Differentially passive circuits that switch and oscillate

F.A. Miranda-Villatoro * **F. Forni** * **R. Sepulchre** *

* *Department of engineering, University of Cambridge, UK*
(e-mail: fam48@cam.ac.uk, f.forni@eng.cam.ac.uk, r.sepulchre@eng.cam.ac.uk).

Abstract: The concept of passivity is central to analyze circuits as interconnections of passive components. We illustrate that when used *differentially*, the same concept leads to an interconnection theory for electrical circuits that switch and oscillate as interconnections of passive components with operational amplifiers (op-amps). The approach builds on recent results on dominance and p-passivity aimed at generalizing dissipativity theory to the analysis of non-equilibrium nonlinear systems. Our paper shows how those results apply to basic and well-known nonlinear circuit architectures. They illustrate the potential of dissipativity theory to design and analyze switching and oscillating circuits quantitatively, very much like their linear counterparts.

Keywords: switches, oscillators, differential passivity

1. INTRODUCTION

The concept of passivity originates in circuit theory. It characterizes circuit elements that can possibly store and dissipate the energy provided by the environment, but not the other way around. Passivity is inherently an interconnection concept: passive interconnections of passive components model passive circuits. Dissipativity theory, the system theoretic generalization of passivity theory, has become a cornerstone of system theory. It provides an interconnection theory to design and analyze stable dynamical systems. Such systems dissipate the energy stored internally and provided externally. In short, dissipativity theory is an interconnection theory for Lyapunov stability analysis.

In recent years, many researchers have pointed to the relevance of studying stability *incrementally* or *differentially* when addressing questions that go beyond the stability analysis of isolated equilibria. Differential stability concepts include contraction theory (Lohmiller and Slotine, 1998), convergence theory (Pavlov et al., 2005), or differential Lyapunov theory (Forni and Sepulchre, 2014). They have proven relevant in a number of areas, most prominently in questions pertaining to nonlinear observers (Aghannan and Rouchon, 2003), oscillator synchronization (Stan and Sepulchre, 2007), or regulation theory (Jouffroy and Fossen, 2010). Differential dissipativity is to differential stability what dissipativity is to Lyapunov stability. It was introduced in the recent papers (Forni et al., 2013; Forni and Sepulchre, 2013; van Der Schaft, 2013).

The present paper aims at illustrating the potential of differential passivity as an interconnection theory of circuits that can switch an oscillate. In contrast with classical dissipativity theory, such a theory must cope with the stability analysis of dynamical systems that posses multiple equilibria or limit cycles. It is based on the concept of p-dominance and p-passivity recently introduced in Forni and Sepulchre (2017a,b). Intuitively, the attractors of a p-dominant system are the attractors of a p-dimensional system: a unique equilibrium for $p = 0$, but possibly multiple equilibria for $p = 1$, and limit cycles for $p = 2$. We interpret classical differential dissipativity theory as an interconnection theory for 0-dominance, that is, differential stability. The extension to $p = 1$ and $p = 2$ is motivated by the analysis of multistability or limit cycles in interconnected systems.

Nonlinear circuit theory provides a realm of switching and oscillatory behaviors designed from the simple elements of linear circuit theory interconnected with operational amplifiers (op-amp) (Clayton and Winder, 2003). Our aim in the present paper is to illustrate that those building blocks are the natural building blocks of p-passive circuits. System theoretic tools have lacked so far for the quantitative analysis and design of such circuits. Their analysis normally rests on simplifying assumptions, time-scale separation arguments leading to asymptotic analysis, or reductions to two-dimensional phase portraits. In contrast, we are aiming at quantitative and computationally tractable certificates such as those used in the theory of linear time-invariant systems. Such tools have made the success of robustness and performance analysis of linear time-invariant systems.

A pillar of passivity theory is the passivity theorem, which states that the negative feedback interconnection of two passive systems is passive. It is also well-known that switches and oscillatory circuits require both positive and negative feedback interconnections of op-amps with passive elements. We stress in the present paper that such interconnections fall in the category of the p-passivity theorem, which states that the negative feedback intercon-

⋆ The research leading to these results has received funding from the European Research Council under the Advanced ERC Grant Agreement Switchlet n. 670645.

nection of a p_1-passive with a p_2-passive circuit is $p_1 + p_2$ passive.

The differential analysis in this paper assumes smooth systems. A companion paper shows that an analog framework exists for non-smooth systems. To account for the lack of differentiability, *differential* concepts have then to be replaced by *incremental* concepts. Many of the nonlinear circuits discussed in the present paper have a non-smooth analog that falls in the category of linear complementarity systems studied in Miranda-Villatoro et al. (2018).

