

# MINOR-MINIMAL NON-PROJECTIVE PLANAR GRAPHS WITH AN INTERNAL 3-SEPARATION

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# MINOR-MINIMAL NON-PROJECTIVE PLANAR GRAPHS WITH AN INTERNAL 3-SEPARATION

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*To my parents, for educating me in what truly matters.*

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## SUMMARY

The property that a graph has an embedding in the projective plane is closed under taking minors. Thus by the well known Graph Minor theorem of Robertson and Seymour, there exists a finite list of minor-minimal graphs, call it  $\Omega$ , such that a given graph  $G$  is projective planar if and only if  $G$  does not contain any graph isomorphic to a member of  $\Omega$  as a minor. Glover, Huneke and Wang found 35 graphs in  $\Omega$ , and Archdeacon proved that those are all the members of  $\Omega$ , but Archdeacon's proof never appeared in any refereed journal. In this thesis we develop a modern approach and technique for finding the list  $\Omega$ , independent of previous work.

Our approach is based on conditioning on the connectivity of a member of  $\Omega$ . Assume  $G$  is a member of  $\Omega$ . If  $G$  is not 3-connected then the structure of  $G$  is well understood. In the case that  $G$  is 3-connected, the problem breaks down into two main cases, either  $G$  has an internal separation of order three or  $G$  is internally 4-connected. In this thesis we find the set of all 3-connected minor minimal non-projective planar graphs with an internal 3-separation. For proving our main result, we use a technique which can be considered as a variation and generalization of the method that Robertson, Seymour and Thomas used for non-planar extension of planar graphs. Using this technique, besides our main result, we also classify the set of minor minimal obstructions for  $a$ -,  $ac$ -,  $abc$ -planarity for rooted graphs. (A rooted graph  $(G, a, b, c)$  is  $a$ -planar if there exists a split of the vertex  $a$  to  $a'$  and  $a''$  in  $G$  such that the new graph  $G'$  obtained by the split has an embedding in a disk such that the vertices  $a', b, a'', c$  are on the boundary of the disk in the order listed. We define  $b$ - and  $c$ -planarity analogously. We say that the rooted graph  $(G, a, b, c)$  is  $ab$ -planar if it is  $a$ -planar or  $b$ -planar, and we define  $abc$ -planarity analogously.)

# CHAPTER I

## INTRODUCTION

In this chapter we will provide the graph theoretic context of the results to follow. In Section 1.1 we describe terminology used for our results. In Section 1.2 we explain the basic concepts and terminology for embedding a graph in a surface. In Section 1.3 we present an overview of the history of excluding a set of graphs as a subdivision or as a minor for classifying some structure or property in graphs. In Section 1.4 we briefly explain the method of Robertson, Seymour and Thomas for classifying non-planar extensions of planar graphs. Their method is the main motivation for the results presented in Chapters 2 and 3. In Section 1.5 we present four applications of the set of minor minimal non-projective planar graphs. In Section 1.6 we briefly sketch the idea and approach used by Glover, Huneke and Wang, and Archdeacon for describing the set of minor minimal non-projective planar graphs. In Section 1.8 we state the main results of this thesis. In Section 1.9 we provide an outline of the proof of the main results.

We should acknowledge that part of this thesis was joint work with Luke Postle. An expository version of this result appeared in [6].

### ***1.1 Graph Theoretic Preliminaries***

We use standard graph theory notation and terminology, as found in [7, 12]. A graph is an ordered pair  $(V(G), E(G))$  consisting of a nonempty finite set  $V(G)$  of vertices and a set  $E(G)$  of edges, which are two-element subsets of  $V(G)$ . So, graphs have no loops or multiple edges. In some part of this thesis, we use a more general notion, called *multigraph*, which is an ordered pair  $(V(H), E(H))$  consisting of a nonempty set  $V(H)$  of vertices and a multiset  $E(H)$  of edges, which are two-element subsets of

$V(G)$ . So, multigraphs have no loops but they could have parallel edges.

If  $e = \{u.v\}$  is an edge where  $u, v \in V(G)$ , then we write  $e = uv$  and say that  $u$  and  $v$  are the *ends* of  $e$ . If  $u$  is an end of  $e$  then we say that  $e$  is *incident* with  $u$  and vice versa. If  $u, v \in V(G)$  such that there exists an edge  $e \in E(G)$  with  $e = uv$ , then we say that  $u$  and  $v$  are *adjacent*. *Path* and *cycle* are defined as usual, in particular, they do not have repeated vertices.

Graphs or multigraphs are usually represented in a pictorial manner with vertices appearing as points or a solid circle and edges represented by lines or curves connecting the two vertices associated with the edge.

The following are two classes of graphs. The first class is *complete* graphs which consists of graphs with vertex set  $V$  where each pair of distinct vertices in  $V$  is connected by an edge, and the second class is *bipartite* graphs which consists of graphs with vertex set  $V$  where  $V$  is partitioned into two sets  $A, B$  where if  $e$  is an edge in the edge set  $E$  then  $e$  has one end in  $A$  and the other in  $B$ .

For a graph  $G = (V, E)$ , if  $V' \subseteq V$ ,  $E' \subseteq E$  and for every edge  $e' \in E'$  both ends of  $e'$  belong to  $V'$ , then the graph  $G' = (V', E')$  is a *subgraph* of  $G$ . Given a graph  $G = (V, E)$ , if  $V'$  is a subset of vertices, we denote by  $G[V']$  the subgraph with vertex set  $V'$  and edge set containing all edges of  $G$  with both ends contained in  $V'$ . Then graph  $G[V']$  is called the graph *induced* by  $V'$ . For a graph  $G = (V, E)$ , and  $S \subset V$ , we denote the graph obtained by removing the vertices in  $S$  by  $G - S$  or  $G \setminus S$ .

A graph  $G$  is *connected* if there exists a path between any two vertices of  $G$ , and *disconnected* otherwise. We say a graph  $G$  is *k-connected* for  $|V(G)| \geq k + 1$  and  $k \in \mathbb{N}$  if for any  $S \subset V$  where  $|S| \leq k - 1$ ,  $G - S$  is connected. A subgraph  $H$  of  $G$  is a *connected component* of  $G$  if  $H$  is a maximal connected subgraph of  $G$ . A vertex  $v$  of a connected graph  $G$  is a *cut vertex* if  $G - \{v\}$  is disconnected. Similarly, if  $G = (V, E)$ , a set  $S \subset V$  is a *cutset* if  $G - S$  is disconnected. We say that  $(G_1, G_2)$  is a *separation* of  $G$  if  $(G_1, G_2)$  are edge-disjoint subgraphs of  $G$  whose union is  $G$ . If



$k = |V(G_1) \cap V(G_2)|$  then we say the *order* of the separation  $(G_1, G_2)$  is  $k$ , or  $(G_1, G_2)$  is a  $k$ -*separation* of  $G$ .

Let  $G$  and  $H$  be two graphs, we say  $G$  and  $H$  are *isomorphic* if there exists a bijection  $f$  between  $V(G)$  and  $V(H)$  such that any two vertices  $u$  and  $v$  are adjacent in  $G$  if and only if  $f(u)$  and  $f(v)$  are adjacent in  $H$ . If  $v \in V(G)$ , the *neighborhood* of  $v$ , denoted by  $N(v)$ , is the set of all vertices in  $G$  adjacent to  $v$ . The degree of a vertex  $v \in V(G)$ , denoted by  $d(v)$ , is equal to the size of its neighborhood.

Let  $G$  and  $H$  be two graphs. The *disjoint union* of  $G, H$  denoted by  $G + H$  is the graph with vertex set  $V(G) \cup V(H)$  and edge set  $E(G) \cup E(H)$ .

Let  $G$  be a graph. We say that a function  $f : V(G) \rightarrow X$  is a *coloring* of  $G$  if for all  $e = uv \in E(G)$ ,  $f(u) \neq f(v)$ . We say that  $L$  is a *list-assignment* for a graph  $G$  if  $L(v)$  is a set of colors for every vertex  $v \in V(G)$ . Let  $G$  be a graph with a list-assignment  $L$ . We say  $G$  has an  $L$ -*coloring* if there exists a coloring  $f$  such that  $f(v) \in L(v)$  for all  $v \in V(G)$ . We say an  $L$ -coloring of a graph  $G$  is a  $k$ -*list coloring* if  $|L(v)| = k$  for all  $v \in V(G)$ .

Let  $G$  be a graph. The *line graph* of  $G$ , denoted by  $L(G)$ , is a graph that each vertex of  $L(G)$  represents an edge of  $G$ , and two vertices of  $L(G)$  are adjacent if and only if their corresponding edges are adjacent in  $G$ .

Let  $G$  be a graph and  $e = uv$  be an edge of  $G$ . A *contraction* of the edge  $e$  is the operation of removing the vertices  $u, v$  from  $G$  and replacing them with a new vertex  $w \notin V(G)$  such  $w$  is adjacent to all vertices of  $(N(u) \cup N(v)) \setminus \{u, v\}$ . We denote the resulting graph by  $G/e$ . A graph  $H$  is called a *minor* of the graph  $G$  if  $H$  is *isomorphic* to a graph that can be obtained by zero or more edge contractions from a subgraph of  $G$ . A graph  $S$  is a *subdivision* of a graph  $G$  if  $S$  is obtained from  $G$  by replacing its edges by internally disjoint nonzero length paths with the same ends, called *segments*.

Let  $S$  be a subgraph of a graph  $G$ . An  $S$ -*bridge* in  $G$  is a connected subgraph

$B$  of  $G$  such that  $E(B) \cap E(S) = \emptyset$  and either  $E(B)$  consists of a unique edge with both ends in  $S$ , or for some component  $C$  of  $G \setminus V(S)$  the set  $E(B)$  consists of all edges of  $G$  with at least one end in  $V(C)$ . The vertices in  $V(B) \cap V(S)$  are called the *attachments* of  $B$ .

The distance between two vertices  $u$  and  $v$  of  $G$ , denoted by  $d(u, v)$ , is the length of the shortest path between them.

We say a graph  $H$  is a *cover* of a graph  $G$  if there exists a surjective function  $f : V(H) \rightarrow V(G)$  such that for each vertex  $u \in V(H)$ , the restriction of  $f$  to  $N(u)$  is a bijection between  $N(u)$  and  $N(f(u))$ . We say a graph  $H$  is an *emulator* of  $G$  if there exists a surjective function  $f : V(H) \rightarrow V(G)$  such that for each vertex  $u \in V(H)$ , the function  $f_u : N(u) \rightarrow N(f(u))$  induced by  $f$ , is surjective.

## 1.2 Graphs on Surfaces

We follow standard terminology in topology, as can be found in Hatcher [21]. A surface is a 2-dimensional topological manifold without boundary. For studying graphs on surfaces, we follow the exposition of Mohar and Thomassen [37].

Two surfaces are *homeomorphic* if there exists a bijective continuous mapping between them such that the inverse is also continuous. Let  $X$  be a topological space. A *curve* (*arc*) in  $X$  is the image of a continuous function  $f : [0, 1] \rightarrow X$ . We say a curve is *simple* if  $f$  is one to one. We say the curve  $J = f([0, 1])$  connects the points  $f(0)$  and  $f(1)$ . We refer to the points  $f(0), f(1)$  as the *endpoints* of the curve  $J$ , and the set  $I = f((0, 1))$  as the *interior* of  $J$ .

A topological space  $X$  is *arcwise connected* if for any two points in  $X$  there exists a simple arc connecting them. Note that existence of a simple arc between two points of  $X$  defines an equivalence relation on the points in  $X$ . The equivalence classes are called *arcwise connected components*. We say a set  $C \subseteq X$  *separates*  $X$  if  $X \setminus C$  is not arcwise connected. For a set  $C \subseteq X$ , we refer to each arcwise connected component

of  $X \setminus C$  as a *face*.

We say a graph  $G'$  is embedded in a topological space  $X$ , if the vertices of  $G'$  are distinct points of  $X$  and every edge of  $G'$  is a simple arc in  $X$  connecting the two vertices, such that its interior is disjoint from other edges and vertices. An embedding of a graph  $G$  in a topological space  $X$  is an isomorphism  $\Sigma$  of  $G$  to a graph  $G'$  embedded in  $X$ . If there is an embedding of  $G$  into  $X$ , we say that  $G$  *can be embedded* into  $X$ . We denote such embedding of  $G$  by  $\Sigma(G)$ . For an embedding  $\Sigma(G)$  in a surface  $S$ , we say a cycle  $C$  of  $\Sigma(G)$  is *essential* if  $C$  is a non-null-homotopic cycle, or equivalently,  $C$  is not contractible in the surface  $S$ .

By the *torus* we mean the product of two circles,  $S_1 \times S_1$ . By the *projective plane*,  $\mathbb{RP}^2$ , we mean the topological space obtained from a closed disk by identifying diagonally opposite points on the boundary of the disk.

Our main concern in this thesis are graphs which can be embedded in the *plane*,  $\mathbb{R}^2$ , and projective plane,  $\mathbb{RP}^2$ . If  $G$  is a graph embedded in the plane then we say that  $G$  is a *planar graph*; in which case, there exists an infinite face of  $G$ . If  $G$  is connected, we call the infinite face of an embedding  $\Sigma(G)$  of  $G$  the *outer face* of  $\Sigma(G)$ . If  $G$  can be embedded in the plane or projective plane, we say  $G$  is *planar*, or *projective planar*, respectively. For more details about surfaces and their characterization, we refer the reader to [21, 37].

### ***1.3 Embedding and Excluding Subgraphs and Minors***

Graphs on surfaces and their properties and especially determining whether a graph can be embedded in a given surface have been studied for decades. The best known result for planar graphs is the famous theorem of Kuratowski [31] which characterizes planar graphs in terms of forbidden subgraphs.

**Theorem 1.3.1** ([31]). *A graph is planar if and only if it does not contain a subdivision of  $K_{3,3}$  or a subdivision of  $K_5$  as a subgraph.*

The forbidden graphs  $K_{3,3}$  and  $K_5$  are called *Kuratowski graphs*. The result of Theorem 1.3.1 was independently discovered by Frink and Smith [17] and Pontrjagin [8]. Some short proofs of Theorem 1.3.1 are published by Dirac and Schuster [15], Makarychev [32] and Thomassen [52, 53]. A natural generalization of Kuratowski's theorem could be characterizing which graphs are embeddable in other surface in terms of forbidding some other structures.

Erdős and König [30] raised the question:

*Question 1.3.2* ([30]). For any surface  $S$ , is there a finite list of graphs  $\mathbf{Forb}(S)$ , such that exclusion of these graphs and all their subdivisions as subgraphs characterizes the graphs embeddable in  $S$ ?

Kuratowski's theorem, Theorem 1.3.1, settled the question for the plane, i.e.  $S = \mathbb{R}^2$ . Glover and Huneke [19] settled the question for the projective plane by proving that  $\mathbf{Forb}(\mathbb{RP}^2)$  is finite. Glover, Huneke and Wang [18] proposed a list of 103 graphs, called  $\Omega'$ , for  $\mathbf{Forb}(\mathbb{RP}^2)$  and Archdeacon [1, 2] showed that the list  $\Omega'$  is a complete list, i.e.  $\Omega' = \mathbf{Forb}(\mathbb{RP}^2)$ . We refer the reader to Appendix A in [37] to see the pictures of graphs in  $\mathbf{Forb}(\mathbb{RP}^2)$ . Archdeacon and Huneke [4] found a proof of finiteness of  $\mathbf{Forb}(S)$  for all non-orientable surfaces. Robertson and Seymour [46, 43, 44] answered the general question by showing that  $\mathbf{Forb}(S)$  is finite for any  $S$ . Seymour [50] and Thomassen [54] found a shorter proof of finiteness of  $\mathbf{Forb}(S)$  for any surface  $S$ . Mohar [33, 34] presented a linear time algorithm where for a given graph  $G$  and surface  $S$ , it either finds an embedding of  $G$  in  $S$  or identifies a subgraph of  $G$  that is homeomorphic to a member of  $\mathbf{Forb}(S)$ . A side result of his proof yields a constructive proof of finiteness of  $\mathbf{Forb}(S)$  for any surface  $S$ . We refer the reader to [37] for more information about excluding subgraphs.

A related approach for charactering whether a graph can be embedded in a given surface  $S$  is to exclude a finite set of graphs as a minor. We start with a well known theorem of Robertson and Seymour [48], formerly known as Wagner's conjecture [57].

**Theorem 1.3.3** ([48]). *For any family of graphs  $\mathcal{F}$  which is closed under taking minors, (i.e. if  $G \in \mathcal{F}$  then every minor of  $G$  belongs to  $\mathcal{F}$ ) there exists a finite list of graphs  $\mathcal{L}$  such that  $G \in \mathcal{F}$  if and only if there exists no  $H \in \mathcal{L}$  such that  $G$  has a minor isomorphic to  $H$ .*

Note that for any surface  $S$ , the property that a graph can be embedded in  $S$  is closed under taking minors. Thus, as a consequence of Robertson and Seymour's theorem, for any surface  $S$ , there exists a finite list of graphs  $\mathbf{Forb}_0(S)$  such that  $G$  can be embedded in  $S$  if and only if there exists no  $H \in \mathbf{Forb}_0(S)$  such that  $G$  has a minor isomorphic to  $H$ . Wagner [57] showed that  $\mathbf{Forb}_0(\mathbb{R}^2) = \{K_{3,3}, K_5\}$ .

**Theorem 1.3.4** ([57]). *A graph is planar if and only if neither  $K_{3,3}$  nor  $K_5$  is a minor of  $G$ .*

In his Ph.D. thesis [1], Archdeacon proved the list  $\Omega \subseteq \mathbf{Forb}(\mathbb{RP}^2)$  found by Glover, Huneke and Wang [18] satisfies. The list  $\Omega$  is shown in Figure 1.1.

**Theorem 1.3.5** ([1]). *A graph can be embedded in the projective plane if and only if it has no minor isomorphic to a member of  $\Omega$ .*

Unfortunately, a proof of Theorem 1.3.5 never appeared in a refereed journal. An announcement was published by Archdeacon [2], but so far a proof appeared only in Archdeacon's Ph.D. thesis [1].

In this thesis we propose a new proof of Theorem 1.3.5 based on the connectivity of a graph in the set  $\Omega$ . It consists of two steps and we carry out the first step.

There is a related result of Ding and Iverson [13]. Since in most applications the graphs under consideration are almost always well-connected, it is desirable to refine the set  $\Omega$  for graphs which are well-connected. The first attempt along these lines is a result of Robertson, Seymour, and Thomas (unpublished) which says a  $k$ -connected graph  $G$ ,  $k = 2, 3$  is projective planar if and only if it does not contain any  $k$ -connected

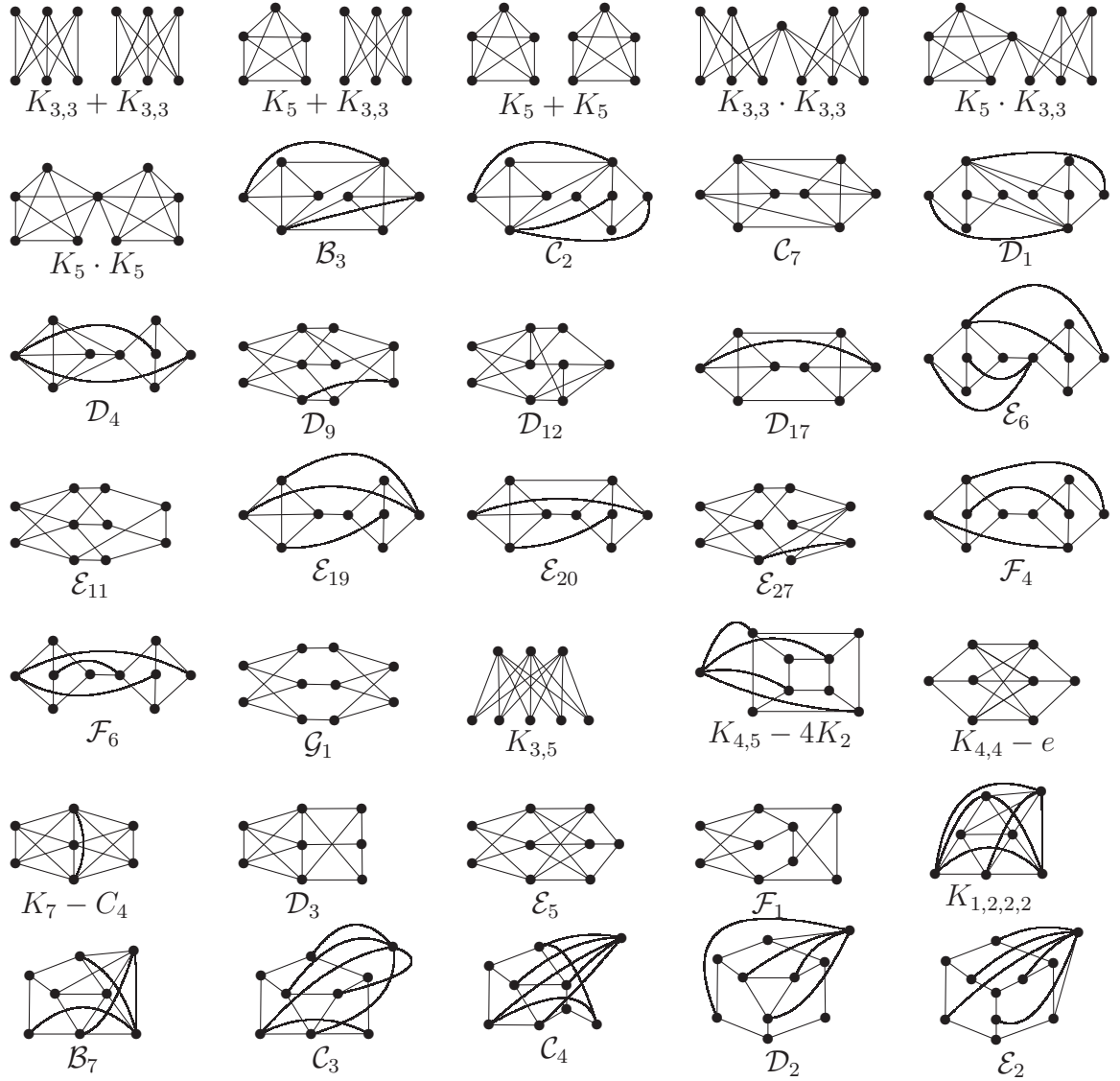


Figure 1.1: The set of minor minimal non-projective planar graphs

member of  $\Omega$  as a minor. There are several attempts to establish similar results for internally 4-connected graphs. We define the set  $\Omega_{i4c}$  to be the set of minor minimal internally 4-connected non-projective planar graphs. There are several attempts for determining the set  $\Omega_{i4c}$ . In particular, Maharry and Slilaty (unpublished) showed that internally 4-connected projective planar graphs can be characterized by excluding a subset of  $\Omega$  (some of which are not internally 4-connected). Thomas (unpublished) observed that in addition to the eleven internally 4-connected members of  $\Omega$  which belong to  $\Omega_{i4c}$ , there are at least two other minor-minimal internally 4-connected

non-projective graphs in the set  $\Omega_{i4c}$ . Finally Ding and Iverson [13], by making use of the set  $\Omega$ , found the set  $\Omega_{i4c}$  (it has size 23) and they proved that these are the only members of  $\Omega_{i4c}$ . So they completely determined the set  $\Omega_{i4c}$ .

