"Quantumness" versus "Classicality" of Quantum States and Quantum Protocols

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Entanglement is one of the pillars of quantum mechanics and quantum information processing, and as a result the *quantumness* of non-entangled states has typically been overlooked and unrecognised until the last decade. We give a robust definition for the classicality versus quantumness of a single multipartite quantum state, a set of states, and a protocol using quantum states. We show a variety of non-entangled (separable) states that exhibit interesting quantum properties, and we explore the "zoo" of separable states; several interesting subclasses are defined based on the diagonalizing bases of the states, and their non-classical behavior is investigated.

1. Introduction

The topic of this paper is the *quantumness* of single party and multipartite quantum states, ensembles of quantum states, and quantum protocols. The core (Secs. 1-6) was written in 2007 and made available as a pre-print on the arXiv[1] but remained unpublished until now.

Consider an isolated discrete classical system with N distinguishable states. The most general state of the classical system is a probabilistic distribution over these distinguishable states. Now consider its counterpart, an isolated discrete *quantum* system. Its most general state is a probabilistic mixture of pure states drawn from an N-dimensional Hilbert space. Yet, in various special cases, the quantum state seems to be identical to a classical probability distribution. Similarly, in various special cases, a quantum protocol using a set of quantum states seems to be practically identical to a classical protocol which is using a classical set of states. Our first goal is to define such special quantum states that are equivalent to classical probability distributions; we also define sets of classical states and classical protocols.

Quantumness of states (for instance, their "quantum correlations") is often associated with their entanglement, and it is sometimes even assumed (explicitly or implicitly) that non-entangled states can be considered "classical". We argue that this is not the case, because some (actually, most) non-entangled states do exhibit non-classical features. Intuitively speaking, only quantum states that correspond *exactly* to a classical probability distribution can potentially be considered classical; most non-entangled states can only be written as a probability distribution over tensor-product quantum states, e.g., for bipartite systems $\rho_{\rm sep} = \sum_i p_i |\phi_i\rangle_A |\psi_i\rangle_B \langle\phi_i|_A \langle\psi_i|_B$, hence do not usually resemble any conventional distribution over classical states. While entanglement is extensively analyzed and quantified (see,[2,3] and references therein), the "quantumness" of non-entangled (separable) states has often been overlooked until recently.[4] Our second goal is to present the quantumness exhibited by various separable states, and to explore the "zoo of separable states". Our last goal is to define (and make use of) measures of quantumness $\mathcal{Q}(\rho)$ that vanish on any classical state $\rho_{\rm classical}$.

The structure of the rest of the article is as follows. In Sec. 2 we provide definitions of classical bases, states and protocols. Sec. 3 we give various examples of separable states that do not fit our definition of classicality and emphasizes quantum aspects of these states. In Sec. 4 we explore different types (a "zoo") of *separable states* using definitions from Sec. 2. In Sec. 5 we discuss convertibility of states between classes under local operations without discarding subsystems. In Sec. 6 we present some candidates for a measure of the quantumness of states. In Sec. 7 we prove that the proposed measures are monotonic under certain class of operations. Appendix A is our original 2007 summary, whereas Sec. 8 summarises our results obtained shortly after the original article had become available as a pre-print. Sec. 9 discusses work by other groups since 2007. Section 10 addresses the question relative 'quantumness'.

2. Classicality of Quantum States and Quantum Protocols

If a quantum state or a quantum protocol has an exact classical equivalent, it cannot present any interesting non-classical properties nor any advantage over its analogous classical counterpart. The state(s) of the quantum system can then potentially be considered "classical". For instance, if a single quantum system is prepared in one of the orthogonal states $|0\rangle$, $|1\rangle$, $|2\rangle$, etc., and is then measured in this computational basis, there is nothing genuinely quantum in that process. Tensor product states of multipartite system can also be considered classical. Consider a set of states in the computational basis, e.g., $\{|00\rangle; |01\rangle; |10\rangle; |11\rangle\}$; this set has a strict classical analogue — the classical states $\{00; 01; 10; 11\}$. As long as no other quantum states are added to the set (or appear in a protocol which is using these states), the analogy is kept, so these quantum states can be considered classical. Tensor product states such as $|-\rangle|0\rangle|+\rangle$ (where $|\pm\rangle = [|0\rangle \pm |1\rangle]/\sqrt{2}$) can also be considered classical as we soon explain.

First, we define classical bases. We justify our claim that any such basis presents no quantumness, and we justify (via many examples) why bases that do not follow our "classicality" definition are "quantum".

We start with a single system and then move to bipartite and multipartite systems:

Definition 1. Let A be a quantum system. Any orthonormal basis $\{|i\rangle_A\}$ of A can be considered as a *classical basis* of the system.

For example, the computational basis $\{|0\rangle; |1\rangle\}$ of a single qubit is obviously classical. The Hadamard basis $\{|+\rangle; |-\rangle\}$ is also classical.

