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IRREDUCIBILITY AND ASYMPTOTICS OF STOCHASTIC BURGERS EQUATION DRIVEN BY α-STABLE PROCESSES

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Abstract

The irreducibility, moderate deviation principle and ψ-uniformly exponential ergodicity with $\psi(x) := 1 + \|x\|_0$ are proved for stochastic Burgers equation driven by the α-stable processes for $\alpha \in (1,2)$, where the first two are new for the present model, and the last strengthens the exponential ergodicity under total variational norm derived in [21].

Keywords: stochastic Burgers equation; α-stable noises; Irreducibility, ψ-uniformly ergodicity, moderate deviation

Mathematics Subject Classification (2000): 60F10, 60H15, 60J75.

1. INTRODUCTION

In [21], the strongly Feller property and exponential ergodicity have been proved for the stochastic Burgers equation driven by rotationally symmetric α-stable processes with $\alpha \in (1,2)$. In this paper, we prove a stronger ψ-uniformly exponential ergodicity, the irreducibility, and the moderate deviation principle for occupation measures. Before state our main results, we briefly recall the framework of the study and results derived in [21].

Let $\mathbb{H}$ be the space of all square integrable functions on the torus $\mathbb{T} = [0, 2\pi)$ with vanishing mean values. Let $Au = -u''$ be the second order differential operator. Then A is a positive self-adjoint operator on $\mathbb{H}$. Let $\lambda_{2k} := \lambda_{2k+1} := k^2$ and

$$e_{2k}(x) := \pi^{-\frac{1}{2}} \cos(kx), \ \ e_{2k+1}(x) := \pi^{-\frac{1}{2}} \sin(kx).$$

It is easy to see that $\{e_k, k \in \mathbb{N}\}$ forms an orthogonal basis of $\mathbb{H}$ and

$$Ae_k = \lambda_k e_k, \ k \in \mathbb{N}.$$

The norm in $\mathbb{H}$ is denoted by $\| \cdot \|_0$.

For $\gamma \rangle 0$, let $\mathbb{H}^\gamma$ be the domain of the fractional operator $A^{\frac{\gamma}{2}}$:

$$\mathbb{H}^\gamma := A^{-\frac{\gamma}{2}}(\mathbb{H}) = \left\{ \sum_k \lambda_k^{-\frac{\gamma}{2}} a_k e_k : (a_k)_{k \in \mathbb{N}} \subset \mathbb{R}, \sum_k a_k^2 < +\infty \right\}.$$

It is a separable Hilbert space with the inner product

$$\langle u, v \rangle_\gamma := \langle A^{\frac{\gamma}{2}} u, A^{\frac{\gamma}{2}} v \rangle_0 = \sum_k \lambda_k^\gamma \langle u, e_k \rangle_0 \langle v, e_k \rangle_0.$$

For $u \in \mathbb{H}$, let $\|u\|_\gamma = \sqrt{\langle u, u\rangle_\gamma}$ if $u \in \mathbb{H}^\gamma$, and $\|u\|_\gamma = \infty$ otherwise. The C_0-contraction semigroup e^{-tA} generated by $-A$ reads

$$\mathrm{e}^{-tA}u := \sum_k \mathrm{e}^{-t\lambda_k}\langle u, e_k\rangle_0 e_k, \ \ t \geq 0.$$

Obviously,

$$\|A^\gamma \mathrm{e}^{-tA}u\|_0 \leq \sup_{x>0}(x^\gamma \mathrm{e}^{-x})t^{-\gamma}\|u\|_0 = \gamma^\gamma \mathrm{e}^{-\gamma}t^{-\gamma}\|u\|_0, \ \ \gamma > 0. \tag{1.1}$$

Let $\{W_t^k, t \geq 0\}_{k\in\mathbb{N}}$ be a sequence of independent standard one-dimensional Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The cylindrical Brownian motion on $\mathbb{H}$ is defined by

$$W_t := \sum_k W_t^k e_k.$$

For $\alpha \in (0, 2)$, let S_t be an independent $\alpha/2$-stable subordinator, i.e., an increasing one dimensional Lévy process with Laplace transform

$$\mathbb{E}\mathrm{e}^{-\eta S_t} = \mathrm{e}^{-t|\eta|^{\alpha/2}}, \ \ \eta > 0.$$

The subordinated cylindrical Brownian motion $\{L_t\}_{t\geq 0}$ on $\mathbb{H}$ is defined by

$$L_t := W_{S_t}.$$

Notice that in general L_t does not belong to $\mathbb{H}$.

We are concerned about the following stochastic Burgers equation in the Hilbert space $\mathbb{H}$:

$$\mathrm{d}X_t \ = \ [-AX_t - B(X_t)]\mathrm{d}t + Q\mathrm{d}L_t, \quad X_0 = x \in \mathbb{H}, \tag{1.2}$$

where $B(u) := B(u, u)$ for the bilinear operator b defined by $B(u, v) := uv'$ for $v \in \mathbb{H}^1$ and $u \in \mathbb{H}$, and $Q \in \mathcal{L}(\mathbb{H})$ is given by

$$Qu : = \sum_{k=1}^\infty \beta_k\langle u, e_k\rangle_0 e_k, \quad u \in \mathbb{H},$$

with $\beta = (\beta_k)_{k\in\mathbb{N}}$ such that there exist some $\delta \in (0, 1)$ and $\frac{3}{2} < \theta' \leq \theta < 2$ satisfying

$$\delta\lambda_k^{-\frac{\theta}{2}} \ \leq \ |\beta_k| \ \leq \ \delta^{-1}\lambda_k^{-\frac{\theta'}{2}}, \qquad k \in \mathbb{N}. \tag{1.3}$$

By [25, Lemma 2.1], we have

$$\langle B(u, v), w\rangle_0 \leq C\|u\|_{\sigma_1}\|v\|_{\sigma_2+1}\|w\|_{\sigma_3}, \ \ \sigma_1 + \sigma_2 + \sigma_3 > 1/2, u, w \in \mathbb{H}, v \in \mathbb{H}^1. \tag{1.4}$$

Moreover, let

$$Z_t := \int_0^t \mathrm{e}^{-(t-s)A}Q\mathrm{d}L_s \quad t \geq 0 \tag{1.5}$$

satisfies $Z_\cdot \in \mathcal{D}([0, \infty); \mathbb{H}^1)$ and

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\|Z_t\|_1\right] < \infty, \ \ T > 0, \tag{1.6}$$

see e.g. [21, (4.5)]. Recall that for a topology space E, $\mathcal{C}([0, \infty); E)$ (resp. $\mathcal{D}([0, \infty); E)$) stands for the space of the continuous (resp. right continuous with left limits) maps from

$[0,T]$ to E. The following result is due to [21, Theorem 4.2]. For a σ-finite measure μ on E we denote $\mu(f)=\int_E f\mathrm{d}\mu$, $f\in L^1(\mu)$.

Theorem 1.1 ([21]). *Let $\alpha\in(1,2)$ and the assumption (1.3) hold for some $\delta\in(0,1)$ and $\frac{3}{2}<\theta'\le\theta<2$.*

(1) *For any $x\in\mathbb{H}$, (1.2) has a unique solution $(X_t^x)_{t\ge0}$ starting at x, and*

$$X_\cdot^x - Z_\cdot \in \mathcal{C}([0,\infty),\mathbb{H})\cap\mathcal{C}((0,\infty),\mathbb{H}^1).$$

In particular, $(t,x)\mapsto X_t^x$ is a Markov process on $\mathbb{H}$.

(2) *The Markov semigroup P_t for X_t^x is strong Feller, and has a unique invariant probability measure μ_0 such that*

$$\sup_{|f|\le1}|P_t\Phi(x)-\mu_0(f)|\le C(1+\|x\|_0)\mathrm{e}^{-\gamma t},\quad t\ge0, x\in\mathbb{H} \tag{1.7}$$

holds for some constants $C,\gamma>0$.

In this paper, we prove the following two theorems on the irreducibility, moderate deviation principle of occupation measures for solutions to (1.2), and the ψ-uniformly exponential ergodicity for $\psi(x):=1+\|x\|_0$. The first two properties are new for the present model, and the third strengthen the exponential ergodicity (1.7) with $|f|\le\psi$ replacing $|f|\le1$.

Theorem 1.2. *In the situation of Theorem 1.1, for any $x\in\mathbb{H}$, the solution $(X_t^x)_{t\ge0}$ of (1.2) is irreducible in $\mathbb{H}$, i.e.*

$$\mathbb{P}\left(\|X_T^x-a\|_0<\varepsilon\right)>0,\quad \varepsilon>0, T>0, a\in\mathbb{H}.$$

To state our second result, we recall the notion of moderate deviations (MDP). Let $\mathcal{M}_b(\mathbb{H})$ be the space of signed σ-additive measures of bounded variation on H, equipped with the τ-topology $\tau:=\sigma(\mathcal{M}_b(\mathbb{H}),\mathcal{B}_b(\mathbb{H}))$ of convergence against all bounded Borel functions, which is stronger than the usual weak convergence topology $\sigma(\mathcal{M}_b(\mathbb{H}),C_b(\mathbb{H}))$. We denote $\mathcal{M}_1(\mathbb{H})$ the space of probability measures on $\mathbb{H}$. Given a $\psi:\mathbb{H}\to\mathbb{R}_+$, define

$$\mathcal{B}_\psi:=\mathcal{B}_\psi(\mathbb{H},\mathbb{R})=\{f\in\mathcal{B}(\mathbb{H},\mathbb{R}):|f(x)|\le\psi(x)\}.$$

Let $b(t):\mathbb{R}^+\to(0,+\infty)$ be an increasing function verifying

$$\lim_{t\to\infty}b(t)=+\infty,\quad \lim_{t\to\infty}\frac{b(t)}{\sqrt{t}}=0, \tag{1.8}$$

and let

$$\mathfrak{M}_t:=\frac{1}{b(t)\sqrt{t}}\int_0^t(\delta_{X_s}-\mu)\mathrm{d}s.$$

To characterize *moderate deviations* of X_t from its *asymptotic limit* μ, one estimate the long time behaviours of

$$\mathbb{P}_\mu\left(\mathfrak{M}_t\in A\right), \tag{1.9}$$

where $A\in\tau$ is a given domain of deviation, and $\mathbb{P}_\mu$ is the probability measure taken for the system X with initial distribution μ. This problem refers to the central limit theorem for $b(t)=1$, the large deviation principle (LDP) for $b(t)=\sqrt{t}$, and the moderate deviation principle (MDP) for $b(t)$ satisfying (1.8), see [4]. We say that $\mathbb{P}_\mu\left(\mathfrak{M}_t\in\cdot\right)$ satisfies the MDP with a rate function I on $\mathcal{M}_1(\mathbb{H})$, if the following three properties hold for any b satisfying (1.8):

(a1) for any $a \geq 0$, $\{\nu \in \mathcal{M}_1(\mathbb{H}); I(\nu) \leq a\}$ is compact in $(\mathcal{M}_1(\mathbb{H}), \tau)$;
(a2) (the upper bound) for any closed set F in $(\mathcal{M}_1(\mathbb{H}), \tau)$,

$$\limsup_{T\to\infty} \frac{1}{b^2(T)} \log \mathbb{P}_\mu(\mathfrak{M}_T \in F) \leq -\inf_F I;$$

(a3) (the lower bound) for any open set G in $(\mathcal{M}_1(\mathbb{H}), \tau)$,

$$\liminf_{T\to\infty} \frac{1}{b^2(T)} \log \mathbb{P}_\mu(\mathfrak{M}_T \in G) \geq -\inf_G I.$$

Theorem 1.3. *In the situation of Theorem 1.1, let $\psi(x) = 1 + \|x\|_0$. Then the following statements hold.*

(1) *The Markov semigroup P_t associated with (1.2) has a unique invariant measure μ_0 with $\mu_0(\|\cdot\|_0) := \int_{\mathbb{H}} \|x\|_0 \mu_0(\mathrm{d}x) < \infty$ and*

$$\sup_{f\in\mathcal{B}_\psi} |P_t f(x) - \mu_0(f)| \leq C\mathrm{e}^{-\gamma t}(1 + \|x\|_0), \quad x \in \mathbb{H}, t \geq 0$$

holds for some constants $C, \gamma > 0$.

