makes his surprising third claim in which he appears to retreat from an outright rejection of aesthetic considerations in mathematics. As Kant goes on to argue,

(iii) [o]ne could rather call a *demonstration* of such properties beautiful, since by means of this the understanding, as the faculty of concepts, and the imagination, as the faculty for exhibiting them a priori, feel strengthened (which together with the precision which is introduced by reason, is called its elegance): for here at least the satisfaction, although its ground lies in concepts, is subjective, whereas perfection is accompanied with an objective satisfaction (*CJ*, V 366).

Kant's claim raises a number of questions. Why should mathematical demonstrations but not mathematical properties be called beautiful? What exactly is the difference between mathematical properties and their demonstration that makes the latter, but not the former, suitable for aesthetic consideration? And, more specifically, why do the demonstrations of mathematical properties rather than the properties themselves elicit a subjective rather than objective satisfaction, as they would have to, if the satisfaction at issue is genuinely aesthetic?[13]

On initial consideration, one might be tempted to construe Kant as suggesting that only mathematical demonstrations *loosely speaking* can be regarded as beautiful. That is, when Kant speaks of a 'demonstration of mathematical properties' one might understand him as referring to some kind of representation of mathematical objects and their qualities. According to this reading, the depiction of circles and triangles in paintings or their display on the facade of buildings, for example, could be judged aesthetically. Thus, one might argue that we can reflect on the outer form or shape of such objects aesthetically while abstracting from any conceptual representations, for instance their conduciveness to mathematical problem solving. Representations of geometrical figures such as circles and triangles would then be regarded as the object of non-conceptual reflection that grounds an aesthetic pleasure.[14]

This reading is tempting, I think, because Kant suggests that a pure judgment of taste is possible even in the case of an object for which we have a determinate concept, 'if the person making the judgment [...] abstracted from it [i.e. the concept] in his judgment' (*CJ*, V 231). Kant thus leaves room for the possibility of non-conceptual, aesthetic judgment by abstracting from concepts that may otherwise be the basis of determining judgments about the object.[15] As a reading of the above passage, however, this initial proposal is unconvincing. Kant never uses the term 'demonstration' (*Demonstration*) in the loose or non-technical sense as referring to a representation of some form or other. Rather, he unambiguously employs the term in order to denote 'a proof which is the ground of mathematical certainty' (*Logic*, IX 71), as he defines it in the *Logic Lectures*.[16] Kant's third statement, then, is not concerned with representations of mathematical objects such as geometrical shapes on wallpapers, but with mathematical proofs. Far from focussing on judgments that abstract from concepts, and hence from cognitive claims, Kant is concerned with proofs that are the very basis of mathematical cognition. But this only raises the questions with which we were confronted above in a new form. For why should Kant allow for the aesthetic appreciation of mathematical proofs if he so adamantly rejects judgments of beauty about mathematical objects and their properties?

I suggest that in order to answer this question we need to pay closer attention to the specific character of mathematical proofs on Kant's account. Since mathematical certainty, for Kant, is intuitive certainty, he also characterises a demonstration in the *Critique of Pure Reason* as 'an apodictic proof, insofar as it is intuitive' (*CPR*, A 734/B 762).[17] This is important for understanding Kant's difficult third claim. Indeed, I think that if we take account of the intuitive nature of mathematical demonstrations, we can explain Kant's conception of beauty in mathematics. A closer look at what Kant says in §62 about the admiration we may experience for mathematical reasoning will begin to throw light on the connection between the intuitive character of mathematical demonstrations and the aesthetic appreciation of mathematics.

Having described how, in the history of mathematics, the geometers 'delighted' in the purposiveness of geometrical figures, Kant points out that this purposiveness may arouse in us a sense of admiration:

> For in the necessity of that which is purposive and so constituted as if it were intentionally arranged for our use, but which nevertheless seems to pertain originally to the essence of things, without any regard to our use, lies the ground for the great admiration of nature [...] (*CJ*, V 363).

Kant suggests that we experience a sense of admiration when objects seem to us as if they were created for the realisation of our purposes, yet are not in fact so created.[18] This general account of admiration, however, immediately raises a question about the particular case of our admiration for mathematical objects. In the case of natural objects whose origin is independent of our purposes, it is indeed unsurprising that we may feel a sense of astonishment if those objects turn out to be conducive to our purposes. And yet, the case of mathematical objects is different insofar as these objects, as Kant also argues, are products of the human mind.[19] They are not things whose character is contingent and discovered by us only empirically, but objects whose particular nature is determined by a priori concepts. This is why objects of mathematics can be constructed, that is, exhibited in a priori intuition.[20] We can, in other words, produce sensible representations of mathematical concepts without recourse to any particular experience. Thus, while it may, of course, be true that we do not always construct mathematical objects *in order to* solve certain problems in mathematics, it is not obvious why we should be astonished at their apparent fit with the needs of human reason. For, in so far as they are the products of our own intellectual activity, they cannot but be in conformity with, and in this respect also purposive for, that activity. It seems, rather, that in the case of mathematical objects we ourselves '*introduce* the *purposiveness* into the figure that [we] draw *in accord with a concept*' (*CJ*, V 365).[21] Why, then, should we be surprised by, and indeed feel admiration for, the apparent purposiveness of objects that we have constructed ourselves?