One may argue that our definition is too flexible and that Nature allows only one basis to be classical [a]. For instance an alternative for Def. 1 is

> Let A be a quantum system with a single *preferred* orthonormal basis $\{|i\rangle_A\}$, in the sense that measurements can only be performed in this basis. Only this basis can be considered as a *classical basis* of the system.

While this narrower definition is valuable for some physical scenarios, there is nothing in conventional quantum theory that favors one of the system's bases over any other. In *relativistic quantum field theory* it is commonly believed that Nature generally provides a preferred basis, however, on time-scales that are sufficiently short for performing quantum computation, all bases are equivalent. We therefore adopt the more general Def. 1.

We now move to defining classical bases for bipartite and multipartite systems.

Definition 2. Let A and B be two single party quantum subsystems with orthonormal bases $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$ respectively. The tensor-product basis $\{|i\rangle_A \otimes |j\rangle_B\}$ is a *classical basis* of the bipartite system AB.

Definition 3. (recursive) Let A be a (bipartite or multipartite) quantum subsystem with a *classical* basis $\{|i\rangle_A\}$, and let B be a single party quantum subsystem with an orthonormal basis $\{|j\rangle_B\}$. The tensor-product basis $\{|i\rangle_A \otimes |j\rangle_B\}$ is a *classical basis* of the composite AB system.

The redundancy in Defs. 2–3 is kept for readability.

Let us see a few examples. For two qubits, the computational basis is classical, as well as the basis $\{|++\rangle; |+-\rangle; |-+\rangle; |--\rangle\}$. On the other hand, the Bell basis $\{|\Phi_\pm\rangle; |\Psi_\pm\rangle\}$ is obviously non-classical, and more interestingly, even the basis $\{|00\rangle; |01\rangle; |1+\rangle; |1-\rangle\}$ is non-classical.

Having identified classical bases, we proceed to define a classical state and a set of classical states.

[a] This approach is essential for the theory of *coherence* which is currently attracting a lot of interest.[32]

Definition 4. A state ρ is a classical state, iff there exists a classical basis $\{|v_i\rangle\}$ in which ρ is diagonal.

Following our definition, any single state ρ (either pure or mixed) of a single system S *can always be considered classical*. A joint state of two or more quantum systems can also either be pure or mixed. If it is pure it is either a tensor product state or an entangled state. Following the classicality definitions, any such tensor-product state is classical while any such entangled state is non-classical. For mixed bipartite or multipartite states the situation is much more complicated: Tensor-product mixed states are obviously still classical as each subsystem can be diagonalized in a classical basis of its own. Entangled mixed states are obviously non-classical. Between these two extremes we can find a zoo of separable—yet quantum—states.

We made this definition independently of a similar definition in Refs. 5 and 6 (see Sec. 8); they use the name "(properly) classically correlated states" which is more precise, yet longer, than our term "classical states".

Prior to dealing with separable quantum states we provide two additional useful definitions.

Definition 5. A set of states $\rho_1 \dots \rho_k$ is a classical set iff all ρ_i are diagonalizable in a single classical basis.

If a quantum protocol (be it computational, cryptographic, or any other physical process) is limited to a classical set of states, the process has an exact classical equivalent, and cannot present any advantage over an analogous classical protocol. More formally:

Definition 6. A protocol (in quantum information processing) is *classical* iff all states involved in it belong to a single classical set of states.

One extremely simple example of a non-classical protocol is when Alice sends Bob a single qubit in the computational basis, and Bob applies a Hadamard transform and then measures the qubit in the computational basis. Another similar example is when Alice sends to Bob a single qubit in the computational basis, and Bob applies a Hadamard transform and then measures it in the Hadamard basis.

If a protocol involves two or more pure nonorthogonal states it cannot be considered classical (see Ref. 7 for a thorough analysis of the quantumness of protocols involving only pure states). Yet following our definitions, even protocols involving only pure orthogonal product-states might be highly quantum; and similarly, even a single bipartite mixed separable state can be highly non-classical.

3. Non-classicality of Separable States: Examples

Let us prove the quantumness of several interesting separable states.

3.1. *Pseudo-pure states*

A state of the form $\epsilon|\psi\rangle\langle\psi| + \frac{1-\epsilon}{N}I$ with $\epsilon > 0$ is called a *pseudo-pure state* (PPS) as the part with the coefficient ϵ transforms as if the state was a pure state. PPSs focus wide interest based on theoretical and experimental grounds. It has been shown[8] that there is a volume of *separable* PPSs around the totally-mixed state I/N; So every PPS (of a multipartite system) with low-enough ϵ is separable. This fact was even used to argue that experiments which produce such low-ϵ states (as in room temperature liquid state NMR) are not truly quantum. It was later argued, however, that albeit being separable, these states do exhibit non-classical effects.[9] Using our definitions we see that:

Proposition 1. *A PPS* $\rho_\epsilon = \epsilon\rho + \frac{1-\epsilon}{N}I$ *is quantum iff* ρ *is, for any* $\epsilon > 0$.