(2) *For any initial distribution ν with $\mu(\|\cdot\|_0) < +\infty$ and any measurable function f with $|f\psi^{-1}\|_\infty := \sup_{\mathbb{H}} |f\psi^{-1}| < \infty$, the limit*

$$\sigma^2(f) := \lim_{t\to\infty} \frac{1}{t}\mathbb{E}^\mu \left(\int_0^t (f(X_s) - \mu(f))\mathrm{d}s \right)^2 \in \mathbb{R}$$

exists. Moreover, the family $\{\mathbb{P}_\mu(\mathfrak{M}_t \in \cdot) : t \geq 0\}$ satisfies the MDP with rate function

$$I(\mu) := \sup\left\{ \mu(f) - \frac{1}{2}\sigma^2(f) : f \in \mathcal{B}_b(\mathbb{H}) \right\}.$$

To prove the irreducibility using a standard argument developed in [] for SDEs driven by cylindrical α-stable process, we will solve A control problem for the associated deterministic system in Section 2, and establish a maximum inequality for stochastic convolution in Section 3. Unlike the cylindrical α-stable process where components processes are independent, the rotationally α-stable process we considered has strong correlations between any two components, which leads to essential difficulty to follow the line of []. To overcome the difficulty, we propose a new procedure including the following three steps: taking a sample path of $\alpha/2$-stable subordinator ℓ, solving a new control problem by mollifying ℓ as in [], and proving the irreducibility by showing that for the stochastic systems driven by W_{ℓ_t}. With these preparations, Theorems 1.2 and 1.3 will be proved in Sections 4 and 5 respectively.

2. A CONTROL PROBLEM FOR THE ASSOCAITED DETERMINISTIC SYSTEM

Consider the path space of the subordinator S_t:

$$\mathcal{S} = \{\ell : [0,\infty) \to [0,\infty); \ell \text{ is strictly increasing, right continuous and has left limit}\}.$$

For any $\ell \in \mathcal{S}$, the set of jumps

$$\mathcal{J}(\ell) := \{t \geq 0 : \ell_{t-} \neq \ell_t\}$$

is at most countable. Let

$$\gamma_t = \inf\{s \geq 0 : \ell_s \geq t\}, \quad t \geq 0.$$

Consider the following deterministic system in $\mathbb{H}$:

$$\mathrm{d}x_t^\ell + \left[Ax_t^\ell + B\left(x_t^\ell\right)\right]\mathrm{d}t = Q\mathrm{d}u_{\ell_t}, \quad x_0^\ell = x_0, \tag{2.1}$$

where $u : [0,\infty) \to \mathbb{H}$ is the controller to be chosen later. Let

$$z_t^\ell = \int_0^t e^{-A(t-s)} Q\mathrm{d}u_{\ell_s}, \ \ y_t^\ell = x_t^\ell - z_t^\ell, \ \ t \geq 0. \tag{2.2}$$

Then

$$\frac{\mathrm{d}y_t^\ell}{\mathrm{d}t} + Ay_t^\ell + B(y_t^\ell + z_t^\ell) = 0, \quad x_0^\ell = x_0. \tag{2.3}$$

Define

$$t_{\mathrm{e}}(a,T) \ = \ \sup\left\{t < \frac{T}{2} : \ \|e^{-At}a - a\|_0 < \frac{\varepsilon}{2}\right\}, \ \ T>0, \varepsilon>0, a \in \mathbb{H}. \tag{2.4}$$

It is easy to see that $t_{\mathrm{e}}(a,T) \in (0, T/2]$. For notational simplicity, we often write $t_{\mathrm{e}} = t_{\mathrm{e}}(a,T)$. The main result in this section is the following.

Proposition 2.1. *Let $\ell \in \mathcal{S}$ and $x_0 \in \mathbb{H}^1$. For any $\varepsilon > 0$, $T > 0$ and $a \in \mathbb{H}$, there exist $u \in \mathcal{C}([0,\ell_T];\mathbb{H}^2)$ with bounded total variation and $x^\ell \in D([0,T];\mathbb{H}^1)$ solving (2.1) such that*

$$\|\, x_T^\ell - a\|_0 \ \leq \ \varepsilon, \qquad T \notin \mathcal{J}(\ell).$$

Moreover,

$$\|z_t^\ell\|_2 \ \leq \ C_T(1 + \|\mathrm{e}^{-At_\varepsilon}a\|_6^2 + \|\, x_{t_\varepsilon}\|_6^2), \quad 0 \leq t \leq T,$$

where t_ε is defined by (2.4) and x_{t_ε} is determined by (2.1) with $u_{\ell_t} = 0$ for $t \in [0, t_\varepsilon]$.

To prove this result, we regularize $\ell \in \mathcal{S}$ by

$$\ell_t^\delta = \frac{1}{\delta}\int_0^\delta \ell_{t+r}\mathrm{d}r, \qquad t \geq 0, \delta > 0,$$

and prove the assertion for ℓ_t^δ replacing ℓ. It is clear that ℓ_t^δ is strictly increasing and continuous. Let γ_t^δ be the inverse of ℓ_t^δ.

Lemma 2.2. *For all $\delta > 0$, we have*

$$\gamma_t^\delta \leq \gamma_t \leq \gamma_t^\delta + \delta, \quad \forall \ t \geq 0.$$

Proof. Denote $t_0 = \gamma_t$ and $t_1 = \gamma_t^\delta$, it is easy to see $\ell_{t_1}^\delta = t$ and $\ell_{t_0} \geq t$. Observe $\ell_{t_0}^\delta = \frac{1}{\delta}\int_0^\delta \ell_{t_0+r}\mathrm{d}r > t$ since $\ell_{t_0+r} > t$ for $r > 0$. If $t_0 < t_1$, then $t < \ell_{t_0}^\delta < \ell_{t_1}^\delta = t$. Contradiction. If $t_0 > t_1 + \delta$, we have $\ell_{t_1+\delta} < t$, otherwise $t_0 \leq t_1 + \delta$. Consequently, $\ell_{t_1}^\delta = \frac{1}{\delta}\int_0^\delta \ell_{t_1+r}\mathrm{d}r < t$ since $\ell_{t_1+r} < t$ for all $r \in [0,\delta]$, but $\ell_{t_1}^\delta = t$, contradiction. Hence, $t_0 \in [t_1, t_1+\delta]$. □

Lemma 2.3. *For any $T > 0, \varepsilon > 0, \delta > 0, a \in \mathbb{H}$, let $t_\varepsilon = t_\varepsilon(a,T)$ is defined by (2.4) and take*

$$u_t := 1_{[\ell_{t_\varepsilon}^\delta, \ell_T^\delta]}(t) Q^{-1} F(\gamma_t^\delta), \ \ t \in [0, \ell_T^\delta], \tag{2.5}$$

where γ_t^δ is the inverse function of ℓ_t^δ and

$$F(t) := x_t^{\ell^\delta} - x_{t_\varepsilon}^{\ell^\delta} + \int_{t_\varepsilon}^t Ax_s^{\ell^\delta}\mathrm{d}s + \int_{t_\varepsilon}^t B(x_s^{\ell^\delta})\mathrm{d}s, \quad t \in [t_\varepsilon, T]\,. \tag{2.6}$$

Then $u \in \mathcal{C}([0, \ell_T^\delta]\,; \mathbb{H}^2)$ *and* $F \in \mathcal{C}([t_\varepsilon, T]\,; \mathbb{H}^4)$ *with*

$$\|F(t)\|_4 \le C_T(1 + \|\mathrm{e}^{-At_\varepsilon} a\|_6^2 + \|x_{t_\varepsilon}^{\ell^\delta}\|_6^2) < \infty, \quad t \in [t_\varepsilon, T]\,, \tag{2.7}$$

$$\|F(t_1) - F(t_2)\|_4 \le C_T(1 + \|\mathrm{e}^{-At_\varepsilon} a\|_6^2 + \|x_{t_\varepsilon}^{\ell^\delta}\|_6^2)|t_1 - t_2|, \quad t_1, t_2 \in [t_\varepsilon, T]\,. \tag{2.8}$$

Moreover, let $x^{\ell^\delta} \in \mathcal{C}([0, T]\,; \mathbb{H}^1)$ *solve the system* (2.1) *for* ℓ^δ *replacing* ℓ*. Then*

$$\| x_T^{\ell^\delta} - a\|_0 < \varepsilon/2.$$

Proof. We first observe that $x_t^{\ell^\delta}$ has the representation

$$x_t^{\ell^\delta} = \mathrm{e}^{-At} x_0 + \int_0^t \mathrm{e}^{-A(t-s)} B(x_s^{\ell^\delta}) \mathrm{d}s, \quad 0 \le t \le t_\varepsilon, \tag{2.9}$$

$$x_t^{\ell^\delta} = \frac{t - t_\varepsilon}{T - t_\varepsilon} \mathrm{e}^{-At_\varepsilon} a + \frac{T - t}{T - t_\varepsilon} x_{t_\varepsilon}^{\ell^\delta}, \quad t_\varepsilon \le t \le T. \tag{2.10}$$

Indeed, by (2.5), $u_t = 0$ for all $t \in [0, \ell_{t_\varepsilon}^\delta]$, the system (2.1) is a deterministic Burgers equation, which admits a unique solution $x^{\ell^\delta} \in \mathcal{C}([0, t_\varepsilon]\,; \mathbb{H}^1)$ given by (2.9). On the other hand, for $t \in [t_\varepsilon, T]$, substituting $x_t^{\ell^\delta}$ with the form (2.10) into the left hand of the system (2.1), we obtain

$$Qu_{\ell_t^\delta} = F(t), \quad t \in [t_\varepsilon, T]\,,$$

where $F(t)$ is defined by (2.6). Taking

$$u_t = Q^{-1} F(\gamma_t), \quad t \in [\ell_{t_\varepsilon}^\delta, \ell_T^\delta]\,,$$

we immediately obtain that (x, u) solves the system (2.1) for $t \in [t_\varepsilon, T]$.