Even though mathematical objects are products of the human mind, Kant claims that they nevertheless elicit a 'rightful' (*CJ*, V 364) sense of admiration in us. I believe that what is important for understanding this experience of admiration is not, as Kant first seems to suggest, that we may mistake the a priori rules of construction for empirical ones.[22] What is important is rather that, on Kant's account, mathematical construction and, indeed, demonstrations that rely on such construction are dependent on intuition. In order to understand why we may feel a sense of admiration in mathematics, we thus need to appreciate Kant's characterisation of mathematical knowledge as synthetic a priori.[23] Although the concept of a circle, for instance, is an a priori concept, Kant argues that we cannot know of the particular properties of the circle and its conduciveness to the solution of difficult mathematical problems without constructing it in a priori intuition:

> The many rules, the unity of which (from a principle) arouses this admiration, are one and all synthetic, and do not follow from a *concept* of the object, e.g., from that of a circle, but need this object to be given in intuition (*CJ*, V 364).

Similarly, 'curves yield […] purposive solutions that were not thought of at all in the rule that constitutes their construction' (*CJ*, V 363). We therefore cannot, according to Kant, infer the properties of a circle or a curve that are conducive to the solution of geometrical problems analytically but only synthetically by recourse to intuitive representation. And it is this dependency on intuition, I suggest, that ultimately grounds the experience of admiration in mathematics on Kant's account.

What, then, is it about the a priori synthetic nature of mathematical constructions and demonstrations that elicits such admiration? Given Kant's conception of mathematics, what is special about mathematical objects is not that they appear to us as if they were designed even though they are not. What generates surprise is rather that, through reflection on such objects, our conceptual understanding is in fact purposefully aided by the activities of imagination. Even against the background of Kant's critical conception of mathematical objects as products of the human mind, it is astonishing that we can construct these objects in pure intuition and, by reflecting on them, extend our knowledge of them and their relation to other mathematical objects purely a priori. What is surprising, in other words, is the unlikely fit of the knowledge gained through sensible representations in a priori intuition 'with the principles already grounded in the mind', that is, our conceptual understanding of the object (ibid.). What provokes surprise, then, is not simply the fact that geometrical figures or numbers are conducive to the solution of a host of difficult mathematical problems. What evokes astonishment is rather, more generally, that in mathematical demonstrations we can solve difficult conceptual problems by recourse to a priori sensible representation. It is this 'harmony' between our concepts given by the understanding and 'space, by the determination of which (by means of the imagination, in accordance with a concept) the object is alone possible' that is made evident in mathematical proofs (*CJ*, V 365). And it is this harmony, I suggest, that is the ground of our admiration in mathematics.

Returning to the passage in which Kant characterises his general conception of the admiration of nature, it is then important not to overlook a final qualification of this conception:

> For in the necessity of that which is purposive and so constituted as if it were intentionally arranged for our use, but which nevertheless seems to pertain originally to the essence of things, without any regard to our use, lies the ground for the great admiration of nature, *not outside of us so much as in our reason*' (*CJ*, V 363; italics added).

What is considered as worthy of admiration, in the end, is not so much the nature of things in the external world but rather our own intellects. In the specific case of mathematics, it is not the objects of mathematics and their properties but our own intellectual capacities involved in mathematical demonstrations, understanding and imagination, that elicit in us a sense of admiration. The admiration we feel in mathematics is thus only indirectly a response to a particular proof, and directly linked to the fit of our conceptual capacity with the capacity of imagination.

5. Beauty in mathematical demonstrations

As Kant argues in §62, we naturally feel a sense of admiration in mathematics, not because of a formal purposiveness of mathematical objects for problem solving but because of the surprising harmony of our intuitive and conceptual capacities. Does this account help to answer the questions raised in the previous section about Kant's third claim? In particular, should the admiration we may feel in the face of mathematical proofs really be characterised as an aesthetic response?