Proof. Any diagonalizing basis of ρ_ϵ also diagonalizes ρ, independently of ϵ. Since ρ is quantum, it is not diagonalizable in a classical basis, and so is ρ_ϵ. □

This is true for any system dimension. As a special case for $N = 4$, a separable Werner state[10] $\chi = \epsilon|\Psi_-\rangle\langle\Psi_-| + \frac{1-\epsilon}{4}I$ is non-classical for any $0 < \epsilon \le \frac{1}{3}$ (see also Ref. 11 for a different demonstration of non-classicality of the Werner states). Note that the Werner state is also separable and non-classical for any $-\frac{1}{3} \le \epsilon < 0$.

3.2. *States used for quantum key distribution*

The original quantum key distribution protocol, BB84 involves qubits in four different states: $|0\rangle$, $|1\rangle$, $|+\rangle$, and $|-\rangle$, sent from Alice to Bob. The protocol may also be described in a less conventional manner,[12] where Alice sends in two steps either the state $\rho_{0_{(\mathrm{BB84})}} = \frac{1}{2}\left[|00\rangle\langle 00| + |1+\rangle\langle 1+|\right]$ to represent '0' or $\rho_{1_{(\mathrm{BB84})}} = \frac{1}{2}\left[|01\rangle\langle 01| + |1-\rangle\langle 1-|\right]$ to represent '1'; the right-hand-qubit is sent first and the left-hand-qubit is sent later on in order to reveal the basis of the first qubit.

Proposition 2. *Neither* $\rho_{0_{(\mathrm{BB84})}}$ *nor* $\rho_{1_{(\mathrm{BB84})}}$ *is 'classical'.*

Proof. Any diagonalizing product basis of $\rho_{0_{(\mathrm{BB84})}}$ includes $|0\rangle_A \otimes |0\rangle_B$ and $|1\rangle_A \otimes |+\rangle_B$. That basis cannot be classical, as Bob's parts, $|0\rangle_B$ and $|+\rangle_B$, are not orthogonal and hence cannot be members of a single classical basis. The same reasoning applies to $\rho_{1_{(\mathrm{BB84})}}$, too. □

Thus, although all the four states involved in the protocol $|00\rangle$, $|1+\rangle$, etc. are mutually orthogonal tensor-product states, the protocol is highly "quantum".

3.3. *States that present nonlocality without entanglement*

Various sets of states proposed in Refs. 13, 14 define processes that exhibit non-local quantum behavior although none of the participating states is entangled. In particular, spatially separated parties cannot reliably distinguish between different members of the set (albeit comprising of mutually orthogonal direct product states!) without assistance of entanglement. For instance, the set $\{|01+\rangle; |1+0\rangle; |+01\rangle; |---\rangle\}$ is non-classical.

3.4. *The Bernstein-Vazirani Algorithm*

The Bernstein-Vazirani algorithm[15] generates no entanglement (see Ref. 16). However, it is clearly a quantum algorithm, with no classical equivalent. It makes use of states from the computational *and* Hadamard bases, which are not simultaneously diagonalizable in a single classical basis.

4. A Zoo of Separable States

Within the set of all separable states we identify some interesting subsets based on their diagonalizing bases.

First let us consider the classical states:

Definition 7. *Classic* is the set of the states diagonalized in a classical basis: A bipartite state (this argument easily extends to multipartite states) is classical if, and only if, Alice and Bob can perform a measurement in its (classical) diagonalizing basis via local orthogonal measurements, without exchanging any message (classical or quantum), and such a measurement can be performed without disturbing the state.

The notion of diagonalizing basis is now used to define more subsets of the separable states. Ref. 14 defines a *complete product basis* (CPB) as follows: A CPB is a complete orthonormal basis of a multipartite Hilbert space, where each basis element is a (tensor) product state. We define the set of CPB-states as follows:

Definition 8. A state ρ is a CPB-state iff it is diagonalizable in a CPB.

Clearly, all classical states are CPB-states; but not vice versa. Thus, in a multipartite finite-dimensional Hilbert space $Classic \subset CPB \subset Sep \subset \mathcal{H}_{\text{total}}$. For example, $\rho_{0_{(\text{BB84})}}$ and $\rho_{1_{(\text{BB84})}}$ are non-classical CPB-states diagonalized in the CPB $\{\rho_{00}; \rho_{01}; \rho_{1+}; \rho_{1-}\}$. For additional examples of CPB sets of states see Sec. 5.

Let V be an orthonormal basis of a subspace of a multipartite Hilbert space $\mathcal{H}$, where each basis element is a (tensor) product state. Ref. 14 defines that V is an *unextendible product basis* (UPB) if the subspace $\mathcal{H} - \text{span}\{V\}$ contains no product state. We define the set of UPB-states as follows:

Definition 9. A separable state ρ is a UPB-state if it can be diagonalized in a UPB and its kernel is spanned by a basis that contains no product states.