The rest of the paper is organized as follows. Section 2 provides a brief summary of the concepts of dominance and p-passivity. Section 3 revisits standard properties of the operational amplifier its differential passivity properties. Section 4 revisits the basic architectures of circuits that switch and oscillate, analyzing those systems as both positive and negative feedback interconnections of operational amplifiers with passive linear circuits. The discussion in Section 5 suggests that those architectures are robust and amenable to regulation.

2. DOMINANCE AND DIFFERENTIAL PASSIVITY

We consider the nonlinear system

$$\dot{x} = f(x) \tag{1}$$

where $x \in \mathbb{R}^n$ and f is a smooth vector field. The *prolonged system* consists of (1) augmented with the linearized equation $\dot{\delta x} = \partial f(x)\delta x$, where $\partial f(x)$ denotes the Jacobian linearization of f. By construction $\delta x \in \mathbb{R}^n$ (identified with the tangent space of $\mathbb{R}^n$).

An important notion for this paper is the inertia $(p, 0, n-p)$ of a symmetric matrix, meaning that the matrix has p eigenvalues in the open left half-plane, 0 eigenvalues on the imaginary axis, and $n-p$ eigenvalues in the right half-plane. The following definition is taken from (Forni and Sepulchre, 2017a).

Definition 1. A nonlinear system (1) is *p-dominant with rate* $\lambda \geq 0$ if there exist a constant symmetric matrix P with inertia $(p, 0, n-p)$ and $\varepsilon \geq 0$ for which the prolonged system satisfies

$$\begin{bmatrix} \dot{\delta x} \\ \delta x \end{bmatrix}^\top \begin{bmatrix} 0 & P \\ P & 2\lambda P + \varepsilon I \end{bmatrix} \begin{bmatrix} \dot{\delta x} \\ \delta x \end{bmatrix} \leq 0 \tag{2}$$

for all $(x, \delta x) \in \mathbb{R}^n \times \mathbb{R}^n$ The property is strict if $\varepsilon > 0$. ⌟

Solving (2) is equivalent to finding a uniform solution P to the linear matrix inequalities $\partial f(x)^\top P + P\partial f(x) + 2\lambda P \leq -\epsilon I$ for all $x \in \mathbb{R}^n$. For a linear system $\dot{x} = Ax$, the inequality reduces to $(A+\lambda I)^\top P + P(A+\lambda I) \leq -\epsilon I$, which is feasible for linear systems whose eigenmodes can be split into p *dominant* modes and $n-p$ *transient* modes, separated by the rate λ, (Forni and Sepulchre, 2017b, Proposition 1). For nonlinear systems p-dominance captures the property that the asymptotic behavior of the system is p-dimensional. This intuitive characterization is made precise in the following result (Forni and Sepulchre, 2017a, Corollary 1).

Theorem 1. Let (1) be a strictly p-dominant system with rate $\lambda \geq 0$. Then, every bounded solution of (1) asymptotically converge to

- a unique fixed point if $p = 0$;
- a fixed point if $p = 1$;
- a simple attractor if $p = 2$, i.e. a fixed point, a set of fixed points and their connected arcs, or a limit cycle.

In what follows we will study p-dominant systems as interconnections of open systems. We consider open systems of the form

$$\dot{x} = f(x) + Bu\,, \quad y = Cx \tag{3}$$

where $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^q$ define the input and the output to the system, respectively. B and C are matrices of appropriate dimension. The prolonged system to (3) is obtained by augmenting (3) with the linearized equations $\dot{\delta x} = \partial f(x)\delta x + B\delta u$, $\delta y = C\delta x$. The following definition is taken from (Forni and Sepulchre, 2017a).

Definition 2. A nonlinear system (3) is p-passive from u to y with rate $\lambda \geq 0$ if there exist a constant symmetric matrix P with inertia $(p, 0, n-p)$ and $\varepsilon \geq 0$ for which the prolonged system satisfies

$$\begin{bmatrix} \dot{\delta x} \\ \delta x \end{bmatrix}^\top \begin{bmatrix} 0 & P \\ P & 2\lambda P + \varepsilon I \end{bmatrix} \begin{bmatrix} \dot{\delta x} \\ \delta x \end{bmatrix} \leq \begin{bmatrix} \delta y \\ \delta u \end{bmatrix}^\top \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \delta y \\ \delta u \end{bmatrix} \tag{4}$$

for all $(x, \delta x) \in \mathbb{R}^n \times \mathbb{R}^n$ and all $(u, \delta u) \in \mathbb{R}^m \times \mathbb{R}^m$. The property is strict if $\varepsilon > 0$. ⌟