As we mentioned earlier, the result of Ding and Iverson [13] is similar to our work in the sense that in both works, we focused on non-projective planar graphs with certain connectivity. However, our work tries to identify elements of  $\Omega$  while Ding and Iverson [13] use the set  $\Omega$  for characterizing the set  $\Omega_{i4c}$ .

## 1.4 *Non-Planar Extensions of Planar Graphs*

In this section, we are going to briefly mention a beautiful result and technique of Robertson, Seymour and Thomas [49], which is the main motivation toward the technique that we used to get our main result in this thesis.

If  $S$  is a subgraph of a graph  $G$ , then an  $S$ -path in  $G$  is a path with at least one edge, both ends in  $S$ , and otherwise disjoint from  $S$ . Robertson, Seymour and Thomas [49] proved the following result.

**Theorem 1.4.1.** *Let  $G$  be an internally 4-connected planar graph, and let  $H$  be an internally 4-connected non-planar graph containing a subdivision of  $G$  as a subgraph. Then there exists a subgraph  $S$  of  $H$  isomorphic to a subdivision of  $G$  such that one of the following conditions holds:*

- (i) *There exists an  $S$ -path in  $H$  such that no boundary of a face of  $S$  contains both ends of the path, or*
- (ii) *There exist two disjoint  $S$ -paths with ends  $s_1, t_1$  and  $s_2, t_2$ , respectively, such that the vertices  $s_1, s_2, t_1, t_2$  belong to some region boundary of  $S$  in the order listed. Moreover, for  $i = 1, 2$  the vertices  $s_i$  and  $t_i$  do not belong to the same segment of  $S$ , and if two segments of  $S$  include all of  $s_1, t_1, s_2, t_2$ , then those segments are vertex-disjoint.*

Let a non-planar graph  $H$  have a subgraph  $S$  isomorphic to a subdivision of a planar graph  $G$ . By using the above theorem, we can get useful information about the structure of minimal subgraphs of  $H$  that have a subgraph isomorphic to a subdivision of  $G$  and are non-planar.

We would like to mention some direct applications of the above theorem. Ding, Oporowski, Thomas and Vertigan [14] using Theorem 1.4.1 showed that except for one well-defined infinite family, there are only finitely many minimal graphs of crossing number at least two. By the Petersen graph we mean the complement of the line graph of  $K_5$ . Thomas and Thomson [51] showed that every internally 4-connected non-planar graph of girth at least five has a subgraph isomorphic to a subdivision of  $P_{10}^-$ , the Petersen graph with one edge deleted. This result implies that Tutte's 4-flow conjecture [55] holds for graphs with no subdivision isomorphic to  $P_{10}^-$ . Kawarabayashi, Norine, Thomas and Wollan [28] applied Theorem 1.4.1 to make a major step toward proving Jørgensen's conjecture [27]. By using their result [28], they [29] succeeded in proving Jørgensen's conjecture [27] for graphs with sufficiently large number of vertices.

In Chapters 2 and 3 we generalize a variation of Theorem 1.4.1 to certain projective planar graphs, and we use the developed theory to obtain our main results in this thesis.

### ***1.5 Application of the list of minor minimal non-projective planar graphs***

Despite the fact that the size of the set  $\Omega$ , the set of minor minimal non-projective planar graphs, is fairly large, it has useful application for attacking questions and conjectures related to graphs which can be or can not be embedded in the projective plane.

We start by showing an application of the set  $\Omega$  in settling the Hanani-Tutte's Conjecture [20, 56] for the projective plane. There are two versions of the conjecture:



the weak version, which has been settled, so we call it the weak Hanani-Tutte's Theorem, see Theorem 1.5.1, and strong Hanani-Tutte's Conjecture, see Conjecture 1.5.2. Note that the strong version implies the weak version.

**Theorem 1.5.1.** *If a graph can be drawn in a surface  $S$  so that every two edges cross an even number of times, then the graph can be embedded in the surface  $S$ .*

Hanani [20] and Tutte [56] proved the Conjecture for planar graphs, i.e. the surface  $\mathbb{S}^2$ . Using homology theory, Cairns and Nikolayevsky [9] showed that if a graph can be drawn on an orientable surface so that every pair of non-adjacent edges crosses an even number of times, then the graph can be embedded in that surface. Pelsmajer, Schaefer, and Stefankovic [40] gave a proof of the weak Hanani-Tutte's theorem (Theorem 1.5.1) for the projective plane, which also established the result for any non-orientable surface. These results completely established Theorem 1.5.1.

*Conjecture 1.5.2.* If a graph can be drawn in a surface  $S$  so that every two non-adjacent edges cross an even number of times, then the graph can be embedded in the surface  $S$ .

Pelsmajer, Schaefer, and Stefankovic [41] established Conjecture 1.5.2 for the projective plane. They used the list of minimal forbidden minors for the projective plane and their proof heavily depends on the list  $\Omega$  and it can not be extended to other surfaces in a natural way. This shows an application and importance of the list  $\Omega$ . For more details and history about the Hanani-Tutte's conjecture, we refer the reader to [40, 41].

The second application of the set  $\Omega$  would be a helpful tool to answer the following question:

*Question 1.5.3.* Determine the graphs which have two disjoint essential cycles in every embedding in a surface.

Mohar and Robertson [35] found a sufficient and necessary condition for graphs

embedded in the torus to have two disjoint essential cycles. They solve this problem by characterizing graphs embedded in the torus which do not have two disjoint essential cycles. They raised Question 1.5.3 in [35] as a possible generalization of their result. By using the list  $\Omega$ , Mohar and Robertson [36] answered Question 1.5.3. They showed that graphs which have two disjoint essential cycles in every embedding in surfaces, are precisely the graphs that cannot be embedded in the projective plane with exception of the graphs  $K_{3,k}$ ,  $k \geq 5$  and simple extensions of these graphs. We refer the reader to [36] to see more details and more precise statement of the result.

As a third application of the list  $\Omega$ , we should mention its application toward proving of a beautiful conjecture of Negami [39]:

*Conjecture 1.5.4.* A connected graph is projective planar if and only if it has a planar cover.

There was a lot of effort by various authors [3, 5, 22, 23, 24, 26, 38] in trying to prove the conjecture; however Conjecture 1.5.4 is still open. Previous results proved the following theorem:

**Theorem 1.5.5.** *Negami's conjecture (Conjecture 1.5.4) holds if and only if the graph  $K_{1,2,2,2}$  does not have a planar cover.*

The main resource for proving Theorem 1.5.5 was the list  $\Omega$ , since all authors [3, 5, 22, 23, 24, 26, 38] try to show that if  $G \in \Omega$  then  $G$  does not have a planar cover. For more details and history on planar covers, we refer the reader to a survey paper by Hliněný [25]

Fellows [16] conjectured that Conjecture 1.5.4 should hold for planar emulators. To be more precise, he conjectured that a connected graph  $G$  has a planar emulator if and only if  $G$  has a planar cover. One direction of the above statement holds trivially since any planar cover of  $G$  is also a planar emulator. Surprisingly, Rieck and Yamashita [42] found planar emulators for the graphs  $K_{1,2,2,2}$  and  $K_{4,5} - M_4$ .

Note that  $K_{1,2,2,2}, K_{4,5} - 4K_2 \in \Omega$ . The property of having a planar emulator is closed under taking minors. Thus, by the well known theorem of Robertson and Seymour [48], Theorem 1.3.3, there exists a finite set of graphs called  $\Lambda$ , such that  $G$  has a planar emulator if and only if  $G$  does not include any graph isomorphic to a member of  $\Lambda$  as a minor. Various authors [11, 25, 42] try to identify members of  $\Lambda$ . Their approach was to start from a graph  $G \in \Omega$  and study whether  $G$  has a planar emulator. For more details and results on planar emulators, we refer the reader to [10, 25].

Finally, we briefly mention a result of Robertson and Seymour [47], where they used the list  $\Omega$  to characterize which graphs are a minor of a graph that can drawn in the plane with at most one crossing.

**Theorem 1.5.6** ([47]). *A graph  $G$  is a minor of a graph  $H$  where  $H$  can be drawn in the plane with at most one crossing, if and only if  $G$  can be embedded in the projective plane in such a way that some non-null homotopic closed curve intersects the graph at most twice.*

## ***1.6 Previous approaches for finding the list of minor minimal non-projective planar graphs***

Toward settling the question of Erdős and König, Question 1.3.2, Glover, Huneke and Wang [18] proposed a list of 103 graphs, called  $\Omega'$ , for  $\mathbf{Forb}(\mathbb{R}\mathbb{P}^2)$ . We briefly explain some of their techniques and approaches here. First we need to define some basics.

A subgraph  $K$  of  $G$  is a  $K$ -graph if either  $K$  is a subdivision of a graph  $H$  isomorphic to  $K_{2,3}$  where there is a  $K$ -bridge with attachments on all vertices of  $K$  that correspond to a vertex of  $H$  of degree two, or  $K$  is a subdivision of a graph  $H$  isomorphic to  $K_4$ , where there is a  $K$ -bridge with attachments on all degree three vertices of  $H$ . It is easy to see that if  $G$  is embedded in the projective plane then any

$K$ -graph of  $G$  contains an essential cycle. Therefore, we have the following lemma:

**Lemma 1.6.1** ([18]). *If  $G$  contains two disjoint  $K$ -graphs, then  $G$  cannot be embedded in the projective plane.*

The above lemma was one of the key tools for the result of Glover, Huneke and Wang [18]. Let  $G$  be a graph,  $v \in V(G)$  and  $N(v) = \{v_1, v_2, \dots, v_j, v_{j+1}, \dots, v_n\}$ . We define a new graph  $S_{v:(1,\dots,j)}(G)$ , to be the graph obtained from  $G$  by deleting the edges  $vv_k$  for  $k = j + 1, \dots, n$  and adding a new vertex  $v'$  and edges  $vv'$  and  $v'v_k$  for  $k = j + 1, \dots, n$ . Observe that by contracting the edge  $vv'$  in  $S_{v:(1,\dots,j)}(G)$ , we get a graph isomorphic to the graph  $G$ . We call the above procedure *vertex splittings and edge deletions*. Note that we reserve the terminology of vertex splitting for some other definition which comes across in Section 1.8

**Lemma 1.6.2** ([18]). *For any surface  $S$ , if  $G$  cannot be embedded in  $S$  then  $S_{v:(1,\dots,j)}(G)$  cannot be embedded in  $S$ , for any vertex  $v \in V(G)$  and  $1 \leq j \leq |N(v)|$ .*

For any surface  $S$ , we can define a partial order relation  $\preceq$  in  $\mathbf{Forb}(S)$  as follow: for  $G, G' \in \mathbf{Forb}(S)$ , we say  $G' \preceq G$  if there exists  $v \in V(G)$  and  $1 \leq j \leq |N(v)|$  such that  $S_{v:(1,\dots,j)}(G)$  contains a subdivision of  $G'$  as subgraph. Glover, Huneke and Wang [18] proposed a set  $\mathcal{M}$  of five graphs as a maximal set for  $\mathbf{Forb}(\mathbb{RP}^2)$  with respect to the relation  $\preceq$  and they showed that all 103 graphs in  $\Omega'$  can be obtained from elements of  $\mathcal{M}$  using the vertex splittings and edge deletions procedures. We refer the reader to Appendix A in [37] to see the pictures of graphs in  $\mathbf{Forb}(\mathbb{RP}^2)$  and Figure 1.1 for graphs in  $\Omega$ .

Archdeacon [1] verified that the set  $\mathcal{M}$  is maximal and he checked that graphs in the set  $\Omega'$  are the only graphs that can be obtained from  $\mathcal{M}$  using the vertex splittings and edge deletions procedure. As we mentioned in Section 1.3, unfortunately, Archdeacon's work never appeared in a refereed journal. An announcement

was published by Archdeacon [2], but so far a proof appeared only in Archdeacon's Ph.D. thesis [1].

### ***1.7 Minor Minimal Non-Projective Planar Graphs***

In this section, we briefly explain our approach for finding the list  $\Omega$ . Suppose  $G$  is a non-projective planar minor minimal graph. We consider five different cases depending on the connectivity of  $G$ .

In the first case, if  $G$  is disconnected then  $G$  is a disjoint union of two Kuratowski graphs. Therefore, there are three possibilities: either  $G$  is isomorphic to  $K_{3,3} + K_{3,3}$ ,  $K_{3,3} + K_5$  or  $K_5 + K_5$ .

In the second case, if  $G$  has a separation of size one, then  $G$  is obtained by identifying two Kuratowski graphs along a single vertex. Therefore, there are three possibilities: either  $G$  is isomorphic to  $K_{3,3} \cdot K_{3,3}$ ,  $K_{3,3} \cdot K_5$  or  $K_5 \cdot K_5$ .

In the third case, if  $G$  has a separation of order two then  $G$  is obtained by identifying two Kuratowski graphs along two vertices and removing the edge between the two vertices if it exists. Therefore, there are six possibilities: either  $G$  is isomorphic to  $\mathcal{B}_3$ ,  $\mathcal{C}_2$ ,  $\mathcal{D}_1$ ,  $\mathcal{F}_6$ ,  $\mathcal{D}_4$  or  $\mathcal{E}_6$ .

The last two cases deal with the cases when  $G$  is 3-connected with an internal separation of size three and  $G$  is internally 4-connected, respecting the later two cases are more complicated. We solve the former and we outline an approach for the latter in the last Chapter.

The main idea for dealing with the last two cases is a generalization of the theory developed by Robertson, Seymour and Thomas [49], Theorem 1.4.1 for classification of all minor minimal non-planar extensions of a planar connected graph. In Section 1.8, we outline the procedure for finding members of  $\Omega$  which are 3-connected but have an internal separation of size three. This thesis is devoted to settling this case. As a result, if  $G \in \Omega$  is 3-connected with an internal separation of size three then there

are twelve possibilities, either  $G$  is isomorphic to  $\mathcal{C}_7$ ,  $\mathcal{D}_3$ ,  $\mathcal{D}_9$ ,  $\mathcal{D}_{12}$ ,  $\mathcal{E}_5$ ,  $\mathcal{E}_{11}$ ,  $\mathcal{E}_{19}$ ,  $\mathcal{E}_{27}$ ,  $\mathcal{F}_1$ ,  $\mathcal{G}_1$ ,  $K_7 - C_4$  or  $K_{3,5}$ . This is the statement of Theorem 1.8.1.

We briefly outline our plan to settle the case when  $G$  is internally 4-connected in Chapter 6.

## 1.8 Main Results

Let us now state the main results of this thesis.

**Theorem 1.8.1.** *Let  $G$  be a minor-minimal graph that cannot be embedded in the projective plane such that  $G$  is 3-connected but not internally 4-connected. Then  $G$  is isomorphic to one of the graphs  $\mathcal{C}_7$ ,  $\mathcal{D}_3$ ,  $\mathcal{D}_9$ ,  $\mathcal{D}_{12}$ ,  $\mathcal{E}_5$ ,  $\mathcal{E}_{11}$ ,  $\mathcal{E}_{19}$ ,  $\mathcal{E}_{27}$ ,  $\mathcal{F}_1$ ,  $\mathcal{G}_1$ ,  $K_7 - C_4$ ,  $K_{3,5}$ , depicted in Figure 1.2.*

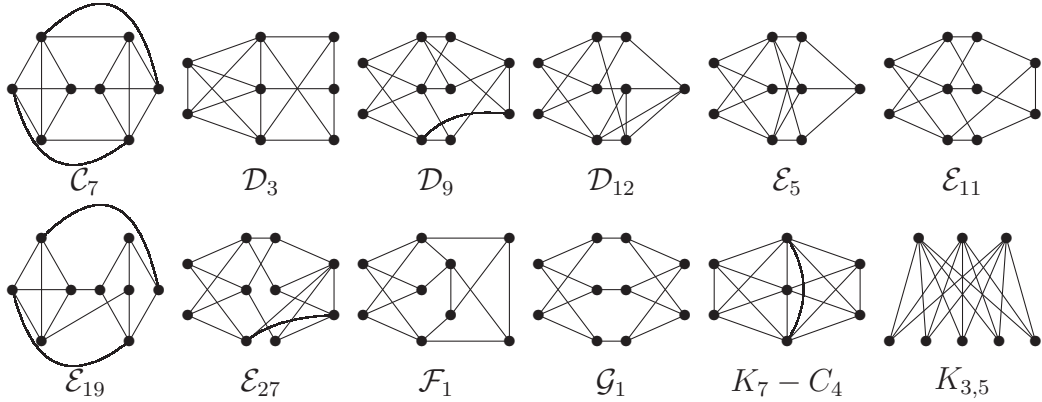


Figure 1.2: 3-connected minor-minimal non-projective planar graphs with an internal 3-separation.

Here, we briefly highlight the procedure of proving Theorem 1.8.1. In this procedure, we prove some theorems which may be of independent interest.

As we mentioned in Section 1.4, we are going to generalize Theorem 1.4.1 as a tool for further applications. First, we would like to motivate why an extension of Theorem 1.4.1 is useful.

Let  $G$  be a graph and let  $x_1, x_2, x_3$  be distinct vertices of  $G$ . We say that  $(G, x_1, x_2, x_3)$  is a *rooted graph*. We refer to  $x_1, x_2, x_3$  as *terminal vertices*. We say

two rooted graphs  $(G, x_1, x_2, x_3)$  and  $(H, y_1, y_2, y_3)$  are *isomorphic* if there exists an isomorphism  $\phi$  from  $V(G)$  to  $V(H)$  such that  $\phi(x_i) = y_i$  for all  $i \in \{1, 2, 3\}$ .

We say  $G'$  is obtained from  $G$  by *splitting*  $v$ , if there exist  $v_1, v_2 \in V(G')$  such that  $v_1$  and  $v_2$  are not adjacent in  $G'$  and  $G$  is isomorphic to the graph obtained from  $G'$  by identifying  $v_1, v_2$ , where  $v$  corresponds to the vertex obtained from identifying  $v_1, v_2$ .

We say a rooted graph  $(G, x_1, x_2, x_3)$  is  $x_1$ -*planar* if there exists a split of the vertex  $x_1$  to  $x'_1$  and  $x''_1$  in  $G$  such that the new graph  $G'$  obtained by the split has an embedding in a disk such that the vertices  $x'_1, x_2, x''_1, x_3$  are on the boundary of the disk in the order listed. We define  $x_2$ - and  $x_3$ -planarity analogously. We say that the rooted graph  $(G, x_1, x_2, x_3)$  is  $x_1x_2$ -*planar* if it is  $x_1$ -planar or  $x_2$ -planar, and we define  $x_1x_3$ -,  $x_2x_3$ - and  $x_1x_2x_3$ -planarity analogously. We say that the rooted graph  $(G, x_1, x_2, x_3)$  is *3-connected* if there exists no separation  $(G_1, G_2)$  of  $G$  of order at most two such that  $x_1, x_2, x_3 \in V(G_1)$  and  $V(G_2) - V(G_1) \neq \emptyset$ . Similarly we say that  $(G, x_1, x_2, x_3)$  is *internally 4-connected* if it is 3-connected and there exists no separation  $(G_1, G_2)$  of  $G$  of order three such that  $x_1, x_2, x_3 \in V(G_1)$ ,  $|V(G_1)| \geq 4$  and  $|E(G_2)| \geq 4$ .