Note that a UPB-state (built from UPB elements suggested in Ref. 14) such as $\rho_{\epsilon UPB} = (1-6\epsilon)\rho_{01-} + \epsilon\rho_{1-0} + 2\epsilon\rho_{-01} + 3\epsilon\rho_{---}$ (when $\epsilon \neq 0$ and $\epsilon \neq 1/6$), demonstrates that there are UPB-states that are not in CPB. The reason is that it has a unique diagonalization in that relevant subspace. (Note also that with $\epsilon \to 0$, this state is infinitesimally close to a classical state.) In fact, the corollary of definitions 8 and 9 is that UPB and CPB are disjoint sets. A CPB-state cannot be a UPB-state and vice versa. As an example consider states built from a probability distribution over the eight CPB states[13] $\{|01\pm\rangle; |1\pm0\rangle; |\pm01\rangle; |000\rangle; |111\rangle\}$. A CPB-state does no have to span the entire CPB, i.e. a state formed from $\{|01-\rangle; |1-0\rangle; |-01\rangle\}$ is a CPB-state, but *not* a UPB-state.

We identified another class of UPB-states that can be proven to be non-classical:

Proposition 3. *The uniform mixture of UPB elements* $\rho_{\rm UPB} = (\rho_{01+} + \rho_{1+0} + \rho_{+01} + \rho_{---})/4$ *is non-classical.*

Proof. Assume that $\rho_{\rm UPB}$ is classical. The same classical basis that diagonalizes it, also diagonalizes the state $I/4 - \rho_{\rm UPB}$. However, this contradicts the fact that it is *bound-entangled*[14] and therefore quantum. □

More generally, we argue that since UPB and CPB are disjoint, all UPB states are non-classical.

The last set we define is the set of states with an entangled basis which we call EB:

Definition 10. A state ρ is a EB state if it cannot be diagonalized in any product basis.

As we had already seen, many separable states belong to this EB set, e.g., various PPS and Werner states. Obviously, all non-separable states also belong to this set.

5. Convertibility into a Classical State

Classification of multipartite quantum states is usually made in the framework of allowed types of communication between the parties. The traditional *local operations and classical communication* (LOCC) is relevant for distinguishing the class of entangled states from separable states because entanglement is a LOCC monotone, i.e. LOCC cannot map states from the *Sep* subset to the entangled subset - the amount of entanglement in a state can be only reduced by LOCC. However, LOCC is a too broad class of operations to be useful for classification inside *Sep*, because general unrestricted LOCC can generate any state inside *Sep*.

Classical messages exchanged between the parties and/or classical results of local measurements can be interpreted as kept in some classical registers - either systems with classical degrees of freedom or encoded in classical sets of states of quantum systems. If these registers are allowed to be discarded, i.e. corresponding degrees of freedom traced out, then the information encoded in them will be lost. The ability to discard subsystems is the core ingredient of LOCC, that allows it to generate *any* state in *Sep*.

Note that local operations and unidirectional classical communication, but even *without* adding the ability to discard subsystems, are sufficient for converting the BB84 states into classical states. A slightly more complicated (qubit plus qutrit) state, $\rho = \frac{1}{3}[|00\rangle\langle 00| + |1+\rangle\langle 1+| + |+2\rangle\langle +2|]$ requires local operations (again, without discarding subsystems) and bidirectional classical communication in order for it to be converted into a classical state. We call these two types of CPB-states "unidirectional CPB-states" and "multi-directional CPB-states" respectively.

Interestingly, there are CPB-states that belong to neither subset: consider a state built from a probability distribution over *all* the eight states[13] $\{|01\pm\rangle; |1\pm 0\rangle; |\pm 01\rangle; |000\rangle; |111\rangle\}$; although it is a CPB-state, such a state cannot be converted into a classical states unless quantum communication is allowed. Thus, we specify also a third subset of the CPB states — "Q-convertible CPB-states".

6. Measures of quantumness:

A measure of non-classicality (quantumness), $\mathcal{Q}(\rho)$, of a state ρ has to satisfy two conditions; (a) $\mathcal{Q}(\rho) = 0$ iff ρ is classical, (b) $\mathcal{Q}(\rho)$ is invariant under local unitary operations. One might also expect a third condition; (c) $\mathcal{Q}(\rho)$ is monotonic under local operations (without classical communication) [b]; Yet, condition (c) is not always satisfied by quantum states: The classical state $\frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|13\rangle\langle 13|$ of a 2×4 system can be converted to the non-classical CPB-state $\rho_{0_{(\mathrm{BB84})}}$ just by discarding a subsystem—Bob redefines his qu-quadrit as two qubits with $|0\rangle_{\mathrm{quad}} = |00\rangle$ and $|3\rangle_{\mathrm{quad}} = |1+\rangle$, and discards his first qubit. Thus, quantumness is not a LO (local operation) monotone. It is natural to conjecture that quantumness is a monotone under LO (without CC) without the ability to discard subsystems (see Sec. 7 for discussion of this conjecture).