Kant's account of admiration in mathematics shows some obvious dissimilarities with his account of the appreciation of beauty. Indeed, these dissimilarities may lead one to suspect that the former should not be identified with the latter. While aesthetic judgment is non-conceptual, in mathematical constructions the imagination provides a priori representations of concepts. Moreover, even though mathematical proofs are the basis of intuitive certainty, they also lead to conceptual cognition. While the proof of Thales' Theorem, for example, relies on intuition, this intuition itself grounds the knowledge that all points on the circle are the missing points of all possible right-angled triangles. One might therefore think that the harmony of imagination and understanding, occasioned by mathematical demonstrations, is not the harmony of free play, but rather the harmony associated with cognitive judgments, and judgments of perfection in particular, a harmony brought about by the guidance of a determinate concept.

And yet, while mathematical demonstrations are ultimately the ground of determinate knowledge claims, on Kant's account the imagination must nevertheless make a spontaneous and non-conceptual contribution to the processes that lead to such knowledge. As we have seen in the previous section, according to Kant it is through a priori intuitive representation that we gain insight into the properties of mathematical objects that could not be analytically inferred from their concepts. This process begins with the construction of an object according to a concept and ends with the cognitive judgment that the object has such-and-such properties. It is in between these two acts, I suggest, that the imagination makes a contribution that is free from conceptual determination. In order to substantiate this claim, we need to look more closely at the role imagination plays in learning something new about mathematical objects that was not entailed 'in the rule that constitutes their construction' (*CJ*, V 363).

Thus, in mathematical construction the imagination produces a representation in a priori intuition that provides an instance of a mathematical concept. While the result is an 'individual object', the construction nevertheless provides a 'universal' representation of all those objects that fall under the concept (*CPR*, A 713/B 741). It does so by representing the rule-governed 'act of construction' (*CPR*, A 714/B 742) or, as Kant puts it in the 'Schematism' chapter of the first *Critique*, the 'universal procedure' of producing the object (*CPR*, A 140/B 179). This characterisation of construction in mathematics indicates that the imagination, while generating non-conceptual and intuitive unities, does so in accordance with concepts.[24] Mathematical demonstrations do not, however, end but begin with the construction of mathematical objects. It is only through subsequent reflection on these objects that we can find out about their further properties such as, for example, the relation between a circle and right-angled triangles. Subsidiary steps are thus required in order to prove a mathematical claim.

It is at this point, I suggest, that the activity of the imagination makes its original contribution. Insofar as mathematical knowledge is, on Kant's account, synthetic a priori, the subsidiary steps in a proof can consist neither in the analysis of the mathematical concepts thus instantiated nor in simply reading off hitherto unknown properties from empirically given data. Rather than constructing an intuitive representation of what was already analytically entailed in the original concepts, and rather than considering sensory data provided to us empirically, the imagination spontaneously offers sensory unities that are produced independently of the determination of conceptual rules. We may think of this as imaginatively playing around with the mathematical objects constructed in a priori intuition before, subsequently, recognising – conceptually – that they have particular properties or stand in certain relations to one another. The possibility of making the next step in a proof, I suggest, is grounded in this free play of our imaginative activities that offer different ways of combining the sensory manifold by, for instance, drawing new connections between mathematical objects that were not originally thought in the concepts of those objects.

In the proof of Thales' Theorem, for example, we start by constructing a circle and a triangle whose three points A, B and C lie on the circle and whose baseline AC is a diameter of the circle. Subsidiary steps involving auxiliary principles and constructions are then needed in order to prove the theorem. Thus, we can show that by dividing the triangle through a line drawn from the centre of the circle to point B, we obtain two isosceles triangles, that is, triangles whose base angles are equal. This turns out to be a crucial move in the demonstration, which enables us, by operations of addition and division, to prove that ABC is a right angle.[25] That the construction of two isosceles triangles will turn out to be a useful step in the proof cannot be known in advance, however. As Béatrice Longuenesse points out, the capacities involved in choosing the next step in a proof should rather be understood in the context of Kant's famous notion of 'mother wit'.[26] On this conception, the choice of intermediary steps in a proof depends for Kant 'on that particular aspect of the power of judgment' which is 'a talent that no learning of scholarly rule can replace'. That is, the choice of intermediary steps is not learnt, or deduced from given principles, but is dependent on one's aptitude for making the right judgment.

