Extensions of the shapley value in weighted voting systems

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EXTENSIONS OF THE SHAPLEY VALUE IN
WEIGHTED VOTING SYSTEMS

BY

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Dedicated to my Parents

H.R.H. John W. Ellah II and his Wife

Christiana N. Ellah

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ABSTRACT

EXTENSIONS OF THE SHAPLEY VALUE IN WEIGHTED VOTING SYSTEMS

The present work reviews the concept of values in the theory of games with particular reference to political games.

A model based on the Shapley value concept is developed and applied to simulated and practical voting situations. In particular it is shown how numerical expressions can be obtained for the values of each group or party given their sizes and with a knowledge of their previous voting patterns.

Data based on the Nigerian political set up as well as other political systems, including the U.N., E.E.C. etc. was used for calculating the values of the different participants.

CONTENTS

NOTATION

The following notation is used all through the Thesis.

1. B, C, D, L, L*, M, O, S, S*, T, ... = subsets of the set of players
2. $B_i(V)$ = Banzhaf value
3. d_i = real vectors
4. E = Expectation operator
5. {i} = set notation for i as only member
6. i, j, k, m, n, Px, ... = number of players or individual players or as specifically defined
7. N = set of players or finite carrier or as defined
8. [P] = partition
9. (P_i) = Negotiation group
10. Pr. = Probability
11. S^m = Set of minimal winning coalitions
12. U = Set of players or the universe of players or as specifically defined
13. $\underline{U}(S)$ = Standard of Fairness value
14. V or V* or V** = Characteristic function
15. V(S) = value of S in characteristic function form
16. W(S) = Number of votes or weight of S
17. ω.r.t. = With respect to
18. $\ell(N)$ = Simple games or as specifically defined
19. $n_i(V)$= Number of swings
20. $\bar{n}(V)$ = total number of swings
21. ϕ_i, $\phi_i[V]$, $\phi_i(V)$ = Shapley value
22. Z_i = Shapley value - multilinear extensions
23. Σ = Summation operator
24. π = multiplication operator

25. $\int$ = integral operator

26. $\gamma_n(S) = \frac{(S-1)!\,(n-1)!}{n!}$ = Probability measure

27. μ = mean

28. σ^2 = variance

29 ω, μ, λ = angle measures or operators as the case may be

30. $\exists$ = There exists, $\nexists$ = There does not exist

31. $\in$ = member of, $\notin$ = not member of

32. $\forall$ = for every

33. $>$ = greater than

34. $\geqslant$ = greater than or equal to

35. $\gg$ = much greater than

36. $<$ = less than

37. $\leqslant$ = less than or equal to

38. $\ll$ = much lesser than

39. $\simeq$ = in equilibrium

40. iff = if and only if

41. $\bigcup$ = Union

42. $\bigcap$ = Intersection

43. $\subset$ = Subset

44. $\subseteq$ = proper subset

45. $\triangleright$ = delta

CHAPTER 0

INTRODUCTION

Voting occupies a central position in democratic theory and practice. It helps to provide a tool for helping democratic societies consisting of different individuals with desperate preferences to decide on one course of action.

Voting is not a simple process as would be expected. Early works by Jean-Charles de Borda and Marquis de Condorcet in the late 18th century succeeded in revealing that certain voting methods could in fact hide surprising logical subtleties. This is further exposed when an attempt is made to evaluate the voting systems with respect to the equitability of the principles of proportional representation or when individuals are faced with the problem of deciding on one course of action when they have more than two alternatives. The problem is made clearer still when one evaluates the voting powers possessed by different individuals or groups of individuals in a voting system.

The problem of equitability in proportional representation and the validity of different voting schemes have been studied in some detail by many game theorists including, Fishburn, P.C. as contained in his paper on "Paradoxes of Voting" Fishburn, P.C. (1974) and also his paper on monotonicity Paradoxes in the theory of Elections, Fishburn, P.C. (1982). Also Gibbard, G. (1973) has a good coverage on the manipulation of election schemes as well as Niemi G.R. and Riker, W.H. on "The Choice of Voting Systems" Niemi, G.R. and Riker, W.H. (1976) to name a few. They all seem to conclude that "any voting system can lead to paradoxical results where losers are preferred to winners and winners become losers. In certain situations, however, some voting systems are better than others", Niemi and Riker (1976, p.21).

The aim of this thesis as indicated by its title is to look at the problem of measuring power in a weighted voting body which is involved in making yes or no decisions when faced with two alternatives. In order to do this, the already well known Shapley value is extended to cover areas where weighted voting, coupled with social, political and economic bias of the players (participants), play major roles in determining the outcome of the game and thus its value. A model which gives result like the Shapley value is developed and applied.

The Shapley value according to Aumann,R.J. (1978) "is an a priori measure of a games utility to its players; it measures what each player can expect to obtain, "on the average", by playing the game. Other concepts of cooperative game theory, such as the Core, Bargaining set and N-M solution predict outcomes (or sets of outcomes) that are in themselves stable, that cannot be successfully challenged or upset in some appropriate sense. Almost invariably, they fail to define a unique result; and in a significant proportion of the cases, they do not define any result at all. The Shapley value, although it is not in any formal sense defined as an average of such "stable" outcomes, nevertheless can be considered a mean which takes into account the various power relationships and possible outcomes." It is clear therefore that the Shapley value is a better tool for predicting outcomes than most of the other concepts we are familiar with in game theory.

The Banzhaf index as will be shown later is one of the most prominent value concepts in connection with political games. It has close relationship with most value concepts, namely Coleman index, Rae index, Dahlingham index etc. Aumann in the same article as quoted above saw the Banzhaf index or Banzhaf value as a variant of the Shapley value and he went on to say that a "variant of the Shapley value called the Banzhaf value has achieved some prominence in connection with political models", and he concluded by saying that "In general, it is not efficient".

It seems therefore that the Shapley value is more prominent because of its efficiency in connection with political games value models due perhaps to its mathematical derivation and properties. Also, it seems that no superior value measure has yet been developed with respect to political games. An extension of such a model would, I believe, constitute a major contribution to the clearer understanding of games' theory as applied to political models.

In the theory of games a variety of optimality principles are studied and these principles are derived by stipulating the necessary and/or sufficient conditions they have to satisfy. This is an axiomatic approach to the problem and in the study of the Shapley value and its extensions we are indeed considering an axiomatic description of a principle of optimality which is characterised as the principle of a fair subdivision of payoffs.

This thesis is concerned therefore with the consideration of an axiomatic description of a principle of optimality. The first chapter will be devoted to a survey of the different models developed and applied to voting games with particular reference to political games.

The second chapter will cover a detailed analysis of the Shapley value and the extensions done by Owen, G. In the third chapter the theoretical base of our work will be presented, while chapter four will contain the results of the applications of these models as well as comparisons and deductions based on simulated data.

The fifth chapter will contain results of the applications as in Chapter Four but based on data from practical voting situations and conclusions. The Appendix, which follows Chapter Five, will contain an alternative approach to the value concept, some mathematical derivations, an extension of Owen's technique to Oceanic games and a few major computer programs developed and used for the course.

CHAPTER ONE

REVIEW OF VALUE CONCEPTS

1.1 BRIEF HISTORICAL BACKGROUND

The mathematical analysis of voting is carried out through the Theory of Games known as voting games. Voting games are classified as "Simple Games". A simple game can be defined as a cooperative/competitive enterprise in which the major goal of the players (participants) is "Winning" and the rule guiding this is a specification of the coalitions capable of achieving this desire to win. This abstract definition which will be rigorously expanded under the heading of simple games covers most of the familiar examples of constitutional political machinery, including direct majority rule, weighted voting, direct or indirect election of a President or Prime Minister, bicameral or multicameral legislatures, committees and subcommittees, veto situations etc.

The modern mathematical approach to the theory of conflict resolution which voting belongs to can be traced back to the invention of the modern theory of games by Von Neumann and Morgenstern as contained in their 1944 classic, "Theory of Games and Economic Behaviour", which was based on Von Neumann's earlier papers of 1928 and 1937.

It was in the 1950s that most of the more useful analytic tools of voting games were achieved through the efforts of Kenneth Arrow, Martin Shubik, Duncan Black and Robin Farquharson.

Von Neumann and Morgenstern, nevertheless explored the mathematical structure of simple games and to provide a solution they applied the concept of "Stable Set" which they had already developed for a general class of coalition games - simple games although they never used the words "Stable Set" for their concept. (VN-M 1944 ch.10) . This solution concept was very logical and they were able to construct an economic vote-selling model from it, where vote-selling implies trading of votes in a market game involving the exchange of money or goods other than "Power" as is the case in political games. Their "equilibrium price' implied the share of spoils which each player (participant) was expected to receive if he belonged to the winning coalition. Unfortunately, as pointed out in the introduction with respect to Aumann's comments, only a very small insignificant class of simple games yielded a solution via this approach. We can rightly regard this VN-M price vector as an early form of "Power Index" which in itself constituted a major step forward in the quest for a quantitative analysis of the power of voters in an abstract voting system. Shubik and Weber, R.S. (1978) Young, H.P. (1978) and Wilson, R. (1969) have done some work using the Vote Selling approach while Gurk and Isbell, S.R. (1959) Vickrey, W.S. (1959) and Wilson, R. (1971) have a good coverage of the so-called "main simple solutions".

In 1954 Shapley and Shubik, M. (1954) published a paper entitled "A method for Evaluating the Distribution of Power in a Committee System" where they succeeded in adapting a general-purpose solution concept developed in their 1953 paper, the so-called "Shapley value" to the case of simple games. Their new technique yielded some numerical indices capable of being directly interpreted in terms of the a priori ability of the players to affect an outcome. The major and important advantage these indices had over the VN-M equilibrium prices was that they were well defined for all classes of simple games.

1.2 SIMPLE GAMES

As defined earlier simple games are cooperative/competitive in nature and the major goal of the players where players stand for participants including politicians, board members etc. is to belong to the 'winning' coalition. They constitute a distinguished class of multiperson or N-person games, namely those in which each coalition that might form is either all-powerful or completely ineffectual and powerless. These classes of games are well suited for the study of organisations, committees, legislatures or any system that has a common "Political" structure where power and authority rather than monetary payoff is the fundamental goal and major driving force.

Simple games are by their unique structure relatively independent of most of the restrictive and sometimes controversial assumptions that underlie the more general theory of games. Thus, for several reasons, including methodology and practice, the theory of simple games requires a self-contained, independent analysis.

For a formal definition of simple games; Let N denote a set of players, and let S denote the set of subsets of N,
Let $N = \{1, 2, 3, 4, \ldots, n\}$ be the players in N. Then S a subset of N is called a coalition of players $n_i \in N$.

In a game G, S is a winning coalition if $S \geq C$ where C is the required number for winning. C is referred to as the "quota".
If S is a winning coalition then L = N-S is a losing coalition
S^m is called a minimal winning coalition if $S^m = C$
Let S^{mu} = Union of the set of all minimal winning coalitions then
P is a dummy if $P \notin S^{mu}$, also P is a dictator if $S = \{\{P\}\}$ for some $P \in N$.
$B = L \cap L^*$ = Blocking coalition where L^* is the compliment of L.
We note that Blocking coalitions neither win nor permit their compliments to win.
For a straight majority simple game S is winning if

$$S \geq \tfrac{1}{2}N + 1 \text{ if N is even}$$

and

$$S \geq \tfrac{1}{2}N + \tfrac{1}{2} \text{ if N is odd.}$$

This class of straight majority simple games is what we are interested in. Shapley (1962) and Lucas, W.F. (1972) have a rigorous coverage of other properties of simple games, as well as other definitions and proofs. We shall now define some common terms which we shall refer to constantly throughout this work. They include the characteristic function of a game, an imputation, the core of a game etc.

1.3 CHARACTERISTIC FUNCTION

A major factor in multiperson cooperative games, as would be expected, is the urge to form coalitions and thus the maximum amount or payoff obtainable by such a coalition is therefore the primary concern of the players. The starting point for most studies of co-operative N-person games should therefore be the "characteristic function". The characteristic function formulation was suggested by von Newman in 1928 and later presented in their 1944 classic. An N-person game (N,v) in characteristics function form consists of a set N of players as defined in 1.2 but with characteristic function V which assigns the real number V(S) to each nonempty subset S of players. (players in our model will represent politicians). In some other models they could represent board members, business executives, organisations etc. The value V(S) is therefore a measure of the worth or power of coalition S and is regarded as the 'expected value' of such a coalition, thus the members of coalition S expect V(S) between them. The characteristic function can then be defined as a set D of n-dimensional real vectors $d = (d_1, d_2, d_3, ---, dn)$ which represents the realizable distribution of 'spoil', 'wealth' or patronage among the N players. Player j therefore expects dj.

We note that the specification of the game might reasonably be required to satisfy the following :

(1) $V(\phi) = 0$ which implies that the set of non players should realize nothing.

(2) $V(B \cup C) \geqslant V(B) + V(C)$ (2) implies the super-additivity property of the game which means that the value realised by two different sets while playing together should not be less than the values due them before the union.

also (3) $V(di) \geqslant 0$ This condition of the game quarantees individual rationality or pareto optimality. No player should earn a negative value, but a zero value is allowable, in which case one does not get paid just for playing the game.

and (4) $\Sigma d_i = V(N)$ $\forall i \varepsilon N$ This implies group rationality. The winning coalition shares the whole value of the game among themselves.

The set D above defined as n-dimensional real vectors representing the realizable distribution of wealth is usually referred to in the language of game theory as an "imputation". It consists of all d_j which satisfy (4) above as well as (5) below

(5) $d_j \geqslant V(\{j\})$ for every $j \varepsilon N$

We require a few more definitions before we discuss value concepts. Let S be a coalition (winning) then S is 'effective' for imputation d or d is S-effective if $\sum_{j \varepsilon S} d_j \leqslant V(S)$ which implies that the value of the coalition should not be less than the values of the individual players. Thus let x and y ε D, the set of imputations, then x "dominates" y if $\exists$ a nonempty set S such that x is S-effective and each member of S would prefer x_i to y_i for every i ε S. A subset L of D is a "stable set" if no x ε L dominates any y ε L. This is necessary for the existence of internal and external stability of a set of imputations. The existence of stable sets led to the concept of the "core" of a game.

The "CORE" is a subset of any 'stable set' as defined above. It is therefore a set of imputations such that no imputation belonging to it is dominated by some other imputation. This precisely implies that it is a set of all undominated imputations.
It could be defined formally as

$$C = \{d \; \varepsilon \; D : \Sigma_{i \varepsilon S} \, d_i \geqslant V(S) \text{ for all non empty } S \subseteq N$$

This further implies that no coalition S can protest against or have the ability to block an outcome x in C on the grounds that such a coalition can expect more.

Donald B. Gilles (1959) and Shapley and Shubik (1969) have carried out an extensive and detailed analysis of the CORE concept. Most of its applications as a characteristic function value as will be seen later are in the area of market games. We shall now carry out a detailed survey of various value concepts, including those applied to political games.

1.4 VALUE CONCEPTS

The search for a rigorous way(s) of determining the payoff vector led to different definitions and approaches to the problem of value determination. Different models were therefore developed and proposed, including the Shapley Value or Shapley Shubik power index, the Banzhaf power index and its associate the Coleman index, etc., the standard of fairness concept, the ψ stability, the α-power model (alpha power model), the graph approach, the Kernel; Also the Bargaining Set model as well as the core. We shall give a brief summary of each of them but we shall extend the discussion of the Shapley value into Chapter two in order to expose most of its properties and derivation force.

It must be pointed out that only a few of the value concepts mentioned above have yielded successful results in political games, namely, the Shapley Value, Banzhaf index and to a small extent, Coleman index. The others have been more successful in the areas of economics and market games where the payoff is usually tangible e.g. money instead of power and authority, nevertheless, a brief survey of most of them is necessary for a proper understanding of the problem.

1.4.1 The Shapley Value or Shapley Shubik index

The Shapley Value or Shapley Shubik index as the names imply was put forward by Shapley and Shubik in their 1954 paper based on a model Shapley developed in his 1953 paper. References to these papers will be in chapter two.

The Shapley value, according to Aumann (1978) is an a priori measure of a games utility to its players; it measures therefore the average expectation of a player while playing the game. It is based on a system of coalition formations and is defined as

$$\phi_i\,[V] = E\,[V\,(S(i,\rhd) \cup \{i\}) - V(S(i,\rhd))]$$ where $\rhd$

defines a given ordering of the players and $S(i, \rhd)$ is the set of players preceding player i under the ordering $\rhd$. E is the expectation operator or expected value under the given randomization scheme. If all coalitions are equally likely, then each order on N, the number of players has probability $^1/_{|N|!}$. A proof of the uniqueness of this value has been given by Dubey, P. (1975).

As mentioned earlier the Shapley value will be rigorously defined in the next chapter but it is necessary to have this brief definition meantime since a few of the other values we intend to survey presently make some references to it.

1.4.2 The Bargaining Set

The 'Bargaining set' concept is based on the CORE as defined in 1.3. It is therefore connected with the idea of a "stable set". The aim of the bargaining set is to try to define what payoff vectors are stable once a coalition is formed. An individual outcome is "stable" if there is no objection to it and where there is any, there is sure to be a counter objection. A player i in set S can object to another player j in S if a payoff vector d is proposed, if it is possible for him to join a new coalition M without j and find a realizable vector d* where every one in M gets more. Player j can also counter object if he too can find a coalition S* containing himself and without i having a realizable vector d** in which all members of M get their original amount d and everyone in M∩S* gets at least what he would have realized in the objection d*. Sets where the above bargains and counter bargains can take place would be referred to as bargaining sets. From the nature of bargaining sets, we note that they are best suited for market games where the payoff is not restricted to power and authority, nevertheless, it is possible for politicians and political parties to constitute themselves into bargaining sets if no single party succeeded in winning an overall majority. Bargaining sets are really not suited for the situations we are interested in where yes and no answers are required for decision making. An extensive coverage of the bargaining sets is contained in Nash, J.F. (1950), Nash, J.F. (1953) and Harasanyi, J.C. and Selten, R. (1972) The bargaining set concept is precisely concerned with locating stable sets and predicting the coalitions that could be formed from it, bearing in mind the tendency of players to seek for optimal payoffs. We shall survey some other models that are concerned with the equitable way to share payoffs in order to ensure the stability of a coalition.

1.4.3 Standard of Fairness

The standard of fairness model incorporates the "psychology" of the players in an $\underset{\sim}{N}$-person game by giving consideration to the players' bargaining abilities, moral codes, roles in other coalitions and their a priori expectations. All the above information is necessary for the adequate definition of the "standard of fairness" of the players. It is of course difficult and even impossible to get all the required information. The standard of fairness is defined using Thrall's partition function to determine the "power" of a Coalition and thus its value. This is done by regarding the game as being played among various coalitions who have pure strategies (A strategy can be defined as a complete description of how one would be expected to behave under every possible circumstance) of breaking themselves into negotiating groups. The power derived from this approach has been shown to be a new characteristic function which reduces every game to a constant-sum game. Maschler, M. (1963).

The mathematical definition of "Standard of fairness" is hereby given: Let an N-person game be defined in characteristic function form as in 1.3 above.

In addition to satisfying conditions 1 - 4 of 1.3 it is further required that

(1) $V(N) > V(S) + V(N-S)$ $\forall$ Coalition $S \varepsilon N$ except N and 0

We note that this extra requirement is an impossible condition to satisfy hence for a "fair" split of the payoff accruing to Coalition S the standard of fairness concept recommends that

(2) $V(N) - \{ V(1) + V(2) + --- + V(K) \}$

For $j = 1, --- K \varepsilon S$

be split equally among each player j. Player j receiving his original value V(j) in addition.

Standard of fairness could then be defined as a vector function

$\phi ([P]) \equiv \{\phi_1([P]), \phi_2([P]), \ldots, \phi_K([P]) \}$

defined for each partition $[P] \equiv (P_1, P_2, ----, P_k)$ of N into negotiating groups (negotiating groups mean intermediate sub Coalitions) and satisfying the following:-

(2) $\phi_i([P]) \geqq V(P_i)$, $i = 1,2 --- K$ (rationality within negotiating group)

and(3) $\phi_1([P]) + \phi_2([P]) + --- + \phi_k([P]) = v(N)$

(3) implies that all the negotiating groups will share the amount v(N).

The pair (ϕ([P]); N) where N stands for set of players in game (V;N) and ϕ([P]) stands for the standard of fairness satisfying (2) and (3) is known as a game space and a game space is therefore Thrall's game in partition function form. Lucas W.F. (1963) and Thrall, R.M. and Lucas, W.F. (1963) have more details.

The 'standard of fairness' can also be defined in terms of the Shapley value as follows.

Let the players in partition [P] regard the partition as final. Also let the negotiating groups in [P] namely P_1, P_2, --- P_k consider themselves as involved in a K-person game (V**, [P]) having the characteristic function

$$V^{**}(P_{j_1}, P_{j_2} ----, P_{jm}) = V(P_{j_1} \cup P_{j_2} \cup --- \cup P_{jm})$$

since the Shapley value is regarded as an a priori measure of a player's value and since it is necessary for a negotiating group to evaluate itself in any partition that it belongs to, then, it would be in order to have the Shapley value of (V**; [P]) as the evaluation of the game.

Thus the standard of fairness based on the Shapley value is

$$\phi_j([P]) = \sum_{S_\alpha} \frac{(t_\alpha - 1)!\,(K - t_\alpha)!}{K!} [V^{**}(S_\alpha) - V^{**}(S_\alpha - P_j)]$$

$$\phi_j([P]) = \sum_S \gamma_K(t) [V^{**}(S) - V^{**}(S - P_j)] \qquad \text{For } j = 1, 2, ---K$$

K = number of negotiating groups in [P]

S_α = All possible coalitions of the negotiation groups and

t_α = Number of negotiation groups in S.

$$\gamma_K(t) = \frac{(t_\alpha - 1)!\,(K - t_\alpha)!}{K!}$$

Maschler, M. (1963) contains a detailed treatment of the "standard of fairness" concept.

It is important to note that like the bargaining set concept the application of the "Standard of fairness" model is not in the area of political games but would serve as a useful tool in such a system where physical exchange of spoils is possible, for example, market games.

The concept is concerned primarily with the way the excess over the contribution made to a coalition is to be split. This model suggests that since everyone played a part in the realization of the excess, such an excess should be split equally, each player receiving this equal share in addition to his personal value which could be regarded as his own contribution. It is expected that a player would remain in a coalition where he has more excess accruing to him. We shall survey in the next section the α-power model which is similar in conception to the standard of fairness idea.

1.4.4 The α-power model

The α-power model is very similar in conception to the standard of fairness approach except that it proposes a free parameter "α" as a tool for defining the way two complementary groups will participate in distributing their excess.

The model constructs a standard of fairness function $\underline{U}$, defined for all Coalitions S of N players as a function of the grand Coalition, the Coalition value and the value of the complement of S, S* in conjunction with a free parameter α,$(0 \leqslant \alpha \leqslant 1)$. We illustrate by considering a 3-person game in detail.

Let A,B,C be the players in a characteristic function game (V;N) and let the proposed or actual outcome of the game (V;N) be represented by a payoff configuration $(X\ [P\ ]) = (X_A, X_B, X_C;\ \ S_1,\ S_2,\ ---S_m)$

where $X = (X_A, X_B, X_C)$ represents a 3-dimensional real vector known as the payoff vector which stands for a realizable disbursement of points among the players. [P] = coalition structure is a partition of the players into m mutually disjoint coalitions and for this case $(1 \leqslant m \leqslant 3)$.

Then (1) $\underline{U}(S) \geqslant V(S)$ The standard of fairness value must not be less than the value of the characteristic function for any coalition.

(2) $\underline{U}(A,B,C) = V(A,B,C)$ - for the grand coalition

and $\underline{U}(S) + \underline{U}(\tilde{S}) = V(A,B,C)$ where $\underline{U}(\tilde{S})$ = the complement of $\underline{U}(S)$.

Thus the free parameter $(0 < \alpha < 1)$ defines the 'fair' way that the two complementary groups

S and S* will partition their excess $V(A,B,C) - V(S) - V(S^*)$ in order to assess

their power, where V(S) stands for the coalition value of S and V(S*) stands for the value of the complement of S, S*.
The parameter α is further assumed to be independent of which coalition forms.

The standard of fairness function can then be derived as follows:-

$$\underline{U}(A,B) = V(A,B) + \alpha\,[V(A,B,C) - V(A,B) - V(C)] : \underline{U}(C) = \underline{U}(A,B,C) - \underline{U}(A,B)$$

(1) $$\underline{U}(A,C) = V(A,C) + \alpha\,[V(A,B,C) - V(A,C) - V(B)] : \underline{U}(B) = \underline{U}(A,B,C) - \underline{U}(A,C)$$

$$U(B,C) = V(B,C) + \alpha\,[V(A,B,C) - V(B,C) - V(A)] : \underline{U}(A) = \underline{U}(A,B,C) - \underline{U}(B,C)$$

and
$$\underline{U}(A,B,C) = V(A,B,C)$$

The above standard of fairness function has been suggested as a more realistic representation of the value of the coalitions.

Stability in payoff disbursement has been defined by Rapoport and Kahan (1980) as a set of payoff configurations (PCs) in which the differences between a player's payoff x_i and his power $\underline{U}_i$ in (1) above are equal for all members within each coalition S_j, $S_j \varepsilon [P]$.

A set of all stable payoff configurations in the α-power model for the three person game has been derived by Rapoport and Kahan in their paper on "Coalition formation in the triad", Kahan, J.P. and Rapoport, A.(1980) p.16). We note that the choice of α is not easy to make and could affect the efficiency of the above model. We also note that the model incorporates the standard of fairness technique and also the value of the complementary coalition set in an attempt to determine a stable payoff configuration(s).

We shall look at a slightly different approach in connection with the use of graph theory for value analysis.

1.4.5 Graphs in Cooperative Games

There has been some attempts to apply graph theory to analyse cooperation in games by incorporating certain allocation rules for selecting a payoff for every possible cooperation structure. Cooperation in this sense means coalition formation. A brief analysis of one of such concepts which links graph theory with coalition formation is as follows:

Let N be a set of players. A graph GR on N is a set of unordered pairs of distinct members of the set of players N = (1, 2, 3, , , , n)

Let these unordered pairs be called links. Then g defines a set of links on N and g^N is a complete graph of all the links.

Let GR = set of all graphs on N

then (1) $GR = \{g/g \subseteq g^N$

(2) $g^N = \{n : m/n \varepsilon N$ and $m \varepsilon N$, $n \neq m$ where $n : m$ defines a link between n and m.

Players are linked if there exist bilateral agreements between them which for our purposes we refer to as coalitions.

Let $S \subseteq N$, $g \varepsilon GR$, $n \varepsilon S$ and $m \varepsilon S$. Then n_i and m_i are connected in S by g iff there is a path in g which links n_i to m_i and remains in S. g therefore defines a unique partition of S which groups players together if they are connected in S by g.

Let S/g ("S divided by g) denote such a partition, then $S/g = \{\{i/i$ and j are connected in S by $g\}/j \varepsilon S\}$

Let y be an allocation rule which maps the graph g unto the allocation vector from the values V(C) of a coalition C.

We require

(3) $\sum_{n \varepsilon C} Y_n(g) = V(C)$ which means that the values of the individuals sum to the value of the coalition

and (4) $Y_n(g) - y_n(g') = y_m(g) - y_m(g') \geqslant 0$

where g' is g if n is not linked to m . This means that the value of a coalition is diminished by the same amount at both ends if the links between them are removed, i.e. where the coalition breaks up.

Myerson, R.B. (1977) has been able to prove that there is at most one such allocation rule. Also $y(g) = \phi(V/g)$, for every $g \varepsilon GR$ establishing a relationship between the allocation rule and the Shapley value operator $\phi(.)$. Examples for the use of this approach can be found in Myerson's paper on Graphs and Cooperation in Games.

(Maths. of Oper. Res. (1977) Vol. 2 page 227). We shall carry our survey to those value concepts that have been used in connection with political games or have been so suggested.

1.5 POLITICAL GAMES VALUE CONCEPTS

In addition to the Shapley Shubik index a few other indices have been developed and applied to political games. Some have been quite successful, for example, the Banzhaf power index, the Coleman index, Dalingham index and the Rae index. Some have not been as successful, for example, the ψ-stability concept and the Kernel, nevertheless we shall carry out a survey of all starting with the less successful ones.

1.5.1 ψ-STABILITY

ψ-stability concept can be traced back to Luce and Rogow (1956). In this concept a legislative scheme is supposed to be describable by the characteristic function of a simple game as defined in 1.3. The payoff for passing a bill is considered to be the "Power" due to the winning group, while the power distribution scheme among the winning players is an imputation. Furthermore "Power" is supposed to be "-- a divisible and transferable commodity--" Luce and Raiffa (1957). The problem then is to determine the power distribution with respect to the coalition structures which are considered to be stable. It is important to note that in the development of the ψ-stability concept, "stability" as such is not defined but ψ-stability is defined below.

The analysis so far made have been restricted to the stable distribution of power in a two party state, namely, the United States presidential system. Coalitions in equilibrium are considered unlike the Shapley value which gives an a priori measure since the coalitions that might form are not yet known.

In the calculation for the power distribution, pairs consisting of an imputation and a corresponding arrangement of players are isolated and these are tested for stability using the definition of ψ-stability which will be given later. Thus a pair $[d, S]$ are isolated, where d is an imputation and S is a coalition structure which remain in equilibrium when described in characteristic function form V and when changes in coalition arrangements are limited by a function ψ. A pair $[d, S]$, where d is an imputation and S a coalition structure is ψ-stable for the game (V,N) and given "boundary condition" ψ, if

(a) $\quad V(S) \leq \sum_{i \in S} d_i \quad$ for every S in $\psi(S)$

and

(b) $\quad d_i = V(\{i\}) \quad$ for coalition structure S player i is alone in S.

(It is important to note that the CORE as defined in 1.3 is a special class of the ψ-stability scheme) i,e, if d is an imputation in the core then the pair $[d, \{1\}, \{2\}, ----, \{n\}]$ is ψ-stable for ψ.

The function ψ has the set of all coalition structures as its domain and the range is the class of all sets of subsets of the players. Thus if T is in S, then $T \in \psi(S)$. Also S^* is a possible change from coalition structure S if $S^* \in \psi(S)$.

ψ-stability for a coalition therefore guarantees that no admissible change insures any profit to the participants.

In practice, the calculation and power distribution via this technique involves the division of players in a group (party) into two distinct non overlapping subsets of potential defectors and diehards in the event of a bill. The model allows potential defectors to defect and vote on the side of the other party but forbids the formation of a coalition by defectors from two different parties. ψ is then chosen to represent the limitations on defections from the party structure. Different cases of defection are then considered and for each case the resulting coalition is examined for stability, using the definition for ψ-stability. Thus if $\psi(S) = [T]$ either $\exists$ i such that $T \cup \{i\} \in S$ or L and $L' \in S$ such that $T = L \cup L'$ The distribution of power is then determed by analysing the role of potential defectors. We note also that a pair $[d,S]$ is ψ-unstable if there exists T in $\psi(S)$ such that $v(T)$ is greater than the sum of the payments in T as given by d.

Luce and Rogow (1956) have some relatively simple calculations based on the ψ-stability concept. The choice of ψ and most of the underlying suppositions have generated a lot of disagreement among game theorists. The technique is therefore considered as not being very efficient. Luce and Raiffa (1957, pp.223-231) contain a summary of some of these criticisms.

This model can be applied to political games but its application will become acceptable when most of the underlying assumptions are removed, especially with respect to the choice of ψ.

1.5.2 The Kernel of a cooperative game

The kernel is a subset of the bargaining set as defined in 1.4.2. It therefore has its base on the 'stable set' concept like the 'CORE' of section 1.3.

In order to define the 'kernel' of a game formally we need some preliminary definitions about the sort of cooperative games usually associated with the kernel.

Let Γ be a cooperative N-person game in characteristic function form satisfying all the conditions as stated in 1.3 except that V(S) is not assumed to be a superadditive function. $N = (1, 2, 3, \cdots n)$

Also let $(d,S) = (d_1, d_2, d_3, \text{---} d_n; S_1, S_2, \text{---} S_n)$ define an outcome of the game where di denotes the payoff to player i and $S = \{S_1, S_2, \text{---} S_n\}$ represent the coalition structures that were formed. Also let [S] be a partition of N satisfying individual and group rationality as in 1.3. Then,

(d, S) = an individually rational payoff configuration (i.r . p. c) Also let S be fixed, then, d is the set of all payoffs satisfying conditions (3) and (4) of 1.3 and is a cartesian product of m simplices. Thus

(1) $\quad d \equiv d(S) \equiv P_1 \times P_2 \times \ldots \times P_m$

and (2) $\quad P_j = [\{ di \}_{i \in S_j} / di \geqslant 0, \sum_{i \in S_j} di = V(S_j)]$

$j = 1, \text{----}, m$ coalitions

Further let D* be an arbitrary coalition. The "excess" of D* with respect to (d,S) is

(3) $\quad e(D^*) \equiv V(D^*) - \sum_{i \in D^*} di$

Thus e(D*) represents the total gain of members of D* if they should withdraw from (d;S) and form D*, thereby making $e(S_j) = 0$, $j = 1, \text{---}, m$.

Also let m* and n be two distinct players in S_j of S. We denote

$T_{m^*,n}$ = set of all coalitions which contain player m* but not n

Thus (4) --- $T_{m^*,n} \quad \{ D/D \subset N, m^* \in D, n \notin D \}$

The maximum surplus of m* over n is given as

(5) --- $P_{m^*,n} \equiv \max_{D \in T_{m^*,n}} e(D)$ This therefore represents the

maximum surplus (or minimum loss) due to m^* by withdrawing from (d,S) and joining D without the consent of n.

Also player m^* is said to outweigh player n denoted by $m^* >> n$ or $n << m^*$ if

$$P_{m^*,n} > P_{n,m^*} \quad \text{and } d_n \neq 0$$

Also if neither $m^* >> n$ nor $n >> m^*$ then both m^* and n are in equilibrium. We note that a player is in equilibrium with himself and that two distinct players in two disjoint sets are also in equilibrium. Also, if player i had 0 in (d,S) no player outweighs him. (special rule).

It therefore follows that a coalition S_j of S is "balanced" w.r.t. (d,S) if each pair of players of S_j are in equilibrium, denoted by $m^* \approx n$.

Now, the kernel "K" of a game Γ is a set of all individually rational payoff configurations that have only balanced coalitions. Thus $(d,S) \in K$ iff each pair of players are in equilibrium with respect to (d,S)

It follows therefore that

$$(6) \qquad m^* >> n \quad \text{if } (P_{m^*,n} - P_{n,m^*})\, d_n > 0$$

and also $m^* \approx n$ iff

$$(7) \qquad (P_{m^*,n} - P_{n,m^*})\, d_n \leq 0 \text{ and } (P_{n,m^*} - P_{m^*,n}) d_{m^*} \leq 0$$

In order to determine the shares among the players in the game a pseudo pure bargaining N-person game is defined based on the following properties of the kernel

(8) $V(N) \geqslant V(1) + V(2) + \ldots + V(n)$

(9) $(d,S) \in K$ for $S \neq \{N\}$ iff $d_i = v(i), i = 1,2, - \ldots n$.
and $(d,N) \in K$ iff

(10) $d_i = v_i + [V(N) - V(1) - V(2) - \ldots - v(n)]/n$

where K = Kernel for $i = 1,2, \ldots n$

To derive the shares then we require the following. Let the triplet $(Q;N; w_1, w_2 \ldots w_n)$ define a pseudo pure bargaining N-person game.

Then define $V(N)$ based on $w_1, w_2 \ldots w_n = (d;N)$s.t.

(11) $d_i = wi + [Q(N) - w_1 - w_2 - \ldots -w_n]/n$ for $i = 1,2, \ldots n$

provided that

(12) $d_i \geqslant 0$

We note that where (12) does not hold the technique applied would be to isolate player i who has the smallest d_i (d_i would be negative) and assign 0 to him. Then base the share of the other members on the pseudo pure bargaining set. Peleg (1963) has succeeded in proving that for each coalition there exists at least one stable payoff vector in the bargaining set concept.

Davis and Maschler (1965) have done a detailed analysis of the kernel. Their work includes opinions expressed by game theory experts with respect to the applicability of the kernel technique to real life situations.

A practical application of the kernel to the political coalitions found in Europe can be found in Schofield, N. (1977), pp.29-49). He observed that the kernel predictions performed well for countries with a low degree of political polarization and fragmentation. He also compared it to his resource/reward regression relationship. He noted,"the relationship between polarization, fragmentation and payoffs appears to be most complex. The notion of the kernel happily appears to be of some use in exploring these relations."

1.5.3 Common property - Successful Political Power indices

We shall continue our survey of the political games power indices by looking through the successful power indices, except the Shapley Shubik index which we mentioned briefly earlier and hope to mention again in detail in chapter 2.

These include the Banzhaf index, the Rae index, Coleman index and the Dahlingham index. In addition to the properties of games in characteristic function form they also have the following properties in common.

$$(1) \qquad V(S) = \begin{cases} 1 \text{ if } S \text{ is a winning coalition} \\ 0 \text{ if } S \text{ is a losing coalition} \end{cases}$$

We shall now survey all of them one after the other, noting the similarities between them.

1.5.4 The Banzhaf Index

The attraction of the Banzhaf index lies in its easy and straightforward verbal definition which has resulted to the Banzhaf index having a greater appeal to the legal mind than the rest. Thus it has been cited in cases involving the distribution of power in committees and also cases involving the problem of political representation more often than any of the other power indices; nevertheless, the Shapley Shubik index appeals more to the game theorists due to its underlying mathematical properties.

The principal word in the Banzhaf model is "SWING". We now define swing for player i as a pair of sets $(S, S - \{i\})$ such that S is winning and $S - \{i\}$ is losing.

Let $\eta_i(V)$ denote the number of swings for player i in the game $V \varepsilon \ell(N)$ where

$\ell(N)$ denotes simple games on N.

also let $\bar{\eta}(V)$ = Total number of swings i.e.

$$(1) \qquad \bar{\eta}(V) = \sum_{i \, \varepsilon N} \eta_i(V)$$

We note that $\eta_i(V) = 0$ implies that player i is a dummy, thus his vote makes no difference either way, also $\eta_i(V) = \bar{\eta}(V)$ implies that player i is a dictator, thus his vote is all that is necessary and sufficient.

$\eta_i(V)$ = swing number is known as the "raw" Banzhaf index. These were the numbers Banzhaf (1965) used for his calculations.

To derive the Banzhaf index one has to consider all the situations when the vote of player i would definitely cause coalition S to win or Bill B to be passed and would cause coalition S to lose if i leaves the coalition resulting in a defeat for Bill B.

These are regarded as swing situations and the player of interest is the one that actually determines the swing. As expected these "raw" Banzhaf indices will have different magnitudes, yet our principal interest lies in the ratio of these numbers, therefore, it has been common practice to normalize them by making them add up to 1.

This could be done by dividing the swings for player i by the total number of swings in the game.

Thus (2) $B_i(V) = \eta_i(V) \big/ \bar{\eta}(V) \quad i = 1, 2, \ldots, n.$

Where $B(V)$ = Banzhaf index

$\eta_i(V)$ = Swing number for player i

and $\bar{\eta}(V)$ = Total number of swings.