A class of measures of quantumness of ρ is defined as

$$\mathcal{Q}_D(\rho) = \min_{\rho_c} D(\rho, \rho_c) \tag{1}$$

where D is any measure of distance between two states such that the conditions (a)-(b) are satisfied, and the minimum is taken over all classical states ρ_c. One of the natural candidates for D is the relative entropy $S(\rho\|\rho_c) = \mathrm{tr}\,\rho\log\rho - \mathrm{tr}\,\rho\log\rho_c$, in which case we refer to it as $\mathcal{Q}_{\mathrm{rel}}(\rho)$ — the *relative entropy of quantumness.* The benefit of using the relative entropy as a measure is that it was extensively studied for measuring entanglement[2] (relative to the closest separable state). Thus, we can adopt and make use of some known results, and we can also monitor the connection between the quantumness of states and their entanglement. There are other measures (and their variants) that can potentially be very useful for studying quantumness, such as the *fidelity of quantumness* and *Von Neumann mutual information.*

[b] These conditions resemble the line of thought used in searching for the measure of entanglement.[2]

For bipartite pure states, the relative entropy of quantumness equals its entropy of entanglement. In other words, a pure state is as quantum as it is entangled. Any bipartite entangled state $|\Psi\rangle$ can be written in a Schmidt decomposition $|\Psi\rangle = \sum_{i=1}^{d} c_i |ii\rangle_{AB}$, where $c_i \geq 0$ and $d = \min[d_A, d_B]$, d_A, d_B are dimensions of local Hilbert spaces. If we use the relative entropy of entanglement then the closest separable state[2] is

$$\sigma_{cl} = \sum_{i=1}^{d} (c_i)^2 |ii\rangle\langle ii|_{AB}. \tag{2}$$

This state happens to be also classical, and thus the relative entropy of quantumness (which is equal to its relative entropy of entanglement) is $\mathcal{Q}_{\rm rel}(\Psi) = -\sum_i (c_i)^2 \log[(c_i)^2]$. [The classical state σ_{cl} lies on entangled-separable boundary.] Note that the quantumness of a maximally entangled state is $\mathcal{Q}_{\rm rel}(\Psi_{ME}) = \log d$.

Let us present some mixed states for which their quantumness can easily be calculated: According to [2, Th. 4], σ_{cl} is the separable state that minimizes $S(\rho_p \| \sigma_{cl})$ for any state of the form $\rho_p = p\ |\Psi\rangle\langle\Psi| + (1-p)\sigma_{cl}$, too. Therefore, the relative entropy of entanglement of ρ_p equals to its relative entropy of quantumness.

Given any bipartite state ρ_{AB}, let its *Schmidt basis* be the (classical) basis diagonalizing $\mathrm{tr}\,_B\rho_{AB} \otimes \mathrm{tr}\,_A\rho_{AB}$. Let $\rho_{\rm Sch}$ be produced from ρ_{AB} by writing it in its Schmidt basis and having all off-diagonal elements zeroed. The state ρ_{AB} and its Schmidt state yield identical classical correlations if measured in the Schmidt basis.

The Schmidt state can be found very useful for defining quantumness for any state ρ_{AB}, as ρ_c is usually unknown; instead of using Eq. (1) as a measure, one can directly refer to the distance between a state ρ_{AB} and its corresponding Schmidt state:

$$\mathcal{Q}_D(\rho) = D(\rho_{AB}, \rho_{\rm Sch}) \tag{3}$$

as a measure of quantumness of a state. If we now use the relative entropy, the resulting measure satisfies conditions (a) and (b).

We saw above, that for a pure bipartite state the Schmidt state $\rho_{\rm Sch} = \sigma_{cl}$ is the closest classical state. One might conjecture that for any bipartite state ρ, the closest classical state (using relative entropy measure) is its Schmidt-state $\rho_{\rm Sch}$. This however, is not true. For instance, we checked the CPB-state $\rho_{0_{\rm (BB84)}}$ which is useful in quantum key distribution; it is interesting to note that either the classical state $\frac{1}{2}\rho_{00} + \frac{1}{4}\rho_{10} + \frac{1}{4}\rho_{11}$ or the classical state $\frac{1}{4}\rho_{0+} + \frac{1}{4}\rho_{0-} + \frac{1}{2}\rho_{1+}$, are actually closer to $\rho_{0_{\rm (BB84)}}$ than its Schmidt state — a state diagonal in the classical basis (known as the Breidbart basis) $\{\rho_{0b_0}; \rho_{0b_1}; \rho_{1b_0}; \rho_{1b_1}\}$ (where $|b_0\rangle = \cos\frac{\pi}{8}|0\rangle - \sin\frac{\pi}{8}|1\rangle$,$|b_1\rangle = \sin\frac{\pi}{8}|0\rangle + \cos\frac{\pi}{8}|1\rangle$). We verified numerically that the above two states are the closest ones (among all classical states) to $\rho_{0_{\rm (BB84)}}$, hence can be used for calculating its relative entropy of quantumness. The entropy of quantumness relative to the Schmidt state is different in this case of course.

