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Inhomogeneous Initial Data and Small-Field Inflation

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Abstract. We consider the robustness of small-field inflation in the presence of scalar field inhomogeneities. Previous numerical work has shown that if the scalar potential is flat only over a narrow interval, such as in commonly considered inflection-point models, even small-amplitude inhomogeneities present at the would-be onset of inflation at $\tau = \tau_{\rm i}$ can disrupt the accelerated expansion. In this paper, we parametrise and evolve the inhomogeneities from an earlier time $\tau_{\rm IC}$ at which the initial data were imprinted, and show that for a broad range of inflationary and pre-inflationary models, inflection-point inflation withstands initial inhomogeneities. We consider three classes of perturbative pre-inflationary solutions (corresponding to energetic domination by the scalar field kinetic term, a relativistic fluid, and isotropic negative curvature), and two classes of exact solutions to Einstein's equations with large inhomogeneities (corresponding to a stiff fluid with cylindrical symmetry, and anisotropic negative curvature). We derive a stability condition that depends on the Hubble scales $H(\tau_{\rm i})$ and $H(\tau_{\rm IC})$, and a few properties of the pre-inflationary cosmology. For initial data imprinted at the Planck scale, the absence of an inhomogeneous initial data problem for inflection-point inflation leads to a novel, lower limit on the tensor-to-scalar ratio.

Contents

1 Introduction

Two striking features of the Cosmic Microwave Background (CMB) radiation are its average smoothness and the non-trivial correlations of its small anisotropies on scales significantly larger than the Hubble radius when it decoupled from electrons. The approximate scale-invariance of the underlying primordial perturbations cannot be explained by causal dynamics in the standard hot big bang model, and would be a surprising outcome from any quantum gravitational initial state. The success of the theory of inflation [1–3] springs from its ability to explain these observations within a simple theoretical framework that permits detailed observational tests.

Inflation drives the universe towards a locally highly homogeneous state, and small quantum fluctuations, stretched beyond the Hubble radius by inflation, provide the candidate seeds for large-scale structure in the universe [4–8]. The generic predictions of inflation are in excellent agreement with all current observations [9]; however, much remains to be understood about how and why inflation happened, and what might have prevented it.

A puzzling aspect about inflation is the question of how it got started: while the enormous expansion during this era smooths any pre-existing inhomogeneities, sufficiently large initial inhomogeneities can prevent the energy density from becoming dominated by the potential energy, and cause the expansion to remain decelerating. In the simplest estimates, the onset of the accelerated phase requires a homogeneous patch extending several (or even many) Hubble radii, which arguably suggests fine-tuning of the pre-inflationary initial state. This is the problem of inhomogeneous initial data for inflation.

The question of the robustness of inflation is important as it affects the naturalness of the inflationary framework. For ardent critics of inflation, the inhomogeneous initial data problem is interpreted as a serious challenge to the entire inflationary paradigm (see e.g. [10]). However, research by several groups over the past few decades have shown that the simplest formulation of the initial data problem can be misleading [11–44], and inflation is less sensitive to inhomogeneities than one might naively expect. Perhaps most importantly, numerical simulations of general relativity coupled to a scalar field with a flat inflationary potential have shown that even initial configurations dominated by inhomogeneities typically lead to inflation [15, 16, 21, 24–26, 34, 39, 41], at least as long as the amplitude of the inhomogeneities of the scalar field, $\delta\phi$, is smaller than the width of the inflationary region of the potential, $\Delta\phi$. For large-field models with $\Delta\phi > 1$,[1] this solves the problem of inhomogeneous initial data without requiring any smoothness of the initial patch [39]. Moreover, if chaotic inflation is realised, a single smooth Planck sized domain can result in inflation and lead to an eternal process of self-reproduction [45].

For models with $\Delta\phi < 1$, the solution to the inhomogeneous initial data problem is less immediate. It can be avoided in models with non-trivial topology [37, 46–48], and ameliorated in inflationary potentials with extended plateau regions [39, 41] (which can be achieved, for example through a non-trivial kinetic terms [43, 49]). Moreover, if the universe went through phases of both high-scale and low-scale inflation, or got stuck in a false metastable vacuum before the final period of inflation, the inhomogeneities present at the onset of the final phase of inflation are expected to be small (cf. e.g. [36, 42]).

Nevertheless, for commonly considered small-field models with $\Delta\phi \ll 1$, the initial data problem can still appear quite severe. Examples of such models include low-scale potentials that are flat only near an inflection point. Inflection-point models are becoming increasingly popular as they are automatically consistent with observational upper limits on the tensor-to-scalar ratio, and appear to admit comparatively simple ultraviolet completions into a string theory (see e.g. [50–55], or [56] for a recent review). Recently, reference [41], numerically studied the impact of inhomogeneities present at the potential onset of inflation in small-field inflection-point models (see also [39, 44]), and found that even highly sub-dominant gradient energy densities of the scalar field can spoil inflation. At face value, these results may be taken to suggest that the simplest inflection-point models suffer severely from the inhomogeneity problem. In this paper, we show that such a conclusion would be premature.

The conformal time $\tau_{\rm IC}$ at which the initial data were imprinted (e.g. when four-dimensional general relativity first gave an appropriate description of the dynamics) may in general have far preceded the onset of inflation at conformal time $\tau_{\rm i}$. During the pre-inflationary era between $\tau_{\rm IC}$ and $\tau_{\rm i}$, the comoving Hubble radius grew, and the most dangerous modes for disrupting inflation had wavelengths far longer than the Hubble radius at $\tau_{\rm IC}$. The power spectrum of inhomogeneities at

[1]Throughout this paper, we set the reduced Planck mass to one: $M_{\rm Pl} = 1/\sqrt{8\pi G} = 2.4 \times 10^{18}\,{\rm GeV} = 1$.

$\tau_{\rm IC}$ is not in general expected to be scale-invariant, but should go to zero as the wavelength goes to infinity.

In this paper, we parametrise the spectrum of inhomogeneities at $\tau_{\rm IC}$ and show that the initial data problem depends on four properties of the scenario: *i)* the energy scales $H_{\rm IC} = H(\tau_{\rm IC})$ and $H_{\rm i} = H(\tau_{\rm i})$, *ii)* the amount of expansion between $\tau_{\rm IC}$ and $\tau_{\rm i}$, *iii)* the spectral index of the initial inhomogeneities on super-horizon scales, and *iv)* the narrowing of the plateau region as the energy scale of inflation is lowered. We then exemplify our results analytically by considering pre-inflationary epochs dominated by either scalar-field kinetic energy, a general relativistic fluid with equation-of-state w, or negative curvature (which may be isotropic or anisotropic). Using these models, we assess the severity of the inhomogeneous initial data problem.

We find qualitatively different scenarios depending on whether the expansion is decelerating (e.g. as for fluid domination), or displays constant comoving Hubble parameter, $\mathscr{H} = Ha$, (as for negative isotropic curvature domination), and, in the former case, if the fall-off of the initial power spectrum on large scales is 'steep' or 'moderate'. This leads us to three main results:

- If the pre-inflationary expansion is decelerating and the initial data are imprinted at the Planck scale, so that $H_{\rm IC} = 1$, the inhomogeneous initial data problem only ever becomes relevant for very low energy models. For 'moderate' initial power spectra, we derive a lower limit on the tensor-to-scalar ratio r, above which one should not expect an initial data problem. This limit implies that a large fraction of interesting inflection-point models are robust against inhomogeneities, with the detailed bound depending on a combination of parameters of the pre-inflationary cosmology. For example, if the pre-inflationary epoch is radiation dominated and the inhomogeneities at $\tau_{\rm IC}$ have $\delta\phi \sim \mathscr{O}(1)$ on the horizon scale and a spectral index of $n_{\rm IC} = 3$ on larger scales, we find the limit: $r > 2.5 \times 10^{-22}$. For 'steep' initial spectra, no models are expected to have a problem with inhomogeneities.

- If the initial data are imprinted at energies much below the Planck scale with a 'moderate' power-spectrum, the inhomogeneous initial data problem becomes more severe. We show that this leads to a lower bound on the initial energy scale, $H_{\rm IC} \gtrsim 4 \times 10^{-8}$, for models robust against initial inhomogeneities. By contrast, models with 'steep' initial power spectra can be robust against inhomogeneities for smaller $H_{\rm IC}$.

- If the pre-inflationary dynamics is dominated by isotropic negative curvature so that the comoving Hubble expansion rate is constant, there is no inhomogeneous initial data problem for $H_{\rm IC} \gtrsim 4 \times 10^{-8}$, independently of $H_{\rm i}$.

We conclude that even the simplest inflection-point models with narrow inflationary plateaux do not in general exhibit an inhomogeneous initial data problem. While our results are derived for a particular class of small-field inflection-point models (that directly generalise those considered in [41]), we expect similar arguments to hold also for other small-field scenarios. Moreover, our main results in §4 apply to pre-inflationary cosmologies in which the expansion is on average non-accelerating and the superhorizon scalar field inhomogeneities are not significantly sourced. We expect these conditions to be satisfied by many interesting classes of both perturbative and non-perturbative pre-inflationary cosmologies.

This paper is organised as follows: In §2, we introduce the family of inflection-point models that we consider, review how inflationary models can fail due to inhomogeneities. We also parametrise their initial spectrum at $\tau_{\rm IC}$. In §3, we discuss the pre-inflationary era and derive some simple but illuminating analytic results for the evolution of perturbative inhomogeneities during inflation. We also briefly discuss the cases of positive and anisotropic negative curvature. In §4, we analyse the severity of the problem of initial inhomogeneities, and draw together the main results of this paper. We conclude by discussing possible future directions in §5. In Appendix A, we provide details of the perturbative calculations in §3, and in Appendix B we discuss solutions with cylindrical symmetry and anisotropic negative curvature in more detail.

2 The initial data problem for inflection-point inflation

In this section, we briefly review the class of inflationary models we consider and the wavelength-dependence of 'dangerous' inhomogeneities. We also discuss and parametrise the initial spectrum of inhomogeneities, which will feature in our bounds derived in §4.

2.1 Inflection-point inflation

In order to investigate the dependence of the inhomogeneous initial data problem on the energy scale $H_{\rm i}$, we construct a one-parameter family of models with various values for the inflationary potential, V_0. These models directly generalise the 'typical small-field model' of reference [41].

The initial data problem is expected to be most severe in models with very narrow inflationary plateaux, $\Delta\phi \ll 1$. In homogeneous cosmology, the minimum required width of the inflationary plateau shrinks as V_0 is decreased: the lower the energy scale, the narrower the plateau, cf. Figure 1. In this section, we briefly review the relevant properties of inflection-point potentials and show why modes with $k \approx \mathscr{H}_{\rm i} = a_{\rm i} H_{\rm i}$ are the most dangerous for inhibiting inflation.

The models we consider have an inflection point at $\phi = 0$ and for $0 < \phi \lesssim \mu$ are described by,

$$V(\phi) = V_0 \left(1 - \left(\frac{\phi}{\mu}\right)^4\right) \quad . \tag{2.1}$$

For $\mu < 1$, this potential supports small-field, slow-roll inflation. Clearly, equation (2.1) captures the plateau region adjacent to the inflection point, and should be joined on either side by suitable smooth extensions. In §4, we consider both sharply rising and flat extensions of the potential to negative values, however, we note that the former class are likely to suffer from the (homogeneous) 'overshoot problem', as discussed in e.g. [57, 58] (see also [52, 54] for a discussion in the context of a multi-field string theory embedding of inflection-point inflation). We will not address the overshoot problem here, but note that it motivates focussing on rather flat extensions of the plateau. In this section, we assume that the average value of the field is close to the inflection point when $H = H_{\rm i}$, with a small kinetic energy, so that the relevant part of the potential is captured by equation (2.1).