Also the swing probabilities for player i could be defined as

$$(3) \quad B_i(V) = \eta_i(V) \big/ 2^{n-1} \qquad i = 1, 2, \ldots n$$

Finally, let a_i stand for the probability that player i will vote "yea" to a bill and $1-a_i$, the probability that he will vote "nay"

Where $a_i = 0 < a_i < 1 \quad i \in N$

Then the generalized Banzhaf probability index can be given by

$$(4) \quad B_i' [a] = \sum_{S, i \in S \subseteq N} a_{S,i} \, [V(S) - V(S - \{i\})]$$

where $a_{S,i}$ = The probability that $Y = S - \{i\}$
(and Y stands for "yea" voters)

We note that (4) is similar in structure to the Shapley Shubik index

$$(5) \quad a_{s,i} = \left(\prod_{j \in S - \{i\}} a_j\right) \left(\prod_{j \in N - S} (1-a)\right)$$

The proof of the above as well as the derivation of the swing probabilities, including other calculations involving the Banzhaf index with respect to its upper and lower bounds, extensions and applications to weighted majority games can be found in Dubey and Shapley (1979)

1.5.5 Relationship with Shapley

The brief analysis of the Banzhaf index in Section 1.5.4 confirms to us that mathematically the Banzhaf index is based on the equiprobable combinations of the N players, while a closer look at the brief sketch on the Shapley value (or Shapley Shubik index) would remind us that the Shapley value is mathematically based on the equiprobable permutations of the N players. They both seem to be similar in some mathematical sense and this can be clearly portrayed if they are presented in their generalized form.

A generalization of the Shapley index as will be seen in the first part of the 2nd chapter leads to the Shapley value

$$\phi_i [V] = \sum_{S: i \in S \subseteq N} \frac{|S - \{i\}|! \; |N - S|!}{|N|!} [V(S) - V(S - \{i\})]$$

(regarded as player i's marginal contribution to all possible coalitions)

Also since Banzhaf regards every coalition as equally likely, it follows that for simple games the Banzhaf index could be converted to the Banzhaf value. Thus

$$B_i' [V] = \sum_{S: i \in S \subseteq N} \frac{1}{2^{|N - \{i\}|}} [V(S) - V(S - \{i\}]$$

For every $S \in N$

We note that the Banzhaf value like the Shapley value is symmetrical, linear, possses the dummy properties but fails to satisfy the efficiency criterion which is satisfied by the Shapley value as will be shown in Chapter 2. We also note that $B_i'[V]$ can be normalized as was done for $B_i(V)$ earlier. The conversion formula is fairly messy but for purposes of reference it shall be given thus

$$B_i[V] = \frac{[\,V(N) - \sum_j V(\{j\})\,]\;\; B'_i\,[V] + [\,\overline{B'[V]} - V(N)\,]\;\; V(\{\,i\,\})}{\overline{B'[V]} - \sum_j V(\{\,j\,\})}$$

Straffin, D (1977) has carried out some comparisons between the Shapley index and the Banzhaf index based on his practical application of both indices to real life voting situations. He concluded by recommending that "The Banzhaf index should be used for situations in which voters, vote completely independently, the Shapley Shubik index for situations in which a common set of values tends to influence the choices of all voters." This confirms that they constitute the same tool but perhaps suited for slightly different circumstances.

1.5.6 The Rae Index

Douglas Rae (1969) considered the problem of comparing the responsiveness of different voting systems to the general will of the electorate. He approached it by counting the number of ways the average voter can find his vote in agreement with the outcome of the voting, i.e. being on the winning side.

He thus defined an index of agreement to be $= a_j = \{\, Y \subseteq N:\ j \in Y \in W \ \text{ or } j \notin Y \notin W \}$ where W = set of all winning coalitions in a simple game $V \in \mathcal{L}(N)$. Thus the overall responsiveness of the voting system is the sum $\bar{a}$ or its average $\bar{a}/n$ or better the average probability of responsiveness which is given by $\bar{a}/(n\,2^n)$. where n is the number of elements in N.

It has been shown that the "Rae Index" is the Banzhaf index stated differently, Dubey and Shapley (1979) .

The following identity confirms it

$$a_i \equiv 2^{n-\ell} + \eta_i$$

where η_i = swing number

1.5.7 Coleman Index

James Coleman (1973) considered the two different types of power that can be exercised by a player in simple games. For simple games he used the word "collectivity". He stated that such a player can either initiate or prevent action. He carried out his calculations for "initiating action" by considering the fraction of all the losing coalitions where by his joining such a coalition would turn it to winning. And the power to "prevent action" he calculated by considering the fraction of all the winning coalitions that would lose should he leave the coalition.

Thus, let w = total number of winning coalitions

and σ = total number of losing coalitions

Then for player i, the power to prevent action is

$$a_{pi} = \eta_{i/w}$$

and the power to initiate action is

$$a_{ni} = \eta i/\sigma$$

We note that η_i = swing number.

We also note that $\frac{1}{B'_i} = 1/2 \left(\frac{1}{a_{pi}} + \frac{1}{a_{ni}} \right) \Longrightarrow$ the harmonic mean of a_{pi} and a_{ni} = B'_i where B'_i = the swing probabilities of player i as calculated in the Banzhaf model. It is clear therefore that the Coleman index is a re-definition of the Banzhaf Index.

1.5.8 Dahlingham Index

In an attempt to carry the power survey of voters some way further, Robert Dahl (1957) gave a definition of the power of one individual over the other as

$$a_i = wi/2^{n-1} - \bar{w}_i/2^{n-1} \quad \text{where}$$

w_i = winning coalition containing i.

$\bar{w}_i$ = w - w_i = winning coalition not containing i
n = number of elements in N the set of players
We note that $\eta_i = w_i - \bar{w}_i$ = the swing number

Thus $a_i \equiv B_i'$

where B_i' = swing probability of player i as given in 1.5.4.

We then note that the Dahlingham index is also a redefinition of the Banzhaf index, Alingham (1975).

The brief survey above covers a few of those techniques developed in mathematics and applied to political science, especially with respect to political games. A few other techniques exist also which were not covered here but all the most important ones have been discussed. Brams, S.J. (1978) contains a great deal of work on the use of mathematical techniques in political games. Some of his techniques are fairly different from the ones mentioned here, especially his calculations on the U.S. presidential primaries, nevertheless they all have the same underlying principles and nearly always use the same mathematical and statistical tools.

As shown above all the political games indices surveyed are re-definitions of the Banzhaf index and in the introduction it was pointed out that Aumann, R.J. sees the Banzhaf index as a variant of the Shapley value, Aumann, R.J. (1968, p.999). It therefore follows that an extension to the Shapley value will also constitute an extension to all the other political game indices discussed above. We shall then devote the next chapter to a detailed analysis of the Shapley value and some of its extension, namely, those done by Owen, G. A tabulated summary of the values discussed is given in the next page.

1.5.9 TABULATED SUMMARY

We hereby give a tabulated summary of all the value concepts we have analysed, including some of their special requirements.

CORE	C
Imputation	I
Stable Set	SS
Characteristic function (0,1) normalization	CF
Characteristic function not in (0,1) normalization	CF′
Political games	Pg
Market games	Mg
Psychology of Players	PSY
Partition	P
Division of excess	eD
Complement Set	S*
Connectedness	Cn
Related to Shapley Value	RSV
Related to Banzhaf Value	RBV
Swing Numbers	SN
Ordinary Winning Numbers	SN′
Special Requirement	SR

VALUE CONCEPT	C	I	SS	CF	CF′	Pg	Mg	PSY	℘	eD	S*	Cn	RSV	RBV	SN	SN′	Special Requ. SR	Page
Bargaining Set	✓	✓	✓	-	✓	-	✓	-	-	-	-	-	-	-	-	-	Concerned with locating SS.	12
Standard of Fairness	-	-	-	-	✓	-	✓	✓	✓	✓	-	-	✓	-	-	-	Con.with equitable way of splitting excess	13
α-Power Model	-	-	-	-	✓	-	✓	-	✓	✓	✓	-	-	-	-	-	Def. of for fair split. ex.	16
Graphs in Coop.games	-	-	-	-	✓	-	-	-	-	-	-	✓	✓	-	-	-	Def. Connectedness	18
ψ-Stability	-	✓	✓	-	-	✓	✓	-	-	-	-	-	-	-	-	-	-limits changes in coal.structures	20
Kernel	✓	✓	✓		✓	✓	✓	-	✓	✓	-	-	-	-	-	-	Splitting ex.	22
Banzhaf Index	-	-	-	✓	-	✓	-	-	-	-	-	-	✓	-	✓	-	Counting of swings	26
Rae Index	-	-	-	✓	-	✓	-	-	-	-	-	-	✓	✓	-	✓	Counting of votes on winning side	30
Coleman Index	-	-	-	✓	-	✓	-	-	-	-	-	-	✓	✓	✓	-	Power to win compared to lose	31
Dahlingham Index	-	-	-	✓	-	✓	-	-	-	-	-	-	✓	✓	✓	-	Power of i over j	32
Shapley Value	-	-	-	✓	-	✓	-	-	-	-	-	-	✓	✓	-	✓	Concerned with the Pivot number	11

CHAPTER TWO

THE SHAPLEY VALUE

In this chapter we shall carry out a detailed survey of the Shapley value and some of its extensions as stated earlier except the new model we are proposing which we shall give in Chapter Three.

A study of the Shapley value as stated in the introduction involves the consideration of an axiomatic description of a principle of optimality. Most of what is to follow is therefore concerned with defining these axioms and describing an abstract model that satisfies these axioms. Optimality will be defined in terms of payoff vectors and the search for optimality would then involve the search for stable and equitable payoff vectors.

2.1 DETAILED ANALYSIS OF THE SHAPLEY VALUE

We need some important definitions as well as all or most of the other definitions given in Chapter One for a realistic approach to our analysis of the Shapley value. The following definitions are therefore necessary.

A. Coalition : Let U be a finite set of players. Any subset S of U ($S \subset U$) will be known as a coalition.

Let V be the characteristic function for game G. We required that the coalitions in U satisfy the following axioms. Let π be any permutation of players. Then π is called an automorphism of the characteristic function V if for any coalition $S \subset U$

(1) $\quad V(\pi S) = V(S)$

This implies a mapping of each player i into πi and thus each coalition $S = (i_1 \ldots . i_s)$ into $\pi_S = (\pi_{i_1}, \ldots . \pi_{is})$. To obtain our axiom of symmetry we only require that for any automorphism of the game V.

(2) $\quad \phi_i(V) = \phi_{\pi i}(V)$ where $\phi(V)$ denotes

the value. (2) only tells us that the value is essentially a property of the abstract game.

B. Dummy : A player in a cooperative game with characteristic function $\mathbb{V}$ is called a "dummy" if $\forall$ coalition S not containing player i

(3) $\mathbb{V}(S \cup i) = \mathbb{V}(S) + \mathbb{V}(i)$ which implies that a dummy contributes only what he can win on his own while playing independently. He could therefore be deleted and the game will still be unchanged. The set of players consisting of all non-dummies is called the "support" or "carrier" of the game.

Thus if N is the support of game V

(4) $$V(S) = V(N \cap S) + \sum_{i \in U/N} V(i)$$

This leads us directly to the axiom of effectiveness which requires that if N is the support of V then

(5) $$\sum_{i \in N} \phi_i(V) = \mathbb{V}(N)$$

We also desire that the vector $\phi(V)$ be an "imputation" as defined in Chapter One, thus implying that the value represents the full yield of the game. (This axiom is not satisfied by the Banzhaf value, see Dubey, P. and Shapley L.S. (1979 page 128) and Dubey, P. (1975)

If we have two games hence two characteristic functions with the same set of players participating, it will be fair to assume that the payoffs of the players should be a combination of their payoffs in each of the games.

Thus if V* and W* are the characteristic functions for the two games

(6) $$\phi[V^* + W^*] = \phi[V^*] + \phi[W^*]$$

Thus (6) represents our last required axiom, the "axiom of aggregation" which could be interpreted to mean that when two independent games are combined the values must be added player by player.

As expected any system satisfying the three axioms given here will be non-contradictory and complete. The vector satisfying those three axioms we call the Shapley vector or Shapley value.

Let us recast the three axioms so as to have a direct link with our derivation of the Shapley value.

Axiom A : Let U be the set of players. For every π in $\pi(U)$

$$\phi_{\pi i}[\pi V] = \phi_i[V] \rightarrow \text{ Symmetry}$$

Axiom B : For every carrier N of V

$$\sum_{j \in N} \phi_j[V] = V(N) \rightarrow \text{ Efficiency}$$

Axiom C : For every two games V^* and W^*

$$\phi[V^* + W^*] = \phi[V^*] + \phi[W^*] \qquad \text{Super-additivity or Law of Aggregation}$$

Let V be a characteristic function game in $[0,1]$ normalization. Let N be a finite carrier of V; $\forall$ $i \notin N$, we define

$$\phi_i[V] = 0 \text{ giving zero to any dummy player}$$

Let $R \subseteq U$, $R \neq 0$, we define

$$(7) \qquad V_R(S) = \begin{cases} 1 & \text{if } S \supseteq R \\ 0 & \text{if } S \not\supseteq R \end{cases}$$

Where $S \subset U$. The function CV_R is a symmetric game; For every non-negative C, and R is a carrier of V. We let r, s, n,....., be the numbers of the elements in R, S, N, ..., respectively where R, S, N are subsets of $\mathbb{U}$, the Set of players.

Then for $C \geqslant 0, \quad 0 < r < \infty$

$$(8) \qquad \phi_i\,[CV_R] = \begin{cases} C/r & \text{if } i \in R \\ 0 & \text{if } i \notin R \end{cases}$$

By Axiom B, which guarantees efficiency

$$(9) \qquad C = CV_R(R) = \sum_{j \in R} \phi_j\,[CV_R] = r\,\phi_i\,[CV_R]$$

$$\forall\ i \in R$$

Also any game with a finite carrier is a linear combination of symmetric games V_R:

Thus

$$(10) \qquad V = \Sigma_{\substack{R \subseteq N \\ R \neq 0}} C_R(V) V_R$$

We note that N being a finite carrier of V, the coefficients are independent of N, and could be given by

$$(11) \qquad C_R(V) = \Sigma_{T \subseteq R} (-1)^{r-t}\, V(T) \qquad (0 < r < \infty)$$

Also it could be verified that

$$(12) \qquad V(S) = \sum_{\substack{R \subseteq N \\ R \neq 0}} C_R(V) V_R(S) \quad \text{For every } S \subseteq U$$

Also for every finite carrier N of V, if $S \subseteq N$ then it follows that from (7) and (11) we get

$$(13) \qquad V(S) = \Sigma_{R \subseteq S}\, \Sigma_{T \subseteq R}\, (-1)^{r-t}\, V(T)$$

$$= \sum_{T \subseteq S} \Big[\sum_{r=t}^{s} (-1)^{r-t} \binom{s-t}{r-t} \Big] V(T)$$

We note that the expression within the brackets vanishes except for the case S = t, thus we are left with the identity V(S) = V(S)

Generally therefore we have

$$V(S) = V(N \cap S) = \Sigma_{R \subseteq N} C_R(V) V_R(N \cap S) = \Sigma_{R \subseteq N} C_R(V) V_R(S)$$

It could also be shown that $C_R(V) = 0$ if R is not contained in every carrier of V as assumed.

From axiom C we note that $\phi[V^* - W^*] = \phi[V^*] - \phi[W^*]$ if V^*, W^* and $V^*\text{-}W^*$ are all games.

As a result we can apply the definition in (8) to the derivation at (10) and obtain

$$\phi_i(V) = \Sigma_{\substack{R \subseteq N \\ i \in R}} C_R(V)/r \quad \forall i \in N$$

if we insert (11) and simplify we get

$$\phi_i(V) = \Sigma_{\substack{S \subseteq N \\ i \in S}} \frac{(S-1)!\,(n-S)!}{n!} V(S) - \Sigma_{\substack{S \subseteq N \\ i \notin S}} \frac{S!\,(n-S-1)!}{n!} V(S)$$

$$\forall i \in N$$

Now, let $\gamma_n(S) = (S-1)!\,(n-S)!/n!$

We get

$$(14) \quad \phi_i[V] = \Sigma_{S \subseteq N} \gamma_n(S)\,[V(S) - V(S - (i))]$$

$\forall\, i \in U$ where N is a finite carrier of V.

Expression (14) is therefore the required Shapley Value.

The uniqueness of the Shapley value has been proven by Dubey, P (1975) as pointed out in Chapter One.

The following properties are true of expression (14)

(a) $\phi_i[V] \geqslant V((i)) \quad i \varepsilon U$

The equality condition results iff

(b) $V(S) = V(S-(i)) + V((i)) \quad i \varepsilon S$ and (b) is true iff i is a dummy.

Other properties including the rigorous derivation of all the axioms and theorems, together with their proofs are all covered in Shapley, L.S. (1953), Vorob'ev, N.N. (1977) Luce and Raiffa (1957) and other literature on games and values.

We may point out that the Shapley value given in (14) can also be reached via a bargaining model. Assuming the players that constitute a finite carrier N arrange to play game V in a grand coalition fashion as presupposed by Shapley. Furthermore, if they also agree that the order of admission of any member into a coalition or the order of joining a coalition is determined by chance, then all arrangements or orderings of players are equally probable. Also, suppose that on admission or on joining a coalition a player makes a demand and is promised the full amount his participation contributed to the value of the coalition as defined by the function V. All the players then play the game efficiently with the aim of realising the total amount V(N), enough to meet all their promises.

The expectations of each player would then be worked out thus. Let $p^{(i)}$ be the set of players preceding player i, then for any $i \varepsilon S$ the payment to i if $S-(i) = p^{(i)}$ is

$$M = V(S) - V(S - (i))$$

Now let the probability of such a contingency be $\gamma_n(S)$ thus the total expectation of i in the scheme is

$$M_i = \Sigma_{S \subseteq N} \gamma_n(S) [V(S) - V(S-(i))] = \phi_i[V] \text{ of } \quad (14)$$

We also note that since all members can occupy all positions with equal probability then the value of the game for each ordering should be allocated to the decisive player whose votes determined the result of that particular voting situation. Such a player we refer to as the "PIVOT' player. We then see that in simple games in which V is monotomic and assumes only the values 0 and 1, the pivot player is the only one that receives a non zero gain of any ordering.

The Shapley value therefore gives a unique result which for our purposes could serve as a good a priori measure of a games utility to the players. Classical Shapley has had a few extensions to the original model and has successfully been used for analysis in real life political situations, either directly or via a few of its extensions, namely, weighted majority games, oceanic games, and through some extensions that could be found in Owen, G. (1971) and Owen, G. (1972).

We shall carry out a brief survey of each of these application modifications and extensions of the Shapley value.

2.2 WEIGHTED MAJORITY GAMES

The classical Shapley approach as analysed in Section 2.1 needs some slight modifications to take care of many real life situations. In real life,voting situations exist whereby voting strength differs with respect to the number of votes each player has or a group of voters have, for example, The U.N. Security Council, the U.S. electoral College, State and National legislatures, multi party parliaments, shares in corporations and companies, etc. Our research, as will be seen later, is concerned mainly with weighted majority games.

The Shapley value is applied to weighted majority games in a slightly different way from the situation where all voters have equal weights.

Let this class of simple games be denoted by [C:W] where C is a real number and W is a measure on R. [We let R be a Boolean ring] . Let W(S) be the total number of votes of coalition S and let C represent the number of votes needed to "win" w.r.t. the characteristic function V.

We have

$$V(S) = \begin{cases} 0 & \text{if } W(S) < C \\ 1 & \text{if } W(S) \geqslant C \end{cases}$$

We note that a carrier of W is also a carrier of V. Let N be a finite carrier of V. We may now denote the game as $[\, C; W_1, W_2, \ldots, W_n \,]$. We assume that the players in N are matched with the natural numbers 1, 2, 3, 4,....n and that Wi = W({i}). The numbers Wi are the weights of the players while C stands for the "quota" i.e. the number required to win. With the above re-definitions we can then apply expression (14) of section 2.1 to determine the value. From the classical Shapley point of view the weights of the players and their values are closely related. Although it is possible also to find players of unequal weights having exactly the same value yet it has been shown in Shapley, L.S. (1953) example 5) that a player's value is a monotonic, non-decreasing function of his weight if the other players' weights are held fixed while their quota either is held fixed or adjusted while preserving the ratio $^{C}/W(N)$.

It will be shown in practical examples in Chapter 3 that the classical Shapley approach produces the obvious result i.e. power proportional to voting strength for the weighted majority game, when the players play the game as separate individuals with groups.

As mentioned earlier the extensions we shall present are applicable to the weighted majority class of games since they could be easily identified with practical political situations where voters belong to different parties with different identities and different voting patterns. The number of representatives from each party constitute the voting strength or weight of the party and each party is regarded as a player since for homogeneous parties the party leader should be able to determine or indicate the direction his party members should vote in the event of a bill.

We also have another class of games, known as oceanic games which have some peculiar properties that differentiate it from both the ordinary classical Shapley one-man, one-vote approach and the weighted majority games approach.

2.3 OCEANIC GAMES

Oceanic games constitute a special class of weighted majority games, but unlike the type discussed in 2.2, here we have a situation where a block of votes is broken up and distributed among a very large number (continuum) of players which we call the ocean while a few major players called atoms control fairly large numbers of votes among themselves. If we denote this by $[C; W_1, W_2, W_3, \ldots W_m; \alpha]$ where

C	=	The quota required to win
Wi	=	The weight of the atoms
α	=	The total weight of the ocean

In most cases we may require that $W_i \leqslant C \leqslant \alpha$. We note that the direct formular applicable to finite person game as stated in (14) of Section 2.1 is not readily applicable here.

The question now is to determine the values of the major (atomic) players from where the values of the ocean can be calculated. We also note that the earlier approach whereby players are randomly shuffled in order to locate the pivot player is not easy to extend to a continuum of oceanic players because the notion of randomly shuffling oceanic players, even without the major players is not easy to formulate. Nevertheless, we note that the ocean is symmetric so we can limit ourselves to inserting the major playersinto the ocean in a properly random way.

Let us consider a sequence of $(m + n_\ell)$ - person weighted majority game. $\perp_\ell = [C; W_1, W_2, W_3, \ldots, W_m, a_{1,\ell}, a_{2,\ell}, \ldots, a_{n,\ell}]$

$$\ell = 1, 2, \ldots.$$

Such that we have

$$(1) \quad \sum_{j=1}^{n_\ell} a_{j,\ell} = \alpha \quad \ell = 1,2, \ldots$$

with α being a positive constant and such that

$$(2) \quad \max_j a_{j,\ell} \equiv a_{max,\ell} \xrightarrow{\quad} 0 \quad \text{as} \quad \ell \to \infty$$

Now conditions (1) and (2) require that $n_\ell \to \infty$ i.e. the minor players tend to a continuum of oceanic players.

Now let $\phi_{i,\ell}$ denote the value of game $\mathbb{L}_\ell$ to the i^{th} major player, $i = 1,\ldots,n$.

Let $<\ >$ define the following

$$<x> = \text{median } g(0,x,1) = \begin{cases} 0 & \text{if } x \leqslant 0, \\ x & \text{if } 0 \leqslant x \leqslant 1, \\ 1 & \text{if } x \geqslant 1. \end{cases}$$

Also let $M = \{1,\ldots\ldots,m\}$ major players, $S = |S|$ and $W(S) = \Sigma_{i\varepsilon S} W_i$. also let $M_i = M - \{i\}$.

It has been shown that for each major player $i\varepsilon M$, the value $\phi_{i,\ell}$ of the game $\mathbb{L}_\ell$ convergers to a limit, thus

$$(3) \quad \phi_{i,\infty} = \sum_{S \subseteq M_i} \int_{<(C-W(S\cup\{i\}))/\alpha>}^{<(C-W(S))/\alpha>} t^S (1-t)^{m-S-1} dt$$

We can recast (3) as

$$(4) \quad \phi_{i,\infty} = \sum_{S \subseteq M_i} \int_{t_1}^{t_2} t^S \ (1-t)^{m-S-1} dt$$

$$t_1 = \left\langle \frac{C-W(S\cup\{i\})}{\alpha} \right\rangle$$

$$t_2 = \left\langle \frac{C - W(S)}{\alpha} \right\rangle$$

In order to calculate the values of the major players in an oceanic game so as to determine also the minor players value we shall resort to formulation (4) above instead of the derivation we had in Section 2.1. We would therefore be expected to resort to this new formulation if we were to determine the values or powers held by major shareholders in a corporation with, say, three major shareholders and (a continuum) a very large number of people having very negligible shares each.

A detailed analysis of Oceanic games as well as a rigorous derivation of the above formulations can be found in Shapiro and Shapley (1978), Milnor and Shapley (1978)

2.4 EXTENSIONS/APPLICATIONS OF THE SHAPLEY VALUE

There have been a number of extensions and applications of the Shapley value to practical voting situations, especially weighted majority games by a few game theorists other than Shapley himself. Lucas, W.F. (1976) has a good survey on its application to such weighted bodies such as the U.N. Security Council, U.S. Electoral College etc. Also Littechild, S.C. and Owen, G (1973) have used it in determining the cost of airport landing fees by different types of aircrafts. In this case landing fee is computed from the maintenance charge (M_i) for aircraft type i plus a capital charge ϕ_i. The capital charge is then computed from the Shapley value, or put differently, a game V is defined by considering the players to be individual aircraft landings with V(S) the hypothetical cost of building a facility that can accommodate a set S of landings. Thus each landing attracts a fee equal to its Shapley value.
Owen, G. (1975) carried out an evaluation of the presidential election game using both the Banzhaf value as compared to the

classical Shapley value. We find in Owen, G (1972) a multilinear extension of the Shapley value which constitutes a fairly good generalization of the value.

We shall survey two of Owen's extensions in some detail since one of the extensions we carried out was based on Owen's extension and modification with respect to political games.

2.4.1 Multilinear Extensions by Owen, G

In Owen's multilinear extensions an N-person game V is defined as a function on the N-cube I^N that is linear in each variable and also coincides with V at the corners of the cube satisfying
$f(x) = V(\{i/x_i = 1\})$

In order to derive the generalized Shapley value, the following initial conditions defined for the classical Shapley also hold.

We let V be a characteristic function of an N- person game as defined in 1.3. Now consider $\alpha = \{0,1\}$, then the domain of V is a subset of the unit N-cube I^N where $I = [0,1]$. To extend V to this cube we write

$$(4) \qquad f(X, \ldots, X_n) = \sum_{S \subseteq N} \{\Pi_{j \in S} X_j \, \Pi_{j \notin S} (1-X_j)\} \; V(S),$$

for $0 \leqslant x_i \leqslant 1$, $i = 1, \ldots, n$. Let α^S represent the S-corner of the cube and n be the number of elements in N.

$$\text{Thus (5)} \quad \alpha_i^S = \begin{cases} 1 \text{ if } i \in S \\ 0 \text{ if } i \notin S \end{cases}$$

We see then that $f(\alpha^S) = V(S)$ because

$$(6) \qquad f(\alpha^S) = \sum_{T \subseteq N} \{\Pi_{j \in T} \alpha_j^S \, \Pi_{j \notin T} (1 - \alpha_j^S)\} \; V(T)$$

We note that the braces vanish except for T = S when it will equal unity, thus f is an extension of v.

An interesting interpretation of (6) is that if player i has probability X_i of joining a coalition, then the probability that coalition S exactly will form assuming independence of the players will be given by (4). f is then thought of as an expected value.

If we let $\phi_i(t)$, i = 1,...,n be a continuous monotone function with $\phi_i(0) = 0$, $\phi_i(1) = 1 \ \forall\, i$

Then $X_i = \phi_i(t)$, $0 \leqslant t \leqslant 1$ will then represent a monotone path from the origin (0,0,0,...,0) to the unit corner (1, 1, 1, 1,....,1)

We write

$$(7) \qquad Z_i = \int_0^1 fi(g(t))d\,\phi_i(t)$$ where f_i is the i^{th} partial derivative of the function f then

$$(8) \qquad \sum_{i=1}^{n} Z_i = \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{dx}{dt} i\, dt = \int_0^1 \frac{df}{dt}\, dt$$

$$= f(\phi(1) - f(\phi(o))$$

$$= f(\alpha^N) - f(\alpha^\phi)$$

Therefore we get

$$(9) \qquad \sum_{i=1}^{n} Z_i = V(N)$$

If we let $f_i(t,t,...t) = \sum_{S \subset N: i \notin S} t^S(1-t)^{n-S-1} [V(S \cup \{i\}) - V(S)]$

Thus $$(10) \qquad Z_i = \sum_{S \subset N: i \notin S} \int_0^1 t^S(1-t)^{n-S-1} dt\, [V(S \cup \{i\}) - V(S)]$$

$$(11) \qquad Z_i = \sum_{S \subset N: i \notin S} \frac{S!(n-S-1)!}{n!} [V(S \cup \{i\}) - V(S)]$$

Thus we have the multilinear extension of the Shapley value.
A detailed derivation of the above is contained in Owen, G. (1972)
We can then apply the above to weighted majority games by bearing in mind the following modifications and representations.
We represent the weighted majority game by $[C : W_1, W_2, W_3 \ldots\ldots W_n]$ with $C \geqslant W_i$, then

$$V(S) \begin{cases} = 1 \text{ if } \sum_S W_i \geqslant C \\ = 0 \text{ if } \sum_S W_i < C \end{cases}$$

where

C is the quota as in Section 2.2

The partial derivative $f_i(X)$ could then be interpreted as the expected marginal value of player i to the coalition he will join, given that j has probability X_j of being there as well. Thus
$f_i(X) = 1$ if

$$(12) \qquad C - W_i \leqslant W(S) < C \quad \text{and 0 otherwise.}$$

If we regard the random variable W(S) as the sum of N-1 independent random variables (each of the remaining players having one), thus the jth can have values of 0 and W_j with probabilities $1 - X_j$ and X_j respectively. It will then have a mean $X_j W_j$ and variance $X_j(1-X_j)W_j^2$ for the point on the diagonal of the cube I^N where $X_j = t$.

$$(13) \qquad \mu = t \, \Sigma_{j \neq i} W_j \rightarrow \text{ mean and}$$

$$(14) \qquad \sigma^2 = t(1-t) \sum_{j \neq i} W_j^2 \qquad \rightarrow \text{ variance}$$

Under normal conditions we assume that the distribution will be normal. Thus the only required calculations would be for t, i.e. the probability that a normal variable with mean (13), variance (14), satisfies (12).

The multilinear extensions of the Shapley value have some similarity with the model we are proposing at least in concept.

2.4.2 Owen and a Modification of the Shapley Value

In Guillermo Owen (1971) a suggestion for the modification of the Shapley value to take care of situations where different affinities between players can give rise to certain coalitions orderings being more probable than others is made. This modification he claims is better suited for political games than the classical Shapley approach and goes on to give some practical examples. In Owen, G. (1971 page 845) he states that "it is a well known fact, in most games, the players do not behave as one would expect from an abstract study of the game. That is the characteristic function or even the normal extensive forms of the games are not sufficient to determine the coalitions which will form, since these depend to a large extent, on personal affinities of the players." He considered the players for the political games to be the political parties since for a homogeneous party, the party leader should have some control over the way the parliamentarians in his party would vote. From observations he notes that the voting power of certain parties only had slight relationships to their payoffs as calculated via the classical Shapley approach, where payoffs are represented by say, the number of cabinet positions held by a party.

In the discussion on the Shapley value all the orderings of players are given the same probability since the only property assumed known about the game was the characteristic function and the value is only a function of the characteristic function.

In some sets of simple games, this line of thought is very adequate but in real life situations and for political games some knowledge exists of the affinities among the players which to a great extent would determine the way the players would vote and thus would affect their Shapley values. We know that a right wing party and a left wing party will hardly ever vote on the same side in the face of a bill and as a result a randomization scheme that assigns equal probability to the way they would vote such as the classical Shapley model may be inadequate to explain the occurrences that take place. Owen therefore goes on to suggest a randomization scheme which takes into account the different affinities among the players and assigns different probabilities to the formation of different coalitions and then uses the Shapley formula to calculate the Shapley values. It is expected that coalitions with higher probabilities would have higher Shapley values.

The scheme was derived as follows:
Let $N = (1, 2, \ldots, n)$ be players in game v. Now, consider all possible $N!$ orderings of the N-players and assign probability $1/N!$ to each. Let $\mathcal{R}$ represent an ordering and let $S(i, \mathcal{R})$ represent the set of players preceding player i under the ordering $\mathcal{R}$. Then the Shapley value would be given by

$$(1) \qquad \phi_i\,[V] = E\,[\,V\,(S(i,\mathcal{R}) \cup \{i\}) - V(S(i,\mathcal{R}))\,]$$

Where E denotes the expected value under the given ordering. The above guarantees all the properties that are known for the Shapley value. Thus ϕ will be additive, a carrier K of the game V will obtain the amount V(K) and for super-additive games ϕ will be an imputation.

The major reason for the modification is to assign some probabilities to the different orderings with respect to their desire for coalition formation. We require that the assignment of these probabilities must possess some properties not contradictory to the Shapley value requirements.

The following two properties are therefore desired.

(A) An ordering and the reverse ordering should have the same probability, for example if 1, 2, 3 is an ordering for a 3-person game where the order of listing defines the way the coalition was formed with 1 starting the coalition, then joined by 2 and then 3 (note: as pointed out earlier the calculation of the Shapley value envisages the formation of grand coalitions with the pivot player casting the winning or blocking vote) If such an ordering has probability t_1 then the ordering 3, 2, 1 which starts with 3, followed by 2 and then 1 should also have probability t_1 . If this is a simple game in [0,1] normalization 2 would be the pivot player.

(B) The removal of a subset, S, of the players should not affect the probabilities assigned to the remaining set, N-S, of players which implies that the addition or removal of dummies should not affect the probabilities assigned.

A scheme was developed satisfying the two properties above and the Shapley value calculated as defined by (1) above. The scheme proposed by Owen assigned to each player a point in space and then the distances between pairs of points so defined was considered as the probabilities of having both points together in one coalition arrangement. This geometric framework was based on an

N-dimensional sphere and it seemed to have satisfied properties (A) and (B) and at the same time non-contradictory to the Shapley value model.

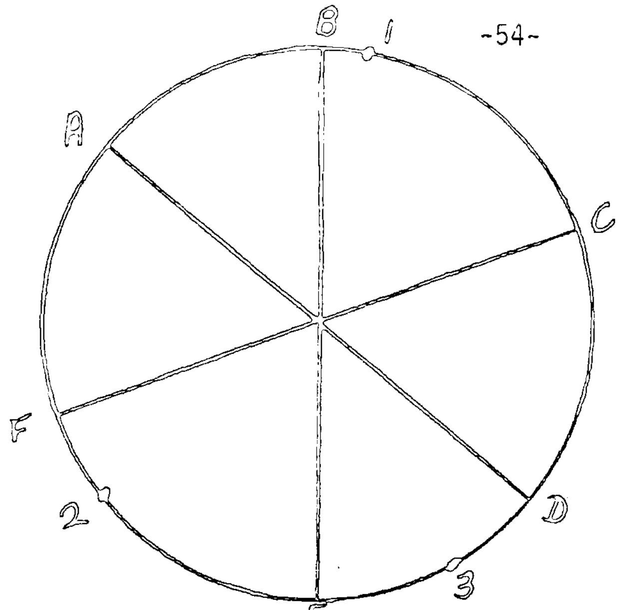
To gain an insight into the working of this scheme; consider several players in an N-person game, each assigned a point x in a Euclidean space of high dimension. Two points (parties) will normally be placed close together if they have some high affinity for each other. If these points are in general position, they would normally determine an (N-2)-sphere, T.

We note that an arbitrary point $z \varepsilon T$ determines an ordering of the players $N=(1,2..n)$ which is the order of increasing distance of the points $x^1, x^2, \ldots x^n$ from z. We also note that ties between points would form a set of measure zero. Each ordering of the players will have different probabilities assigned to them determined by the measure of all z which determine the ordering.

It is theoretically possible to place N players on the surface of an (N-2)-sphere but the relationship between them with respect to their distances from a point z will neither be easy to be related effectively well to political affinities, nor will their representation be easily possible on a two dimension paper except perhaps the (N-2)-sphere is reduced to a circle.

Owen gave some examples based on a circle.
Pairs of points on a circle of course satisfy properties (A) and (B) and would not contradict Shapley's initial assumptions.
Now consider 3 points on the circumference of a circle as shown in Figure (1)

Fig. 1.



Let the 3 points be split into 6 arcs each determining an ordering of the three players or parties. Since it is a circle, we note that antipodal sets on a (sphere or) circle have the same measure which satisfies property (A).

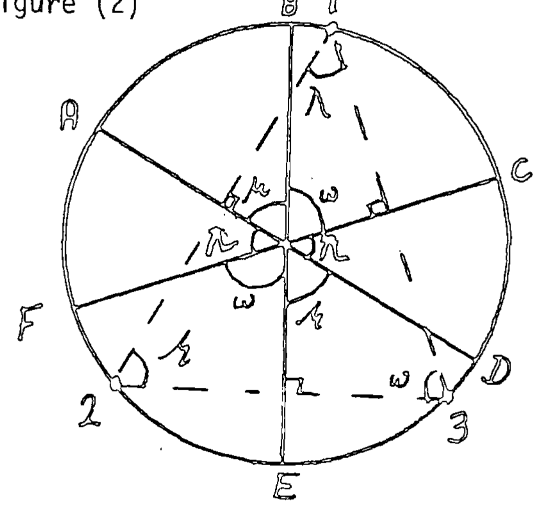
Also if we remove one of the players, we may have to replace the sphere with another of a lower dimension except in degenerate cases.

It has been shown by Owen, G (1971, 348-349) that the relative orderings of the remaining players will have the same probabilities in the reduced game. Also, if the points $x^1, \ldots, x^n$ are the vertices of a regular n-simplex, all the orderings would have the same probability and we have the usual Shapley value.

The following analysis of a 3 party legislature will serve as a very good example.

Let the 3 parties be represented by the three points as shown in figure (2)

Fig. 2.



Thus the players are represented as the three vertices of an inscribed triangle with angles λ, μ and ω. We note that the three perpendicular bisectors of the triangle cut the circle at six points A, B, C, D, E, F.

(2) $\quad FA = CD = \lambda$

(3) $\quad AB = DE = \mu$

(4) $\quad AB = EF = \omega$

We note that the relative distances between each pair of points defines the relative probabilities of coalition formation (ordering).

The six arcs and the associated angles give the ordering probabilities

(5) $\quad FA = CD = \lambda = 312$

(6) $\quad AB = DE = \mu = 123$

(7) $\quad BC = EF = \omega = 231$

We note that $P(3,1,2) = P(2,1,3) = \lambda/2\pi$

and $P(1,2,3) = P(3,2,1) = \mu/2\pi$

and $P(2,3,1) = P(1,3,2) = \omega/2\pi$

The modified Shapley value for player (1) then becomes

$$\phi_1 = \frac{1}{2\pi} \{(\mu+\omega) [V(\{1\})+V(\{1,2,3\}) - V(\{2,3\})]$$

$$+\lambda [V(\{1,2\}) + V(\{1,3\}) - V(\{2\}) - V(\{3\})]$$

We have similar expressions also for the other two players. We note that for the constant sum three person game in (0,1) normalization we get

(8) $\quad \phi_1 = \lambda/\pi, \quad \phi_2 = \mu/\pi$ and $\phi_3 = \omega/\pi$.