The concept of p-passivity is related to p-dominance in the same way as passivity is related to stability. Differential (Forni and Sepulchre, 2013) or incremental (Pavlov and L, 2008) passivity are synonyms of 0-passivity. For a static differentiable nonlinearity $y = \varphi(u)$, 0-passivity simply means monotonicity, that is positivity of its derivative: if $\partial\varphi(u) \geq 0$, then $\delta y^T \delta u = (\partial\varphi(u)\delta u)^\top \delta u \geq 0$ for all δu.

The following p-passivity theorem is the natural extension of the classical passivity theorem. It is taken from (Forni and Sepulchre, 2017a, Theorem 4).

Theorem 2. Let Σ_1 and Σ_2 be (strictly) p_1 and p_2 passive, respectively, from input u_i to output y_i, $i \in \{1, 2\}$, both with rate $\lambda \geq 0$, Then, the negative feedback interconnection

$$u_1 = -y_2 + v_1, \quad u_2 = y_1 + v_2$$

of Σ_1 and Σ_2 is (strictly) $(p_1 + p_2)$-passive from $v = (v_1, v_2)$ to $y = (y_1, y_2)$, with rate λ. The interconnection is (strictly) $(p_1 + p_2)$- dominant.

We observe that negative feedback preserves p-passivity only if the two components share a common rate λ.

For linear systems of the form $\dot{x} = Ax + Bu$, $y = Cx$, p-passivity has a useful frequency domain characterization in terms of the shifted transfer function $G(s - \lambda) = C(sI - (A + \lambda I))^{-1}B$, as shown by the next theorem from Miranda-Villatoro et al. (2017).

Theorem 3. A linear system is p-passive with rate λ if and only if the following two conditions hold,

(1) $\Re\{G(j\omega - \lambda)\} \geq 0$, for all, $\omega \in \mathbb{R} \cup \{+\infty\}$.
(2) $G(s - \lambda)$ has p poles on $\mathbb{C}_+$.

The property is strict if $G(s-\lambda)$ has p poles in the interior of $\mathbb{C}_+$

3. THE OPERATIONAL AMPLIFIER IS 0-PASSIVE

Figure 1 represents a classical model of operational amplifiers (Karki, 2000; Noseek, 2009): the RC network is

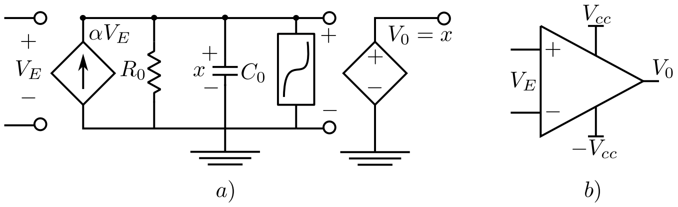

Fig. 1. First order op-amp model with saturation: a) internal structure, b) symbolic representation.

a circuit realization of a first order linear model in parallel with a voltage-controlled current source αV_E, where $\alpha \in (0, +\infty)$, (Noseek, 2009). The RC network models the finite bandwidth property of a real device and it is connected to a static nonlinear element

$$i = \varphi(V) \tag{5}$$

to account for the bounded range of $V_0 = x$. The static nonlinear element is a smooth and odd nonlinearity modelled for instance as follows:

$$i = \varphi(V) := \eta \sinh(\beta V) \tag{6}$$

where $\eta > 0$ and $\beta > 0$ are suitable parameters.

The results of this paper hold for any stiffening nonlinearity φ satisfying the following assumption (Stan and Sepulchre, 2007).

Assumption 4. The static nonlinearity $\varphi : \mathbb{R} \to \mathbb{R}$ is an odd function such that $\frac{\partial \varphi(y)}{\partial y} \in [0, +\infty)$. Furthermore, for any $k > 0$, there exists a $r > 0$ such that

$$y\varphi(y) - ky^2 > 0, \quad \text{for all } |y| > r \ . \tag{7}$$

For example, $\varphi(y) = y^{2n+1}$, for $n \in \mathbb{N}$, $\varphi(y) = \text{arctanh}(y)$ and $\varphi(y) = \sinh(y)$ all satisfy Assumption 4.