Next, since $x_T^{\ell^\delta} = \mathrm{e}^{-At_\varepsilon} a$ and $\|\mathrm{e}^{-At_\varepsilon} a - a\|_0 \le \varepsilon/2$, we have $\| x_T^{\ell^\delta} - a\|_0 \le \varepsilon/2$. It remains to verify the claimed properties of u and F. By the regularity of Burgers equation (see the appendix below) and $\mathrm{e}^{-At_\varepsilon}$ respectively, $x_{t_\varepsilon}^{\ell^\delta} \in \mathbb{H}^6$ and $e^{-At_\varepsilon} a \in \mathbb{H}^6$. For all $t \in [t_\varepsilon, T]$, we have

$$\|x_t^{\ell^\delta}\|_4 \le \|\mathrm{e}^{-At_\varepsilon} a\|_6 + \|x_{t_\varepsilon}^{\ell^\delta}\|_6^2,$$

$$\|B(x_t^{\ell^\delta})\|_4 \;\le\; C\|x_t^{\ell^\delta}\|_6^2 \;\le\; C\left(\| e^{-At_\varepsilon} a\|_6^2 + \|x_{t_\varepsilon}^{\ell^\delta}\|_6^2\right),$$

$$\|Ax_t^{\ell^\delta}\|_4 \;\le\; C\left(\|\mathrm{e}^{-At_\varepsilon} a\|_6 + \|x_{t_\varepsilon}^{\ell^\delta}\|_6\right) \;\le\; C\left(1 + \|\mathrm{e}^{-At_\varepsilon} a\|_6^2 + \|x_{t_\varepsilon}\|_6^2\right),$$

where the second inequality is by [25, Lemma 2.1]. Combining the above inequalities, we immediately get (2.7) and (2.8), as desired. Therefore, $F \in \mathcal{C}([t_\varepsilon, T]\,; \mathbb{H}^4)$, which, together with the assumption of Q and (2.5), yields $u \in \mathcal{C}([0, \ell_T^\delta]\,; \mathbb{H}^2)$.

Finally, it is easy to see that $\|x_{t_\varepsilon}^{\ell^\delta}\|_6 < \infty$. Below we present a proof for completeness. Noting that $x_t^{\ell^\delta} \in \mathbb{H}^1$ for all $t \in [0, t_\varepsilon]$, letting $t_1 = t_\varepsilon/3, t_2 = 2t_\varepsilon/3$, $t_3 = t_\varepsilon$ and taking

$\delta\in(0,\frac{1}{4})$, we have

$$\begin{aligned}\|x^{\ell^\delta}_t\|_2 &\le \|\mathrm{e}^{-At}x_0\|_2+\int_0^t\|A^{1-\delta}\mathrm{e}^{-A(t-s)}\|\|B(x^{\ell^\delta}_s)\|_{2\delta}\mathrm{d}s\\ &\le Ct^{-\frac12}\|x_0\|_1+C\int_0^t(t-s)^{-1+\delta}\|x^{\ell^\delta}_s\|_1^2\mathrm{d}s\\ &\le C\left(t^{-\frac12}\|x_0\|_1+t^\delta\sup_{0\le t\le t_3}\|x^{\ell^\delta}_s\|_1^2\right),\ \ t\in(0,t_3],\end{aligned}\tag{2.11}$$

where the last inequality is by (1.1) and (1.4). Now taking $x^{\ell^\delta}_{t_1}$ as the initial data, we obtain

$$\begin{aligned}\|x^{\ell^\delta}_t\|_4 &\le \|\mathrm{e}^{-A(t-t_1)}x^{\ell^\delta}_{t_1}\|_4+\int_{t_1}^t\|A^{1-\delta}\mathrm{e}^{-A(t-t_1-s)}\|\|B(x^{\ell^\delta}_s)\|_{2+2\delta}\mathrm{d}s\\ &\le C(t-t_1)^{-1}\|x^{\ell^\delta}_{t_1}\|_2+C\int_{t_1}^t(t-s)^{-1+\delta}\|x^{\ell^\delta}_s\|_2^2\mathrm{d}s\\ &\le C\left((t-t_1)^{-1}\|x^{\ell^\delta}_{t_1}\|_2+(t-t_1)^\delta\sup_{t_1\le t\le t_3}\|x^{\ell^\delta}_s\|_2^2\right),\ \ t\in(t_1,t_3].\end{aligned}\tag{2.12}$$

Similarly, taking $x^{\ell^\delta}_{t_2}$ as the initial data we get

$$\|x^{\ell^\delta}_t\|_6 \le C\left((t-t_2)^{-1}\|x^{\ell^\delta}_{t_1}\|_4+(t-t_2)^\delta\sup_{t_2\le t\le t_3}\|x^{\ell^\delta}_s\|_4^2\right),\ \ t\in(t_2,t_3].\tag{2.13}$$

This completes the proof. □

Lemma 2.4. *For all $t>0$, let*

$$z^\ell_t=\int_0^t\mathrm{e}^{-A(t-s)}Q\mathrm{d}u_{\ell_s},\quad z^{\ell^\delta}_t=\int_0^t\mathrm{e}^{-A(t-s)}Q\mathrm{d}u_{\ell^\delta_s}.$$

Then

$$\|z^{\ell^\delta}_t-z^\ell_t\|_2 \le C_T(1+\|\mathrm{e}^{-At_\varepsilon}a\|_6^2+\|x_{t_\varepsilon}\|_6^2)\delta,\qquad t\in[0,T]\setminus\mathcal{J}(\ell).\tag{2.14}$$

Proof. By (2.5), we have $u_t=0$ for all $0\le t\le\ell^\delta_{t_\varepsilon}$. Since $\ell_t\le\ell^\delta_t$,

$$z^\ell_t=z^{\ell^\delta}_t=0,\qquad t\in[0,t_\varepsilon].\tag{2.15}$$

Using integration by parts, we get

$$z^\ell_t = Qu_{\ell_t}-\int_0^tA\mathrm{e}^{-A(t-s)}Qu_{\ell_s}\mathrm{d}s.\tag{2.16}$$

It is easy to see by (2.5) and (2.7) that for all $0\le t\le T$,

$$\|Qu_{\ell_t}\|_2 = \|F(\gamma^\delta_{\ell_t})\|_2 \le \sup_{0\le t\le T}\|F(\gamma^\delta_{\ell_t})\|_2 \le C_T(1+\|\mathrm{e}^{-At_\varepsilon}a\|_6^2+\|x^{\ell^\delta}_{t_\varepsilon}\|_6^2),$$

and that for all $0\le t\le T$ and $0\le s\le t$,

$$\begin{aligned}\|A\mathrm{e}^{-A(t-s)}Qu_{\ell_s}\|_2 &= \|\mathrm{e}^{-A(t-s)}Qu_{\ell_s}\|_4 \le \|Qu_{\ell_s}\|_4 = \|F(\gamma^\delta_{\ell_s})\|_4\\ &\le C_T(1+\|\mathrm{e}^{-At_\varepsilon}a\|_6^2+\|x^{\ell^\delta}_{t_\varepsilon}\|_6^2).\end{aligned}\tag{2.17}$$

Hence,

$$\|z_t^\ell\|_2 \;\le\; C_T(1+\|\mathrm{e}^{-At_\varepsilon}a\|_6^2+\|x_{t_\varepsilon}\|_6^2),\quad 0\le t\le T.$$

Similarly,

$$\|\, z_t^{\ell^\delta}\|_2 \le C_T(1+\|\mathrm{e}^{-At_\varepsilon}a\|_6^2+\|x_{t_\varepsilon}\|_6^2),\quad 0\le t\le T.$$

Using integration by parts again, we further get

$$z_t^{\ell^\delta}-z_t^\ell = Q(u_{\ell_t^\delta}-u_{\ell_t})-\int_0^t A\mathrm{e}^{-A(t-s)}Q(u_{\ell_s^\delta}-u_{\ell_s})\mathrm{d}s$$

which, together with (2.5) and (2.8), yields

$$\begin{aligned}
\|z_t^{\ell^\delta}-z_t^\ell\|_2 \;&\le\; \|F(\gamma_{\ell_t^\delta})-F(\gamma_{\ell_t})\|_2+\int_0^t\|Q(u_{\ell_s^\delta}-u_{\ell_s})\|_4\mathrm{d}s\\
&\le\; \|F(\gamma_{\ell_t^\delta})-F(\gamma_{\ell_t})\|_2+\int_0^t\|F(\gamma_{\ell_s^\delta})-F(\gamma_{\ell_s})\|_4\mathrm{d}s\\
&\le\; C_T(1+\|\mathrm{e}^{-At_\varepsilon}a\|_6^2+\|x_{t_\varepsilon}\|_6^2)\left[|\gamma^\delta_{\ell_t^\delta}-\gamma^\delta_{\ell_t}|+\int_0^t|\gamma^\delta_{\ell_s^\delta}-\gamma^\delta_{\ell_s}|\mathrm{d}s\right]\\
&=\; C_T(1+\|\mathrm{e}^{-At_\varepsilon}a\|_6^2+\|x_{t_\varepsilon}\|_6^2)\left[|t-\gamma^\delta_{\ell_t}|+\int_0^t|s-\gamma^\delta_{\ell_s}|\mathrm{d}s\right],
\end{aligned}$$

where the last equality is by $\gamma^\delta_{\ell_t^\delta}=t$ for all $t\ge 0$. By the definition of $\gamma_\cdot$, if $t\notin\mathcal{J}(\ell)$, i.e. t is a continuous point of ℓ, we have $\gamma_{\ell_t}=t$. Therefore, by Lemma 2.2, we have

$$|t-\gamma^\delta_{\ell_t}|\;\le\;|t-\gamma_{\ell_t}|+\left|\gamma^\delta_{\ell_t}-\gamma_{\ell_t}\right|\;\le\;|t-\gamma_{\ell_t}|+\delta\;\le\;\delta,\qquad t\in[0,T]\setminus\mathcal{J}(\ell).$$

Since $\ell_\cdot$ has at most countably infinite jump points, Lebesgue measure of $\mathcal{J}(\ell)$ is zero. Thus,

$$\int_0^t|s-\gamma^\delta_{\ell_s}|\mathrm{d}s\;\le\;T\delta,\qquad t\in[0,T]$$

and

$$\|z_t^{\ell^\delta}-z_t^\ell\|_2\;\le\;C_T(1+\|\mathrm{e}^{-At_\varepsilon}a\|_6^2+\|x_{t_\varepsilon}\|_6^2)\delta,\qquad t\in[0,T]\setminus\mathcal{J}(\ell).$$