We shall present some examples based on these models in Chapter Four.

We shall now proceed to Chapter Three to propose a model which would incorporate the psychology of the players which includes their different affinities resulting in a predetermined pairwise probability of association from where we can calculate the Shapley value directly.

CHAPTER THREE

THE NEW APPROACH

3.1 THE DIRECT APPROACH MODEL - AN EXTENSION TO THE SHAPLEY VALUE

L.S. Shapley as pointed out earlier proposed a set of 'values' which we regard as an 'a priori' evaluation of players positions in a game by ignoring completely any social or economic structure or the psychology of the players, including their standard of behaviour. He got his results by imagining the random formation of coalitions of all the players, starting from one player and adding one at a time. Each time a player joins he is assigned the advantage gained by the coalition due to his admission; as a result the player whose admission causes the coalition to become a winning coalition as defined in Chapter 1, Section 1.5.3 is assigned the total value of that coalition. He is known as the 'Pivot' player as defined in Chapter 2, Section 2.1 This process is carried out over all coalitions since in the scheme all coalitions are equally likely. These are later normalised so that if we had N players with all orderings equally likely the pivot player would get $\frac{1}{|N|!}$ as mentioned earlier.

This technique was extended by Shapley to cover political games where parties are regarded as distinct groups that form coalitions with other groups in the weighted majority game model as discussed in the last chapter. In the weighted majority games model the assumption that all coalitions were equally likely was still present and calculations were carried out similar to the simple majority games except that in this case, pivot players were the distinct units. He also extended his model to the case involving a few major players and a continuum of minor players in his oceanic games model. In the oceanic games model a few players have fairly heavy weights attached to them while the minor players' weights tend to 0 as their number tend to infinity. The value for the major players was therefore calculated via a limiting process as shown in Chapter 2, Section 2.3 , while the ocean of minor players are assumed to share the remnant of the weight equally.

G. Owen modified the Shapley value by incorporating the psychology of the players which he described by assigning definite probabilities of cooperation for coalition formation to the players which were regarded as homogeneous groups by placing them round a circle at predefined intervals. He computed the Shapley value for the players by associating the common angles (or arcs) to a particular ordering of players as the probability of having such an ordering and hence derived their value as described in Chapter 2, Sub-Section 2.4.2.

A modification of this technique was considered during this work and is described in Appendix B. It is based on the probability of a particular ordering occurring, given positions of points placed at random but in restricted positions on a straight line. The calculations are straight forward but lengthy and this model was finally rejected in favour of that to be described.

G. Owen also carried out a multilinear extension in an attempt to take care of situations where the players were many, since the initial modification model could not easily take care of many players. His multilinear extension included a set of approximations by computing the partial derivatives of $f_i(x)$ where $f_i(x)$ is a probability measure for the derivation of the Shapley value. $f_i(x)$ is then the weighted average of the terms $[V(S \cup \{i\} - V(S)]$ G.Owen (1972) as described in Chapter 2,Sub-Section 2.4.1. The basic idea here is to take account of probabilities of orderings but to sum over varying probabilities.

Similarly in the model to be described we consider distinct homogeneous groups, the i^{th} group having size n_i where the number n_i is large these groups represent political parties or distinct units of players in committees. To determine the value we considered the probability that members of group i vote yea or nay together with members of group j, K, . . . in order to constitute a winning coalition. Each event in which this occurs i.e. given that a minimal winning coalition exists containing members of party i would then contribute to the value for party i.

We know that in practice it is not possible to say with certainty that parties will vote together on any particular issue. Calculating the probabilities of such occurrences could then give a measure of how often members of party i would belong to winning coalitions and hence the value v_i for party i. More precisely we define the value of group i as the expected proportion of group i in a minimal winning coalition given that such a minimal winning coalition exists.

It is simple to carry out an exact calculation using the above reasoning for the case when there are only three players, as will be shown shortly. In order to take care of large numbers of distinct parties including large numbers of distinct players it is possible, under some circumstances, to use the normal approximation to the binomial distribution (since the voting behaviour within groups is assumed to be strictly binomial). When this approximation does not hold, the binomial formulae themselves can be used. In any case we are therefore concerned with the conditional expectation that groups (parties) will vote yea together or nay together in order to constitute a winning coalition as will be shown later. The probabilities can then be varied in order to get a better understanding of any voting system.

Calculations of the formulae follow; these are compared in the next chapter with the simulated results of many voting situations. These simulated results are also analysed by Owen's technique and the results compared in the next chapter.

Finally the model is applied to practical voting situations, namely, the Nigerian Senate etc.The basic assumptions are tested against a small set of actual voting situations and conclusions are drawn about the Shapley values of the various parties. Some comparison is made with the powers of the parties in government as measured by their representation in the Cabinet and other important government offices.

We now carry out the calculation of the Shapley value based on this concept.

3.2 ASSUMPTIONS MADE

The assumptions made in the model will now be listed and the consequences derived. Expressions for the value will be obtained first in the simple case of three players only (3.2.1); then the case of a number of groups will be discussed and an approximation to the value obtained under simplifying assumptions (3.2.4) and finally it will be shown how the value could be calculated when this approximation does not hold (3.2.5).

The assumptions made are -

(i) a player i votes yes to a question with probability a_i and yes to its converse with probability $(1-a_i)$.

(ii) players within group i, and between groups i and j, vote independently.

(iii) there is an equal chance of the question or its converse being asked.

This is clearly a very simplified version of the voting process. More correctly the response should be a function f of the question q, so that $f(-q) = 1-f(q)$ and q should vary according to some probability distribution on the whole real axis. The simplification made is to take q as concentrated at ± 1, equally likely to take either value and to define $f(1) = a_i$, $f(-1) = 1 - a_i$. In any practical situation it is only the difference between the voting behaviour of different groups which is known so the absolute values of the a_i's are irrelevant. Methods of estimating the a_i's are considered in the next chapter.

3.2.1 Three Person game - In The New Approach

Let 3 participants in game V be denoted by 1, 2, 3 and let $12\bar{3}$ imply coalition 1 and 2 together and 3 on the other side. Also let Pr. be the abbreviation for probability. Now let a_i be the probability that party i votes yea and $1-a_i$ be the probability that party i votes nay to the question (+1). Then Pr. of 123 voting together $= a_1a_2a_3 + (1-a_1)(1-a_2)(1-a_3)$

Also Pr. $12\bar{3}$ i.e. 1 and 2 voting together and 3 voting against

$$= a_1a_2(1-a_3) + (1-a_1)(1-a_2)a_3$$

and Pr. $1\bar{2}3$ $= a_1(1-a_2)a_3 + (1-a_1)a_2(1-a_3)$

and Pr. $\bar{1}23$ $= (1-a_1)a_2a_3 + a_1(1-a_2)(1-a_3)$

So value for player 1 is the probability that 1 and 2 vote together and 3 apart, plus the probability that 1 and 3 vote together and 2 apart given that one of the patterns $12\bar{3}$, $1\bar{2}3$ or $\bar{1}23$ happens. But in either case two participants would be present, thus we require that the two share the value hence we take $\frac{1}{2}$ of the probability worked out as above and assign it as the Shapley value for 1 and the same for the other participants.

Thus value 1 = $\frac{1}{2}$ [Pr. $12\bar{3}$ + Pr. $1\bar{2}3$]/[Pr. $12\bar{3}$ + Pr. $1\bar{2}3$ + Pr. $\bar{1}23$]

$$= \tfrac{1}{2}\{a_1a_2(1-a_3) + a_1(1-a_2)a_3 + (1-a_1)(1-a_2)a_3 + (1-a_1)a_2(1-a_3)\} \Big/ \{1-a_1a_2a_3 - (1-a_1)(1-a_2)(1-a_3)\}$$

i.e. $V_1 = \frac{1}{2}(a_2 + a_3 - 2a_2a_3)/(a_1 + a_2 + a_3 - a_2a_3 - a_3a_1 - a_1a_2)$

and V_2 = $\frac{1}{2}$ [Pr. $12\bar{3}$ + Pr. $\bar{1}23$]/[Pr. $12\bar{3}$ + Pr. $1\bar{2}3$ + Pr. $\bar{1}23$]

i.e. $V_2 = \frac{1}{2}(a_1 + a_3 - 2a_1a_3)/(a_1 + a_2 + a_3 - a_2a_3 - a_3a_1 - a_1a_2)$

Similarly V_3 = $\frac{1}{2}$ [Pr. $\bar{1}23$ + Pr. $1\bar{2}3$] /[Pr. $12\bar{3}$ + Pr. $1\bar{2}3$ + Pr. $\bar{1}23$]

So $V_3 = \frac{1}{2}(a_1 + a_2 - 2a_1a_2)/(a_1 + a_2 + a_3 - a_2a_3 - a_3a_1 - a_1a_2)$

If $a_1 = a_2 = a_3$ then $V_1 = V_2 = V_3 =$ which is the ordinary Shapley value for 3 equal participants. Variation of values with different a_i's are shown in graphs G1, G2, G3, G4.

Now let us consider a special case where $a_2 = a_3$; implying that players 2 and 3 have the same voting behaviour, then

$$V_1 = \tfrac{1}{2}(a_2 - a_2^2)/(a_1 + 2a_2 - a_2^2 - 2a_1a_2)$$

$$= (a_2 - a_2^2)/(a_1 + 2a_2 - a_2^2 - 2a_1a_2)$$

$$V_2 = \tfrac{1}{2}(a_2 + a_1 - 2a_1a_2)/(a_1 + 2a_2 - a_2^2 - 2a_1a_2)$$

$$V_3 = \tfrac{1}{2}(a_2 + a_1 - 2a_1a_2)/(a_1 + 2a_2 - a_2^2 - 2a_1a_2)$$

V_2 and V_3 are clearly equal and $V_1 = 1-2V_2$.

If both of them consistently vote together and in a directly opposite way to player 1 (i.e. if $a_2 = a_3 = 0$ while $a_1 = 1$ or $a_2 = a_3 = 1$ while

$a_1 = 0$ or near those values) then player 1's value will vanish. The variation of values V_1 with other values of a_2 is shown in graph G5.

Consider another special case; Let $a_1 = \frac{1}{2}$. This implies that player one will vote with either 2 or 3 on an equal proportion of times. In this case

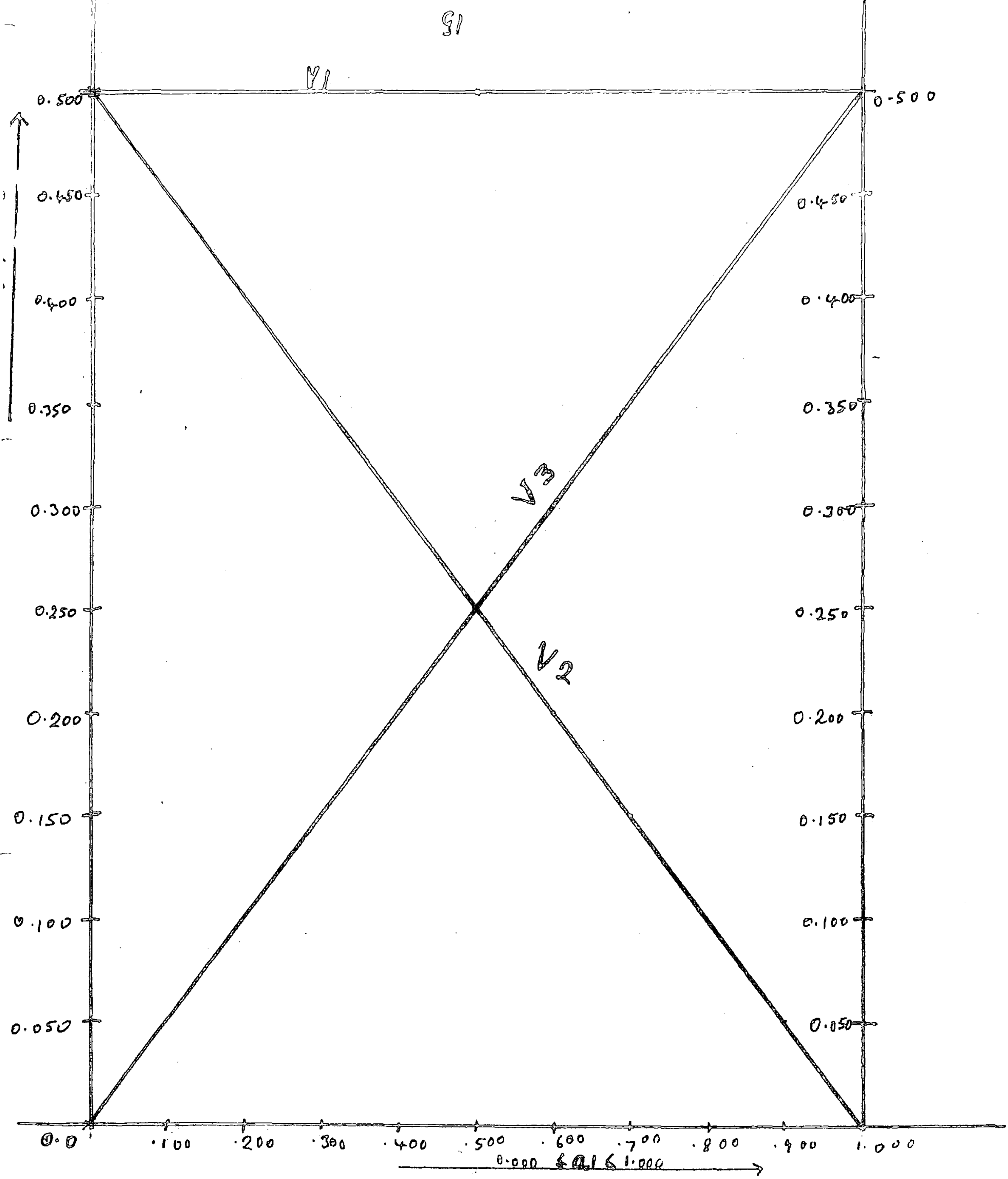
$$V_1 = \tfrac{1}{2}(a_2 + a_3 - 2a_2 a_3) \big/ \{ \tfrac{1}{2}(1 + a_2 + a_3) - a_2 a_3 \}$$

i.e. $V_1 = (a_2 + a_3 - 2a_2 a_3) \big/ (1 + a_2 + a_3 - 2a_2 a_3)$

and $V_2 = V_3 = \tfrac{1}{2} \big/ (1 + a_2 + a_3 - 2a_2 a_3)$

again $V_2 = V_3$ and $V_1 \equiv 1 - 2V_2$.

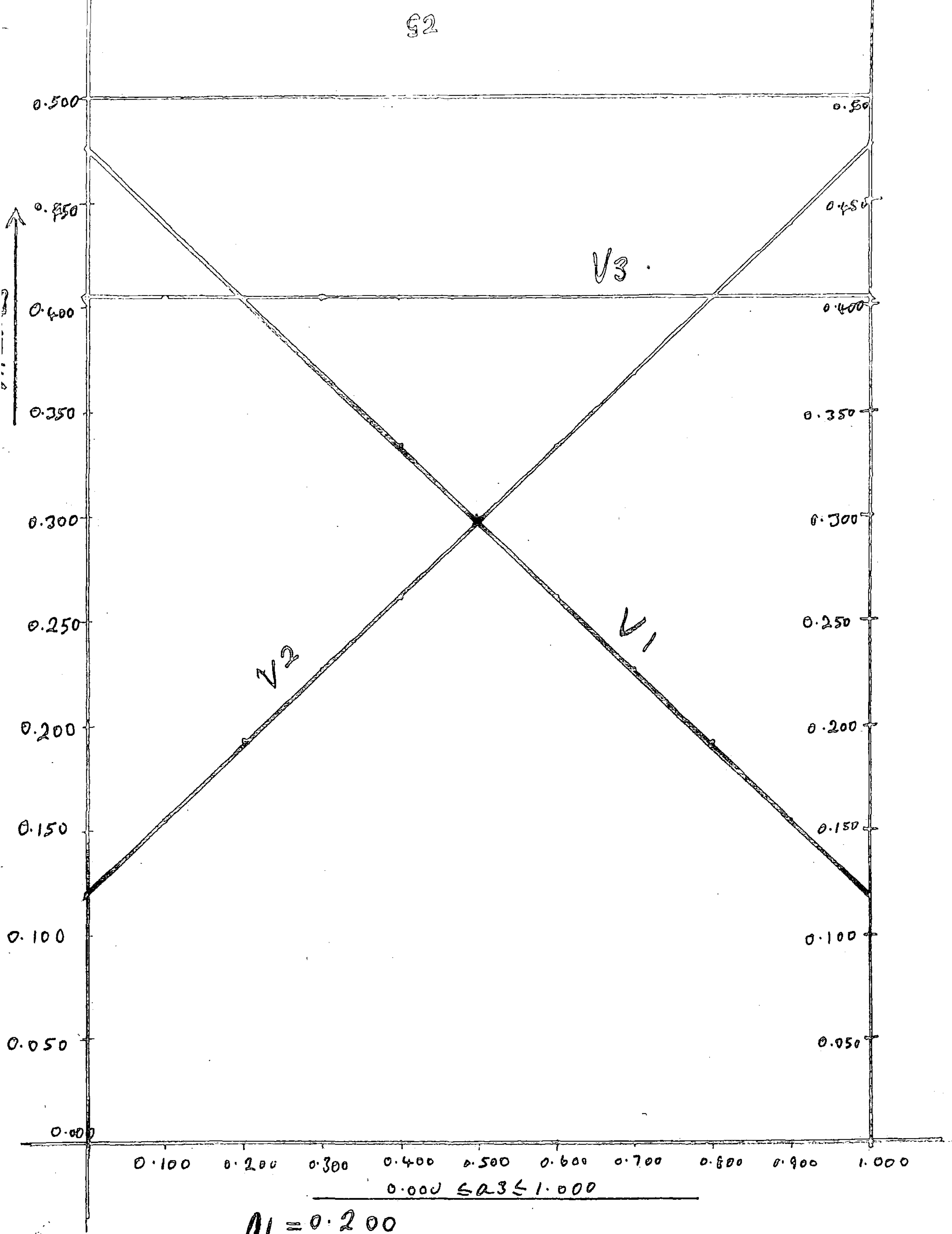
This result implies that if player 1 votes with either player 2 or 3 in equal proportion of times then the value of player one depends on the voting behaviour of players 2 and 3. The more players 2 and 3 vote alike, the more the value of player one diminishes but the more players 2 and 3 differ in voting behaviour the more the value of player one appreciates. The variation of V_1 with a_2 for various values of a_3 is shown in Graph G6.



$0.000 \leq a1 \leq 1.000$

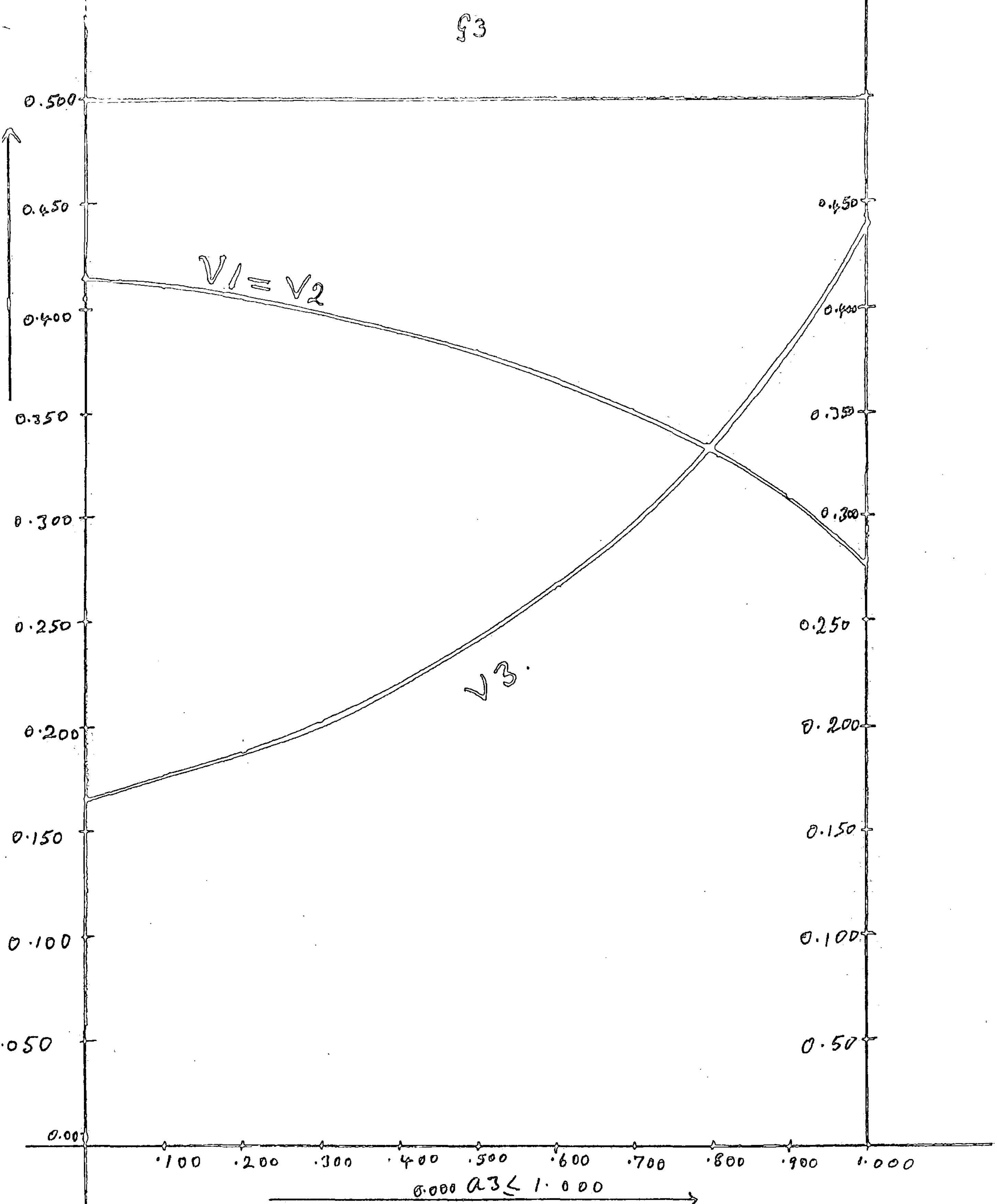
$a2 = 0.000$

$a3 = 1.000$



$a_1 = 0.200$
$a_2 = 0.800$

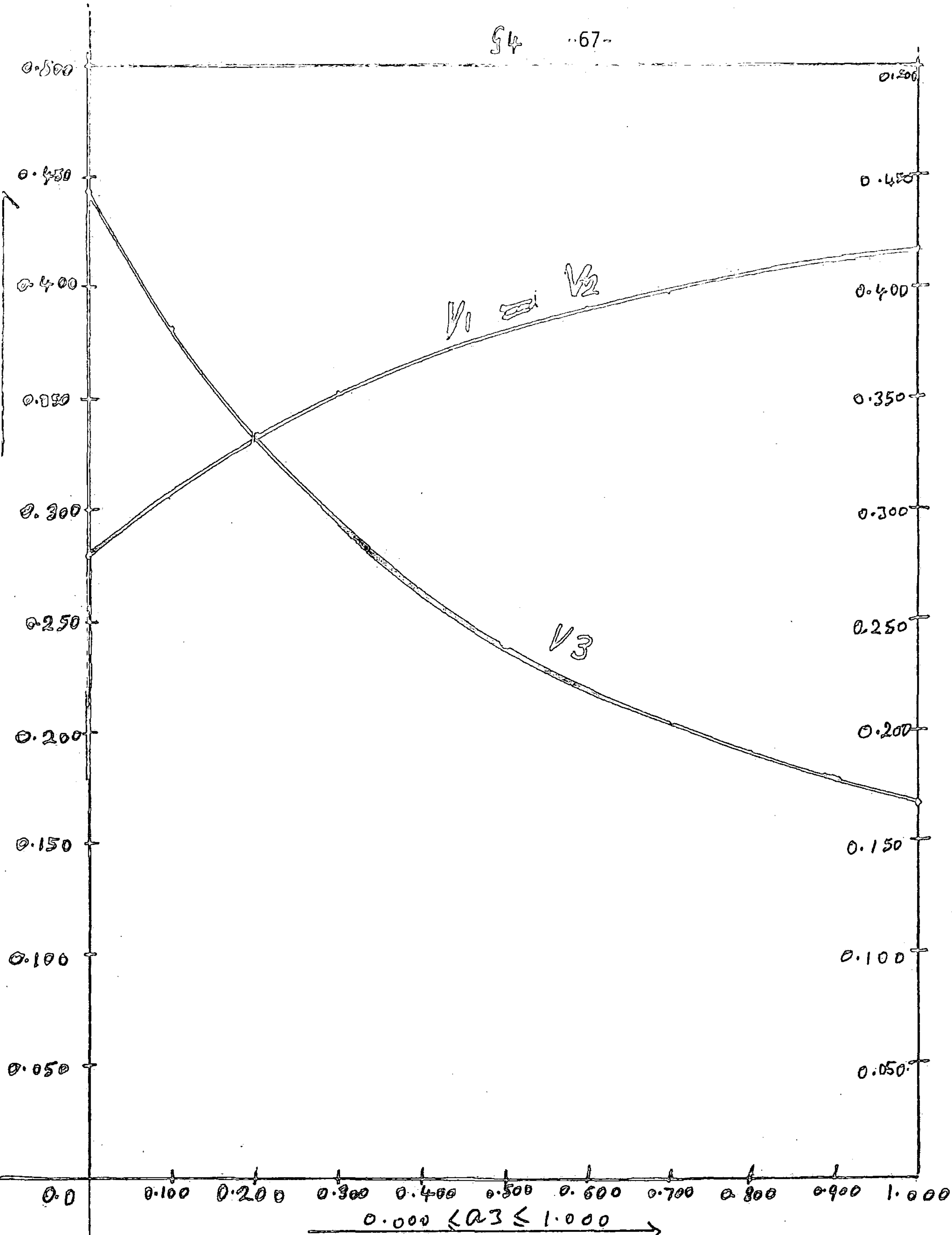
$0.000 \leq a_3 \leq 1.000$



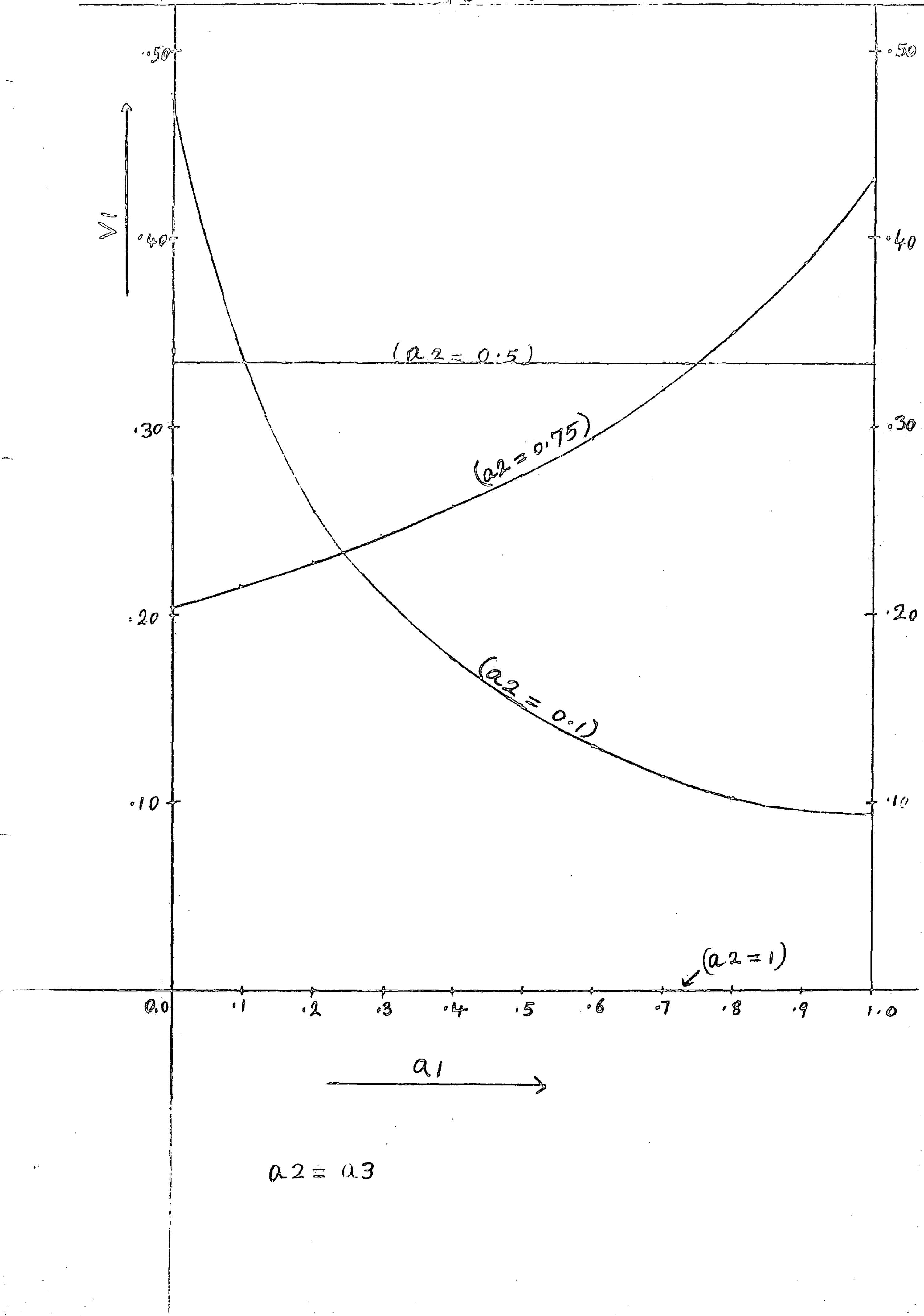
$a_1 = .800$

$a_2 = .800$

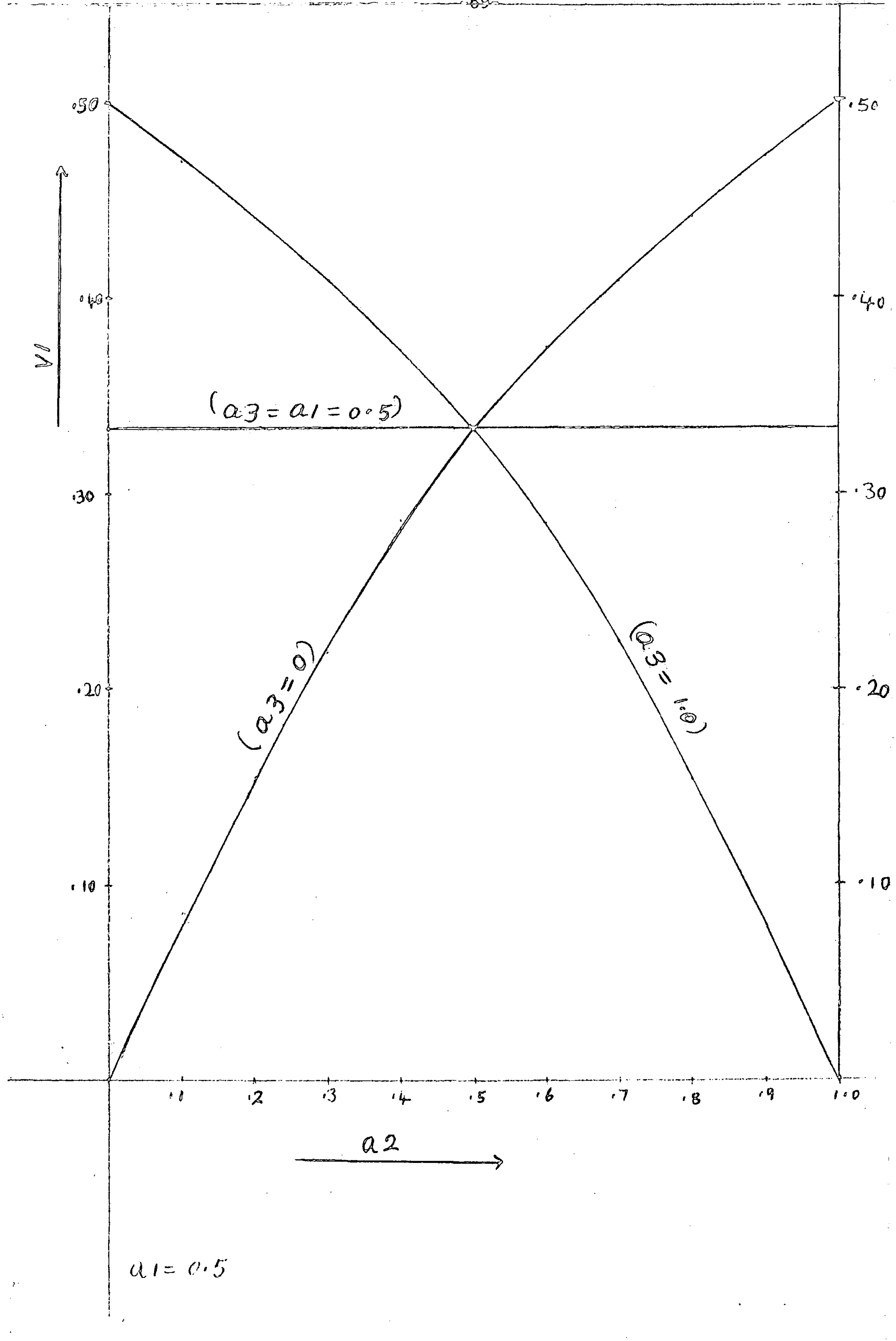
$0.0 \leq a_3 \leq 1.000$



a1 = 0.200
a2 = 0.200
0.000 ≤ a3 ≤ 1.000

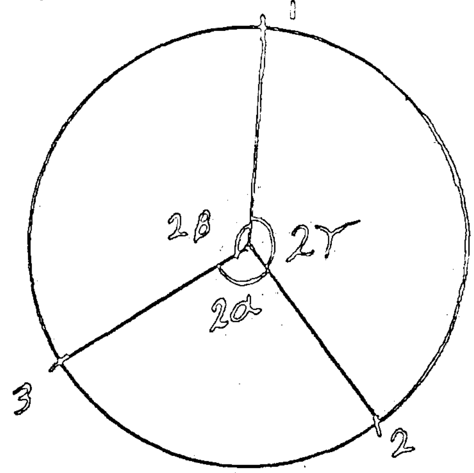


a2 = a3


·50
·40
·30
·20
·10
V1
(a3 = a1 = 0·5)
(a3 = 0)
(a3 = 1·0)
·1
·2
·3
·4
·5
·6
·7
·8
·9
1·0
a2
a1 = 0·5

3.2.2 COMPARISON WITH OWEN

G. Owen considered the three person game in his paper on political games, Owen (1971) pp.351-352). His results were as follows:



$P(123) = P(321) = \beta/2\pi$

$P(132) = P(231) = \gamma/2\pi$

$P(312) = P(213) = \alpha/2\pi$

These ordering probabilities which he used in calculating the values show that if $\beta = \gamma = \alpha$ then $V_1 = V_2 = V_3$ and $\beta = \gamma = \alpha$ implies that the angles (or arcs) that define those orderings are equal in which case the values for players 1, 2 and 3 are $1/3$ each. In his scheme, that will correspond to having the three players equally spaced around a circle. For the three-person game in (0,1) normalization (Majority game - M_3) these reduce to

$$V_1 = \alpha/\pi, \quad V_2 = \beta/\pi, \quad V = \gamma/\pi$$

We see that if α is large, then V_1 appreciates while V_2 and V_3 diminish and V_2 and V_3 appreciate as α becomes smaller. This is equivalent to our case where player 1's value depends on the voting behaviour of players 2 and 3 where $a_1 = \frac{1}{2}$.

The case where $a_2 = a_3 = 1 \text{ or } 0$ would correspond to players 2 and 3 occupying the same position in Owen's case, so that both of them will have all the values while player 1 would have a zero value agreeing with the result above. The general results in 3.2.1 correspond to Owen's formulation with the points allowed to vary in position.

3.2.3 HOMOGENEOUS GROUP MODEL

We shall present an extension of the above model to the case when we have more than three participants. We shall consider here the case where we have many participants formed into distinct groups which are homogeneous enough to satisfy Owen's assumption whereby he regards the political parties as being homogeneous in the sense that "each party leader can control to some degree at least, the manner in which his parliamentarians vote; otherwise it is not really a homogeneous grouping" Owen, G. (1971, p.345) This is similar in a sense to the classical Shapley weighted majority game concept where parties and not individuals occupy pivot positions.

In order to investigate such a situation with respect to our own model we let n_1, $n_2, \ldots n_k$ represent homogeneous political groupings, thus all members in each party vote together each time. Now let G = Grouping (comprising coalition of distinct groups such that $n_i \leqslant N + 1$, thus

(1) $G_i = (n_{i1}, n_{i2}, 0_{i3}, 0_{i4} \ldots 0_{ik})$ implies that in G_i parties n_1 and n_2 by voting yea (nay) together to question k^+ or nay (yea) together to question k^- constitute a minimal winning grouping $G_i \geqslant N+1$ where N+1 = the minimum required to win. Also G_i must be such that $G_i - n_i \leqslant N+1$ thus turning into a losing coalition.

Take for example the case of five distinct groups with weights attached as follows n1 = 36, n2 = 28, n3 = 16, n4 = 8, n5 = 7. The following minimal winning groups are possible.

G1 = (36y, 28y, 16N, 8N, 7N)

G2 = (36y, 28N, 16y, 8N, 7N)

Question k+ G3 = (36y, 28N, 16N, 8y, 7y)

G4 = (36N, 28y, 16y, 8y, 7N)

G5 = (36N, 28y, 16y, 8N, 7N)

$$\text{Question } K^{+} \quad \begin{aligned} G1 &= (36N,\ 28N,\ 16y,\ 8y,\ 7y) \\ G2 &= (36N,\ 28y,\ 16N,\ 8y,\ 7y) \\ G3 &= (36N,\ 28y,\ 16y,\ 8N,\ 7N) \\ G4 &= (36y,\ 28N,\ 16N,\ 8N,\ 7y) \\ G5 &= (36y,\ 28N,\ 16N,\ 8y,\ 7N) \end{aligned}$$

where y = yea, and N = Nay

Let a_i = Probability of group i voting yea (Nay) to K+. Thus probability of having G1 is

$$P_1 = a_1 \times a_2 \times (1-a_3)(1-a_4)(1-a_5) + (1-a_1)(1-a_2) \times a_3 \times a_4 \times a_5$$

Similarly all other minimal winning coalitions can be enumerated and their probabilities derived

Thus

$$(2) \qquad \mu = \sum_j P_j \quad \text{for all minimal winning groupings}$$

$$j = 1, \ldots . m \text{ (minimal winning groupings)}$$

Thus μ = sum of the probabilities of all minimal winning groupings = Prob. of just winning groupings

$$T_{ji} = \frac{n_i}{N + 1 + \lambda_j}$$ = The proportion of the weight of group i in the minimal winning coalition (grouping) G_j.