This proposal hints at the irreducible originality of the choice of moves that lead to the discovery of a proof in mathematics. I believe, however, that the capacity for judgment, or mother wit, ultimately relies on the contribution made by the free activity of the imagination. For the ability to make the right judgment consists precisely in recognising that a universal principle or concept is suitable for subsuming a particular given in sensibility. It is the imagination, I suggest, which first produces sensible unities in light of which the use of an auxiliary principle, such as the principle that all isosceles triangles have equal base angles, will appear more or less adequate. Rather than picking randomly between a set of principles, which are then presented in concreto through constructions in a priori intuition, in mathematical demonstrations we first produce sensible unities in imagination, which are subsequently subsumed under suitable principles. That is, free from the guidance of concepts, we imaginatively play around with the sensory manifold, combining and recombining it with the unities already constructed, thereby offering new intuitive unities to be recognised as falling under conceptual rules. In spontaneously offering new combinations of the sensory manifold in a priori intuition, the imagination thus makes possible the instantiation of subsidiary principles required for proving such mathematical claims as Thales' Theorem.

In this way we can arrive at a proof, conceptually spelt out as a series of inferences. But we do so only insofar as the imagination has already provided us with sensible unities that were not originally thought under the concepts with which we started. The presentation of a 'chain of inferences' (*CPR*, A 717/B 745), in other words, is possible only retrospectively and as a systematic presentation of a mathematical proof already discovered. The intellectual processes involved in the original discovery of the proof by the mathematician – as well as the subsequent recreation of such discovery processes by the mathematics student – however, are 'guided by intuition' (ibid.) and essentially involve the free activity of the imagination.[27]

If this suggestion is correct, then the cognition achieved through mathematical proofs is made possible by a spontaneous act of the imagination. Just as in aesthetic judgment, in mathematical demonstrations the imagination acts freely, unconstrained by, and yet in harmony with, the understanding. It would follow that the resulting fit between the spontaneity of the imagination, on the one hand, and the conceptual determination of the understanding, on the other, is itself undetermined by further conceptual rules and can therefore be regarded as free. There is, in other words, no further rule that can account for the harmony between the imaginative production of synthetic unities in a priori intuition and the conceptual cognition to which our imaginative activity leads. It is for this reason, I believe, that Kant describes the harmony of imagination and understanding in this context as ultimately 'inexplicable' (*CJ*, V 365). Indeed, it is because of our astonishment at the inexplicable fit between intuition and understanding that, as Kant claims, we are led to 'have a presentiment [*ahnen*] of something lying beyond those sensible representations, in which, although unknown to us, the ultimate ground of that accord could be found' (*CJ*, V 365). Even though we neither can nor need to know about this ultimate ground, we are evoked to think of it as the condition of possibility of the agreement of our cognitive faculties.[28] Insofar as the harmony that is taken to lie in such an unknowable ground is itself inexplicable, moreover, it cannot be the source of an intellectual pleasure. The pleasure thus occasioned by mathematical demonstrations cannot be identified with that elicited by the discovery of a solution to a problem but consists in a subjective pleasure, a satisfaction felt in response to the inexplicable fit of imagination and understanding.[29]

Can the pleasure experienced in response to mathematical demonstrations then be classed as an instance of aesthetic appreciation? As we have seen, Kant's account of admiration in mathematics shows some striking similarities with his account of the appreciation of beauty. In both cases, we encounter a surprising fit of our intellectual capacities. The pleasure that is expressed in judgments of taste arises as a result of an unexpected agreement that we experience between formal aspects of the object and the requirements of our cognitive faculties. Similarly, in the case of our feeling of admiration for mathematical demonstrations, we experience an unexpected agreement between our imaginative play with the sensory manifold in a priori intuition and the conceptual insight gained thereby. In the case of aesthetic judgment, Kant describes this harmony of the faculties as leading to an 'animation' or 'quickening' [*Belebung*] 'of both faculties (the imagination and the understanding)' (*CJ*, V 219). Similarly, in the case of mathematical demonstrations, Kant argues that in their interaction imagination and understanding 'feel strengthened' (*CJ*, V 366). In both cases, the objects of appreciation may be considered as subjectively purposive, that is, as purposive for our intellectual capacities. In both cases, it is not the objective purposiveness of the objects themselves, but the subjective purposiveness we become aware of, respectively, on the occasion of mathematical demonstrations and beautiful objects that is the source of our admiration. And in both cases, finally, we are led to an intimation of the super-sensible as a ground of the marvellous harmony of our intellectual capacities.

According to these parallels, the admiration we may feel in the face of mathematical proofs can thus quite plausibly be regarded as an aesthetic appreciation. For although mathematical demonstrations lead to determinate judgments about mathematical objects that leave no room for the free play of the imagination, what elicits the experience of beauty in mathematics are not those objects or their properties themselves, or indeed their conduciveness to the solution of mathematical problems, but rather the free activity of our intellectual faculties that is employed in mathematical demonstrations. On Kant's account, judgments about the beauty of mathematical proofs, I therefore suggest, are a form of aesthetic judgment grounded in the creative process of the imagination that leads to such knowledge.