Let  $(G, x_1, x_2, x_3)$  and  $(H, y_1, y_2, y_3)$  be two rooted graphs. We say  $(G, x_1, x_2, x_3)$  is a *rooted minor* of  $(H, y_1, y_2, y_3)$  if there exists a rooted graph  $(K, y_1, y_2, y_3)$  isomorphic to  $(G, x_1, x_2, x_3)$  obtained from a subgraph of  $H$  by contracting edges with the proviso that an edge with both ends in  $\{y_1, y_2, y_3\}$  cannot be contracted, if an edge with one end  $y_i$  is contracted, then the resulting new vertex will be labeled  $y_i$ , and no edge contraction produces a new edge between  $y_i$  and  $y_j$ ,  $1 \leq i < j \leq 3$ , such that  $(G, x_1, x_2, x_3)$  is isomorphic to  $(K, y_1, y_2, y_3)$ . For simplicity, we sometimes say  $H$  contains  $G$  as a minor where  $x_1, x_2, x_3$  correspond to  $y_1, y_2, y_3$ , respectively.

The following theorem is not hard to prove, although its proof is presented in Section 4.

**Theorem 1.8.2.** *Let  $G$  be a 3-connected minor-minimal non-projective planar graph with an internal 3-separation  $(G_1, G_2)$ . Let  $V(G_1) \cap V(G_2) = \{a, b, c\}$ . Then there does not exist  $x \in \{a, b, c\}$  such that the rooted graphs  $(G_1, a, b, c)$  and  $(G_2, a, b, c)$  are  $x$ -planar.*

Theorem 1.8.2 implies that if  $G, G_1, G_2, a, b, c$  are as in the theorem, then, after possibly permuting  $a, b, c$  and possibly swapping  $G_1$  and  $G_2$  we have that either

- $G_1$  is not  $a$ -planar and  $G_2$  is not  $bc$ -planar, or
- $G_2$  is not  $abc$ -planar.

Unfortunately, the converse does not hold: gluing together two graphs  $G_1, G_2$  as above along a cutset of size three may result in a graph that is projective planar. The difficulty lies in edges with both ends in  $\{a, b, c\}$ . Such edges may “migrate” from  $G_1$  to  $G_2$ , causing the aforementioned problem. This is the main reason why in the definition of minor minimal rooted graphs that are not  $a$ -,  $ab$ - or  $abc$ -planar, no edge contraction is allowed if it produces some new edges between terminal vertices. Thus, we need to understand graphs that are not  $c$ -planar, not  $ac$ -planar and not  $abc$ -planar. Therefore, we should study not “ $c$ -planar, not  $ac$ -planar and not  $abc$ -planar” extensions of planar graphs. This is the main motivation for generalizing Theorem 1.4.1 for our application. Theorem 2.2.10, and Theorem 3.0.17 in Chapter 2 and Chapter 3, respectively, are analogues of Theorem 1.4.1. By use of Theorem 2.2.10, and Theorem 3.0.17, we characterize minor minimal obstructions for not  $c$ -planar, not  $ac$ -planar and not  $abc$ -planar rooted graphs as follows.

**Theorem 1.8.3.** *Let  $(G, a, b, c)$  be a 3-connected rooted graph. If  $(G, a, b, c)$  is not  $c$ -planar then there exists a graph  $J \in \Omega_c$ , where  $\Omega_c = \{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4, \mathcal{O}_5, \mathcal{O}_6, \mathcal{O}_7\}$ , shown in Figure 1.3, such that  $(G, a, b, c)$  contains  $J(1, 2, 3)$  or  $J(2, 1, 3)$  as a rooted minor. Moreover, the rooted graphs in  $\Omega_c$  are minor minimal with respect to the not  $c$ -planarity property.*



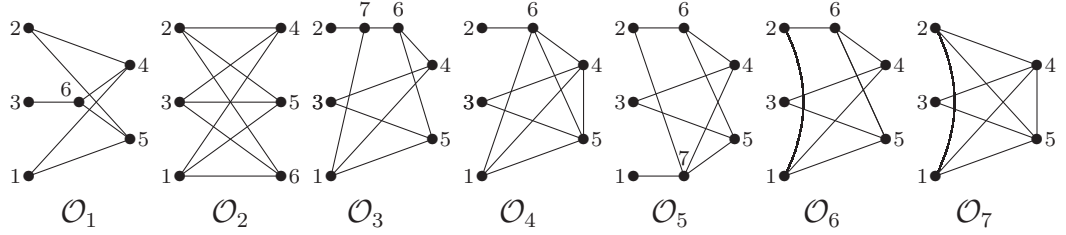


Figure 1.3: The set of minor minimal not 3-planar graphs satisfying the conditions mentioned in the statements of Theorem 4.1.4 and 1.8.3.

**Theorem 1.8.4.** *Let  $(G, a, b, c)$  be a 3-connected rooted graph. If  $(G, a, b, c)$  is not ac-planar then there exists a graph  $J \in \Omega_{ac}$ , where  $\Omega_{ac} = \{\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4, \mathcal{O}_5, \mathcal{O}_6, \mathcal{O}_8, \mathcal{O}_9, \mathcal{O}_{10}, \mathcal{O}_{11}, \mathcal{O}_{12}, \mathcal{O}_{15}, \mathcal{O}_{23}, \mathcal{O}_{24}, \mathcal{O}_{26}, \mathcal{O}_{28}\}$ , shown in Figure 1.4, such that  $(G, a, b, c)$  contains  $J(2, 1, 3)$  or  $J(3, 1, 2)$  as a rooted minor. Moreover, the rooted graphs in  $\Omega_{ac}$  are minor minimal with respect to the not ac-planarity property.*

**Theorem 1.8.5.** *Let  $(G, a, b, c)$  be a 3-connected rooted graph. If  $(G, a, b, c)$  is not abc-planar then there exists a graph  $J \in \Omega_{abc}$ , where  $\Omega_{abc} = \{\mathcal{O}_2, \mathcal{O}_5, \mathcal{O}_{12}, \mathcal{O}_{13}, \mathcal{O}_{14}, \mathcal{O}_{15}, \mathcal{O}_{16}, \mathcal{O}_{17}, \mathcal{O}_{18}, \mathcal{O}_{19}, \mathcal{O}_{20}, \mathcal{O}_{23}, \mathcal{O}_{24}, \mathcal{O}_{25}, \mathcal{O}_{26}, \mathcal{O}_{27}, \mathcal{O}_{28}\}$ , shown in Figure 1.5, such that  $(G, a, b, c)$  contains  $J(\alpha, \beta, \gamma)$  as a rooted minor for some  $\alpha, \beta, \gamma$  such that  $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$ . Moreover, the rooted graphs in  $\Omega_{abc}$  are minor minimal with respect to the not abc-planarity property.*

## 1.9 Outline of the Proof

In Section 2.1 in Chapter 2, we start with definitions which help us to state our results. For a graph  $G$  and a vertex  $c \in V(G)$ , we define a notion of  $c$ -disk system, Definition 2.1.1, where instead of face boundaries we work with a collection of cycles that cover every edge twice. This definition will be extremely useful for working with embeddings in the projective plane.

In Section 2.2 we prove some basic lemmas and we reduce the embedding problem to 2-list coloring of certain auxiliary graphs. By studying the obstructions of 2-list

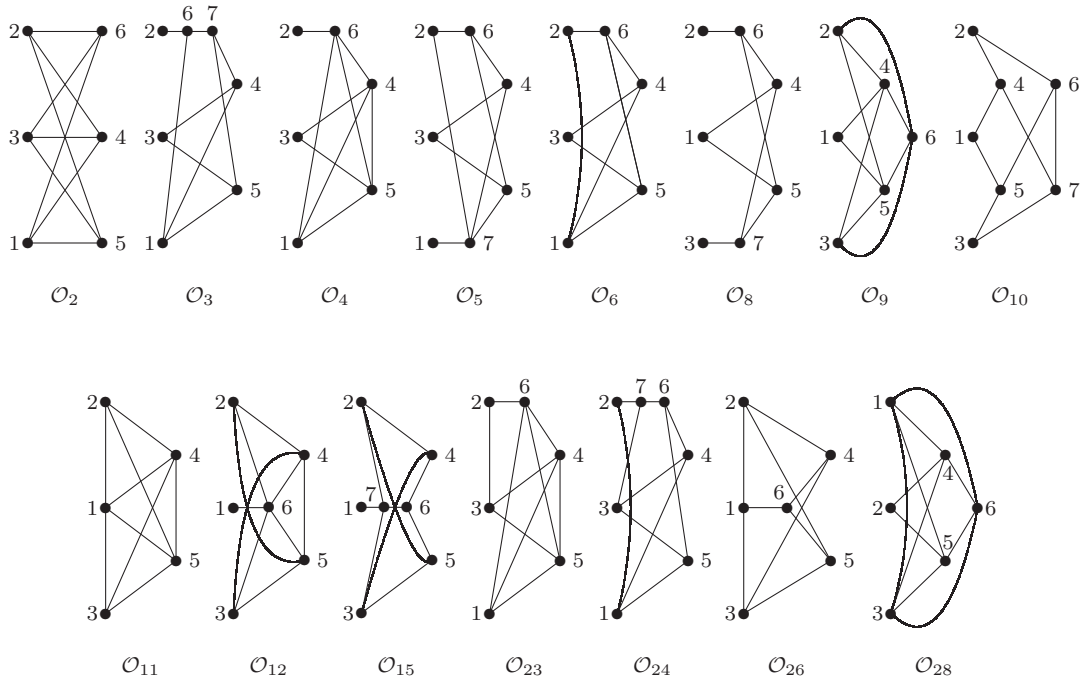


Figure 1.4: The set of minor minimal not 23-planar graphs satisfying the conditions mentioned in the statement of Theorem 1.8.4.

coloring of the auxiliary graph, we are able to build enough theory such that at the end of Chapter 2, we are able to prove Theorem 2.2.10, which is an analogue of Theorem 1.4.1.

In Chapter 3 we are going to focus on application of Theorem 2.2.10 to internally 4-connected rooted graphs in chapter 3. We will see some of the outcomes of Theorem 2.2.10 eliminated or substituted by other outcomes which are easier to deal with in the application. At the end of chapter 3 we present Theorem 3.0.17, which is another analogue of Theorem 1.4.1.

In Chapter 4, using the definition of  $c$ -disk system, Theorem 2.2.10 and Theorem 3.0.17, we are ready to find obstructions for  $c$ -,  $ac$ - and  $abc$ -planarity. We also present a proof of Theorem 1.8.2 from the beginning of this chapter.

In Section 4.1 we find the set of obstructions for  $c$ -planarity. First, by use of

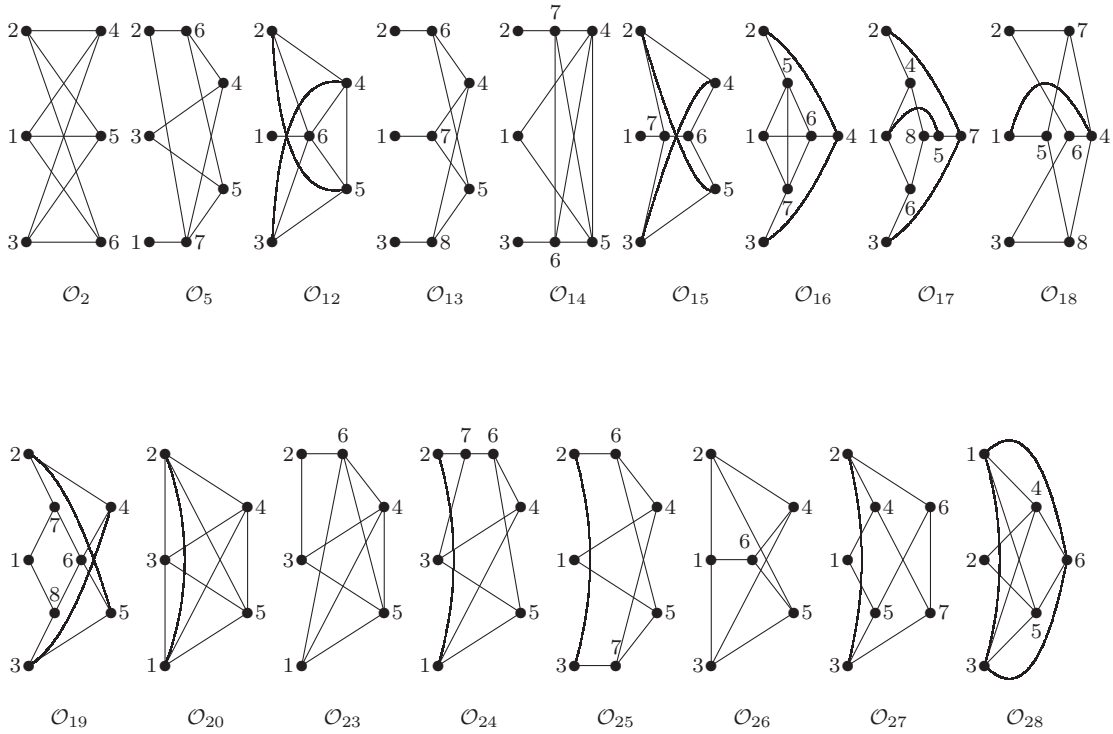


Figure 1.5: The set of minor minimal not 123-planar graphs satisfying the conditions mentioned in the statement of Theorem 1.8.5.

Theorem 3.0.17, we find the set of minor minimal rooted graphs such that an internally 4-connected rooted graph  $(G, a, b, c)$  must contain one if  $(G, a, b, c)$  is not  $c$ -planar. This is the statement of Lemma 4.1.3. Second by use of Lemma 4.1.3 and some beneficial lemmas presented in Section 4.1, we reduce the connectivity condition and at the end of Section 4.1, we prove Theorem 1.8.3 which gives the set of minor minimal obstructions for  $c$ -planarity for 3-connected rooted graphs.

In Section 4.2, for a rooted graph  $(G, a, b, c)$ , we repeat the same procedure as we did in Section 4.1 with the further assumption that the rooted graph  $(G, a, b, c)$  already contains a subdivision of one of the graphs listed in the statement of Theorem 1.8.3 as a subdivision. Therefore  $(G, a, b, c)$  is not  $c$ -planar and we try to prevent  $a$ -planarity. At the end of Section 4.2, we prove Theorem 1.8.4 which gives the set of minor minimal obstructions for  $ac$ -planarity of 3-connected rooted graphs.

In Section 4.3, for a rooted graph  $(G, a, b, c)$ , we repeat the same procedure as we did in Section 4.2 with the further assumption that the rooted graph  $(G, a, b, c)$  already contains a subdivision of one of the graphs listed in the statement of Theorem 1.8.4 as a subdivision. Therefore  $(G, a, b, c)$  is not  $ac$ -planar and we try to prevent  $b$ -planarity. At the end of Section 4.3, we prove Theorem 1.8.5 which gives the set of minor minimal obstructions for  $abc$ -planarity of 3-connected rooted graphs.

In the final chapter, Chapter 5, by applying Theorem 1.8.2 and having the list of minor minimal obstructions for  $c$ -,  $ac$ -,  $abc$ -planarity, obtained in Chapter 4, we start to glue the graphs on their terminal vertices to produce the list of minor minimal non-projective planar graphs. This will prove the main theorem.

## CHAPTER II

### NON- $c$ -PLANAR EXTENSIONS OF A $c$ -DISK SYSTEM

#### 2.1 *Definitions and Preliminaries*

Let  $S$  be a multigraph which is a subgraph of a multigraph  $H$ . We say a cycle  $C$  in the multigraph  $S$  is *proper* if its length is at least three. An  $S$ -*bridge* in  $H$  is a connected subgraph  $B$  of  $H$  such that  $E(B) \cap E(S) = \emptyset$  and either  $E(B)$  consists of a unique edge with both ends in  $S$ , or for some component  $C$  of  $H \setminus V(S)$  the set  $E(B)$  consists of all edges of  $H$  with at least one end in  $V(C)$ . The vertices in  $V(B) \cap V(S)$  are called the *attachments* of  $B$ .

Let  $G$  be a multigraph with minimum degree three. A multigraph  $S$  is a *subdivision* of a multigraph  $G$  if  $S$  is obtained from  $G$  by replacing its edges by internally disjoint nonzero length paths with the same ends, called *segments*. For an edge  $uv \in E(G)$  we denote the segment connecting  $u, v$  in  $S$  by  $\text{seg}(u, v)$ . We denote  $\text{seg}[u, v] \setminus \{u\}$ ,  $\text{seg}[u, v] \setminus \{v\}$ ,  $\text{seg}[u, v] \setminus \{u, v\}$  by  $\text{seg}(u, v]$ ,  $\text{seg}[u, v)$ ,  $\text{seg}(u, v)$ , respectively.

Let  $G, S, H$  be multigraphs such that  $S$  is a subgraph of  $H$  and is isomorphic to a subdivision of  $G$ . In that case we say that  $S$  is a  $G$ -*subdivision* in  $H$ . Suppose  $V(G) = \{v_1, \dots, v_n\}$  and  $V(H) = \{u_1, \dots, u_m\}$  and  $H$  contains  $G$  as a subdivision. It is easy to see that for showing this fact it is enough to identify the branch vertices of  $G$  in  $H$ . For doing that we use the notation  $(v_1, v_2, \dots, v_n) \hookrightarrow (u_{i_1}, \dots, u_{i_n})$ , implying  $u_{i_j}$  corresponds to  $v_j$ ,  $1 \leq j \leq n$ , and we call it *signature*.

Let  $P$  be a path in  $H$ . Let  $\text{Int}(P)$  be an empty set if the length of  $P$  is at most one; otherwise  $\text{Int}(P)$  denotes the path obtained from  $P$  by removing its end vertices.

We say a set of cycles  $\mathcal{C}$  is a *double cycle cover* of  $G$  if for any edge  $e \in E(G)$ ,

there exist exactly two cycles  $C_1, C_2 \in \mathcal{C}$  both containing  $e$ .

**Definition 2.1.1.** *Let  $G$  be a multigraph and  $c$  be a vertex of  $G$  with degree at least three. We say a double cycle cover  $\mathcal{C}$  is  $c$ -disk system if it has the following properties:*

- (i) *For any two distinct cycles  $C_1, C_2 \in \mathcal{C}$ , either  $C_1 \cap C_2$  has only one component and it is a segment, null or a vertex, or  $C_1 \cap C_2$  has exactly two components where one of its components is exactly the single vertex  $c$  and the other one is a segment or a vertex, in which case for any cycle  $C \in \mathcal{C} \setminus \{C_1, C_2\}$ ,  $C \cap C_1$  and  $C \cap C_2$  are a segment, null or a vertex.*
- (ii) *If  $e_1, e_2, e_3$  are three distinct edges incident with a vertex  $v$  of  $G$  and there exist cycles  $C_1, C_2, C_3 \in \mathcal{C}$  such that  $e_i \in E(C_j)$  for all  $1 \leq i \neq j \leq 3$ , then  $v$  has degree three.*
- (iii) *For  $C_1, C_2 \in \mathcal{C}$  with  $C_1 \cap C_2 = P \cup \{c\}$  where  $P$  is a segment or vertex and  $c \notin V(P)$ , there exists an edge  $e$  from  $c$  to one end of  $P$ , and there is no segment other than  $e$  containing  $c$  and the other end of  $P$ .*
- (iv) *For any three distinct proper cycles  $C_1, C_2, C_3 \in \mathcal{C}$ , we have  $|V(C_1) \cap V(C_2) \cap V(C_3)| \leq 1$ .*

A trivial example of a  $c$ -disk system is the set of facial cycles of any 3-connected plane graph  $G$  for any choice of  $c \in V(G)$ . Figure 2.1 illustrates a non-trivial example of a  $c$ -disk system.

We refer to an element of a  $c$ -disk system as a disk. We say  $v_1, v_2 \in V(G)$  are *cofacial* in  $\mathcal{C}$  if there exists  $C \in \mathcal{C}$  such that  $\{v_1, v_2\} \subseteq V(C)$ .

Let  $c$  be a vertex of  $G$  and  $\mathcal{C}$  is a  $c$ -disk system of  $S$ .

If  $B$  is an  $S$ -bridge of  $H$ , then we say that  $B$  is *unstable* if some segment of  $S$  includes all the vertices of attachments of  $B$ , and otherwise we say that  $B$  is *rigid*.