$N + 1 + \lambda_j$, = The exact size of the coalition j

$$\lambda_j \geq 0$$

Thus the contribution to the value of group (party) i from the grouping (coalition) G_j is

(3) $$\frac{T_{ji} \times P_i}{\mu}$$

Thus the value of group (party)i is the sum of its contributions in all the minimal winning groupings, thus

(4) $$V_i = \sum_j \left(T_{ji} \times \frac{P_j}{\mu}\right)$$

j = 1,....m Groupings where party i contributed in bringing about the "WIN".

(4) is similar to (3) of 3.2.4 as will be seen later.

We state (3) of 3.2.4 in advance for comparison.

$$V_i = \frac{\Sigma \frac{xi}{N+1} \times \text{Pr. } \{x_i \text{ and in just winning coalition}\}}{\text{Prob. of having just winning coalitions}}$$

Thus $T_{ji} = \dfrac{n_i}{N+1+\lambda_j}$ replaces $\dfrac{x_i}{N+1}$

and P_j/μ is the conditional probability of this particular minimal winning coalition.

The addition of λ_j is due to the fact that in minimal winning groupings, the quota varies according to the size of the grouping formed.

If all a_0°'s are equal to 0.5 the results from this approach should reduce to the ordinary Shapley value for a weighted majority game. With unequal a_i the results should be comparable to Owen for some point spacing.

3.2.4 THE GENERAL DIRECT APPROACH MODEL

We shall now extend this to a more general case where all individuals in a party are not necessarily required or expected to vote on one side each time. That is the usual happening in most real life situations.

In this model therefore we are concerned with large numbers of players n_i whose behaviour is strictly binomial and since the number n_i(i.e. the number of people in group i (party)) is large, their behaviour can be approximated to the normal distribution provided that the probabilities of each group voting yea together or nay together is not near 1 or 0 because at those points the binomial approximation to the normal fails. Some methods of taking care of these cases will be shown later. The value for party i via this model would then be the conditional expectation of the proportion of party i voting yea or nay together with parties j, k,... in order to constitute a just winning coalition.

We know that if X_1, X_2... are random normal variates independently distributed with mean m_i and variance σ_i^2. then the conditional expectation,

(1) $E\ \{\ X_i/X_1 + X_2 + \ldots = \Sigma\, m_i + K\ \} = m_i + \dfrac{K\sigma_i^2}{\Sigma\sigma_i^2}$ (The derivation is in Appendix A)

Hence if there are groups whose probabilities of voting yes are a_i and sizes n_i such that $\Sigma n_i = 2N+1$ and if $m_i = n_i a_i$, $\sigma_i^2 = n_i a_i\ (1-a_i)$, then if X_i are numbers voting yes, we have that ΣX_i is distributed normally (using the normal approximation to the binomial) about $\Sigma m_i = M$ with variance $\Sigma\sigma_i^2 = S^2$

Hence we have

$$Pr(\Sigma X_i = N+1) \text{ is } \phi\left(\frac{N+1-M}{S}\right) = \phi\left\{\left(\frac{N+\frac{1}{2}-M}{S}\right) + \frac{1}{2S}\right\} \text{ and}$$

$$Pr(\Sigma X_i = N) \text{ is } \phi\left(\frac{N-M}{S}\right) = \phi\left(\frac{N+\frac{1}{2}-M}{S} - \frac{1}{2S}\right)$$

where ϕ is the normal probability ordinate

$$\text{Put } \alpha = \frac{N+\frac{1}{2}-M}{S}$$

then the probability of having a just winning result

(Just winning implies minimal winning) is

$$\phi\left(\alpha + \frac{1}{2S}\right) + \phi\left(\alpha - \frac{1}{2S}\right)$$

If $\frac{1}{2S}$ is small and α is small enough for the normal approximation to be valid, this can be written as

$$(2) \qquad 2\phi(\alpha) + \frac{1}{4S^2} \phi''(\alpha)$$

and the value of group i is

$$(3) \qquad V_i = \sum \frac{X_i}{N+1} \times \text{Pr. } (X_i \text{ and in Just Winning}) / \text{Pr.(Just Winning Coalition)}$$

$$(4) \qquad V_i = \frac{1}{N+1} \sum \frac{X_i \times \text{Pr. } \{X_i \text{ in N+1 voting yes}\} + (n_i - X_i) \times \text{Pr.}\{X_i \text{ in N voting yes}\}}{\text{Pr. (just winning)}}$$

$$\text{and we know that } \frac{\Sigma X_i \times \text{Pr. } \{X_i \text{ in N+1}\}}{\text{Pr.(N+1 voting yes)}}$$

$$= E(X_i / \Sigma X_i = N+1) = \text{expectation of } X_i$$

$$= m_i + \frac{(N+1-M)\sigma_i^2}{S^2}$$ using the result for normality

$$= m_i + \frac{(\alpha S + \frac{1}{2})\sigma_i^2}{S^2}$$

and similarly

$$\frac{\Sigma x_i \times Pr. \{x_i \text{ in } N\}}{Pr.(N \text{ voting yes})} = E(x_i / \Sigma x_i = N)$$

$$= m_i + \frac{(N-M)}{S^2}\sigma_i^2$$

$$= m_i + \frac{(\alpha S - \frac{1}{2})\sigma_i^2}{S^2}$$

So we have

$$(5) \quad V_i = \frac{1}{N+1}\left[\frac{\{m_i + \frac{(\alpha S+\frac{1}{2})\sigma_i^2}{S^2}\} Pr.(N+1 \text{ vote yes}) - \{m_i + \frac{(\alpha S-\frac{1}{2})\sigma_i^2}{S^2}\} Pr.(N \text{ vote yes}) + n_i\, Pr.(N \text{ vote yes})}{Pr.(\text{ just winning})}\right]$$

Thus

$$(6) \quad V_i = \frac{1}{N+1}\left[\frac{\{m_i + (\alpha S+\frac{1}{2})\frac{\sigma_i^2}{S^2}\}\phi(\alpha+\frac{1}{2S}) + \{n_i - m_i - \frac{(\alpha S-\frac{1}{2})\sigma_i^2}{S^2}\}\phi(\alpha-\frac{1}{2S})}{2\phi(\alpha) + \frac{1}{4S^2}\phi''(\alpha)}\right]$$

Now σ^2, S^2, m, n are all of order n, α is of order $\sqrt{n}$

So

$$(7) \quad V_i = \frac{1}{(N+1)}\left[\frac{(m_i + \frac{\alpha}{S}\sigma_i^2)\phi(\alpha+\frac{1}{2S}) + \frac{1}{2}\frac{\sigma_i^2}{S^2}\phi(\alpha+\frac{1}{2S}) + (n_i - m_i - \frac{\alpha}{S}\sigma_i^2)\phi(\alpha-\frac{1}{2S}) + \frac{1}{2}\frac{\sigma_i^2}{S^2}\phi(\alpha-\frac{1}{2S})}{2\phi(\alpha) + \frac{1}{4S^2}\phi''(\alpha)}\right]$$

So regarding $\frac{1}{2}$ as smaller than n, $\frac{1}{2S}$ smaller than α, and keeping 1st order terms only gives

$$(8)\quad V_i = \frac{1}{N+1}\left[\frac{(m_i+\frac{\alpha}{S}\sigma_i^2)(\phi(\alpha)+\frac{1}{2S}\phi'(\alpha)\cdots)+\frac{1}{2}\frac{\sigma_i^2}{S^2}\phi(\alpha)\cdots+(n_i-m_i-\frac{\alpha}{S}\sigma_i^2)(\phi(\alpha)-\frac{1}{2S}\phi'(\alpha)\cdots)+\frac{1}{2}\frac{\sigma_i^2}{S^2}\phi(\alpha)}{2\phi(\alpha)+\ldots\ldots}\right]$$

which reduces to

$$(9)\quad V_i = \frac{1}{N+1}\left[\frac{1}{2}(n_i+\frac{\sigma_i^2}{S^2}) + \frac{\phi'(\alpha)}{4S\phi(\alpha)}(2m_i+\frac{2\alpha}{S}\sigma_i^2 - n_i)\right]$$

Now we know that

n_i = number in group or party

σ_i^2 = $n_i a_i\ (1-a_i)$

m_i = $n_i a_i$

M = Σm_i

S^2 = $\Sigma \sigma_i^2$

2N+1 = Total number of players in the game

$$\alpha = \frac{N+\frac{1}{2}-M}{S}$$

Also $\phi(\alpha) = \frac{1}{\sqrt{2\pi}S}\exp^{-\frac{1}{2}\alpha^2}$, the normal ordinate corresponding to deviation $S\alpha$ with variance S^2 and

$$\phi'(\alpha) = -\frac{\alpha}{\sqrt{2\pi}S}\exp^{-\frac{1}{2}\alpha^2} = -\alpha\phi(\alpha)$$

So with normal approxmation to binomial expression (9) reduces to

$$V_i = \frac{1}{N+1}\left[\frac{1}{2}\left(n_i+\frac{\sigma_i^2}{S^2}\right) - \frac{\alpha}{4S}\left(2m_i+\frac{2\alpha}{S}\sigma_i^2 - n_i\right)\right]$$

$$(10)\quad V_i = \frac{1}{2(N+1)}\left[n_i\left(1+\frac{\alpha}{2S}\right) - \frac{\alpha m_i}{S} + \frac{\sigma_i^2}{S^2}(1-\alpha^2)\right]$$

Thus expression (10) is the extended Shapley value vector which incorporates the probabilities of association and cooperation among players from different groups.

If we allow the probability of every party to vote together with each other to be $\frac{1}{2}$ = .5 then the above expression reduces further to

$$(11) \qquad V_i = \frac{1}{2(N+1)}\left[n_i + \frac{\sigma_1{}^2}{S^2}\right]$$

3.2.5 <u>THE DIRECT APPROACH MODEL - THE EXTREME CASE</u>

In (1) of (3.2.4) we invoked the statistical conditional expectation formula for random variates X_i independently distributed with mean m_i and variance $\sigma_i{}^2$,

$$E\left\{X_i / X_1 + X_2 \ldots = \sum m_i + K\right\} = m_i + \frac{K\sigma_i^2}{\Sigma\sigma_i^2}$$

We went further to assume that ΣX_i was distributed normally about $\Sigma m_i = M$ with variance $\Sigma\sigma^2{}_i = S^2$, and we further carried out a normal approximation to the binomial in (10) of (3.2.4)

It therefore follows that for some voting situations the formula (10) of (3.2.4) will be inaccurate.

We can determine the value by using the same concept with no approximations and carry out our calculations term by term. We shall define the procedure by using a simple example, thus - Let three parties, 1, 2, and 3 be represented as follows, n1 = 37, n2 = 28 and n3 = 30 and let them be associated with probabilities a1, a2, a3. Let the probability that XL from Party 1 votes yea to question K+ = a1 = 0.1 and the probability that X_2 from Party 2 votes year to question K+ $= a_2 = 0.0$

and the probability that X_3 from Party 3 votes year

to question K+ $= a_3 = 0.5$

We have the total number of players to be 95

Thus we want Prob. yea = 48 = Pr.{ $X_1 + X_3 = 48$} (The situation is simplified since party 2 always votes nay to K+ here)

and Prob. Nay = 48 = Prob. {year = 47} = Prob. { $X_1 + X_3 = 47$}

We know the X's are binomial, so let P_i = Probability of having $X_1 = i$ and P_i' = Prob. of having $X_3 = i$. We note that X_2 would hardly join any coalitions with X_1 and X_3.

So (1) Pr.{ $X_1 + X_3 = 48$} $= P_{18}P_{30}' + P_{19}P_{29}' + \ldots + P_{37}P_{11}'$ i.e.

the total Probability of having 48 from players belonging to the two different parties. We can refer each of these to the product

e.g. $P_{20}P_{30}'$.

P_{20} implies the Probability of having 20 players out of 37 in X_1 voting yea i.e. $\frac{37!}{20!17!}(.1)^{20}(.9)^{17} = P_{20}$

$$P_{19} = \frac{37!}{19!18!}(.1)^{19} \times (.9)^{18} = \frac{20}{18} \times \frac{.9}{.1} \times P_{20}$$

$$P_{18} = \frac{37!}{18!19!}(.1)^{18} \times (.9)^{19} = \frac{20}{18} \times \frac{.9^2}{.1^2} \times P_{20}$$

$$P_{17} = \frac{37!}{17!20!}(.1)^{17} \times (.9)^{20} = \frac{.9^3}{.1^3} \times P_{20}$$

$$P_{21} = \frac{37!}{21!16!}(.1)^{21} \times (.9)^{16} = \frac{17}{21} \times \frac{.1}{.9} \times P_{20}$$

$$P_{22} = \frac{37!}{22!15!}(.1)^{22} \times (.9)^{15} = \frac{17 \times 16}{22 \times 21} \times \frac{.1^2}{.9^2} \times P_{20}$$

.
. — — — — — — etc.
.
.

also

$P'_{30} = (.5)^{30}$

$P'_{29} = 30P'_{30}$

$P'_{28} = \frac{30!}{28!\,2!} \times (.5)^{28} \times (.5)^2 = \frac{30 \times 29}{2 \times 1} \times P'_{30}$

$P'_{27} = \frac{30!}{27!\,3!} \times (.5)^{27} \times (.5)^3 = \frac{30 \times 29 \times 28}{3 \times 2 \times 1} \times P'_{30}$

$\vdots$ – – – – – – – – etc

Similarly (2) $\Pr\{X_1 + X_3 = 47\} = P_{17}P'_{30} + P_{18}P'_{29} + \ldots + P_{37}P'_{10}$

We need (3) Pr. yes = 48

(4) Pr. yes = 47

We note that from (4) of (3.2.4)

$$V_1 = \frac{1}{N+1}\sum \frac{X_i \times \Pr.\{X_1 \text{ in } N+1 \text{ voting yea}\} + (n_i - X_1) \times \Pr\{X_1 \text{ in } N \text{ voting yea}\}}{\Pr.\ (\text{Just winning})}$$

Thus sum of (3) and (4) above give us the denominator and for this case N+1 = 48.

For Party X_1

$$\Sigma X_i \Pr(X_1 \text{ in just winning})$$

$$= \Sigma X_i \Pr(X_1 \text{ in 48 voting yea}) + \Sigma(37 - X_1)\mathrm{PR}(X_1 \text{ in 47 voting yea})$$

$$= 18P_{18}P'_{30} + 19P_{19}P'_{29} + \text{---} + 37P_{37}P'_{11} + 20P_{17}P'_{30} + 19P_{18}P'_{29} + \text{---} + 0P_{37}P'_{10}$$

$$\text{Thus } V_\emptyset = \frac{1}{48}\ \frac{\Sigma X_i \times \Pr X_1 \text{ in 48 voting yea} + \Sigma(37-X_1)\Pr(X_1 \text{ in 47 voting yea}}{\Pr\{X_1 + X_3 = 48\} + \Pr\{X_1 + X_3 = 47\}}$$

Since the product term cancels through, the calculation is not invalidated by the small value of the denominator. The successive terms must be calculated until they become insignificant in both numerator and denominator.

3.2.6 SUMMARY OF DIRECT APPROACH CALCULATION TECHNIQUE

In order to calculate the value of a participant in a voting situation we therefore need the following:

(A) The quota = N + 1 = minimal winning coalition

(B) n_i = Number of players from any distinct party or group.

(C) a_i = measure of the degree of cooperation or affinity for others. This measure we interpret as the probability of voting on the same side with other players from other parties. This measure is similar to Owen's distance criterion which places players around a circle or as points on a sphere whereby the distances between any pair of points would determine the affinity among the players of the distinct points. This probability measure can be calculated from past events. A pairwise relationship is established by considering a number of voting situations and determining how often and how many players from different parties have voted on the same side. For example let the proportion of people in party i who voted yea to bill K in some past voting situation be b_i, then $1-b_i$ is the proportion voting nay on the same voting situation. Also let b_j be the proportion of people from party j that voted yea on that same event and $1-b_j$ the proportion that voted nay.

Thus the probability that party i and j would vote on the one side in some future voting session =

$$(1) \qquad b_{ik}b_{jk} + (1-b_{ik})(1-b_{jk}) = 2b_{ik}b_{jk} + 1-b_{ik}-b_{jk}$$

A similar calculation is carried out for all pairs of parties for the available set of voting situations. We then use these probabilities in estimating the a_i's via a least square technique to be discussed in Chapter 4. With the numbers namely a_i, n_i, $N+1$ we then calculate $m_i = n_i a_i$, $M = \Sigma\, m_i$, $\sigma_i^2 = m_i(1-a_i)$, $S^2 = \Sigma\, \sigma_i^2$, and $\alpha = \frac{N+1-M}{S}$ and finally the value V_i for any group or party can then be calculated from (10) of (3.2.4)

$$V_{(i)} = \frac{1}{2(N+1)} \left[n_i\left(1+ \frac{\alpha}{2S}\right) - \frac{\alpha\, m_i}{S} + \frac{\sigma_i^2}{S^2}(1-\alpha^2) \right]$$

Where the distribution of the voting behaviour of the players fails to hold with respect to the use of normal approximation to the binomial we then employ our term by term calculations as illustrated in (3.2.5).

The new model is dynamic in the sense that we can vary our set of a_i's, $i = 1, \ldots m$ parties in order to study the behaviour of the values with respect to the parties.

The model satisfies all the axioms put forward by L.S.Shapley as stated in Chapter 2, except the axiom on symmetry. Shapley's axiom on symmetry requres that no matter where one was positioned the likelihood of one joining a coalition remains unchanged, but our concept is based on the proposition that social factors, time, political and economic factors have definite influence over the behaviour of players in political games and should therefore make the formation of certain coalitions more likely than others, hence the inclusion of the probability factor.

We note that Owen's multilinear extension bears some resemblance to our model but while he tried to determine the value of a player in a game by considering the probabilities that other players form a coalition excluding the player of interest we propose that in calculating a player's value, consideration must be given to his probability of belonging to a coalition with a set of other players and also the probability that the other players can form a coalition with him. We shall present the results from this model and other models discussed in this paper in the next two chapters as applied to simulated and practical voting situations.

CHAPTER 4

APPLICATIONS TO SIMULATED VOTING SITUATIONS

The different approaches which have been described will now be applied to a special political situation, that of majority voting in an Assembly made up of separate political parties, the i^{th} party being of size n_i. In particular it is shown how numerical expressions can be obtained for the values of each party given their sizes and with a knowledge of their previous voting patterns, under the models which have been described.

In 4.1 the formulae from the original Shapley model are reviewed. In 4.2 it is shown how the Owen formulation could be applied and in 4.3 similarly how the direct approach model could be used. In 4.4 are presented the results of analysing a large number of simulated voting situations and comparisons are made with the theory and between the different models.

4.1 ANALYSIS VIA CLASSICAL SHAPLEY

For the purposes of continuity we restate (14) of 2.1 as follows.

$$\phi_i[V] = \sum_{S \subseteq N} \gamma_n(S) \quad [V(S) - V(S-(i)]$$

$\forall_i \in U$ where N is a finite carrier of V and U the set of players.

We note that $\gamma_n(S) = \dfrac{(S-1)!\ (n-S)!}{n!}$

The definition involves the N! permutations of the N finite carrier. If all vote individually, the value of party i of size $n_i = \dfrac{n_i}{\Sigma n_i}$

If they vote together as a group then we have to treat the case like that of a weighted majority game whereby parties are regarded as being 'pivotal' instead of individuals by determining the number of ways the parties can be rearranged and picking out the pivotal party each time. This will therefore require a permutation of the parties as unified homogeneous entities. If the votes are taken in separate legislative bodies whereby winning implies being pivotal in more than one body then the calculations would follow the technique used by Shapley and Shubik in "A method for Evaluating the Distribution of power in a Committee System", Shapley L.S. and Shubik, M (1954, p.792). The technique involves firstly determining the number of ways the different parties can be rearranged and then determining the number of ways an individual player can be rearranged within his own party whereby he becomes the pivotal player in his party and his party becomes the pivotal party within that arrangement. The 2nd and 3rd techniques are similar since they involve a rearrangement of the parties (groups) and determining which party occupied a pivotal position.

Values obtained in both these ways for the particular situation discussed will be given in Section 4.4 after describing the other methods of analysis which are used.

4.2 APPLICATIONS OF OWEN'S FORMULATION

The model presented by Owen as discussed in 2.4.2 is based on the geometry of a sphere or circle. Points P_0 , P_1 ,...P_n representing homogeneous political groupings were placed round a circle as shown in Fig. 1 and Fig. 2 of 2.4.2. The measure of any ordering $\ell_1\ \ell_2 \ldots \ell_n$ was defined as being the length of arc containing all points P whose distances round the circle to the base points were in the order $PP_{\ell_1} \leqslant PP_{\ell_2} \cdots \leqslant PP_{\ell_n}$; the distances between all pairs of points determine this arc and its associated angle.

A comparable definition is possible in terms of areas on a sphere. The probability of having any ordering is assigned to the pivotal player of that ordering. A player's value would then be a summation of all the probabilities of all the orderings where he is pivotal. We note that such an ordering defines a coalition and also that an ordering and its reverse would have the same probability as guaranteed by the geometry of a circle or sphere.

Owen built his theory on a sphere but gave his example using a circle; the computations on an n-dimensional sphere would be difficult. On a circle it is impossible to obtain good results for more than three points; for example 4 distinct groups placed at equal intervals round a circle e.g. points equally spaced in order 1234, give probabilities $\frac{1}{4}$ each for orderings 1243, 2134, 2314, 3214 (or their reverses) and zero for the rest. This can be overcome by resorting to a rotation technique whereby all players are allowed to occupy all positions once at a time. The lack of rotation may be the cause of the many '0' values as will be seen later although Owen did not suggest so.

In carrying out a computerisation project, we required (i) a set of points at angles between 0° and 360° on a circle representing the overall relationship of players in a political game with respect to the affinities among members of different parties.

(ii) A scheme for determining possible orderings and the associated angle or arc which would define the probability of having such an ordering.

To determine the points which represent the overall measure of affinity among all the players we resorted to a multidimensional scaling procedure. For our purpose we required that the scaling be done in one dimension. The program used for the multidimensional scaling is called SPACES, a special analysis package developed at the Centre for Political Studies, Institute of Social Research, University of Michigan, version 3.10 of April 1977 (Numac Oct.1977) This involves a standard procedure, details are given in the appendix.

The input data is required to be in a correlation matrix form. To obtain a correlation matrix we used a program package designed for cluster analysis called "CLUSTAN", Wishart, D.(1978). The program carried out bivariate measures of association between different sets with respect to specific variables. Everitt, B. (1974) contains a good general introduction to the principles of cluster analysis.

Our input to the "CLUSTAN" program is data from voting situations. The data was supplied as strings of binary variables represented by "0", "1", 1 = situation where $X_i > \frac{1}{2} n_i$ which implies that members of party i who voted yea to bill K are more than half the number of people from that party who were present during the voting and "0" otherwise; exact proportions could also be used but a lot of computer time is saved by the use of the binary variables 0,1. The matrix of correlation coefficients produced by the Clustan package on the lower right-hand triangle off-diagonal position is automatically converted to the upper right-hand off-diagonal position via a program designed for that purpose. Then invoke the 'SPACES' program with all the necessary commands and options as presented in the appendix and what we get is a set of points that have been through the multidimensional process and presented in Euclidean one dimension scale.

The output of the scaling procedure is a set of points with the associated distances which represent some measure of affinity among the groups. These distances can be converted into a set of points spaced round a circle representing the degree of cooperation among different parties. A computer program was designed to calculate the probability of having any ordering which is the Shapley value of the pivotal player in that ordering. The search for possible orderings is done over all possible orderings of the N players, N = total number of distinct political groups (parties). The scheme therefore calls for a permutation of all the N players in line with the concept of the Shapley value.

Details of the program are given in the Appendix with a flow diagram.

4.3 APPLICATION OF THE DIRECT APPROACH MODEL AND THE ESTIMATION OF a_i'S

In sub-Section 3.2.6 of Chapter 3 we stated the required set of numbers necessary for calculating the Shapley value via the direct approach model. These were the quota necessary for the formation of a minimal winning coalition, N+1; the number of distinct players in any distinct party or group, n_i; and a measure of the degree of cooperation or affinity for others, a_i, which in this case is the probability that party x_i votes yea (nay) to question K+ and yea (nay) to question K^-. We can write down the quota N+1 and the number n_i directly from the set of data of the players. We therefore require a method for estimating the a_i's. To do this we firstly calculate the proportion of people in the different parties that voted yea or nay to a bill in any particular voting situation as given in (1) of (3.2.6) of Chapter 3. We restate (1), the probability that party i and j would vote on the one side in a voting session k as

$$(1) \qquad b_{ik}b_{jk} + (1-b_{ik})(1-b_{jk}) = 2b_{ik}b_{jk} + 1 - b_{ik} - b_{jk}$$

We stated that similar calculations would be carried out for all pairs of parties, for say T number of voting situations.

These are then summed over all voting sessions thus

$$(2) \qquad \mu_{ij} = \frac{1}{T}\left\{\Sigma_{k}\; b_{ik}b_{jk} + \Sigma\, (1-b_{ik})(1-b_{ik})\right\}$$

T = number of voting situations i, j = 1,...n = number of parties.

Assuming we had five parties, then we shall have ten μ_{ij}'s. Each μ_{ij} gives us then the probability that any pair of parties would vote together on the one side. What we are interested in is the overall probability of the parties voting on one side so as to get an overall measure of relationship between all the parties involved.

This we do by minimising the following.

Let a_i the probability that party i votes yea to question k+, $(1-a_i)$ the probability of voting nay to the same question as calculated from (2) above. The a_i's can be estimated by minimising

$$(3) \qquad \sum_{ij} \{\mu_{ij} - [2a_ia_j + 1 - a_i - a_j]\}^2 \text{ subject to } \begin{array}{l} 0 \leqslant a_i \leqslant 1 \\ 0 \leqslant a_j \leqslant 1 \end{array}$$

This is done using a constrained least square minimisation program described in the Appendix. Thus we have a method for estimating the a_i's. The objective has local minima and is clearly symmetric about $(\frac{1}{2}.\frac{1}{2})$ since the value is unaltered by replacing each a_i by $1-a_i$.

4.4 PRESENTATION OF RESULTS

The methods described in the previous section will now be applied to a large set of simulated voting data. The simulation was done by assuming the voting behaviour to be that described in 4.3 and this assumption is consistent with the practical voting data to be discussed in the next chapter.

The simulated values then serve to

(i) provide a large data set on which the different methods of analysis can be tested and compared and also to

(ii) confirm the validity of the approximations made in the theory.

The basic situation considered was that of five political parties with the following sizes $n1 = 37$, $n2 = 28$, $n3 = 15$, $n4 = 8$, $n5 = 7$. (The reason was that the Nigerian Senate which provided the practical data has five political parties with similar group sizes except that in place of $n1 = 37$ it has $n1 = 36$ and $n3 = 15$ it has $n3 = 16$. This was due to a slight error as contained in Okion Ojigbo (1980) but was amended on the practical application.

The political system was analysed via the classical Shapley approach, firstly by regarding the parties as distinct homogeneous groups where a party occupies a pivotal position as described in 4.2. Secondly, the case where individuals occupied pivotal positions were considered via the technique employed by Shapley and Shubik in "A method for evaluating the Distribution of Power in a Committee System", Shapley and Shubik (1954). In the second case we find that power is proportional to voting strength but in the first case that is not quite true due to the indivisible nature of the assumed homogeneous groups (parties). The results were as follows

		PARTY AS PIVOT		INDIVIDUALS AS PIVOT	
Party	Seats	Value of Party	Value of Individuals in Party	Value of Party	Value of Individuals in Party
PARTY A	37	0.4001	0.0108	0.3890	0.0105
PARTY B	28	0.2333	0.0083	0.2950	0.0105
PARTY C	15	0.2333	0.0155	0.1576	0.0105
PARTY D	8	0.0667	0.0083	0.0842	0.0105
PARTY E	7	0.0667	0.0095	0.0736	0.0105

The above Shapley values would then provide the initial set of numbers for the comparisons that follow from the other models.

Two sets of simulations were carried out. In the first, a set of a_i's were specified and 100 voting situations were generated. The results were analysed on the Owen model and values were calculated. Also it was verified that the actual a_i's could be recovered from this data for the direct approach model. Subsequently up to 5000 voting situations were simulated for given sets of a_i's and only the minimal winning cases were retained. The approximations made in the theory of the direct approach method were compared with these results. Finally, the values given by classical Shapley individual and weighted voting models are compared with the values obtained from the Owen model and from the direct approach model.

4.4.1 a_i's Used and Assessment of Procedure

The following a_i's were used to generate the initial 100 voting situations

$$a_1 = 0.852$$
$$a_2 = 0.059$$
$$a_3 = 0.487$$
$$a_4 = 0.436$$
$$a_5 = 0.524$$

(a) RESULTS FROM OWEN'S MODIFICATION

A matrix of 1's and 0's were generated from the 100 voting situations and analysed using cluster analysis via the package Clustan and multidimensional scaling via 'SPACES' as explained in (4.2). The resulting euclidean one dimensional scale was placed around one half of a circle at the ordinates shown below. This is in line with Owen's application to the Knesset where the parties were "assumed to occupy approximately one half of a circle," G. Owen (1971,p.354)

Party	Point on Circle	Seats	Value of Party	Value of Individual Members
PARTY A	0.0°	37	0.2533	0.0068
PARTY B	180.0°	28	0.2467	0.0088
PARTY C	88.8°	15	0.5000	0.0333
PARTY D	176.8°	8	0.0	0.0
PARTY E	3.0°	7	0.0	0.0

The assignment of zero values is a disadvantage in the Owen's technique caused by the nature of a model based on a circle or one half of a circle. In his example on the Knesset as quoted above, out of 11 parties 5 had zero values and one party had 0.700 while the remaining 5 together had 0.300, G. Owen (1971, p.354) A rotation of the points might give better results but that would negate Owen's ideas of fixed positions.

(b)

The following a_i's were derived from the least squares estimates of the simulated voting situations: a1 = 0.840, a2 = 0.059, a3 = 0.501, a4 = 0.428 and a5 = 0.541.

Another set of a_i^o's were recovered indicating the presence of 2 local minima and they were as follows : a1 = 0.160, a2 = 0.940, a3 = 0.499, a4 = 0.573 and a5 = 0.459.

These are the complementary set of a_i's. The above set of a_i's served as an assessment of the accuracy of the procedure of (4.3).

4.4.2 DIRECT APPROACH RESULTS AND SUMMARY

5,000 voting situations were generated using the following set of a_i's.

Cases \ Players	1	2	3	4	5
1	0.852	0.059	0.487	0.436	0.524
2	0.500	0.500	0.500	0.500	0.500
3	0.852	0.001	0.487	0.436	0.524
4	0.900	0.001	0.500	0.500	0.500
5	0.750	0.250	0.250	0.500	0.500
6	0.750	0.250	0.500	0.250	0.250

Case 1 corresponds to a practical voting situation

- the Nigerian Senate

Case 2 corresponds to the classical Shapley model where

individuals vote independently.

The other cases are used to test the dynamic nature of the model and study the political system adequately.

The following table gives a comparison of the number of minimal winning coalitions predicted from the formula of (3.2.4) of Chapter 3 which can be calculated from the denominator of expression (7) of (3.2.4) with the number of actual minimal winning coalitions recorded in the course of the generation of the 5000 voting situations with respect to the cases considered above.

CASES	1	2	3	4	5	6
Predicted Ratio of minimal winning coalitions	0.214	0.163	0.227	0.239	0.176	0.183
Actual Ratio of minimal winning coalitions	0.209	0.164	0.215	0.226	0.167	0.175

The following tables give an adequate comparison between the values calculated from the theoretical formula derived in Chapter 3 with the values derived from simulating 5000 voting situations and determining the values from a calculation based on the minimal winning situations only.

CASE 1: PLAYERS (PARTIES)	1	2	3	4	5
a_i's	0.852	0.059	0.487	0.436	0.524
Values: Simulation					
Parties	0.3946	0.2848	0.1603	0.0839	0.0764
Individuals	(0.0107)	(0.0102)	(0.0107)	(0.0105)	(0.0109)
Values : Formula					
Parties	0.3903	0.2916	0.1591	0.0847	0.0743
Individuals	(0.0105)	(0.0104)	(0.0106)	(0.0106)	(0.0106)

CASE 2: PLAYERS (PARTIES)	1	2	3	4	5
a_i's	0.500	0.500	0.500	0.500	0.500
Values: Simulation					
Parties	0.3898	0.2963	0.1572	0.0838	0.0728
Individuals	(0.0105)	(0.0106)	(0.0105)	(0.0105)	(0.0104)
Values: Formula					
Parties	0.3894	0.2947	0.1578	0.0842	0.0737
Individuals	(0.0105)	(0.0105)	(0.0105)	(0.0105)	(0.0105)

CASE 3: PLAYERS (PARTIES)	1	2	3	4	5
a_i's	0.852	0.001	0.487	0.436	0.524
Values: Simulation					
Parties	0.3741	0.3056	0.1614	0.0841	0.0740
Individuals	(0.0101)	(0.0109)	(0.0108)	(0.0105)	(0.0106)
Values: Formula					
Parties	0.3721	0.3094	0.1591	0.0850	0.0739
Individuals	(0.0100)	(0.0110)	(0.0106)	(0.0106)	(0.0106)

CASE 4: PLAYERS (PARTIES)	1	2	3	4	5
a_i's	0.900	0.001	0.500	0.500	0.500
Values: Simulation					
Parties	0.4088	0.2742	0.1584	0.0857	0.0728
Individuals	(0.0110)	(0.0098)	(0.0106)	(0.0107)	(0.0104)
Values: Formula					
Parties	0.4001	0.2805	0.1596	0.0851	0.0744
Individuals	(0.0108)	(0.0100)	(0.0106)	(0.0106)	(0.0106)

CASE 5: PLAYERS (PARTIES)	1	2	3	4	5
a_i's	0.750	0.250	0.250	0.500	0.500
Values: Simulation					
Parties	0.3763	0.3012	0.1620	0.0858	0.0745
Individuals	(0.0102)	(0.0108)	(0.0108)	(0.0107)	(0.0106)
Values: Formula					
Parties	0.3811	0.3000	0.1607	0.0843	0.0737
Individuals	(0.0103)	(0.0107)	(0.0107)	(0.0105)	(0.0105)

CASE 6: PLAYERS (PARTIES)	1	2	3	4	5
a_i's	0.750	0.250	0.500	0.250	0.250
Values: Simulation					
Parties	0.3745	0.3036	0.1581	0.0871	0.0765
Individuals	(0.0101)	(0.0108)	(0.0105)	(0.0109)	(0.0109)
Values: Formula					
Parties	0.3811	0.3000	0.1580	0.0857	0.0750
Individuals	(0.0103)	(0.0107)	(0.0105)	(0.0107)	(0.0107)

The extreme case where the normal approximation to the binomial fails due to the set of a_i's (Probabilities) attached to the distribution of players within the voting system, the term by term calculation of Section 3.2.5 Chapter three is recommended, e.g.

CASE 7:	1	2	3	4	9
a_i's	0.1	0.000	0.500	0.500	0.500
VALUES: Direct Term by Term calculation					
Parties	0.4020	0.2847	0.1566	0.0835	0.0730
Individuals	(0.0109)	(0.0102)	(0.0104)	(0.0104)	(0.0104)

A comparison of values deduced for each party, from classical Shapley, Owen (from analysis of the 100 voting situations generated as case 1), direct approach (theoretical values) follows. The individual values of members of each party are in brackets.

CASE 1 : PLAYERS (PARTIES)	1	2	3	4	5
Shapley (individual Pivot)	0.3890 (0.0105)	0.2950 (0.0105)	0.1576 (0.0105)	0.0842 (0.0105)	0.0736 (0.0105)
Shapley (Parties Pivot)	0.4001 (0.0108)	0.2333 (0.0083)	0.2333 (0.0156)	0.0667 (0.0083)	0.0667 (0.0095)
Owen	0.2533 (0.0068)	0.2467 (0.0088)	0.5000 (0.0333)	0.0000 (0.000)	0.0000 (0.000)
Direct Approach	0.3903 (0.0105)	0.2916 (0.0104)	0.1591 (0.0106)	0.0847 (0.0106)	0.0743 (0.0106)

The above values show a remarkable difference in the Owen model with the presence of rather extreme values.

A summary of the values for each party from the Direct Approach model will now be given as calculated from the Theoretical Formula

CASE	1	2	3	4	5
2	0.389	0.295	0.158	0.084	0.074
3	0.372	0.309	0.159	0.085	0.074
4	0.400	0.281	0.160	0.085	0.074
5	0.381	0.300	0.161	0.084	0.074
6	0.381	0.300	0.158	0.086	0.075
7	0.402	0.285	0.157	0.084	0.073

The value changes are fairly small but very reasonable since the set of a_i's with the minimal winning criterion would not let any party have extreme values. The extreme values from the above model can of course be estimated from the formula:

$$\text{Since } V_i = \frac{1}{N+1} \; E\left(X_i \text{ and in minimal winning coalition} \Big/ \text{minimal winning coalitions}\right)$$

The largest possible value is $\frac{n_i}{N+1}$ i.e. about twice the Shapley individual value, this will only occur in the extreme situation where party i must always be in the winning coalition. The smallest value is zero.

The graphs and tables which follow illustrate the changes in V_i with a_i.

G1 illustrates the effect of the changes in attitude (a_i) of members of the largest party on the values of the players when it has a powerful opposition party and the minor parties cling together, while G2 illustrates the effect of the changes in attitude of members of a strong opposition party on the values of the players when the minor parties cling together (i.e. bind themselves together).

In G3 the members of the most important middle party vary their attitude towards the other players while the two major parties stay apart with the two minor parties clinging together. G4 and G5 illustrate the effect of the changes in attitude of members of the most important middle party when the two major parties stay in opposition while the minor parties tend to align with either of the major parties.

Graphs G1, G2, G3, G4 and G5 now follow, after which we have tables T1, T2, T3, and T4. In the tables the three minor parties cling together and thus have the same a_i while the effect of the changes in the a_i's of the major parties on the values are tested. For each table the a_i of one major party is fixed while the a_i of the other varies.

