Philosophia Mathematica

Introduction to Special Issue on Aesthetics in Mathematics

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1. Aesthetics in mathematics

Mathematicians often appreciate the beauty and elegance of particular theorems, proofs, and definitions, attaching importance not only to the truth but also to the aesthetic merit of their work. As Henri Poincaré (1930: 59) put it, mathematical beauty is a 'real aesthetic feeling that all true mathematicians recognise'. Others went further, regarding mathematical beauty as a key motivation driving the formulation of mathematical proofs and even as a criterion for choosing one proof over another. As Hermann Weyl famously and provocatively declared, 'My work always tried to unite the true with the beautiful, but when I had to choose one or the other, I usually chose the beautiful' (cited in Chandrasekhar 1987: 52).

Talk of the beauty of mathematical theorems, proofs, and definitions may thus be commonplace. And yet the tendency among mathematicians to judge mathematical work according to aesthetic standards raises a number of difficult questions:

(1) What is mathematical beauty? What, if anything, distinguishes it from other kinds of beauty? Is it a feature of abstract objects or grounded in sensible properties? Is it a genuine aesthetic category or can it be reduced to non-aesthetic, possibly epistemic, criteria?

(2) What is the status of aesthetic judgments in mathematics? Are they objective judgments grounded, for instance, in the mathematician's cognition of such properties as symmetry or simplicity? Do they rely on subjective responses

particular to individual mathematicians? Or are they, perhaps, grounded in other kinds of mental processes?

(3) Can aesthetic considerations play any legitimate role in mathematical or scientific theorising? Does the beauty of a proof stand in any non-contingent relation to its truth? And can any connection be drawn between the elegance of a mathematical formalism – the differential forms employed to express Maxwell's equations or the group theory used in quantum mechanics – and the truth of the scientific theory that contains the formalism?

(4) Does the phenomenon of aesthetics in mathematics reveal any important analogies between mathematical and artistic practice? In particular, what is the role of imagination in mathematics, and how does it compare to the role of imagination in the arts?

In the recent philosophical literature one finds only a handful of attempts to develop sustained answers to these questions. Thus, a number of authors in aesthetics and the philosophy of mathematics have tried to shed light on mathematical beauty by highlighting its relation to such factors as order, harmony, unity, symmetry and simplicity (see Osborne 1984, Engler 1990 and, more recently, Inglis and Aberdein 2014 whose careful analysis sheds doubt on the connection between beauty and simplicity). Others have argued that judgments about the beauty of mathematics are related to the understanding or enlightenment that the mathematics affords (see Rota 1997 and Cellucci 2015; cf. also Hardy's 1940 classic 'mathematician's apology'). Yet others have argued along Kantian lines that aesthetic judgments in mathematics are grounded in the spontaneous reasoning processes that lead to mathematical cognition (Breitenbach 2015; cf. Wenzel's 2001 more sceptical Kantian account). But there have also been critical voices, questioning whether explicit judgments about the beauty or elegance of mathematics have genuine aesthetic status and suggesting instead that they are only 'quasi-aesthetic' claims (see, in particular, Harré 1958, Zangwill 2001, and Todd 2008).[1]

Some philosophers of mathematics and theoreticians in the field of mathematics education have furthermore stressed the need to take seriously the aesthetic dimension of mathematical practices. Some have argued that, by analogy with art, aesthetic judgments play a major role in the development of mathematics research, for example, by determining which results to include in ongoing research programmes or research monographs (Tymoczko 1993). Others have moreover spoken of a 'generative aesthetic', which 'operates in the actual process of inquiry, in the discovery and invention of solutions' (Sinclair 2004: 270; see also McAllister 2005, Sinclair 2011, and Montano 2014 who, building on McAllister's work, provides the most extensive study to date of the role of aesthetics in shaping mathematical knowledge). Finally, related work, more specifically focussed on set theory, suggests that aesthetic value bears on, and may even serve as evidence for, the truth of mathematical statements (see Kennedy and Väänänen 2015).

While there is a small literature on the theme of this special issue, it has received much less attention than other topics in aesthetics or the philosophy of mathematics. We believe that this is due in part to the fact that the two philosophical sub-disciplines of aesthetics and the philosophy of mathematics are often perceived to lie at opposite ends of the philosophical spectrum and that interaction between philosophers specialising in these apparently distant fields has been sparse. Topics in aesthetics such as the nature of art, beauty and aesthetic experience simply seem to have little connection with such problems in philosophy of mathematics as the logical structure of formal arguments or the ontological status of abstract objects. And yet we believe that the phenomenon of aesthetics in mathematics and the

[1] Although the focus in these critical accounts is primarily on the aesthetics of science, the arguments can easily be extended to the aesthetics of mathematics.

pervasive appeal of aesthetic criteria to mathematicians raise questions that are of importance to core concerns in both areas, for example, about the relation of aesthetic judgment to cognition and about the nature of mathematical reasoning. We believe, moreover, that these questions are best answered by taking into account the insights and concerns of aestheticians as well as philosophers of mathematics. A real dialogue between specialists in the two fields provides the right context for a rigorous analysis of aesthetics in mathematics.