7. The monotonicity of Q_{rel}

In Sec. 6 we conjectured that our measure of quantumness is monotonic under local operations without discarding subsystems. Here we formalize that statement and prove it for the case of Q_{rel} and other possible measures where the distance D has the following additional properties.

- D is invariant under unitary operations: $D(\rho,\tau) = D(U\rho U^\dagger, U\tau U^\dagger)$
- D is additive: $D(\rho\otimes\varrho, \tau\otimes\sigma) = D(\rho,\tau) + D(\varrho,\sigma)$

We now consider measures of quantumness Q that use a distance measure D which satisfies the conditions above, e.g Q_{rel}. Note that these types of quantities are not monogamous.[30]

Let $\mathcal{L}$ be the class of completely positive trace preserving (CPTP) maps (i.e. quantum operations) that can be decomposed into adding a subsystem locally and applying a local unitary. From the conditions on D it is invariant under all $L \in \mathcal{L}$, i.e $D(\rho,\tau) = D[L(\rho), L(\tau)]$.

Proposition 4. *If $D(\rho,\tau)$ is invariant under unitary operations and additive then $Q(\rho)$ is non-increasing under $\mathcal{L}$.*

Proof. We start with a few simple observations.

(1) ρ is classical in a basis $\mathbf{b}$ if and only if it is a fixed point of the dephasing operation in the basis $\mathbf{b}$.
(2) If ρ is classical, so is $U\rho U^\dagger$ - this follows from 1 above.
(3) $Q(\rho) = Q(U\rho U^\dagger)$ for all unitary operations - This is a simple consequence of 2 above and the fact that D is invariant under unitary operations.
(4) $Q(\rho) \geq Q(\rho\otimes\tau)$ for all local states τ - From 1 above we see that this is true for $\rho \in \mathit{Classic}$. From the fact that D is invariant under the addition of a local subsystem we have some classical state ρ_c such that $Q(\rho) = D(\rho,\rho_c) = D(\rho\otimes\tau, \rho_c\otimes\tau) \geq Q(\rho\otimes\tau)$.

Since Q is invariant under unitary operations and it is non-increasing under the addition of local subsystems then it is non-increasing under any composition of these two operations. □

However we note that it is unclear if Q is really monotonically decreasing or simply invariant under $\mathcal{L}$. We do, however know that there are quantum CPB-states that cannot be converted to *Classic* under $\mathcal{L}$. Consider for example $\rho_{0_{(\mathrm{BB84})}}$. Transforming it to a *Classic* using $\mathcal{L}$ will allow unambiguous discrimination between the non-orthogonal states $|0\rangle$ and $|+\rangle$. Furthermore when $D(\rho,\tau)$ is a difference of mutual information it is already known that $Q(\rho)$ is invariant under any reversible local operation.[28]

8. Comparison with previous work

In this section we compare our results with previous works on quantumness and classicality of states which predate our original publication.[1] Our Definition 4 of a classical state is equivalent to *classically correlated states* defined in [5, Eq. (7)]. Our classical states are also equivalent to *locally diagonalizable states* in [17, exercise 15.6, p. 413]. In Ref. 5 the *work deficit* was introduced and it was proposed to be a suitable measure of quantumness of correlations in a multipartite state. The work deficit indeed nullifies on classical states, however it fails to fulfill our requirements from a measure of quantumness, as it can be zero also for some non-classical CPB-states, e.g. ρ_{BB84}. Thus, the work deficit does not distinguish between classical and non-classical states as we define them here. The main motivation of Ref. 5 was to answer the question of how much physical work can be drawn from given multipartite quantum state under restricted class of operations, LOCC. The main motivation of our current work is to sub-classify separable states according to their algebraic description and then support this classification using operational reasoning.

Ref. 11 defines the *quantum discord* $\delta(A : B)_{\{\Pi_j^B\}}$ between one part of a bipartite system to the other. It is a discrepancy between mutual information(s) calculated according to two different, but classically equivalent, expressions. When its minimum (taken over the measurement basis of B) is nonzero, it means that B cannot recover locally all the correlation within the bipartite state. We suggest a symmetrized version of the (minimal) quantum discord

$$\delta(A,B) = \min_{\{\Pi_i^A\},\{\Pi_j^B\}} \left(\delta(B : A)_{\{\Pi_i^A\}} + \delta(A : B)_{\{\Pi_j^B\}} \right) .$$

When $\delta(A,B) = 0$, *each* subsystem can locally recover the correlation between the parts. In such case, the joint state is classical according to our definition, where its classical diagonalizing basis is that defined by A and B's optimal projection operators. Conversely, a bipartite classical state has $\delta(A,B) = 0$. Therefore, $\delta(A,B) = 0$ is equivalent to the classicality of ρ_{AB}. However, $\delta(A,B)$ is limited to bipartite states, while our definition easily extends to multipartite systems.