6. Conclusion

The foregoing discussion sheds light on the question of what, from a Kantian position, can be said about beauty in mathematics. For Kant, beauty is not a feature attributable to mathematical objects and their apparently purposive properties, but is experienced rather in the process of demonstrating mathematical theorems through a creative act of the imagination. Insofar as this creative activity involves the free and spontaneous use of imagination which, though unconstrained by conceptual rules, nevertheless leads to conceptual insight, it points to the fit between our intellectual capacities. And it is the awareness of this fit that, on Kant's account, is experienced with a feeling of pleasure. Thus, the experience of mathematical beauty, according to this account, does not consist in a response to the surprising applicability of our mathematical concepts and theorems to empirical reality. Instead, it is generated by the awareness that our capacities for imaginative synthesis fit together with our conceptual capacities in a way that makes it possible for us to learn something genuinely new about a priori concepts by pure acts of imagination. It is the awareness of this harmony, elicited by the process of mathematical demonstration rather than the finished product, that is the basis of aesthetic experience in mathematics on Kant's account.

A discussion of the statements presented in §62 of the *Critique of Judgment* thus shows that Kant is indeed justified in making his rather suggestive claim (iii) in which he allows for a qualified conception of beauty in mathematics. The Kantian position draws out, in particular, the inherent connection between the creativity of mathematical proofs and the attribution of beauty to such proofs. And yet, this conclusion also raises a number of questions concerning, first, the scope of the Kantian proposal within the context of Kant's theory of mathematics and, second, the possibility of extending the Kantian suggestion beyond Kant's transcendental philosophy. I shall raise two such questions, before concluding with a suggestion for how the Kantian account I have proposed might be developed further.

First, one may wonder whether the proposed conclusion is equally valid, on Kant's account, for different areas of mathematics. While in the *Critique of Judgment* Kant speaks of mathematical objects in general, and explicitly refers to geometrical figures and numbers, his discussion of mathematical beauty focuses on geometry rather than arithmetic. This accords with a related asymmetry between Kant's discussion of geometry and arithmetic in the *Critique of Pure Reason*. What is important about this asymmetry is that Kant seems to provide different accounts of the dependency on intuition of geometry on the one hand and arithmetic on the other.[30] More generally, commentators have disagreed about whether for Kant all mathematical proofs are equally dependent on intuition. That is, they have disagreed about whether intuition is required only for the construction of mathematical objects and the evidence of the truth of the axioms, or also for moving between the individual steps of a proof.[31] Settling such disagreement is beyond the scope of the present discussion. That the disagreement exists may nevertheless be cause for concern for the Kantian account of mathematical beauty I have offered. For if some mathematical proofs are not dependent on intuition, and hence do not require the creative involvement of imagination as I have suggested, then such proofs cannot, on Kant's theory, elicit an aesthetic response. Beauty could only be found in proofs where the imaginative synthesis of intuitive content is experienced as freely harmonising with conceptual understanding. Such a restriction of mathematical beauty to some mathematical demonstrations, however, would seem altogether arbitrary and incompatible with the actual phenomenology of aesthetic pleasure in response to mathematical proofs. What, then, is the scope of Kant's account of mathematical beauty within the wider context of his theory of mathematics?

Second, one may wonder how far it is possible to extend the Kantian proposal beyond Kant's own account of mathematics. For, independently of the question of how to read Kant's conception of the importance of intuition in mathematical demonstrations, few mathematicians today would want to ascribe as extensive a role to intuition as did Kant. Even if one does not adhere to a strictly logicist conception of mathematics as reducible to logic, or to a strictly formalist account of higher mathematics as concerned with statements that are nothing more than uninterpreted strings of symbols, it is undeniable that many mathematical proofs work without recourse to intuition. Again, it would seem ad hoc to rule out in principle, as the Kantian position appears to do, the possibility of ascribing beauty to mathematical proofs that are far removed from anything that can be represented intuitively. Can the Kantian account of mathematical beauty tell us anything more general about aesthetics in mathematics, including attributions of aesthetic features to purely formal demonstrations?