It was this conviction that motivated us to organise an international conference on the topic that brought together aestheticians, philosophers of mathematics and mathematicians at the University of East Anglia in December 2014. The papers included in this special issue are a selection from the many original and illuminating contributions that were presented at the conference and that, in a range of different ways, shed light on the different questions that arise for the aesthetics of mathematics.

2. The papers

Irina Starikova tackles question (1) in her paper 'Aesthetic Preferences in Mathematics: A Case Study'. She asks whether abstract mathematical objects can be genuinely beautiful and, if so, what features make them beautiful. Is their beauty solely a matter of their diagrammatic visualisation? Or does it have to do with their abstract mathematical properties? Starikova develops her answer by considering an illustrative example, the Petersen graph, a highly symmetric object whose properties have been extensively studied. It is possible to represent the Petersen graph set-theoretically as a set of vertices plus a set of edges connecting some of the vertices. This set-theoretical representation can in turn be rendered diagrammatically in a number of different ways. One of these diagrammatic renderings is often singled out by graph-theorists as distinctively beautiful. However, Starikova also observes that mathematicians record emotions characteristic of the experience of beauty not only in response to this specific visual representation of the Petersen graph but also in connection with the graph conceived independently of any diagrammatic representation. She explains this diversity in mathematicians' aesthetic appreciation in the following way. First, she draws attention to the fact that this graph has more symmetries than any representation can make visible. Thus, she suggests that the beauty of the Petersen graph considered as an abstract object has its source in this rich family of symmetries that connects the simplicity and regularity of the graph with a wide variety of other properties that are of interest to mathematicians. Second, she argues that mathematicians single out a particular diagrammatic representation of the graph as especially beautiful because of the comparative ease with which it enables the mathematician to grasp the aesthetically relevant graph-theoretical properties. On Starikova's account, intellectual beauty is thus a simple coordination of significant properties and fruitful consequences, which can be inherited by perspicuous visualisations. The beauty we find in mathematics is thus not simply a matter of pleasing visualisations but also, importantly, 'an aesthetics of the abstract'.

Manya Raman-Sundström and Lars-Daniel Öhman's paper 'Mathematical Fit: A Case Study' further contributes to answering question (1) by exploring the phenomenon of mathematical fit – a property that, they suggest, relates to the cognitive aspects of a proof as well as to its beauty. The authors develop their account of mathematical fit through a series of examples. Distinguishing three different, though possibly interdependent, types of fit, they analyse the features a mathematical proof needs to possess in order to be fitting in one or more ways. First, a proof has direct fit, or fits the theorem it proves, if it is stated in the same terms as the theorem (*coherence*) and uses a tool with the right level of technical power (*specificity*). Second, a proof has presentational fit if the underlying ideas are presented with the appropriate amount of detail (*level of detail*) and the proof makes clear its argumentative structure (*transparency*). Third, a proof has familial fit, that is, fits within a family of proofs, if the idea of the proof generalises to a larger class of theorems (*generality*) and connects to proof ideas of other theorems (*connectedness*). With this analysis in hand, Raman-Sundström and Öhman illustrate why some proofs are more fitting than others, for example, why Euclid's geometrical proof fits the Pythagorean Theorem better than a contemporary trigonometric proof. They conclude by suggesting that mathematical fit has a cognitive as well as an aesthetic dimension because of its close connection with mathematical explanation and mathematical beauty. Mathematical fit is related to mathematical explanations, since both are grounded in either the *coherence* or the *connectedness* of a proof; and mathematical fit is furthermore related to mathematical beauty, since both are characteristics of proofs with the right *level of detail*, *transparency*, *connectedness*, and even *specificity* and *generality*.

Cain Todd addresses questions (2) and (3) in his paper 'Fitting Feelings and Elegant Proofs: On the Psychology of Aesthetic Evaluation in Mathematics'. He does so by shedding light on the relation between aesthetic and epistemic criteria in mathematical reasoning. To this end, he examines the nature of the psychological experience that underpins mathematicians' aesthetic judgments about mathematical proofs and theorems. His core claim is that *bona fide* aesthetic judgments in mathematics are expressions of 'aesthetic-epistemic' feelings – feelings that serve a genuine epistemic function while also having aesthetic attributes. To substantiate this claim, Todd surveys the results of psychological research on epistemic feelings such as the feelings of knowing and understanding and of rightness and certainty. As he notes, the psychological research suggests that epistemic feelings play a genuine cognitive role in reliably indicating the accuracy of one's own mental performance. For example, experiencing fluency, that is, the felt ease with which a cognitive task is performed, plays a crucial role in endorsing mathematical reasoning that is simple to follow and rejecting mathematical proofs that are difficult to understand. Moreover, Todd argues that the same epistemic feelings also have an aesthetic character if they manifest what he calls 'cognitive consonance'. That is, epistemic feelings have aesthetic attributes if they represent the relation between our cognitive processes and the properties of the stimuli at which those processes are directed in a way that appears fitting, or in harmony. According to Todd, mathematicians' aesthetic judgments about theorems or proofs thus express epistemic-aesthetic feelings of fittingness.