The first-order model of the op-amp in Figure 1 has the state-space model

$$\Sigma_{op} : \begin{cases} \dot{x} = -\dfrac{1}{R_0 C_0} x - \dfrac{1}{C_0} \varphi(x) + \dfrac{\alpha}{C_0} V_E \\ V_0 = x \end{cases} \tag{8}$$

which, notably, admits the block diagram representation of the Lur'e system in Figure 2.

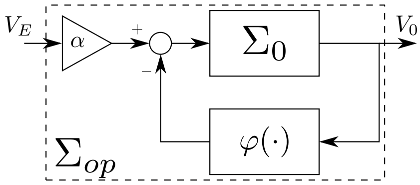

Fig. 2. Open-loop operational amplifier model

The transfer function of the linear part Σ_0

$$G_0(s) = \frac{\frac{1}{C_0}}{s + \frac{1}{R_0 C_0}} \tag{9}$$

is a strictly 0-passive network (by Theorem 3) with rate $\lambda \in [0, \frac{1}{R_0 C_0})$. The op-amp is thus given by the negative feedback interconnection of a strictly 0-passive linear system with a static 0-passive nonlinearity (under Assumption 4). Therefore, by Theorem 2, the closed loop is a strictly 0-passive system from V_E to V_0 with rate $\lambda \in [0, \frac{1}{R_0 C_0})$. The same representation would hold if the RC circuit was replaced by any 0-passive network.

4. SWITCHES AND OSCILLATORS VIA FEEDBACK AMPLIFIERS

4.1 Feedback and boundedness

The great versatility of the op-amp comes from its interconnection properties. The device allows for a wide range of behaviors, enabled by the interconnection of the op-amp with suitable linear networks.

The circuits in this paper will only include interconnections of op-amps with linear stable networks

$$\dot{z} = Az + Bu, \quad y = Cz \ . \tag{10}$$

where $z \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$ are the state, input, and output of a generic linear network, respectively. A, B and C are constant matrices of appropriate dimensions.

Such interconnections always lead to bounded behaviors:

Theorem 5. Suppose that A is a Hurwitz matrix. Under Assumption 4, the trajectories of the system defined by (8), (10), and the interconnection rule

$$V_E = \pm Cz + V_r \quad u = V_0 \tag{11}$$

are all bounded, for any constant voltage $V_r \in \mathbb{R}$.

Proof. Let $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be the positive definite function

$$V(x, z) = \frac{1}{2}x^2 + \frac{1}{2}z^\top P z, \tag{12}$$

where $P = P^\top > 0$ satisfies $A^\top P + PA \leq -Q$, for some $Q = Q^\top > 0$. Then, taking $\eta = \frac{1}{R_0 C_0}$, $\beta = \frac{1}{C_0}$, $\rho = \alpha\|C\| + 2\|PB\|$, $\varepsilon > 0$, and μ_Q given by the smallest eigenvalue of Q, we have

$$\begin{aligned} \dot{V}(x, z) &= -\eta x^2 - z^\top Q z - \beta x \varphi(x) \pm \alpha x C z + 2z^\top P B x \\ &\leq -\eta x^2 - \mu_Q \|z\|^2 - \beta x \varphi(x) + \rho |x| \|z\| \\ &\leq -\eta x^2 - \mu_Q \|z\|^2 - \beta x \varphi(x) + \frac{\rho}{\varepsilon}|x|^2 + \rho\varepsilon\|z\|^2 \ . \end{aligned}$$

Setting $\varepsilon = \frac{\mu_Q}{2\rho}$ yields

$$\dot{V}(x, z) \leq -\eta x^2 - \frac{\mu_Q}{2}\|z\|^2 - \beta x \varphi(x) + \frac{\rho}{\varepsilon}|x|^2.$$

From (7), there exists $r > 0$ such that, for all $|x| > r$, $\dot{V}(x, z) \leq -\eta x^2 - \frac{\mu_Q}{2}\|z\|^2$. Boundedness of solutions follows. ■

In the next sections we will design particular interconnections based on Theorems 1 and 2.

4.2 p-Passivity and interconnections

We consider the interconnection of op-amp (8) and linear networks of the form (10) typically in positive or negative feedback, as show in Figure 3.

Passivity is a theory of negative feedback. In order to apply Theorem 2 to *positive* feedback interconnections, we consider the reverted output $\bar{y} = -y$ and interpret the positive feedback interconnection $V_E = +y + V_r$ as negative feedback interconnection of the reverted output:

$$V_E = -\bar{y} + V_r \,, \quad u = V_0 \ . \tag{13}$$

We note that the network in Figure 4 is strictly 0-passive from u to $y = z$ and strictly 1-passive from u to $\bar{y} = -z$.