□

We are now at the position to prove Proposition 2.1. t

Proof of Proposition 2.1. Let $\delta>0$ be small enough to be chosen. By Lemma 2.3, the equation

$$\mathrm{d}x_t^{\ell^\delta}+\left[Ax_t^{\ell^\delta}+B(x_t^{\ell^\delta})\right]\mathrm{d}t=Q\mathrm{d}u_{\ell_t^\delta},\quad x_0^{\ell^\delta}=x_0 \tag{2.18}$$

is solved by $u\in\mathcal{C}(\left[0,\ell_T^\delta\right];\mathbb{H}^2)$ and $x^{\ell^\delta}\in\mathcal{C}([0,T];\mathbb{H}^1)$, which have the forms (2.9)-(2.6) and

$$\|\, x_T^{\ell^\delta}-a\|_0\le\varepsilon/2.$$

We will compare Eq. (2.18) with d the following equation:

$$\mathrm{d}x_t^\ell+\left[Ax_t^\ell+B(x_t^\ell)\right]\mathrm{d}t=Q\mathrm{d}u_{\ell_t},\quad x_0=x_0. \tag{2.19}$$

Denote $y_t^\ell = x_t^\ell - z_t^\ell$ and $y_t^{\ell^\delta} = x_t^{\ell^\delta} - z_t^{\ell^\delta}$. Then

$$\frac{\mathrm{d}y_t^{\ell^\delta}}{\mathrm{d}t} + Ay_t^{\ell^\delta} + B(x_t^{\ell^\delta}) = 0, \quad y_0^{\ell^\delta} = x_0,$$

$$\frac{\mathrm{d}y_t^{\ell}}{\mathrm{d}t} + Ay_t^{\ell} + B(x_t^{\ell}) = 0, \quad y_0^{\ell} = x_0.$$

By (2.15), we have

$$y_t^{\ell^\delta} - y_t^\ell = 0, \quad t \in [0, t_\varepsilon].$$

Write $\Delta y_t^\ell = y_t^\ell - y_t^{\ell^\delta}$, $\Delta x_t^\ell = x_t^\ell - x_t^{\ell^\delta}$ and $\Delta z_t^\ell = z_t^\ell - z_t^{\ell^\delta}$ for $t \in [t_\varepsilon, T]$. Then

$$\|\Delta y_t^\ell\|_0^2 + 2\int_{t_\varepsilon}^t \|\Delta y_t^\ell\|_1^2 \mathrm{d}s \le 2\left|\int_{t_\varepsilon}^t \langle \Delta y_t^\ell, B(x_s^{\ell^\delta}) - B(x_s^\ell)\rangle_0 \mathrm{d}s\right|. \tag{2.20}$$

Noting that

$$\begin{aligned} B(x_s^\ell) - B(x_s^{\ell^\delta}) =& \ B(x_s^\ell, \Delta x_s^\ell) + B(\Delta x_s^\ell, x_s^{\ell^\delta}) \\ =& \ B(\Delta x_s^\ell) + B(\Delta x_s^\ell, x_s^{\ell^\delta}) + B(x_s^{\ell^\delta}, \Delta x_s^\ell) \\ =& \ B(\Delta y_s^\ell) + B(\Delta z_s^\ell) + B(\Delta y_s^\ell, \Delta z_s^\ell) + B(\Delta z_s^\ell, \Delta y_s^\ell) + B(\Delta x_s^\ell, x_s^{\ell^\delta}) + B(x_s^{\ell^\delta}, \Delta x_s^\ell), \end{aligned}$$

and that $\langle x, B(x,x)\rangle_0 = 0$ for $x \in \mathbb{H}^1$, we obtain

$$\begin{aligned} |\langle \Delta y_s^\ell, B(x_s^\ell) - B(x_s^{\ell^\delta})\rangle_0| \le& \ \|\Delta y_s^\ell\|_0 \Big[\|B(\Delta z_s^\ell)\|_0 + \|B(\Delta y_s^\ell, \Delta z_s^\ell)\|_0 + \|B(\Delta z_s^\ell, \Delta y_s^\ell)\|_0 \\ & + \|B(\Delta x_s^\ell, x_s^{\ell^\delta})\|_0 + \|B(x_s^{\ell^\delta}, \Delta x_s^\ell)\|_0\Big]. \end{aligned}$$

Combining this with (1.4) and the inequality $2ab \le a^2 + b^2$ for $a \ge 0$ and $b \ge 0$, we arrive at

$$\begin{aligned} |\langle \Delta y_s^\ell, B(x_s^\ell) - B(x_s^{\ell^\delta})\rangle_0| \le& \ C\|\Delta y_s^\ell\|_0 \Big[\|\Delta z_s^\ell\|_1^2 + \|\Delta y_s^\ell\|_1 \|\Delta z_s^\ell\|_1 + \|\Delta x_s^\ell\|_1 \|x_s^{\ell^\delta}\|_1\Big] \\ \le& \ C\|\Delta y_s^\ell\|_0 \Big[\|\Delta z_s^\ell\|_1^2 + \|\Delta y_s^\ell\|_1 \|\Delta z_s^\ell\|_1 + \|\Delta y_s^\ell\|_1 \|x_s^{\ell^\delta}\|_1 + \|\Delta z_s^\ell\|_1 \|x_s^{\ell^\delta}\|_1\Big] \\ \le& \ \|\Delta y_s^\ell\|_1^2 + C\|\Delta y_s^\ell\|_0^2 \left(\|\Delta z_s^\ell\|_1^2 + \|x_s^{\ell^\delta}\|_1^2\right) + C\|\Delta z_s^\ell\|_1^2. \end{aligned}$$

This, together with (2.20) and (2.14), implies

$$\begin{aligned} \|\Delta y_t^\ell\|_0^2 \le& \ C\int_{t_\varepsilon}^t \|\Delta y_s^\ell\|_0^2 \left(\|\Delta z_s^\ell\|_1^2 + \|x_s^{\ell^\delta}\|_1^2\right) \mathrm{d}s + C\int_{t_\varepsilon}^t \|\Delta z_s^\ell\|_1^2 \mathrm{d}s \\ \le& \ C\int_{t_\varepsilon}^t \|\Delta y_s^\ell\|_0^2 \left(\|\Delta z_s^\ell\|_1^2 + \|x_s^{\ell^\delta}\|_1^2\right) \mathrm{d}s + C_T(1 + \|\,\mathrm{e}^{-At_\varepsilon} a\|_6^4 + \|x_{t_\varepsilon}\|_6^4)\delta^2, \quad t \in [t_\varepsilon, T]. \end{aligned}$$

By Gronwall's inequality, we obtain

$$\|\Delta y_T^\ell\|_0^2 \le C_T \exp\left[C\int_{t_\varepsilon}^T \left(\|\Delta z_s^\ell\|_1^2 + \|x_s^{\ell^\delta}\|_1^2\right) \mathrm{d}s\right] (1 + \|\mathrm{e}^{-At_\varepsilon} a\|_6^2 + \|x_{t_\varepsilon}\|_6^2)\delta^2.$$

On the orther hand, (2.10) implies

$$\|x_t^{\ell^\delta}\|_1 \le \|\mathrm{e}^{-At_\varepsilon} a\|_1 + \|x_{t_\varepsilon}^{\ell^\delta}\|_1 \le C\left(\|\mathrm{e}^{-At_\varepsilon} a\|_6 + \|x_{t_\varepsilon}^{\ell^\delta}\|_6\right), \quad t \in [t_\varepsilon, T],$$

which, together with (2.14), leads to

$$\int_{t_\varepsilon}^T \Big(\|\Delta z_s^\ell\|_1^2 + \|x_s^{\ell^\delta}\|_1^2 \Big) \, \mathrm{d}s \;\le\; C_T(1 + \|\mathrm{e}^{-At_\varepsilon} a\|_6^4 + \|x_{t_\varepsilon}^{\ell^\delta}\|_6^4)$$

Hence,

$$\|\Delta y_T^\ell\|_0^2 \;\le\; C_T \exp\Big[C_T(1 + \|\mathrm{e}^{-At_\varepsilon} a\|_6^4 + \|x_{t_\varepsilon}^{\ell^\delta}\|_6^4) \Big] (1 + \|\mathrm{e}^{-At_\varepsilon} a\|_6^4 + \|x_{t_\varepsilon}\|_6^4)\delta^2.$$

Combining this with (2.14), as long as $\delta > 0$ is chosen to be sufficiently small we obtain

$$\|\Delta x_T^\ell\|_0^2 \;\le\; 2\|\Delta y_T^\ell\|_0^2 + 2\|\Delta z_T^\ell\|_0^2 \;\le\; \frac{\varepsilon^2}{4}, \qquad T \notin \mathcal{J}(\ell).$$

Therefore, it follows from Lemma 2.3 that

$$\|x_T^\ell - a\|_0 \;\le\; \|\Delta x_T^\ell\|_0 + \|x_T^{\ell^\delta} - a\|_0 \;\le\; \varepsilon, \qquad T \in \mathcal{J}(\ell).$$

The proof is then complete. □

3. Estimate of convolutions

For $\ell \in \mathcal{S}$, $T > 0$ and $u \in \mathcal{C}([0, \ell_T])$, let z_t^ℓ be given in (2.2), and define

$$Z_t^\ell := \int_0^t \mathrm{e}^{-(t-s)A} Q \mathrm{d}W_{\lambda_s} \qquad t \ge 0. \tag{3.1}$$

Lemma 3.1. *For any $T > 0$, $\gamma \in \big[1, \theta' - \frac{1}{2}\big)$ and $p \ge 1$, there exists a constant $C > 0$ such that*

$$\mathbb{E}\Big[\sup_{0 \le t \le T} \| Z_t^\ell\|_\gamma^p \Big] \le C \ell_T^{p/2}, \;\; \ell \in \mathcal{S}. \tag{3.2}$$

Proof. Using integration by parts, we have

$$Z_t^\ell = \int_0^t e^{-A(t-s)} Q \mathrm{d}W_{\ell_s} = QW_{\ell_t} + \int_0^t Ae^{-A(t-s)} QW_{\ell_s} \mathrm{d}s.$$

By (1.3) and the martingale inequality, we obtain

$$\begin{aligned} \mathbb{E} \sup_{0 \le t \le T} \|QW_{\ell_t}\|_\gamma^p &\le \mathbb{E} \sup_{0 \le t \le \ell_T} \|QW_t\|_\gamma^p \\ &\le C_{\gamma,\theta'} \mathbb{E} \sup_{0 \le t \le \ell_T} \|W_t\|_{\gamma-\theta'}^p \\ &\le C_{\gamma,\theta',p} \mathbb{E} \|W_{\ell_T}\|_{\gamma-\theta'}^p \le C_{\gamma,\theta',p} \ell_T^{p/2}. \end{aligned}$$

For $\gamma' \in (\gamma, \theta' - \frac{1}{2})$, (2.1) implies

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}\left\|\int_0^t Ae^{-A(t-s)}QW_{\ell_s}\mathrm{d}s\right\|_\gamma^p &\le \mathbb{E}\sup_{0\le t\le T}\left(\int_0^t \|Ae^{-A(t-s)}QW_{\ell_s}\|_\gamma \mathrm{d}s\right)^p \\ &= \mathbb{E}\sup_{0\le t\le T}\left(\int_0^t \|A^{1+\gamma-\gamma'}e^{-A(t-s)}QA^{\gamma'-\gamma}W_{\ell_s}\|_\gamma \mathrm{d}s\right)^p \\ &\le C_{\gamma,\gamma'}\mathbb{E}\sup_{0\le t\le T}\left(\int_0^t (t-s)^{-1-\gamma+\gamma'}\|QA^{\gamma'-\gamma}W_{\ell_s}\|_\gamma \mathrm{d}s\right)^p \\ &\le C_{\gamma,\gamma',\theta'}\mathbb{E}\sup_{0\le t\le T}\left(\int_0^t (t-s)^{-1-\gamma+\gamma'}\|W_{\ell_s}\|_{\gamma'-\theta'} \mathrm{d}s\right)^p.\end{aligned}$$