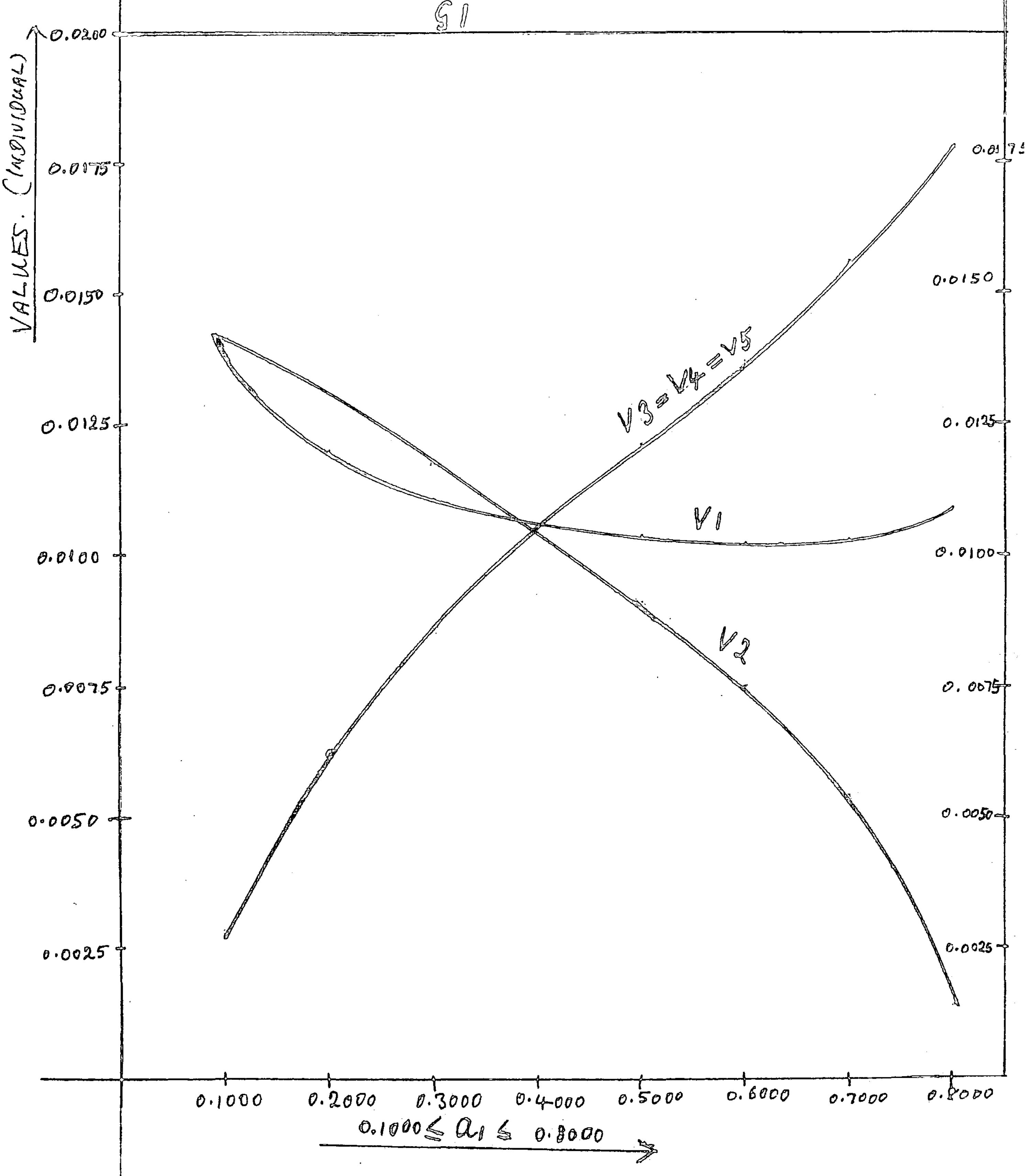
Tables for G1.

0.100 ⩽ a1 ⩽ 0.800

a2 = 0.100

a3 = a4 = a5 = 1.000

Party	a_i	V_i(Party)	V_i(Individual)	a_i	V_i(Party)	V_i(Individual)
1	0.1000	0.5221	.0141	0.5000	0.3836	.0104
2		0.3951	.0141		0.2538	.0091
3		0.0413	.0028		0.1813	.0121
4		0.0221	"		0.0967	"
5		0.0193	"		0.0846	"
1	0.2000	0.4435	.0120	0.6000	0.3794	.0102
2		0.3713	.0133		0.2089	.0075
3		0.0925	.0062		0.2059	.0137
4		0.0494	"		0.1098	"
5		0.0432	"		0.0960	"
1	0.3000	0.4101	.0111	0.7000	0.3824	.0103
2		0.3314	.0118		0.1501	.0054
3		0.1292	.0086		0.2338	.0156
4		0.0689	"		0.1247	"
5		0.0603	"		0.1091	"
1	0.4000	0.3932	.0106	0.8000	0.4023	.0109
2		0.2929	.0105		0.0640	.0023
3		0.1569	.0105		0.2668	.0178
4		0.0837	"		0.1423	"
5		0.0732	"		0.1245	"



$0.1000 \leq a_1 \leq 0.8000$

$a_2 = 0.1000$

$a_3 = a_4 = a_5 = 1.000$

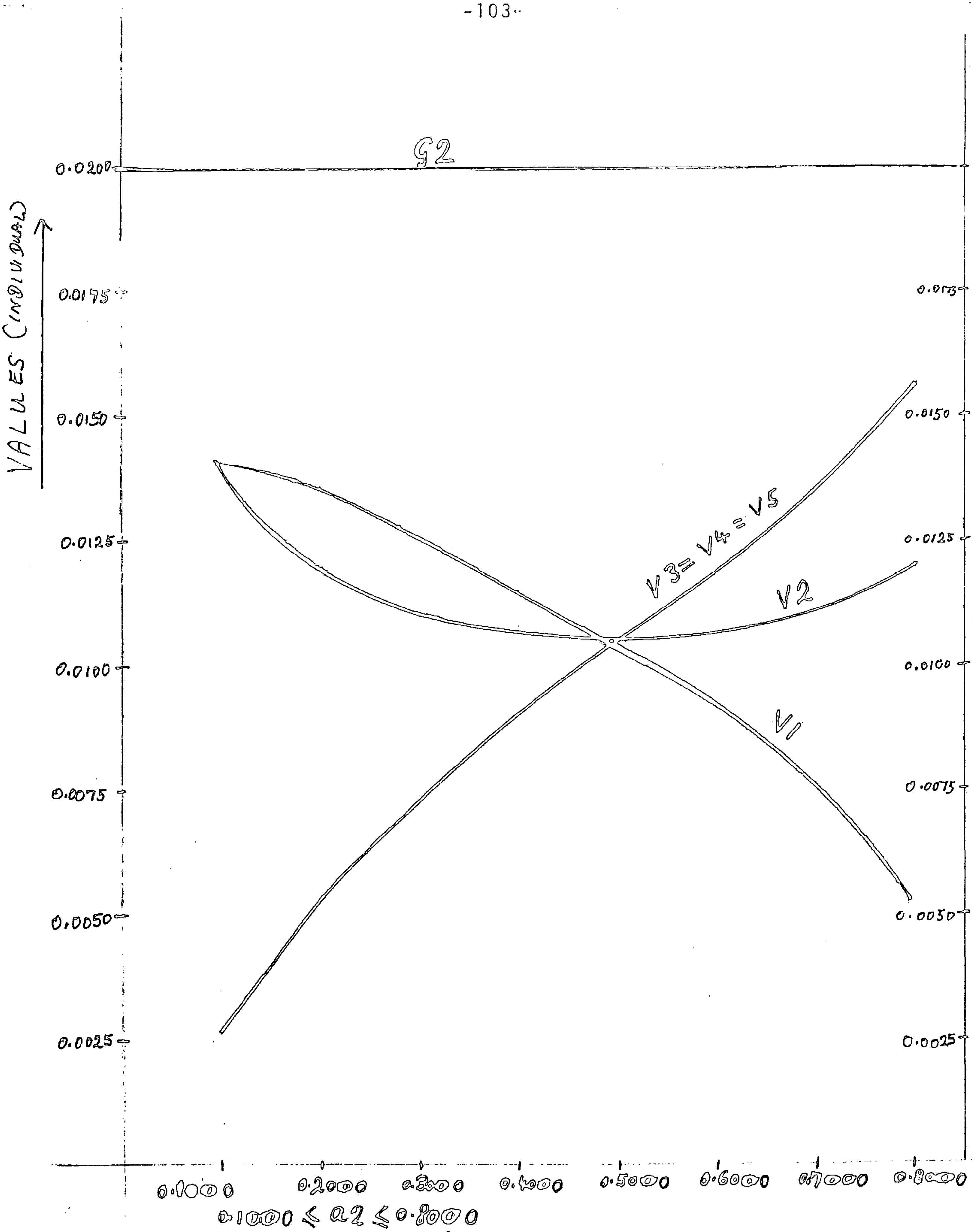
Tables for G2

a1 = 0.1000

0.1000 $\leqslant$ a2 $\leqslant$ 0.8000

a3 = a4 = a5 = 1.0000

Party	a_i	V_i(Party)	V_i(Individual)	a_i	V_i(Party)	V_i(Individual)
1		0.5221	.0141		0.3857	.0104
2	0.1000	0.3951	.0141	0.5000	0.2980	.0106
3		0.0413	.0027		0.1578	.0105
4		0.0221	"		0.0841	"
5		0.0193	"		0.0736	"
1		0.5032	.0136		0.3401	.0092
2	0.2000	0.3347	.0119	0.6000	0.3011	.0108
3		0.0810	.0054		0.1794	.0120
4		0.0432	"		0.0957	"
5		0.0378	"		0.0837	"
1		0.4654	.0125		0.2819	.0076
2	0.3000	0.3111	.0111	0.7000	0.3104	.0111
3		0.1112	.0074		0.2039	.0136
4		0.0595	"		0.1087	"
5		0.0521	"		0.0951	"
1		0.4262	.0115		0.1971	.0053
2	0.4000	0.3015	.0108	0.8000	0.3338	.0120
3		0.1362	.0091		0.2345	.0156
4		0.0726	"		0.1251	"
5		0.0635	"		0.1094	"



$a_1 = 0.1000$

$0.1000 \leq a_2 \leq 0.8000$

$a_3 = a_4 = a_5 = 1.0000$

Tables for G3

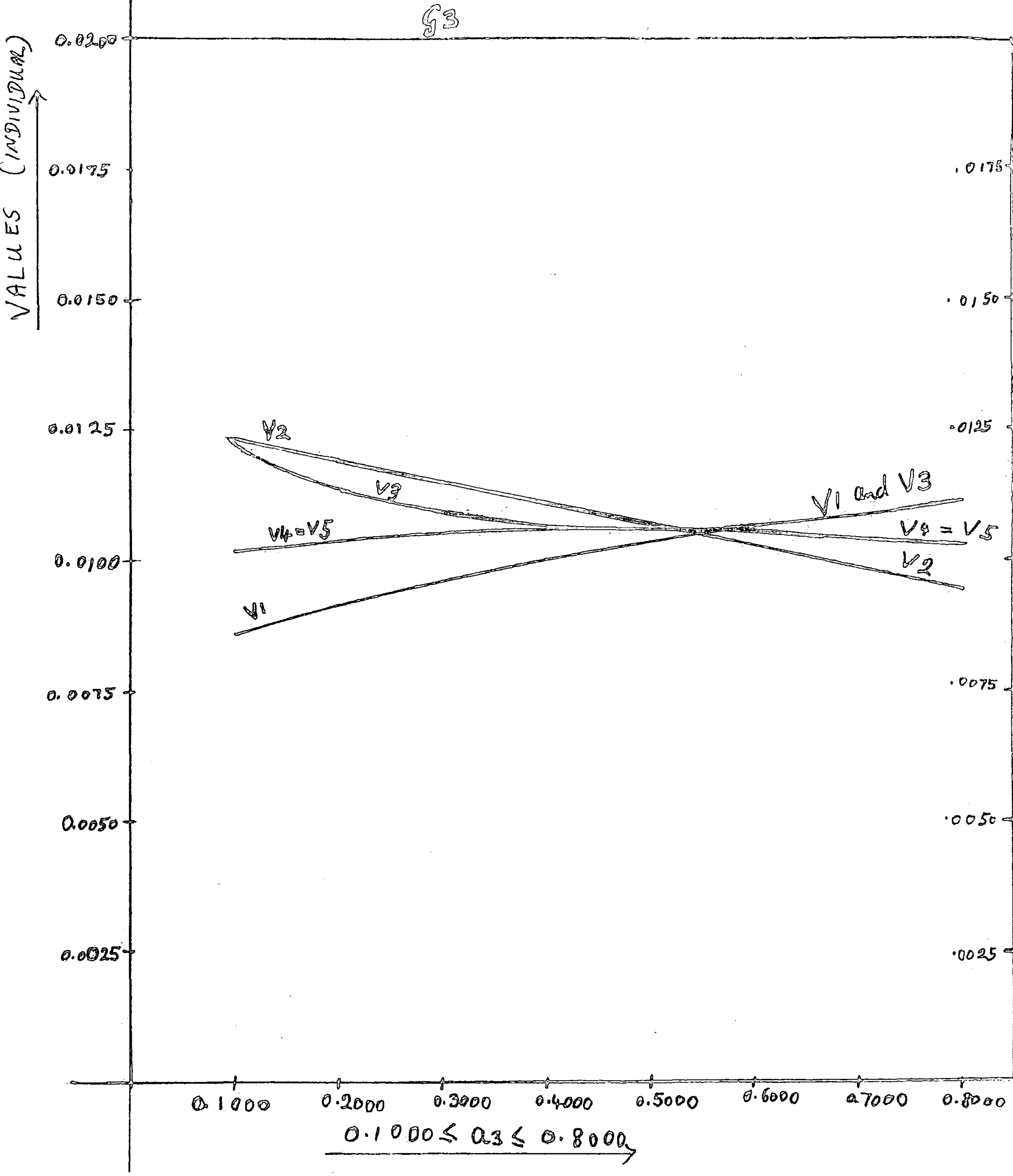
a1 = 0.8000

a2 = 0.1000

0.1000 ≤ a3 ≤ 0.8000

a4 = a5 = 0.4500

Party	a_i	V_i(Party)	V_i(Individual)	a_i	V_i(Party)	V_i(Individual)
1		.3770	.0086		.3830	.0104
2		.3450	.0123		.2995	.0107
3	0.1000	.1848	.0123	0.5000	.1586	.0105
4		.0817	.0102		.0848	.0106
5		.0715	.0102		.0742	.0106
1		.3394	.0092		.3940	.0106
2		.3324	.0119		.2885	.0103
3	0.2000	.1717	.0114	0.6000	.1592	.0106
4		.0835	.0104		.0844	.0105
5		.0731	.0104		.0739	.0105
1		.3566	.0096		.4044	.0109
2		.3210	.0115		.2765	.0099
3	0.3000	.1641	.0110	0.7000	.1621	.0108
4		.0844	.0105		.0837	.0104
5		.0739	"		.0732	"
1		.3707	.0100		.4146	.0112
2		.3101	.0111		.2627	.0094
3	0.4000	.1601	.0107	0.8000	.1681	.0112
4		.0848	.0106		.0824	.0103
5		.0742	"		.0721	"



$a1 = 0.8000$

$a2 = 0.1000$

$0.1000 \leq a3 \leq 0.8000$

$a4 = a5 = 0.4500$

Tables for G4

$a1 = 0.1000$

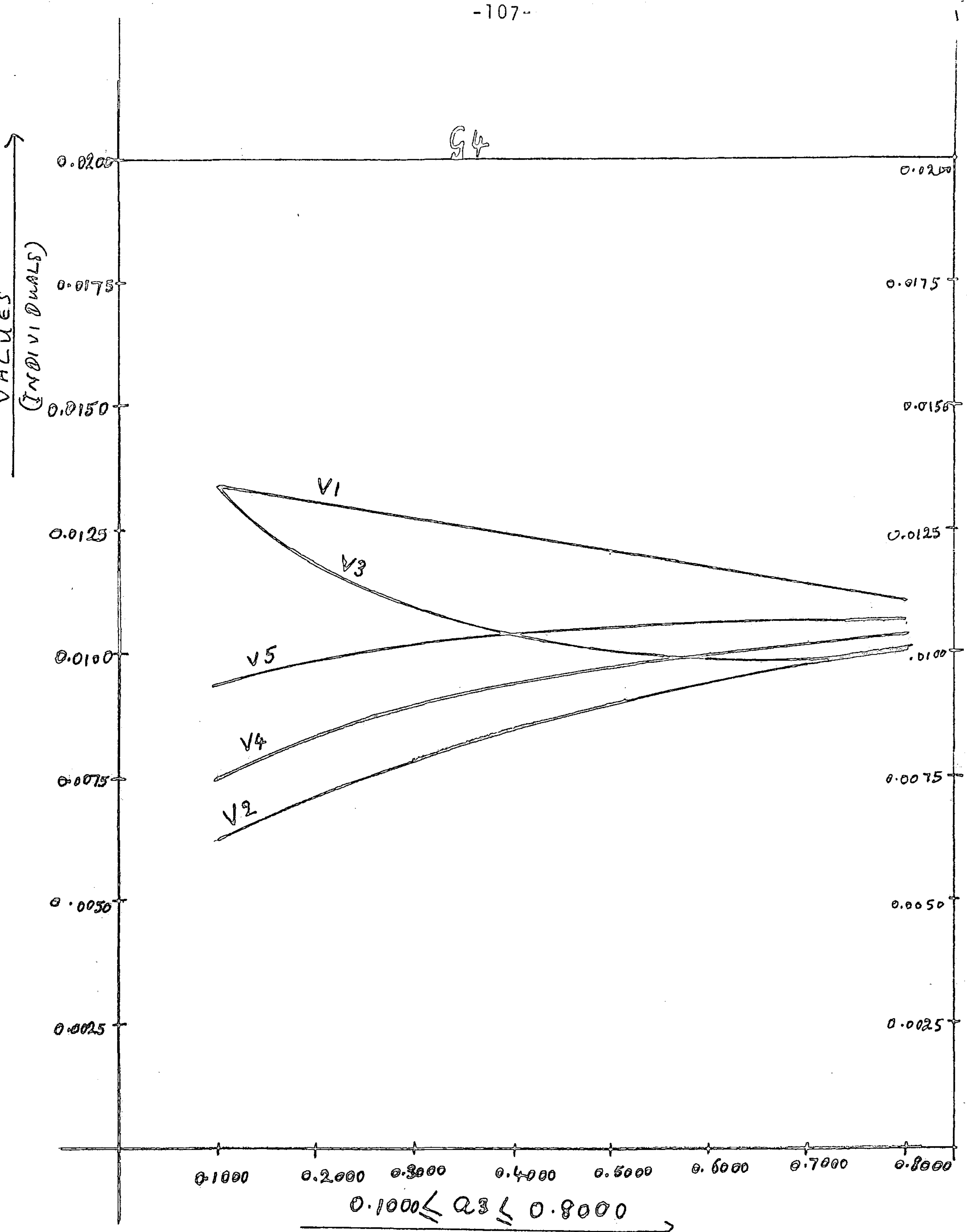
$a2 = 0.8000$

$0.1000 \leqslant a3 \leqslant 0.8000$

$a4 = 0.6000$

$a5 = 0.4000$

Party	a_i	V_i(Party)	V_i(Individual)	a_i	V_i(Party)	V_i(Individual)
1	0.1000	.4963	.0134	0.5000	.4450	.0120
2		.1767	.0063		.2510	.0090
3		.2012	.0134		.1522	.0101
4		.0602	.0075		.0778	.0097
5		.0657	.0094		.0741	.0105
1	0.2000	.4825	.0130	0.6000	.4331	.0117
2		.2027	.0072		.2623	.0094
3		.1784	.0119		.1500	.0100
4		.0669	.0084		.0800	.0100
5		.0694	.0099		.0746	.0107
1	0.3000	.4694	.0127	0.7000	.4208	.0114
2		.2224	.0079		.2727	.0097
3		.1649	.0110		.1498	.0099
4		.0717	.0089		.0818	.0102
5		.0717	.0102		.0749	.0107
1	0.4000	.4569	.0123	0.8000	.4074	.0110
2		.2380	.0085		.2829	.0101
3		.1568	.0104		.1515	.0101
4		.0751	.0094		.0834	.0104
5		.0732	.0104		.0748	.0107



$a_1 = 0.1000$

$a_2 = 0.8000$

$0.1000 \leq a_3 \leq 0.8000$

$a_4 = 0.6000$

$a_5 = 0.4000$

Tables for G5

$a1 = 0.8000$ $a5 = 0.6000$

$a2 = 0.1000$

$0.1000 \leqslant a3 \leqslant 0.8000$

$a4 = 0.4000$

Party	a_i	V_i(Party)	V_i(Individual)	a_i	V_i(Party)	V_i(Individual)
1		.3244	.0088		.3878	.0105
		.3412	.0122		.2948	.0105
3	0.1000	.1828	.0122	0.5000	.1587	.0106
4		.0844	.0106		.0847	.0106
5		.0672	.0096		.0739	.0106
1		.3459	.0093		.3985	.0108
2		.3284	.0117		.2835	.0101
3	0.2000	.1705	.0114	0.6000	.1597	.0106
4		.0853	.0107		.0838	.0105
5		.0699	.0100		.0745	.0106
1		.3625	.0098		.4086	.0110
2		.3167	.0113		.2711	.0097
3	0.3000	.1635	.0109	0.7000	.1630	.0109
4		.0855	.0107		.0824	.0103
5		.0718	.0103		.0748	.0107
1		.3761	.0102		.4184	.0113
2		.3057	.0109		.2567	.0092
3	0.4000	.1599	.0107	0.8000	.1696	.0113
4		.0853	.0107		.0805	.0101
5		.0730	.0104		.0747	.0107

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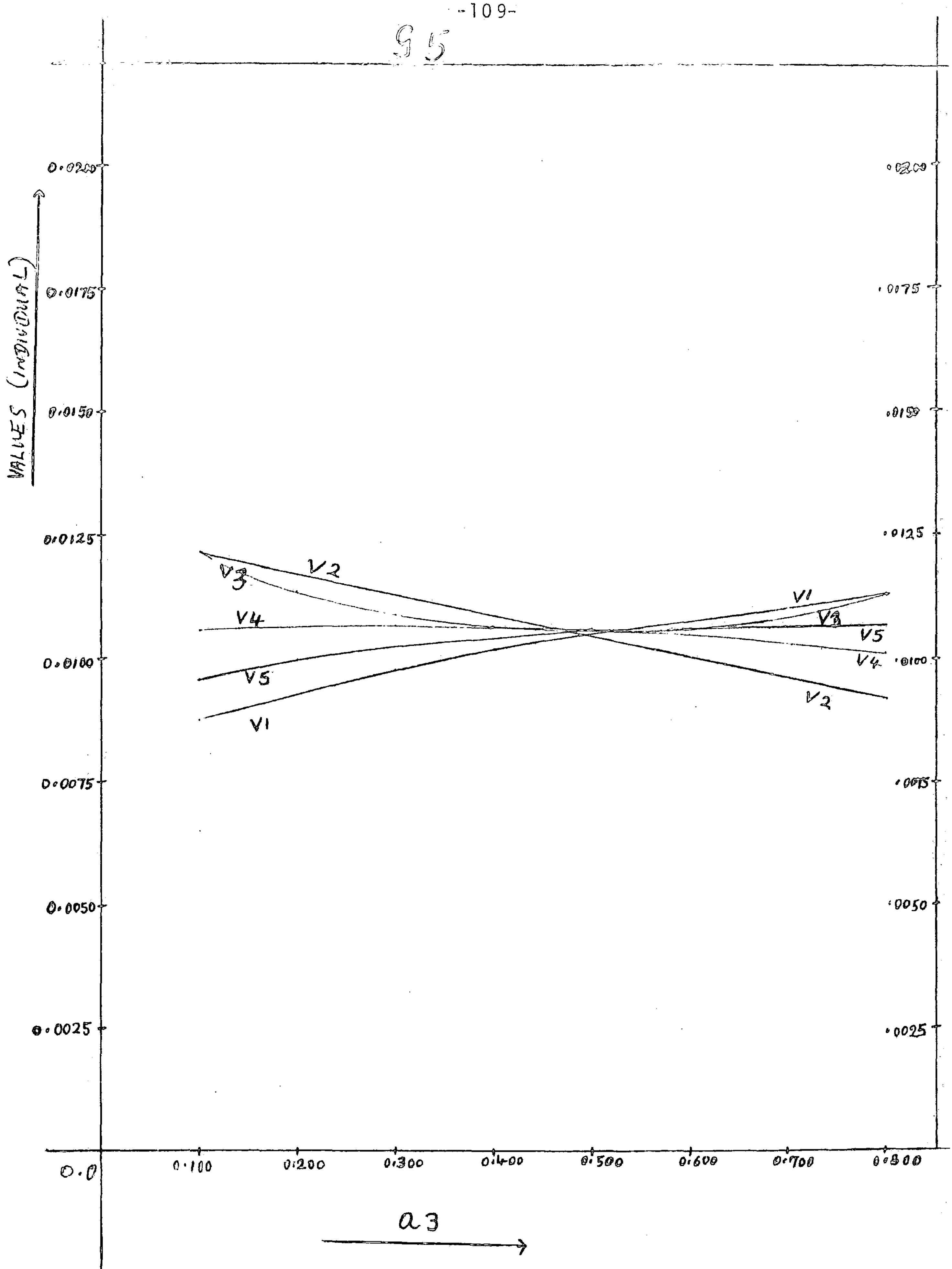


$a_1 = 0.800$
$a_2 = 0.100$
$0.100 < a_3 < 0.800$
$a_4 = 0.400$

Table II

$0.1000 \leq a1 \leq 0.8000$

$a2 = 0.1000$

$a3 = a4 = a5 = 0.5000$

Party		a_i	V_i(Party)	V_i(Individual)	Party		a_i	V_i(Party)	V_i(Individual)
	1	0.1000	.5180	.0140		1	0.5000	.3588	.0097
	2		.3920	.0140		2		.3504	.0125
(A)	3		.0450	.0030	(E)	3		.1454	.0097
	4		.0240	"		4		.0776	"
	5		.0210	"		5		.0679	"
	1	0.2000	.4247	.0115		1	0.6000	.3608	.0098
(B)	2		.3899	.0139	(F)	2		.3347	.0120
	3		.0927	.0062		3		.1522	.0101
	4		.0494	"		4		.0812	"
	5		.0433	"		5		.0710	"
	1	0.3000	.3827	.0103		1	0.7000	.3699	.0100
(C)	2		.3788	.0135	(G)	2		.3168	.0113
	3		.1193	.0080		3		.1567	.0104
	4		.0636	"		4		.0836	"
	5		.0557	"		5		.0731	"
	1	0.4000	.3845	.0104		1	0.8000	.3886	.0105
	2		.3650	.0130		2		.2940	"
(D)	3		.1352	.0090	(H)	3		.1587	.0106
	4		.0721	"		4		.0846	"
	5		.0631	"		5		.0741	"

Table T2

$0.1000 \leqslant a1 \leqslant 0.8000$

$a2 = 0.1000$

$a3 = a4 = a5 = 0.2500$

Party	a_i	V_i(Party)	V_i(Individual)	Party	a_i	V_i(Party)	V_i(Individual)
1	0.1000	.4834	.0131	1	0.5000	.2892	.0078
2		.3658	"	2		.3800	.0136
(A) 3		.0754	.0050	(E) 3		.1654	.0110
4		.0402	"	4		.0882	"
5		.0352	"	5		.0772	"
1	0.2000	.3662	.0099	1	0.6000	.2914	.0079
2		.3894	.0139	2		.3707	.0132
(B) 3		.1222	.0081	(F) 3		.1689	.0113
4		.0652	"	4		.0901	"
5		.0570	"	5		.0788	"
1	0.3000	.3154	.0085	1	0.7000	.2988	.0081
2		.3927	.0140	2		.3601	.0129
(C) 3		.1460	.0097	(G) 3		.1705	.0114
4		.0778	"	4		.0910	"
5		.0681	"	5		.0796	"
1	0.4000	.2949	.0080	1	0.8000	.3112	.0084
2		.3880	.0139	2		.3476	.0124
(D) 3		.1586	.0106	(H) 3		.1706	.0114
4		.0846	"	4		.0910	"
5		.0740	"	5		.0796	"

Table T3

$$a1 = 0.1000$$
$$0.1000 \leqslant a2 \leqslant 0.8000$$
$$a3 = a4 = a5 = 0.5000$$

Party	a_i	V_i(Party	V_i(Individual)	Party	a_i	V_i(Party)	V_i(Individual)
1		.5180	.0140	1		.4858	.0131
2	0.1000	.3920	"	2	0.5000	.2482	.0089
(A) 3		.0450	.0030	(E) 3		.1330	"
4		.0240	"	4		.0709	"
5		.0210	"	5		.0621	"
1		.5177	.0140	1		.4733	.0128
2	0.2000	.3162	.0113	2	0.6000	.2449	.0087
(B) 3		.0831	.0055	(F) 3		.1409	.0094
4		.0443	"	4		.0751	"
5		.0388	"	5		.0657	"
1		.5092	.0138	1		.4600	.0124
2	0.3000	.2774	.0099	2	0.7000	.2459	.0088
(C) 3		.1067	.0071	(G) 3		.1470	.0098
4		.0569	"	4		.0784	"
5		.0498	"	5		.0686	"
1		.4980	.0135	1		.4452	.0120
2	0.4000	.2576	.0092	2	0.8000	.2506	.0090
(D) 3		.1222	.0081	3		.1521	.0101
4		.0652	"	4		.0811	"
5		.0570	"	5		.0710	"

Table T4

$$a1 = 0.1000$$
$$0.1000 \leqslant a2 \leqslant 0.8000$$
$$a3 = a4 = a5 = 0.2500$$

Party	a_i	V_i(Party)	V_i(Individual)	Party	a_i	V_i(Party)	V_i(Individual)
1		.4834	.0131	1		.5152	.0139
2	0.1000	.3658	.0131	2	0.5000	.1732	.0062
(A) 3		.0754	.0050	(E) 3		.1558	.0104
4		.0402	"	4		.0831	"
5		.0352	"	5		.0727	"
1		.5101	.0138	1		.5103	.0138
2	0.2000	.2639	.0094	2	0.6000	.1677	.0060
(B) 3		.1130	.0075	(F) 3		.1610	.0107
4		.0602	"	4		.0859	"
5		.0527	"	5		.0751	"
1		.5185	.0140	1		.5052	.0137
2	0.3000	.2122	.0076	2	0.7000	.1659	.0059
(C) 3		.1346	.0090	(G) 3		.1644	.0110
4		.0718	"	4		.0877	"
5		.0628	"	5		.0767	"
1		.5187	.0140	1		.5005	.0135
2	0.4000	.1859	.0066	2	0.8000	.1660	.0059
(D) 3		.1477	.0098	(H) 3		.1668	.0111
4		.0788	"	4		.0889	"
5		.0689	"	5		.0778	"

We note that a_i is restricted to $0.1000 \leq a_i \leq 0.8000$ for most of the above calculations because the approximations of Chapter 3 Section 3.2.4 guarantee the best results when extreme values such as 0.000 and 1.000 are avoided with respect to the Direct Approach formula; nevertheless, the term by term calculations of Chapter 3 Section 3.2.5 could be used for extreme values if need be. Extreme values, (0.000, 1.000) imply that every member of a party vote together on one side all the time which is not usually the case in practical voting situations.

The above analysis portrays in a clear fashion the effect the different sets of a_i's have on a voting system made up of two large opposition parties with three or more smaller parties.

It is therefore clear that the new model is dynamic as claimed in Sub-Section (3.2.6) of Chapter 3. It has been shown that it is possible to incorporate the psychology of the players with respect to their affinity for voting with other players from other parties as shown by the technique for estimating the a_i's.

We have therefore succeeded in carrying out a valid extension to the Shapley value which has been successfully applied to simulated voting situations.

The application to some practical voting situations will now follow in Chapter Five as well as a comparison with classical Shapley and Owens' modification.

CHAPTER 5

APPLICATIONS TO PRACTICAL VOTING SITUATIONS

In this Chapter different models will be applied to practical voting situations with emphasis on the Nigerian voting situations, since most of the original data was from there.

Section 5.1 will contain a summary of the Nigerian political set-up. In 5.2, 5.3 and 5.4 the Nigerian Senate will be analysed and values calculated via the Direct Approach Model, Classical Shapley and Owen's modification. A summary of value calculations for the House of Representatives and the different houses of assembly will also be carried out.

In 5.5 the effect of these values on the Nigerian political situation will be discussed. Application of the new model to other voting systems namely, United States, E.E.C, and the U.N. will be made in 5.6 while 5.7 will contain the concluding remarks.

5.1 THE NIGERIAN POLITICAL SET-UP

On October 1st 1983, Nigeria will be celebrating 23 years of independence. It became independent on October 1st 1960 and with a population of 80 million plus it is the fourth largest democracy in the world, Guardian, (Oct. 4th, 1982). Since its independence it has experienced many strains which afflict large countries with diverse populations and aspirations in their march towards democracy. The country has about 200 tribal units, Robertson, J. (1974). Regional rivalries based on economic, ethnic and religious differences erupted into a sessionist movement which led to a civil war in 1967 coupled with periodic unrest.

Nigeria derives its name from the River Niger. The Nigerian plateau in the area around Jos is regarded as the focal point in early Nigerian history. Agriculture must have been practised in the plateau region about 3000 B.C. and since then Nigerian history has been characterised by the pressure of northern peoples on the Southern forest belt, Foreign and Commonwealth Office (1981 page 281). Contact with Europe began in the fifteenth century with the Portuguese, and much later with the British who subsequently colonised the area and in 1914 Nigeria was united administratively by the British into one dependency. A Nigerian Council consisting of six African and thirty European members was set up but had no executive or legislative authority. In 1922 a new constitution provided for a legislative Council of 46 members, of whom ten were Africans, four of these being elected. This Council had powers to legislate for the Southern provinces while the governor legislated by proclamation for the northern provinces, Foreign and Commonwealth Office (1981, page 283). This seems to be the beginning of the history of early elections in Nigeria. Political situation changed gradually until 1951 with the introduction of "Richards Constitution" the policy of regionalisation was established. There were three regions, North, East and West each with a regional House of Assembly and a House of Representatives whose members were elected via electoral colleges. The political growth continued gradually. The 1951 Constitution was revised in 1953 and early 1954 and a new Constitution came into force. The changes contained in the new document included the granting of more powers to the Regions and the declaration that Nigeria was a federation, Foreign and Commonwealth Office (1981, page 284). The political system continued to mature until independence on 1st October 1960. Details about Nigeria's early history and march towards independence can be found in Crowther, M.(1962), Dike, K.O. (1956) and Davis, H.O. (1961).

At independence a completely indigeneous government came into power and on October 1st 1963, Nigeria became a Republic within the Commonwealth. In the same year Nigeria created a fourth Region, the Midwest Region, but this first Republic lasted briefly. Nigeria's first civilian Government led by Prime Minister Abubakar Tafawa Balewa was toppled on Jan. 13th 1966 when the Nigerian army mutinied as a reaction to widespread unrest and violence caused by regional rivalries. The first military Government was toppled six months later and the second military government, led by Gen. Yakubu Gowon lasted nine years. During Gowons regime, in May 1967, 12 states were created from the four regions based either entirely on the old provinces created by the British Government or a group of provinces. A third military Government took power in a bloodless coup in 1975 mainly because the Gowon Government appeared to be making very little progress towards returning the country to civilian rule. The new Government led by General Muhammed in 1975 announced a four year programme that would terminate with a return to a democratically elected government. He subdivided Nigeria into 19 States in 1976 and shortly after was killed in an abortive coup in the same year and was replaced by General Obasanjo. General Obasanjo successfully led the country to a democratically elected civilian Government and then retired from public life in October, 1979.

In 1976 a constitution drafting committee was appointed by General Obasanjo charged with identifying a constitutional form better suited to Nigeria's ethnic (tribal) and economic problems. The committee eventually decided to model the new constitution on that of the U.S. - Guardian (Oct. 4th, 1982). The Constitution created a National Assembly with two Houses, the Upper House - Senate which would have five legislators from each State, irrespective of the State's size and a Lower House - the House of Representatives where seats would be allocated according to the population of the States. Each State would also have a legislative body, the House of Assembly which contains three times the total number of seats in the National Assembly's House of Representatives. In addition, each State would have a State Governor and a Deputy, while the country will be run by an executive President and Vice-President. Also created was the Council of State whose members include the following, the President and Vice-President of Nigeria, all former federal Presidents and Heads of Governments, all former Chief Justices holding Nigerian Citizenship; the President of the Senate, the Speaker of the House of Representatives, all the State governors, the federal attorney general and one person appointed by each State Council of Chiefs. The Council mainly advises the President on some matters specified by the Constitution as a consultative body. For more details see NIGERIAN CONSTITUTION (1979); also see Keesings Contemporary Archives, (Dec. 19th 1980, 30621 - 30624).

Except for the Council of States, the other bodies are elective and elections are held along party lines, conducted by the Federal Electoral Commission (FEDECO) whose duty it is to register or reject a Party. At the time of election in 1979 five parties were registered by the FEDECO for the elections, NIGERIA PEOPLES PARTY, (NPP), UNITY PARTY OF NIGERIA (UPN), NATIONAL PARTY OF NIGERIA (NPN), GREAT NIGERIA PEOPLES PARTY (GNPP) and the PEOPLE'S REDEMPTION PARTY (PRP). As would be expected, most of the Parties were organised along tribal lines as reflected by the results, e.g. the Metropolitan Lagos State dominated by the Yorubas had all their legislators from one Party (UPN) led by a veteran Yoruba politician Chief Obafemi Awolowo.

The following tables show a summary of the results of the last 1979 elections. Also shown is a run-down of the population according to States based on the 1963 Census (The Statesman's Year Book 1979/80). There is considerable uncertainty over the total population but estimates based on the electoral registration puts it at 95 million, while the World Bank gave an estimate of 81,039,000 (The Statesman's Year Book, 1981/82, page 929). The population figures given below are based on the 1963 census because that is the one on which all the election data were based.

TABLE 1

STATE HOUSES OF ASSEMBLY RESULTS

Parties / State	GNPP	NPN	NPP	PRP	UPN	Tot.No. of seats	Population (in 1 m)
Anambra	1	10	75	-	-	86	3.6
Bauchi	9	45	4	2	-	60	2.4
Bendel	-	22	3	-	35	60	2.5
Benue	4	44	3	-	-	51	2.4
Borno	60	11	-	1	-	72	3.0
Cross River	16	57	3	-	8	84+	3.5
Gongola	26	17	1	1	18	63	2.6
Imo	2	8	80	-	-	90	3.7
Kaduna	11	68	4	11	5	99	4.1
Kano	2	13	-	123	-	138	5.8
Kwara	2	25	-	-	15	42	1.7
Lagos	-	-	-	-	36	36	1.4
Niger	2	28	-	-	-	30	1.2
Ogun	-	-	-	-	36	36	1.6
Ondo	-	1	-	-	65	66	2.7
Oyo	-	9	-	-	117	126	5.2
Plateau	3	11	34	-	-	48	2.0
Rivers	-	29	13	-	-	42	1.7
Sokoto	26	85	-	-	-	111	4.5

+ a seat (AWa constituency) may not have been contested.

TABLE 2

THE NATIONAL ASSEMBLY AND THE GOVERNORS' RESULT. Members of House of Representatives in brackets ( ) and Governors in square brackets [ ], Senators no brackets.