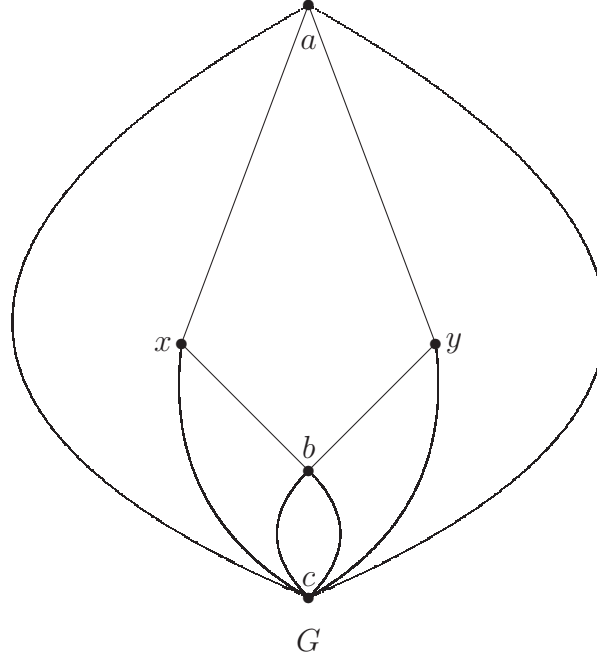


Figure 2.1: The set of facial cycles of the graph  $G$ ,  $\mathcal{C} = \{axc, axby, xcb, bc, ycb, ayc, ac\}$ , is a  $c$ -disk system.

We say a segment  $P$  is a *special segment* if  $c \notin V(P)$  and there exist  $C_1, C_2 \in \mathcal{C}$  such that  $\{c\} \cup V(P) = V(C_1) \cap V(C_2)$ .

Let  $P$  be a path in  $H$  and  $x, y \in V(P)$ , then we denote the subpath of  $P$  connecting  $x$  and  $y$  by  $xPy$ . Suppose  $P$  and  $P'$  are two internally disjoint paths connecting  $x, y$  and  $y, z$ , respectively. We denote the unique path  $P'' \subseteq V(P) \cup V(P')$  connecting  $x$  to  $z$  by  $xP'yP'z$ . Let  $|P|$  denote the length of the path  $P$ ; i.e. the number of edges in  $P$ . The  $P$ -distance of  $x, y$  is defined by the length of the sub-path  $xPy$ .

Let  $B, B'$  be two distinct  $S$ -brides with the set of attachments  $A$  and  $A'$ , respectively. Suppose there exists a cycle  $C \in \mathcal{C}$  such that  $A \cup A' \subseteq V(C)$ . We say  $B$  and  $B'$  are *crossing* in  $C$ , if  $B \cup C \cup C'$  does not have an embedding in the plane such that  $C$  bounds a region.

A path  $P$  in  $H$  is an  $S$ -path if it has at least one edge, and only its ends belong to  $S$ .

An  $S$ -path  $P$  with ends  $u, v$  is an  $S$ -jump if no cycle of  $\mathcal{C}$  includes both ends of  $P$ .

Let  $v_1, v_2, v_3 \in V(S)$  and  $v \in V(H) \setminus V(S)$ , and let  $P_1, P_2, P_3$  be three paths in  $H$  such that  $P_i$  has ends  $v$  and  $v_i$ , they are pairwise disjoint except for  $v$ , and each is disjoint from  $V(S) \setminus \{v_1, v_2, v_3\}$ . Moreover  $v_1, v_2, v_3$  do not belong to the same segment of  $S$ . In those circumstances we say that the triple  $(P_1, P_2, P_3)$  is an *S-fork*, the vertices  $v_1, v_2, v_3$  are its *feet* and the vertex  $v$  is its *center*. If each pair  $v_i, v_j$ ,  $1 \leq i < j \leq 3$ , are cofacial but no cycle in  $\mathcal{C}$  contains all of  $v_1, v_2, v_3$ , then we say the *S-fork* is an *S-triad*.

Let  $v$  be a vertex of degree three in  $S$  and  $Q_1, Q_2, Q_3$  be the segments of  $S$  with one end  $v$  and the other ends  $v_1, v_2, v_3$ , respectively. Let  $(P_1, P_2, P_3)$  be an *S-triad* with feet  $\{x_1, x_2, x_3\}$  where  $x_i \in V(Q_i)$ ,  $1 \leq i \leq 3$ . In these circumstances we say the *S-triad* is *local around v*. We say the sub-paths  $x_1Q_1v_1$ ,  $x_2Q_2v_2$  and  $x_3Q_3v_3$  are the *legs* of the local triad. Let  $S'$  be a  $G$ -subdivision obtained from  $S$  by removing  $V(vQ_1v_1) \cup V(vQ_2v_2) \cup V(vQ_3v_3)$  and adding  $V(P_1) \cup V(P_2) \cup V(P_3)$ . We say  $S'$  is obtained from  $S$  by a *triad exchange*.

Let  $P_1, P_2, P_3$  and  $Q_1, Q_2, Q_3$  be two *S-forks* with feet on  $x_1, x_2, x_3$  and centers  $u, v$ , respectively, such that  $V(P_i) \cap V(Q_j) = \emptyset$  for all  $1 \leq i \neq j \leq 3$ , and  $P_i \cap Q_i$  is a path with an end  $x_i$ . In this circumstance we say that  $(P_1, P_2, P_3; Q_1, Q_2, Q_3)$  is a *double fork* in  $H$ . We say a double fork is *connected* if  $u, v$  are in the same *S-bridge* of  $H$ .

Let  $P_1$  and  $P_2$  be two disjoint *S-paths* with ends  $u_1, v_1$  and  $u_2, v_2$ , respectively, such that  $u_1, u_2, v_1, v_2$  belong to  $V(C)$  and occur on  $C$  in the order listed. In these circumstances we say that the pair  $(P_1, P_2)$  is an *S-cross* on  $C$ . We call  $u_1, u_2, v_1, v_2$  the *feet* of the cross in the order listed, and  $P_1, P_2$  the *arms* of the cross. We say a cross is *solid* if either none of its feet is the vertex  $c$  or if  $c$  is an end of  $P_i$  for some  $i \in \{1, 2\}$  then the other end of  $P_i$  does not belong to a special segment. We say the cross  $(P_1, P_2)$  is *weakly free* on  $C$  if for  $i = 1, 2$  no segment of  $C$  includes both ends of  $P_i$ . We say a weakly free cross is *free* if no two segments of  $C$  that share a vertex



include all the feet of the cross. A cross, denoted by  $(P_1, P_2, P)$ , is called *connected* if there exists a path  $P$  with ends  $u, v$  distinct from  $u_1, u_2, v_1, v_2$  such that  $u \in V(P_1)$  and  $v \in V(P_2)$  and  $P$  is disjoint from  $P_1, P_2$  and  $S$  except at  $u$  and  $v$ . We call the vertices  $u, v$  the *connections* of the connected cross. We say a connected  $S$ -cross is *c-blocking* if  $P_i$ , for some  $i \in \{1, 2\}$ , has one end  $c$  and the other end on a special segment. Note that a  $c$ -blocking  $S$ -cross is weakly free and it is contained in a rigid bridge, by its definition.

Let  $(P_1, P_2)$  be a cross as described in the previous paragraph. We say  $(P_1, P_2, P)$  is a *degenerate*  $S$ -cross if there exists a special segment  $Q$  in  $C$  such that  $u_1 = c, v_1 \in V(Q)$ , and  $u_2 \in V(C) \setminus (V(Q) \cup \{c\}), v_2 \in V(Q)$  and there exists a path  $P$  from  $u \in V(Int(P_1))$  to  $u_2$  which is disjoint from  $S, P_1, P_2$  except at its ends. See Figure 2.2 (A) for an illustration of it.

Let  $(P_1, P_2)$  be an  $S$ -cross as defined above and  $Q_1, Q_2$  be two segments of  $S$  sharing a vertex  $x \neq c$  of degree at least four such that  $u_1, u_2 \in V(Q_1), v_1, v_2 \in V(Q_2)$ . Suppose there exists a cycle  $C, C_1, C_2 \in \mathcal{C}$  such that  $Q_1 \cup Q_2 \subseteq C$  and  $Q_1 \subset C_1 \cap C, Q_2 \subset C_2 \cap C$ , and  $c \notin V(Q_1) \cup V(Q_2)$ . Assume  $u_2, u_1, x, v_2, v_1$  appear in  $C$  as the order listed and there exists a path  $P$  from vertex  $u \in (V(P_1) \cup V(P_2) \cup V(xQ_1u_2) \cup V(xQ_2v_1))$  to  $c$  which is disjoint from  $S \cup P_1 \cup P_2$  excepts at its ends. In these circumstances we say  $(P_1, P_2, P)$  is an  *$S$ -cross anchored at  $c$* . See Figure 2.2 (C) for an illustration of it.

Let  $Q$  be a special segment of  $S$  and  $C_1, C_2 \in \mathcal{C}$  be such that  $V(C_1) \cap V(C_2) = V(Q) \cup \{c\}$ . Let  $P_1$  and  $P_2$  be two disjoint  $S$ -paths with ends  $u_1, v_1 \in V(C_j)$  and  $u_2, v_2 \in V(C_{3-j}), j \in \{1, 2\}$ , respectively, such that  $c \notin \{u_1, v_1, u_2, v_2\}$ . Moreover assume that  $P_1, P_2$  each has exactly one end on  $V(Q)$  and there exists an  $S$ -path  $P$  with one end on  $c$  and the other one on  $u \in V(Q)$  such that  $(P, P_1)$  and  $(P, P_2)$  are  $S$ -crosses. In these circumstances, we say that the triple  $(P, P_1, P_2)$  is a *double facial  $S$ -cross*. See Figure 2.2 (B) for an illustration of it.

Let  $P$  be a segment of  $S$  with ends  $z, w$ . Let  $P_1, P_2$  be two  $S$ -paths with ends  $x_1, y_1$  and  $x_2, y_2$ , respectively such that  $z, x_1, x_2, y_1, w$  appear on  $P$  in the order listed. Assume that  $y_2 \notin V(P)$  and there exists a path  $P_3$  from  $x \in \text{Int}(P_1)$  to  $y \in V(P_2) \setminus \{x_2\}$ . In these circumstances we say that the triple  $(P_1, P_2, P_3)$  is a *tripod* based at  $P$ . We say  $x_1, y_1, x_2, y_2$  are its *feet* in the order listed, and  $zPx_1, y_1w$  and  $yP_2y_2$  are its *legs*.

Let  $Q$  be a special segment of  $S$  with ends  $x, y$  such that  $cx \in E(S)$  by property(iii) in Definition 2.1.1. Suppose  $C_1, C_2 \in \mathcal{C}$  are such that  $V(C_1) \cap V(C_2) = V(Q) \cup \{c\}$ . Let  $(P_1, P_2, P_3)$  be an  $S$ -fork with feet on  $c, u, v$  and center  $o$  where  $\{u, v\} \subseteq V(Q)$  and  $x, u, v, y$  appear on  $Q$  in the order listed. Let  $P$  be an  $S$ -path disjoint from  $P_1, P_2, P_3$  except at  $c$  connecting  $c$  to  $w$ , where  $w \in V(\text{Int}(vQu))$ . Let  $\alpha : c = v_1, v_2, \dots, v_n = y$  be the path contained in  $V(C_1)$  between  $c, y$  and internally disjoint from  $Q$ . Assume  $R$  is an  $S$ -path disjoint from  $P, P_1, P_2, P_3$  between  $v' \in V(\text{Int}(\alpha))$  and  $u' \in V(uQx)$ . In these circumstances we say  $H$ -contains a *blocking interlaced  $S$ -fork of type I*. See Figure 2.2 (D) for an illustration of it.

Let  $P, P_1, P_2, P_3$  be as in the previous paragraph. Let  $(R_1, R_2, R_3)$  be a fork with feet on  $c, u, v$  and center  $o'$  disjoint from  $P, P_1, P_2, P_3$  except at  $c, u, v$ . In these circumstances we say  $H$  contains a *blocking interlaced  $S$ -fork of type II*. We call  $o$  and  $o'$  the centers and  $w$  the connections of a *blocking interlaced  $S$ -fork of type II*. See Figure 2.2 (E) for an illustration of it.

Let  $P$  be a segment of  $S$  of length at least two, and let  $Q$  be a path in  $H$  with ends  $x, y \in V(P)$  and otherwise disjoint from  $S$ . Let  $S'$  be obtained from  $S$  by replacing the path  $xPy$  by  $Q$ ; then  $S'$  is a  $G$ -subdivision in  $H$  and we say that  $S'$  is obtained from  $S$  by *I-rerouting*  $P$  through  $Q$ , or simply *I-rerouting*. Let  $v$  be a vertex of  $S$  of degree  $k$ , let  $P_1, P_2, \dots, P_k$  be the segments of  $S$  incident with  $v$ . let  $k = 3$ ,  $x \in V(P_1) \setminus \{v\}, y \in V(P_2) \setminus \{v\}$  and let  $P$  be an  $S$ -path joining  $x, y$ . Let  $S'$  be obtained from  $S$  by removing the internal vertices of the path  $yP_2v$  and replacing

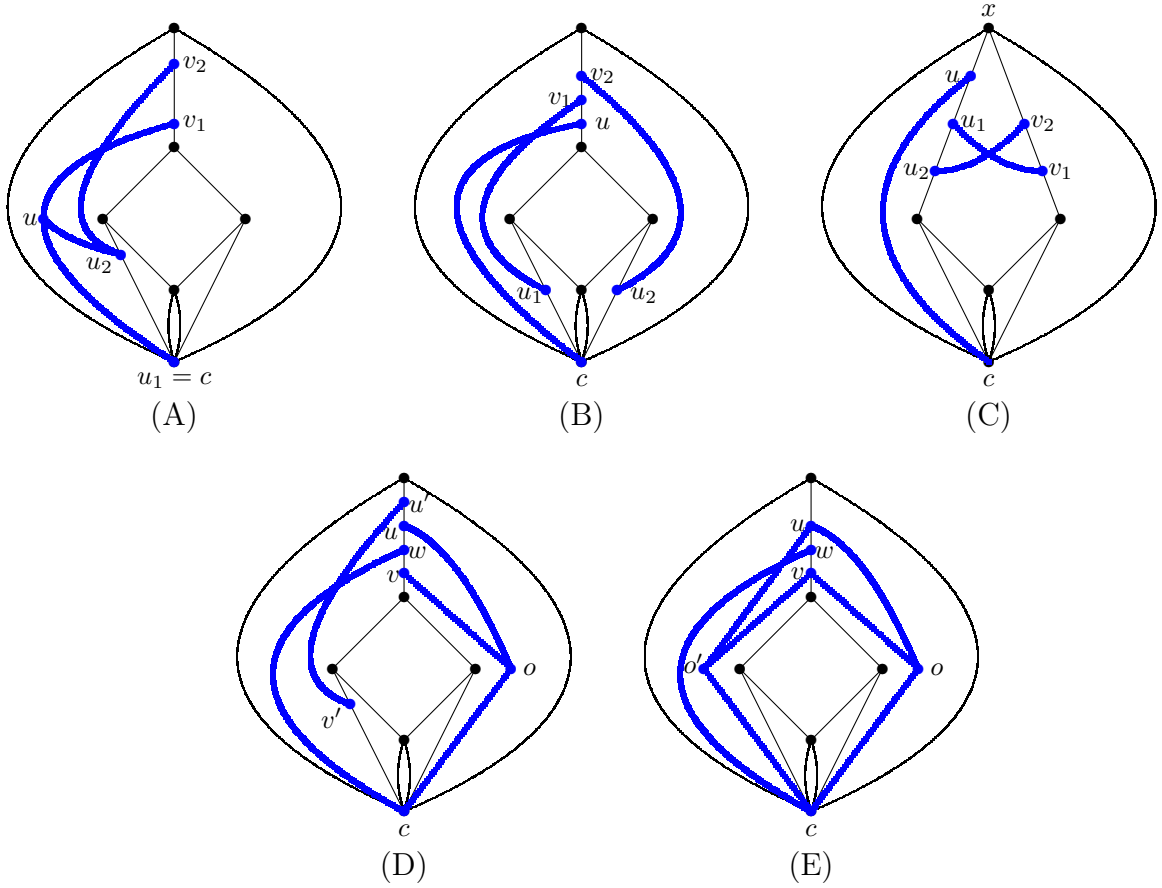


Figure 2.2: The above figures show illustrations of (A): degenerate cross, (B): double facial cross, (C): weakly free cross anchored at  $c$ , (D): blocking interlaced fork of type I, (E): blocking interlaced fork of type II.

by the path  $xPy$ . In those circumstances  $S'$  is called a  $T$ -rerouting of  $S$ . Now, let  $k \geq 4$  and  $u_1, u_2 \in V(P_1) \setminus \{v\}$  and  $v_1, v_2 \in V(P_2) \setminus \{v\}$  be distinct vertices such that the vertices  $u_1, u_2, v, v_1, v_2$  appear on the path  $P_1 \cup P_2$  in the order listed, and for  $i = 1, 2$  let  $Q_i$  be an  $S$ -path in  $H$  with ends  $u_i$  and  $v_i$  such that  $Q_1$  and  $Q_2$  are disjoint. Moreover assume that there exists a cycle  $C \in \mathcal{C}$  containing  $P_1 \cup P_2$ . Let  $S'$  be obtained from  $V(S) \cup V(Q_1) \cup V(Q_2)$  by deleting the edges and internal vertices of the paths  $u_1P_1u_2$  and  $v_1P_2v_2$ . Then  $S'$  is a  $G$ -subdivision in  $H$ , and we say that  $S'$  is obtained by an  $X$ -rerouting from  $S$ .

Let  $v_1, v_2, v_3, v_4 \in V(S)$  be branch vertices of  $S$  such that  $v_4 = c$ ,  $v_1, v_2, v_3$  are distinct from  $c$ . Suppose  $C$  is a connected component of  $H \setminus S$  and  $P_1, P_2, P_3, P_4 \subset (V(C) \cup \{v_1, v_2, v_3, v_4\})$  are four internally disjoint paths such that  $P_i$  has ends on

$v_i$  and a vertex in  $V(C)$  for all  $1 \leq i \leq 4$ . Assume that for each  $\{v_i, v_j, v_k\} \subset \{v_1, v_2, v_3, v_4\}$  there exists a cycle in  $\mathcal{C}$  containing  $v_i, v_j, v_k$ , but no cycle in  $\mathcal{C}$  contains all of  $v_1, v_2, v_3, v_4$ . Moreover suppose  $v_i$  and  $v_j$ ,  $1 \leq i \neq j \leq 3$  are connected by a segment in  $S$ . In those circumstances we say that the quadruple  $(P_1, P_2, P_3, P_4)$  is an *S-pyramid* with *feet* on  $v_1, v_2, v_3, c$  in the order listed.

**Definition 2.1.2.** Let  $\mathcal{C}$  be a *c-disk system* in  $S$ . We say a vertex  $v \neq c$  of  $S$  is *ambiguous* if either

(i)  $v$  belongs to the intersection of two disks both containing  $c$ , or

(ii)  $v$  belongs to a segment whose ends satisfy condition (i) and the segment is a subgraph of a disk containing  $c$ .

**Lemma 2.1.3.** Let  $G$  be a graph with minimum degree three and  $c$  be a vertex of  $G$ . Let  $H$  be a graph and  $S$  be a  $G$ -subdivision in  $H$  with a *c-disk system*  $\mathcal{C}$ . If  $H$  contains an *S-pyramid* then the graph induced by ambiguous vertices of  $S$  is not a forest.

*Proof.* Suppose  $H$  contains an *S-pyramid* with feet on  $v_1, v_2, v_3, c$ . The definition of *S-pyramid* implies that  $v_1, v_2, v_3, c \in V(S)$  and  $v_1, v_2, v_3$  are ambiguous. By (i) in Definition 2.1.2, if  $v$  is a vertex on the segment connecting  $v_i$  to  $v_j$ ,  $1 \leq i \neq j \leq 3$  then  $v$  belongs to a cycle of  $\mathcal{C}$  which contains  $c$ . So by (ii) in Definition 2.1.2,  $v$  is ambiguous. This immediately implies that the graph induced by ambiguous vertices of  $S$  have a cycle. This completes the proof of the lemma.  $\square$

A *separation* in a graph  $G$  is a pair of subgraphs  $(G_1, G_2)$  such that  $E(G_1) \cap E(G_2) = \emptyset$  and  $G_1 \cup G_2 = G$ . The *order* of the separation is  $|V(G_1) \cap V(G_2)|$ .

We say a separation  $(G_1, G_2)$  is *internal* if  $|E(G_1)|, |E(G_2)| \geq 4$ . We say a graph or multigraph  $G$  is *internally 3-connected* if it is 2-connected, has at least five vertices, and has no internal separation of order two. Similarly, we say a graph or multigraph  $G$  is *internally 4-connected* if it is 3-connected, has at least five vertices, and has no internal separation of order three.

## 2.2 Useful lemmas

**Theorem 2.2.1.** *Let  $G$  be a multigraph with minimum degree three,  $H$  be a 3-connected multigraph, and let  $S$  be a  $G$ -subdivision in  $H$  satisfying the following condition:*

(\*) *if  $e, f$  are parallel edges in  $H$  then  $e, f$  are parallel in  $S$ .*

*Then there exists a  $G$ -subdivision  $S'$  in  $H$  obtained from  $S$  by a sequence of  $I$ -reroutings such that every  $S'$ -bridge is rigid.*

*Proof.* Choose a  $G$ -subdivision  $S'$  obtained from  $S$  by a sequence of  $I$ -reroutings such that the number of vertices that belong to rigid  $S'$ -bridges is maximum. Then  $S'$  satisfies condition (\*). We show that all  $S'$ -bridges are rigid.