In Ref. 18, the quantumness of correlations of an *ensemble* of states was characterized by the *minimal entropy produced* when measured in LOCC-distinguishable basis. An orthonormal basis that is distinguishable under LOCC, is not necessarily classical in the sense of Def. 5. For example, some non-classical complete product basis (CPB) can be distinguished under LOCC and an ensemble made of corresponding CPB-states, e.g. $\{\rho_{0_{(\mathrm{BB84})}}, \rho_{1_{(\mathrm{BB84})}}\}$, has zero minimum entropy production, though it is non-classical according to our definition. Although the authors do suggest the possibility of generalizing their approach to any set of allowed operations (denoted by Λ), the arguments of the paper are based on LOCC. When considering an "ensemble" of a single (mixed) state, it seems that their measure of quantum correlation $\mathcal{Q}_\Lambda(\{\rho\})$ coincides with our $\mathcal{Q}_{\mathrm{rel}}(\rho)$ if only Λ is the set of local operations.

Ref. 6 extensively uses the *quantum information deficit* measure of quantumness, and the relative entropy of quantumness (which we use independently). Sec. 5 in Ref. 6 provides very interesting subclasses — yet, different from ours — of the separable states. Their "pseudo-classically correlated states" seem to be identical to our uni- and multidirectional CPB-states and their class of "informationally nonlocal" states seems to be identical to our two subclasses — the UPB-states and the Q-convertible CPB-states.

As we already mentioned in Introduction, the main difference between our approach and those described above is that they start from defining certain operational aspect of quantumness and then test different states on how they fit into that aspect. We go in the opposite direction: we look at the mathematical structure of a state, define different classes accordingly and only then check the practical/operational implications of our classification.

9. Developments since 2007

Many of the ideas presented in our original work were independently reported and developed further in the years since Ref. 1, mostly in the context of quantum discord[11] (for a review including applications quantum information protocols see Ref. 4). Modi et al.[19] adopted and further developed a method similar to the one suggested in sec 6, motivated by the relation of quantum correlations to the relative entropy of entanglement.[2] Their set of classical states corresponds to CPB states. This was taken a step further in Ref. 20 for correlation measures based on more general 'distance' functions. However in that work the set of classical states emerges from a set of local measurements, i.e the approach is more 'operational'. The *fidelity of quantumness* first suggested in Ref. 1 (here Sec. 6) is related to the *geometric discord*,[20,21] while the *Von Neumann mutual information* is related to *symmetric discord* introduced in Ref. 19.

The measures of quantumness based on the Schmidt basis (see Eq. 3) were explored in Ref. 22 where they were given the name *measurement induced disturbance* (although they were considered even earlier in Ref. 23). This was further developed in the context of relative entropy[19] and later in a more general framework in Ref. 20 where it was shown that they are not continuous. The one sided version was introduced in the context of discord in Ref. 26, and more recently explored in much detail in Refs. 34, 35. The all the entropic versions of the quantities presented have been related to thermodynamics in Refs. 20, 26, 31.

Work related to quantum ensembles (Def. 5) was reported in Ref. 24 where the quantumness of ensembles was related to data hiding. A different approach was used by Ref. 25 where an ensemble $\{\rho_1 ... \rho_k\}$ was defined to be classical when the state $\frac{1}{k}|i\rangle\langle i| \otimes \rho_i$ is classical (with $\{|i\rangle\}$ representing orthogonal states). Further examples of quantum ensembles and their relations to quantum states were given in Refs. 26, 27, however these did not include a formal definition for quantum ensembles.

To the best of our knowledge there have been no definitions of classical protocols in the spirit of Def. 6, although much work has been done in this direction Ref. 4[Sec.

5-6]. Similarly definitions of subclasses of quantum states similar to 8 and 9 and the associated Hierarchy of states have not been reported elsewhere.

Proposition 1 was recently used in Ref. 29 to prove general results about quantum correlations in certain types of physical systems related to quantum computing.

10. Relative quantumness

Resource theories[33] are currently attracting a lot of attention as useful tools for understanding the interrelation between states and operations. Roughly speaking the theories define a set of *free operations* and a corresponding set of *free states* that remains invarient under these operations. That is, the set of free operations converts free states to free states. Resource states are those that do not belong to the set of free states and one can use similar ideas to those presented above to quantify the amount of resources using a function which is monotonically decreasing under *free operations*.

There is no canonical way to define a resource theory for 'quantumness'. Recently Baumgratz et al.[32] constructed a resource theory for coherence (roughly corresponding to fixing the basis in our definition 1), but such a resource theory does not work in the case where the basis is not fixed. In principle one could develop a resouce theory for quantumness based on the operations $\mathcal{L}$ presented in Sec. 7 above, but it remain unclear if the resource states could be 'used up' in the theory, i.e if $Q(\rho)$ can decrease under a subset of the free operations $\mathcal{L}$. A different approach is to see quantumness as an 'obstacle' rather than a resource,[28] or to work within a theory for resource destruction.[35] Our results above lead to a different way to view the problem. We identify the relative quantumness of a state with respect to a protocol and/or and ensemble by using the the framework of restricted distributed gates.[27,28]

Definition 11. Let $\mathcal{S}$ be a set of bipartite states and G be some quantum operation (e.g a quantum gate) and let $\mathcal{G}_{\mathcal{S}}$ be the set of quantum channels that take each $\rho \in \mathcal{S}$ to $G(\rho)$. We say that $\mathcal{S}$ is classical with respect to G if $\mathcal{G}_{\mathcal{S}}$ contains a channel that can be implemented using LOCC.