Parties / States	GNPP	NPN	NPP	PRP	NPN	No. of Seats
Anambra	- -	(2) -	(27) 5[1]	- -	- -	(29) 5
Bauchi	(1) -	(18) 5[1]	(1) -	- -	- -	(20) 5
Bendel	- -	(6) 1	(2) -	- -	(12) 4[1]	(20) 5
Benue	- -	(18) 5[1]	(1) -	- -	- -	(19) 5
Borno	(22) 4[1]	(2) 1	- -	- -	- -	(24) 5
Cross River	(4) 2	(22) 3[1]	- -	- -	(2) -	(28) 5
Gongola	(8) 2[1]	(5) 1	(1) -	- -	(7) 2	(21) 5
Imo	- -	(2) -	(28) 5[1]	- -	- -	(30) 5
Kaduna	(1) -	(19) 3	(2) -	(10) 2[1]	(1) -	(33) 5
Kano	- -	(7) -	- -	(39) 5[1]	- -	(46) 5
Kwara	(1) -	(8) 3[1]	- -	- -	(5) 2	(14) 5
Lagos	- -	- -	- -	- -	(12) 5[1]	(12) 5
Niger	- -	(10) 5[1]	- -	- -	- -	(10) 5
Ogun	- -	- -	- -	- -	(12) 5[1]	(12) 5
Ondo	- -	- -	- -	- -	(22) 5[1]	(22) 5
Oyo	- -	(4) -	- -	- -	(38) 5[1]	(42) 5
Plateau	- -	(3) 1	(13) 4[1]	- -	- -	(16) 5
Rivers	- -	(10) 3[1]	(4) 2	- -	- -	(14) 5
Sokoto	(6) -	(31) 5[1]	- -	- -	- -	(37) 5
Total Numbers:	(43) 8[2]	(167) 36[7]	(79) 16[3]	(49) 7[2]	(111) 28[5]	449 95
Percentages:	(9.6)8.4 [10.5]	(37.2)37.9 [36.8]	(17.6)16.8 [15.8]	(10.9)7.4 [10.5]	(24.7)29.5 [26.3]	

The above tables show that the National Party of Nigeria (NPN) won the elections and thus their presidential candidate Alhaji Shehu Shagari became Nigeria's first executive president under the new presidential system of Government. He nevertheless did not succeed in obtaining a majority in either the Senate or the House of Representatives and so he postponed the inaugural session of the National Assembly, originally scheduled by the military Government for October 2nd 1979 until October 9th 1979. Within this period a "co-operation" agreement was worked out between the NPN and NPP whereby the two parties undertook to "work together in the interest of the Unity, peace, stability and progress of the country" Keesings Contemporary Archives (Dec. 19th, 1980, page 30627). Although this did not constitute a formal coalition yet it gave the NPN federal administration an effective working majority of 52 out of 95 in the Senate and 246 out of 449 in the House of Representatives at the beginning of the National Assembly's term. As a result of the above"quasi-coalition", Dr. Joseph Wayas (NPN) was elected president of the Senate while Mr. Edwin Ume-Ezeoke (NPP) was elected Speaker of the House of Representatives. Also an NPP deputy President of the Senate was elected, as well as an NPN deputy Speaker for the House of Representatives. See Keesings Contemporary Archives as quoted above for details. It was therefore possible to pass most of the President's bills and as will be pointed out later this "quasi-coalition" did not last until the end of the National Assembly's term.

Another interesting result was Kaduna State where the Governor was elected from a minority party in the State's House of Assembly. As would be expected he enjoyed a difficult time and was finally impeached and removed from office before the end of his term. Details of Governor Alhaji Balarabe Musa's impeachment and subsequent removal from office can be found in most Nigerian daily papers.

A detailed analysis of the Nigerian Senate with respect to calculating the values of the players now follows.

5.2 THE NIGERIAN SENATE

The Nigerian Senate is therefore a relatively new voting body since it only came into existence in October 1979, as a result data with respect to proceedings have been quite scanty. Our sources of data for the tables given above and the Senate proceedings were Okion, Ojigbo (1980); West Africa 24/31 December (1979); Federal Republic of Nigeria, National Assembly Debates, Dec. (1979-1981); and Keesing's Contemporary Archives, (1980, page 30621 - 30628).

There are five political parties represented in the Nigerian Senate and they were represented in the Senate as follows:

National Party of Nigeria	(NPN)	=	36
Unity Party of Nigeria	(UPN)	=	28
Nigerian Peoples Party	(NPP)	=	16
Great Nigeria Peoples Party	(GNPP)	=	8
Peoples Redemption Party	(PRP)	=	7

Although Okion Ojigbo summarised the positions of the parties as stated above, Okion Ojigbo (1980, page 318) yet while enumerating the number of senators for each party the following was the case

National Party of Nigeria	(NPN)	=	37
Unity Party of Nigeria	(UPN)	=	28
Nigeria Peoples Party	(NPP)	=	15
Great Nigeria Peoples Party	(GNPP)	=	8
Peoples Redemption Party	(PRP)	=	7

This was due to the fact that one senator, Mr. George Baba Hoomkwap was listed as a member of the National Party of Nigeria (NPN) while in fact he belonged to the caucus of the Nigeria Peoples Party (NPP)

National Assembly debates, (Vol. 4, No. 32, Column 1961 page 7). Nevertheless, this slight error did not result in any significant change. The voting situations were firstly analysed as above with NPN = 37 and NPP = 15 and later with NPN = 36 and NPP = 16. The difference in values as would be seen was negligible. 19 voting situations were recovered from the Federal Republic of Nigeria, National Assembly Debates covering specific voting sessions from December 1979 until 1981. The number is small but these are actual situations; the simulated data discussed in the last chapter provided extensive material, but there is value in analysing real data.

The proportions of voters from each party who voted yea and nay during each voting situation was recorded as presented in Table 3. These proportions were then used to estimate the a_{ij}'s via formulae (1), (2) and (3) of Section (4.3) of Chapter 4. A 0-1 matrix was constructed from the data as indicated in the same table and this was used for determining the affinity of association between parties via the cluster analysis package and the multi-dimensional package (SPACES) described in Section (4.1) of Chapter 4.

TABLE 3

5.2.1

VOT-ING SES-SION	NPN % yes Prop. 0-1	 % no Prop.	UPN % yes Prop. 0-1	 % no Prop.	NPP % yes Prop. 0-1	 % no Prop.	GNPP % yes Prop. 0-1	 % no Prop.	PRP % yes Prop. 0-1	 % no Prop.	
A	100 1.000 1	0.0 0.000	0.0 0.000 0	100 1.000	53.8 .538 1	46.1 .461	14.2 .142 0	85.8 .858	100 1.000 1	0.0 0.000	% Prop. 0-1
B	100 1.000 1	0.0 0.000	0.0 0.000 0	100 1.000	61.5 .615 1	38.5 .385	0.0 0.000 0	100 1.000	50 .500 1	50 .500	% Prop. 0-1
C	4.3 .043 0	95.6 .956	100 1.000 1	0.0 0.000	50 .500 1	50 .500	100 1.000 1	0.0 0.000	100 1.000 1	0.0 0.000	% Prop. 0-1
D	4.5 .045 0	95.5 .955	100 1.000 1	0 0.000	80 .800 1	20 .200	100 1.000 1	0.0 0.000	100 1.000 1	0.0 0.000	% Prop. 0-1
E	86.4 .864 1	13.6 .136	0.0 0.000 0	100.0 1.000	75 .750 1	25 .250	25 .250 0	75 .750	100 1.000 1	0.0 0.000	% Prop. 0-1
F	100 1.000 1	0.0 0.000	0.0 0.000 0	100 1.000	58.3 .583 1	41.7 .417	14.2 .142 0	85.8 .858	66.6 .666 1	33.3 .333	% Prop. 0-1
G	95.0 .950 1	5.0 .050	0.0 0.000 0	100 1.000	50 .500 0	50 .500	100 1.000 1	0.0 0.000	100 1.000 1	0.0 0.000	% Prop. 0-1
H	90 .900 1	10 .100	0.0 0.000 0	100 1.000	75 .750 1	25 .250	100 1.000 1	0.0 0.000	40 .400 0	60 .600	% Prop. 0-1
I	3.2 .032 0	96.8 .968	100 1.000 1	0.0 0.000	53.8 .538 1	46.2 .462	80 .800 1	20 .200	100 1.0 1	0.0 0.0	% Prop. 0-1

Table 3 (Cont.)

VOTING SESsion	NPN % yes Prop. 0-1	 % no Prop.	UPN % yes Prop. 0-1	 % no Prop.	NPP % yes Prop. 0-1	 % no Prop.	GNPP % yes Prop. 0-1	 % no Prop.	PRP % yes Prop. 0-1	 % no Prop.	
J	8.69 .087 0	91.3 .913	0.0 0.0 0	100 1.0	64.2 .642 1	35.8 .358	25 .250 0	75 .750	66.6 .666 1	33.3 .333	% Prop. 0-1
K	4.5 .045 0	95.5 .955	100 1.000 1	0.0 0.000	60.0 .600 1	40.0 .400	75 .750 1	25 .250	100 1.000 1	0.0 0.000	% Prop. 0-1
L	100 1.0 1	0.0 0.0	4.40 .044 0	95.6 .956	100 1.000 1	0.000 0.000	100 1.000 1	0.000 0.000	100 1.0 1	0.0 0.000	% Prop. 0-1
M	100 1.0 1	0.0 0.0	94.7 .947 1	5.3 .053	100 1.0 1	0.0 0.0	100 1.0 1	0.0 0.0	83.3 .833 1	16.7 .167	% Prop. 0-1
N	3.3 .033 0	96.7 .967	96.2 .962 1	3.84 .038	66.6 .666 1	33.3 .333	16.7 .167 0	83.3 .833	0.0 0.0 0	100 1.0	% Prop. 0-1
O	6.66 .066 0	93.3 .933	100 1.0 1	0.0 0.0	76.9 .769 1	23.0 .230	0 0.0 0	100 1.0	0 0.0 0	100 1.0	% Prop. 0-1
P	93.1 .931 1	6.89 .069	0.0 0.0 0	100 1.0	14.2 .142 0	85.8 .858	0.0 0.0 0	100 1.0	0 0.0 0	100 1.0	% Prop. 0-1
Q	96.1 .961 1	3.84 .038	0.0 0.0 0	100 1.0	60 .600 1	40 .400	100 1.0 1	0 0.0	100 1.000 1	0 0.0	% Prop. 0-1
R	95.6 .956 1	4.34 .043	0.00 0.000 0	100 1.000	100 1.000 1	0.0 0.000	100 1.000 1	0.0 0.000	50 .500 0	50 .500	% Prop. 0-1
S	100 1.000 1	0.0 0.000	0.0 0.000 0	100 1.000	25 .250 0	75 .750	0.0 0.000 0	100 1.000	100 1.000 1	0 0.000	% Prop. 0-1

The following a_i's for the different parties were recovered which indicate an overall measure of the tendency of the parties to vote together in order to constitute a minimal winning coalition.

A	NPN	=	a1	=	0.852
	UPN	=	a2	=	0.0
	NPP	=	a3	=	0.487
	GNPP	=	a4	=	0.436
	PRP	=	a5	=	0.524

Another set of a_i's was also recovered which indicate the presence of different local minima in the least squares approximation, the two sets were approximately complementary as would be expected since as already stated the objective is unchanged by replacing each "a" by 1-a.

B	NPN	=	a1	=	0.148
	UPN	=	a2	=	1.000
	NPP	=	a3	=	0.513
	GNPP	=	a4	=	0.564
	PRP	=	a5	=	0.470

The above a_i's determined our choice for the initial probabilities that were used for the simulation exercise of Chapter 4.

5.2.2 CLASSICAL SHAPLEY RESULTS

Both the Classicial Shapley results already obtained in Chapter 4 are listed again here.

		PARTY AS PIVOT		INDIVIDUAL AS PIVOT	
Party	Seats	Val. of Party	Val. of indiv. in Party	Val. of Party	Val. of indiv. in Party
NPN	37	0.400	0.0108	0.389	0.0105
UPN	28	0.233	0.0083	0.295	0.0105
NPP	15	0.233	0.0155	0.158	0.0105
GNPP	8	0.067	0.0083	0.085	0.0105
PRP	7	0.067	0.0095	0.074	0.0105

The calculations were also carried out with the amended number of Senate seats for the NPN and NPP and the following results were obtained.

		PARTY AS PIVOT		INDIVIDUAL AS PIVOT	
Party	Seats	Val.of Party	Val.of Indiv. in Party	Val.of Party	Val.of Indiv. in Party
NPN	36	0.400	0.0111	0.378	0.0105
UPN	28	0.233	0.0083	0.294	0.0105
NPP	16	0.233	0.0145	0.168	0.0105
GNPP	8	0.067	0.0083	0.084	0.0105
PRP	7	0.067	0.0095	0.074	0.0105

Classical Shapley results indicate that the middle party and to a small extent the largest party would be more powerful if they remained completely homogeneous voting each way each time. We hope to validate this suggestion in our summary of the political situation in Nigeria.

5.2.3 OWEN'S MODIFICATION RESULTS

TABLE 4

Party	Point on the Circle	Seats	Value of Party	Value of Individual members
NPN	180	37 or 36	0.2694	0.0072
UPN	0.0	28	0.2306	0.0082
NPP	97.0	15 or 16	0.4999	0.0333
GNPP	119.0	8	0.0	0.0
PRP	54.0	7	0.0	0.0

The same results were obtained for NPN = 37 or 36 and NPP = 15 or 16. The above result is similar to the result obtained from the simulation exercise giving some indication of the reproducibility of values. It indicates that the middle party is rather more powerful due to its tendency to vote with either of the two major opposition parties with a probability of about .5. This model though makes some useful predictions but is inclined to exaggerate the values of the players as a result of the numerous "0" values which is caused by the theoretical base of the model which is the circle.

5.2.4 RESULTS FROM DIRECT APPROACH MODEL

Party	a_i's	Seats	Value of Party	Value of Indiv. in Party	Seats	Value of Party	Value of Indiv. in Party
NPN	0.852	37	0.372	0.0101	36	0.358	0.0099
UPN	0.0	28	0.309	0.0110	28	0.313	0.0112
NPP	0.487	15	0.159	0.0106	16	0.169	0.0106
GNPP	0.436	8	0.085	0.0106	8	0.085	0.0106
PRP	0.524	7	0.074	0.0106	7	0.074	0.0106

Calculations were done with data based on both NPN = 37, and NPN = 36 and also NPP = 15 and NPP = 16. The difference in values was very insignificant. The results are as reported above.

The results show that the middle parties have gained slightly more power than their values via the Classical Shapley approach, while the major parties lose or gain according to their attitude to the minor parties.

5.2.5 Direct Approach - Group Concept

When these parties were analysed through the group concept of 3.2.3 of Chapter 3 with same a_i's the following values were calculated.

Party	a_i's	Seats	Value of Party	Value of Indiv in Party	Seats	Value of Party	Value of Indiv. in Party
NPN	0.852	37	0.351	0.0094	36	0.348	0.0096
UPN	0.0	28	0.291	0.0103	28	0.286	0.0102
NPP	0.487	15	0.221	0.0147	16	0.233	0.0147
GNPP	0.436	8	0.077	0.0096	8	0.078	0.0098
PRP	0.524	7	0.57	0.0081	7	0.057	0.0081

The values of the major middle party namely NPP due to its voting tendencies is seen to appreciate considerably. We shall now present a summary of the values for the House of Representatives and the local Houses of Assembly.

5.3 RESULTS FROM THE HOUSE OF REPRESENTATIVES

From the election results presented in 5.1 it is clear that the voting pattern in Nigeria is along ethnic and tribal lines, thus a state supports a party in all the legislative bodies on the same scale so that the ratio of legislators in the House of Representatives from a party is similar to the ratio in the Senate as also reflected by the number of Governors from the different parties. Analysis of the House of Representatives was done using the same a_i's as calculated from the Senate. The results were as follows:

HOUSE OF REPRESENTATIVES

Parties	Seats	Direct Approach Vi's (Individual)	Direct Approach Vi's (Grouping)	Vi's from Shapley (weighted)
NPN	167	0.373	0.360	0.400
UPN	111	0.245	0.300	0.233
NPP	79	0.176	0.174	0.233
GNPP	43	0.096	0.073	0.067
PRP	49	0.109	0.093	0.067

Owen's model gave the same results as the Senate and for Shapley (Individual Pivot), the value would be proportional to the weights. The trend is similar to the Senate.

5.4 LOCAL HOUSES OF ASSEMBLY - VALUES

To complete the picture a distribution of powers between the various parties in all the states is presented as calculated via the Direct Approach (General) model using the following a_i's NPN = a1 = 0.160 UPN = a2 = 0.940, NPP = a3 = 0.449, GNPP = a4 = 0.572 and PRP = a5 = 0.459 as in (b) of 4.4.1. These were estimated using the simulated data since data on voting situations from the different states is very difficult to come by. The values in some cases could be very different if the a_i's from the States voting situations were used because some local parties remain in direct opposition at the state level, while their counterparts at the national level cooperate. These instances are of course not very common. Data for the seats in the different Houses of Assembly was as contained in Table 1 of 5.1 (State Houses of Assembly results). The distribution now follows.

PARTIES' VALUES - Individual values in brackets ( )

STATES	NPN	UPN	NPP	GNPP	PRP
Anambra	.1212 (.0121)	0.0 (0.0)	.8551 (.0114)	.0112 (.0112)	0.0 (0.0)
Bauchi	.8591 (0.0191)	0.0 (0.0)	.0333 (0.0083)	.0625 (0.0069)	.0184 (0.0092)
Bendel	.1534 (.0070)	.8074 (.0231)	.0323 (.0108)	0.0 (0.0)	0.0 (0.0)
Benue	.9642 (.0219)	0.0 (0.0)	.0184 (.0061)	.0173 (.0043)	0.0 (0.0)
Borno	0.1497 (.0136)	0.0 (0.0)	0.0 (0.0)	.8230 (.0137)	0.0137 (0.0137)
Cross River	.7971 (0.0140)	.0431 (0.0054)	.0245 (.0082)	.1175 (.0073)	0.0 (0.0)
Gongola	.2289 (.0135)	.3282 (.0182)	.0154 (.0154)	.4123 (.0159)	.0152 (.0152)
Imo	.0916 (.0114)	0.0 (0.0)	.8751 (.0109)	.0216 (.0108)	0.0 (0.0)
Kaduna	.8143 (.0120)	.0209 (.0042)	.0258 (.0065)	.0627 (.0057)	.0763 (.0069)

Cont.

STATES	NPN	UPN	NPP	GNPP	PRP
Kano	.1015	0.0	0.0	.0138	.8765
	(.0078)	(0.0)	(0.0)	(.0069)	(.0071)
Kwara	.6582	.2699	0.0	.0438	0.0
	(.0263)	(.0180)	(0.0)	(.0219)	(0.0)
Lagos	0.0	1.0	0.0	0.0	0.0
	(0.0)	(.0277)	(0.0)	(0.0)	(0.0)
Niger	.9427	0.0	0.0	.0003	0.0
	(.0337)	(0.0)	(0.0)	(.0002)	(0.0)
Ogun	0.0	1.0	0.0	0.0	0.0
	(0.0)	(.0277)	(0.0)	(0.0)	(0.0)
Ondo	0.0	1.0	0.0	0.0	0.0
	(0.0)	(.0151)	(0.0)	(0.0)	(0.0)
Oyo	0.0	1.0	0.0	0.0	0.0
	(0.0)	(.0085)	(0.0)	(0.0)	(0.0)
Plateau	.2466	0.0	.6720	.0577)	0.0
	(.0224)	(0.0)	(.0198)	(.0192)	(0.0)
Rivers	.7900	0.0	.1732	0.0	0.0
	(0.0272)	(0.0)	(0.0133)	(0.0)	(0.0)
Sokoto	.8968	0.0	0.0	.1032	0.0
	(.0106)	(0.0)	(0.0)	(.0040)	(0.0)

The above completes the picture of the political situation in Nigeria. NPN the ruling party is more widespread and wherever they appear they tend to command a lot of power. They may all well succeed in obtaining an overwhelming majority if they are able to build on their present powers.

An analysis of the effect of the values on the political situation based on the Senate calculations now follows. The Senate as pointed out earlier seems to reflect the trend of events in the whole political spectrum of Nigeria.

5.5 EFFECT OF VALUES (SENATE) ON POLITICAL SITUATION IN NIGERIA

PARTY	Seats	Values from Classical Shapley Party-Pivot	Val. from Classical Shapley Indiv. as pivot	Values from Owen	Val. from Direct Approach General	Val. from Direct Approach Grouping	No. of Cabinet positions held	No. of non-Cabinet posts held
NPN	36	0.389	0.378	0.2694	0.358	0.348	19	13
UPN	28	0.295	0.294	0.2306	0.313	0.286	0	0
NPP	16	0.158	0.168	0.4999	0.169	0.233	5	4
GNPP	8	0.084	0.084	0.0	0.085	0.0 78	0	0
PRP	7	0.074	0.074	0.0	0.074	0.057	0	0

As stated earlier in 5.1, the Nigerian President, Alhaji Shehu Shagari postponed the inaugural session of the National Assembly which was originally scheduled for Oct.2nd 1979 until Oct. 9th 1979 in order to give his party the National Party of Nigeria (NPN) the chance of forming a coalition in order to have a working majority in the Senate as well as the House of Representatives. The NPN succeeded in forming a 'quasi-coalition' with the middle party, the Nigerian Peoples Party NPP which they referred to as a "co-operation agreement". This enabled the NPN federal Government to have an effective working majority of 52 out of 95 in the Senate and 246 out of 449 in the House of Representatives as pointed out earlier. Our Direct Approach calculations show that the ratio of the value of NPP to the value of NPN is 0.358 : 0.169 = .472 in the individual calculation technique and 0.348 : 0.233 = 0.670 for the grouping case. In this instance the results obtained from the grouping are more vital since a permanent coalition arrangement is being worked out.

In contrast the ratios for the other models are .406 and .444 for Shapley and 1.856 for Owen, so this model is intermediate in its estimate of a small party's value between Shapley and Owen. It is reasonable to compare the ratios with the distribution of influential positions. The distribution of cabinet positions show that out of 24 Cabinet positions, the NPN with a Direct Approach value of 0.358 had 19 while NPP with 0.169 had only 5 and out of 17 non-Cabinet ministerial positions, NPN had 13, while NPP had only 4. NPP in addition had no special presidential advisers. The ratios are .26 for Cabinet and .31 for non-Cabinet. Such an arrangement where a party receives much less than its value in a coalition arrangement is not expected to last. As a result the cooperation agreement between NPN and NPP came to an end and all NPP Cabinet and non-Cabinet appointees resigned except the few who decided to leave their Party and remain in the NPN Government, for example, Professor Ishaya Audu, External affairs minister. It seems clear therefore that this new model could serve as a useful guide to political parties, Governments, committees and any organisation that has a political structure in helping the players to take decisions with respect to co-operation, coalition, etc.

when determining the allocations due to different individuals, parties or organisations within such scheme.

Further variations of the a_i's were carried out in order to study the effect of changes in attitude on the Nigerian political parties' values with respect to the Nigerian Senate. The results obtained were similar to those obtained in the simulation of Chapter 4.

We can, therefore, infer that the best course of action for a middle party is to remain united and to have a flexibility with respect to association with major opposition parties. Major parties should seek coalition with minor parties in order to achieve working majorities but in doing so must guarantee payoffs not less than the Shapley value of the co-operative minor parties. The minor parties may in fact be given more than their due in order to keep them in the coalition and for the sake of stability. The calculation for such a value can be based on any of the three techniques discussed. Owen would give an exaggerated result which the parties involved should use as their optimum bargaining point. Classical Shapley will give a conservative result which the parties should regard as their minimum bargaining point while the Direct Approach model will give an equitable mid way result since it takes all possible parameters into consideration. Long lasting coalitions should resort to the group concept but a one off coalition which gets dissolved as soon as a bill is passed or their aim achieved should resort to the general concept whereby individual participation is paramount.

Applications of the Direct Approach model to other voting systems now follow .

5.6 Application of Direct Approach Model to other Voting Systems

A. U.S.A.

An attempt was made to apply the Direct Approach model to the situation in the U.S. The major handicap was data but as stated earlier the a_i's can be estimated from different sources, including utterances of the players newspaper reports and sample surveys.
In 1966 during the Presidency of Johnson, the average Democrat voted with the majority of his party against the majority of Republicans only sixty-one per cent of the time, while the average Republican voted with the majority of his party against a majority of Democrats sixty-seven per cent of the time, Vile, M.J.C. (1976 page 149). Some members of the House of Representatives and the Senate were more often in opposition to a majority of their party than in agreement with it. This was the era referred to as the period of 'Conservative coalition' which may still exist to some extent presently. The split in the Democratic party was a split between Northern and Southern Democrats. This made it possible for the Southern Democrats to vote against a majority of northern Democrats in line with the Republicans and thus certain legislations were checked e.g. Civil rights bills, Vile, M.J.C. (1976). The voting attitude of American legislators is controlled by several factors other than party allegiance. These include the attitudes of the constituents towards a particular legislation, loyalty to administration, effect of pressure groups, as well as personality factors. This type of set up produces a fluidity in voting patterns and the slackness of party ties and as a result gives the American political committee system a vitally important role.
From sources such as the percentage of voting pattern quoted above it is possible to estimate our a_i's and in such fluid voting situations

the result would be very close to that where each party had its a_i's to be 0.500 nevertheless the a_i for the Senate and House of Representatives for that year was estimated to be as follows: 0.390 for the Democratic party and 0.670 for the Republican party. In order to study the variation in values had their voting attitudes been different, other sets of a_i's were used and values calculated therefrom.

D = DEMOCRATS : R = REPUBLICANS

ai's \ CASES	1	2	3	4
a_i's				
D:	.390	.500	.390	.900
R:	.670	.500	.900	.330

Case 1 reflects the 1966 Voting situation. Case 2 reflects the situation whereby the legislators voted without bias which is similar to the Classical Shapley individual pivot case while cases 3 and 4 were used to determine what would have happened to the values of the legislators had, (a) the Democrats maintained their 1966 voting attitude while the Republicans voted almost as a block and (b) the Republicans maintained their 1966 voting attitude while the Democrats voted almost as a block as represented by the a_i's for 3 and 4 respectively.

The following values were calculated for the Senate and the House of Representatives, individual values are in brackets.

SENATE : D = 67, R = 33

VALUES	DEMOCRATS	REPUBLICANS
CASES		
1	.6794 (.0101)	.3206 (.0097)
2	.6634 (.0099)	.3267 (.0099)
3	.6323 (.0094)	.3592 (.0109)
4	.9267 (.0138)	.0711 (.0022)

HOUSE OF REPRESENTATIVES : D = 295, R = 140

CASES	VALUES	DEMOCRATS	REPUBLICANS
1		.6833 (.00232)	.3166 (.00226)
2		.6782 (.00230)	.3218 (.00230)
3		.6443 (.00218)	.3557 (.00254)
4		.9576 (.00325)	.0424 (.00030)

As stated earlier the values for case 1 is very close to the Classical Shapley value where individuals occupy pivot positions as calculated via the Direct Approach model by assigning a_i = 0.500 to each of the parties as reflected in case 2 which is used as a yardstick to determine where a party has increased or decreased in value.

Case 3 indicates that the Republicans would have increased their value by voting together on one side more often than they did realising that the party was less than ½ the Democratic party both in the house of representatives and the Senate.

Case 4 indicates that the Democratic party would have succeeded in reducing the Republic party to "dummies" or close to dummies by voting together on one side more often than they did; Because of their voting attitude their power as calculated in Case 1 did not reflect their overwhelming majority. It must be pointed out that only simply majority cases are considered as stipulated in Chapter 3. The above analysis clearly shows how useful the model presented in Chapter 3 can be with respect to the analysis of powers.

B. Application of the Direct Approach model to the EEC

The EEC as at 1973 had 9 member States. Most decisions are expected to be a consensus of all the member States but the members of the Council on proposals from the Commission had different weights attached to them as shown in the following table. Brams,S.J.(1976). The Commission is a collegiate body of 13 individual members, chosen by member states,which serves as the administrative arm of the Council, the main decision-making body. Action by the 1973 Council on proposals from the Commission required a qualified majority of 41 out of 58. Our model is designed for simple majority minimal winning cases but can nevertheless give an idea as to the values of the members. We do require details of voting situations in order to estimate our a_i's but in the absence of that we can make estimates of the a_i's using what we can gather from the interactions between member States e.g. the case of the sale of agricultural products, the fishing rights problem etc. The following a_i's were estimated and the result from the direct approach calculation is given as compared to the results from the Banzhaf model as calculated by Brams, S.J. and contained in Brams, S.J. (1976 page 184).

States	Weight	Banzhaf Index	a_i's	Direct Approach
France	10	.167	0.1	.185
Germany	10	.167	0.5	.169
Italy	10	.167	0.5	.169
Belgium	5	.091	0.5	.085
Netherlands	5	.091	0.5	.085
Luxembourg	2	.016	0.2	.036
Denmark	3	.066	0.5	.057
Ireland	3	.066	0.2	.054
U.K.	10	.167	0.9	.148

The Direct Approach values seem more reasonable than the Banzhaf values. It seems clear from observation that France and Germany command a lot of power in the EEC, so does Italy; and certainly, any model that allocates the same amount of power via value calculations to France, Germany, Italy and the U.K. is not very realistic, thus the values calculated via the Direct Approach model seem to show how powerful and useful the Direct Approach model can be.

C. Application to the U.N.

A vote in the U.N. presently can have more than one meaning. A "yes" vote can mean support, it can also mean that one does not like the bill at all but finds it inconvenient to distinguish himself by voting against it. Only a "no" vote still keeps its unambiguity, Kaufman, J. (1980).

There are some geographical subdivisions in the U.N. which could be regarded as electoral groups since most proposals go through these groups before they come before the floor of the General Assembly. The groups are composed as shown in the table below. It is also difficult to collect enough material and then to estimate the a_i's for the different groups but the U.N. Year Book 1978 provided some material for this. The results from the calculations based on the a_i's estimated from such data is given below.

It must be pointed out that most recent decisions in the U.N. are now being adopted without votes, e.g. in 1978 54% of the decisions taken were done without votes, Kaufman, J. (1980 page 210). The system is now working towards compromise situations which in effect produce consensus rather than voting. Also the values calculated are not very representative of the Powers of the different member states since the existence ofthe Security Council with enormous powers due to the possession of veto power makes the permanent members of the Security Council able to have real powers which are out of proportion as compared to other States. The results from our Direct Approach calculations without consideration for decisions that require 2/3 majorities and Security Councils approvals now follow -

U.N.electoral Group	Number	%	a_i's	Actual number of People Representatives (in millions)	Vi for groups
African Group	50	33	.800	434.78	0.371
Asian Group	39	26	.600	2326.90	0.236
Latin American group (including States of the Carribean area)	29	19	.500	339.52	0.159
Socialist States of Eastern Europe	11	7	0.0	394.35	0.042
Western Europe & others (includes Australia, Canada, New Zealand & USA)	23	15	.890	642.82	0.188

The above results are reasonable. For example, the Socialist States of Eastern Europe are about half the Western European States, precisely 11:23 = .478 having a percentage ratio of 7:15 yet the powers calculated via the Direct Approach model allocates powers in the following ratio .042 : 0.188 = .223. This ultimately gives an indication of how powerful the Western European Countries and the U.S. are in the U.N. The power allocation is very reasonable and the model is useful. The above power allocations are not very representative of the powers of the different member States because the Security Council membership was not considered in these calculations as pointed out above.

5.7 CONCLUSION

It can be concluded that the model just presented is a useful tool for analysing the power or value of any group or individual concerned with a system that involves yes and no votes. It is superior to the other models because it is dynamic as a result of the consequence of varying the a_i's (probabilities of association). It can therefore be used to study in detail and in advance the behaviour of any system that has the political voting character with respect to determining all the possible occurrences that may take place in the event of a bill or a voting situation. The major outstanding advantage is of course the inclusion of the probabilities of all the individual players concerned in its calculation.

The analysis of the practical situations shows that this model can be used in almost all circumstances whether data was available or in short supply. When data is not readily available the small number of situations that can be obtained either by random sampling of opinion or from past voting situations can then be used to calculate the probability parameters.

It is therefore clear that having applied this model successfully to simulated and practical data it can therefore be claimed to be a useful extension to the Shapley value concept since its theoretical base is centred around the classical Shapley concept.

An alternative approach which was evolved in the course of this work will be presented in appendix B. Appendix A will contain the mathematical derivation of the conditional expectation function used in Chapter 3. Appendices C and D will contain details of computer techniques used in the application of Owen's modification and computerised extension. Appendix E will contain an extension of the Owen concept to Oceanic games and the project will be concluded with the computer programs used and the usual bibliography.

APPENDIX A

Derivation of Conditional Expectation Function used in General Direct Approach model of Chapter three.

In the generalised Direct Approach model of Chapter three, the conditional expectation of X_i normally distributed (μ_i, σ_i^2) subject to

$\Sigma X_i = \sum_i \mu_i + K$ was stated to be $\mu_i + \dfrac{K\sigma_i^2}{\sum_i \sigma_i^2}$

The derivation for three variables is given.

Probability density function (Pdf)

$$P(X_1 X_2 X_3) = \frac{1}{(2\pi)^{3/2}\sigma_1\sigma_2\sigma_3} \exp - \frac{1}{2}\left\{\frac{(X_1-\mu_1)^2}{\sigma_1^2} + \frac{(X_2-\mu_2)^2}{\sigma_2^2} + \frac{(X_3-\mu_3)^2}{\sigma_3^2}\right\}$$

The P.d.f of $X_1 + X_2 + X_3 = z$ is

$$\frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2+\sigma_3^2}} \exp - \frac{1}{2(\sigma_1^2+\sigma_2^2+\sigma_3^2)}(z-\Sigma\mu)^2$$

We change the variables to

$$\begin{aligned} U_1 &= X_1 - \mu_1 & X_1 &= U_1 + \mu_1 \\ U_2 &= X_2 - \mu_2 & X_2 &= U_2 + \mu_2 \\ U_3 &= X_1 + X_2 + X_3 - \mu_1 - \mu_2 - \mu_3; & X_3 &= U_3 - U_1 - U_2 + \mu_3 \end{aligned}$$

So
$$\begin{pmatrix} X_1 - \mu_1 \\ X_2 - \mu_2 \\ X_3 - \mu_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}$$

and the exponent
$$(X_1 - \mu_1)(X_2 - \mu_2)(X_3 - \mu_3) \begin{pmatrix} \frac{1}{\sigma_1^2} & & 0 \\ & \frac{1}{\sigma_2^2} & \\ 0 & & \frac{1}{\sigma_3^2} \end{pmatrix} \begin{pmatrix} X_1 - \mu_1 \\ X_2 - \mu_2 \\ X_3 - \mu_3 \end{pmatrix}$$

$$= (\underline{X} - \underline{\mu})^T V^{-1} (\underline{X} - \underline{\mu})$$

because

$$(U_1\ U_2\ U_3)\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{\sigma_1^2} & & 0\\ & \frac{1}{\sigma_2^2} & \\ 0 & & \frac{1}{\sigma_3^2}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -1 & -1 & 1\end{pmatrix}\begin{pmatrix}U_1\\ U_2\\ U_3\end{pmatrix}$$

$$= (U_1\ U_2\ U_3)\begin{pmatrix}\frac{1}{\sigma_1^2}+\frac{1}{\sigma_3^2} & \frac{1}{\sigma_3^2} & -\frac{1}{\sigma_3^2}\\ \frac{1}{\sigma_3^2} & \frac{1}{\sigma_2^2}+\frac{1}{\sigma_3^2} & -\frac{1}{\sigma_3^2}\\ \frac{1}{\sigma_3^2} & -\frac{1}{\sigma_3^2} & \frac{1}{\sigma_3^2}\end{pmatrix}\begin{pmatrix}U_1\\ U_2\\ U_3\end{pmatrix}$$

So probability $(U_1U_2U_3) = \dfrac{1}{(2\pi)^{3/2}\sigma_1\sigma_2\sigma_3}\ exp - \frac{1}{2}\ \underline{U}\bar{V}^{-1}\underline{U}\ du_1\, du_2\, du_3$

and conditional probability $/\ U_3 = K$ is

$$\frac{1}{2\pi}\ \sqrt{\sigma_1^2+\sigma_2^2+\sigma_3^2}\ \ \exp - \frac{1}{2}\ \underline{U}\bar{V}^{-1}\underline{U} + \frac{1}{2}\ \frac{K^2}{(\sigma_1^2+\sigma_2^2+\sigma_3^2}\ )$$

Now consider

$$(U_1\ U_2\ K)\begin{pmatrix}\frac{\sigma_1^2+\sigma_3^2}{\sigma_1^2\sigma_3^2} & \frac{1}{\sigma_3^2} & -\frac{1}{\sigma_3^2}\\ \frac{1}{\sigma_3^2} & \frac{\sigma_2^2+\sigma_3^2}{\sigma_2^2\ \sigma_3^2} & -\frac{1}{\sigma_3^2}\\ -\frac{1}{\sigma_3^2} & -\frac{1}{\sigma_3^2} & \frac{1}{\sigma_3^2}\end{pmatrix}\begin{pmatrix}U_1\\ U_2\\ K\end{pmatrix} - \frac{K^2}{\sigma_1^2+\sigma_2^2+\sigma_3^2}$$

$$= \left(\frac{\sigma_1^2 + \sigma_3^2}{\sigma_1^2\ \sigma_3^2}\right) U_1^2 + \left(\frac{\sigma_2^2 + \sigma_3^2}{\sigma_2^2\ \sigma_3^2}\right) U_2^2 + K^2\left(\frac{1}{\sigma_3^2} - \frac{1}{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}\right) + \frac{2U_1U_2}{\sigma_3^2} - \frac{2U_1K}{\sigma_3^2} - \frac{2U_2K}{\sigma_3^2}$$

The exponent can be written as

$$(U_1 - m_1)(U_2 - m_2) \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} U_1 - m_1 \\ U_2 - m_2 \end{pmatrix} = a_{11}(U_1 - m_1)^2 + 2a_{12}(U_1 - m_1)(U_2 - m_2)$$

$$+ a_{22}(U_2 - m_2)^2$$

So (i) $a_{11} = \dfrac{\sigma_1^2 + \sigma_3^2}{\sigma_1^2 \sigma_3^2}$ $\quad a_{22} = \dfrac{\sigma_2^2 + \sigma_3^2}{\sigma_2^2\ \sigma_3^2}$ $\quad a_{12} = \dfrac{1}{\sigma_3^2}$

$$\det A = \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{\sigma_1^2\ \sigma_2^2\ \sigma_3^2} = a_{11}a_{22} - a_{12}^2 = \Delta$$

(ii) $\left.\begin{aligned} a_{11}m_1 + a_{12}m_2 &= \frac{K}{\sigma_3^2} \\ a_{12}m_1 + a_{22}m_2 &= \frac{K}{\sigma_3^2} \end{aligned}\right\}$ so that $\begin{aligned} \Delta m_1 &= \frac{K}{\sigma_3^2}(a_{22} - a_{12}) = \frac{K}{\sigma_3^2} \cdot \frac{1}{\sigma_2^2} \\ \Delta m_2 &= \frac{K}{\sigma_3^2}(a_{11} - a_{12}) = \frac{K}{\sigma_3^2} \cdot \frac{1}{\sigma_1^2} \end{aligned}$

Hence the conditional expectation of U_1 is m_1, U_2 is m_2

So $E(U_1) = m_1 = \dfrac{K\ \sigma_1^2}{(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)}$ So $E(X_1) = \mu_1 + \dfrac{K\ \sigma_1^2}{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}$

$$E(U_2) = m_2 = \frac{K\sigma_2^2}{\sigma_1^2+\sigma_2^2+\sigma_3^2}\, E(X_2) = \mu_2 + \frac{K\sigma_2^2}{\sigma_1^2+\sigma_2^2+\sigma_3^2}$$

$$U_3 = K, \text{ So } E(X_3) = K - E(U_1) - E(U_2) + \mu_3$$

$$= K\left[1 - \frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2+\sigma_2^2+\sigma_3^2}\right] + \mu_3$$

$$\text{So } E(X_3) = \mu_3 + \frac{K\sigma_3^2}{\sigma_1^2+\sigma_2^2+\sigma_3^2}$$

Thus $E(X_1 + X_2 + X_3) = \mu_1 + \mu_2 + \mu_3 + K$

Hence given $X_1 \quad \mu_1 \sigma_1^2$

$X_2 \quad \mu_2 \sigma_2^2$

- - - - -

$X_n \quad \mu_n \sigma_n^2$

Thus

We have conditional expectations subject to $\sum_i X_i = \sum_i \mu_i + K$ to be

$$\mu_i + \frac{K\sigma_i^2}{\sum_i \sigma_i^2} \qquad \text{as required.}$$

An alternative approach to the model presented in Chapter 3 will be presented in the next Appendix.