For any  $S'$ -bridge  $B$ , let  $A(B)$  denote the set of attachments of  $B$ . Suppose there exists a segment  $P : v_0v_1 \dots v_n$  of  $S'$  and an  $S'$ -bridge  $B^*$  such that  $A(B^*) \subseteq V(P)$ . Let  $\mathcal{B}$  be the set of all  $S'$ -bridges such that their sets of attachments are contained in  $V(P)$ . Note that  $\mathcal{B} \neq \emptyset$  since  $B^* \in \mathcal{B}$ . We claim that there exists a bridge  $B \in \mathcal{B}$ ,  $x, y \in A(B)$  and an  $S'$ -bridge  $B'$  such that  $A(B') \cap V(\text{Int}(xPy)) \neq \emptyset$  and  $A(B') \cap (V(S) \setminus V(P)) \neq \emptyset$ . For any  $B \in \mathcal{B}$ , let  $x_B, y_B \in A(B)$  be such that the  $P$ -distance of  $x_B, y_B$  is maximum among all pairs of vertices in  $A(B)$ . Let  $A = \bigcup_{B \in \mathcal{B}} A(B)$ ,  $x^*, y^* \in A$  be such that the  $P$ -distance of  $x^*, y^*$  is maximum among all pairs of vertices in  $A$ . The fact that  $H$  is 3-connected and condition (\*) imply that there is no  $0 \leq i \leq n - 1$  and  $B \in \mathcal{B}$  such that  $\{x_B, y_B\} = \{v_i, v_{i+1}\}$ . Note that for any  $B \in \mathcal{B}$  since  $\{x_B, y_B\}$  is not a cut set in  $H$  there exists an  $S'$ -bridge  $B'$  such that  $B'$  has an attachment in  $V(\text{Int}(x_BPy_B))$  and an attachment in  $V(S) \setminus V(x_BPy_B)$ . Therefore either there exists a  $B \in \mathcal{B}$  such that the bridge  $B'$  has an attachment on  $V(S) \setminus V(P)$  in which case the the proof of the claim is completed, or for any  $v \in V(\text{Int}(x^*Py^*))$  there exists a an  $S'$ -bridge  $B \in \mathcal{B}$  such that  $v \in V(\text{Int}(x_BPy_B))$ . Since  $H$  is 3-connected  $\{x^*, y^*\}$  is not an  $S'$ -cut set. Thus there exists an  $S'$ -bridge

$B''$  and  $v \in V(\text{Int}(x^*Py^*))$  such that  $v \in A(B'')$  and  $A(B'') \cap (V(S) \setminus V(P)) \neq \emptyset$ . By the above statement there exists an  $S'$ -bridge  $B$  such that  $v \in V(x_BPy_B) \setminus \{x_B, y_B\}$ . This completes the proof of the claim.

Assume  $B$  is such a bridge and let  $Q \subseteq B$  be a path connecting  $x_B, y_B$  and let  $S''$  be a  $G$ -subdivision of  $S'$  obtained by  $I$ -rerouting of  $x_BPy_B$  through  $Q$ . Note that any rigid  $S'$ -bridge  $\tilde{B}$  where  $A(\tilde{B}) \cap (V(x_BPy_B) \setminus \{x_B, y_B\}) = \emptyset$ , is a rigid  $S''$ -bridge, and there exists a rigid  $S''$ -bridge containing  $V(x_BPy_B)$  and  $V(\tilde{B})$  for all  $S'$ -bridge  $\tilde{B}$  where  $A(\tilde{B}) \cap V(\text{Int}(x_BPy_B)) \neq \emptyset$ . This contradicts the choice of  $S'$ .  $\square$

**Lemma 2.2.2.** *Let  $G$  be a multigraph with minimum degree three and  $c \in V(G)$ . Let  $H$  be a 3-connected multigraph, and let  $S$  be a  $G$ -subdivision in  $H$  satisfying condition  $(*)$  in Theorem 2.2.1. Let  $\mathcal{C}$  be a  $c$ -disk system in  $S$ , and  $B$  be an  $S$ -bridge such that no disk includes all attachments of  $B$ . Then one of the following conditions holds.*

- (i) *there exists an  $S$ -jump, or*
- (ii) *there exists an  $S$ -triad, or*
- (iii) *there exists an  $S$ -pyramid, or*
- (iv)  *$G$  is isomorphic to  $K_4$ .*

*Proof.* We may assume that (i) and (ii) do not hold. Let  $A$  be the set of all attachments of  $B$  and  $k$  be the maximum integer such that for every  $k$ -element subset  $A'$  of  $A$  there exists a disk  $C \in \mathcal{C}$  such that  $A' \subseteq V(C)$ . By the lemma's hypothesis and our assumption  $3 \leq k < |A|$ , there exist distinct vertices  $a_1, a_2, \dots, a_{k+1} \in A$  such that  $a_1, a_2, \dots, a_{k+1} \in V(C)$  for no disk  $C \in \mathcal{C}$ . For  $i = 1, 2, \dots, k+1$  let  $C_i \in \mathcal{C}$  be a disk in  $S$  such that  $V(C_i)$  includes all of  $a_1, a_2, \dots, a_{k+1}$  except  $a_i$ . We may assume  $a_1, \dots, a_k$  are distinct from  $c$ . Since  $a_1, a_2$  belong to  $C_3$  and  $C_4$ , there exists a segment  $P_{12}$  which is a subgraph of  $C_3 \cap C_4$  of  $S$  containing  $a_1$  and  $a_2$ . Similarly there exist

segments  $P_{ij} \subset V(C_t)$ ,  $t \in \{1, \dots, k+1\} \setminus \{i, j\}$ ,  $1 \leq i \neq j \leq k$  containing  $a_i, a_j$ . Note that  $a_1, \dots, a_k$  are branch vertices of  $S$ , because otherwise for some  $i \in \{1, \dots, k\}$ ,  $a_i$  belongs to the interior of some segment of  $S$ , hence  $a_1, \dots, a_k, a_{k+1}$  all belong to the cycle  $C_1$ , a contradiction. Since  $P_{12} \cup P_{23} \cup P_{13}$  is a cycle and is a subgraph of  $C_4$ , it is equal to  $C_4$ . Hence,  $k = 3$ . Now consider two cases, either  $a_4 \neq c$ , in which case property (ii) in Definition 2.1.1 implies  $\deg(a_1) = \deg(a_2) = \deg(a_3) = \deg(a_4) = 3$  so  $S$  is isomorphic to a subdivision of  $K_4$  which implies  $G$  is isomorphic  $K_4$  so (iv) holds. Or,  $a_4 = c$ , in which case, there exists an  $S$ -pyramid and (iii) holds.  $\square$

**Lemma 2.2.3.** *Let  $G$  be a multigraph and  $C$  be a cycle in  $G$ . Then one of the following conditions holds:*

- (i) *The multigraph  $G$  has a planar embedding so that  $C$  bounds a region.*
- (ii) *there exists a separation  $(A, B)$  of  $G$  of order at most three such that  $V(C) \subseteq A$  and  $G[B]$  has at least five vertices and it does not have a drawing in a disk with vertices in  $A \cap B$  drawn on the boundary of the disk.*
- (iii) *There exist two disjoint paths in  $G$  with ends  $s_1, t_1 \in V(C)$  and  $s_2, t_2 \in V(C)$ , respectively which are disjoint from  $C$  except at  $s_1, t_1, s_2, t_2$  such that  $s_1, s_2, t_1, t_2$  occur on  $C$  in the order listed.*

**Definition 2.2.4.** *Let  $G$  be a multigraph with minimum degree three and  $c \in V(G)$ . Let  $H$  be a 2-connected multigraph, and let  $S$  be a  $G$ -subdivision in  $H$  satisfying condition (\*) in Theorem 2.2.1. Let  $\mathcal{C}$  be a  $c$ -disk system in  $S$ . An auxiliary graph  $\mathcal{A}(S, H)$  is a graph with vertex-set  $V(\mathcal{A}(S, H))$  corresponding to the set of all  $S$ -bridges of  $H$ . For each  $B \in V(\mathcal{A}(S, H))$ , let  $L(B)$  be the set of all cycles  $C \in \mathcal{C}$  such that  $C$  contains all attachments of  $B$  and  $C \cup B$  has an embedding in the plane such that the cycle  $C$  bounds a region. Two vertices  $B$  and  $B'$  of  $\mathcal{A}(S, H)$  are adjacent by an edge if there exists some  $C \in \mathcal{C}$  such that  $B$  and  $B'$  are crossing in  $C$ .*

Note that since  $H$  is 2-connected then for each  $B \in V(\mathcal{A})$ ,  $|L(B)| \leq 2$ , and if  $L(B) = \{C_1, C_2\}$  then  $c \in C_1 \cap C_2$ . Moreover, if  $B, B'$  are adjacent vertices of  $\mathcal{A}$  with  $L(B) = L(B') = \{C_1, C_2\}$  then  $B$  and  $B'$  are crossing on  $C_1$  and  $C_2$ .

For simplicity, if the graphs  $S$  and  $H$  are easily understandable from the context, we refer to  $\mathcal{A}(S, H)$  by  $\mathcal{A}(S)$  or  $\mathcal{A}$ . Note that we use  $B$  to refer to a vertex of  $\mathcal{A}(S, H)$  and the corresponding a  $S$ -bridge. We say a vertex  $B \in V(\mathcal{A}(S, H))$  is *valid* if  $L(B) \neq \emptyset$ , and we say  $B \in V(\mathcal{A}(S, H))$  is *invalid* if  $L(B) = \emptyset$ . We say a vertex  $B \in V(\mathcal{A}(S, H))$  is *pre-colored*,  $|L(B)| = 1$ . We say an auxiliary graph  $\mathcal{A}$  is *list colorable* if all of its vertices are valid and there exists a mapping  $\varphi : V(\mathcal{A}) \rightarrow \bigcup_{B \in V(\mathcal{A})} L(B)$  such that  $\varphi(B) \in L(B)$  and if  $BB'$  is an edge of  $\mathcal{A}$  then  $\varphi(B) \neq \varphi(B')$ . We say a vertex  $B \in \mathcal{A}(S, H)$  is *rigid* if  $B$  is rigid as an  $S$ -bridge.

Here we should point out that the list coloring of the auxiliary graph is exactly the same as ordinary 2-list coloring, where the number of colors appearing in the union of the lists of colors in each connected component of the auxiliary graph is at most two.

The following lemma follows from Lemma 2.2.2 and Lemma 2.2.3.

**Lemma 2.2.5.** *Let  $G$  be a graph with minimum degree three and  $c$  be a vertex of  $G$ . Let  $H$  be an internally 3-connected multigraph containing  $S$  as a  $G$ -subdivision with a  $c$ -disk system  $\mathcal{C}$  such that all  $S$ -bridges are rigid. If a subgraph of the auxiliary graph  $\mathcal{A}(S, H)$  contains an invalid vertex then one of the following condition holds:*

- (i) *there exists a connected  $S$ -cross contained in a rigid  $S$ -bridge, or*
- (ii) *there exists a separation  $(A, B)$  of  $H$  of order at most three such that  $V(S) \subseteq A$  and  $G[B]$  has at least five vertices and it does not have a drawing in a disk with vertices in  $A \cap B$  drawn on the boundary of the disk, or*
- (iii) *there exists an  $S$ -jump, or*
- (iv) *there exists an  $S$ -triad, or*



(v) there exists an  $S$ -pyramid, or

(vi)  $G$  is isomorphic to  $K_4$ .

*Proof.* It is easy to see that if a bridge  $B$  is invalid either there is no cycle in  $\mathcal{C}$  containing all of its attachments in which case by Lemma 2.2.2 (iii), (iv), (v) or (vi) occurs, or there exists a cycle  $C \in \mathcal{C}$  such that  $C$  contains all attachments of  $B$ , but  $C \cup B$  does not have an embedding such that the cycle  $C$  bounds a region, thus by Lemma 2.2.3, and the fact that all of  $S$ -bridges are rigid either (i) or (ii) holds.  $\square$

**Lemma 2.2.6.** *Let  $G$  be a graph and for all  $v \in V(G)$ , let  $L(v)$  be a nonempty subset of  $\{1, 2\}$ ,  $v \in V(G)$ . Then  $G$  has no  $L$ -coloring if and only if one of the following conditions holds:*

(i) there exists an odd cycle in  $G$ , or

(ii) there exists a path  $v_0, v_1, \dots, v_{2k}$ ,  $k \geq 1$  in  $G$  such that  $L(v_0) = \{i\}$  and  $L(v_{2k}) = \{3 - i\}$  for some  $i \in \{1, 2\}$ , or

(iii) there exists a path  $v_0, v_1, \dots, v_{2k+1}$ ,  $k \geq 0$  in  $G$  such that  $L(v_0) = L(v_{2k+1}) = \{i\}$  for some  $i \in \{1, 2\}$ .

**Lemma 2.2.7.** *Let  $G$  be a multigraph with minimum degree three and  $c \in V(G)$ . Let  $H$  be a 3-connected multigraph, and let  $S$  be a  $G$ -subdivision in  $H$  satisfying condition (\*) in Theorem 2.2.1. Let  $\mathcal{C}$  be a  $c$ -disk system in  $S$  such that all  $S$ -bridges are rigid. Let  $\Delta$  be a subgraph of the auxiliary graph  $\mathcal{A}(S, H)$  such that for each  $S$ -bridge  $B$  in  $\Delta$  there exist exactly two cycles in  $\mathcal{C}$  containing all attachments of  $B$  and  $L(B) \neq \emptyset$ . Then  $H$  has a  $G$ -subdivision  $S'$  obtained from  $S$  by repeated  $I$ -reroutings such that  $S'$  and the  $c$ -disk system  $\mathcal{C}'$  induced in  $S'$  satisfy one of the following conditions:*

(i) there exists a  $c$ -blocking  $S'$ -cross in  $H$ , or

(ii) there exists a special segment  $P$  and a connected double  $S$ -fork such that its feet are contained in  $V(P) \cup \{c\}$ , or

(iii) there exists blocking interlaced  $S$ -fork of type II, or

(iv)  $\Delta$  is list colorable.

*Proof.* Note that if all components of  $\Delta$  are list colorable then (iv) holds, so assume that there exists a connected component of  $\Delta$  which is not list colorable, or without loss of generality we may assume  $\Delta$  is connected and it is not list colorable. The property (i) in Definition 2.1.1 implies that if  $B_1$  and  $B_2$  are two bridges and  $C_1, C_2, C_3 \in \mathcal{C}$  are such that  $C_1, C_3$  each contains all attachments of  $B_1$  and  $C_2, C_3$  each contains all attachments of  $B_2$ , then  $C_1 = C_2$ . So the union of all lists associated to vertices of  $\Delta$  has size exactly two. So for each vertex  $B \in \Delta$ ,  $L(B) = \{C_1, C_2\}$ , and  $P$  is the special segment where  $C_1 \cap C_2 = P \cup \{c\}$ . Note that since (iv) does not hold the intersection of  $C_1$  and  $C_2$  contains a special segment. Since (iv) does not hold and for each vertex  $B \in \Delta$  we have  $L(B) = \{C_1, C_2\}$ , by Lemma 2.2.6, there exists an odd cycle in  $\Delta$ .

Let  $S'$  be obtained from  $S$  by a sequence of I-reroutings such that the length of an induced odd cycle consisting of rigid bridges with list size exactly two in  $\mathcal{A}(S', H)$  is minimum.

*Claim 1.* The length of the shortest induced odd cycle consisting of rigid bridges with list size exactly two in  $\mathcal{A}(S', H)$ , is three.

For proving Claim 1, let  $\gamma : B_0 B_1 B_2 \dots B_{2k} B_0$ ,  $k \in \mathbb{N}$  be the shortest induced odd cycle consisting of rigid vertices in  $\mathcal{A}(S', H)$ , where  $k \geq 2$ . Note that  $B_1 B_2 B_3$  is an induced path in  $\mathcal{A}(S', H)$ . Let  $A_i$  be the set of attachments of  $B_i$ ,  $0 \leq i \leq 2k$ . By the lemma's assumption,  $\bigcup_{i=1}^3 A_i \subseteq V(P) \cup \{c\}$ . Since  $B_1 B_2 B_3$  is an induced path,  $|A_2 \cap V(P)| \geq 2$ . Let  $x_1, \dots, x_k$  be attachments of  $B_2$  on  $P$  listed in the order they appeared on  $P$ . Moreover, suppose  $a, b$  are the end vertices of  $P$  such that  $a P x_1$  and

$bPx_k$  are disjoint sub-paths. First, we are going to show that  $A_i \cap V(\text{Int}(x_1Px_k)) \neq \emptyset$ ,  $i = 1, 3$ . By symmetry assume  $A_1 \cap V(\text{Int}(x_1Px_k)) = \emptyset$ . Since  $B_1$  crosses  $B_2$ ,  $|A_1 \cap V(aPx_1)| \geq 1$  and  $|A_1 \cap V(x_kPb)| \geq 1$ . The fact that  $B_3$  crosses  $B_2$  implies that either  $|A_3 \cap V(\text{Int}(x_1Px_k))| \geq 1$ , or  $|A_3 \cap V(aPx_1)| \geq 1$  and  $|A_3 \cap V(x_kPb)| \geq 1$ . It is not hard to see that in either case,  $B_3$  crosses  $B_1$ , a contradiction.

Now suppose that  $a = v_1, \dots, v_t = b$  are the vertices of  $P$  listed in the order in which they appear in  $P$ . Note that  $B_i$  crosses  $B_j$  if and only if  $\min\{\ell : v_\ell \in A_j\} < \max\{\ell : v_\ell \in A_i\}$  and  $\min\{\ell : v_\ell \in A_i\} < \max\{\ell : v_\ell \in A_j\}$ . Now it is easy to see that the fact that  $B_0$  crosses  $B_1$  and  $B_{2k}$  implies that  $B_0$  crosses  $B_2, \dots, B_{2k-1}$ , a contradiction. This completes the proof of Claim 1.

*Claim 2.* The multigraph  $H$  has a  $G$ -subdivision  $S''$  obtained from  $S'$  by at most one  $I$ -rerouting, an  $S'$ -path  $Z$  with ends  $c, z$  where  $z$  is in interior of the special segment of  $S''$ , called  $Q$ , and two forks  $F_1, F_2$  with feet  $c, x_1, x_2$  and  $c, y_1, y_2$  such that  $x_1, x_2, y_1, y_2 \in V(Q)$  and  $z \in \text{Int}(x_1Qx_2) \cap \text{Int}(y_1Qy_2)$ .

As Claim 1 implies, let  $\gamma : B'_1B'_2B'_3$  be a triangle in  $\mathcal{A}(S', H)$  where  $B'_i$ ,  $1 \leq i \leq 3$  are rigid. Let  $A'_i$  be the set of attachments of  $B'_i$ ,  $1 \leq i \leq 3$ . Let  $a, b$  be the end vertices of  $P$ . Note that there exist  $1 \leq m \neq n \leq 3$  such that  $|A'_m \cap V(P)| \geq 2$  and  $|A'_n \cap V(P)| \geq 2$ . Without loss of generality,  $m = 1, n = 2$  and  $x'_1, \dots, x'_k$  and  $y'_1, \dots, y'_\ell$  are attachments of  $B'_1$  and  $B'_2$  on  $P$  listed in the order in which they appear on  $P$ . Moreover assume that  $x'_i$ ,  $1 \leq i \leq k$  and  $y'_j$ ,  $1 \leq j \leq \ell$  are such that  $aPx'_1$  and  $x'_kPb$  are disjoint sub-paths of  $P$  and, similarly  $aPy'_1$  and  $y'_\ell Pb$  are disjoint sub-paths of  $P$ . Note that  $B'_1$  contains a fork  $F'_1$  with feet on  $c, x'_1, x'_k$  and similarly,  $B'_2$  contains a fork  $F'_2$  with feet on  $c, y'_1, y'_\ell$ .

Without loss of generality, we may assume that  $a, x'_1, y'_1$  appear on  $P$  in the order listed, where  $x'_1$  could be equal to  $y'_1$ . First assume that  $|A'_3 \cap V(P)| = 1$ . The fact that  $B'_3$  crosses both  $B'_1$  and  $B'_2$  implies that there exists  $z \in A'_3$  such that  $z \in \text{Int}(x'_1Px'_k) \cap \text{Int}(y'_1Py'_\ell)$ , which shows that Claim 2 holds.

So assume that  $|A'_3 \cap V(P)| \geq 2$ . Let  $z'_1, \dots, z'_t$  be the attachments of  $B'_3$  on  $P$  in the order listed such that  $aPz'_1$  and  $z'_tPb$  are disjoint subpaths of  $P$ . We assume that the  $P$ -distance of  $a$  to  $z'_1$  is at least as large as the  $P$ -distance of  $a$  to  $y'_1$ , otherwise by switching the role of  $B'_2, B'_3$  or  $B'_1, B'_3$  and  $B'_3, B'_2$  we could maintain the condition. If  $z'_1 \neq y'_1$  then Claim 2 holds by setting  $z = z'_1, F_1 = F'_1$  and  $F_2 = F'_2$ .