The properties of this family of inflection-point models are well-known and reviewed in e.g. [59]. If slow-roll inflation begins at $0 < \phi_{\rm min} \ll \mu$, the number of e-folds of inflation is well-approximated by,

$$N = \int_{\phi_{\rm min}}^{\phi_{\rm max}} \frac{1}{\sqrt{2\epsilon_V}} {\rm d}\phi \approx \frac{1}{8}\frac{\mu^4}{\phi_{\rm min}^2} \quad , \tag{2.2}$$

where $\epsilon_V = \frac{1}{2}(V'^2/V^2)$. The spectral index of the curvature perturbations, evaluated around $\phi_{\rm min}$, is given by,

$$n_s - 1 = -\frac{3}{N} \quad , \tag{2.3}$$

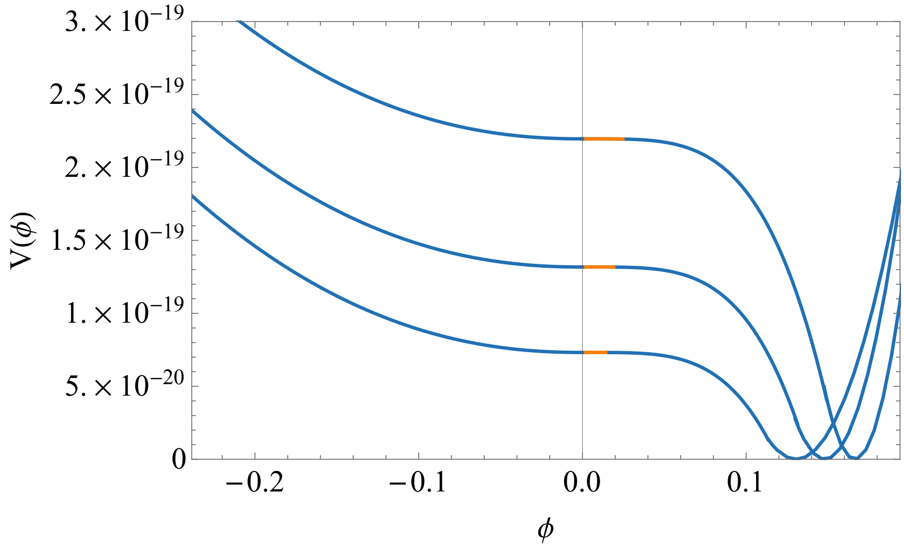

Figure 1. Examples of inflection-point potentials. Most of the inflationary expansion occurs when the field is in a narrow interval of length $\Delta\phi \sim \mathcal{O}(\mu^2)$, here marked in orange.

so that observational compatibility requires $N \approx 100$ e-folds of inflationary expansion, independently of the energy scale of inflation. Assuming that inflation ends soon after the moment when $|\eta_V| = |V''/V| = 1$, the distance traversed by the field during inflation is approximately given by,

$$\phi_{\max} - \phi_{\min} \approx \mathcal{O}(\mu^2)\,. \tag{2.4}$$

Clearly, models with small field excursions have $\mu < \mathcal{O}(1)$.

Finally, the amplitude of the curvature perturbations generated from quantum fluctuations during inflation scales like,

$$A_s \sim N^3 \frac{V_0}{\mu^4}\,. \tag{2.5}$$

Imposing the observationally inferred normalisation of the CMB anisotropies then leads to a one-parameter family of models.

Starting from an observational compatible reference model with $V_0 = V_{\rm ref}$ and the inflationary field-displacement $\Delta\phi = \Delta\phi_{\rm ref}$, it follows from equations (2.4) and (2.5) that other observationally compatible models in this family have field displacements $\Delta\phi$ and energy scales V given by the scaling relation,

$$\left(\frac{\Delta\phi}{\Delta\phi_{\rm ref}}\right)^2 = \frac{V_0}{V_{\rm ref}} = \left(\frac{H_{\rm i}}{H_{\rm ref}}\right)^2\,. \tag{2.6}$$

We use the 'typical small-field model' of [41] for our reference parameters:[2]

$$\begin{aligned} \mu_{\rm ref} &= 0.12\,, & V_{\rm ref} &= 7.3\times10^{-20}\,,\\ H_{\rm ref} &= 1.6\times10^{-10}\,, & \Delta\phi_{\rm ref} &= 5.0\times10^{-3}\,, \end{aligned} \tag{2.7}$$

where we have used the field excursion during all but the last e-fold of inflation as our measure of $\Delta\phi$. For these parameters, the tensor-to-scalar ratio is $r = 2.6\times10^{-12}$.

[2] These numerical values differ from those of [41] which sets the Planck mass, as opposed to the reduced Planck mass, to unity.

Small-field models satisfy $\Delta\phi < 1$. For this class of inflection-point models, this translates into the limits $H_{\rm i} < 3.2 \times 10^{-8}$ corresponding to $r < 10^{-7}$.

2.2 Dangerous inhomogeneities

Some inhomogeneities are more dangerous to inflation than others. Inflation is destabilised if, in some part of space, the scalar field fluctuates towards the minimum and pre-maturely ends inflation in that region, with gradients dragging the field in the rest of spacetime down towards the minimum. However, gradients also have a stabilising effect [39, 41], as we now review.

For a potentially destabilising fluctuation to become energetically favourable, the energy gain from the potential energy, ΔV, must overcome the gradient energy, $\frac{1}{2a^2}(\nabla\phi)^2$, that will (at least initially) attempt to pull the scalar field fluctuation back towards the plateau. Clearly then, for a fixed k-independent amplitude of scalar field inhomogeneities, low-k modes are more dangerous than high-k modes. Inhomogeneities with $k < \mathscr{H}_{\rm i}$ can locally be viewed as renormalisations of the homogeneous cosmology, and do not cause the entire universe to collapse [39]. Thus, the modes most dangerous for inflation have $k \approx \mathscr{H}_{\rm i} = H_{\rm i} a_{\rm i}$. As we will discuss quantitatively below, reference [41] found numerically that the probability of destabilisation decreased markedly for wavelengths a factor of two smaller than the Hubble radius. These results are consistent with earlier work [39].

For non scale-invariant inhomogeneities, the long-wavelength modes remain the most dangerous as long as the potential energy gain is dominated by linear or quadratic terms. If the potential energy gain becomes dominated by cubic or higher order terms, it is possible for short-wavelength modes to become dangerous: in the cubic case, this only happens for rather steep spectra of inhomogeneities at $\tau_{\rm i}$, corresponding to a spectral index of $\gtrsim 5$.

The sub-horizon evolution of high-k inhomogeneities mitigates the risk they pose for inflation: perturbative inhomogeneities decay like $\delta\phi_k \sim 1/a$ (as we show in §3.2), and non-perturbatively large inhomogeneities can trigger gravitational collapse into black holes promptly after horizon entry [60]. The black holes are not expected to disrupt inflation [41]. For these reasons, we here focus on the limited range of 'dangerous modes' that have wave numbers in a small interval around $\mathscr{H}_{\rm i}$, cf. Figure 2. This drastically simplifies the problem of inhomogeneous initial data.

Inflation is robust against the effects of sufficiently small inhomogeneities. The exact limit on their amplitude depends on the width of the inflationary plateau, the steepness of the potential beyond it, the admixture of wavelengths of the modes, and the criterion for robustness. We here adopt the criteria that a model is safe from inhomogeneities if it yields 60 or more e-folds of inflation. Extrapolating the numerical results of reference [41],[3] we express the corresponding bound on the total amplitude of scalar field inhomogeneities with $k \approx \mathscr{H}_{\rm i}$ as a fraction f, of the width $\Delta\phi_{\rm ref}$:

$$|\delta\phi_{k\approx\mathscr{H}_i}| < f\, \Delta\phi_{\rm ref}\,. \tag{2.8}$$

Here $\Delta\phi_{\rm ref}$ is as in equation (2.7). We consider two values for the fraction f: if the scalar potential is flat for negative values of the potential, $f = 1.6$, while if it raises sharply beyond the plateau, $f = 0.17$. The latter type of potential suffers from the overshoot problem already for homogeneous cosmology [58], and the reduction of f is directly related to this problem; inhomogeneities can pick up excess kinetic energy by fluctuating to negative values. We expect that potentials without an overshoot problem give $f > 0.17$.

[3] We are grateful to Eugene Lim, Josu Aurrekoetxea, Katy Clough, and Raphael Flauger for discussions on this point.

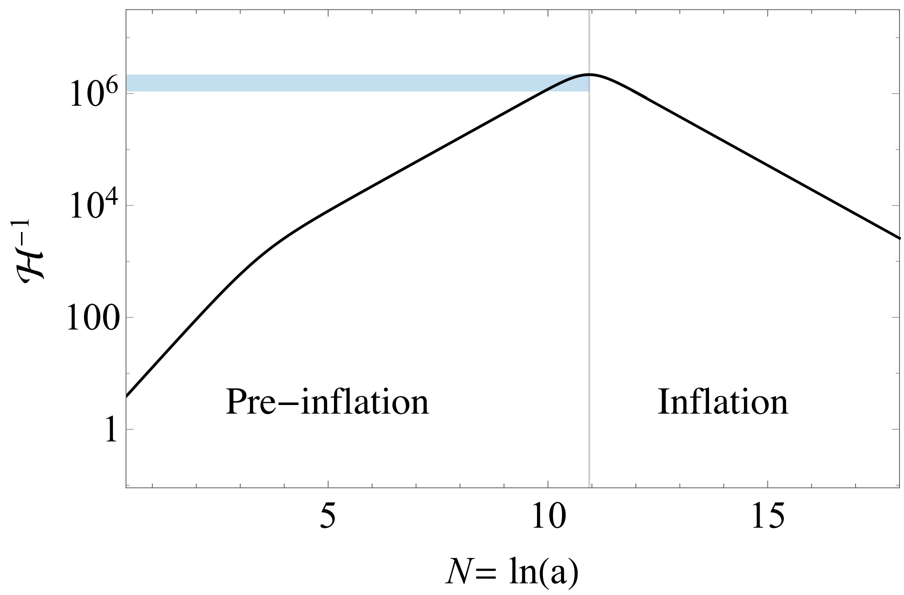

Figure 2. Evolution of the comoving Hubble radius, $\mathscr{H}^{-1}$, in a pre-inflationary epoch of decelerated expansion driven by scalar field kinetic energy and radiation (solid) and during inflation (dashed). Here $a(\tau_{\rm IC}) = 1$. The perturbations most dangerous for inflation (shaded blue) have $k^{-1} \gg \mathscr{H}_{\rm IC}^{-1}$.

These values for f are obtained from numerical simulations in which all inhomogeneities were concentrated in a superposition of three modes with comoving wavelengths $\mathscr{H}_{\rm i}^{-1}$. Reference [41] also found that if, in addition, modes with half the wavelength were included, the corresponding values were increased to $f = 2.6$ and $f = 0.21$ for extended flat and steep potentials, respectively. This suggests that f is a k-dependent function, consistent with the argument that long wavelength modes are the most dangerous. Due to the scarcity of numerical data, we will not attempt to model $f(k)$ here. Moreover, we will not consider other sources of inhomogeneity, e.g. those in tensor modes, that only indirectly impact the stability of inflation [44].

Using the scaling equation (2.6), the condition for stability (2.8) can be extended to other energy scales in our class of inflection-point models:

$$|\delta\phi_{k\approx\mathscr{H}_i}| < f\Delta\phi = f\Delta\phi_{\rm ref}\left(\frac{H_i}{H_{\rm ref}}\right) . \tag{2.9}$$

In §4 we use this inequality to derive a bound on $H_{\rm i}$ and $H_{\rm IC}$ from the absence of an inhomogeneous initial data problem.