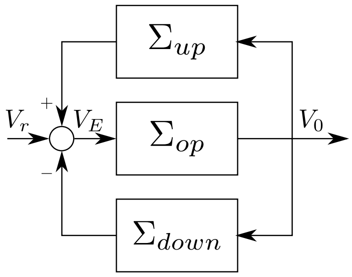

Fig. 3. Feedback loops of a circuit with an operational amplifier

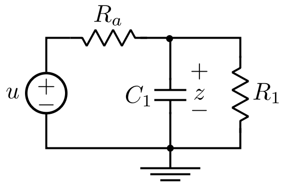

Fig. 4. *RC* network that is both 0-passive and 1-passive from different ports

Indeed, define $a = \frac{R_1+R_a}{R_1 R_a C_1}$, $b = \frac{1}{R_a C_1}$, and $c = 1$. Then the transfer functions from u to y reads

$$G(s) = \frac{cb}{s+a} \tag{14}$$

whereas the transfer function from u to $\bar{y}$ reads

$$\bar{G}(s) = -\frac{cb}{s+a} \tag{15}$$

By Theorem 3, $G(s)$ is strictly 0-passive for any rate $\lambda \in [0, a)$. On the other hand, $\bar{G}(s)$ is strictly 1-passive for any rate $\lambda \in (a, +\infty)$.

4.3 1-*Passive circuits*

By Theorem 1, multistable circuits that switch among several fixed points may arise from the interconnection of the op-amp with strictly 1-passive networks.

As an illustration, consider the positive feedback of the op-amp with the *RC* network in Figure 4.

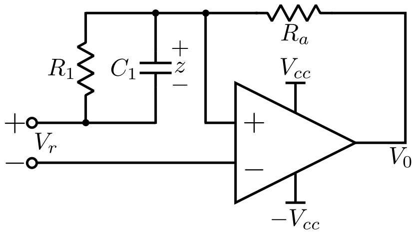

Fig. 5. An op-amp in positive feedback with a passive network.

The *RC* network is strictly 1-passive from u to $\bar{y} = -z$ with rate $\lambda \in (a, +\infty)$, and takes the role of Σ_{up} in Figure 3. The op-amp is 0-passive with rate $\lambda \in [0, \frac{1}{R_0 C_0})$. Thus, for

$$\frac{1}{R_0 C_0} > a, \tag{16}$$

the two systems share a common interval for their λ rates [1].

By Theorem 2 the closed loop system in Figure 5 is strictly 1-passive from V_r to V_0 with rate $\lambda \in (a, \frac{1}{R_0 C_0})$.

Theorem 5 guarantees boundedness of the closed-loop trajectories for any constant input V_r. Therefore, Theorem 1 guarantees asymptotic convergence of all trajectories to some fixed point. In particular, taking $V_r = 0$ for simplicity, the closed loop is bistable for

$$\left.\frac{\partial \varphi(x)}{\partial x}\right|_{x=0} < -\frac{1}{R_0} + \frac{\alpha R_1}{R_1 + R_a}, \tag{17}$$

which guarantees the existence of three equilibrium points (two stable nodes and a saddle).

4.4 2-*Passive circuits*

By Theorem 1, oscillatory circuits may arise from the interconnection of the op-amp with strictly 2-passive networks.

As an illustration, consider the negative feedback of the op-amp with the *RC* network in Figure 6.

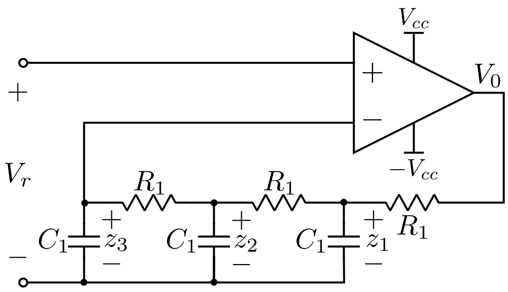

Fig. 6. RC oscillator circuit.

The *RC* network admits the state-space representation

$$A = \frac{1}{R_1 C_1}\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix},\ B = \frac{1}{R_1 C_1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ C = [0\ 0\ 1].$$

The transfer function $G(s)$ from u to y reads

$$G(s) = \frac{\frac{1}{(R_1 C_1)^3}}{s^3 + \frac{5}{R_1 C_1}s^2 + \frac{6}{(R_1 C_1)^2}s + \frac{1}{(R_1 C_1)^3}}. \tag{18}$$

Denoting by p_i the i-th pole of G and by $\beta_i = |\Re\{p_i\}|$ the magnitude of the real part of the poles of G (without loss of generality, we assume $0 \leq \beta_1 \leq \beta_2 < \beta_3$), the *RC* network is strictly 2-passive from u to y with rate $\lambda \in \left(\max\left\{\beta_2, \frac{\beta_1+\beta_2+\beta_3}{3}\right\}, \beta_3\right)$. The network is thus constrained to a negative feedback interconnection, taking the role of Σ_{down} in Figure 3.