Therefore  $z'_1 = y'_1$ . Without loss of generality, assume that  $a, z'_t, y'_t$  appear on  $P$  in the order listed. Note that  $B'_3$  contains a fork  $(W, W', W'')$  with feet on  $c, z'_1$  and  $z'_t$ , and center  $w$ . Let  $S''$  be obtained from  $S'$  by I-rerouting  $z'_1Pz'_t$  through  $z'_1W'wW''z'_t$ . By setting  $z = w, F_1 = F'_1$  and  $F_2 = F'_2$ , Claim 2 holds. This completes the proof of Claim 2.

So by applying Claim 2, assume  $S'', Q, Z, F_1, F_2, z, x_1, x_2, y_1$  and  $y_2$  are as stated in the statement of Claim 2. Let  $F_1 = (X, X', X'')$  with feet on  $c, x_1, x_2$  and center  $x$  and,  $F_2 = (Y, Y', Y'')$  with feet on  $c, y_1, y_2$  and center  $y$ . Without loss of generality, assume that  $aQx_1$  and  $x_2Qb$  are disjoint sub-paths of  $Q$  and, similarly  $aQy_1$  and  $y_2Qb$  are disjoint sub-paths of  $Q$ . Moreover  $a, x_1, y_1$  appear on  $Q$  in the order listed.

- $y_2, x_2, b$  appear on  $Q$  in the order listed.

If  $x_1 = y_1$  and  $x_2 = y_2$  then it is easy to see that (iii) holds. So by symmetry assume that  $x_2 \neq y_2$ . Let  $S^*$  be obtained from  $S''$  by I-rerouting  $x_1Qx_2$  through  $x_1X'xX''x_2$ . It is not hard to see that the fork  $(\alpha, \alpha', \alpha'')$  where  $\alpha : zZc, \alpha' : zQy_1Qx_1, \alpha'' : zQy_2Qx_2$  and the fork  $(\beta, \beta', \beta'')$  where  $\beta : yYc, \beta' : yY'y_1Qx_1, \beta'' : yY''y_2Qx_2$  form a connected double  $S^*$ -fork. Thus (ii) holds.

- $x_2, y_2, b$  appear on  $Q$  in the order listed.

Note that in this case,  $x_1 \neq y_1$  and  $x_2 \neq y_2$ , otherwise we are in the previous case. Let  $S^*$  be obtained from  $S''$  by I-rerouting  $x_1Qx_2$  through  $x_1X'xX''x_2$ . It is not hard to see  $x_1Qy_1Y'yY''y_2, cZzQx_2, y_1Qz$  form a  $c$ -blocking  $S^*$ -cross. Thus (i) holds.

This completes the proof of the lemma.  $\square$

**Lemma 2.2.8.** *Let  $G$  be a multigraph with minimum degree three,  $H$  be a 3-connected multigraph, and let  $S$  be a  $G$ -subdivision in  $H$  satisfying condition  $(*)$  in Theorem 2.2.1. Let  $\mathcal{C}$  be a  $c$ -disk system in  $S$ . Let  $B_1$  and  $B_2$  be two rigid  $S$ -bridges crossing in  $C$ , where for  $i = 1, 2$ ,  $C \in \mathcal{C}$  is the unique cycle containing the attachments of  $B_i$ . Then one of the following conditions holds:*

- (i) *there exists a solid  $S$ -cross with its arms contained in  $S$ -rigid bridges, or*
- (ii) *there exists a connected  $S$ -cross contained in a rigid  $S$ -bridge, or*
- (iii) *there exists a degenerate  $S$ -cross, or*
- (iv) *there exists a separation  $(H_1, H_2)$  of  $H$  of order at most three such that  $V(S) \subseteq H_1$ , and  $H_2$  has at least five vertices and it does not have a drawing in a disk with vertices in  $V(H_1) \cap V(H_2)$  drawn on the boundary of the disk. Furthermore, there is no special segment  $P \subset C$  such that  $H_1 \cap H_2 \subset V(P) \cup \{c\}$ .*

*Proof.* By assumption of the lemma for the  $S$ -bridge  $B_i$ ,  $1 \leq i \leq 2$ , there is no special segment  $Q$  such that  $V(Q) \cup \{c\}$  contains all attachments of  $B_i$ . Now assume that (iv) does not hold. Since  $B_1$  and  $B_2$  are crossing in  $C$ , by Lemma 2.2.3, there exist two  $S$ -paths  $P_1, P_2 \subseteq B_1 \cup B_2$  with ends  $x_1, y_1$  and  $x_2, y_2$ , respectively, such that  $x_1, x_2, y_1, y_2$  appear in  $C$  in the order listed. If  $P_1$  and  $P_2$  belong to the same  $S$ -bridge then (ii) holds. Now assume that  $V(P_1) \subseteq V(B_1)$  and  $V(P_2) \subseteq V(B_2)$ . Assume that  $(P_1, P_2)$  is not a solid  $S$ -cross. So by symmetry, we may assume that  $x_1 = c$  and  $y_1$  belong to a special segment, called  $Q$ . Since  $C$  is the unique disk containing all attachments of  $B_1$ , there exists a path  $P$  from  $x \in V(Int(P_1))$  to a vertex  $y \in V(C) \setminus (V(Q) \cup \{c\})$  disjoint from  $S$  and  $P_1$  and  $P_2$  except at its ends. Now by symmetry, we consider three cases: either  $\{x_2, y_2\} \cap V(Q) = \emptyset$ , or  $x_2 \notin V(Q), y_2 \in V(Q)$ , or  $\{x_2, y_2\} \subseteq V(Q)$ . Note that the first case does not happen by property (iii) in Definition 2.1.1. In the second case if  $y \neq x_2$  then it is easy to see that there exists a solid  $S$ -cross, so (i)

holds, and if  $y = x_2$  there exists a degenerate  $S$ -cross, thus (iii) holds. In the last case note that since  $C$  is the unique cycle containing all attachments of  $B_2$ , there exists a path  $P'$  from  $x' \in V(\text{Int}(P_2))$  to a vertex  $y' \in V(C) \setminus (V(Q) \cup \{c\})$  disjoint from  $S$  and  $P_1$  and  $P_2$  except at its ends. Note that  $(P_1, y_2 P_2 x' P' y')$  form an  $S$ -cross. Now it is easy to see that the same argument presented for settling the second case applies to this case. This completes the proof of the lemma.  $\square$

**Lemma 2.2.9.** *Let  $G$  be a multigraph with minimum degree three,  $H$  be a 3-connected multigraph, and let  $S$  be a  $G$ -subdivision in  $H$  satisfying condition  $(*)$  in Theorem 2.2.1. Let  $\mathcal{C}$  be a  $c$ -disk system in  $S$  such that all  $S$ -bridges are rigid. Let  $\Delta$  be a subgraph of  $\mathcal{A}(S, H)$  such that for any  $S$ -bridge  $B \in \Delta$ ,  $1 \leq |L(B)| \leq 2$ . Then  $H$  has a  $G$ -subdivision  $S'$  obtained from  $S$  by repeated  $I$ -reroutings such that  $S'$  and the  $c$ -disk system  $\mathcal{C}'$  induced in  $S'$  satisfy one of the following conditions:*

- (i) *there exists an  $S'$ -jump, or*
- (ii) *there exists a degenerate  $S'$ -cross, or*
- (iii) *there exists a connected  $S'$ -cross contained in a rigid  $S'$ -bridge, or*
- (iv) *there exists a separation  $(H_1, H_2)$  of  $H$  of order at most three such that  $V(S) \subseteq H_1$ , and  $H_2$  has at least five vertices and it does not have a drawing in a disk with vertices in  $V(H_1) \cap V(H_2)$  drawn on the boundary of the disk. Furthermore, there is no special segment  $P \subset C$  such that  $H_1 \cap H_2 \subset V(P) \cup \{c\}$ , or*
- (v) *there exists a solid  $S'$ -cross with its arms contained in rigid  $S$ -bridges, or*
- (vi) *there exists a double facial  $S'$ -cross, or*
- (vii) *there exists a blocking interlaced  $S$ -fork of type I, or*
- (viii) *there exists a blocking interlaced  $S'$ -fork of type II, or*

(ix) there exists a special segment  $P$  and a connected double  $S$ -fork such that one of its feet is on  $c$  and the other two are contained in  $V(P)$ , or

(x)  $\Delta$  is list colorable.

*Proof.* Suppose (x) does not hold, i.e.  $\Delta$  is not list colorable. Note that if all components of  $\Delta$  are list colorable then (x) holds, so assume that there exists a connected component of  $\Delta$  which is not list colorable, or without loss of generality we may assume  $\Delta$  is connected and it is not list colorable. Now Lemma 2.2.6 implies that there exist three possible cases: either there exist two pre-colored vertices with different lists connected by a path of even length in  $\Delta$ , there exist two pre-colored vertices with the same list connected by a path of odd length in  $\Delta$ , or there exists an odd cycle consisting of not pre-colored vertices in  $\Delta$ . Let  $a, b$  be ends of  $P$  such that  $bc \in E(S)$  as Definition 2.1.1 (iii) implies. Suppose  $\beta : c = v_1, v_2, \dots, v_n = a$  is the path connecting  $c$  and  $a$  in  $C_1$  and that it is internally disjoint from  $P$ .

- Case 1: Suppose  $\alpha : B_0 B_1 B_2 \dots B_{2k}$ ,  $k \in \mathbb{N}$  is an even path of rigid bridges where  $B_0, B_{2k}$  are pre-colored vertices with different lists and  $B_i, 1 \leq i \leq 2k - 1$  are not pre-colored with list size exactly two.

Note that  $L(B_0) = \{C_j\}, L(B_{2k}) = \{C_{-j+3}\}, j \in \{1, 2\}$  and  $L(B_i) = \{C_1, C_2\}, 1 \leq i \leq 2k - 1$ . Let  $S'$  be a  $G$ -subdivision obtained from  $S$  by a sequence of I-reroutings such that there exists an even length path consisting of rigid bridges which connects two pre-colored bridges with different lists using not pre-colored vertices of  $\mathcal{A}(S', H)$  with list size exactly two, and with respect to this condition the length of the path is minimum.

*Claim 1.* The length of the shortest even path consisting of rigid not pre-colored vertices with list size exactly two connecting two pre-colored bridges with different lists in  $\mathcal{A}(S', H)$ , is two, or (iii) holds.

Let  $\alpha : B_0 B_1 B_2 \dots B_{2k+1} B_{2k+2}$ ,  $k \in \mathbb{N}$  be the shortest even path connecting  $B_0$  and  $B_{2k+2}$ . Note that  $B_1 B_2 B_3$  is an induced path in  $\mathcal{A}(S', H)$ . Let  $A_i$  be the set of attachments of  $B_i$ ,  $1 \leq i \leq 3$ . By the lemma's assumption,  $\bigcup_{i=1}^3 A_i \subseteq V(P) \cup \{c\}$ . It is easy to see that  $|A_2 \cap V(P)| \geq 2$ . Let  $y_1, \dots, y_\ell$  be the attachments of  $B_2$  on  $P$  listed in the order they appeared on  $P$ . Moreover, suppose  $aPy_1$  and  $bPy_\ell$  are disjoint subpaths of  $P$ . Similarly as in the proof of Claim 1 in Lemma 2.2.7, we have  $A_i \cap V(\text{Int}(y_1Py_\ell)) \neq \emptyset$ ,  $i = 1, 3$ . Assume  $A_1 \cap V(\text{Int}(y_1Py_\ell)) = \emptyset$ . Since  $B_1$  crosses  $B_2$ ,  $|A_1 \cap V(aPy_1)| \geq 1$  and  $|A_1 \cap V(y_\ell Pb)| \geq 1$ . The fact that  $B_3$  crosses  $B_2$  implies that either  $|A_3 \cap (V(y_1Py_\ell) \setminus \{y_1, y_\ell\})| \geq 1$ , or  $|A_3 \cap V(aPy_1)| \geq 1$  and  $|A_3 \cap V(y_\ell Pb)| \geq 1$ . It is not hard to see that in either case,  $B_3$  crosses  $B_1$ , a contradiction.

Suppose  $u \in A_1$  and  $v \in A_3$  are such that  $y_1 \in V(aPu)$  and  $y_\ell \in V(bPv)$ . An immediate consequence of the above argument is that since neither  $B_0$  nor  $B_4$  crosses  $B_2$ ,  $A_3 \cap V(y_\ell Pb) \setminus \{y_\ell\} \neq \emptyset$  and  $A_1 \cap V(aPy_1) \setminus \{y_1\} \neq \emptyset$ . Let  $x_1, \dots, x_k, z_1, \dots, z_t$  be attachments of  $B_1, B_3$  on  $P$  listed in the order they appear on  $P$ , respectively. Assume that  $x_i$ ,  $1 \leq i \leq k$  and  $z_j$ ,  $1 \leq j \leq t$  are such that  $aPx_1, bPx_k$  and  $aPz_1, bPz_t$  are disjoint subpaths of  $P$  and  $a, x_1, y_1, z_1, b$  appeared on  $P$  as the order listed. Let  $Q_2 \subseteq V(B_2) \setminus (A_2 \setminus \{y_1, y_\ell\})$  be a path connecting  $y_1, y_\ell$ . Let  $S''$  be obtained from  $S'$  by I-rerouting  $y_1Py_\ell$  through  $Q_2$ . Since  $\alpha$  is an induced path,  $B_j \cap V(\text{Int}(y_1Py_\ell)) = \emptyset$ ,  $j \in \{0, \dots, 2k+2\} \setminus \{1, 2, 3\}$ . Note that  $B_j$  becomes an  $S''$ -bridge and it remains rigid for  $j \in \{0, \dots, 2k+2\} \setminus \{1, 2, 3\}$ . Note that  $B_3$  contains a fork  $(W, W', W'')$  with feet on  $c, z_1$  and  $z_t$ , respectively and center  $w$ .

Let  $B$  be the  $S''$ -bridge such that  $(V(B_1) \cup V(B_3) \cup V(y_1Py_\ell) \cup \{c\}) \subseteq V(B)$ . Now either for all  $S'$ -bridges,  $B'$ , with the property  $V(B') \cap V(\text{Int}(y_1Py_\ell)) \neq \emptyset$  we have  $V(B') \cap ((V(C_1) \cup V(C_2)) \setminus (V(P) \cup \{c\})) = \emptyset$ , in which case  $B$  is a valid not pre-colored  $S''$ -bridge and  $\alpha' : B_0, B, B_4, \dots, B_{2k+2}$  is an induced even path consisting of rigid vertices which has shorter length than  $\alpha$ . Or, there exists a  $S'$ -bridge  $B'$



containing a path  $O$  from  $o' \in V(y_1Py_\ell) \setminus \{y_1, y_\ell\}$  to  $o \in (V(C_1) \cup V(C_2)) \setminus (V(P) \cup \{c\})$ . By symmetry assume that  $o \in V(C_1)$  and  $a, b, o, c$  are listed in the order they appear on  $C_1$ . Now, either  $o' \in V(y_1Px_k)$ ,  $o' \in V(x_kPz_1)$  or  $o' \in V(z_1Py_\ell)$ . We just investigate the case that  $o' \in V(x_kPz_1)$  since the other cases are completely similar. It is not hard to see that if  $o' \in V(x_kPz_1)$  then  $oOo'Px_kPy_1, cWwW''z_\ell$  and  $wW'z_1Po'$  is a  $c$ -blocking  $S''$ -cross, so (iii) holds. This proves Claim 1.

As Claim 1 implies assume  $B_0B_1B_2$  is a path such that  $B_0, B_2$  are pre-colored and  $B_1$  is not pre-colored. Let  $A_i$  be the set of attachments of  $B_i$ ,  $0 \leq i \leq 2$ . Let  $x_1, \dots, x_k$  be the attachments of  $B_1$  on  $P$  listed in the order they appear on  $P$  ( $k$  could be equal to 1). Assume  $x_1Pa$  and  $x_kPb$  are internally disjoint paths. Similarly to  $\beta$  assume  $\lambda : c = v'_1, v'_2, \dots, v'_n = a$  is a path contained in  $V(C_2)$  between  $c, a$  and internally disjoint from  $P$ . Since  $B_0, B_2$  are pre-colored,  $A_0 \cap V(Int(\beta)) \neq \emptyset$  and  $A_2 \cap V(Int(\lambda)) \neq \emptyset$ . Let  $u' \in A_0 \cap V(Int(\beta))$  and  $v' \in A_2 \cap V(Int(\lambda))$ . Since  $B_0, B_2$  are crossing  $B_1$ , there exist  $u, v \in V(x_1Pb) \setminus \{x_1\}$  and  $S'$ -paths  $U$  connecting  $u, u'$  and  $V$  connecting  $v, v'$  contained in  $B_0$  and  $B_2$ , respectively. This immediately shows that (vi) holds.

- Case 2: Suppose  $\alpha : B_0B_1B_2 \dots B_{2k+1}$ ,  $k \in \mathbb{N}$  is an odd path of rigid bridges where  $B_0, B_{2k+1}$  are pre-colored vertices with the same list and if  $B_i, 1 \leq i \leq 2k + 1$  are not pre-colored with list size exactly two.

Let  $S'$  be a  $G$ -subdivision obtained from  $S$  by a sequence of I-reroutings such that there exists an odd length path consisting of rigid bridges which connects two pre-colored bridges with the same list using not pre-colored vertices of  $A(S', H)$  with list size exactly two, and with respect to this condition the length of the path is minimum.

*Claim 2.* The length of the shortest odd path consisting of rigid not pre-colored vertices with list size exactly two connecting two pre-colored bridges with the same

list, is three, or either (ii), (iii), (iv) or (v) holds.

For proving Claim 2 note that if  $\alpha : B_0B_1$  is a path connecting two pre-colored bridges with the same list, call it  $C_1$ , then the fact that  $B_0$  crosses  $B_1$  and they are pre-colored and rigid, implies that by applying Lemma 2.2.8 either (ii), (iii), (iv) or (v) holds. So assume that  $\alpha : B_0B_1B_2 \dots B_{2k+1}$  and  $k \in \mathbb{N}$ . In this case the rest of the proof of Claim 2 is similar to the proof of Claim 1 in Case 1, so we skip its proof here.

As Claim 2 implies, we assume  $\alpha : B_0B_1B_2B_3$ . Let  $A_i, 0 \leq i \leq 3$  be the set of attachments of  $B_i$ . Without loss of generality, assume  $L_{B_0} = L_{B_3} = \{C_1\}$ . We consider two possibilities, either  $B_0 \neq B_3$  or  $B_0 = B_3$ .

First assume  $B_0 \neq B_3$ . Similarly to the argument presented in Case 1, we have  $|A_i \cap V(P)| \geq 2, i = 1, 2$ . Let  $x_1, \dots, x_k, y_1, \dots, y_\ell$  be attachments of  $B_1, B_2$  on  $P$  listed in the order they appeared on  $P$ , respectively. Assume that  $a, x_1, y_1, x_k, y_\ell, b$  are listed in the order they appeared on  $P$ . Since  $\alpha$  is induced,  $|\{x_1, y_1, x_k, y_\ell\}| = 4$ . Note that since  $\alpha$  is an induced path,  $B_0$  crosses  $B_1$ ,  $B_2$  crosses  $B_3$ , and  $B_0$  and  $B_3$  are pre-colored, there must exist vertices  $v \in V(Int(\beta)), u \in V(C_1) \setminus (V(\beta) \cup V(P)), v' \in A_0 \cap V(x_1Py_1), u' \in A_3 \cap V(x_kPy_\ell)$  and paths  $V \subseteq V(B_0)$  and  $U \subseteq V(B_3)$  connecting  $v, v'$  and  $u, u'$ , respectively. However, as the property (iii) of  $c$ -disk system in Definition 2.1.1 implies  $V(C_1) \setminus (V(\beta) \cup V(P)) = \emptyset$ , so the first case does not happen.

Thus  $B_0$  must be equal to  $B_3$ . It is easy to see that either  $|A_1 \cap V(P)| \geq 2$  or  $|A_2 \cap V(P)| \geq 2$ . By symmetry, we may assume  $|A_1 \cap V(P)| \geq 2$ . Let  $x_1, \dots, x_k$  be attachments of  $B_1$  on  $P$  listed in the order they appear on  $P$  such that  $aPx_1$  and  $bPx_k$  are disjoint subpaths of  $P$ . Let  $Q_1 \subseteq (V(B_1) \setminus (A_1 \setminus \{x_1, x_k\}))$  be the path connecting  $x_1$  and  $x_k$ . Suppose  $\beta : c = v_1, v_2, \dots, v_n = a$  is the path contained in  $V(C_1)$  between  $c, x_1$  and it is internally disjoint from  $P$ . By switching the roles of  $B_1$  and  $B_2$ , we may assume  $A_2 \cap V(P) \subseteq V(x_1Pb)$ .