Finally, Adam Rieger in his contribution 'The Beautiful Art of Mathematics' addresses questions (2) and (4). He argues for two distinct claims: first, the aesthetic vocabulary employed by mathematicians should be taken literally and, second, in certain respects, mathematical practice can be regarded as an art. Thus, supported by a range of examples, Rieger first argues that the typical object of aesthetic evaluation is the propositional content of a theorem or proof, thought of as a finite sequence of propositions. It follows from this, he suggests, that sensory properties are not necessary for aesthetic properties and that aesthetic evaluation in mathematics should not just be seen as a disguised form of epistemic evaluation, concerning, e.g., the fruitfulness of a proof method. Aesthetic judgments can be genuine, he claims, even if they are about the propositional content of a mathematical theorem or proof. Rieger then advances his second claim by highlighting salient traits that are shared by artistic and mathematical practice, notably their common attempts at telling us how things are in an aesthetically valuable way, their common concern with a selection of leading motifs and themes, as well as their organisation and composition. Rieger concludes on the basis of this analogy that some parts of mathematics can thus be regarded as an art.

3. Further questions

Each paper collected in this special issue offers a specific way of situating aesthetic considerations within mathematics. We also find each paper suggestive, explicitly or implicitly, of further goals and avenues for enquiry.

Thus, Starikova focusses on the notion of intellectual beauty in mathematics, which she likens to 'a power, causing pleasure to mathematicians while they are intellectually engaged with the mathematical entity'. Her account suggests that, to understand what intellectual beauty is, we need to understand how the properties of abstract mathematical entities trigger an aesthetic response in the subject. Starikova's paper prompts us to think further not only about the objective character of beautiful mathematical objects but also about the subject's intellectual engagement with the mathematical entity that is required for the relevant pleasurable response to occur. Does such intellectual engagement consist in the process of understanding, for example, of what a particular graph entails and what consequences it has? Or is the subject's intellectual engagement with the mathematical entity comparable to a form of perception, for instance, the perception of a graph's visual representation, which is involved in experiencing the graph's perceptual beauty? And how, more generally, does such intellectual engagement compare to the cognitive processes involved in the aesthetic appreciation of artworks? In raising these questions, Starikova's discussion furthermore points to an interesting link of the aesthetics of mathematics with the visual aspects of mathematical thinking and the epistemic benefits thereof. We believe that this link may well be mobilised in future studies of the relationship between aesthetics and mathematics.[2]

We furthermore find two interesting but very different approaches to the concept of fit, or fittingness, discussed in Raman-Sundström and Öhman's paper and in Todd's contribution. Raman-Sundström and Öhman focus on the properties that make mathematical proofs more or less fitting with the proven theorem. Todd, by contrast, is interested in the feeling of fittingness, where such fittingness is understood as a relation between our cognitive processes and the objects at which such processes are directed. One might wonder whether there is any

[2] In this context, it will be worth paying specific attention to the different but related literature on the role of visualisation and diagrammatic reasoning in mathematics (see, e.g., Giaquinto 2007 and De Toffoli and Giardino 2014).

significant relation between the feeling of fittingness Todd describes and the phenomenon of mathematical fit that concerns Raman-Sundström and Öhman. One might ask, in other words, whether there is any non-contingent connection between the objective relation of fit holding between different mathematical entities and the feeling of fittingness that expresses features of the subject's psychology. Is it reasonable to expect that being aware of the phenomenon of mathematical fit will be correlated with the subjective feeling of fittingness? If it could be shown, for example, that our cognitive processes appear to be in harmony with the stimuli at which they are directed whenever they follow through a proof that fits with the proven theorem, might we be able to say more about why fitting proofs appear beautiful to us? These questions remain rather speculative. But answers to them might shed further light on the relation between the features of beautiful mathematical entities and the response of the subject that is involved in aesthetic appreciation.

Finally, Adam Rieger begins to outline an account of the artistic dimension of mathematical practice, thus pointing to a type of investigation that is almost absent in the existing literature and deserves to be pursued further. His proposal calls for an exploration of the possible points of contact between the intellectual processes involved in artistic and mathematical construction, which might contribute to identifying similarities between the creative effort characteristic of each activity. One might ask, for example, how such intellectual processes relate to the cognitive processes of reasoning, understanding, knowing; and one might wonder in what way, if any, they represent creative or imaginative activities. We believe that a fuller elaboration on these questions would shed important light on the intellectual contributions to the creation of art as well as mathematics.

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