For

$$\frac{1}{R_0 C_0} > \max\left\{\beta_2, \frac{\beta_1+\beta_2+\beta_3}{3}\right\} \tag{19}$$

the op-amp and the RC network share a common interval for their λ rates. By Theorem 2 the closed loop system in

[1] (16) requires that the op-amp dynamics is faster than the dynamics of the external network, as usual in applications.

Figure 6 is thus strictly 2-passive from V_r to V_0 with rate $\lambda \in \left(\max\left\{\beta_2, \frac{\beta_1+\beta_2+\beta_3}{3}\right\}, \min\left\{\beta_3, \frac{1}{R_0 C_0}\right\}\right)$.

Theorem 5 guarantees boundedness of the closed loop trajectories for any constant input V_r, thus Theorem 1 guarantees that the trajectories of the four dimensional closed-loop circuit all converge to a simple attractor.

For $R_1 = 3.3K\Omega$ and $C_1 = 200\mu F$, $G(s)$ has poles $p_1 = -0.3, p_2 = -2.35$ and $p_3 = -4.92$. Therefore, strict 2-passivity holds for $\lambda \in (2.52, 4.92)$. For op-amp parameters $\alpha = 0.1, R_0 = 1M\Omega, C_0 = 15.9nF, \varphi(x) = (\frac{x}{12})^5$ condition (19) holds. These specific parameters also ensure that the unique fixed point at the origin is unstable. Thus, by Theorem 1, every trajectory converges asymptotically to a limit cycle. The steady-state is an oscillation, as shown in Figure 7.

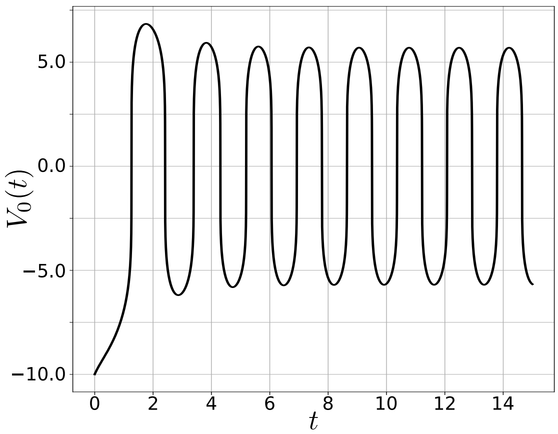

Fig. 7. Output of the RC circuit of Figure 6, with $R_1 = 3.3K\Omega$ and $C_1 = 200\mu F$.

5. MIXED FEEDBACK AND MODULATION

The mixed feedback amplifier is a classical device of nonlinear circuit theory (Chua et al., 1987). It combines positive and negative feedback around an operational amplifier to create nonlinear behaviors. We illustrate this flexibility with the simple system of Figure 8.

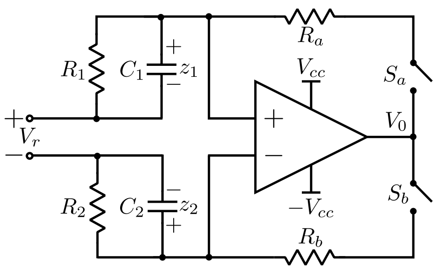

Fig. 8. Mixed feedback with switches.

The two linear networks correspond to two copies of the RC network in Figure 4. The behavior of the closed loop is dictated by the interconnection pattern of the switches S_a and S_b. If S_a is closed and S_b is open then the network reduces to the one in Figure 5; the closed loop is 1-passive. If S_a is open and S_b is closed, then the closed loop is 0-passive. When both switches are closed, the feedback circuit is not necessarily 1-passive because the rates of the two networks may not be compatible. In fact, a suitable selection of the network parameters lead to richer behaviors.