Now if  $A_2 \cap V(\text{Int}(x_1Px_k)) \neq \emptyset$  then let  $y \in A_2 \cap V(\text{Int}(x_1Px_k))$  be the vertex in  $A_2 \cap V(\text{Int}(x_1Px_k))$  which has the minimum  $P$ -distance to  $x_1$ . Let  $O \subseteq V(B_2)$  be a path from  $y$  to  $c$ . The fact that  $B_0$  is pre-colored and  $B_0$  crosses  $B_2, B_1$  implies that there exist  $v \in V(\text{Int}(\beta))$  and  $v' \in V(yPb) \setminus \{y\}$ . If  $v' \in V(x_kPb)$  then (vii) holds. If  $v' \in V(\text{Int}(yPx_k))$  then  $S''$  obtained from  $S'$  by I-rerouting  $x_1Px_k$  through  $Q_1$ , has the property that  $vVv'Px_k, cOyPx_1, yPv'$  form a  $c$ -blocking  $S''$ -cross. Therefore, (iii) holds.

Now assume that  $A_2 \cap V(\text{Int}(x_1Px_k)) = \emptyset$ . Since  $B_1$  crosses  $B_2$  implies that  $|A_2 \cap V(P)| \geq 2$ . We may assume  $x_1, x_k \in A_2$  since otherwise by the switching roles of  $B_1$  and  $B_2$ , as the above argument shows either (iii) or (vii) holds. Moreover, we can also infer that  $A_2 \cap V(P) = \{x_1, x_k\}$ . Note that  $B_1$  contains a fork  $(W, W', W'')$  with feet on  $c, x_1, x_k$ , respectively and center  $w$ . Similarly  $B_2$  contains a fork  $(O, O', O'')$  with feet on  $c, x_1, x_k$ , respectively and center  $o$ . Since  $B_0$  is pre-colored, there exist  $v \in V(\text{Int}(\beta))$  and  $v' \in V(x_1Pb) \setminus \{x_1\}$ . If  $S''$  is obtained from  $S'$  by I-rerouting  $x_1Px_k$  through  $Q_1$  then it is easy to see that (vii) holds.

- Case 3: Suppose  $\gamma : B_0B_1B_2 \dots B_{2k}B_0$ ,  $k \in \mathbb{N}$  is an odd induced cycle consisting of rigid bridges where  $B_i, 0 \leq i \leq 2k$  are not pre-colored and have list size exactly two.

In this case Lemma 2.2.7 implies either (iii), (viii), (ix) holds.

This completes the proof of the lemma.

□

**Theorem 2.2.10.** *Let  $G$  be a multigraph with minimum degree three,  $H$  be a 3-connected multigraph, and let  $S$  be a  $G$ -subdivision in  $H$  satisfying condition (\*) in Theorem 2.2.1. Let  $\mathcal{C}$  be a  $c$ -disk system in  $S$  such that all  $S$ -bridges are rigid. Then  $H$  has a  $G$ -subdivision  $S'$  obtained from  $S$  by repeated I-reroutings such that  $S'$ , the*

$c$ -disk system  $C'$  induced in  $S'$  and the auxiliary graph  $\mathcal{A}(S', H)$  satisfy one of the following conditions:

- (i) there exists an  $S'$ -jump, or
- (ii) there exists a degenerate  $S'$ -cross, or
- (iii) there exists a connected  $S'$ -cross contained in a rigid  $S'$ -bridge, or
- (iv) there exists a separation  $(H_1, H_2)$  of  $H$  of order at most three, and there is no special segment  $P \subset C'$  such that  $V(C') \subseteq H_1$ , and  $H_2$  has at least five vertices and it does not have a drawing in a disk with vertices in  $V(H_1) \cap V(H_2)$  drawn on the boundary of the disk, or
- (v) there exists a solid  $S'$ -cross with its arms contained in rigid  $S$ -bridges, or
- (vi) there exists a double facial  $S'$ -cross, or
- (vii) there exists a blocking interlaced  $S$ -fork of type I, or
- (viii) there exists a blocking interlaced  $S'$ -fork of type II
- (ix) there exists a special segment  $P$  and a connected double  $S$ -fork such that one of its feet is on  $c$  and the other two are contained in  $V(P)$ , or
- (x) there exists an  $S'$ -triad, or
- (xi) there exists an  $S'$ -pyramid, or
- (xii)  $G$  is isomorphic to  $K_4$ , or
- (xiii)  $\mathcal{A}(S', H)$  is list colorable.

*Proof.* By Theorem 2.2.1, we may assume that all the  $S$ -bridges are rigid. Suppose (xiii) does not happen. If  $\mathcal{A}(S, H)$  contains an invalid vertex then by Lemma 2.2.5 either (i), (iii), (x), (xi) or (xii) holds. Note that in the case that  $\mathcal{A}(S, H)$  contains an

invalid vertex, the outcome (ii) of Lemma 2.2.5 combined with Lemma 2.2.7 implies that either (iv) or (iii), (viii), (ix) holds. So suppose that all of the vertices of  $\mathcal{A}(S, H)$  are valid.

Note that if  $\mathcal{A}(S, H)$  is list colorable then (xiii) holds, a contradiction. The property (i) in Definition 2.1.1 implies that if  $B_1$  and  $B_2$  are two bridges and  $C_1, C_2, C_3 \in \mathcal{C}$  are such that  $C_1, C_3$  each contains all attachments of  $B_1$  and  $C_2, C_3$  each contains all attachments of  $B_2$ , then  $C_1 = C_2$ . So for each  $B \in V(\mathcal{A}(S, H))$ ,  $|L(B)| \leq 2$ .

Now Lemma 2.2.8 and Lemma 2.2.9 implies either (i), (ii), (iii), (iv), (v), (vi), (vii), (viii) or (ix) holds. This completes the proof of the theorem.  $\square$

## CHAPTER III

### SOME APPLICATIONS OF THE THEORY TO ROOTED GRAPHS

Let  $G$  be a graph and let  $x_1, x_2, x_3$  be three distinct vertices of  $G$ . We say that  $(G, x_1, x_2, x_3)$  is a *rooted* graph.

We say  $(G, x_1, x_2, x_3)$  is a *subgraph* of  $(H, x_1, x_2, x_3)$ , if  $(G, x_1, x_2, x_3)$  can be obtained from  $(H, x_1, x_2, x_3)$  by deleting some vertices in  $V(H) \setminus \{x_1, x_2, x_3\}$  and edges of  $H$ .

Let  $(G, x_1, x_2, x_3)$  and  $(H, y_1, y_2, y_3)$  be two rooted graphs. We say  $(G, x_1, x_2, x_3)$  is a *isomorphic* of  $(H, y_1, y_2, y_3)$  if  $G$  and  $H$  are isomorphic and there exists an isomorphism  $\phi$  from  $V(G)$  to  $V(H)$  such that  $\phi(x_i) = y_i$ . For simplicity, we sometimes say  $G$  and  $H$  are isomorphic where  $x_1, x_2, x_3$  correspond to  $y_1, y_2, y_3$ , respectively.

We say  $(G, x_1, x_2, x_3)$  is a *subdivision* of  $(H, y_1, y_2, y_3)$ , if there exists a rooted graph  $(K, y_1, y_2, y_3)$  obtained from  $(H, y_1, y_2, y_3)$  by replacing edges of  $H$  by segments such that  $(K, y_1, y_2, y_3)$  is isomorphic to  $(G, x_1, x_2, x_3)$ . For simplicity, we sometimes say  $G$  contains  $H$  as a subdivision where  $x_1, x_2, x_3$  correspond to  $y_1, y_2, y_3$ , respectively.

**Lemma 3.0.11.** *Let  $(G, a, b, c)$  be a rooted multigraph with minimum degree three,  $(H, a, b, c)$  be an internally 4-connected rooted multigraph, and let  $(S, a, b, c)$  be a  $(G, a, b, c)$ -subdivision in  $(H, a, b, c)$  satisfying condition  $(*)$  in Theorem 2.2.1. Let  $\mathcal{C}$  be a  $c$ -disk system in  $S$ . If there exists a special segment  $P$  and a connected double  $S$ -fork such that one of its feet is on  $c$  and the other two are contained in  $V(P)$  then either there exists an  $S$ -jump,  $S$ -triad or a  $c$ -blocking  $S$ -cross.*

*Proof.* Let  $C_1, C_2 \in \mathcal{C}$  be such that  $C_1 \cap C_2 = V(P) \cup \{c\}$ . Let  $x, y$  be ends of  $P$ , and  $cx \in E(S)$ . Let  $(P_1, P_2, P_3)$  and  $(Q_1, Q_2, Q_3)$  be two forks with feet on  $c, u, v$  where

$u, v \in V(P)$  and centers  $o, o'$ , respectively. Suppose  $uPx$  and  $vPy$  are disjoint.

Since  $(H, a, b, c)$  is internally 4-connected,  $\{c, u, v\}$  is not a cut set. Thus there exists a path  $Z$  from  $z_1 \in X \setminus \{c, u, v\}$  to  $z_2 \in V(S) \setminus X$  such that  $Z$  is internally disjoint from  $X \cup S$  except at  $z_1, z_2$ . We consider two cases, either  $z_2 \notin C_1 \cup C_2$  or  $z_2 \in C_1 \cup C_2$ .

In the first case, i.e.  $z_2 \notin C_1 \cup C_2$  then either there exists a vertex  $z \in \{c, u, v\}$  such that there is no disk in  $\mathcal{C}$  containing  $z, z_2$ , in which case there exists an  $S$ -jump with one ends at  $z$  and the other at  $z_2$ . So assume that for all  $z \in \{c, u, v\}$ ,  $z, z_2$  are co-facial. If there is no  $C \in \mathcal{C}$  such that  $u, v, z_2 \in C$ , then there exists an  $S$ -triad with feet on  $z, u, v$ , so the assertion of the lemma holds. So there exists  $C \in \mathcal{C}$  containing  $u, v, z_2$ . Note that  $u, v \in C_1 \cap C_2 \cap C_3$ , so by property (iv) of Definition 2.1.1,  $C = C_1$  or  $C = C_2$ , implying  $z_2 \in C_1 \cup C_2$ , a contradiction with the assumption of the first case.

In the second case, i.e.  $z_2 \in C_1 \cup C_2$  it is easy to see that  $P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2 \cup Q_3 \cup Z$  contains a  $c$ -blocking  $S$ -cross. The fact that  $(P_1, P_2, P_3)$  and  $(Q_1, Q_2, Q_3)$  are connected  $S$ -forks implies the  $S$ -cross is connected.  $\square$

**Lemma 3.0.12.** *Let  $(G, a, b, c)$  be a rooted multigraph and  $(H, a, b, c)$  be an internally 4-connected rooted multigraph. Let  $(S, a, b, c)$  be a  $(G, a, b, c)$ -subdivision in  $(H, a, b, c)$  satisfying condition  $(*)$  in Theorem 2.2.1. Let  $\mathcal{C}$  be  $c$ -disk system of  $(S, a, b, c)$ . Suppose  $(H, a, b, c)$  contains a blocking interlaced fork of type II. Then  $H$  has a  $G$ -subdivision  $S'$  obtained from  $S$  by an  $I$ -rerouting such that  $S'$ , the  $c$ -disk system  $\mathcal{C}'$  induced in  $S'$  satisfy one of the following conditions:*

- (i) *there exists an  $S'$ -jump, or*
- (ii) *there exists an  $S'$ -triad, or*
- (iii) *there exists a blocking interlaced  $S'$ -fork of type I, or*
- (iv) *there exists a  $c$ -blocking  $S'$ -cross.*

*Proof.* Since  $(H, a, b, c)$  contains a blocking interlaced fork of type II, there exist a special segment  $Q$  of  $S$  with ends  $x, y$  where  $cx \in E(S)$ , two  $S$ -forks  $(P_1, P_2, P_3)$  and  $(R_1, R_2, R_3)$  with feet on  $c, u, v$  and centers  $o$  and  $o'$ , respectively, where  $\{u, v\} \subseteq V(Q)$  and  $x, u, v, y$  appear on  $P$  in the order listed, and an  $S$ -path  $P$  disjoint from  $P_1, P_2, P_3, R_1, R_2, R_3$  except at  $c$  connecting  $c$  to  $w$ , where  $w \in V(\text{Int}(vPu))$ . Note that the two forks only share three vertices of their feet.

Since  $(H, a, b, c)$  is an internally 4-connected rooted graph, by symmetry, there exists a path  $Z$  from  $z_1 \in V(R_1 \cup R_2 \cup R_3) \setminus \{c, u, v\}$  to  $z_2 \in V(S) \setminus \{c, u, v\}$  disjoint from  $V(P_1, P_2, P_3)$ . Similarly as in the proof of Lemma 3.0.11, we consider two cases, either  $z_2 \notin C_1 \cup C_2$  or  $z_2 \in C_1 \cup C_2$ .

In the first case, i.e.  $z_2 \notin C_1 \cup C_2$  either there exists a vertex  $z \in \{c, u, v\}$  such that there is no disk in  $\mathcal{C}$  containing  $z, z_2$ , in which case there exists an  $S$ -jump with one ends at  $z$  and the other at  $z_2$ . So assume that for all  $z \in \{c, u, v\}$ ,  $z, z_2$  are co-facial. If there is no  $C \in \mathcal{C}$  such that  $u, v, z_2 \in C$ , then there exists an  $S$ -triad with feet on  $z, u, v$ , so the assertion of the lemma holds. So there exists  $C \in \mathcal{C}$  containing  $u, v, z_2$ . Note that  $u, v \in C_1 \cap C_2 \cap C_3$ , so by property (iv) of Definition 2.1.1,  $C = C_1$  or  $C = C_2$ , implying  $z_2 \in C_1 \cup C_2$ , a contradiction with the assumption of the first case.

In the second case, i.e.  $z_2 \in C_1 \cup C_2$ , if  $V(Z) \cap V(\text{Int}(P)) \neq \emptyset$  then it is easy to see that  $P \cup R_1 \cup R_2 \cup R_3 \cup Z$  contains a  $c$ -blocking  $S$ -cross, thus (iv) holds. So we assume that  $V(Z) \cap V(\text{Int}(P)) = \emptyset$ . If  $z_2 \notin V(Q)$ , then  $R_1 \cup R_2 \cup R_3 \cup Z$  contains a path  $W$  from  $u$  to  $z_2$  such that the fork  $(P_1, P_2, P_3)$ , and paths  $W$  and  $P$  form a blocking interlaced fork of type I, hence (iii) holds.

So assume  $z_2 \in V(Q)$ . If  $z_2 \in V(\text{Int}(uQv))$  then let  $S'$  be obtained from  $S$  by  $I$ -rerouting of  $uQv$  through  $uP_2oP_3v$ . We can see that the  $S$ -forks  $(R_1 \cup R_2 \cup R_3)$  and  $(wQu, wQv, wPc)$  and the path  $Z$  from a connected  $S'$ -fork. If  $z_2 \notin V(\text{Int}(uQv))$  then there exists a path  $W \subset R_1 \cup R_2 \cup R_3 \cup Z$  from  $u$  to  $z_2$ . Let  $S'$  be obtained from  $S$  by  $I$ -rerouting of  $uQz_2$  through  $W$ . It is easy to see that the two  $S'$ -forks



$(oP_1c, oP_2u, oP_3vQz_2)$  and  $(wPc, wQu, wQvQz_2)$  form a connected double  $S'$ -fork. Now by applying Lemma 3.0.11 either (i) or (iv) happens. This completes the proof of the lemma.  $\square$

The following lemma, i.e. Lemma 3.0.13 only has a use in the proof of Lemma 3.0.14. Here we also define and describe three configurations which become useful in explaining the proof of Lemma 3.0.14. They only have application in the proof of Lemma 3.0.14 and the statement of Lemma 3.0.13.

Let  $(P_1, P_2, P_3)$  be an  $S$ -fork with feet on  $x, u, v$  and  $P$  be an  $S$ -path with ends  $w, z$  disjoint from  $P_1, P_2, P_3$  such that there exists a disk  $C \in \mathcal{C}$  containing  $x, w, u, z, v$ , listed in the order they appear on  $C$ . Moreover assume that there exists a segment  $Q$  such that  $u, z, v \in V(Q)$  and  $x, w \notin V(Q)$ . In these circumstances, if  $x = c$  and  $Q$  is a special segment then we say the quadruple  $(P_1, P_2, P_3, P)$  is an *interlaced  $S$ -fork of type I*, and if  $x = c$ ,  $Q$  is not a special segment, and there exists a special segment  $Q'$  such that  $Q$  and  $Q'$  have the vertex  $u$  in common (note that  $w$  must be on the special segment) then we say the quadruple  $(P_1, P_2, P_3, P)$  is an *interlaced  $S$ -fork of type II*, and if  $Q$  is not a special segment and there exists a special segment  $Q'$  such that  $x \in V(Q')$  and  $u = c$  then we say the quadruple  $(P_1, P_2, P_3, P)$  is an *interlaced  $S$ -fork of type III*. The vertices  $x, w, u, z, v$  are called *feet* of the corresponding interlaced  $S$ -fork in the order listed. See Figure 3.1 (A), (B), (C) for their illustrations.

**Lemma 3.0.13.** *Let  $G$  be a multigraph with minimum degree three,  $H$  be a 3-connected multigraph, and let  $S$  be a  $G$ -subdivision in  $H$  satisfying condition (\*) in Theorem 2.2.1. Let  $\mathcal{C}$  be a  $c$ -disk system in  $S$ . Let  $(P_1, P_2, P_3, P)$  be an interlaced  $S$ -fork with feet on  $x, w, u, z, v$ , and  $C \in \mathcal{C}$  be the unique disk such that  $x, w, u, z, v \in C$ . If  $B_1, B_2$  are  $S$ -bridges such that  $V(P_1) \cup V(P_2) \cup V(P_3) \subseteq V(B_1)$ ,  $V(P) \subseteq V(B_2)$ ,  $C$  is the only disk containing all attachments of  $B_1$  and  $C$  is the only disk containing all attachments of  $B_2$ , then*

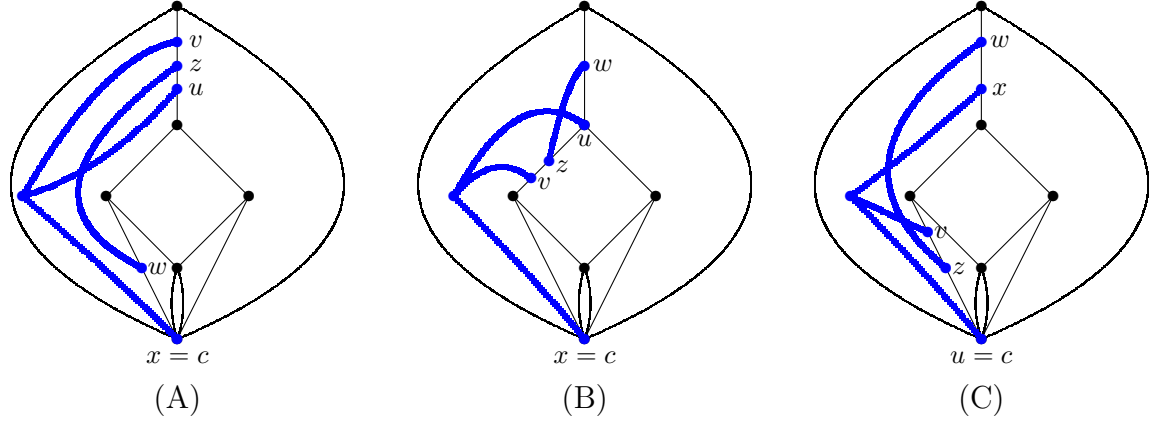


Figure 3.1: The above figures show illustrations of (A): interlaced fork of type I, (B): interlaced fork of type II, (C): interlaced fork of type III.

- (i) If  $(P_1, P_2, P_3, P)$  is an interlaced  $S$ -fork of type I then there exists either a degenerate  $S$ -cross or a weakly free solid  $S$ -cross.
- (ii) If  $(P_1, P_2, P_3, P)$  is an interlaced  $S$ -fork of type II or III, then  $H$  has a  $G$ -subdivision  $S'$  obtained from  $S$  by  $I$ -reroutings such that  $S'$  and the  $c$ -disk system  $\mathcal{C}'$  induced in  $S'$  contains a weakly free solid  $S$ -cross.

*Proof.* First assume that  $(P_1, P_2, P_3, P)$  is an interlaced  $S$ -fork of type I. So  $x = c$  and there exists a special segment  $Q$  with ends  $a, b$  such  $bc \in E(S)$  and  $a, u, z, v, b$  appear on  $Q$  in the order listed. Let  $\alpha : c = u_0, u_1, \dots, u_m = a$  be the path in  $\mathcal{C}$  connecting  $c$  to  $a$  and internally disjoint from  $Q$ . Since  $C \in \mathcal{C}$  is the only cycle containing all attachments of  $B_1$ , there exists a path  $Z$  from a vertex  $x \in V(P_1) \cup V(P_2) \cup V(P_3)$  to a vertex  $y \in V(\text{Int}(\alpha))$ . If  $y \in V(\text{Int}(c\alpha w))$  then there exists an  $S$ -path  $W$  in  $B_1$  with one end on  $u$  and the other end on  $y$  such that  $(W, P)$  form a weakly free solid  $S$ -cross on  $C$ . Similarly, If  $y \in V(\text{Int}(w\alpha a))$  then there exists an  $S$ -path  $W$  in  $B_1$  with one end on  $v$  and the other end on  $y$  such that  $(W, P)$  form a weakly free solid  $S$ -cross on  $C$ . At the end if  $y = w$  then there exists a degenerate  $S$ -cross. This completes the proof of (i).