While a detailed treatment of these questions centring on Kant's controversial account of the synthetic a priori status of mathematics will have to be deferred to another occasion, I would like to end by suggesting a way in which the Kantian proposal may be extended to contemporary discussions. As we have seen, the crucial insight of the Kantian account is that the experience of beauty in mathematics presents a non-conceptual response felt in light of our own creative imaginative activities involved in mathematical demonstrations. A more general lesson to be learnt from the Kantian account is possible, I believe, if we understand the creative practices and reasoning processes in mathematics that elicit an aesthetic response in a wider sense than the one proposed by Kant's account of our intellectual faculties. More specifically, our mathematical practices and reasoning processes may go beyond what Kant describes as the free and harmonious activity of the faculties of imagination and understanding and include, for instance, the interaction of visual and conceptual thinking as well as the purely formal manipulation of symbols. Even where intuitive representation in the Kantian sense is not necessary for mathematical demonstrations, visual representation may nevertheless guide mathematical thinking.[32] Visual representations may, for instance, provide alternative evidence for, and indeed first suggest, a mathematical theorem that can also be proven purely formally. It seems plausible that, in such a case, the experience of beauty is a response to our awareness of the creative activity of visual representation, which is subsequently recognised as fitting our conceptual understanding of the theorem.[33] Moreover, in cases where mathematical proofs cannot be related to intuitive representation at all, and where only a purely formal proof of a theorem is possible, the discovery of such a proof may nevertheless involve reasoning processes that can be regarded as original and creative rather than governed by conceptual rules. In other words, while the mathematical proof may be strictly formal, the way in which the mathematician first came to think of it may not. In such a case, it still seems plausible to suggest that it is our awareness of the apparent creativity of thought in the actual practice of mathematics that can be appreciated aesthetically.[34]

Extending the Kantian account of beauty in mathematics in the proposed direction promises to offer resources for a robust account applicable beyond the limits of the Kantian starting point. According to this account, mathematical beauty turns out to consist in a non-conceptual response generated by our own reasoning processes rather than a rational insight into mind-independent truths, as the Platonist holds. It is the Kantian characterisation of mathematical beauty as an emotional response to the creative intellectual processes that lead to mathematical knowledge, I believe, which makes sense of what seems so special about something striking us as beautiful amidst our attempts to gain mathematical cognition.[35]

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Notes

[1] Poincaré (1930: 59). Others have gone further in arguing for the motivational function of beauty in mathematics (cf. Wigner, 1960). Similarly, many theoretical physicists, including Paul Dirac, Albert Einstein, Hermann Weyl and Werner Heisenberg, explicitly recognised mathematical beauty as one of the key motivations behind the formulation of physical theories (see Chandrasekhar, 1987: 60 ff.).

[2] For conflicting approaches to these questions see, for instance, Hardy (1940), Rota (1977), and Osborne (1984) and, with relevance to the physical sciences, Mamchur (1987), McAllister (1996) and the collection of papers in McAllister (2002). For some striking examples of beauty and elegance in the mathematical and natural sciences see Glynn (2010).

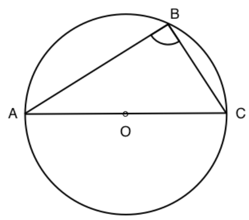
[3] See in particular *Critique of Judgment* (*CJ*), V 308 f., and *Anthropologie Nachschriften*, XXV 1061 and 1310 f. References to Kant's texts are made by citing the volume and page number of the Academy edition (Kant 1900 ff.), with the exception of the *Critique of Pure Reason* (*CPR*), which is referred to by citing the page numbers of the original A and B versions. Translations are my own, but in the case of the *Critique of Judgment* (*CJ*) are guided by Kant (2000).

[4] See, for instance, Rueger (1997), Koriako (1999: 154 ff.) and Wenzel (2001 and 2005: 133 ff.). Wenzel argues, in particular, that Kant changed his mind, excluding the possibility of beauty in mathematics only in the 1790s. Cf. also Giordanetti's (1995) discussion of the relationship of genius, artist and scientist in Kant. It is similarly characteristic of the prevalent attitude, I believe, that other commentators on Kant's account of mathematics and his conception of aesthetics simply remain silent on the question of mathematical beauty. See the collection of classic papers on Kant's account of mathematics in Posy (1992), as well as the influential discussions of Kant's account of aesthetics in Crawford (1974), Guyer (1997) and Allison (2001). Crawford (1982) and Winterbourne (1988) present exceptions in analysing the analogies between Kant's accounts of mathematics and art. They do not, however, address the more specific question of the possibility of beauty in mathematics.

[5] Rueger (1997: 315).

[6] On the prevalence of broadly Platonist conceptions among mathematicians and natural scientists, see Chandrasekhar (1987).

[7] According to Thales' Theorem, if A, B and C are points on a circle where the line AC is a diameter of the circle, then the angle ABC is a right angle.



[8] I shall come back to the nature of the subjective purposiveness of the objects of aesthetic judgment in more detail in the next section.

[9] Cf. Wolff (1720), Baumgarten (1750/1973), Meier (1748), and Leibniz (1969/1690).