Take S_a and S_b both closed and define $a_1 = \frac{1}{R_a C_1}$, $a_2 = \frac{1}{R_b C_2}$, and $b_i = a_i + \frac{1}{R_i C_i}$, $i = 1, 2$. With these data, the upper network is strictly 1-passive from V_0 to $-z_1$ and the lower network is strictly 0-passive from V_0 to z_2, respectively with rates $\lambda_{up} \in (b_1, +\infty)$ and $\lambda_{down} \in [0, b_2)$. A common rate λ can be found for $b_1 < \min\{b_2, \frac{1}{R_0 C_0}\}$, since the op-amp is 0-passive with rate $\lambda_{op} \in [0, \frac{1}{R_0 C_0})$. In this case, the feedback circuit is 1-passive, by Theorem 2.

In contrast, Theorem 2 cannot be used for $b_1 > b_2$ because of the absence of a common rate. However, the aggregate transfer function reads

$$G(s) = \frac{(a_2 - a_1)s + a_2 b_1 - a_1 b_2}{(s + b_1)(s + b_2)}, \tag{20}$$

which has positive real part if and only if there exist $\lambda \in [0, b_2) \cup (b_1, +\infty)$ such that

$$(a_2 - a_1)\lambda < a_2 b_2 - a_1 b_1 \tag{21}$$
$$(a_2 - a_1)\lambda < a_2 b_1 - a_1 b_2. \tag{22}$$

Indeed,

(1) $G(s)$ is strictly 0-passive[2] with rate $\lambda \in \left[0, \frac{a_2 b_2 - a_1 b_1}{a_2 - a_1}\right)$ for $a_2 > a_1$ and $\frac{a_2 b_2 - a_1 b_1}{a_2 - a_1} > 0$;

(2) $G(s)$ is strictly 2-passive with rate $\lambda \in \left(\frac{a_2 b_2 - a_1 b_1}{a_2 - a_1}, +\infty\right)$ for $a_1 > a_2$.

The conditions above reveal that mixed feedback allows for both 0-passivity and 2-passivity. The network behavior can be modulated from monostable to oscillatory via parameter variations. Figure 9 shows the degree of p-passivity of the closed loop for different values $0\Omega < R_a, R_b < 3K\Omega$. Indeed, transitions from monostable to oscillatory regimes are obtained by the variation of one of the two resistances.

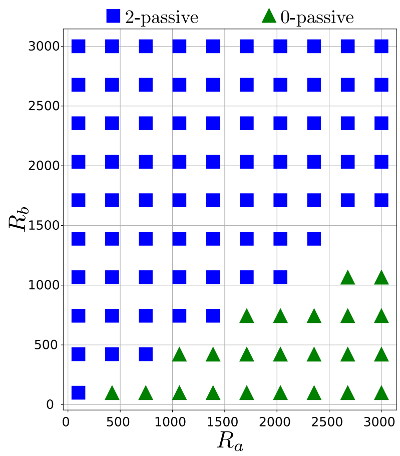

Fig. 9. p-passivity of the circuit of Figure 8 with both switches closed, as a function of the resistors R_a and R_b. Gaps correspond to lack of p-passivity.

[2] Indeed, since $b_1 > b_2$, it follows that $a_1 b_1 > a_1 b_2$ and $a_2 b_1 > a_2 b_2$. Hence, $a_2 b_2 - a_1 b_1 < a_2 b_1 - a_1 b_2$ and conditions (21) and (22) reduce to $\lambda < \frac{a_2 b_2 - a_1 b_1}{a_2 - a_1}$. Moreover, since $b_1 > b_2$ it follows that $\lambda < b_2$. From this last observation, together with $\lambda \in [0, b_2) \cup (b_1, +\infty)$ and Proposition 3 it follows that $G(s)$ is 0-passive.

As a final illustration, we consider parameters $R_1 = R_2 = 3.3K\Omega$, $R_a = R_b = 1K\Omega$, $C_1 = 100\mu F$, $C_2 = 200\mu F$, $R_0 = 1M\Omega$, $C_0 = 15.9nF$ and $\alpha = 1$. For these parameters $G(s)$ in (20) is strictly 2-passive. When both switches are closed the origin is the only equilibrium point and is unstable. Hence, by Theorems 1 and 5 we conclude the existence of a limit cycle. Figure 10 shows transitions among different behaviors, driven by the switches.