Since

$$\begin{aligned}\int_0^t (t-s)^{-1-\gamma+\gamma'}\|W_{\ell_s}\|_{\gamma'-\theta'}\mathrm{d}s &\le \sup_{0\le t\le T}\|W_{\ell_s}\|_{\gamma'-\theta'}\int_0^t (t-s)^{-1+\gamma+\gamma'}\mathrm{d}s \\ &\le C_{\gamma,\gamma',T}\sup_{0\le t\le T}\|W_{\ell_s}\|_{\gamma'-\theta'},\end{aligned}$$

by the same argument as the above we get

$$\mathbb{E}\sup_{0\le t\le T}\left\|\int_0^t Ae^{-A(t-s)}QW_{\ell_s}\mathrm{d}s\right\|_\gamma^p \le C_{\gamma,\gamma',\theta',p,T}\ell_T^{p/2}.$$

Collecting the above inequalities, we obtain the desired estimate. □

Lemma 3.2. *For any* $\ell \in \mathcal{S}$, $T > 0$ *and* $\mathrm{e} > 0$,

$$\mathbb{P}\left(\sup_{0\le t\le T}\|Z_t^\ell - z_t^\ell\|_1 \le \varepsilon\right) > 0.$$

Proof. For any $N \in \mathbb{N}$, let $\mathcal{H}_N = \mathrm{span}\{e_i : i \le N\}$ and let $\mathcal{H}^N$ be its orthogonal complementary. Let $\Pi_N : \mathbb{H} \to \mathcal{H}_N$ and $\Pi^N : \mathbb{H} \to \mathcal{H}^N$ to be the corresponding orthogonal projections. We have

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\le t\le T}\|Z_t^\ell - z_t^\ell\|_1 \le \varepsilon\right) \\ &\ge \mathbb{P}\left(\sup_{0\le t\le T}\|\Pi_N(Z_t^\ell - z_t^\ell)\|_1 \le \frac{\varepsilon}{2}, \sup_{0\le t\le T}\|\Pi^N(Z_t^\ell - z_t^\ell\|_1 \le \frac{\varepsilon}{2}\right) \\ &= \mathbb{P}\left(\sup_{0\le t\le T}\|\Pi_N(Z_t^\ell - z_t^\ell)\|_1 \le \frac{\varepsilon}{2}\right)\mathbb{P}\left(\sup_{0\le t\le T}\|\Pi^N(Z_t^\ell - z_t^\ell)\|_1 \le \frac{\varepsilon}{2}\right),\end{aligned}$$

where the last inequality follows from the independence of $\Pi_N Z_t^\ell$ and $\Pi^N Z_t^\ell$. Below, we estimate these two probabilities respectively.

For the first one, using integration by parts, we get

$$Z_t^\ell - z_t^\ell = Q(W_{\ell_t} - u_{\ell_t}) + \int_0^t A\mathrm{e}^{-A(t-s)}Q(W_{\ell_s} - u_{\ell_s})\mathrm{d}s.$$

Obviously, there exist a constant $C_N > 0$ such that

$$\|\,\Pi_N\left[Q(W_{\ell_t} - u_{\ell_t})\right]\|_1 \le C_N\|\Pi_N\left[W_{\ell_t} - u_{\ell_t}\right]\|_0,$$

and

$$\left\|\Pi_N \int_0^t Ae^{-A(t-s)}Q(W_{\ell_s}-u_{\ell_s})\mathrm{d}s\right\|_1 \le \int_0^t \left\|\Pi_N \int_0^t Ae^{-A(t-s)}Q(W_{\ell_s}-u_{\ell_s})\right\|_1 \mathrm{d}s$$
$$\le C_N \int_0^t \|\Pi_N\left[W_{\ell_s}-u_{\ell_s}\right]\|_0 \mathrm{d}s$$
$$\le TC_N \sup_{0\le t\le \ell_T} \|\Pi_N\left[W_t-u_t\right]\|_0.$$

Hence,

$$\sup_{0\le t\le T}\|\Pi^N(Z_t^\ell - z_t^\ell\|_1 \le TC_N \sup_{0\le t\le T}\|\Pi_N\left[W_{\ell_t}-u_{\ell_t}\right]\|_0$$
$$\le TC_N \sup_{0\le t\le \ell_T}\|\Pi_N\left[W_t-u_t\right]\|_0.$$

It is clear $(\Pi_N W_t)_{t\ge0}$ and $(\Pi_N u_t)_{t\ge0}$ can be identified with an N dimensional standard Wiener process and a continuous function in $\mathcal{C}([0,\infty);\mathbb{R}^N)$. Since the support of a Brownian motion is the whole continuous function space, we have

$$\mathbb{P}\left(\sup_{0\le t\le \ell_T}\|\Pi_N\left(W_t-u_t\right)\|_0\le\delta\right)>0,\ \ \delta>0.$$

Therefore,

$$\mathbb{P}\left(\sup_{0\le t\le T}\|\Pi_N(Z_t^\ell-z_t^\ell)\|_1\le\frac{\varepsilon}{2}\right)>0. \tag{3.3}$$

On the other hand, by (3.2) with $\gamma\in(1,\theta'-\frac12)$, Chebyshev's inequality and the spectral inequality $\|\Pi^N x\|_1\le\lambda_N^{\gamma-1}\|x\|_\gamma$ for $x\in\mathbb{H}^\gamma$, we have

$$\mathbb{P}\left(\sup_{0\le t\le T}\|\Pi^N(Z_t^\ell-z_t^\ell)\|_1\ge\frac{\varepsilon}{2}\right)\le\mathbb{P}\left(\sup_{0\le t\le T}\|\left(Z_t^\ell-z_t^\ell\right)\|_\gamma\ge\frac{\varepsilon}{2}\lambda_N^{\gamma-1}\right)$$
$$\le\frac{2\mathbb{E}\left[\sup_{0\le t\le T}\|\,Z_t^\ell\|_\gamma\right]+2\sup_{0\le t\le T}\|z_t^\ell\|_\gamma}{\varepsilon\lambda_N^{\gamma-1}}.$$

From the previous inequality and (3.2), choose a sufficiently large N, we get

$$\mathbb{P}\left(\sup_{0\le t\le T}\|\Pi^N(Z_t^\ell-z_t^\ell)\|_1\ge\frac{\varepsilon}{2}\right)<1,$$

equivalently,

$$\mathbb{P}\left(\sup_{0\le t\le T}\|\Pi^N(Z_t^\ell-z_t^\ell)\|_1<\frac{\varepsilon}{2}\right)>0. \tag{3.4}$$

Combining (3.3), (3.3) and (3.4), we finish the proof. □

4. Proof of Theorem 1.2

For $\ell \in \mathcal{S}$, let Z_t^ℓ be in (3.1), and let X_t^ℓ solve

$$\mathrm{d}X_t^\ell = [-AX_t^\ell - B(X_t^\ell)]\mathrm{d}t + Q\mathrm{d}W_{\ell_t}, \quad X_0^\ell = x_0 \in \mathbb{H}. \tag{4.1}$$

Then $Y_t^\ell := X_t^\ell - Z_t^\ell$ satisfies

$$\frac{\mathrm{d}Y_t^\ell}{\mathrm{d}t} + AY_t^\ell + B(Y_t^\ell + Z_t^\ell) = 0, \quad Y_0^\ell = x_0. \tag{4.2}$$

Proof of Theorem 1.2. Since $S_\cdot \in \mathcal{S}$ a.s., it suffices to show that for each $\ell \in \mathcal{S}$,

$$\mathbb{P}(\|X_T^\ell - a\|_0 \le \varepsilon) > 0. \tag{4.3}$$

Since $X_t^\ell \in \mathbb{H}^1$ for $t > 0$, by the Markov property, we may and do assume that $x_0 \in \mathbb{H}^1$. Below, we prove (4.3) for $x_0 \in \mathbb{H}^1$.

By Proposition 2.1, there exist $u \in \mathcal{C}([0,T]\,;\mathbb{H}^4)$ with bounded total variation and $x^\ell \in \mathcal{D}([0,T]\,;\mathbb{H}^1)$ solving

$$\mathrm{d}x_t^\ell + \left[Ax_t^\ell + B(x_t^\ell)\right]\mathrm{d}t = Q\mathrm{d}u_{\ell_t}, \quad x_0^\ell = x_0,$$

such that

$$\|x_T^\ell - a\|_0 \le \varepsilon/2, \qquad T \notin \mathcal{J}(\ell).$$

So, when $T \notin \mathcal{J}(\ell)$ we have

$$\begin{aligned}\mathbb{P}(\|X_T^\ell - a\|_0 \le \varepsilon) \ge\ & \mathbb{P}\Big(\|\, X_T^\ell - x_T^\ell\|_0 \le \frac{\varepsilon}{2}, \|X_T^\ell - a\|_0 \le \frac{\varepsilon}{2}\Big)\\ = \mathbb{P}\Big(\|\, X_T^\ell - x_T^\ell\|_0 \le \frac{\varepsilon}{2}\Big) &\ge \mathbb{P}\Big(\|\, Y_T^\ell - y_T^\ell\|_0 \le \frac{\varepsilon}{4}, \|\, Z_T^\ell - z_T^\ell\|_0 \le \frac{\varepsilon}{4}\Big)\\ \ge \mathbb{P}\Big(\|\, Y_T^\ell - y_T^\ell\|_0 \le \frac{\varepsilon}{4}, &\sup_{0\le t\le T}\|\, Z_t^\ell - z_t^\ell\|_0 \le \varepsilon'\Big), \quad \varepsilon' \in (0, \varepsilon/4),\end{aligned} \tag{4.4}$$

where $z_t^\ell = \int_0^t e^{-A(t-s)}Q\mathrm{d}u_{\ell_s}$ and y_t^ℓ are in (2.2).