2.3 Parametrisation of the initial inhomogeneities

Presumably, inhomogeneities present at $\tau_{\rm IC}$ were fashioned by quantum gravitational dynamics, about which little is known.[4] In lieu of a complete theory of the initial data, we here merely parametrise the statistical distribution of initial inhomogeneities, and briefly highlight the most relevant properties of their spectrum.

[4] In the context of string theory – the leading candidate theory of quantum gravity – the question of inflationary initial data is likely to involve the properties of the effective theories at energies below the compactification scale, and the state of the universe at energies at or above the string scale. While substantial, yet partial, progress have been made on the former issue, the latter still remains a significant challenge (see e.g. [56] for a review).

We assume that the inhomogeneities present at $\tau_{\rm IC}$ are classical and statistically homogeneous, isotropic and Gaussian. We define the power spectrum of inhomogeneities at $\tau_{\rm IC}$ by,

$$\langle \delta\phi_{\mathbf{k}}(\tau_{\rm IC})\delta\phi_{\mathbf{k}'}(\tau_{\rm IC})\rangle = \frac{2\pi^2}{k^3}\delta^{(3)}(\mathbf{k}+\mathbf{k}')P_{\delta\phi(\tau_{\rm IC})}(k)\,. \tag{2.10}$$

In general, $P_{\delta\phi(\tau_{\rm IC})}(k)$ can be a complicated function, however, we expect the power spectrum to be suppressed on wavelengths greater than the largest dynamically relevant length scale. In particular, as the wavelength of the modes goes to infinity (and $k \to 0$), we expect $P_{\delta\phi(\tau_{\rm IC})}(k) \to 0$. For modes with $k < \mathscr{H}_{\rm IC}$, it may be appropriate to describe the power-spectrum by a simple power-law,[5]

$$P_{\delta\phi(\tau_{\rm IC})}(k\,|\,k<\mathscr{H}_{\rm IC}) = A\left(\frac{k}{\mathscr{H}_{\rm IC}}\right)^{n_{\rm IC}-1}\,, \tag{2.11}$$

in terms of which the assumption of $P_{\delta\phi(\tau_{\rm IC})}(0) = 0$ gives $n_{\rm IC} > 1$. Reasonably, this is consistent with a finite total power in modes with $k < \mathscr{H}_{\rm IC}$. Here A sets the amplitude of inhomogeneities with $k = \mathscr{H}_{\rm IC}$. The energy density contribution from these modes is $\rho_{\delta\phi(\tau_{\rm IC})|_{k=\mathscr{H}_{\rm IC}}} = H^2 A$, which is clearly substantial for $A = 1$.

The spectral index of the initial inhomogeneities is an important parameter in our analysis, and the condition $n_{\rm IC} > 1$ can be motivated by several additional arguments. Obviously, were we to assume that classical cosmology with decelerating expansion would hold up until the singularity at $\tau = 0$, no causal mechanism could generate perturbations with comoving wavelengths $\gg \mathscr{H}_{\rm IC}^{-1}$, and there would be no power in modes with $k \ll \mathscr{H}_{\rm IC}$. Moreover, quantum fluctuations can lead to suppressed perturbations on scales larger than the Hubble radius, as we now show.

Neglecting gravitational back-reaction around a homogeneous FRW background with constant $\epsilon_{\rm H}$, a massless scalar field perturbation $\varphi_k = \delta\phi_k/a$ satisfies the linear-order equation,

$$\frac{{\rm d}\varphi_k}{{\rm d}\tau^2} + \left(k^2 - \frac{c}{\tau^2}\right)\varphi_k = 0\,. \tag{2.12}$$

Here, we have used that $\mathscr{H} = \frac{1}{\epsilon_{\rm H}-1}\frac{1}{\tau}$, and introduced the constant,

$$c = \frac{2-\epsilon_H}{(\epsilon_H - 1)^2}\,, \tag{2.13}$$

for $\epsilon_{\rm H} \neq 1$. In the case where the background geometry is de Sitter spacetime, we have $c = 2$ and the solutions to equation (2.12) include the familiar Bunch-Davies wavefunction,

$$\varphi_k = \frac{e^{ik\tau}}{\sqrt{2k}}\frac{(k\tau - i)}{k\tau}\,. \tag{2.14}$$

In this case, long-wavelength modes have a power spectrum,

$$P_\varphi = \frac{k^3}{2\pi^2}\left|\varphi_k\right|^2 \sim \frac{1}{\tau^2}\,, \tag{2.15}$$

independently of k (for sufficiently small k) so that the power spectrum of ϕ is scale-invariant: $P_{\delta\phi} \sim H^2$ [61]. This is no longer the case in a more general FRW background with constant $\epsilon_H > 1$.

[5] If the dynamics responsible for the initial inhomogeneities includes multiple length scales, the power-law parametrisation should be appropriate for wavelengths larger than the largest dynamical scale, which may be longer than $\mathscr{H}_{\rm IC}^{-1}$. We do not discuss this modified scenario here, but our analysis can be straightforwardly applied also to this scenario.

Specialising to the case in which the energy density is dominated by a relativistic fluid with equation-of-state parameter $0 < w < 1$, we have $3/2 < \epsilon_H < 3$ and consequently $-\frac{1}{4} < c < 2$. The solution to equation (2.12) is now given by a combination of Bessel functions,

$$\varphi_k(\tau) = c_1\sqrt{\tau}J_\nu(k\tau) + c_2\sqrt{\tau}Y_\nu(k\tau)\,, \tag{2.16}$$

with

$$\nu = \tfrac{1}{2}\sqrt{1+4c} = \frac{1}{2}\frac{|3-\epsilon_H|}{|\epsilon_H - 1|}\,. \tag{2.17}$$

The slowest fall-off in the long-wavelength limit $k \ll \tau$ is obtained from the 'Bunch-Davies-like' solutions that scale as,

$$\varphi_k(\tau) \sim \frac{1}{k^\nu \tau^{\nu-1/2}}\,. \tag{2.18}$$

The spectral index for the perturbations of ϕ is then given by,

$$n_{\rm IC} - 1 = 3 - 2\nu = 3 - \frac{|3-\epsilon_H|}{|\epsilon_H - 1|} > 0\,. \tag{2.19}$$

In the entire range $0 < w < 1$, we have $n_{\rm IC} > 1$ and thus a suppression of the power in modes with wavelength far longer than the Hubble radius. For example, a radiation-dominated period corresponds to $n_{\rm IC} = 3$.

We note in closing that since the appropriate theory of inflationary initial data is unknown, arguments about its properties are necessarily heuristic. In this section we have pointed out that scale invariance of the inhomogeneities is certainly not guaranteed, and should perhaps not be expected.

3 The pre-inflationary epoch

On general grounds, there is no reason to expect that the time $\tau_{\rm IC}$ should coincide with the onset of inflation at $\tau_{\rm i}$. This means that inflation was preceded by a pre-inflationary era of non-accelerated expansion.[6] Since inflation by construction is very efficient at erasing any traces of the pre-inflationary state, not much is known about this epoch.

Our main results in §4 depend only on a few properties of the pre-inflationary era, and apply to both small and large inhomogeneities. For concreteness and to be able to make analytic progress, in this section we parametrise the pre-inflationary spacetime by a homogeneous and isotropic Friedmann-Robertson-Walker (FRW) metric, and treat the inhomogeneities as linear-order perturbations. This approach has some obvious limitations, but also several benefits. While the full problem of determining how inhomogeneities affect the duration of inflation is clearly non-linear, the *pre-inflationary* evolution of small perturbations with $\delta\phi \approx \Delta\phi \ll 1$ can still be well-described by perturbation theory. For small-field models with a very narrow plateau region, such small perturbations may suffice to destabilise inflation (at which point the perturbative treatment ceases to be valid).

Moreover, as noted in §2.2, the most dangerous modes for inflation had wavelengths far larger than the Hubble radius during most of the pre-inflationary era. These modes are governed by a coarse-grained effective theory obtained by integrating out short-wavelength inhomogeneities. This results in a prescription similar to the 'stochastic inflation' framework [62] (see also [63]), and includes a

[6]More exotically, it is possible that the pre-inflationary era involved a contracting phase. We will not discuss this scenario here.

small sourcing induced by short-wavelength inhomogeneities. Our bounds derived in §4 applies to pre-inflationary cosmologies in which this sourcing is negligible. In all models that we study analytically in this section, the superhorizon modes are constant.

This section is organised as follows: in §3.1, we review the salient features of homogeneous pre-inflationary cosmology, and in §3.2 we discuss the evolution of perturbative inhomogeneities in the three classes of pre-inflationary scenarios that we consider. In §3.3 and §3.4 we respectively review the cases of positive and anisotropic negative curvature. More details of the perturbative calculations are deferred to Appendix A, and some special, non-linear solutions are further discussed in Appendix B.

3.1 Homogeneous pre-inflationary cosmology

In this section, we discuss the homogeneous limit of the pre-inflationary cosmology and derive simple expressions for the duration of the pre-inflationary era and the growth of the comoving Hubble radius. The Friedmann-Robertson-Walker (FRW) metric is given by,

$$\mathrm{d}s^2 = a^2(\tau)\left(-\mathrm{d}\tau^2 + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j\right) , \tag{3.1}$$

with the spatial components,

$$\gamma_{ij} = \frac{\delta_{ij}}{\left(1+\frac{\mathscr{K}}{4}(x^2+y^2+z^2)\right)^2} . \tag{3.2}$$

We set $a(\tau_{\rm IC}) = 1$ so the (isotropic) curvature parameter $\mathscr{K}$ is in general not an integer. We here consider three-dimensional spaces that are flat or open with $\mathscr{K} = 0$ or $\mathscr{K} < 0$, and discuss the case of positive curvature in §3.3. A useful parameter is,

$$\epsilon_H = -\frac{\frac{\mathrm{d}H}{\mathrm{d}t}}{H^2} = 1 - \frac{\mathscr{H}_\tau}{\mathscr{H}^2} , \tag{3.3}$$

which, in the pre-inflationary era of non-accelerated expansion, satisfies $\epsilon_H \geq 1$. Here, and subsequently, we denote derivatives with respect to conformal time with the sub-script τ.

We consider universes containing a homogeneous inflaton field, $\bar{\phi}$, with potential $V(\bar{\phi})$ and a perfect, isentropic fluid with energy density $\rho_f(\tau)$ and pressure $p_f = w\rho_f$. In the homogeneous limit, the Einstein equations are given by,

$$3(\mathscr{H}^2+\mathscr{K}) = \frac{1}{2}\bar{\phi}_\tau^2 + a^2 V(\bar{\phi}) + a^2\rho_f , \tag{3.4}$$

$$-(2\mathscr{H}_\tau+\mathscr{H}^2+\mathscr{K}) = \frac{1}{2}\bar{\phi}_\tau^2 - a^2 V(\bar{\phi}) + a^2 p_f . \tag{3.5}$$

The Klein-Gordon equation and momentum conservation equation for the fluid are given by,

$$\bar{\phi}_{\tau\tau} + 2\mathscr{H}\bar{\phi}_\tau + a^2 V'(\bar{\phi}) = 0 , \quad (\rho_f)_\tau + 3\mathscr{H}(\rho_f + p_f) = 0 . \tag{3.6}$$

We derive analytic results for three classes of cosmological backgrounds:

$$\begin{array}{lllll} i) & \text{Kinetic energy domination :} & \frac{1}{2a^2}\bar{\phi}_\tau^2 & \gg \max\left(V(\bar{\phi}),\ \rho_f, -3\frac{\mathscr{K}}{a^2}\right) & \\ ii) & \text{Fluid domination :} & \rho_f & \gg \max\left(\frac{1}{2a^2}\bar{\phi}_\tau^2,\ V(\bar{\phi}), -3\frac{\mathscr{K}}{a^2}\right) & (3.7) \\ iii) & \text{Curvature domination :} & -3\frac{\mathscr{K}}{a^2} & \gg \max\left(\frac{1}{2a^2}\bar{\phi}_\tau^2,\ V(\bar{\phi}),\ \rho_f\right) . & \end{array}$$

During kinetic energy domination (considered also in [64, 65]), $\epsilon_{\rm H} = 3$, and the energy density decreases as $\rho = \rho(\tau_{\rm IC})/a^6$. During fluid domination (considered recently in [66]),[7] $\epsilon_{\rm H} = \frac{3}{2}(1+w)$ and $\rho(\tau) = \rho(\tau_{\rm IC})/a^{3(1+w)}$. For isotropic curvature domination (considered e.g. in [67]), $\epsilon_{\rm H} = 1$ and the isotropic curvature contribution decays like $-\mathscr{K}/a^2$. In general, the energy density at $\tau_{\rm IC}$ may receive contributions from multiple sources. The different decay rates of the energy density then often lead to a separation of the pre-inflationary era into distinct epochs during which one source dominates.