A quantum operation in $G_{\mathcal{S}}$ can be seen as a version of G restricted to the elements in $\mathcal{S}$. What is rather surprising is that it is possible to find examples where $\mathcal{S}$ is classical with respect to G, but adding a single separable state ρ such that $G(\rho)$ is also separable, is sufficient to make $\mathcal{S}' = \mathcal{S} \cup \{\rho\}$ non classical with respect to G.[27] A two quibt example is as follows:

Let G be the CNOT gate and $\mathcal{S}$ be set of states $\{|00\rangle, |11\rangle\}$. The corresponding $\mathcal{G}_{\mathcal{S}}$ includes a multitude of operations including the CNOT but also the following LOCC operation: Alice (first qubit) measures in the computational basis and sends the result to Bob (second qubit). Bob then applies σ_x if necessary.

Now consider a new set of states $\mathcal{S}'$ which includes all states in $\mathcal{S}$ and one additional state $|++\rangle$. The CNOT takes $|++\rangle$ to itself, Yet, it is possible to show that there is no LOCC channels in $\mathcal{G}_{\mathcal{S}'}$.

Assume that there is some LOCC channel $\Phi \in \mathcal{G}_{\mathcal{S}'}$. First note that Φ requires some communication since Bob's final state depends on Alice's initial state. Second note that whatever operation Bob performs must depend on the information he collects and the messages he receives from Alice. This operation must fall into one of two distinct catagories. Either it leaves $|0\rangle$ invariant (up to a phase) or it takes $|0\rangle$ to $|1\rangle$. If Bob takes $|0\rangle$ to $|1\rangle$ he can rule out the fact that the initial state was $|00\rangle$, similarly if Bob leaves $|0\rangle$ as is, he can rule out the initial state $|10\rangle$. Since Alice can act with a σ_x locally and switch between $|00\rangle$ and $|10\rangle$ while leaving $|++\rangle$ invariant, they can use many repetitions of an LOCC Φ to distinguish between the three non-orthogonal states deterministically and unambiguously. But that should be impossible, ruling out possibility of an LOCC channel $\Phi \in \mathcal{G}_{\mathcal{S}'}$.

The fact that $\mathcal{G}_{\mathcal{S}'}$ cannot be used to distinguish between non-orthogonal states is useful in a cryptographic setting. Using the example above, consider a situation where Alice wants Bob to implement $\mathcal{G}$ only on computational basis states, but does not want him to know what she is sending. By extending the set of input states to include $|++\rangle$, Alice can be assured that Bob does not look at her state, moreover, she can verify that Bob is making the correct transformation deterministically using single qubit measurements.

11. Conclusions

This work provides definitions of classical bases, states, ensembles and protocols in the context of quantum information processing. These definitions were used to sub-classify the domain of separable states into a "zoo of separable states". This classification was supported by operational reasoning using examples of different types of separable states and their significance. We demonstrated that our definition of classicality is robust in the sense that those separable states that do not fit our definition of classicality, necessarily exhibit quantum properties within some context. Finally we discussed the concept of relative quantumness in the context of quantum ensembles.

The work we presented in the 2007 arXiv version of this manuscript has been used in a number of contexts related to quantum information processing and thermodynamics. However many open questions remain, in particular with regards to understanding what makes a particular process quantum and what is the source of the quantum advantage. This question is central to current quantum information research, and we hope that the tools provided here can help progress in that direction.

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Sec 1-6 and Appendix A were mostly written by BG, DK and TM in 2007. Sec. 8 - 11 were written by AB, BG and TM.

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Appendix A. The original summary (2007)

The results presented in Secs. 2-6 of this paper were originally reported on arXiv.[1] This paper contains some subsequent modifications which have been done around that time, mainly the addition of Sec. 8 and clarification on the conversion of states from any state in *Sep* to *Classic* (now collected into a Section 5). For completeness we include the original summary of the work which appeared in 2007 below.

Summary

This paper gives definitions for classical states and protocols in quantum information processing. We explored the "zoo" of separable states, we gave a good number of examples and we defined some useful measures for the quantumness of non-classical states. Our measures and our analysis are mainly based on the notions of "diagonalizing basis" and the "Schmidt basis" (which are identical in the case of pure entangled states). Other measures of quantumness have been defined and used previously: Ref.[11] defines the *quantum discord* between the parts of a bipartite state. Ref.[6] extensively uses the *quantum information deficit* measure of quantumness, and the relative entropy of quantumness (which we use independently). Section 5 in[6] provides very interesting subclasses — yet, different from ours — of the separable states. Their class of "informationally nonlocal" states seems to be identical to our two subclasses — the UPB-states and the unconvertible CPB-states.

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