Write $\Delta Y_t^\ell = Y_t^\ell - y_t^\ell, \Delta X_t^\ell = X_t^\ell - x_t^\ell$ and $\Delta Z_t^\ell = Z_t^\ell - z_t^\ell$. Then (2.3) and (4.2) yield

$$\frac{\mathrm{d}\Delta Y_t^\ell}{\mathrm{d}t} + A\Delta Y_t^\ell + B(X_t^\ell) - B(x_t^\ell) = 0, \quad \Delta Y_0^\ell = 0,$$

which clearly implies

$$\|\Delta Y_t^\ell\|_0^2 + 2\int_0^t \|\Delta Y_t^\ell\|_1^2 \mathrm{d}s \le 2\int_0^t |\langle \Delta Y_s^\ell, B(X_s^\ell) - B(x_s^\ell)\rangle_0|\mathrm{d}s.$$

Since $\langle x, B(x,x)\rangle_0 = 0$ for $x \in \mathbb{H}^1$, we have

$$\begin{aligned}&|\langle \Delta Y_s^\ell, B(X_s^\ell) - B(x_s^\ell)\rangle_0|\\ &= \langle \Delta Y_s^\ell, B(\Delta X_s^\ell)\rangle_0 + \langle \Delta Y_s^\ell, B(\Delta X_s^\ell, x_s^\ell)\rangle_0 + \langle \Delta Y_s^\ell, B(x_s^\ell, \Delta X_s^\ell)\rangle_0\\ &= \langle \Delta Y_s^\ell, B(\Delta Y_s^\ell, \Delta Z_s^\ell)\rangle_0 + \langle \Delta Y_s^\ell, B(\Delta Z_s^\ell, \Delta Y_s^\ell)\rangle_0 + \langle \Delta Y_s^\ell, B(\Delta Z_s^\ell, \Delta Z_s^\ell)\rangle_0\\ &\quad + \langle \Delta Y_s^\ell, B(\Delta X_s^\ell, x_s^\ell)\rangle_0 + \langle \Delta Y_s^\ell, B(x_s^\ell, \Delta X_s^\ell)\rangle_0,\end{aligned}$$

which, together with (1.4) and the inequality $2ab \le a^2 + b^2$ for $a, b \ge 0$, implies

$$\begin{aligned}
&|\langle Y_s^\ell, B(X_s^\ell) - B(x_s^\ell)\rangle_0| \\
&\le C(\|\Delta Y_s^\ell\|_0 \|\Delta Y_s^\ell\|_1 \|\Delta Z_s^\ell\|_1 + \|\Delta Y_s^\ell\|_0 \|\Delta Z_s^\ell\|_1^2 + \|x_s^\ell\|_1 \|\Delta Y_s^\ell\|_0 \|\Delta X_s^\ell\|_1) \\
&\le C(\|\Delta Z_s^\ell\|_1^2 + \|x_s^\ell\|_1^2)\|\Delta Y_s^\ell\|_0^2 + C\|\Delta Z_s^\ell\|_1^2 + \left(\frac{1}{2}\|\Delta Y_s^\ell\|_1^2 + \frac{1}{4}\|\Delta X_s^\ell\|_1^2\right) \\
&\le C(\|\Delta Z_s^\ell\|_1^2 + \|x_s^\ell\|_1^2)\|\Delta Y_s^\ell\|_0^2 + \|\Delta Y_s^\ell\|_1^2 + C\|\Delta Z_s^\ell\|_1^2
\end{aligned}$$

for some constant $C > 0$. Hence,

$$\begin{aligned}
\|\Delta Y_t^\ell\|^2 &\le C\int_0^t (\|\Delta Z_s^\ell\|_1^2 + \|x_s^\ell\|_1^2)\|\Delta Y_s^\ell\|_0^2 \mathrm{d}s + C\int_0^t \|\Delta Z_s^\ell\|_1^2 \mathrm{d}s \\
&\le C(\sup_{0\le t\le T} \|\Delta Z_t^\ell\|_1^2 + \sup_{0\le t\le T} \|x_t^\ell\|_1^2)\int_0^t \|\Delta Y_s^\ell\|_0^2 \mathrm{d}s + CT \sup_{0\le t\le T} \|\Delta Z_t^\ell\|_1^2, \quad 0 \le t \le T.
\end{aligned}$$

When $\sup_{0\le t\le T} \|\Delta Z_t^\ell\|_0 \le \varepsilon'$, we have

$$\|\Delta Y_t^\ell\|^2 \le C((\varepsilon')^2 + \sup_{0\le t\le T} \|x_t^\ell\|_1^2)\int_0^t \|\Delta Y_s^\ell\|_0^2 \mathrm{d}s + CT(\varepsilon')^2.$$

By Gronwall's inequality,

$$\|\Delta Y_T^\ell\|^2 \ \le\ CT\exp\left[C(\varepsilon' + \sup_{0\le t\le T} \|x_t\|_1)T\right](\varepsilon')^2, \ \text{ if } \sup_{0\le t\le T} \|\Delta Z_t^\ell\|_0 \le \varepsilon'.$$

Since $\sup_{0\le t\le T} \|x_t^\ell\|_1 < \infty$, when ε' is sufficiently this implies

$$\|\Delta Y_T^\ell\|_0 \le \frac{\varepsilon}{4}, \ \text{ if } \sup_{0\le t\le T} \|\Delta Z_t^\ell\|_0 \le \varepsilon'.$$

Hence, for small enough $\varepsilon' > 0$,

$$\mathbb{P}\left(\| Y_T^\ell - y_T^\ell\|_0 \le \frac{\varepsilon}{4}, \sup_{0\le t\le T} \| Z_T^\ell - z_T^\ell\|_0 \le \varepsilon'\right) = \mathbb{P}\left(\| Z_T^\ell - z_T^\ell\|_0 \le \varepsilon'\right) > 0.$$

This and (4.4) yield that (4.3) holds for $T \notin \mathcal{J}(\ell)$. Since X_t is right continuous and the set $[0,\infty) \setminus \mathcal{J}(\ell)$ is dense, (4.3) holds for all $T > 0$. Then the proof is finished. □

5. ψ-UNIFORMLY EXPONENTIAL ERGODICITY AND MODERATE DEVIATION

5.1. **Galerkin approximation.** Recall that $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $\mathbb{H}$. For any $m \in \mathbb{N}$, let $\mathcal{H}_m := \mathrm{span}\{e_k : k \le m\}$ with orthogonal projection $\Pi_m : \mathbb{H} \to \mathcal{H}_m$. Then the Galerkin approximation of (1.2) reads

$$\mathrm{d}\tilde{X}_t^m + [A\tilde{X}_t^m + B^m(\tilde{X}_t^m)]\mathrm{d}t = Q\mathrm{d}L_t^m, \quad \tilde{X}_0^m = x^m, \tag{5.1}$$

where $x^m = \Pi_m x$, $B^m(x) = \Pi_m[B(x)]$ for $x \in \mathbb{H}$, and $L_t^m = \Pi_m L_t = W_{S_t}^m$ with W_t^m being an m-dimensional standard Brownian motion.

Since the Lévy measure of W_{S_t} can not be approximated by those of $W_{S_t}^m$, the approximation procedure in [] does not apply. Alternatively, we show that $\Delta X_t^m = \tilde{X}_t^m - X_t^m$ converges to zero. The advantage of this new procedure is that the approximation of W_{S_t} is avoided.

Theorem 5.1. *For all $t>0$, $\mathbb{P}$-a.s.*

$$\lim_{m\to\infty}\|\tilde X_t^m - X_t\|_1 \;=\; 0. \tag{5.2}$$

Proof. Let X_t solve (1.2) with $X_0=x$, and denote $X_t^m=\Pi_m X_t$. Then

$$\mathrm{d}X_t^m + [AX_t^m + B^m(X_t)]\mathrm{d}t = Q\mathrm{d}L_t^m, \quad X_0^m = x^m. \tag{5.3}$$

By (1.6) and Theorem 1.1,

$$\lim_{m\to\infty}\|X_t^m - X_t\|_1 = 0, \qquad t>0.$$

Combining this with Lemma 5.2 below, we finish the proof. □

Lemma 5.2. *Let $\Delta X_t^m = \tilde X_t^m - X_t^m$. Then $\mathbb{P}$-a.s.*

$$\lim_{m\to\infty}\|\Delta X_t^m\|_1 = 0, \qquad t\geq 0.$$

Proof. (1) We first prove that for some constant $C>0$,

$$\sup_{0\le t\le T, m\in\mathbb{N}}\|\tilde X_t^m\|_0^2 \le A_T, \quad T>0, m\in\mathbb{N}, \tag{5.4}$$

holds for

$$A_T := 2\exp\left(C\int_0^T(1+\|Z_s\|_1^2)\mathrm{d}s\right)\left[\|x\|_0^2 + T\sup_{0\le t\le T}|Z_t\|_1^4\right] + 2\sup_{0\le t\le T}\|Z_t\|_1^2.$$

For $\ell\in\mathcal{S}$, let

$$Z_t^{m,\ell} \;=\; \int_0^t \mathrm{e}^{-A(t-s)}Q\mathrm{d}W_{\ell_s}^m.$$

Then

$$\|Z_t^{m,\ell}\|_\gamma \;\le\; \|Z_t^\ell\|_\gamma, \qquad \gamma\in\mathbb{R}.$$

By (3.2) with $\gamma=1$, we have $\mathbb{P}$-a.s.

$$\sup_{0\le t\le T, m\in\mathbb{N}}\|Z_t^{m,\ell}\|_0 \;\le\; \sup_{0\le t\le T, m\in\mathbb{N}}\|Z_t^{m,\ell}\|_1 \;\le\; \sup_{0\le t\le T}\|Z_t^\ell\|_1 \;<\; \infty. \tag{5.5}$$

It is easy to see that $\tilde Y_t^{m,\ell} := \tilde X_t^{m,\ell} - Z_t^{m,\ell}$ solves the equation

$$\partial_t \tilde Y_t^{m,\ell} + A\tilde Y_t^{m,\ell} + B^m(\tilde Y_t^{m,\ell} + Z_t^{m,\ell}) = 0, \quad \tilde X_0^{m,\ell} = x^m. \tag{5.6}$$

Applying the chain role to $\|\tilde Y_t^{m,\ell}\|_0^2$ gives

$$\|\tilde Y_t^{m,\ell}\|_0^2 + 2\int_0^t\|\tilde Y_s^{m,\ell}\|_1^2\mathrm{d}s \;=\; \|x^m\|_0^2 + 2\int_0^t\langle\tilde Y_s^{m,\ell}, B^m(\tilde Y_s^{m,\ell} + Z_s^{m,\ell})\rangle\mathrm{d}s. \tag{5.7}$$

Letting $\tilde B^m(x,y) = B^m(x,y) + B^m(y,x)$, the relation $\langle\tilde Y_s^{m,\ell}, B^m(\tilde Y_s^{m,\ell})\rangle = 0$ implies

$$\begin{aligned}
&\big|\langle\tilde Y_s^{m,\ell}, B^m(\tilde Y_s^{m,\ell} + Z_s^{m,\ell})\rangle\big| \\
&= \big|\,\langle\tilde Y_s^{m,\ell}, \tilde B^m(\tilde Y_s^{m,\ell}, Z_s^{m,\ell}) + B^m(Z_s^{m,\ell})\rangle\big| \\
&\le\; C\|\tilde Y_s^{m,\ell}\|_0\|\tilde Y_s^{m,\ell}\|_1\|Z_s^{m,\ell}\|_1 + C\|\tilde Y_s^{m,\ell}\|_0\|Z_s^{m,\ell}\|_1^2 \\
&\le\; C(1+\|Z_s^{m,\ell}\|_1^2)\|\tilde Y_s^{m,\ell}\|_0^2 + \|\tilde Y_s^{m,\ell}\|_1^2 + \|Z_s^{m,\ell}\|_1^4 \\
&\le\; C(1+\|Z_s^{\ell}\|_1^2)\|\tilde Y_s^{m,\ell}\|_0^2 + \|\tilde Y_s^{m,\ell}\|_1^2 + \|Z_s^{\ell}\|_1^4,
\end{aligned}$$

for some constant $C>0$ independent of m and T. Combining this with (5.7) and $\|x^m\|_0 \le \|x\|_0$, we arrive at

$$\|\tilde{Y}_t^{m,\ell}\|_0^2 \le \|x\|_0^2 + C\int_0^t \left(1+\|Z_s^\ell\|_1^2\right)\|\tilde{Y}_s^{m,\ell}\|_0^2 \mathrm{d}s + \int_0^t \|Z_s^\ell\|_1^4 \mathrm{d}s.$$

By Gronwall's lemma this implies

$$\|\tilde{Y}_t^{m,\ell}\|_0^2 \le \exp\left(C\int_0^t (1+\|Z_s^\ell\|_1^2)\mathrm{d}s\right)\|x\|_0^2 + \int_0^t \exp\left[C\int_s^t (1+\|Z_r^\ell\|_1^2)\mathrm{d}r\right]|Z_s^\ell\|_1^4 \mathrm{d}s,$$

so that (5.4) holds.