The total amount of expansion during a pre-inflationary era lasting from $\tau_{\rm IC}$ (when $H = H_{\rm IC}$) to $\tau_{\rm i}$ (when $H = H_{\rm i}$) is given by,

$$\frac{a(\tau_{\rm i})}{a(\tau_{\rm IC})} = \left(\frac{H_{\rm IC}}{H_{\rm i}}\right)^{\frac{2}{3(1+w)}} , \tag{3.8}$$

where $w = 1$ corresponds to kinetic energy domination like an 'ultra-stiff' fluid, and $w = -1/3$ also corresponds to curvature domination. For $w > -1/3$, the comoving Hubble radius, $\mathscr{H}^{-1}$, grows during the pre-inflationary era by,

$$\frac{\mathscr{H}_{\rm i}^{-1}}{\mathscr{H}_{\rm IC}^{-1}} = \left(\frac{H_{\rm IC}}{H_{\rm i}}\right)^{\frac{1+3w}{3(1+w)}} . \tag{3.9}$$

This is a substantial growth for natural values of the parameters: for example, a radiation-dominated pre-inflationary era lasting from the Planck scale ($H_{\rm IC} = 1$) to the energy-scale of our reference model (cf. equation (2.7)) generates an expansion and growth of the comoving Hubble radius of $a(\tau_{\rm i})/a(\tau_{\rm IC}) = \mathscr{H}_{\rm i}^{-1}/\mathscr{H}_{\rm IC}^{-1} = \exp(11.3) = 8.0 \times 10^4$. Moreover, the energy scale of inflation may be much lower than that of the reference model. Successful primordial nucleosynthesis and thermalisation of the neutrinos requires a hot big bang cosmology with $T \gtrsim 4$ MeV [68], which in the extreme case of instant reheating immediately following the end of inflation is consistent with $H_{\rm i} \approx 10^{-42}$ (corresponding to $r \approx 10^{-76}$). In this case a radiation-dominated pre-inflationary era from the Planck scale would generate 48 e-folds of expansion.

In the kinetic-energy dominated era, the homogeneous inflaton satisfies the equation $\bar{\phi}'' = 0$, where prime denotes a derivative with respect to the number of e-foldings: $X' = {\rm d}X/{\rm d}N = {\rm d}X/(\mathscr{H}{\rm d}\tau)$. The speed of the field is then constant, with $\bar{\phi}' = \pm\sqrt{6}$. In the other scenarios of equation (3.7), the additional Hubble friction from the fluid or the curvature term leads to the equation of motion,

$$\bar{\phi}'' + \frac{3}{2}(1-w)\bar{\phi}' = 0 \,, \tag{3.10}$$

and a rapid slow-down of the background scalar field.[8]

The growth of the co-moving Hubble radius, cf. equation (3.9), is not unique to homogeneous FRW universes. In §3.4 we show that similar results hold also for universes dominated by negative, anisotropic spatial curvature.

3.2 Perturbative pre-inflationary inhomogeneities

It is instructive to examine the pre-inflationary evolution of small inhomogeneities in the three classes of cosmologies of equation 3.7. To do so, we consider the linear scalar perturbations in conformal-

[7] In highly inhomogeneous universes dominated by scalar field gradient energy with wavelengths smaller than the Hubble radius (defined by spatial averaging), the average energy density decreases approximately like radiation, $w = 1/3$ [39].

[8] This effect ameliorates the overshoot problem for small-field inflation.

Newtonian gauge. The line element is given by,

$$\mathrm{d}s^2 = a^2(\tau)\left(-(1+2\Phi)\mathrm{d}\tau^2 + (1-2\Psi)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j\right) . \tag{3.11}$$

The benefit of this gauge is that the potentials Φ, Ψ agree with the gauge invariant Bardeen potentials and the scalar field perturbation $\delta\phi(\tau,\mathbf{x}) = \phi(\tau,\mathbf{x}) - \bar{\phi}(\tau)$ directly corresponds to the gauge invariant scalar field perturbation [69]. We assume that gravitational waves can be neglected. Inhomogeneous gravitational waves do not by themselves shorten the duration of inflation, and pre-inflationary cosmologies involving both tensor and scalar inhomogeneities was recently studied in [44]. In the absence of anisotropic stress, the $i \neq j$ components of the Einstein equation gives,

$$\Phi = \Psi\,, \tag{3.12}$$

and we will henceforth only use the Newtonian potential, Φ. The remaining Einstein equations are then given by (for reviews, see e.g. [69, 70]),

$$\nabla^2\Phi - 3\mathscr{H}\Phi_\tau - 3(\mathscr{H}^2 - \mathscr{K})\Phi = \frac{1}{2}\left(a^2V'(\bar{\phi})\delta\phi + \bar{\phi}_\tau\delta\phi_\tau - \Phi\bar{\phi}_\tau^2 + a^2\delta\rho_f\right) , \tag{3.13}$$

$$\partial_i\left(\mathscr{H}\Phi + \Phi_\tau\right) = \frac{1}{2}\left(\bar{\phi}_\tau\partial_i\delta\phi - a^2(1+w)\bar{\rho}_f v_i\right) , \tag{3.14}$$

$$\Phi_{\tau\tau} + 3\mathscr{H}\Phi_\tau + (2\mathscr{H}_\tau + \mathscr{H}^2 - \mathscr{K})\Phi = \frac{1}{2}\left(-\Phi\bar{\phi}_\tau^2 + \bar{\phi}_\tau\delta\phi_\tau - a^2V'(\bar{\phi})\delta\phi + a^2w\delta\rho_f\right) . \tag{3.15}$$

Here $v_i = \partial_i v$ for the fluid's velocity potential v. The leading-order Klein-Gordon equation for the field perturbation is given by,

$$\delta\phi_{\tau\tau} + 2\mathscr{H}\delta\phi_\tau - \nabla^2\delta\phi - 4\Phi_\tau\bar{\phi}_\tau + 2a^2\Phi V'(\bar{\phi}) + a^2V_{\phi\phi}(\bar{\phi})\delta\phi = 0\,. \tag{3.16}$$

We now state the results of this perturbative analysis; more details can be found in Appendix A. For all three classes of pre-inflationary cosmologies of equation (3.7), we find that the amplitude of scalar field inhomogeneities stays constant on super-horizon scales, and undergo pressure-damped oscillatory decay on sub-horizon scales. In all three cases the amplitude of the scalar field inhomogeneities evolve like,

$$\delta\phi_{\mathbf{k}} \sim \begin{cases} \text{constant} & k \ll \mathscr{H}\,, \\ e^{-N} & k \gg \mathscr{H}\,. \end{cases} \tag{3.17}$$

In Appendix B.1, we review how the result of [71] implies that equation (3.17) also holds in the case of large, non-linear inhomogeneities with cylindrical symmetry.

3.3 Positive curvatures

Inflation cannot occur in regions that collapse before inflation can commence. Such regions behave rather like local closed universes. Closed universes are those with compact space sections. In the FRW case this requires positive 3-curvature but in the most general closed universes (for example those of Bianchi type IX) the curvature can change sign with time and is mostly negative during any period of chaotic 'mixmaster' dynamics: it only becomes positive when the dynamics are close to isotropy. The fate of critically overdense regions in the pre-inflationary period mirrors that of the fate of closed universes. The most overdense regions may collapse to form primordial black holes after they enter the horizon [60], and then evaporate primarily into massless and relativistic particles by the Hawking effect. However, in any period of expansion that is dominated by the kinetic energy of a scalar field the corresponding Jeans length equals the horizon size and there is little scope for overdensities to collapse into black holes before they can be supported by pressure. Therefore the most pronounced overdensities will be filtered out by gravitational collapse. They can collapse before inflation can begin and leave the remaining lower-density regions to undergo inflation. Thus, in an inhomogeneous chaotic inflationary scenario, only the lower density regions that avoid premature gravitational collapse will be able to inflate and become candidate regions to contain our visible universe.

Another scenario might ensue if overdense regions collapse and bounce through a sequences of growing oscillations because of entropy increase [79]. These oscillating regions will approach flatness asymptotically unless a strong energy condition violating matter source, like a potential-dominated scalar field, comes to dominate. In that case the oscillations will cease and be replaced by unending inflationary expansion [80]. However, studies of the fate of anisotropic closed universes have shown that the sequence of growing oscillations demanded by the Second Law of thermodynamics becomes increasingly anisotropic [81, 82].

The issue of whether a closed region could envelop an open region in inflationary universes which are still very close to the critical density at late times was first posed by Zeldovich and Grishchuk [83] in the setting of a spherically symmetric model with S^3 spatial topology. This raises the question of the conditions for closed universes to collapse. It is difficult to answer in general because (unlike, for the singularity theorems) it involves properties of the general Einstein equations rather than simply of the geodesic equations. The general fate of closed universes is addressed by the *closed-universe recollapse conjecture* [84, 85]. It depends upon the spatial topology of the universe. Only spaces with S^3 or $S^2 \times S^1$ (and products thereof) can possess maximal hypersurfaces and hence have an expansion maximum and so collapse. These results were recently confirmed in reference [40]. This is a necessary condition but it is far from sufficient because various conditions must also apply to the matter content. Surprisingly, closed FRW universes can avoid recollapse even when $\rho > 0$ and $\rho + 3P > 0$ because they can experience finite-time infinities in the acceleration of the scale factor before a maximum is reached, [85] (so called 'sudden' singularities [86]). This can be avoided by imposing a matter regularity condition, like $|P| < C\rho$, with $C > 0$ constant or by continuity of $dP/d\rho$. Similar unusual behaviours for higher time-derivatives of the scale factor are also possible for scalar fields with fractional power-law potentials [87].[9]

3.4 Anisotropic spatial curvature

In our discussion of a possible phase of curvature-dominated expansion prior to inflation in §3.2, we focussed on the simple case of isotropic negative spatial curvature, which is a familiar ingredient in the Friedman equation for FRW universes with hyperbolic space sections. However, if we are interested in the effects of significant levels of inhomogeneity we need to take into account the effects of anisotropy as well. Cosmological anisotropies can be due to simple anisotropies in the expansion rate, with no anisotropy in the spatial curvature, and these are typified by the Kasner-like behaviour. They contribute an anisotropy energy density, σ^2, to the Friedman equation that falls off as a^{-6}, where a is the mean expansion scale factor. For these cosmological models the 3-curvature is of constant sign.

[9] A proof for collapse of closed Bianchi type IX universes was given by Lin and Wald [88] and other cases with S^3 and $S^2 \times S^1$ topologies in refs. [84, 85].