[10] While, for Wolff the aesthetic perception of perfection is only a less than optimal, sensory cognition, inferior to the rational cognition of that perfection (cf. Wolff, 1720, § 404), Baumgarten conceives of aesthetic experience as the result of a more complex activity of a range of mental capacities involved in the sensible representation of perfection (cf. Baumgarten, 1750/1973, § 14). For a helpful comparison of different perfectionist accounts of aesthetics see Guyer (2007).

[11] Cf. Kant's discussion of perfection in § 15 of the 'Critique of Aesthetic Judgment' (*CJ*, V 226 ff.).

[12] Cf. *CPR*, A 78/B 103.

[13] These questions presuppose that Kant's claim (iii) should be taken at face value. Given Kant's tentative formulation in the passage quoted, however, one may wonder whether he is really advocating unconditionally that we consider the demonstration of mathematical properties as beautiful, or whether he might be suggesting, more cautiously, that *if* – perhaps improperly – we were to speak of beauty in mathematics, *then* the demonstration of mathematical properties would be a more suitable object of appreciation than the properties themselves. Before passing a judgment on the question of whether or not Kant's claim should be taken at face value, I shall examine how much of what Kant is claiming here can be justified, given his conception of aesthetics on the one hand and his account of mathematics on the other. Moreover, regarding the particular passage quoted above, I believe that although Kant's mode of expression is, indeed, tentative, Guyer and Matthews' translation (in Kant 2000) inadequately over-emphasises Kant's cautiousness when rendering 'Eher würde man eine Demonstration solcher Eigenschaften [...] schön nennen können' as 'It would be better to be able to call a demonstration of such properties beautiful [...]' (*CJ*, V 366). Unlike the translation, the German does not imply any evaluative judgment on whether or not it would be good to call mathematical demonstrations beautiful, but states that we could justifiably do so.

[14] A reading of this kind is suggested by Giordanetti (2008: 220) who argues that Kant denies the ascription of beauty to arithmetical formulas, yet allows it in the case of music, in which the relation between tones is a representation of arithmetical proportions. This reading is not only problematic given the reasons spelt out below, but it is also highly speculative. In the paragraph in which claim (iii) appears, Kant is concerned with the properties of both numbers and geometrical figures but makes no mention of music.

[15] On the difficult question of the criteria for abstraction from concepts, see Guyer (1997: 220ff.). On the possibility of aesthetic judgments in the presence of conceptual judgments, see the discussion of free and adherent beauty in Guyer (1997: 184 ff., and 2002), Allison (2001: 119 ff.), and Rueger (2007).

[16] A second reason that speaks against the initial reading can be found in the 'General Remark on the Analytic of Aesthetic Judgment'. There, Kant argues that mathematical objects are not promising candidates for aesthetic consideration because they can be conceived only as the representation of a concept and are therefore not candidates for non-conceptual reflection (cf. *CJ*, V 241).

[17] Cf. *Logic*, IX, 70.

[18] Similarly, in the introduction to the *Critique of Judgment* Kant claims that we feel pleasure in the purposiveness we experience in the unity of nature, and displeasure in its disunity (*CJ*, V 187 f.). The pleasure Kant refers to here is an intellectual one. It is the satisfaction produced by our finding the world to be conducive for human understanding.

[19] H. W. Cassirer (1970: 318 f.) points out that the significance of this difference between the formal purposiveness of mathematical objects and the formal purposiveness of nature may strike especially the transcendental philosopher. While the purposiveness of mathematical objects can be explained by reference to their construction in conformity with the a priori principles of understanding, the purposiveness of external nature seems entirely underdetermined by those principles. As Cassirer also acknowledges, however, the agreement of intuition and understanding in mathematical construction remains, in the end, inexplicable. I come back to this inexplicability in Section 4 below.

[20] See Kant's discussion of construction in mathematics in the 'Discipline of Pure Reason' (*CPR*, A 713/B 741 ff.).

[21] One may, perhaps, be surprised if a particular mathematical explanation accounts for the character of a particular natural phenomenon rather than another. That the objects of mathematics are conducive to human understanding in general, however, seems entirely

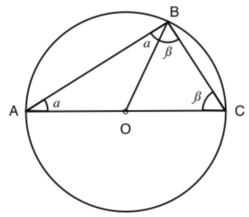
unsurprising because of their constructed nature. See Kant's discussion of the applicability of geometrical principles to the trajectories of celestial bodies in *Prolegomena* (IV 320 ff.).

[22] Kant first suggests that this admiration may be based on our mistakenly regarding the properties and rules of construction of mathematical objects as empirically given, and thus as surprisingly conducive to our own needs (cf. *CJ*, V 364). It is unclear, however, how such a mistaken conception of mathematics could ever be regarded as 'rightful' (ibid.). This error theory of admiration in mathematics therefore should not be, and in fact is not, Kant's last word on the matter.