(2) By the equations (5.6) and (5.3), we have

$$\partial_t \Delta X_t^m + AX_t^m + B^m(\tilde{X}_t^m) - B^m(X_t) = 0, \qquad \Delta X_0^m = 0.$$

Then there exists a constant $C>0$ such that

$$\begin{aligned} \|\Delta X_t^m\|_0 &\le \int_0^t \|e^{-(t-s)}\left[B_m(\tilde{X}_s^m) - B_m(X_s)\right]\|_0 \mathrm{d}s \\ &= \int_0^t \|e^{-(t-s)}\left[B(\tilde{X}_s^m) - B(X_s)\right]\|_0 \mathrm{d}s \\ &\le C\int_0^t (t-s)^{-\frac{5}{6}}\|B(\tilde{X}_s^m) - B(X_s)\|_{-\frac{5}{3}} \mathrm{d}s \end{aligned} \tag{5.8}$$

Since $B(x) = B(x^m + (x-x^m))$ for $x \in \mathbb{H}^1$, it follows that

$$B(\tilde{X}_s^m) - B(X_s) = B(\tilde{X}_s^m) - B(X_s^m) - \tilde{B}(X_s^m, X_s - X_s^m) - B(X_s - X_s^m),$$

where $\tilde{B}(x,y) = B(x,y) + B(y,x)$ for $x,y \in \mathbb{H}^1$. Applying Eq. (1.4) with $\sigma_1 = \frac{5}{3}, \sigma_2 = -1, \sigma_3 = 0$, we obtain

$$\begin{aligned} \|B(\tilde{X}_s^m) - B(X_s^m)\|_{-\frac{5}{3}} &\le \|B(\Delta X_s^m, \tilde{X}_s^m)\|_{-\frac{5}{3}} + \|B(X_s^m, \Delta X_s^m)\|_{-\frac{5}{3}} \\ &\le \|\Delta X_s^m\|_0 \|\tilde{X}_s^m\|_0 + \|\Delta X_s^m\|_0 \|\|X_s^m\|_0 \\ &\le \left(\sqrt{A_T} + \sup_{0\le t\le T}\|X_t\|_0\right)\|\Delta X_s^m\|_0. \end{aligned}$$

Combining this with (5.8) gives

$$\begin{aligned} \|\Delta X_t^m\|_0^2 &\le C\int_0^t (t-s)^{-\frac{5}{6}}\left(\sqrt{A_T} + \sup_{0\le t\le T}\|X_t\|_0\right)\|\Delta X_s^m\|_0 \mathrm{d}s \\ &+ C\int_0^t (t-s)^{-\frac{5}{6}}\left(\|X_s\|_0\|X_s - X_s^m\|_0 + \|X_s - X_s^m\|_0^2\right)\mathrm{d}s. \end{aligned}$$

Noting that

$$\|\Delta X_t^m\|_0 \le \|X_t^m\|_0 + \|\tilde{X}_t^m\|_0 \le \sup_{0\le t\le T}\|X_t\|_0 + \sqrt{A_T} < \infty, \ \ t\in[0,T],$$

by Fatou's lemma we get

$$\limsup_{m\to\infty}\|\Delta X_t^m\|_0^2 \le C\int_0^t (t-s)^{-\frac{5}{6}}\left(\sqrt{A_T} + \sup_{0\le t\le T}\|X_t\|_0\right)\limsup_{m\to\infty}\|\Delta X_s^m\|_0 \mathrm{d}s, \quad 0\le t\le T,$$

so that by Gronwall's inequality,

$$\limsup_{m\to\infty} \|\Delta X_t^m\|_0 \;=\; 0, \qquad t\in[0,T].$$

□

5.2. **ψ-uniformly exponential ergodicity and moderate deviation.** We will use the following exponential ergodicity result in [9].

Theorem 5.3 (Theorem 5.2 (b), [9])**.** *Let $(X_t)_{t\geq 0}$ be an irreducible and aperiodic Markov process on a Polish space E with Markov semigroup P_t, and let $\psi \geq 1$ be a measurable function on E. If*

$$P_t\psi(x) \;\leq\; \lambda(t)\psi(x) + b1_{\mathcal{K}}(x), \quad t\in(0,T], x\in E$$

holds for some constants $T, b>0$, a measurable petite set $\mathcal{K}$ on E, and a bounded function λ on $[0,T]$ with $\lambda(T)<1$, then X_t is ψ-uniformly ergodic, i.e., there exist constants $C,\gamma>0$ such that

$$\sup_{|f|\leq\psi} |P_t f(x) - \mu_0(f)| \;\leq\; C\mathrm{e}^{-\gamma t}\psi(x), \qquad t>0. \tag{5.9}$$

Proof of Theorem 1.3(1). Since $1+\|\cdot\|_0$ is comparable with $\sqrt{M+\|\cdot\|_0^2}$ for any $M\geq 1$, we will take $\psi(x)=\sqrt{M+\|x\|_0^2}$ instead of $1+\|x\|_0$ for $M>1$ large enough to be determined.

(1) We first observe that it suffices to find out a constant $C>0$ such that

$$\begin{aligned} &\left|\int_{\mathcal{H}^m} (\psi(x^m+Qy)-\psi(x^m)-\langle Qy,\nabla\psi(x^m)\rangle_0 1_{\|y\|_0\leq 1})\nu_m(\mathrm{d}y)\right| \\ &\leq\; C\Big(1+\frac{1}{\sqrt{M}}\Big), \quad x^m\in\mathcal{H}^m, \quad x^m\in\mathcal{H}_m := \mathrm{span}\{e_i : i\leq m\}. \end{aligned} \tag{5.10}$$

Let $\mathcal{L}^m$ be the generator of $\tilde{X}_t^m$ given by (5.6). Since $\langle x^m, B_m(x^m)\rangle = 0$, it is easy to see that

$$\begin{aligned} \mathcal{L}^m\psi(x^m) \;&=\; -\langle Ax^m + B_m(x^m), \nabla\psi(x^m)\rangle_0 \\ &\quad + \int_{\mathcal{H}^m} (\psi(x^m+Qy)-\psi(x^m)-\langle Qy,\nabla\psi(x^m)\rangle_0 1_{\|y\|_0\leq 1})\nu_m(\mathrm{d}y) \\ &=\; -\frac{\|x^m\|_1^2}{\psi(x^m)} + \int_{\mathcal{H}^m} (\psi(x^m+Qy)-\psi(x^m)-\langle Qy,\nabla\psi(x^m)\rangle_0 1_{\|y\|_0\leq 1})\nu_m(\mathrm{d}y). \end{aligned}$$

where the last equality is by $\langle x^m, B_m(x^m)\rangle = 0$. Let $\mathcal{K}_m = \{x^m\in\mathcal{H}^m : \|x^m\|_1\leq M\}$. By (5.10) and (5.2), we have

$$\begin{aligned} \mathcal{L}^m\psi(x^m) \;&\leq\; -\frac{\|x^m\|_1^2}{\psi(x^m)} + C\Big(1+\frac{1}{\sqrt{M}}\Big) \\ &\leq\; -\frac{\|x^m\|_1^2+M}{\psi(x^m)} + \frac{M}{\psi(x^m)} + C\Big(1+\frac{1}{\sqrt{M}}\Big) \\ &\leq\; -\psi(x^m) + \sqrt{M} + C\Big(1+\frac{1}{\sqrt{M}}\Big), \quad x^m\in\mathcal{K}_m. \end{aligned}$$

On the other hand, if $x^m \notin \mathcal{K}_m$, then e $\|x^m\|_1 \geq M$ and thus,

$$\begin{aligned}\mathcal{L}^m\psi(x^m) &\leq -\frac{\|x^m\|_1^2}{\psi(x^m)} + C_{\alpha,Q}(1+\frac{1}{\sqrt{M}})\\ &\leq -\frac{\frac{1}{2}(M+\|x^m\|_1^2)}{\psi(x^m)} + C_{\alpha,Q}(1+\frac{1}{\sqrt{M}})\\ &\leq -\frac{1}{2}\psi(x^m) + C_{\alpha,Q}(1+\frac{1}{\sqrt{M}})\\ &\leq -\frac{1}{4}\psi(x^m),\end{aligned} \tag{5.11}$$

as long as we choose $M > 1$ sufficiently large. In conclusion, when $M > 1$ is large enough, there exists a constant $b > 0$ such that

$$\mathcal{L}^m\psi(x^m) \leq -\frac{1}{4}\psi(x^m) + b1_{\mathcal{K}_m}(x^m), \ \ m \geq 1.$$

By [9, Theorem 5.1 (d)], this implies

$$\mathbb{E}[\psi(\tilde{X}_t^m)] \leq \mathrm{e}^{-t/4}\psi(x^m) + b1_{\mathcal{K}_m}(x^m), \quad t \geq 0.$$

. Since $\lim_{m\to\infty}\|x^m - x\|_0 = 0$ and $\lim_{m\to\infty}\|\tilde{X}_t^m - X_t\|_1 = 0$ a.s. for $t > 0$, by letting $m \to \infty$ we obtain

$$\mathbb{E}[\psi(X_t)] \leq \mathrm{e}^{-t/4}\psi(x) + b1_{\mathcal{K}}(x), \quad t \geq 0,$$

where $\mathcal{K} := \{x \in \mathbb{H} : \|x\|_1 \leq M\}$ is a compact (hence petite) set in $\mathbb{H}$. By Theorem (5.3), we prove the ψ-uniformly exponential ergodicity of X_t.