For a massless scalar field, ϕ in a Kasner metric,

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2, \tag{3.18}$$

we have

$$\phi(t) = \phi_0 + \sqrt{\frac{2}{3}}\ln(t), \quad \rho = \frac{\dot{\phi}^2}{2} = \frac{f^2}{2t^2}, \quad \sum_{i=1}^{3} p_i = 1 = \sum_{i=1}^{3} p_i^2 + f^2, \quad 0 \leq f^2 \leq 2/3\,. \tag{3.19}$$

When the constant $f^2 = 0$ this is the vacuum Kasner metric; when $f^2 = 2/3$, it becomes the isotropic FRW universe [89].[10]

We will show in Appendix B.2 that a period of evolution dominated by anisotropic 3-curvature produce a mean scale factor evolution $a \propto t^{1/(1+2\Sigma)}$ where $\Sigma = \sigma/\Theta$ is the constant shear (σ) to volume Hubble rate (Θ). Notice that the anisotropy falls very slowly ($\sigma \propto 1/t$), compared to the situation in Kasner models with isotropic 3-curvature. When $\Sigma = 0$ we reduce to the isotropic curvature dominated analysis of pre-inflation in §3.2. As the 3-curvature anisotropy increases and $\Sigma \to 1$ the average expansion dynamics that control the onset of inflation mimic a perfect fluid universe with an energy density

$$p/\rho \equiv w = \frac{(4\Sigma - 1)}{3}$$

with $-1/3 \leq w \leq 1/3$. The case with non-zero anisotropic curvature, anywhere in the allowed range of values ($0 < \Sigma \leq 1$) corresponds to the effect of a fluid with an equation of state running between that of the massless scalar field itself ($\Sigma = 1$) and that of the isotropic curvature. This reduces it to the analysis of fluid dominated pre-inflation in §3.2.

4 The severity of the problem of inhomogeneous initial data

In this section, we translate the numerical stability bound (equation (2.9)) into a condition on the pre-inflationary cosmology and the inflationary model. In §4.1, we show that this leads to a novel lower bound on the tensor-to-scalar ratio, r, in decelerating cosmologies with inhomogeneities imprinted when $H_{\rm IC} = 1$. In §4.3 we assess the severity of the problem for $H_{\rm IC} \ll 1$, and in §4.4 we consider the special case of pre-inflationary universes dominated by negative curvature.

4.1 The stability condition

Inflation fails if any mode is large enough to trigger destabilisation, so that the probability of a successful period of inflation is given by,

$$\mathscr{P}(\text{inflationary success}) = \prod_{|\mathbf{k}| \geq \mathscr{H}_{\rm i}} \mathscr{P}(|\delta\phi_{\mathbf{k}}| < \delta\phi_{\rm max}(k))\,, \tag{4.1}$$

where $\delta\phi_{\rm max}$ denotes the minimum amplitude of inhomogeneities that spoil inflation, and where $\mathbf{k}$ runs over the discrete set of Fourier modes within the horizon. In §2.2 we reviewed why modes with $k < \mathscr{H}_{\rm i}$ can be neglected, and also why modes with $k \approx \mathscr{H}_{\rm i}$ are expected to more dangerous for inflation than those with shorter wavelengths. For concreteness, we take as our definition of a 'stable'

[10] The ranges of the p_i are not disjoint: $-1/3 \leq p_1 \leq 1/3, 0 \leq p_2 \leq 2/3, 1/3 \leq p_3 \leq 1$.The metric has two free parameters, f and one of the p_i.

model to be one in which, with 95% probability, the total amplitude of field fluctuations in the range $\mathscr{D}_k \equiv [\mathscr{H}_\mathrm{i}, \sqrt{3}\mathscr{H}_\mathrm{i}]$ do not disrupt inflation according to the condition (2.9):

$$\mathscr{P}(\delta\phi_{\mathscr{D}_k} < f\Delta\phi) \geq 0.95\,. \tag{4.2}$$

The upper limit of $\mathscr{D}_k$ is chosen to match the '$N = 1$' simulations of reference [41]. Equation (4.2) neglects the possible failures due to higher-k modes, but as long as these are rarer than the failure due to the $k = \mathscr{H}_\mathrm{i}$ modes, this will result in a small correction to the overall survival probability for inflation. Moreover, equation (4.2) is the probability of inflationary success of each Hubble-sized patch at τ_i, but if the pre-inflationary universe involves a large number of such patches (as is expected in flat and open cosmologies), inflation can succeed globally despite a low probability of success of each patch. This makes the condition (4.2) a conservative one.

In §2.3 we parametrised the inhomogeneities at τ_IC as Gaussian fluctuations, consistent with the assumptions of references [39, 41]. In the continuous-k approximation, the variance of the dangerous modes with $k \in \mathscr{D}_k$ is given by,

$$\sigma^2(\tau_\mathrm{i})\Big|_{\mathscr{D}_k} = \int_{\mathscr{D}_k} \mathrm{d}\ln k\, P_{\delta\phi(\tau_\mathrm{i})}(k)\,. \tag{4.3}$$

We now restrict our discussion to pre-inflationary cosmologies with decelerated expansion. The power spectrum of field inhomogeneities at τ_i is given by,

$$P_{\delta\phi(\tau_\mathrm{i})}(k) = P_{\delta\phi(\tau_\mathrm{IC})}(k)\left(\frac{\mathscr{H}_\mathrm{i}}{k}\right)^q\,, \tag{4.4}$$

where the last factor captures the subhorizon evolution of the inhomogeneities. In the perturbative regime, $q = \frac{1}{2}$ for kinetic-energy domination and $q = \frac{2}{1+3w}$ for fluid domination, cf. equations (A.4) and (A.10). In this section, we focus on a narrow range of modes with $k \approx \mathscr{H}_\mathrm{i}$ and neglect this additional damping.[11] Moreover, in equation (4.4), we have assumed that the scalar field inhomogeneities do not evolve substantially on superhorizon scales (consistent with our findings in §3). The variance is then given by,

$$\sigma^2(\tau_i)\Big|_{\mathscr{D}_k} = A\,b\left(\frac{\mathscr{H}_\mathrm{i}}{\mathscr{H}_\mathrm{IC}}\right)^{n_\mathrm{IC}-1} = A\,b\left(\frac{H_\mathrm{i}\,a_\mathrm{i}}{H_\mathrm{IC}\,a_\mathrm{IC}}\right)^{n_\mathrm{IC}-1}\,, \tag{4.5}$$

where we have defined,

$$b = \frac{\sqrt{3}^{n_\mathrm{IC}-1} - 1}{n_\mathrm{IC} - 1}\,. \tag{4.6}$$

As $n_\mathrm{IC} \to 1$, $b \to \ln\sqrt{3}$; unless the spectrum is very steep, b tends not to be very large. The amplitude A is defined in equation (2.11), and we recall that $A = 1$ (which we will use as a 'generic' value in our estimates below) corresponds to $\delta\phi \sim \mathscr{O}(1)$ for modes with $k = \mathscr{H}_\mathrm{IC}$.

Equation (4.5) depends on the energy scale of inflation and the amount of expansion between τ_IC and τ_i. To make analytic progress, we parametrise the decreasing energy density by a single-fluid equation-of-state parameter w, which we take to be constant during the pre-inflationary epoch. We

[11] As we will discuss in §4.4, in the case of negative isotropic curvature-dominated pre-inflationary cosmologies, this subhorizon damping will become important.

note that this parametrisation includes, but is not limited to, the examples of decelerating cosmologies discussed in §3.2. At $\tau_{\rm i}$, the scalar potential has just begun to dominate the energy density so that,[12]

$$\frac{a_{\rm i}}{a_{\rm IC}} = \left(\frac{\rho_{\rm IC}}{\rho_{\rm i}}\right)^{\frac{1}{3(1+w)}} = \left(\frac{H_{\rm IC}}{H_{\rm i}}\right)^{\frac{2}{3(1+w)}} . \tag{4.7}$$

The variance of the scalar field inhomogeneities then simplifies to,

$$\sigma^2(\tau_i)\Big|_{\mathscr{D}_k} = A\, b \left(\frac{H_{\rm i}}{H_{\rm IC}}\right)^p , \tag{4.8}$$

where,

$$p = (n_{\rm IC} - 1)\frac{1+3w}{3(1+w)} . \tag{4.9}$$

Assuming that the amplitude $\delta\phi^2_{\mathscr{D}_k}$ is the square of a Gaussian with variance $\sigma^2 = \sigma^2(\tau_{\rm i})|_{\mathscr{D}_k}$, then with 95% probability, $|\delta\phi_{\mathscr{H}_{\rm i}}| < 2.0\sigma$.[13] Combined with the stability condition (4.2), this gives,

$$2.0\,\sigma(\tau_i)\Big|_{\mathscr{D}_k} < f\Delta\phi . \tag{4.10}$$

We take equation (4.10) as our condition for when a model is said to have no problem with initial inhomogeneities. Using equations (2.6) and (4.8), this leads to our main inequality:

$$\left(\frac{H_{\rm IC}}{H_{\rm i}}\right)^{2-p} < \frac{f^2\Delta\phi^2_{\rm ref}}{4A\, b}\left(\frac{H_{\rm IC}}{H_{\rm ref}}\right)^2 . \tag{4.11}$$

Equation (4.11) captures the competition between the two relevant effects: the lower the energy scale of inflation, the larger is the ratio $\mathscr{H}_{\rm IC}/\mathscr{H}_{\rm i}$, but also, the narrower is the inflationary plateau. This competition leads to two qualitatively different regimes depending on whether the power spectra are 'steep' with $p \geq 2$, or 'moderate' with $0 < p < 2$. We now discuss the implication of equation (4.11) for each of these possible cases.

4.2 A lower bound on r

For spectra with $p > 2$, equation (4.11) implies an *upper* bound on the inflationary energy scale,

$$\log H_{\rm i} < \frac{1}{p-2}\log\left(\frac{f^2\Delta\phi^2_{\rm ref}}{4AbH^2_{\rm ref}}H^p_{\rm IC}\right) . \tag{4.12}$$

Taking $\Delta\phi_{\rm ref}$ and $H_{\rm ref}$ as in equation (2.7), setting $f = 1.6$ as discussed around equation (2.8), and taking $A = H_{\rm IC} = 1$, and, for concreteness, $b = 1$ (neglecting its parameter dependence), we find,

$$\log H_{\rm i} < \frac{1}{p-2}\log\left(6.2\times 10^{14}\right) . \tag{4.13}$$

Since the right-hand-side is non-negative (for $p > 2$), but $\log H_{\rm i} < 0$ in any inflationary model, this inequality does not constrain $H_{\rm i}$. For $p = 2$, the dependence on the inflationary energy immediately cancels between the two competing effects, and $H_{\rm i}$ is again unconstrained. In other words, regardless of the energy scale of inflation, there is no problem with inhomogeneous initial data if their spectrum is steep and imprinted at the Planck scale.

[12]In highly inhomogeneous cosmologies, the energy density and Hubble parameter can be defined through spatial averaging, cf. [39, 41].

[13]If n independent Gaussian modes contribute to the amplitude, then $\delta\phi^2_{\mathscr{D}_k} \sim \sigma^2\chi^2_n$ so that $\sigma^2 = \frac{1}{n}\sigma^2(\tau_{\rm i})|_{\mathscr{D}_k}$. This leads to marginally stronger bounds that we will not consider here.