[23] A discussion of Kant's argument for this characterisation goes beyond the scope of this paper. See Kant's account of the method of mathematics in the 'Discipline of Pure Reason' (*CPR*, A 713 ff./B 741ff.) and his earlier account of the synthetic character of mathematics in his 1764 essay 'Inquiry Concerning the Distinctness of the Principles of Natural Theology and Morality' (II 276 ff.). For detailed analysis of Kant's account of mathematics, see the papers collected in Posy (1992) as well as Young (1982), Friedman (1992: 55-135), Shabel (1998 and 2006) and Carson (1999).

[24] How exactly we should understand this harmony of imaginative production with the concepts such production instantiates is a matter of serious controversy in the literature. Commentators disagree about whether the imagination is wholly guided by concepts or, alternatively, offers a contribution that, while making concept application possible, is itself free from conceptual determination. Young (1988) and Allison (2004), for instance, hold that in schematising, the imagination is concept-governed, even if it does not follow conceptual rules consciously but only blindly. Bell (1987) and Gibbons (1994), by contrast, argue that the activity of the imagination in schematising is to be understood on the model of the free and creative imagination at work in aesthetic experience. Commentators thus disagree about how much we should read into Kant's famous pronouncement that the schematism is 'a hidden art in the depths of the human soul' whose activity cannot be understood or spelt out conceptually (*CPR*, A 141/B 180). I cannot resolve this conflict in the present paper. While I believe that more could be said about the freedom of the imagination in this context, my argument does not hinge on the extent to which the imagination acts spontaneously in the schematism, and hence the original construction of mathematical objects. The crucial claim is, rather, that the imagination makes a spontaneous contribution to the subsequent reflection on those objects, thereby making possible mathematical demonstrations.

[25] From the fact that triangles AOB and COB are isosceles triangles, and the fact that the sum of the angles in a triangle is equal to 180°, we know that (i) $\alpha + (\alpha + \beta) + \beta = 180°$. By operations of addition and division we then get (ii) $2\alpha + 2\beta = 180°$, and (iii) $\alpha + \beta = 90°$.



[26] Longuenesse (1998: 288). Cf. *CPR*, A 133/B 172.

[27] I thus agree with Crawford (1982: 165), and similarly Winterbourne (1988), who warn against the danger of identifying the (ineffable) 'order of discovery' of a mathematical proof with the (effable) 'order of teaching or systematic presentation of truths already discovered'. This distinction does not imply, however, that beauty can be experienced *only* in the original discovery of the proof. For, in going through a proof presented to us, we may follow through and thus appreciate the creative thought processes that could have led to its discovery.

[28] It is the intimation of the ultimate ground of such harmony that, as Kant argues, has led Plato 'to the enthusiasm that elevated him beyond the concepts of experience to ideas' (*CJ*, V 363). For Kant, such enthusiasm is misguided insofar as it presupposes that we can have cognitive access to ideas that are in principle unknowable. Cf. also Kant's more appreciative mention of Plato's 'philosophical spirit', demonstrated by his 'wonderment' in the face of the great power of pure reason in geometry in the prize essay 'What Progress has metaphysics made in Germany since the time of Leibniz and Wolff?' (XX 324).

[29] I thus disagree with Wenzel's (2001: 421) suggestion that, on Kant's account, only empirical objects can arouse in us a sense of admiration. In the case of mathematics, it is precisely our reflection on objects that can be known purely a priori which elicits in us a subjective, aesthetic response.

[30] See, in particular, the 'Transcendental Aesthetic' (*CPR*, A 23/B 37 ff.) and the 'Discipline of Pure Reason' (*CPR*, A 708/B 736).

[31] See, for example, the discussions in Hintikka (1967), Parsons (1983) and Potter (2000: 53 f.).

[32] On the prevalence and importance of visual thinking in mathematics see Giaquinto (2007).

[33] For some striking examples that seem to fit this suggestion see Brown (1999).

[34] In this context, the growing discussion of mathematical practice is of particular interest. Cf., e.g., Bueno and Linnebo (2009: 137 ff.).

[35] Earlier versions of this paper were presented at Cambridge and St Andrews, and I thank the audiences for very helpful questions and suggestions. I would also like to thank, specifically, John Callanan, John Collins, Marina Frasca-Spada, Nick Jardine, Oskari Kuusela, Sasha Mudd, Davide Rizza and Alexander Rueger for their invaluable feedback on earlier drafts. Finally, I am very grateful to the anonymous referee for this journal whose careful and constructive criticisms have significantly helped to improve the paper.