APPENDIX B

THE STRAIGHT LINE APPROACH

We now present an alternative approach to the Direct approach model. The technique has great flexibility as will be seen shortly and is closely related to the classical Shapley concept, yet the cumbersome calculations involved in deriving the value made it difficult for application to more than three participants, nevertheless, we recommend it for further research.

We looked at the relationship between different parties from the direction of a linear model whereby each party was assigned some length of a straight line within which its members had freedom of movement. Allowance was made for overlaps in order to permit members of one group to cooperate with members of a different group.

Thus, let X_1, X_2 and X_3 be three players such that $X_1 \in [0,1]$, $X_2 \in [\alpha,1+\alpha]$, $X_3 \varepsilon [\beta,1+\beta]$ and $0 \leqslant \alpha \leqslant \beta \leqslant 1$ and $1 + \beta < 2$. With the above arrangement X_1 can be the "Pivot" player if he was somewhere between X_2 and X_3 and the same for X_2 and X_3. Thus the sum of the probability of X_1 being in between X_2 and X_3 would then be the value of X_1. This will correspond to the orderings $2\overset{*}{1}3$ and $3\overset{*}{1}2$. This could be likened to a voting situation whereby either X_2 or X_3 would vote on the same side with X_1 because the view initiated and held by X_1 is acceptable on the average to the views held by either X_2 or X_3 or both. The above probability could then be calculated from the areas occupied by the orderings $2\overset{*}{1}3$ and $3\overset{*}{1}2$ which in this case will be three dimensional resulting in the calculation of the volume.

This implies triple integration, e.g. if X_1 in $[\beta, 1]$ the probability of having the ordering $1\overset{*}{2}3$ with 2 as the pivot is

$$\int_{\beta}^{1} dX_1 \int_{X_1}^{1+\alpha} dX_2 \int_{X_2}^{1+\beta} dX_3$$

Similar integrations would also be carried out for $2\overset{*}{1}3$, $2\overset{*}{3}1$, $3\overset{*}{2}1$, $3\overset{*}{1}2$ and $1\overset{*}{3}2$

For a formal derivation let us restate the above conditions.

Let $X_1 \varepsilon [0,1]$, $X_2 \varepsilon [\alpha, 1+\alpha]$, $X_3 \varepsilon [\beta, 1+\beta]$. and $1 + \beta < 2$

Four cases result.

(1) $0 \leqslant \alpha \leqslant \beta \leqslant 1 < 1 + \alpha$

(2) $0 \leqslant \alpha \leqslant 1 \leqslant 1 + \alpha \leqslant \beta$

(3) $0 \leqslant 1 \leqslant \alpha \leqslant \beta$

(4) $0 \leqslant \alpha \leqslant 1 \leqslant \beta \leqslant 1 + \alpha$

Consider (1) $0 \leqslant \alpha \leqslant \beta \leqslant 1$

We have three contributions to the integral,

(i) $X_1 < \alpha$ which implies that X_1 cannot be in the middle and thus cannot be a pivot player. (i) results in the orderings $1\overset{*}{3}2$ and $1\overset{*}{2}3$.

(ii) $\alpha \leqslant X_1 \leqslant \beta$ this results to three orderings as follows, $1\overset{*}{3}2$, $1\overset{*}{2}3$ and $2\overset{*}{1}3$

(iii) $\beta \leqslant X_1 \leqslant 1$ which results to six orderings $1\overset{*}{2}3$, $2\overset{*}{1}3$, $2\overset{*}{3}1$, $3\overset{*}{2}1$, $3\overset{*}{1}2$ and $1\overset{*}{3}2$. To each ordering we associate the probability of the middle player being the 'pivot' for that particular ordering.

For contribution (i) (Probability we denote by Pr. for brevity).

(1) $\quad$ Pr. $1\overset{*}{3}2 = \frac{1}{2}\alpha + \alpha^2 - \alpha\beta + \frac{1}{2}\alpha^3 - \alpha^2\beta + \frac{1}{2}\alpha\beta^2$

(2) $\quad$ Pr. $1\overset{*}{2}3 = \frac{1}{2}\alpha - \alpha^2 + \alpha\beta - \frac{1}{2}\alpha^3 + \alpha^2\beta - \frac{1}{2}\alpha\beta^2$

(3) Contribution (ii) Pr. $1\overset{*}{2}3 = -\frac{1}{2}\alpha + \frac{1}{2}\beta - \alpha\beta + \frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2 + \frac{3}{2}\alpha\beta^2$

$$- \frac{3}{2}\alpha^2\beta + \frac{1}{2}\alpha^3 - \frac{1}{2}\beta^3$$

(4) $\quad$ Pr. $2\overset{*}{1}3 = \frac{1}{2}\beta^2 - \alpha\beta + \frac{1}{2}\alpha^2$

(5) $\quad$ Pr. $\overset{*}{1}32 = -\frac{1}{2}\alpha + \frac{1}{2}\beta - \alpha^2 - \beta^2 + 2\alpha\beta + \frac{3}{2}\alpha^2\beta - \frac{1}{2}\alpha^3 - \frac{3}{2}\alpha\beta^2 + \frac{1}{2}\beta^3$

(6) Contribution (iii) Pr. $1\overset{*}{2}3 = \frac{1}{6} - \frac{1}{2}\alpha^2 - \frac{1}{2}\beta^2 + \alpha\beta + \frac{1}{2}\alpha^2\beta - \alpha\beta^2 + \frac{1}{3}\beta^3$

(7) $\quad$ Pr. $2\overset{*}{1}3 = \frac{1}{6} - \frac{1}{2}\alpha + \frac{1}{2}\beta - \frac{1}{2}\beta^2 + \frac{1}{2}\alpha\beta^2 - \frac{1}{6}\beta^3$

(8) $\quad$ Pr. $2\overset{*}{3}1 = \frac{1}{6} - \frac{1}{2}\alpha + \alpha\beta - \frac{1}{2}\beta^2 - \frac{1}{2}\alpha\beta^2 + \frac{1}{3}\beta^3$

(9) $\quad$ Pr. $3\overset{*}{2}1 = \frac{1}{6} - \frac{1}{2}\beta + \frac{1}{2}\beta^2 - \frac{1}{6}\beta^3$

(10) $\quad$ Pr. $3\overset{*}{1}2 = \frac{1}{6} + \frac{1}{2}\alpha - \frac{1}{2}\beta + \frac{1}{2}\beta^2 - \alpha\beta - \frac{1}{6}\beta^3 + \frac{1}{2}\alpha\beta^2$

and

(11) $\quad$ Pr. $1\overset{*}{3}2 = \frac{1}{6} + \frac{1}{2}\alpha - \frac{1}{2}\beta + \frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2 - \alpha\beta - \frac{1}{2}\alpha^2\beta + \frac{1}{2}\alpha\beta^2 - \frac{1}{6}\beta^3$

In order to determine the probability of any particular ordering with respect to case (1) we have to sum the probabilities of that particular ordering in all the situations where it contributes to the integral and to determine the probability that any player is pivotal we have to sum over all the situations where that particular player is pivotal as follows.

The probability of 3 as pivot for case 1

$= \text{Pr. } 2\overset{*}{3}1 \quad \text{Contribution (iii)} + \sum_{i=1}^{3} \text{Pr.} 1\overset{*}{3}2 \text{ (i = contributions)}$

$= \text{Pr. } 1\overset{*}{3}2 \quad \text{Contribution (i)} + \text{Pr. } 1\overset{*}{3}2 \quad \text{Contribution (ii)} + \text{Pr. } 1\overset{*}{3}2$
Contribution (iii) + $\text{Pr. } 2\overset{*}{3}1$ Contribution (iii)

(12) $$= \frac{1}{3} + \frac{1}{2}\alpha^2 + \alpha\beta - \beta^2 - \alpha\beta^2 + \frac{2}{3}\beta^3$$

Also Pr. 2 as pivot

$= \text{Pr. } 3\overset{*}{2}1 \quad \text{Contribution (iii)} + \sum_{i=1}^{3} \text{Pr. } 1\overset{*}{2}3 \text{ (i = contributions)}$

(13) $$= \frac{1}{3} - \alpha^2 + \alpha\beta + \frac{1}{2}\beta^2 - \frac{1}{3}\beta^3$$

Similarly Pr. 1 as pivot

$= \text{Pr. } 3\overset{*}{1}2 \quad \text{Contribution (iii)} + \sum_{i=1}^{3} \text{Pr.} 2\overset{*}{1}3$

(14) $$= \frac{1}{3} + \frac{1}{2}\alpha^2 - 2\alpha\beta + \frac{1}{2}\beta^2 + \alpha\beta^2 - \frac{1}{3}\beta^3$$

To determine the overall probability of any ordering therefore we have to consider the remaining three cases thus

Case (2) $0 \leqslant \alpha \leqslant 1 \leqslant 1 + \alpha \leqslant \beta$

We have two contributions to the integral

(i) $X_1 \leqslant \alpha$ and (ii) $1 \geqslant X_1 \geqslant \alpha$

For (i) we have only $\text{Pr. } \overset{*}{1}23 = \alpha$

(ii) we have $\text{Pr. } 2\overset{*}{1}3$ and $\text{Pr. } \overset{*}{1}23$

$\text{Pr } 2\overset{*}{1}3 = \frac{1}{2}(1-\alpha)^2$, So Pr. 1 as Pivot

(15) $$\frac{1}{2}(1-\alpha)^2$$

$\text{Pr. } 1\overset{*}{2}3 = -\frac{1}{2}\alpha^2 + \frac{1}{2}$ Thus

Pr. 2 as Pivot = Pr. $1\overset{*}{2}3$ Contribution (i) + Pr.$1\overset{*}{2}3$ Contribution (ii)
which gives

(16) $$= \alpha - \frac{1}{2}\alpha^2 + \frac{1}{2}$$

Also we have the third case where $0 \leqslant 1 \leqslant \alpha \leqslant \beta$ as stated above.

We can only have Pr. $1\overset{*}{3}2$ and Pr. $1\overset{*}{2}3$

(17) $$\text{Pr. } 123 = 1 - \frac{1}{2}(1 + \alpha - \beta)^2 = \text{Pr. 2 as Pivot}$$

and

(18) $$\text{Pr. } 1\overset{*}{3}2 = \frac{1}{2}(1 + \alpha - \beta)^2 = \text{Pr. 3 as Pivot}$$

Finally we have the Fourth Case when $0 \leqslant \alpha \leqslant 1 \leqslant \beta \leqslant 1 + \alpha$ we have two contributions to the integral

(i) $X_1 \leqslant \alpha$ (ii) $\alpha \leqslant X_1 \leqslant 1$

For (i) we have Pr.$1\overset{*}{3}2$ and Pr. $1\overset{*}{2}3$ and for (ii) we have Pr. $2\overset{*}{1}3$ and Pr.$1\overset{*}{3}2$ again. Thus total for Pr.$1\overset{*}{3}2$ = $\frac{1}{2}(1+ \alpha -\beta)^2$ which is the same as in Case 3 above and total for Pr. $1\overset{*}{2}3$ $= \beta - \alpha^2 + \alpha\beta - \frac{1}{2}\beta^2$ also Pr. $2\overset{*}{1}3 = \frac{1}{2} - \alpha + \frac{1}{2}\alpha^2 = \frac{1}{2}(1 -\alpha)^2$ same as in Case 2 above.

To determine the value of any player we have to consider the probability that such a player occupies a pivotal position in any of the orderings discussed above and work out the players value therefrom.

We summarise the above for clarity

1 as Pivot: we consider

(a) $0 \leqslant \alpha \leqslant \beta \leqslant 1 \leqslant 1 + \alpha$ which gives $\frac{1}{3} + \frac{1}{2}\alpha^2 - 2\alpha\beta + \frac{1}{2}\beta^2 + 2\beta^2 - \frac{1}{3}\beta^3$

(b) $0 \leqslant \alpha \leqslant 1 \leqslant 1 + \alpha \leqslant \beta$ " " $\frac{1}{2}(1-\alpha)^2$

(c) $0 \leqslant 1 \leqslant \alpha \leqslant \beta$ " " NONE

(d) $0 \leqslant \alpha \leqslant 1 \leqslant \beta \leqslant 1 + \alpha$ " " $\frac{1}{2}(1-\alpha)^2$ same as (b)

For 2 as Pivot we consider (a) - (d) as above

and get $\frac{1}{3} - \alpha^2 + \alpha\beta + \frac{1}{2}\beta^2 - \frac{1}{3}\beta^3$ from (a)

$\alpha - \frac{1}{2}\alpha^2 + \frac{1}{2}$ " (b)

$1 - \frac{1}{2}(1 + \alpha - \beta)^2$ " (c)

$\beta - \alpha^2 + \alpha\beta - \frac{1}{2}\beta^2$ " (d) and

similarly for 3 as pivot we get

$$\frac{1}{3} + \frac{1}{2}\alpha^2 + \alpha\beta - \beta^2 - \alpha\beta^2 + \frac{2}{3}\beta^3 \text{ from (a)}$$

None from (b)

$\frac{1}{2}(1 + \alpha - \beta)^2$ from (c)

and also $\frac{1}{2}(1 + \alpha - \beta)^2$ from (d) same as (c)

When $\alpha = \beta = 0$ we consider only (a) and we get $V1 + V2 = V3 = \frac{1}{3}$ = Classical Shapley value for 3 equal participants.

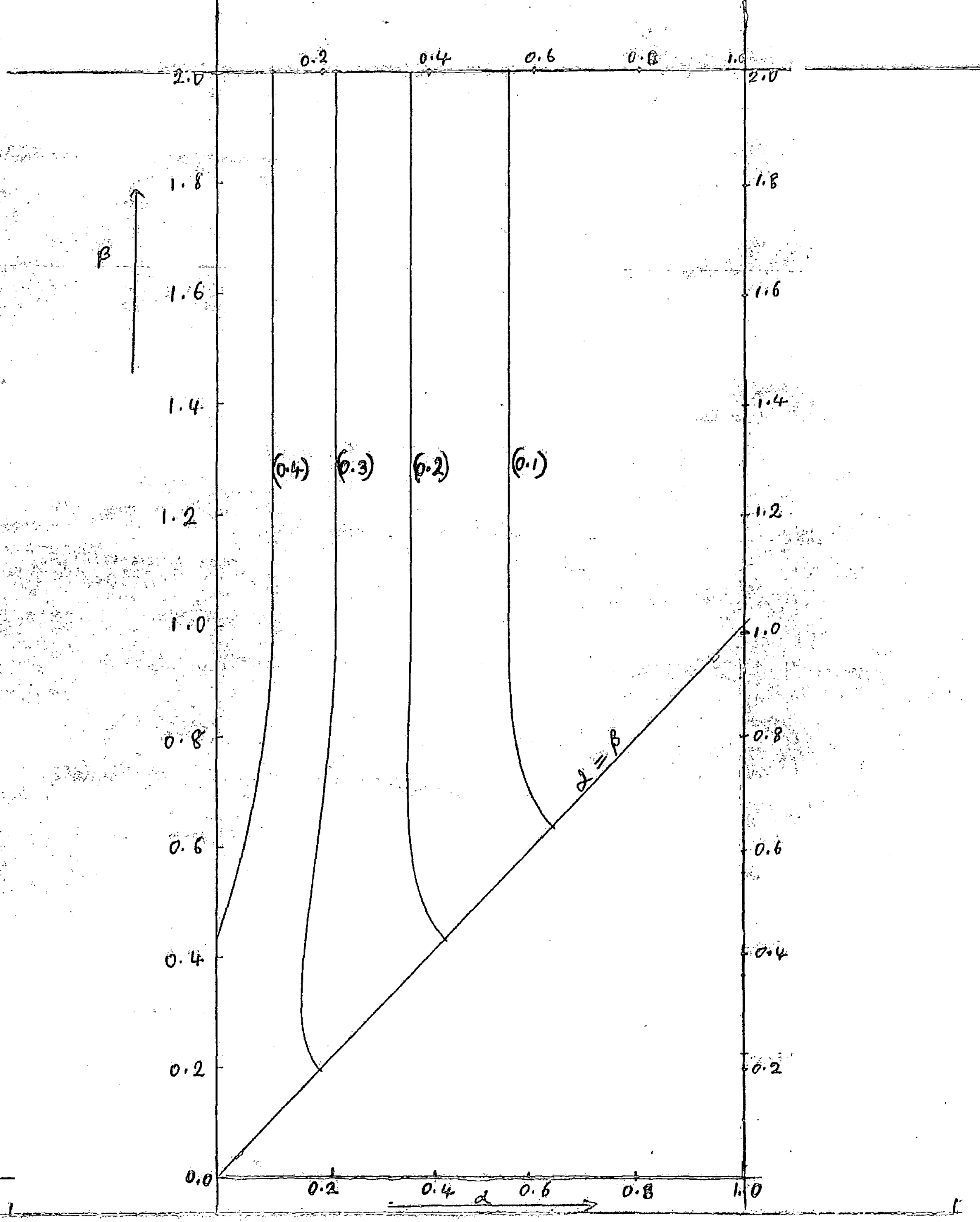
The value in this model will then vary according to the number α and β. The scheme has great flexibility because through α and β the players are allowed a great freedom which can be reflected by a Democratic voting system e.g. U.S. Senate where the politicians have great tendency to hold to their individual views resulting in the tendency to share several views in common with members of other political parties.

We realise that a practical application of this variation linear model will be fairly difficult because of the type of mathematical integrations involved, nevertheless we recommend further research into this line of thought and perhaps it will be possible to devise a numerical technique for tackling the Shapley value concept through this line of thought. A simulation technique is an alternative to integration but the size involved would be quite large.

Graphs showing the variations of the value of the players with variations in α and β for all the cases now follow after which details of the computerisation technique used for the analysis of values via Owen's modification will be presented in Appendix C and D.

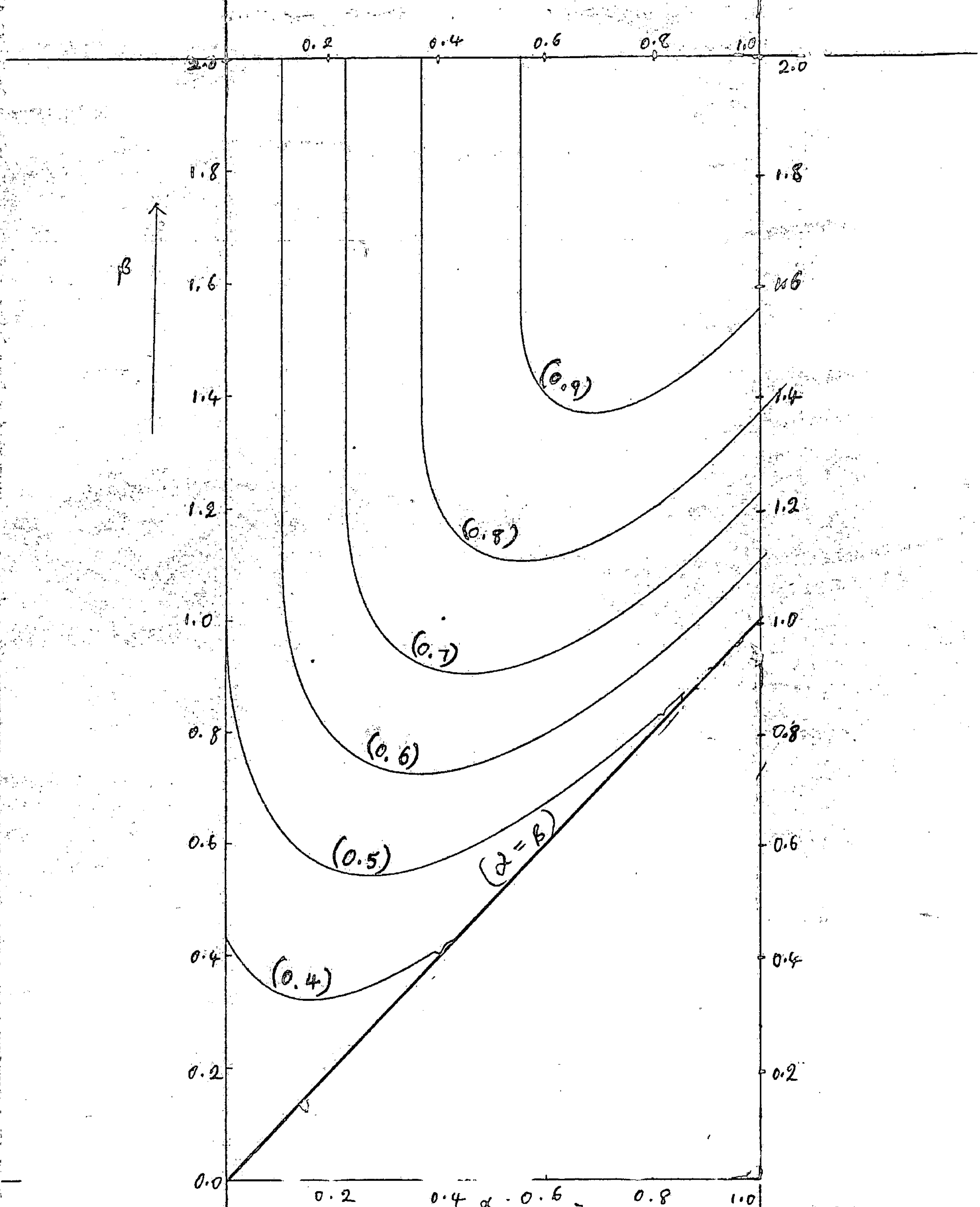
$X_1 \in [0,1]$, $X_2 \in [\alpha, 1+\alpha]$, $X_3 \in [\beta, 1+\beta]$
and $1+\beta < 2.0$

VALUES FOR X_1 AS PIVOT.



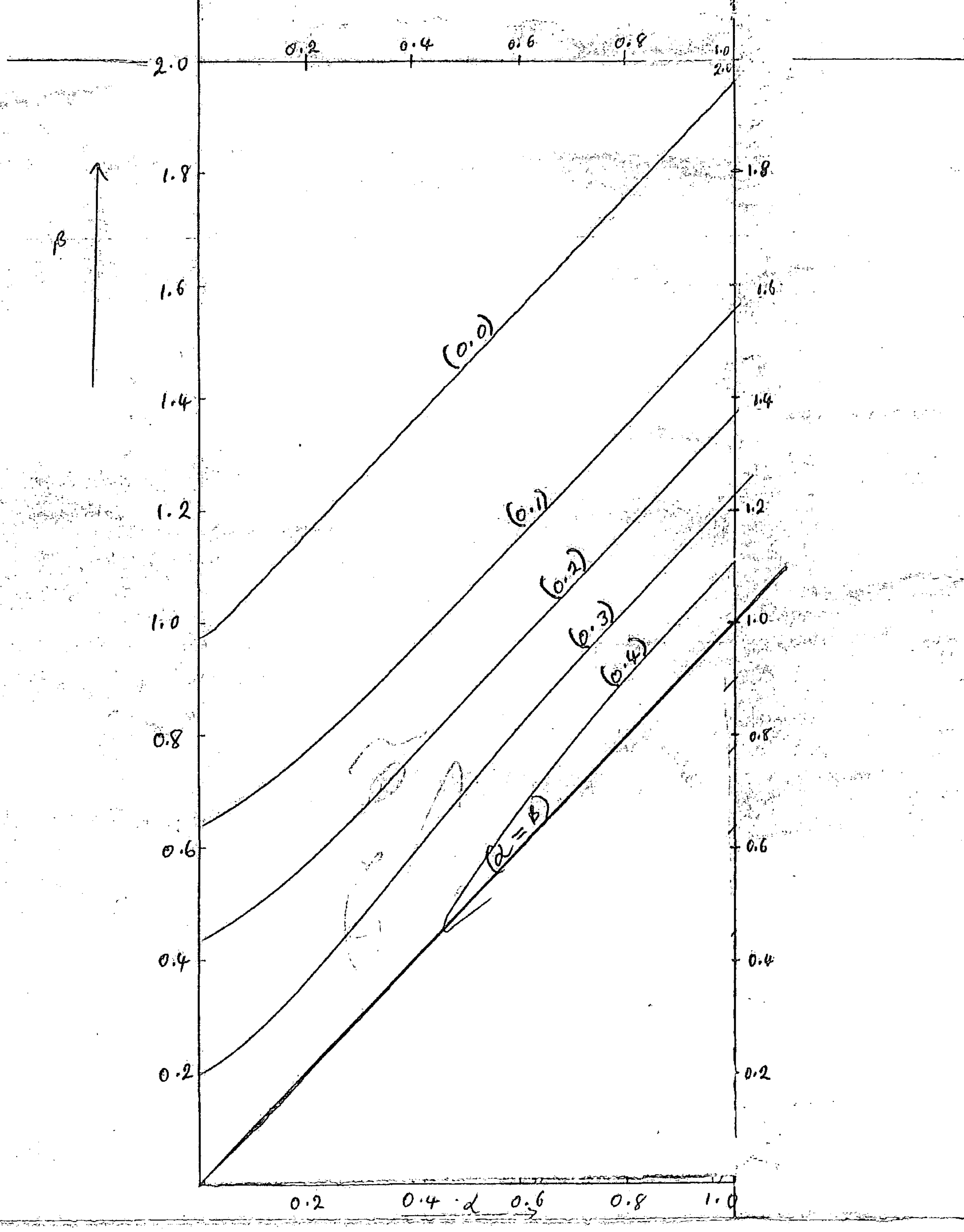
$X_1 \in [0,1]$, $X_2 \in [\alpha, 1+\alpha]$, $X_3 \in [\beta, 1+\beta]$
and $1+\beta < 2$

VALUES FOR X_2 as PIVOT.



$X_1 \in [0,1]$, $X_2 \in [\alpha, 1+\alpha]$, $X_3 \in [\beta, 1+\beta]$
and $1+\beta < 2$

VALUES OF X_3 as PIVOT



APPENDIX C

Details of Options and Commands used on the SPACES PACKAGE

The following are the details of options and commands used on the spaces package with respect to the multidimensional scaling of chapters 4 and 5 in an attempt to determine the overall relationship between different distinct groups of players as required by the Owen approach for input as distinct points round the circle or one half of it.

INIT: The init command helps to create an initial configuration and the option we used was Kruskal's arbitrary starting configuration with an indication that we needed the scaling to be done in one dimension. Kruskal, J.B. (1964b) has a good coverage on the general concepts of multidimensional scaling. The model used was originally invented by C.H. Coombs in 1950, Coombs, C. (1950) and generalised to the multidimensional case by Bennet, J.T. and Hayes, W.L. in the early 1950's. In this type of scaling there are two kinds of points called "subject" points and "stimulus" points. Distances from only one subject point at a time are compared to the different stimulus points. Kruskal's paper on multidimensional scaling by Optimizing goodness of fit to a non metric hypothesis, has the details of the theory Kruskal, J.B. (1964a).

Regr = Diss : we specified Regr = Diss to indicate that the data matrix represented interpoint distances.

Dist: The dist command was finally given for a display of the matrix of interpoint distances.

The 'SPACES' package accepts data in the OSIRIS matrix types. We used OSIRIS type 2 matrix which analysed data only in the upper-right triangular, off-diagonal portion of the array.

The matrix of correlation coefficients produced by the Clustan package (mentioned in Chapter 4) on the lower right-hand triangular off-diagonal position is automatically converted to the upper right-hand off-diagonal portion via a program designed for that purpose. Then invoke the 'SPACES' program with all the necessary commands and options as presented above and what we get is a set of points that have been through multidimensional process and presented in Euclidean one dimension scale.

Details of the program used and a flow diagram for Owen's modification now follow.

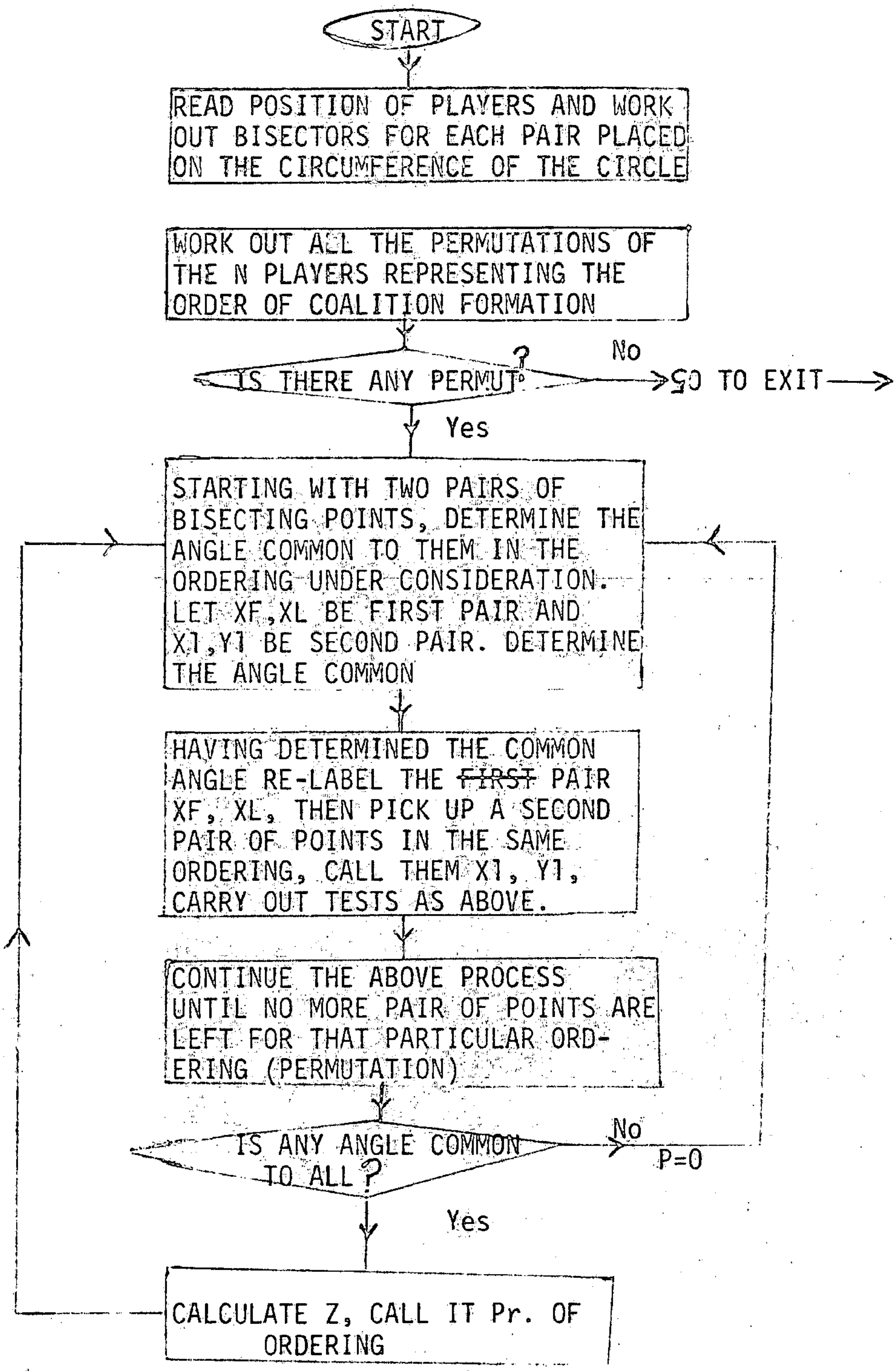
APPENDIX D (1)

PROGRAM FOR CALCULATION OF VALUE VIA OWEN'S METHOD

The program used for calculating the Shapley values via Owen's method calls for a permutation of the N players as grouped into their distinct homogeneous parties (groups) in line with the Shapley value concept as pointed out in Chapter 4. This requirement was accomplished by a program designed for that purpose which will be presented after the next appendix as well as some other programs used during this course. Permutation of the distinct groups was carried out via the "adjacent mark order". Details of this procedure are contained in Page and Wilson's book on "An Introduction to computational Combinatorics", Page, E.S. and Wilson, J.B. (1979) also in Applied Combinatorics, Tucker, A. (1980).

A simplified flow diagram of the program follows.

A SIMPLIFIED FLOW DIAGRAM FOR THE COMPUTERISED OWEN'S MODIFICATION OF THE SHAPLEY VALUE



The above flow diagram contains the processes involved in the modification. A link between Owen's modification and Oceanic games was also established as contained in the next Appendix. The program designed for the application of Owen's technique and the other programs will be presented after the presentation of the link with Oceanic games.

PERMUT = PERMUTATION

Pr. = PROBABILITY

Z = THE COMMON ANGLE

APPENDIX E

Owen's Modification and Oceanic Games

An extension of Owen's modification technique of 2.4.2, Chapter 2, to Oceanic Games was attempted. Oceanic Games as defined in 2.3 of Chapter 2 is a special class of weighted majority games where a few players (atoms) control very large number of votes, while a block of votes is broken up and distributed among a very large number of players. Such situations arise with respect to shares in companies etc.

The structure of the circle makes it difficult to place a continuum of points on it and yet determine any meaningful common orderings defined by common angles or arcs. In order to carry out the link with Oceanic games, the minor players were assigned fixed positions all spaced at equal intervals from each other on the circle while the major players interchanged positions with each other. The necessity for interchanging the positions of the major players is because the position of any particular major player with respect to the minor players can influence the weight of that major player and by allowing the major players to interchange positions no advantage is given to any major player over the other. Some minor players may not have any values at all due to their positions but position in the case of minor players is irrelevant, they all have the same weight and their attitude towards the major players are assumed to be the same. They therefore share equally whatever values that accrue to them, but where definite preference exists between the ocean and the major players then such should be defined by fixing the positions of the major players without interchanging them. In determining these positions the Oceanic players should be treated firstly as a single player after which it should be broken up into the exact number of oceanic players and allowed to occupy a specified portion of the circle as determined from the preference scale.

The results for 2 major players and three major players with an ocean of 12 and 18 minor players respectively is given. The expected Shapley value as calculated using formula (4) of 2.3 Chapter 2 is also given.

(Individual weights and values of Ocean in brackets)

PLAYERS	WEIGHTS	SHAPLEY VALUE OCEANIC	OWEN MODIFICATION EXT.OCEANIC-VALUE
1	2.5	.1220	.1420
2	2.5	.1220	.1420
OCEAN (12)	6 (.500)	.7560 (.0630)	.7160 (.0597)
1	1.666	.1670	.1800
2	1.666	.1670	.1800
3	1.666	.1670	.1800
OCEAN (18)	6 (.333)	.4990 (.0277)	.5400 (.0300)
1	4.0	.1254	.1400
2	2.0	.0784	.0700
3	5.0	.2500	.2620
OCEAN (18)	14 (.777)	.5462 (.0303)	.5280 (.0293)

Extensive computations were carried out but the amount of work involved in determining appropriate intervals and the computer calculations involved in the determination of the different possible orderings suggests that Owen's modification as extended to Oceanic games may be hard to apply when the number of minor players becomes very large.

The technique described above provides a link between Owen's modification based on a sphere (circle) with the concept of Oceanic games. It could be useful in calculating the Shapley value when preference situations exist between the major players (atoms) and minor players (ocean). The above concludes the extensions undertaken in this course.

Every model in this thesis was computerised but we shall present the programs used for Owen's modification and the least squares minimisation as well as an example print out of the result of the simulation of Chapter 4 since it is not necessary to list all the programs used. A listing of the specified programs and the example print out together with necessary comments and descriptions now follows.