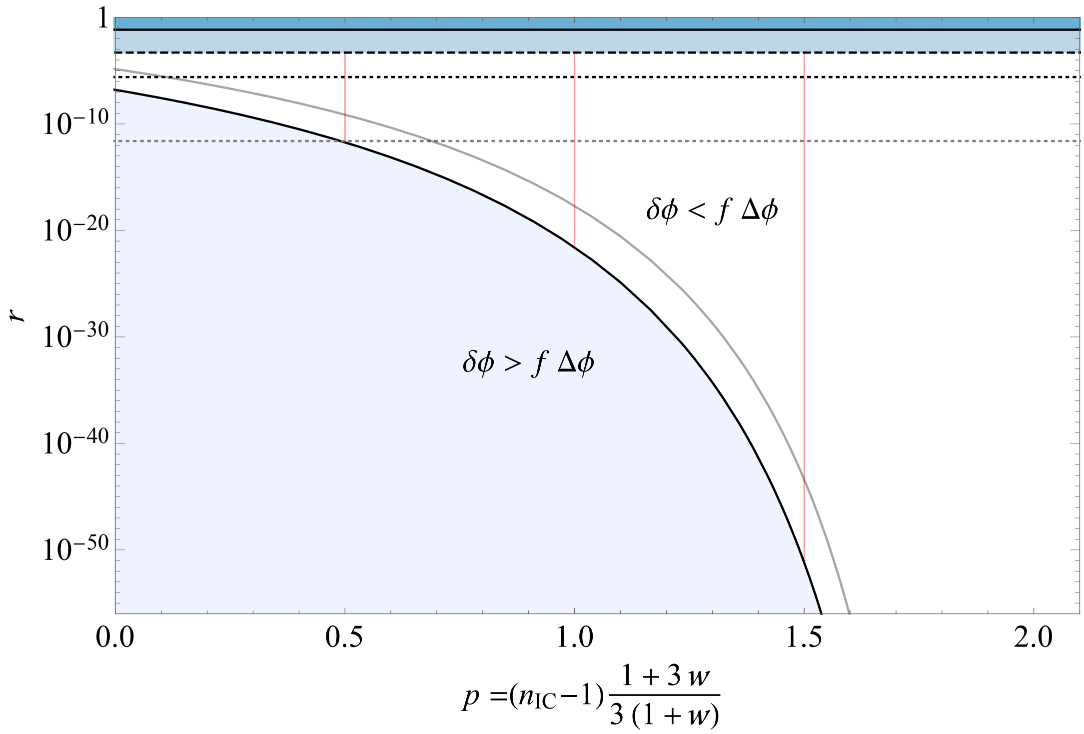

Figure 3. A lower bound on r: models with $\delta\phi < f\Delta\phi$ do not have a problem with inhomogeneous initial data imprinted at the Planck scale. The black curve corresponds to inflection-point potentials with a flat extension to negative field values ($f = 1.6$); the grey curve corresponds to sharply rising potentials ($f = 0.17$). Dotted horizontal lines, from top down, correspond to the boundary of the small-field region and the 'typical' model of reference [41]. Shaded regions on top correspond to current observational constraint from BICEP2/Planck, $r < 0.07$ (dark blue) and possible future constraint from CMB Stage-4 experiment with $\sigma(r) \approx 5 \times 10^{-4}$ (lighter blue). Vertical red lines correspond to $w = 1/3$ and, from left to right, $n_{\rm IC} = 2$, 3, 4.

For spectra with a 'moderate' fall-off, equation (4.11) implies an interesting *lower* limit on $H_{\rm i}$:

$$\log H_{\rm i} > -\frac{1}{2-p} \log\left(\frac{f^2 \Delta\phi_{\rm ref}^2}{4AbH_{\rm ref}^2} H_{\rm IC}^p\right) . \tag{4.14}$$

We may express this as a bound on r by using,

$$\log_{10} r = \log_{10} P_t - \log_{10} P_s = 8.0 + 2 \log_{10} H_{\rm i} \, , \tag{4.15}$$

where we have used the normalisation of the primordial power spectrum, $P_s = 2.2 \times 10^{-9}$, and $P_t = \frac{2}{\pi^2} H_{\rm i}^2$. This gives,

$$\log_{10} r > 8.0 - \frac{2}{2-p} \log_{10}\left(\frac{f^2}{4Ab} \frac{\Delta\phi_{\rm ref}^2}{H_{\rm ref}^2} H_{\rm IC}^p\right) . \tag{4.16}$$

Again taking $\Delta\phi_{\rm ref}$ and $H_{\rm ref}$ as in equation (2.7) and setting $A = H_{\rm IC} = b = 1$ and $f = 1.6$, we find the limit,

$$\log_{10} r > 8.0 - \frac{29.6}{2-p} . \tag{4.17}$$

Equation (4.17) is plotted in Figure 3. Models with tensor-to-scalar ratios above this limit have no problem with inhomogeneous initial data. We note that this includes rather vast regions of parameter space. For example, equation (4.17) implies that initial inhomogeneities with $n_{\rm IC} = 3$ that pass through a radiation dominated pre-inflationary era from the Planck scale are not problematic for inflationary models with,

$$r\big|_{p=1} > 2.5 \times 10^{-22} \,. \tag{4.18}$$

For potentials raising steeply for negative values, $f = 0.17$ as discussed following equation (2.8), and the constraint (4.17) becomes more severe: $\log_{10} r > 8.0 - \frac{25.7}{2-p}$ so that $r|_{p=1} > 2.0 \times 10^{-18}$.

The weakness of these lower bounds indicates that the inhomogeneous initial data problem is not in general severe even for small-field inflection-point models. It should be noted that the current Planck constraint (or even a hypothetical stronger, future constraint from ground-based CMB experiments) does not significantly impact the inhomogeneous initial data problem, contrary to the assertions of [10].

For our 'typical' reference model of equation (2.7), the bound (4.11) can be written as a constraint on p:

$$p > \frac{\log\left(\frac{f^2 \Delta\phi_{\rm ref}^2}{4Ab}\right)}{\log H_{\rm ref}} = 0.49 \,, \tag{4.19}$$

where in the last step we have specialised to $A = b = 1$ and $f = 1.6$. Consequently, for a radiation dominated pre-inflationary era, the reference model is safe from $\delta\phi_{k=\mathscr{H}_{\rm IC}}(\tau_{\rm IC}) \sim 1$ inhomogeneities if $n_{\rm IC} \geq 2.0$.

4.3 Initial data imprinted below the Planck scale

The energy density at the initial time $\tau_{\rm IC}$ is bounded from above by the Planck density, but $H_{\rm i} < H_{\rm IC} \ll 1$ is a general possibility. If the initial inhomogeneities were imprinted at energies much below the Planck scale, the pre-inflationary phase is shortened, and the initial data problem can become more severe.

Using equation (4.11), we see that avoiding the inhomogeneous initial data problem requires an initial Hubble rate satisfying,

$$H_{\rm IC} > \left(\frac{4AbH_{\rm ref}^2}{f^2\Delta\phi_{\rm ref}^2}\right)^{1/p} \frac{1}{H_{\rm i}^{(2-p)/p}} \,, \tag{4.20}$$

Figure 4 shows the bound on $H_{\rm IC}$ for the parameters $A = b = 1$, $f = 1.6$ and $\Delta\phi_{\rm ref}$ and $H_{\rm ref}$ as in equation (2.7). For moderate spectra with $p < 2$, we find that for $H_{\rm IC} < 4.0 \times 10^{-8}$, the entire family of small-field models are susceptible to disruption from inhomogeneities. Models with steep initial spectra ($p > 2$) are more robust. For our reference small-field model and a pre-inflationary epoch with $p = 1$ (e.g. given by a radiation domination and inhomogeneities with $n_{\rm IC} = 3$), the initial data problem is absent as long as $H_{\rm IC} > 1.0 \times 10^{-5}$.

4.4 Isotropic negative-curvature domination

In the case that the pre-inflationary universe is dominated by isotropic negative curvature, we have that $\epsilon_{\rm H} = 1$ and the comoving Hubble radius is constant. The 'dangerous' modes are then within the horizon during the entire pre-inflationary epoch. Using the perturbative damping of equation (A.16),

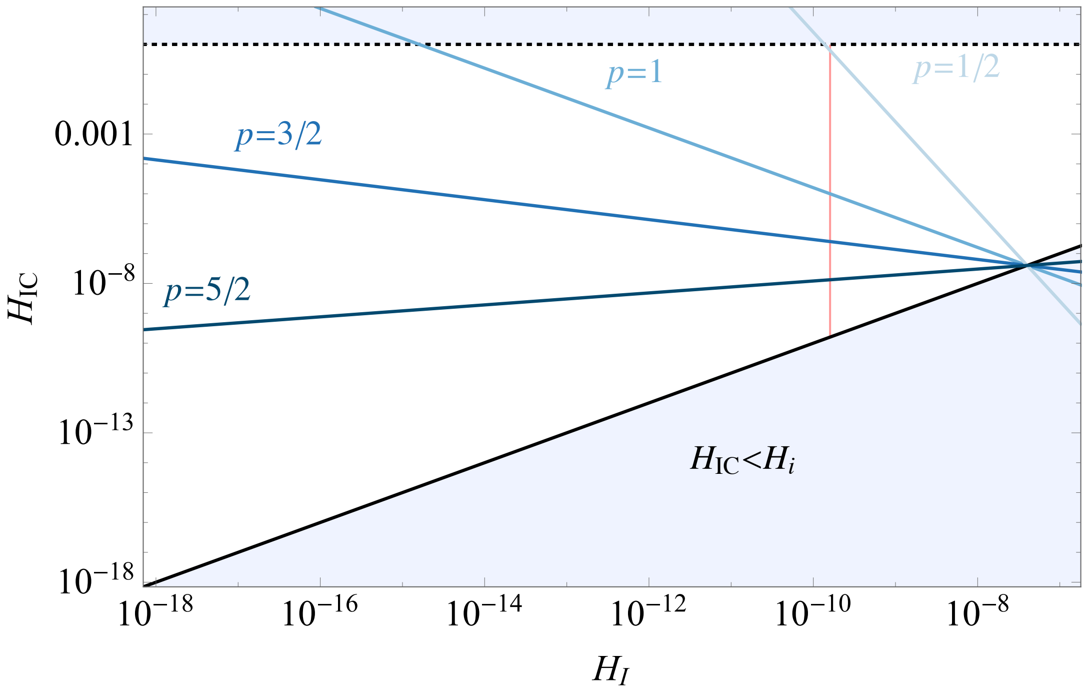

Figure 4. The bounds on $H_{\rm IC}$ above which initial inhomogeneities with unit amplitude fail to disrupt inflation.The vertical red line corresponds to the reference model.

and taking $\sigma^2(\tau_{\rm IC})\big|_{\mathscr{D}_k} = Ab$ with b as in equation (4.6), equation (4.10) gives,

$$H_{\rm IC} > \frac{2\sqrt{Ab}}{f\Delta\phi_{\rm ref}}\, H_{\rm ref}\,. \tag{4.21}$$

This inequality is the same as that obtained for a decelerating pre-inflationary epoch with $p = 2$, and is clearly independent of $H_{\rm i}$. Taking $A = b = 1$, $f = 1.6$, and using (2.7), we again find the bound,

$$H_{\rm IC} > 4.0 \times 10^{-8}\,. \tag{4.22}$$

Initial inhomogeneities imprinted above this energy scale are not expected to disrupt inflation.

5 Conclusions

We have studied the inhomogeneous initial data problem in small-field inflection-point models for inflation. These models only require the potential to be flat over a very limited field range $\Delta\phi \ll 1$, and are sensitive to disruption from scalar-field inhomogeneities with an amplitude $\delta\phi \sim \mathscr{O}(\Delta\phi)$ [39, 41, 44]. The problem of inhomogeneities is expected to be worse for this class of models than those with extended field ranges.