(2) It remains to prove (5.10). Obviously,

$$\begin{aligned}&\left|\int_{\mathcal{H}^m}(\psi(x^m+Qy)-\psi(x^m)-\langle Qy,\nabla\psi(x^m)\rangle_0 1_{\|y\|_0\leq 1})\nu_m(\mathrm{d}y)\right|\\ \leq &\left|\int_{\|y\|_0\leq 1}(\psi(x^m+Qy)-\psi(x^m)-\langle Qy,\nabla\psi(x^m)\rangle_0)\nu_m(\mathrm{d}y)\right|\\ &+\left|\int_{\|y\|_0>1}(\psi(x^m+Qy)-\psi(x^m))\nu_m(\mathrm{d}y)\right|\end{aligned} \tag{5.12}$$

By Taylor's expansion,

$$\begin{aligned}&|\psi(x^m+Qy)-\psi(x^m)-\langle Qy,\nabla\psi(x^m)\rangle_0|\\ \leq &\sup_{\theta\in[0,1]}\left|\frac{\|y\|_0^2}{\psi(x^m+\theta Qy)} - \frac{|\langle y, x^m+\theta Qy\rangle_0|^2}{\psi^3(x^m+\theta Qy)}\right| \leq \frac{2}{\sqrt{M}}\|y\|_0^2.\end{aligned}$$

Since ν_m has a density $\frac{C_m}{\|y\|_0^{m+\alpha}}$ for $y \in \mathcal{H}_m$ with $C_m = \frac{\alpha 2^\alpha\Gamma(\frac{m}{2}+\frac{\alpha}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{2-\alpha}{2})}$, we have

$$\begin{aligned}&\left|\int_{\|y\|_0\leq 1}(\psi(x^m+Qy)-\psi(x^m)-\langle Qy,\nabla\psi(x^m)\rangle_0)\nu_m(\mathrm{d}y)\right|\\ \leq &\frac{2}{\sqrt{M}}\int_{\|y\|_0\leq 1}\|y\|_0^2\frac{C_m}{\|y\|_0^{m+\alpha}}\mathrm{d}y = \frac{2C_m}{\sqrt{M}}\int_0^1\int_{\mathbb{S}_{m-1}} r^{1-\alpha}\mathrm{d}r\mathrm{d}\sigma_{m-1} = \frac{2C_m|\mathbb{S}_{m-1}|}{(2-\alpha)\sqrt{M}},\end{aligned}$$

where $|\mathbb{S}_{m-1}| = \frac{2(\pi)^{m/2}}{\Gamma(m/2)}$ is the volume of $\mathbb{S}_{m-1}$. Moreover,

$$\begin{aligned} C_m|\mathbb{S}_{m-1}| &= \frac{\alpha 2^\alpha \Gamma\left(\frac{m}{2}+\frac{\alpha}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{2-\alpha}{2}\right)} \frac{2\pi^{m/2}}{\Gamma(m/2)} \le \frac{\alpha 2^\alpha \Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{2-\alpha}{2}\right)} \frac{2\pi^{m/2}}{\Gamma(m/2)} \\ &= \frac{\alpha 2^\alpha \frac{m}{2}\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{2-\alpha}{2}\right)} \frac{2\pi^{m/2}}{\Gamma(m/2)} \le \sup_{m\ge 1} \frac{\alpha 2^\alpha m \pi^{m/2}}{\Gamma\left(\frac{2-\alpha}{2}\right)\Gamma\left(\frac{m}{2}\right)} =: C' < \infty. \end{aligned}$$

Hence,

$$\left|\int_{\|y\|_0\le 1} (\psi(x^m+Qy)-\psi(x^m)-\langle Qy, \nabla\psi(x^m)\rangle_0)\nu_m(\mathrm{d}y)\right| \le \frac{C'}{\sqrt{M}}.$$

Similarly, there exist constants $C_Q > 0$ such that

$$\begin{aligned} &\left|\int_{\|y\|_0>1} (\psi(x^m+Qy)-\psi(x^m))\nu_m(\mathrm{d}y)\right| \\ \le &\left|\int_{\|y\|_0>1} \frac{|\langle x^m+\theta Qy, Qy\rangle_0|}{\psi(x^m+\theta Qy)}\nu_m(\mathrm{d}y)\right| \le \left|\int_{\|y\|_0>1} \|Qy\|_0 \nu_m(\mathrm{d}y)\right| \\ \le & C_Q \left|\int_{\|y\|_0>1} \|y\|_0\nu_m(\mathrm{d}y)\right| \le \sup_{m\ge1} C_Q \int_1^\infty \int_{\mathbb{S}_{m-1}} \frac{C_m}{r^\alpha}\mathrm{d}r\mathrm{d}\sigma_{m-1} < \infty. \end{aligned}$$

Therefore, (5.10) holds for some constant $C > 0$. □

Proof of Theorem 1.3(2). We follow the argument in [18, p. 429-431]. Given $f \in \mathcal{B}_b(\mathbb{H})$, consider the following Feynman-Kac formula

$$P_t^{\lambda f} g(x) = \mathbb{E}\left[\exp\left(\lambda \int_0^t f(X_s^x)\mathrm{d}s\right) g(X_t^x)\right], \quad g \in \mathcal{B}_\psi.$$

For any $\delta > 0$ and $|\lambda| \le \delta$, we have

$$\|P_t^{\lambda f} g\|_\psi \le e^{\delta\|f\|t}\|g\|_\psi.$$

So, $\lambda \to P_1^{\lambda f} g \in \mathcal{B}_\psi$ is holomorphic for all $|\lambda| < \delta$.

When $\lambda = 0$, $P_1 g = \mathbb{E}[g(X_1^x)]$ with $g \in \mathcal{B}_\psi$. By the exponential ergodicity result (5.9), we get that 1 is an isolated simple spectrum of P_1 and the constant function is the corresponding eigenfunction. Denote $\mathcal{P}_0$ be the projection with respect to the eigenvalue 1, which is defined by

$$\mathcal{P}_0 g = \mu(g), \quad g \in \mathcal{B}_\psi.$$

The spectrum of the $P_1(I-\mathcal{P}_0)$ has a spectrum radius less than ρ from (5.9).

By Kato's holomorphic perturbation theorem, for any $r \in \left(\rho, \frac{1+\rho}{2}\right)$, there exist some $\tilde{\delta} \in (0,\delta)$ such that for all $D_{\tilde{\delta}} = \{\lambda \in \mathbb{C} : |\lambda| \le \tilde{\delta}\}$ the operator $P_1^{\lambda f}$ acting on $\mathcal{B}_\psi$ has the following properties: (1) $P_1^{\lambda f}$ has a single simple eigenvalue $\sigma(\lambda)$ with the largest modulus of the spectrum, moreover, there exists some number $c \in (\frac{1}{2}, 1)$ such that $|\sigma(\lambda)| \ge c$; (2) $\mathcal{P}_\lambda$ is the projection of $P_1^{\lambda f}$ corresponding to $\sigma(\lambda)$, $\lambda \in D_{\tilde{\delta}} \to \mathcal{P}_\lambda \in \mathcal{L}(\mathcal{B}_\psi)$ is holomorphic and $\|\mathcal{P}_\lambda 1 - \mathcal{P}_0 1\|_\psi \le$ e with some sufficiently small e $\in (0,1)$; (3) the spectral radius of $P_1^{\lambda f}(I-\mathcal{P}_\lambda)$ is strictly less than r.

By (3), the following relation holds

$$N := \sup_{z\in S(\frac{1}{r}),\lambda\in D_{\tilde\delta}} \|(I - zP_1^{\lambda f}(I-\mathcal{P}_\lambda))^{-1}\|_{\mathcal{B}_\psi\to\mathcal{B}_\psi} < \infty,$$

where $S(1/r) = \{z \in \mathbb{C} : |z| = \frac{1}{r}\}$.

By Cauchy integral we have

$$\begin{aligned}(P_1^{\lambda f}(I-\mathcal{P}_\lambda))^n &= \frac{1}{n!}\frac{\partial^n}{\partial^n z}(I - zP_1^{\lambda f}(I-\mathcal{P}_\lambda))^{-1}|_{z=0}\\ &= \frac{1}{2\pi i}\int_{S(\frac{1}{r})}\frac{(I-zP^{\lambda f}(I-\mathcal{P}_\lambda))^{-1}}{z^{n+1}}\mathrm{d}z,\end{aligned}$$

from which we get

$$\|P_n^{\lambda f} - \sigma(\lambda)^n\mathcal{P}_\lambda\|_{\mathcal{B}_\psi\to\mathcal{B}_\psi} = \|(P_1^{\lambda f}(I-\mathcal{P}_\lambda))^n\|_{\mathcal{B}_\psi\to\mathcal{B}_\psi} \le Nr^n.$$

Since $\|P_t^{\lambda f}\|_{\mathcal{B}_\psi\to\mathcal{B}_\psi} \le e^{\lambda\|f\|}$ for $0 \le t \le 1$, by a standard argument and the semigroup property of $P_t^{\lambda f}$, we have

$$\|P_t^{\lambda f} - \exp\left(t\log\sigma(\lambda)\right)\mathcal{P}_\lambda\|_{\mathcal{B}_\psi\to\mathcal{B}_\psi} \le Cr^t. \tag{5.13}$$

For any probability measure ν with $\nu(\psi) < \infty$, by (5.13), for all large t so that $Cr^t < 1$, $\log\int_{\mathbb{H}} P_t^{\lambda f}1\mathrm{d}\nu$ are holomorphic on $D_{\tilde\delta}$. Moreover, by the inequality in (2),

$$\lim_{t\to\infty}\sup_{|\lambda|<\tilde\delta}\sup_{\nu\in A(L)}\left|\frac{1}{t}\log\int_{\mathbb{H}}P_t^{\lambda f}1\mathrm{d}\nu - \log\sigma(\lambda)\right| = 0.$$

By Cauchy's theorem for holomorphic function, for any e $\in (0,\tilde\delta)$ we have

$$\lim_{t\to\infty}\sup_{|\lambda|<\mathrm{e}}\sup_{\nu:\nu(\psi)<\infty}\left|\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\frac{1}{t}\log\int_{\mathbb{H}}P_t^{\lambda f}1\mathrm{d}\nu - \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\log\sigma(\lambda)\right| = 0, \qquad k\in\mathbb{N}.$$

By the C^2-regularity criterion in [, Theorem 1.2], we have

$$\lim_{t\to\infty}\sup_{\nu:\nu(\psi)<\infty}\left|\frac{1}{b^2(t)}\log\mathbb{E}^\nu\exp\left(b^2(t)\mathfrak{M}_t(f)\right) - \frac{1}{2}\sigma^2(f)\right| = 0,$$

where $\mathfrak{M}_t(f) := \frac{1}{b(t)\sqrt{t}}\left(\int_0^t f(X_s)\mathrm{d}s - \mu(f)\right)$ with $b(t)\to\infty$ and $\frac{b(t)}{\sqrt{t}}\to 0$ as $t\to\infty$, and

$$\sigma^2(f) = \lim_{t\to\infty}\left(\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\frac{1}{t}\log\int_{\mathbb{H}}P_t^{\lambda f}1\mathrm{d}\mu\right)|_{\lambda=0} = \lim_{t\to\infty}\frac{1}{t}\mathbb{E}^\mu\left(\int_0^t(f(X_s)-\mu(f))\mathrm{d}s\right)^2.$$

By [4, Chapter 6], we immediately obtain the MDP result in the theorem. □

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