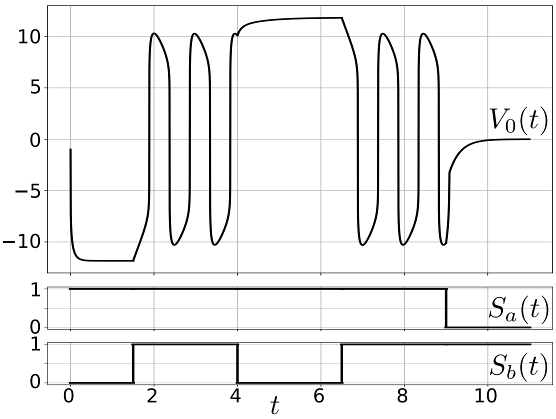

Fig. 10. Transitions among different behaviors of circuit of Figure 8 driven by the switches S_a and S_b; 0 - open switch, 1 - closed switch. Recall that for $(S_a, S_b) = (0, 1)$ the circuit is 0-passive, for $(S_a, S_b) = (1, 0)$ the circuit is 1-passive, and for $(S_a, S_b) = (1, 1)$ the circuit is 2-passive.

6. CONCLUSIONS

We have analyzed basic examples of circuits that switch and oscillate as interconnections of linear circuits and operational amplifiers. The approach builds upon dominance theory and p-passivity. The saturated op-amp model guarantees boundedness of trajectories in closed loop and allows for positive and negative feedback interconnections with linear p-passive networks. Specific interconnections lead to 1- and 2-passive networks, leading to a tractable analysis of bistability and oscillations in possibly high-dimensional models.

The stability analysis in this paper is based on solving linear matrix inequalities very much as in the stability analysis of linear systems. Such a computational framework suggests many possible extensions to analyze the performance and robustness of switching and oscillatory circuits in the same way as for linear systems.

REFERENCES

Aghannan, N. and Rouchon, P. (2003). An intrinsinc observer for a class of Lagrangian systems. *IEEE Transactions on Automatic Control*, 48(6).

Chua, L.O., Desoer, C.A., and Kuh, E.S. (1987). *Linear and nonlinear circuits*. Mc-Graw Hill.

Clayton, G. and Winder, S. (2003). *Operational Amplifiers*. EDN Series for Design Engineers. Elsevier.

Forni, F. and Sepulchre, R. (2013). On differentially dissipative dynamical systems. In *9th IFAC Symposium on Nonlinear Control Systems*, 15–20. Toulouse, France.

Forni, F. and Sepulchre, R. (2014). A differential Lyapunov framework for contraction analysis. *IEEE Transactions on Automatic Control*, 59(3), 614–628.

Forni, F. and Sepulchre, R. (2017a). Differential dissipativity theory for dominance analysis. Submitted to IEEE Transactions on Automatic Control, http://arxiv.org/abs/1710.01721.

Forni, F., Sepulchre, R., and van der Schaft, A.J. (2013). On differential passivity of physical systems. In *52rd IEEE Conference on Decision and Control*, 6580–6585. Florence, Italy.

Forni, F. and Sepulchre, R. (2017b). A dissipativity theorem for p-dominant systems. In *56th IEEE Conference on Decision and Control*. Melbourne, Australia.

Jouffroy, J. and Fossen, T.I. (2010). A tutorial on incremental stability analysis using contraction theory. *Modeling, Identification and Control*, 31(3), 93–106.

Karki, J. (2000). Effect of parasitic capacitance in op amp circuits. Technical Report SLOA013A, Texas Instruments.

Lohmiller, W. and Slotine, J.J.E. (1998). On contraction analysis for nonlinear systems. *Automatica*, 34(6), 683–696.

Miranda-Villatoro, F.A., Forni, F., and Sepulchre, R. (2017). Analysis of Lur'e dominant systems in the frequency domain. Submitted to Automatica, http://arxiv.org/abs/1710.01645.

Miranda-Villatoro, F.A., Forni, F., and Sepulchre, R. (2018). Dominance analysis of linear complementarity systems. Submitted to MTNS 2018, http://arxiv.org/abs/1802.00284.

Noseek, J.A. (2009). Circuit elements, modeling and equation formulation. In *The Circuits and Filters Handbook: Fundamentals of Circuits and Filters*, 13–1, 13–11.

Pavlov, A. and L, M. (2008). Incremental passivity and output regulation. *Systems & Control Letters*, 57, 400–409.

Pavlov, A., Van De Wouw, N., and Nijmeijer, H. (2005). Convergent systems: analysis and synthesis. In *Control and observer design for nonlinear finite and infinite dimensional systems*, 131–146. Springer.

Stan, G.B. and Sepulchre, R. (2007). Analysis of interconnected oscillators by dissipativity theory. *IEEE Transactions on Automatic Control*, 52(2), 256–270.

van Der Schaft, A.J. (2013). On differential passivity. In *9th IFAC Symposium on Nonlinear Control Systems*, 21–25. Toulouse, France.