OWEN'S MODIFICATION EXTENSION PROGRAM

```
LIST PNT4
      C     THIS PROGRAM AUTOMATICALLY PLACES INDIVIDUALS OR
      C     PARTIES ROUND A CIRCLE OR ONE HALF OF IT
      C     AT PRE-DEFINED DISTANCES AND THEN CALCULATES THE
      C     SHAPLEY VALUE OF THE PARTICIPANTS IN LINE WITH
      C     G. OWEN'S EXTENSION MODIFICATIONS.
      C     FIRSTLY THE PARTICIPANTS ARE PLACED ON THE CIRCLE
      C     AND THEN THE PERPENDICULAR BISECTORS OF EACH OF THE
      C     LINES JOINING THE DIFFERENT PARTICIPANTS ARE WORKED OUT.
      C     A PERMUTATION OF ALL THE PARTICIPANTS IS CARRIED OUT
      C     AND EACH PERMUTATION REPRESENTS AN ORDERING OF THE
      C     PARTICIPANTS. A SEARCH IS THEN CARRIED OUT TO
      C     DETERMINE THE ORDERINGS THAT HAVE COMMON ARCS OR
      C     ANGLES WHICH WOULD THEN BE ASSIGNED AS THE
      C     PROBABILITY THAT SUCH AN ORDERING CAN EXIST WHICH IN
      C     TURN BECOMES THE SHAPLEY VALUE FOR SUCH AN ORDERING.
      C
      C
      C
      C
            DIMENSION XN(25,25),YN(25,25),DPN(5),II(3),JJ(25),JF(25)
            DIMENSION TJM(25),PPP(25)
            COMMON /X/ JL(5000,25)
            COMMON /Y/ JT(5000,25)
            COMMON /W/ ITP(2600,10)
            COMMON /P/ IMP(2600,10)
            COMMON /Z/ JJ,JF,TJM
            COMMON /V/ T,IT
            DATA DPN/0.0,90.0,180.0,270.0/
            IZ=4
            NUM=4
            IPP=0
            JPP=0
            AC=0
            K=0
            KKB=0
            IN=NUM
            INN=IZ-1
            KK=1
            JM=1
            DO 5 M=1,INN
            KK=KK+1
            DO 6 J=KK,IZ
            XN(M,J)=AMOD(((DPN(M)+DPN(J))/2.00+180.00),360.00)
            YN(M,J)=AMOD((XN(M,J)+180.00),360.00)
            XN(J,M)=YN(M,J)
            YN(J,M)=XN(M,J)
          6 CONTINUE
          5 CONTINUE
            DO 7 N=1,IZ
            DO 8 NJ=1,IZ
            IF(N.EQ.NJ)GO TO 8
            IF(AC.EQ.0)GO TO 8
            WRITE(6,900)XN(N,NJ),YN(N,NJ)
          8 CONTINUE
          7 CONTINUE
            WRITE(6,990)
            NN=3
            DO 100 I=1,NN
            IF(I.GT.1) GO TO 20
            DO 10 J=1,NN
         10 II(J)=J
            GO TO 30
         20 KK=II(I)
            II(I)=II(I-1)
            II(I-1)=KK
         30 CONTINUE
            K=K+1
            DO 15 L=1,NN
         15 JL(K,L)=II(L)
        100 CONTINUE
            IF(NUM.EQ.3) GO TO 33
         25 NM=NN+1
            CALL PERMUT (K,NN,NM)
            KP=K*NM
            K=KP
            NN=NN+1
            IF(NM.LT.NUM) GO TO 25
            IF(NUM.LT.IZ) GO TO 33
            DO 200 IH=1,K
            DO 800 IM=1,NM
            ID=NM+1-IM
            ITP(IH,ID)=JL(IH,IM)
        800 CONTINUE
        200 CONTINUE
            DO 3 KC=1,K
          3 WRITE(6,350)(JL(KC,KM),KM=1,NM),(ITP(KC,KM),KM=1,NM)
         33 CONTINUE
            IF(NUM.EQ.3) GO TO 99
            KG=K
            IF(NUM.GT.3) GO TO 989
         99 KG=3
```

```
  989 KB=0
      DO 50 IP=1,KG
      P=0
      IIP=IP
      KKG=KG
      DO 60 JI=1,NUM
   60 JJ(JI)=JL(IP,JI)
      XF=XN(JJ(1),JJ(2))
      XL=YN(JJ(1),JJ(2))
      X1=XN(JJ(2),JJ(3))
      Y1=YN(JJ(2),JJ(3))
      P=1
      J1=2
      J2=3
      IC=0
      IF(NUM.EQ.3) GO TO 555
      IF(NUM.LT.IZ) GO TO 16
      IF(AC.EQ.0)GO TO 16
      WRITE(6,901)XF,XL,X1,Y1
   16 IF(XF.LT.XL.AND.X1.LT.Y1)GO TO 111
      IF(XF.LT.XL.AND.X1.GT.Y1)GO TO 222
      IF(XF.GT.XL.AND.X1.LT.Y1)GO TO 333
      IF(XF.GT.XL.AND.X1.GT.Y1)GO TO 444
      P=0
      TR=100
      GO TO 400
  111 IF(XF.EQ.X1.AND.XL.EQ.Y1) GO TO 446
      IF(X1.GT.XF.AND.Y1.GT.XL) GO TO 142
      XL=Y1
      X1=0
      Y1=0
      TR=1
      GO TO 400
  142 XF=X1
      X1=0
      Y1=0
      TR=2
      GO TO 400
  222 IF(Y1.EQ.XF.AND.X1.EQ.XL) GO TO 447
      IF(Y1.GT.XF.AND.X1.GT.XL) GO TO 121
      XF=X1
      X1=0
      Y1=0
      TR=3
      GO TO 400
  121 XL=Y1
      X1=0
      Y1=0
      TR=4
      GO TO 400
  333 IF(Y1.EQ.XF.AND.X1.EQ.XL) GO TO 447
      IF(Y1.GT.XF.AND.X1.GT.XL) GO TO 131
      XF=X1
      X1=0
      Y1=0
      TR=5
      GO TO 400
  131 XL=Y1
      X1=0
      Y1=0
      TR=6
      GO TO 400
  444 IF(XF.EQ.X1.AND.XL.EQ.Y1) GO TO 446
      IF(X1.GT.XF.AND.Y1.GT.XL) GO TO 445
      XL=Y1
      X1=0
      Y1=0
      TR=7
      GO TO 400
  445 XF=X1
      X1=0
      Y1=0
      TR=8
      GO TO 400
  446 X1=0
      Y1=0
      TR=9
      GO TO 400
  447 P=0
      TR=20
      XF=0
      XL=0
  400 IF(NUM.LT.IZ) GO TO 601
      IF(AC.EQ.0)GO TO 601
      WRITE(6,906)XF,XL,TR
  601 CONTINUE
          J1=J2
       J2=J2+1
        X1=XN(JJ(J1),JJ(J2))
       Y1=YN(JJ(J1),JJ(J2))
      IF(NUM.LT.IZ) GO TO 555
      IF(AC.EQ.0)GO TO 555
      WRITE(6,902)XF,XL,X1,Y1
  555 IF(P.EQ.0)GO TO 511
      IF(XF.LT.XL.AND.X1.LT.Y1) GO TO 123
      IF(XF.LT.XL.AND.X1.GT.Y1) GO TO 124
```

```
      IF(XF.GT.XL.AND.X1.LT.Y1) GO TO 125
      IF(XF.GT.XL.AND.X1.GT.Y1) GO TO 126
      P=0
      GO TO 6000
  123 IF(XL.EQ.Y1) GO TO 504
      IF(XF.EQ.X1) GO TO 517
      IF(XL.EQ.X1.OR.XF.EQ.Y1) GO TO 507
      IF(X1.LT.XF.AND.Y1.GT.XL) GO TO 504
      IF(X1.GT.XF.AND.X1.LT.XL) GO TO 505
      IF(Y1.LT.XL.AND.Y1.GT.XF) GO TO 506
      IF(XF.LT.X1.AND.XL.LT.X1) GO TO 507
      IF(XF.GT.Y1.AND.XL.GT.Y1) GO TO 507
      P=0
      GO TO 6000
  504 IC=1
      X1=0
      Y1=0
      IF(J2.LT.NUM) GO TO 601
      P=ABS(XL-XF)/180.00
      CALL WEIGHT (IC,NUM,IZ)
      GO TO 500
  505 IC=2
      XF=X1
      X1=0
      Y1=0
      IF(J2.LT.NUM) GO TO 601
      P=ABS(XL-XF)/180.00
      CALL WEIGHT (IC,NUM,IZ)
      GO TO 500
  506 IC=3
      XL=Y1
      X1=0
      Y1=0
      IF(J2.LT.NUM) GO TO 601
      P=ABS(XL-XF)/180.00
      CALL WEIGHT (IC,NUM,IZ)
      GO TO 500
  507 IC=4
      P=0
      XL=0
      XF=0
      X1=0
      Y1=0
      IF(J2.LT.NUM) GO TO 601
      IF(IIP.EQ.KKG) GO TO 66
      IF(J2.EQ.NUM) GO TO 50
  517 IF(XL.EQ.Y1) GO TO 504
      IF(XL.LT.Y1) GO TO 504
      IF(XL.GT.Y1)P=0
      GO TO 6000
  124 IF(XF.EQ.X1) GO TO 5111
      IF(XL.EQ.Y1) GO TO 5111
      IF(XF.EQ.Y1) GO TO 511
      IF(XL.EQ.X1) GO TO 511
      IF(X1.LT.XF) GO TO 508
      IF(X1.GT.XF.AND.X1.LT.XL) GO TO 509
      IF(Y1.GT.XF.AND.Y1.LT.XL) GO TO 510
      IF(XF.GT.Y1.AND.XL.LT.X1) GO TO 511
      IF(XF.LT.Y1.AND.XL.LT.Y1)GO TO 508
      P=0
      GO TO 6000
  508 IC=5
      X1=0
      Y1=0
      IF(J2.LT.NUM) GO TO 601
      P=ABS(XL-XF)/180.00
      CALL WEIGHT (IC,NUM,IZ)
      GO TO 500
  509 IC=6
      XF=X1
      X1=0
      Y1=0
      IF(J2.LT.NUM) GO TO 601
      P=ABS(XL-XF)/180.00
      CALL WEIGHT (IC,NUM,IZ)
      GO TO 500
  510 IC=7
      XL=Y1
      X1=0
      Y1=0
      IF(J2.LT.NUM) GO TO 601
      P=ABS(XL-XF)/180.00
      CALL WEIGHT (IC,NUM,IZ)
      GO TO 500
  511 IC=8
      P=0
      XL=0
      XF=0
      Y1=0
      X1=0
      IF(J2.LT.NUM) GO TO 601
      IF(IIP.EQ.KKG) GO TO 66
      IF(J2.EQ.NUM) GO TO 50
 5111 IC=9
      X1=0
      Y1=0
```

```
      IF(J2.LT.NUM) GO TO 601
      P=ABS(XL-XF)/180.00
      CALL WEIGHT (IC,NUM,IZ)
      GO TO 500
  125 IF(XF.EQ.Y1) GO TO 511
      IF(XL.EQ.X1) GO TO 511
      IF(XL.LT.X1.AND.XF.GT.Y1) GO TO 511
      IF(X1.LT.XL.AND.XL.LT.Y1) GO TO 512
      IF(Y1.GT.XF.AND.XF.GT.X1) GO TO 513
      P=0
      GO TO 6000
  126 IF(XF.EQ.X1) GO TO 514
      IF(XL.EQ.Y1) GO TO 514
      IF(Y1.GT.XL.AND.X1.LT.XF) GO TO 501
      IF(X1.LT.XF.AND.Y1.LT.XL) GO TO 502
      IF(X1.GT.XF.AND.Y1.GT.XL) GO TO 503
      P=0
      GO TO 6000
  512 IC=10
      XF=X1
      X1=0
      Y1=0
      IF(J2.LT.NUM) GO TO 601
      P=ABS(XL-XF)/180.00
      CALL WEIGHT (IC,NUM,IZ)
      GO TO 500
  513 IC=11
      XL=Y1
      X1=0
      Y1=0
      IF(J2.LT.NUM) GO TO 601
      P=ABS(XL-XF)/180.00
      CALL WEIGHT (IC,NUM,IZ)
      GO TO 500
  501 IC=12
      X1=0
      Y1=0
      IF(J2.LT.NUM) GO TO 601
      P=ABS((360.00-XF)+XL)/180.00
      CALL WEIGHT (IC,NUM,IZ)
      GO TO 500
  502 IC=13
      XL=Y1
      X1=0
      Y1=0
      IF(J2.LT.NUM) GO TO 601
      P=ABS((360.00-XF)+XL)/180.00
      CALL WEIGHT (IC,NUM,IZ)
      GO TO 500
  503 IC=14
      XF=X1
      X1=0
      Y1=0
      IF(J2.LT.NUM) GO TO 601
      P=ABS((360.00-XF)+XL)/180.00
      CALL WEIGHT (IC,NUM,IZ)
      GO TO 500
  514 IC=14
      X1=0
      Y1=0
      IF(J2.LT.NUM) GO TO 601
      P=ABS((360.00-XF)+XL)/180.00
      CALL WEIGHT (IC,NUM,IZ)
      GO TO 500
 6000 WRITE(6,7000)P
 7000 FORMAT(6X,F8.3)
      GO TO 601
  500 KB=KB+1
      DO 23 KA=1,NUM
   23 JT(KB,KA)=JJ(KA)
      IF(NUM.LT.IZ)GO TO 40
      IF(P.EQ.0)GO TO 40
      P=P/2.0
      KKB=KKB+1
      DO 777 KAA=1,NUM
      IDD=NUM+1-KAA
      IMP(KKB,IDD)=JJ(KAA)
  777 CONTINUE
      IPP=IPP+1
      PPP(IPP)=P
      WRITE(6,905)T,IT,P
      WRITE(6,904)(JJ(L),L=1,NUM)
      WRITE(6,907)(TJM(LM),LM=1,NUM)
      IF(AC.EQ.0)GO TO 40
      WRITE(6,903)P,IC,XF,XL,X1,Y1
   40 IF(IIP.LT.KKG)GO TO 50
   66 DO 21 IK=1,KB
      IF(NUM.EQ.IZ)GO TO 50
      DO 22 KA=1,NUM
   22 JL(IK,KA)=JT(IK,KA)
   21 CONTINUE
      NM=NN+1
      K=KB
      CALL PERMUT (K,NN,NM)
      NUM=NUM+1
      K=KB*NUM
```

```
      NN=NN+1
      GO TO 33
   50 CONTINUE
      P=0
      DO 7772 IKP=1,KKB
      DO 7773 JJI=1,NUM
      JJ(JJI)=IMP(IKP,JJI)
 7773 CONTINUE
      CALL WEIGHT(IC,NUM,IZ)
      JPP=JPP+1
      P=PPP(JPP)
      WRITE(6,905)T,IT,P
      WRITE(6,904)(JJ(L),L=1,NUM)
      WRITE(6,907)(TJM(LM),LM=1,NUM)
      IF(AC.EQ.0)GO TO 999
      WRITE(6,903)P,IC,XF,XL,X1,Y1
  999 IF(IKP.EQ.KKB)GO TO 9991
      P=0
 7772 CONTINUE
  990 FORMAT(//,4X,6H 1=UPN,2X,7H 2=GNPP,2X,6H 3=NPP,2X,6H 4=PRP,
     1 2X,6H 5=NPN,//)
  906 FORMAT(3X,F10.6,2X,F10.6,2X,F10.6)
  905 FORMAT(/,2X,F7.3,2X,I3,F10.6)
  904 FORMAT(4X,15I7)
  907 FORMAT(4X,15F7.3)
  903  FORMAT(1H ,1X,F10.6,I3,1X,F10.6,2X,F10.6,2X,F2.1,2X,F2.1)
  900 FORMAT(/,1H ,2X,F10.6,2X,F10.6)
  350 FORMAT(12X,4I2,10X,4I2)
  901 FORMAT(/,1H ,1X,F8.3,2X,F8.3,4X,F8.3,1X,F8.3)
  902 FORMAT(1H ,1X,F10.6,2X,F10.6,3X,F10.6,2X,F10.6)
 9991 STOP
      END
C
C
C     SUBROUTINE FOR CARRYING OUT THE PERMUTATION EXERCISE.
C
C
      SUBROUTINE PERMUT (K,NN,NM)
      COMMON /X/ JL(5000,25)
      COMMON /Y/ JT(5000,25)
      DO 22 KT=1,K
      DO 16 KM=1,NN
   16 JT(KT,KM)=JL(KT,KM)
      JT(KT,NM)=NM
   22 CONTINUE
      KP=0
      DO 27 KC=1,K
      DO 17 KM=1,NM
      IF(KM.EQ.1)GO TO 19
      K2=JT(KC,NM-KM+1)
      JT(KC,NM-KM+1)=JT(KC,NM+2-KM)
      JT(KC,NM+2-KM)=K2
   19 CONTINUE
      KP=KP+1
      DO 18 KS=1,NM
   18 JL(KP,KS)=JT(KC,KS)
      IF(KM.LT.NM) GO TO 17
   17 CONTINUE
   27 CONTINUE
      RETURN
      END
C
C
C     SUBROUTINE FOR ATTACHING WEIGHTS TO THE
C     PARTICIPATING PARTIES OR GROUPS ACCORDING
C     TO THE EXACT NUMBER OF PLAYERS THE PARTY
C     OR GROUP HAS.
C
C
      SUBROUTINE WEIGHT (IC,NUM,IZ)
      DIMENSION JJ(25),JF(25),TJM(25)
      COMMON /X/ JL(5000,25)
      COMMON /Y/ JT(5000,25)
      COMMON /W/ ITP(2600,10)
      COMMON /P/ IMP(2600,10)
      COMMON /Z/ JJ,JF,TJM
      COMMON /V/ T,IT
      DO 31 LM=1,NUM
   31 JF(LM)=JJ(LM)
      DO 4 LM=1,NUM
      IF(JF(LM).EQ.1)TJM(LM)=0.000
      IF(JF(LM).EQ.2)TJM(LM)=0.000
      IF(JF(LM).EQ.3)TJM(LM)=0.000
      IF(JF(LM).EQ.4)TJM(LM)=0.000
      IF(JF(LM).EQ.5)TJM(LM)=0.00
      IF(JF(LM).EQ.6)TJM(LM)=0.000
      IF(JF(LM).EQ.7)TJM(LM)=0.000
      IF(JF(LM).EQ.8)TJM(LM)=0.0000
      IF(JF(LM).EQ.9)TJM(LM)=0.0000
      IF(JF(LM).EQ.10)TJM(LM)=0.0000
      IF(JF(LM).EQ.11)TJM(LM)=0.0000
      IF(JF(LM).EQ.12)TJM(LM)=0.0000
      IF(JF(LM).EQ.13)TJM(LM)=0.0000
      IF(JF(LM).EQ.14)TJM(LM)=0.0000
      IF(JF(LM).EQ.15)TJM(LM)=0.0000
      IF(JF(LM).EQ.16)TJM(LM)=0.0000
```

```
      IF(JF(LM).EQ.17)TJM(LM)=0.0000
      IF(JF(LM).EQ.18)TJM(LM)=0.0000
      IF(JF(LM).EQ.19)TJM(LM)=0.0000
      IF(JF(LM).EQ.20)TJM(LM)=0.0000
      IF(JF(LM).EQ.21)TJM(LM)=0.0000
      IF(JF(LM).EQ.22)TJM(LM)=0.0000
      IF(JF(LM).EQ.23)TJM(LM)=0.0000
      IF(JF(LM).EQ.24)TJM(LM)=0.0000
      IF(JF(LM).EQ.25)TJM(LM)=0.0000
      IF(JF(LM).EQ.26)TJM(LM)=0.000
      IF(JF(LM).EQ.27)TJM(LM)=0.000
      IF(JF(LM).EQ.28)TJM(LM)=0.000
      IF(JF(LM).EQ.29)TJM(LM)=0.000
    4 CONTINUE
      AVE=0
      DO 5 LM=1,NUM
      AVE=TJM(LM)+AVE
      IF(AVE.LT.0.000)GO TO 5
      IF(NUM.LT.IZ)GO TO 6
      T=AVE
      IT=JF(LM)
      GO TO 6
    5 CONTINUE
    6 RETURN
      END
: of file
```

```
IG 9
L2 03:13:35 to 03:49:13 Sat 10-Sep-83
 91.47
351.55
ll disconnected~
```

```
>         C       PROBABILITY THAT ANY TWO PARTIES WILL VOTE
>         C       TOGETHER ON ONE SIDE. THE RESULT IS THEN
>         C       USED BY THE NEXT PROG. FOR FINAL CALCULATIONS OF THE ai's.
>         C
>         C
>         C
>         C
>                 DIMENSION A(100),B(100),C(100),D(100),E(100)
>                 K=0
>                 CC1=0.0
>                 CC2=0.0
>                 CC3=0.0
>                 CC4=0.0
>                 CC5=0.0
>                 CC6=0.0
>                 CC7=0.0
>                 CC8=0.0
>                 CC9=0.0
>                 CB1=0.0
>                 WRITE(6,15)
>               7 DO 1 I=1,100
>                 IF(K.EQ.1) GO TO 222
>                 IF(I.GT.1) GO TO 111
>                 READ(5,10)A(I),B(I),C(I),D(I),E(I)
>                 GO TO 222
>             111 READ(5,100)A(I),B(I),C(I),D(I),E(I)
>             222 IF(K.EQ.1) GO TO 80
>                 AA1=2.0*A(I)*B(I)+1.0-A(I)-B(I)
>                 AA2=2.0*A(I)*C(I)+1.0-A(I)-C(I)
>                 AA3=2.0*A(I)*D(I)+1.0-A(I)-D(I)
>                 AA4=2.0*A(I)*E(I)+1.0-A(I)-E(I)
>                 AA5=2.0*B(I)*C(I)+1.0-B(I)-C(I)
>                 IF(K.EQ.0) GO TO 9
>              80 AA6=2.0*B(I)*D(I)+1.0-B(I)-D(I)
>                 AA7=2.0*B(I)*E(I)+1.0-B(I)-E(I)
>                 AA8=2.0*C(I)*D(I)+1.0-C(I)-D(I)
>                 AA9=2.0*C(I)*E(I)+1.0-C(I)-E(I)
>                 AB1=2.0*D(I)*E(I)+1.0-D(I)-E(I)
>                 IF(K.EQ.1) GO TO 70
>               9 CC1=CC1+AA1
>                 CC2=CC2+AA2
>                 CC3=CC3+AA3
>                 CC4=CC4+AA4
>                 CC5=CC5+AA5
>                 IF(K.EQ.0) GO TO 91
>              70 CC6=CC6+AA6
>                 CC7=CC7+AA7
>                 CC8=CC8+AA8
>                 CC9=CC9+AA9
>                 CB1=CB1+AB1
>                 IF(K.EQ.1) GO TO 94
>              91 WRITE(6,11)A(I),B(I),AA1,A(I),C(I),AA2,A(I),D(I),AA3,
>                1 A(I),E(I),AA4,B(I),C(I),AA5
>                 GO TO 1
>              94 WRITE(6,11)B(I),D(I),AA6,B(I),E(I),AA7,C(I),D(I),
>                1 AA8,C(I),E(I),AA9,D(I),E(I),AB1
>               1 CONTINUE
>                 IF(K.EQ.1) GO TO 96
>                 DD1=CC1/100.0
>                 DD2=CC2/100.0
>                 DD3=CC3/100.0
>                 DD4=CC4/100.0
>                 DD5=CC5/100.0
>                 WRITE(6,13)CC1,DD1,CC2,DD2,CC3,DD3,CC4,DD4,CC5,DD5
>                 IF(K.EQ.0) GO TO 93
>              96 DD6=CC6/100.0
>                 DD7=CC7/100.0
>                 DD8=CC8/100.0
>                 DD9=CC9/100.0
>                 DB1=CB1/100.0
>                 WRITE(6,13)CC6,DD6,CC7,DD7,CC8,DD8,CC9,DD9,CB1,DB1
>                 IF(K.EQ.1) GO TO 333
>              10 FORMAT(////,8X,F5.3,8X,F5.3,8X,F5.3,8X,F5.3,8X,F5.3)
>             100 FORMAT(///,8X,F5.3,8X,F5.3,8X,F5.3,8X,F5.3,8X,F5.3)
>              15 FORMAT(3X,15H NPN  AND  UPN )
>              11 FORMAT(F5.3,1X,F5.3,1X,F5.3,2X,F5.3,1X,F5.3,1X,F5.3,2X,
>                1 F5.3,1X,F5.3,1X,F5.3,2X,F5.3,1X,F5.3,1X,F5.3,2X,
>                1 F5.3,1X,F5.3,1X,F5.3,2X)
>              13 FORMAT(//,F7.3,2X,F7.3,3X,F7.3,2X,F7.3,3X,F7.3,2X,F7.3,3X,
>                1 F7.3,2X,F7.3,3X,F7.3,2X,F7.3,//)
>              93 K=K+1
>                 GO TO 7
>             333 STOP
>                 END
#End of file
#
```

MI

CONSTRAINED LEAST SQUARES MINISATION PROGRAM

```
LIST VOTS2
      C      THIS PROG. CALCULATES THE ai's WHICH ARE
      C      THE OVER ALL PROBABILITY MEASURE OF ASSOCIATION
      C      BETWEEN THE PARTICIPANTS i.e. THE PROBABILITY
      C      THAT ALL THE PARTIES WILL VOTE TOGETHER ON ONE SIDE.
      C
      C
      C
      C
             IMPLICIT REAL*8(A-H,O-Z)
             DIMENSION BL(5),BU(5),W(70),X(5)
             INTEGER IBOUND,IFAIL,J,LIW,LW,N,NOUT
             INTEGER IW(7)
             N=5
      C      INITIAL GUESSES ARE MADE W.R.T. THE FUNCTION VALUES ON EXIT.
             X(1)=(INITIAL GUESS)
             X(2)=
             X(3)=
             X(4)=
             X(5)=
             IBOUND=3
             BU(1)=1.0
             BL(1)=0.0
             LIW=7
             LW=70
             IFAIL=1
             CALL E04JAF(N,IBOUND,BL,BU,X,F,7,LIW,W,LW,IFAIL)
             IF(IFAIL.NE.0)WRITE(6,99998)IFAIL
             IF(IFAIL.EQ.1)GO TO 20
             WRITE(6,99997)F
             WRITE(6,99996)(X(J),J=1,N)
          20 STOP
       99998 FORMAT(//,16H ERROR EXIT TYPE,I3)
       99997 FORMAT(//,27H FUNCTION VALUE ON EXIT IS,F8.4)
       99996 FORMAT(13H AT THE POINT,5F9.4)
             END
             SUBROUTINE FUNCT1(N,XC,FC)
             IMPLICIT REAL*8(A-H,O-Z)
             DIMENSION XC(5)
             INTEGER N
             N=5
             X1=XC(1)
             X2=XC(2)
             X3=XC(3)
             X4=XC(4)
             X5=XC(5)
             FC=((0.143-1+X1+X2-2*X1*X2)**2)
            1 + ((0.545-1+X1+X3-2*X1*X3)**2)
            1 + ((0.526-1+X1+X4-2*X1*X4)**2)
            1 + ((0.635-1+X1+X5-2*X1*X5)**2)
            1 + ((0.542-1+X2+X3-2*X2*X3)**2)
            1 + ((0.625-1+X2+X4-2*X2*X4)**2)
            1 + ((0.520-1+X2+X5-2*X2*X5)**2)
            1 + ((0.608-1+X3+X4-2*X3*X4)**2)
            1 + ((0.594-1+X3+X5-2*X3*X5)**2)
            1 + ((0.789-1+X4+X5-2*X4*X5)**2)
             RETURN
             END
#End of file
#
```

SAMPLE OUT-PUT VOTING SIMULATION

THIS IS A SAMPLE OUT-PUT FROM THE SIMULATION PROGRAM. THE FIRST SET OF NUMBERS ARE THE GIVEN ai's.(THESE ARE RECORDED ONLY ONCE) THEY ARE FOLLOWED BY THE EXACT NUMBERS FROM EACH PARTY THAT VOTED EITHER YES OR NO. NEXT TO THESE ARE THE PROPORTIONS THAT VOTED EITHER WAY. THESE ARE FOLLOWED BY THE CONTRIBUTION MADE BY EACH PARTY TO THAT PARTICULAR MINIMAL WINNING COALITION. IN ORDER TO SAVE SPACE VOTING SITUATIONS ARE RECORDED AFTER EVERY TEN MINIMAL WINNING COALITIONS.THE FINAL SET OF NUMBERS WITHIN EACH SET OF TEN MINIMAL WINNING COALITIONS IS THE VALUE CALCULATED. A SUMMARY OF THESE VALUES WITH THE ASSOCIATED VALUES CALCULATED FOR THE VOTING SYSTEM CAN BE FOUND AT THE END OF THE SIMULATION EXERCISE.(WE NOTE THAT ONLY 2000 VOTING SITUATIONS WERE SIMULATED.)

OT	NPN		UPN		NPP		PRP		GNPP		
	YES	NO	YES	NO	YES	NO	YES	NO	YES	NO	
		0.850		0.010		0.480		0.500		0.530	
10	32	5	0	28	8	7	5	3	3	4	YES=48 NO=47
	0.86	0.14	0.0	1.00	0.53	0.47	0.63	0.38	0.43	0.57	
	.667		.0		.167		.104		.063		
	0.310		0.350		0.167		0.094		0.079		
20	32	5	0	28	8	7	3	5	5	2	YES=48 NO=47
	0.86	0.14	0.0	1.00	0.53	0.47	0.38	0.63	0.71	0.29	
	.667		.0		.167		.063		.104		
	0.281		0.408		0.150		0.083		0.077		
30	34	3	0	28	8	7	3	5	3	4	YES=48 NO=47
	0.92	0.08	0.0	1.00	0.53	0.47	0.38	0.63	0.43	0.57	
	.708		.0		.167		.063		.063		
	0.269		0.402		0.158		0.085		0.085		
40	28	9	0	28	12	3	5	3	3	4	YES=48 NO=47
	0.76	0.24	0.0	1.00	0.80	0.20	0.63	0.38	0.43	0.57	
	.583		.0		.250		.104		.063		
	0.427		0.233		0.169		0.081		0.090		
50	34	3	0	28	6	9	3	5	4	3	YES=47 NO=48
	0.92	0.08	0.0	1.00	0.40	0.60	0.38	0.63	0.57	0.43	
		.063		.583		.188		.104		.063	
	0.229		0.462		0.173		0.071		0.065		
60	32	5	0	28	6	9	5	3	4	3	YES=47 NO=48
	0.86	0.14	0.0	1.00	0.40	0.60	0.63	0.38	0.57	0.43	
		.104		.583		.188		.063		.063	
	0.496		0.175		0.158		0.083		0.087		
70	31	6	0	28	9	6	4	4	3	4	YES=47 NO=48
	0.84	0.16	0.0	1.00	0.60	0.40	0.50	0.50	0.43	0.57	
		.125		.583		.125		.083		.083	
	0.379		0.298		0.162		0.075		0.085		
0	33	4	0	28	6	9	6	2	3	4	YES=48 NO=47
	0.89	0.11	0.0	1.00	0.40	0.60	0.75	0.25	0.43	0.57	
	.688		.0		.125		.125		.063		
	0.327		0.350		0.156		0.085		0.081		
0	33	4	0	28	7	8	4	4	3	4	YES=47 NO=48
	0.89	0.11	0.0	1.00	0.47	0.53	0.50	0.50	0.43	0.57	
		.083		.583		.167		.083		.083	
	0.435		0.231		0.167		0.098		0.069		

0	33	4	1	27	5	10	6	2	3	4	YES=48 NO=47
	0.89	0.11	0.04	0.96	0.33	0.67	0.75	0.25	0.43	0.57	
	.688		.021		.104		.125		.063		
	0.333		0.354		0.140		0.092		0.079		
0	33	4	0	28	6	9	7	1	2	5	YES=48 NO=47
	0.89	0.11	0.0	1.00	0.40	0.60	0.88	0.13	0.29	0.71	
	.688		.0		.125		.146		.042		
	0.277		0.406		0.165		0.087		0.065		
0	33	4	0	28	7	8	3	5	5	2	YES=48 NO=47
	0.89	0.11	0.0	1.00	0.47	0.53	0.38	0.63	0.71	0.29	
	.688		.0		.146		.063		.104		
	0.608		0.062		0.154		0.094		0.081		
0	31	6	1	27	8	7	5	3	2	5	YES=47 NO=48
	0.84	0.16	0.04	0.96	0.53	0.47	0.63	0.38	0.29	0.71	
		.125		.563		.146		.063		.104	
	0.400		0.296		0.162		0.075		0.067		
	34	3	0	28	6	9	5	3	3	4	YES=48 NO=47
	0.92	0.08	0.0	1.00	0.40	0.60	0.63	0.38	0.43	0.57	
	.708		.0		.125		.104		.063		
	0.412		0.292		0.156		0.081		0.058		
	35	2	0	28	7	8	5	3	1	6	YES=48 NO=47
	0.95	0.05	0.0	1.00	0.47	0.53	0.63	0.38	0.14	0.86	
	.729		.0		.146		.104		.021		
	0.398		0.294		0.144		0.092		0.073		
	34	3	0	28	7	8	2	6	5	2	YES=48 NO=47
	0.92	0.08	0.0	1.00	0.47	0.53	0.25	0.75	0.71	0.29	
	.708		.0		.146		.042		.104		
	0.515		0.175		0.154		0.083		0.073		
	30	7	0	28	9	6	5	3	3	4	YES=47 NO=48
	0.81	0.19	0.0	1.00	0.60	0.40	0.63	0.38	0.43	0.57	
		.146		.583		.125		.063		.083	
	0.242		0.465		0.144		0.081		0.069		
	33	4	0	28	6	9	4	4	4	3	YES=47 NO=48
	0.89	0.11	0.0	1.00	0.40	0.60	0.50	0.50	0.57	0.43	
		.083		.583		.188		.083		.063	
	0.365		0.290		0.181		0.083		0.081		
	33	4	0	28	8	7	5	3	2	5	YES=48 NO=47
	0.89	0.11	0.0	1.00	0.53	0.47	0.63	0.38	0.29	0.71	
	.688		.0		.167		.104		.042		
	0.390		0.287		0.171		0.081		0.071		
	31	6	0	28	9	6	4	4	4	3	YES=48 NO=47
	0.84	0.16	0.0	1.00	0.60	0.40	0.50	0.50	0.57	0.43	
	.646		.0		.188		.083		.083		
	0.323		0.348		0.169		0.079		0.081		
	32	5	0	28	8	7	2	6	6	1	YES=48 NO=47
	0.86	0.14	0.0	1.00	0.53	0.47	0.25	0.75	0.86	0.14	
	.667		.0		.167		.042		.125		
	0.548		0.115		0.162		0.094		0.081		
	34	3	0	28	8	7	5	3	0	7	YES=47 NO=48
	0.92	0.08	0.0	1.00	0.53	0.47	0.63	0.38	0.0	1.00	
		.063		.583		.146		.063		.146	
	0.400		0.294		0.160		0.075		0.071		

0.86	0.14	0.0	1.00	0.53	0.47	0.25	0.75	0.86	0.14	
.667		.0		.167		.042		.125		
0.435		0.242		0.162		0.085		0.075		
31	6	2	26	7	8	4	4	3	4	YES=47 NO=48
0.84	0.16	0.07	0.93	0.47	0.53	0.50	0.50	0.43	0.57	
	.125		.542		.167		.083		.083	
0.333		0.346		0.150		0.092		0.079		
33	4	0	28	10	5	2	6	3	4	YES=48 NO=47
0.89	0.11	0.0	1.00	0.67	0.33	0.25	0.75	0.43	0.57	
.688		.0		.208		.042		.063		
0.452		0.229		0.175		0.073		0.071		
28	9	0	28	8	7	6	2	5	2	YES=47 NO=48
0.76	0.24	0.0	1.00	0.53	0.47	0.75	0.25	0.71	0.29	
	.188		.583		.146		.042		.042	
0.398		0.296		0.144		0.096		0.067		
34	3	0	28	6	9	4	4	4	3	YES=48 NO=47
0.92	0.08	0.0	1.00	0.40	0.60	0.50	0.50	0.57	0.43	
.708		.0		.125		.083		.083		
0.369		0.294		0.162		0.090		0.085		
30	7	0	28	7	8	5	3	5	2	YES=47 NO=48
0.81	0.19	0.0	1.00	0.47	0.53	0.63	0.38	0.71	0.29	
	.146		.583		.167		.063		.042	
0.400		0.292		0.158		0.081		0.069		
32	5	1	27	5	10	4	4	5	2	YES=47 NO=48
0.86	0.14	0.04	0.96	0.33	0.67	0.50	0.50	0.71	0.29	
	.104		.563		.208		.083		.042	
0.308		0.344		0.194		0.081		0.073		
32	5	0	28	6	9	5	3	5	2	YES=48 NO=47
0.86	0.14	0.0	1.00	0.40	0.60	0.63	0.38	0.71	0.29	
.667		.0		.125		.104		.104		
0.456		0.233		0.154		0.081		0.075		
33	4	0	28	6	9	6	2	2	5	YES=47 NO=48
0.89	0.11	0.0	1.00	0.40	0.60	0.75	0.25	0.29	0.71	
	.083		.583		.188		.042		.104	
0.450		0.233		0.152		0.092		0.073		
34	3	0	28	7	8	5	3	2	5	YES=48 NO=47
0.92	0.08	0.0	1.00	0.47	0.53	0.63	0.38	0.29	0.71	
.708		.0		.146		.104		.042		
0.335		0.346		0.158		0.085		0.075		
33	4	1	27	5	10	3	5	6	1	YES=48 NO=47
0.89	0.11	0.04	0.96	0.33	0.67	0.38	0.63	0.86	0.14	
.688		.021		.104		.063		.125		
0.535		0.121		0.154		0.096		0.094		
31	6	3	25	5	10	4	4	4	3	YES=47 NO=48
0.84	0.16	0.11	0.89	0.33	0.67	0.50	0.50	0.57	0.43	
	.125		.521		.208		.083		.063	
0.331		0.346		0.175		0.079		0.069		
32	5	0	28	5	10	5	3	6	1	YES=48 NO=47
0.86	0.14	0.0	1.00	0.33	0.67	0.63	0.38	0.86	0.14	
.667		.0		.104		.104		.125		
0.346		0.350		0.169		0.058		0.077		
31	6	0	28	6	9	5	3	5	2	YES=47 NO=48
0.84	0.16	0.0	1.00	0.40	0.60	0.63	0.38	0.71	0.29	
	.125		.583		.188		.063		.042	

	0.452		0.237		0.150		0.079		0.081		
0	34	3	1	27	5	10	1	7	7	0	YES=48 NO=47
	0.92	0.08	0.04	0.96	0.33	0.67	0.13	0.88	1.00	0.0	
	.708		.021		.104		.021		.146		
	0.294		0.408		0.142		0.079		0.077		
0	35	2	3	25	6	9	2	6	2	5	YES=48 NO=47
	0.95	0.05	0.11	0.89	0.40	0.60	0.25	0.75	0.29	0.71	
	.729		.063		.125		.042		.042		
	0.554		0.133		0.144		0.090		0.079		
0	32	5	0	28	9	6	5	3	2	5	YES=48 NO=47
	0.86	0.14	0.0	1.00	0.60	0.40	0.63	0.38	0.29	0.71	
	.667		.0		.188		.104		.042		
	0.379		0.294		0.158		0.100		0.069		
0	31	6	0	28	10	5	4	4	3	4	YES=48 NO=47
	0.84	0.16	0.0	1.00	0.67	0.33	0.50	0.50	0.43	0.57	
	.646		.0		.208		.083		.063		
	0.258		0.410		0.175		0.090		0.067		
0	31	6	0	28	7	8	6	2	3	4	YES=47 NO=48
	0.84	0.16	0.0	1.00	0.47	0.53	0.75	0.25	0.43	0.57	
		.125		.583		.167		.042		.083	
	0.375		0.296		0.171		0.079		0.079		
0	33	4	0	28	10	5	1	7	4	3	YES=48 NO=47
	0.89	0.11	0.0	1.00	0.67	0.33	0.13	0.88	0.57	0.43	
	.688		.0		.208		.021		.083		
	0.275		0.406		0.169		0.077		0.073		

1	0.310	0.350	0.167	0.094	0.079
2	0.281	0.408	0.150	0.083	0.077
3	0.269	0.402	0.158	0.085	0.085
4	0.427	0.233	0.169	0.081	0.090
5	0.229	0.462	0.173	0.071	0.065
6	0.496	0.175	0.158	0.083	0.087
7	0.379	0.298	0.162	0.075	0.085
8	0.327	0.350	0.156	0.085	0.081
9	0.435	0.231	0.167	0.098	0.069
10	0.335	0.354	0.140	0.092	0.079
11	0.277	0.406	0.165	0.087	0.065
12	0.608	0.062	0.154	0.094	0.081
13	0.400	0.296	0.162	0.075	0.067
14	0.412	0.292	0.156	0.081	0.058
15	0.398	0.294	0.144	0.092	0.073
16	0.515	0.175	0.154	0.083	0.073
17	0.242	0.465	0.144	0.081	0.069
18	0.365	0.290	0.181	0.083	0.081
19	0.390	0.287	0.171	0.081	0.071
20	0.323	0.348	0.169	0.079	0.081
21	0.548	0.115	0.162	0.094	0.081
22	0.400	0.294	0.160	0.075	0.071
23	0.435	0.242	0.162	0.085	0.075
24	0.333	0.346	0.150	0.092	0.079
25	0.452	0.229	0.175	0.073	0.071
26	0.398	0.296	0.144	0.096	0.067
27	0.369	0.294	0.162	0.090	0.085
28	0.400	0.292	0.158	0.081	0.069
29	0.308	0.344	0.194	0.081	0.073
30	0.456	0.233	0.154	0.081	0.075
31	0.450	0.233	0.152	0.092	0.073
32	0.335	0.346	0.158	0.085	0.075
33	0.535	0.121	0.154	0.096	0.094
34	0.331	0.346	0.175	0.079	0.069
35	0.346	0.350	0.169	0.058	0.077
36	0.452	0.237	0.150	0.079	0.081
37	0.294	0.408	0.142	0.079	0.077
38	0.554	0.133	0.144	0.090	0.079
39	0.379	0.294	0.158	0.100	0.069
40	0.258	0.410	0.175	0.090	0.067
41	0.375	0.296	0.171	0.079	0.079
42	0.275	0.406	0.169	0.077	0.073

0.383433 0.296280 0.160466 0.084226 0.075595

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