We have emphasised that the time at which the initial data were imprinted, $\tau_{\rm IC}$, may have greatly preceded the time $\tau_{\rm i}$ of the would-be onset of inflation, and that the most dangerous modes for disrupting inflation had wavelengths far bigger than the horizon at $\tau_{\rm IC}$. By parametrising the initial inhomogeneities at $\tau_{\rm IC}$, and the pre-inflationary evolution, we have used the numerically derived condition $\delta\phi\big|_{k\approx\mathscr{H}_{\rm i}} \leq f\Delta\phi$ to find those cosmological models that free from the disruptive inhomogeneous initial data problem. The resulting simple bound depends only on a few parameters of the pre-inflationary cosmology, and the energy scale of inflation.

For inhomogeneities with $\delta\phi(\tau_{\rm IC})\big|_{k=\mathscr{H}_{\rm IC}} \sim 1$ imprinted at the Planck scale, we expressed this bound as a lower limit on the tensor-to-scalar ratio, r. For many pre-inflationary cosmologies, this limit is very weak, which indicates that inflection-point inflation do not *in general* possess a problem with inhomogeneous initial data. In particular, these results give no credence to the envisioned doom of the inflationary scenario, proposed in reference [10], through the unnaturalness of the inhomogeneous initial data problem for small-field inflation.

Unless the initial spectrum of inhomogeneities falls off very steeply for super-horizon wavelengths and the pre-inflationary expansion is decelerating, the initial data problem becomes more severe if $H_{\rm IC}$ is much below the Planck scale. We find that all models in this class fail for $H_{\rm IC} \lesssim 4 \times 10^{-8}$.

Our analysis has a number of caveats. We have focussed on distributions of inhomogeneities that decrease in amplitude on scales far larger than the horizon (i.e. $n_{\rm IC} > 1$), which is physically well-motivated as we discussed in §2.3. For $n_{\rm IC} \leq 1$ (over some large but finite range of scales, so that the total power stays finite), the inhomogeneous initial data problem is expected to be more severe. Such a scenario is conceivable if the primary source of the initial inhomogeneities is much larger the initial Hubble radius.

Moreover, in §3.2 we reviewed the sub-horizon decay of scalar field inhomogeneities for small perturbations around a homogeneous background cosmology, and in Appendix B.1 we reviewed how similar results hold for the special case of nonlinear cylindrically symmetric scalar field inhomogeneities. For decelerating pre-inflationary backgrounds, modes that just enter the horizon at the onset of inflation are most dangerous, and the sub-horizon fall-off rate of small-wavelength inhomogeneities is not very important. However, for isotropic negative curvature dominated cosmologies, the dangerous modes are always inside the horizon. We expect that nonlinear evolution involving caustics, shock waves and gravitational collapse leads to a more rapid decay of sub-horizon inhomogeneities, thereby making inflation more robust than our estimates suggest.

Our estimates neglect the evolution of the scalar field inhomogeneities on super-horizon scales. Since this is a very small effect in both the perturbative cosmologies and the non-linear, cylindrically symmetric case, we expect our results to be rather broadly applicable. However, pre-inflationary cosmologies in which $\delta\phi_k$ is boosted on superhorizon scales can lead to more stringent bounds than those we derive.

Future numerical work could improve our bounds. The chance that modes with $k \ll \mathscr{H}_{\rm i}$ destabilises inflection-point inflation was only briefly studied in [41] (see also [39]). A determination of how the coefficient f in equation (2.9) grows with k could directly improve the determination of the full destabilisation probability. Moreover, references [39] and [41] studied the impact of modes with $k \geq \mathscr{H}_{\rm IC}$ on inflation. A full simulation of the inhomogeneous pre-inflationary phase preceding inflection-point inflation, including modes with $\mathscr{H}_{\rm i} \leq k \leq \mathscr{H}_{\rm IC}$, would provide an independent test of the bounds derived in this paper. Finally, our work has focused exclusively on single-field inflation, and it is possible that multi-field models of small-field inflation are more sensitive to initial inhomogeneities, leading to interesting constraints on the combination of the number of fields and the tensor-to-scalar ratio. Recent advances in constructing explicit inflationary models with many interacting fields [72–75] could allow for this question to be investigated in detail.

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A Perturbative pre-inflationary inhomogeneities

In this appendix, we provide details of the derivation of the general evolution of scalar field inhomogeneities summarised by equation (3.17). We consider in turn each of the cases listed in equation (3.7).

i) Kinetic-energy domination

In the limit where the scalar field's kinetic energy dominates the total energy density (and $\mathscr{K}$ is negligible), the equations for the metric perturbations, cf. (3.13)–(3.15), simplify to,

$$\frac{1}{\mathscr{H}^2}\nabla^2\Phi - 3\Phi' = \frac{\sqrt{6}}{2}\delta\phi' \,, \quad \Phi'' + \Phi' = \frac{\sqrt{6}}{2}\delta\phi' \,, \tag{A.1}$$

where $X' = \mathrm{d}X/\mathrm{d}N \equiv \mathrm{d}X/\mathrm{d}(\mathscr{H}\tau)$ denotes a derivative with respect to the number of e-folds. Re-expressed in flat-space Fourier modes,

$$\Phi_{\mathbf{k}}(\tau) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3x \Phi(\tau, x)\, e^{-ik\cdot x} \,, \tag{A.2}$$

the solutions of equation (A.1) are given by the Bessel functions,

$$\Phi_{\mathbf{k}}(\tau) = \mathscr{H}(\tau) \left(c_1\, J_1\left(\frac{k}{2\mathscr{H}}\right) + c_2\, Y_1\left(\frac{k}{2\mathscr{H}}\right) \right) . \tag{A.3}$$

For small arguments, $\frac{k}{2\mathscr{H}} \ll 1$, $\Phi_{\mathbf{k}}(\tau)$ has a constant and a decaying solution. Dropping the latter, we note that both the Newtonian potential and the scalar field perturbation are constant for $\frac{k}{2\mathscr{H}} \ll 1$.

Upon 'horizon entry' at $k = 2\mathscr{H}$, the modes begin damped oscillations. In the large argument expansion, $J_\nu(x),\ Y_\nu(x) \sim 1/\sqrt{x} \times \cos(x - \vartheta)$, with ϑ depending on ν and the type of Bessel function. It then follows that the envelopes of the gravitational potential and the scalar field decay as,

$$\Phi_{\mathbf{k}} \sim \left(\frac{\mathscr{H}}{k}\right)^{3/2} , \quad \delta\phi_{\mathbf{k}} \sim e^{-N} \sim \left(\frac{\mathscr{H}}{k}\right)^{1/2} . \tag{A.4}$$

ii) Fluid domination

When the fluid dominates the energy density (and $\mathscr{K}$ is negligible), the gravitational potential Φ is governed by the equation,

$$\Phi_{\tau\tau} + 3(1+w)\mathscr{H}\Phi_\tau - w\nabla^2\Phi = 0 \,. \tag{A.5}$$

Since we are considering a perfect, isentropic fluid, $\delta P = w\delta\rho$, and the sound-speed is given by $w = c_s^2$. Expressed in terms of the Fourier modes, $\Phi_k(\tau)$, the solutions to equation (A.5) are the Bessel functions,

$$\Phi_k(\tau) = \mathscr{H}^{\nu}\left(c_1\, J_\nu\left(\frac{2\sqrt{w}}{1+3w}\frac{k}{\mathscr{H}}\right) + c_2\, Y_\nu\left(\frac{2\sqrt{w}}{1+3w}\frac{k}{\mathscr{H}}\right)\right)\,, \tag{A.6}$$

for $\nu = \frac{1}{2}(5+3w)/(1+3w)$. In the long wave-length limit, $k/\mathscr{H} \ll \frac{2\sqrt{w}}{1+3w}$, these solutions are again given by one constant and one decaying mode. By contrast, short wave-length 'sound waves' oscillate and decrease in magnitude as,

$$\Phi_k(\tau) \sim \left(\frac{\mathscr{H}}{k}\right)^{\nu+\frac{1}{2}}\,. \tag{A.7}$$

The exponent is minimised as $w \to 1$, in which case the scaling agrees with the kinetic energy dominated case: $\Phi_\mathbf{k} \sim (\mathscr{H}/k)^{3/2}$. For $w = 1/3$, $\Phi_k \sim (\mathscr{H}/k)^2$. The scalar field perturbations are governed by the Klein-Gordon equation, which in this limit is given by,

$$\delta\phi_{\tau\tau} + 2\mathscr{H}\delta\phi_\tau - \nabla^2\delta\phi = 0\,. \tag{A.8}$$

The solutions for the Fourier modes can be expressed in terms of the Bessel functions,

$$\delta\phi_\mathbf{k} = \mathscr{H}^{-\tilde{\nu}}\left(c_1 J_{\tilde{\nu}}\left(\frac{2}{1+3w}\frac{k}{\mathscr{H}}\right) + c_2 Y_{\tilde{\nu}}\left(\frac{2}{1+3w}\frac{k}{\mathscr{H}}\right)\right)\,, \tag{A.9}$$

where $\tilde{\nu} = \frac{3}{2}\frac{w-1}{1+3w} < 0$. Again, for $k^2 \ll \mathscr{H}^2$, there exists a constant solution for $\delta\phi_k(\tau)$. For $k \gg \mathscr{H}$, $\delta\phi_\mathbf{k}$ goes through damped oscillations and decays like,

$$\delta\phi_\mathbf{k} \sim e^{-N} \sim \left(\frac{\mathscr{H}}{k}\right)^{\frac{2}{1+3w}}\,. \tag{A.10}$$

***iii)* Isotropic curvature domination**

In the negative-curvature dominated universe, the comoving Hubble parameter is constant and given by,

$$\mathscr{H}^2 = -\mathscr{K}\,. \tag{A.11}$$

The gravitational scalar perturbations are governed by the equation,

$$\Phi'' + 6\Phi' - \frac{1}{\mathscr{H}^2}\nabla^2\Phi + 8\Phi = 0\,. \tag{A.12}$$

Since the three-dimensional hypersurfaces are open and negatively curved, we cannot expand the solution in plane waves. Following [76, 77], we instead use the directly analogous expansion of Φ in terms of eigenfunctions of the Laplace operator on negatively curved spaces, which are characterised by their wave-number k satisfying $\nabla^2\Phi_k = -k^2\Phi_k$ for $k^2 \geq -K$. Expressed in terms of this basis of functions, equation (A.12) becomes,

$$\Phi_k'' + 6\Phi_k' + \left(8 + \frac{k^2}{\mathscr{H}^2}\right)\Phi_k = 0\,. \tag{A.13}$$

Since $\mathscr{H}$ is constant during this era, the solutions are simply given by,

$$\Phi_k = e^{-3N}\left(c_1 e^{-iN\sqrt{\tilde{k}^2-1}} + c_2 e^{iN\sqrt{\tilde{k}^2-1}}\right)\,, \tag{A.14}$$

where $\tilde{k}^2 = k^2/\mathscr{H}^2 \geq 1$.

The scalar field perturbations are determined by equation (3.14),

$$\frac{1}{2}\bar{\phi}'\delta\phi_{\mathbf{k}} = \Phi_k + \Phi_k' \,. \tag{A.15}$$

Since $\bar{\phi}' \sim \exp(-2N)$ (cf. equation (3.10) for $w = -1/3$), the scalar field perturbation oscillates with an amplitude that decreases like,

$$\delta\phi_{\mathbf{k}} \sim e^{-N} \sim \frac{H}{H_{\rm IC}} \,, \tag{A.16}$$

for all k.

Therefore, in all of the pre-inflationary cosmologies that we consider, the perturbative scalar field inhomogeneities behave like,

$$\delta\phi_{\mathbf{k}} \sim \begin{cases} \text{constant} & k \ll \mathscr{H} \,, \\ e^{-N} & k \gg \mathscr{H} \,. \end{cases} \tag{A.17}$$

In pre-inflationary cosmologies with successive eras during which the cosmic energy density is dominated by different sources, the super-horizon modes of $\Phi_{\mathbf{k}}$ and $\delta\phi_{\mathbf{k}}$ evolve only mildly during transition periods (in direct analogy to the shift in superhorizon modes of Φ by 9/10 at matter-radiation equality after inflation).

Large inhomogeneities are not captured by this analysis. In some special cases with large and highly symmetric inhomogeneities, the perturbative results extend straightforwardly, as we exemplify in Appendix B.1. More general non-linear inhomogeneities may undergo prompt gravitational collapse after they enter the horizon, leading to a quicker decay of long-wavelength inhomogeneities than equation (A.17) suggests.

B Non-linear pre-inflationary inhomogeneities

In this appendix, we bring together relevant results on the evolution of non-linear inhomogeneous cosmologies that admit exact solutions to Einstein's equations. In Appendix B.1, we show that cylindrical scalar field inhomogeneities evolve much like small perturbations around the FRW solutions, and in Appendix B.2 we provide further details on pre-inflationary universes with anisotropic, negative spatial curvatures.

B.1 Large, cylindrical inhomogeneities

In this section we examine the fate of a special class of nonlinear inhomogeneities in the scalar field. When the metric possesses cylindrical symmetry it is possible to solve the problem exactly, as first discovered for the vacuum problem by Einstein and Rosen [78]. We take the metric to be,

$$ds^2 = -e^{2(\chi-\psi)}(d\tau^2 - dr^2) + a^2(e^{2\psi}dz^2 + e^{-2\psi}(rd\theta)^2) \,, \tag{B.1}$$

where the free functions are $a^2(\tau, r), \psi(\tau, r)$ and $\chi(\tau, r)$. We ignore ψ as it contains the gravitational-wave behaviour.

If we add a massless, homogeneous scalar field $\bar{\phi}(\tau)$, we recover the flat, kinetic-energy dominated FRW universe as the special case $\psi = 0$, $a^2 = \exp(2\chi) = \tau$ so that,

$$p = \rho = \frac{1}{2}e^{-2\chi}\bar{\phi}_\tau^2 = \rho(\tau_{\rm IC})\left(\frac{\tau_{\rm IC}}{\tau}\right)^3 \,.$$

The background value of the field grows like $\bar{\phi} \sim \ln(\tau)$, consistent with equation (3.10).

The inhomogeneous exact solution for the metric (B.1) generalises this to give decoupled Bessel equations for $\phi(r,t)$ and $\psi(r,t)$ The solution for $\phi(r,t)$ is [71]:

$$\phi(\tau, r) = \bar{\phi}(\tau) + \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \mathrm{d}k \, e^{ikr} \phi_k(\tau) \,, \tag{B.2}$$

where the Fourier modes are given by,

$$\phi_k(\tau) = c_1 J_0(k\tau) + c_2 Y_0(k\tau) \,. \tag{B.3}$$

Qualitatively, the evolution is straightforward and identical to that of perturbative inhomogeneities. The ratio of the coordinate size of the inhomogeneity wavelength to the particle horizon coordinate size is $k\tau$. On large (superhorizon) scales, where $k\tau \to 0$, we have $J_0 \to 1$ and $Y_0 \to \ln(k\tau)$, so the evolution is Kasner-like ($\psi \propto \ln(\tau) \propto \phi$) or FRW-like ($\psi = 0$) depending on whether $c_2/(2\pi)^{3/2}$ is bigger or less than the first (homogeneous) term on the right-hand side of equation (B.2). Superhorizon inhomogeneities imprinted at $\tau_{\rm IC}$ in general include a constant and a decaying mode, precisely as we found in §3.2 for perturbative inhomogeneities.

When the inhomogeneity enters the horizon (which equals the Jeans length) the future evolution is given by the small-scale $k\tau \to \infty$ limit of the Bessel functions. Due to its restrictive symmetries, this solution does not admit bound regions and gravitational collapse. Moreover, the scalar field waves propagate at the speed of light so there can be no shocks. Precisely as in the case of perturbative inhomogeneities, the scalar field inhomogeneities are pressure-damped away through oscillatory decay: $\phi_k(\tau) \simeq \frac{1}{\sqrt{k\tau}} \times \sin(k\tau - \vartheta)$. Thus, also in this case,

$$\delta\phi_{\mathbf{k}} \sim \begin{cases} \text{constant} & k \ll \mathscr{H} \,, \\ e^{-N} & k \gg \mathscr{H} \,. \end{cases} \tag{B.4}$$

Further details of the evolution of this exact solution and the cosmological case with radiation can be found in [71].

B.2 Anisotropic spatial curvature

In §3.4 we reviewed how anisotropic negative curvature leads to an average expansion rate that mimics that of a perfect fluid with an effective equation of state in the range $-1/3 \leq w \leq 1/3$. In this section, we provide further details of this analysis.

Apart from anisotropies in the expansion rate (discussed in §3.4), there can also be anisotropies in the 3-curvature. This is the most general form of anisotropy and is non-Newtonian, deriving from the magnetic part of the Weyl curvature. Cosmological models with anisotropic 3-curvature can have very complex evolution, with chaotic dynamics near the initial singularity and the sign of the 3-curvature scalar can be time-dependent. Here we outline the new effects created by anisotropic 3-curvature.

The most general anisotropic Bianchi universes that contain the open Friedmann model as a special subcase are those of type VII_h. The late-time asymptotes for the non-tilted type VII_h spacetimes, with $h \neq 0$ and a matter content that obeys the strong energy condition (so no inflation), evolve towards the vacuum plane-wave metric found by Doroshkevich et al and Lukash [90, 91] that is known as the Lukash metric. These spacetimes describe the most general effects of spatially homogeneous perturbations on open FRW universes. When the strong energy condition is obeyed, then isotropic expansion was found to be stable but not asymptotically stable at late times [84, 92, 93].

The line element of the Lukash metric takes the form

$$ds^2 = -dt^2 + t^2 dx^2 + t^{2r} e^{2rx} \left[(A dy + B dz)^2 + (C dy + A dz)^2 \right] , \tag{B.5}$$

where r is an arbitrary constant parameter in the range $0 < r < 1$, $A = \cos v$, $B = f^{-1} \sin v$, $C = -f \sin v$ and $v = k(x + \ln t)$ [94–96]. Note that f, k and r are related by

$$\frac{k^2}{f^2}(1 - f^2)^2 = 4r(1 - r) \qquad \text{and} \qquad r^2 = hk^2 , \tag{B.6}$$

where h is the associated group parameter. For $r = 1$ and $f^2 = 1$ the Lukash metric reduces to the empty isotropic Milne universe, with scale factor $a = t$. More details can be found in ref. [97] and other universes with anisotropic curvature can be seen, along with their effects on primordial nucleosynthesis in [98].[14]

If we use the average scale factor (a) to define the volume expansion rate via the standard relation $\Theta = 3\dot{a}/a$, then the average volume expansion of the vacuum Lukash universe is described by the following version of the Raychaudhuri equation

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - 2\sigma^2 , \tag{B.7}$$

where $\sigma^2 = \sigma_{ab}\sigma^{ab}/2$ is the magnitude of the shear tensor.

The absence of matter means that the Lukash spacetime is Ricci flat. The curvature of the spatial sections, however, is not zero. In particular, the 3-Ricci tensor ($\mathscr{R}_{ab}$) is completely determined by its scalar and its symmetric and trace-free parts, that is respectively by

$$\mathscr{R} = -\tfrac{2}{3}\Theta^2 + 2\sigma^2 , \tag{B.8}$$

$$\mathscr{S}_{ab} = -\tfrac{1}{3}\Theta\sigma_{ab} + \sigma_{c\langle a}\sigma^{c}{}_{b\rangle} + E_{ab} , \tag{B.9}$$

where $\mathscr{S}_{ab} = \mathscr{R}_{\langle ab \rangle} = \mathscr{R}_{(ab)} - \mathscr{R}h_{ab}/3$.[15] The scalar $\mathscr{R}$ is negative, which means that the model is spatially open. The expression (B.8) is the generalised Friedmann equation.

The magnitude of the shear tensor associated with the Lukash solution is

$$\sigma^2 = \tfrac{1}{2}\sigma_{\alpha\beta}\sigma^{\alpha\beta} = \frac{(1 - r)(1 + 2r)}{3t^2} . \tag{B.10}$$

The mean Hubble volume expansion of the Lukash universe is determined by the scalar

$$\Theta = \frac{1 + 2r}{t} . \tag{B.11}$$

Hence, the average scale factor obeys the simple power law $a \propto t^{(1+2r)/3}$ with $0 < r < 1$.

When measuring the average anisotropy of the expansion, it helps to introduce the following dimensionless and expansion-normalised shear parameter

$$\Sigma \equiv \frac{3\sigma^2}{\Theta^2} . \tag{B.12}$$

[14] If the spatial topology of type VII_h, or even type V, open universes is made compact then they are all constrained to be isotropic [99].

[15] Angled brackets denote the symmetric and trace-free part of orthogonally projected tensors and the orthogonally projected components of vectors.

In the Lukash spacetime the scalars σ^2 and Θ are given by (B.10) and (B.11) respectively. Using these expressions we see that Σ is constant and

$$\Sigma = \frac{1-r}{1+2r}\,. \tag{B.13}$$

Given that $\Sigma > 0$ and $0 < r < 1$, we immediately deduce that $0 < \Sigma < 1$, in accord with $\mathscr{R} < 0$, in (B.8). Thus, although isotropy ($\Sigma = 0$) is not *asymptotically stable* (in the Lyapunov sense), it is *stable* in the sense that any deviations from isotropy never diverge [84, 93, 94] and Σ tends to a constant at large times. Note that when we set $r \to 1$ the Σ-parameter approaches zero and the expansion becomes isotropic (i.e. the Milne universe).

We can therefore also write the power-law evolution of the average scale factor as $a \propto t^{1/(1+2\Sigma)}$. Thus, in the absence of any shear anisotropy we have $a \propto t$, as in the Milne universe. For maximum shear anisotropy as $r \to 0$, we obtain the familiar scale-factor evolution characteristic of the Kasner vacuum or kinetic-dominated scalar field solution 3.2 (i.e. $a \propto t^{1/3}$).

The trace of the 3-Ricci tensor $\mathscr{R}_{\alpha\beta}$ associated with the surfaces of constant time is

$$\mathscr{R} = -\frac{k^2(1-f^2)^2}{2f^2t^2} - \frac{6r^2}{t^2} = -\frac{2r(1+2r)}{t^2} < 0. \tag{B.14}$$

Spatial curvature anisotropies are described via the symmetric and trace-free tensor $\mathscr{S}_{\alpha\beta}$. The only non-zero components of $\mathscr{S}_{\alpha\beta}$ are

$$\mathscr{S}_{11} = -\frac{4r(1-r)}{3t^2}\,, \qquad \mathscr{S}_{22} = \frac{k^2(1-f^2)(2+f^2)}{3f^2t^2}\,, \qquad \mathscr{S}_{33} = -\frac{k^2(1-f^2)(1+2f^2)}{3f^2t^2} \tag{B.15}$$

and

$$\mathscr{S}_{23} = -\frac{kr(1-f^2)}{ft^2}\,. \tag{B.16}$$

According to (B.14)-(B.16), the spatial curvature of the model vanishes at the maximum shear limit, namely as $r \to 0$, hence the approach to the Kasner expansion rate. When $r \to 1$, only the isotropic part of $\mathscr{R}_{\alpha\beta}$ survives as $k^2(1-f^2) = 0$ as $r \to 0$ or 1.

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