Now assume that  $(P_1, P_2, P_3, P)$  is an interlaced  $S$ -fork of type II or III. Let  $Q$

be the segment such that  $\{u, z, v\} \subseteq Q$ . Let  $o$  be the common vertex of  $P_1, P_2, P_3$ . Let  $S'$  be obtained from  $S$  by  $I$ -rerouting of  $uQv$  through  $uP_2oP_3v$ . We claim that  $(xP_1o, vQzPw)$  is a solid weakly free  $S'$ -cross. By definition of interlaced  $S$ -fork, none of  $x$  or  $w$  belongs to  $Q$ . So  $x$  and  $w$  are in different segments of  $S'$  than  $o$ . This shows that the  $S'$ -cross is weakly free. Note that since  $(P_1, P_2, P_3, P)$  is an interlaced  $S$ -fork of type II or III,  $Q$  is not a special segment of  $S$ , therefore the vertex  $o$  does not belong to a special segment of  $S'$  which guarantees that  $(xP_1o, vQzPw)$  is a solid  $S'$ -cross. This completes the proof of (ii) and the proof of the lemma.  $\square$

**Lemma 3.0.14.** *Let  $(G, a, b, c)$  be a rooted multigraph with minimum degree three,  $(H, a, b, c)$  be an internally 4-connected rooted multigraph, and let  $(S, a, b, c)$  be a  $G$ -subdivision in  $H$  satisfying condition  $(*)$  in Theorem 2.2.1. Let  $\mathcal{C}$  be a  $c$ -disk system in  $S$ . Suppose  $H$  contains a solid  $S$ -cross  $(P_1, P_2)$  or a connected  $S$ -cross  $(P_1, P_2, P_3)$ , and the rigid  $S$ -bridges  $B_1, B_2$  containing  $P_1, P_2$ , respectively, such that there exists a unique  $C \in \mathcal{C}$ , where all attachments of  $B_1$  are only contained in  $C$  and all attachments  $B_2$  are only contained in  $C$ . Then  $H$  has a  $G$ -subdivision  $S'$  obtained from  $S$  by repeated  $I, T, X$ -reroutings such that  $S'$  and the  $c$ -disk system  $\mathcal{C}'$  induced in  $S'$  satisfy one of the following conditions:*

- (i) *there exists an  $S'$ -jump*
- (ii) *there exists a weakly free  $S'$ -cross anchored at  $c$ , or*
- (iii) *there exists an  $S'$ -triad, or*
- (iv) *there exists a free solid  $S'$ -cross, or*
- (v) *there exists a free  $c$ -blocking  $S'$ -cross, or*
- (vi) *there exists a degenerate  $S'$ -cross.*

*Proof.* Let  $(P_1, P_2)$  be solid  $S$ -cross or  $(P_1, P_2, P_3)$  be a connected  $S$ -cross with feet  $u_1, v_1$  and  $u_2, v_2$ , respectively. Let  $B_1$  and  $B_2$  be the  $S$ -rigid bridges containing  $P_1, P_2$ , respectively.

For seeking a contradiction assume that none of (i) to (vi) holds.

*Claim 1.*  $H$  has a  $G$ -subdivision  $S^*$  obtained from  $S$  by at most one  $I$ -rerouting such that  $S^*$  and the  $c$ -disk system  $\mathcal{C}^*$  induced in  $S^*$  satisfy one of the following conditions: there exists either a weakly free solid  $S^*$ -cross, a weakly free connected  $S^*$ -cross, or an  $S^*$ -tripod.

Assume that there exists a segment  $Q$  such that  $u_1, u_2, v_1$  appear on  $Q$  in the order listed, otherwise there exists a weakly free  $S$ -cross so Claim 1 holds. The fact that  $B_2$  is rigid implies that there exists a path  $P$  between  $P_2$  and a vertex  $v$  in  $V(S) \setminus V(Q)$ . Thus  $P_1 \cup P_2 \cup P$  includes a cross with one foot outside of  $Q$ . So we may assume  $v_2 \notin V(Q)$ .

Note that if  $B_1 = B_2$  then there exists an  $S$ -tripod and Claim 1 holds.

So assume that  $B_1 \neq B_2$  which implies that the  $S$ -cross  $(P_1, P_2)$  is solid. Since  $B_1$  is rigid there exists a path  $P_3$  with one end  $u_3 \in V(P_1) \setminus \{u_1, v_1\}$  and the other end  $z \in V(S) \setminus V(Q)$ . Note that  $u_2 \neq c$ . If  $z = v_2$  then there exists a tripod so Claim 1 holds. So we assume that  $z \neq v_2$ . By symmetry assume that  $u_1, u_2, v_1, z, v_2$  appear on  $C$  in the order listed. Note that  $z \notin V(Q)$ .

If  $z$  and  $u_1$  do not belong to the same segment then the fact that  $(P_1, P_2)$  is a solid  $S$ -cross implies that  $(P_2, u_1 P_1 u_3 P_3 z)$  forms a weakly free solid  $S$ -cross unless  $u_1 = c$  and  $z$  belongs to a special segment, or  $z = c$  and  $u_1$  belongs to a special segment. Note that since  $(P_1, P_2)$  is solid either  $v_2 \neq c$  or  $Q$  is not a special segment. In the first case, there exists an interlaced  $S$ -fork of type III, so by Lemma 3.0.13(ii), Claim 1 holds. In the second case, if  $v_1$  belongs to the same special segment containing  $u_1$  then there exists an interlaced  $S$ -fork of type I, so by Lemma 3.0.13(i) either there exists a degenerate  $S$ -cross, i.e. (vi) holds, a contradiction, or Claim 1 holds. And, if  $v_1$  does

not belong to the same special segment containing  $u_1$  then  $u_1$  must be a branch vertex of  $S$ , therefore there exists an interlaced  $S$ -fork of type II, so by Lemma 3.0.13(ii), Claim 1 holds.

Now assume that  $z$  and  $u_1$  belong to the same segment, called  $Z$ . Then the fact that  $z \notin V(Q)$  implies that  $u_1$  is a branch vertex of  $S$ . We consider two cases, either  $\deg(u_1) = 3$  or  $\deg(u_1) \geq 4$ . First assume that  $\deg(u_1) = 3$ . Let  $S'$  be obtained from  $S$  by  $T$ -rerouting, i.e. by removing  $u_1Zv_2$  and adding  $P_2$  to  $S$ . Then there exists an  $S'$ -triad with center at  $u_3$  and feet on  $u_1, v_1, z$ , thus (iii) holds, a contradiction. Now assume  $\deg(u_1) \geq 4$  in which case let  $S'$  be obtained from  $S$  by  $I$ -rerouting of  $u_1Zz$  through  $u_1P_1u_3P_3z$ . Note that since  $c$  is a branch vertex of  $S$ ,  $u_3 \neq c$  and  $u_2 \neq c$ . Note that  $u_1Zz$  belongs to a special segment of  $S$  if and only if  $u_1P_1u_3P_3z$  belongs to a special segment of  $S'$ . Now  $(u_3P_1v_1, u_2P_2v_2Zz)$  forms a weakly free solid  $S'$ -cross unless one of these events happens: either  $z = c$  and  $u_2$  belongs to a special segment of  $S'$ , or  $v_1 = c$  and  $u_3$  belongs to a special segment of  $S'$ . Note that neither of these events happens by property (iii) in Definition 2.1.1. This completes the proof of Claim 1.

*Claim 2.* If there exists an  $S$ -tripod then there exists a weakly free solid  $S$ -cross.

Let  $(P_1, P_2, P_3)$  be an  $S$ -tripod based on  $Q$ . Let  $u_1, v_1$  and  $u_2, v_2$  and  $u_3, v_3$  be the ends of  $P_1, P_2, P_3$ , respectively. Let  $u_1, u_2, v_1$  appear on  $Q$  in the order listed and  $u_3 \in V(\text{Int}(P_1))$  and  $v_3 \in V(P_2) \setminus \{u_2\}$ . Assume that the  $S$ -tripod  $(P_1, P_2, P_3)$  has been chosen so that the sum of its legs is minimum.

Note that since  $a, b, c$  are branch vertices,  $\{a, b, c\} \cap ((V(P_1) \cup V(P_2) \cup V(P_3)) \setminus \{u_1, v_1, v_2\}) = \emptyset$ . Let  $X = (V(P_1) \cup V(u_2P_2v_3) \cup V(P_3) \cup V(u_1Qv_1)) \setminus \{u_1, v_1, v_3\}$ . The fact that  $(H, a, b, c)$  is internally 4-connected and  $|X| \geq 2$  implies that there exists a path  $W$  from  $u \in X$  to  $z \in V(S) \setminus (X \cup \{u_1, v_1, v_3\})$ . Note that by  $I$ -rerouting  $u_1Qv_1$  through  $u_1P_1v_1$ , we get another tripod with the same feet and legs as  $(P_1, P_2, P_3)$ . So there exists symmetry between  $u_1Qv_1, P_1$  and  $u_2P_2v_3, P_3$ , respectively. By this

symmetry, we may assume that  $u \in V(P_1) \cup V(P_3) \setminus \{u_1, v_1, v_3\}$ . Since  $C$  is the only cycle in  $\mathcal{C}$  containing attachments of  $B_1$ ,  $z \in V(C)$ . Note that  $z \notin V(Q)$  and  $z \notin V(v_3P_2v_2)$  since otherwise there exists an  $S$ -tripod such that the sum of its legs is smaller than the sum of the legs of  $(P_1, P_2, P_3)$ , contradicting with choice of the  $S$ -tripod  $(P_1, P_2, P_3)$ . By symmetry assume that  $u_1, u_2, v_1, z, v_2$  appear on  $C$  in the order listed.

Now if  $z$  and  $u_1$  do not belong to the same segment then either  $(P_2, u_1P_1uWz)$  or  $(P_2, u_1P_1u_3P_3uWz)$  forms a weakly free solid  $S$ -cross, respectively, unless  $u_1 = c$  and  $z$  belongs to a special segment, or  $z = c$  and  $u_1$  belongs to a special segment or  $v_2 = c$  and  $u_2$  belong to a special segment. Similarly to the argument presented in proof Claim 1, the first two cases cause the existence of a weakly free solid  $S$ -cross, thus Claim 2 holds. In the last case, by property (iii) of Definition 2.1.1, we can see that  $(P_2, u_1P_1u_3PuWz, uP_3v_3)$  forms a free  $c$ -blocking  $S$ -cross. Thus (v) holds, a contradiction.

Note that if  $z$  and  $u_1$  belong to the same segment then the same argument as presented in proof of Claim 1 shows that there exists a weakly free solid  $S$ -cross, so Claim 2 holds.

Now by applying Claim 1 and Claim 2, we may assume that there exists a weakly free solid  $S$ -cross.

Let  $v$  be branch vertex of  $S$ ,  $Q_1, Q_2$  be two segments of  $S$  and  $C \in \mathcal{C}$  be such that  $V(Q_1) \cup V(Q_2) \subseteq V(C)$  and  $Q_1, Q_2$  have the vertex  $v$  in common. Let  $P_1$  and  $P_2$  be the two disjoint  $S$ -paths introduced before with ends  $u_1, v_1$  and  $u_2, v_2$ , respectively, such that  $u_1, u_2 \in V(Q_1)$ ,  $v_1, v_2 \in V(Q_2)$  and  $u_2, u_1, v, v_2, v_1$  appear on  $C$  in the order listed. Assume the other ends of  $Q_1$  and  $Q_2$  are  $w_1$  and  $w_2$ , respectively. Let  $C_1, C_2 \in \mathcal{C}$  be such that  $V(Q_1) \subseteq V(C) \cap V(C_1)$  and  $V(Q_2) \subseteq V(C) \cap V(C_2)$ .

*Claim 3.* The set  $K = \{v, v_1, u_2\}$  is not a cut set in  $H$ .

For proving Claim 3, assume  $K$  is a cut set. Since  $(H, a, b, c)$  is internally 4-connected, for any separation  $(H_1, H_2)$  of  $H$  of order three,  $\{a, b, c\} \not\subseteq V(H_1)$  and  $\{a, b, c\} \not\subseteq V(H_2)$ . Since  $a, b, c$  are branch vertices of  $S$  and each has degree at least three,  $(V(vQ_1u_2) \cup V(vQ_2v_1)) \setminus \{v, v_1, u_2\}$  does not contain  $a, b$  or  $c$ . This contradicts the fact that  $H$  is internally 4-connected. Hence, Claim 3 holds.

We consider two cases, either  $\deg(v) = 3$  or  $\deg(v) \geq 4$ .

If  $\deg(v) = 3$  then let  $S'$  be obtained from  $S$  by  $T$ -rerouting, i.e. by removing the path  $Int(vQ_1u_2)$  and adding  $v_2P_2u_2$ , and  $\mathcal{C}'$  be the  $c$ -disk system induced in  $S'$ . In this circumstance,  $V(P_1) \cup V(vQ_1u_2)$  contains an  $S'$ -triad with center  $u_1$  and feet  $v, u_2, u_1$ . Hence, (iii) holds, a contradiction.

Now, assume  $\deg(v) \geq 4$ . Suppose among all weakly free crosses  $(P'_1, P'_2)$  with feet on  $Q_1, Q_2$  or on segments obtained by a sequence of  $I$ -reroutings of some subpaths of  $Q_1, Q_2$ , the weakly free cross  $(P_1, P_2)$  is chosen so that  $|u_2Q_1w_1| + |v_1Q_2w_2|$  is minimum. As Claim 3 implies,  $K$  is not a cut set. Therefore there exists a path  $P$  from a vertex in  $x \in (V(vQ_1u_2) \cup V(vQ_2v_1) \cup V(P_1) \cup V(P_2)) \setminus \{v, u_2, v_1\}$  to a vertex  $y \in V(S) \setminus (V(vQ_1u_2) \cup V(vQ_2v_1) \cup V(P_1) \cup V(P_2))$  such that  $P$  is disjoint from  $V(P_1) \cup V(P_2) \cup V(vQ_1u_2) \cup V(vQ_2v_1)$ . By symmetry assume  $x \in V(vQ_1u_2) \cup V(P_1)$ .

We claim that  $y \notin (V(u_2Q_1w_1) \cup V(v_1Q_2w_2)) \setminus \{v_1, u_2\}$ . For proving the claim note that if  $y \in V(v_1Q_2w_2) \setminus \{v_1\}$  then either  $x \in V(Int(vQ_1u_2))$  or  $x \in V(P_1) \setminus \{v_1\}$  which implies that either  $(xPy, P_2)$ , or  $(uP_1xPy, P_2)$  forms a weakly free cross such that  $|u_2Q_1w_1| + |v_1Q_2w_2| > |u_2Q_1w_1| + |yQ_2w_2|$ , contradicting the choice of  $(P_1, P_2)$ . If  $y \in V(u_2Q_1w_1) \setminus \{u_2\}$ , then either  $x \in V(P_1) \setminus \{u_1, v_1\}$ ,  $x \in V(vQ_1u_1) \setminus \{v\}$ , or  $x \in V(u_1Q_1u_2) \setminus \{u_1, u_2\}$ . In the first case  $(P_1, v_2P_2xPy)$  form a weakly free  $S$ -cross such that  $|u_2Q_1w_1| + |v_1Q_2w_2| > |yQ_1w_1| + |v_1Q_2w_2|$ , a contradiction. In the remaining two cases, let  $S'$  be obtained from  $S$  by  $I$ -rerouting  $xQ_1y$  through  $xPy$ . It is not hard to see that either  $(P_1, v_2P_2u_2Q_1y)$  or  $(xQ_1u_1P_1v_1, v_2P_2u_2Q_1y)$  forms a weakly free  $S'$ -cross such that  $|u_2Q_1w_1| + |v_1Q_2w_2| > |yQ_1w_1| + |v_1Q_2w_2|$ , a contradiction. This

completes the proof of the claim.

Note that if  $y \in V(C) \setminus (V(Q_1) \cup V(Q_2))$ , then we already know that either  $x \in V(\text{Int}(vQ_1u_2))$  or  $x \in V(P_1) \setminus \{v_1\}$ , which implies that either  $(P, P_2)$  or  $(u_1P_1xPy, P_2)$  forms a free solid  $S$ -cross causing (iv) to hold, unless  $y = c$  and  $Q_1$  is a special segment. In that case since  $Q_1$  is a special segment,  $c \in V(C_1)$ . Note that  $C_1, C_2$  and  $C$  are cycles, and  $v \in C \cap C_1 \cap C_2$ , thus by property (iv) in Definition 2.1.1,  $c \notin V(C_2)$ . Let  $S'$  be obtained from  $S$  by  $X$ -rerouting, i.e. by replacing the paths  $vQ_1u_2, vQ_2v_1$  by paths  $vQ_2v_2P_2u_2, vQ_1u_1P_1v_1$ , respectively, and  $C'$  be the  $c$ -disk system induced in  $S'$ . It is easy to see that either  $x \in V(P_1) \cup V(vQ_1u_1)$  or  $x \in V(u_1Q_1u_2)$ , which implies  $(xPc, v_1Q_2v_2)$  or  $(u_1Q_1xPc, v_1Q_2v_2)$  forms a solid  $S'$ -cross, respectively since  $c \notin V(C_2)$  and  $c \notin V(Q_1) \cup V(Q_2)$ .

Therefore we assume  $y \notin V(C)$ . If  $y \notin V(C_1)$ , then  $P$  is an  $S$ -jump, so (i) holds, a contradiction. So assume that  $y \in V(C_1)$ , now either  $y \notin V(C_2)$  or  $y \in V(C_2)$ . If  $y \notin V(C_2)$ , then let  $S'$  be obtained from  $S$  by  $X$ -rerouting, i.e. by replacing the paths  $vQ_1u_1, vQ_2v_2$  by paths  $vQ_2v_2P_2u_2, vQ_1u_1P_1v_1$ , respectively, and  $C'$  be the  $c$ -disk system induced in  $S'$ . Since  $y \notin V(C_2)$  then  $V(u_1Q_1u_2) \cup V(P)$  contains an  $S'$ -jump, so (i) holds, a contradiction. Therefore, we assume that  $y \in V(C_2)$ . We claim that  $y$  and  $v$  belong to two different connected components of  $C_1 \cap C_2$ . For proving the claim, let  $Q \subseteq C_1 \cap C_2$  be the segment containing  $y, v$ . Note that also  $Q_1 \subseteq C_1 \cap C$  and  $Q_2 \subseteq C_2 \cap C$ , so by property (ii) in Definition 2.1.1,  $\deg(v) = 3$ , a contradiction. Now the property (i) in Definition 2.1.1 implies that  $y = c$ , so (ii) holds, a contradiction. This completes the proof of the lemma.  $\square$

We say  $G'$  is obtained from  $G$  by *splitting*  $v$ , if there exists  $v_1, v_2 \in V(G')$  such that  $v_1$  and  $v_2$  are not adjacent and  $G$  is isomorphic to the graph obtained from  $G'$  by identifying  $v_1, v_2$  where  $v$  corresponds to the vertex obtained from identifying  $v_1, v_2$ .

**Definition 3.0.15.** *Let  $a, b, c$  be three distinct vertices of multigraph  $G$  with two parallel  $ac$  edges and two parallel  $bc$  edges such that these are the only parallel edges*



of  $G$ . Suppose a simple graph  $G'$  is obtained from  $G$  by splitting the vertex  $c$  into  $c_1, c_2$  such that  $G^*$  is a 3-connected planar graph and in the unique embedding of  $G^*$ , the cycle  $J : c_1, a, c_2, b$  bounds a face. Let  $S^*$  be a subdivision of  $G^*$  such that  $c_1a, c_2a, c_1b, c_2b$  are edges in  $S^*$ . Let  $\mathcal{F}^*$  be the set of cycles bounding a face in the unique embedding of  $S^*$ . Let  $S$  be the subdivision of  $G$  obtained from  $S^*$  by identifying  $c_1$  and  $c_2$ . We describe a  $c$ -disk system  $\mathcal{C}$  for  $S$  obtained from  $\mathcal{F}^*$  as follows:

- (i) if there exists  $F^* \in \mathcal{F}^*$  such that  $c_1, c_2 \notin V(F^*)$  then  $F^* \in \mathcal{C}$ , or
- (ii) if there exists  $F^* \in \mathcal{F}^*$  and  $i \in \{1, 2\}$  such that  $c_i \in V(F^*), c_{3-i} \notin V(F^*)$  then  $F \in \mathcal{C}$  where  $F$  is obtained from  $F^*$  by relabeling  $c_i$  by  $c$ .
- (iii) cycles  $C_a : c, a$  and  $C_b : c, b$  belong to  $\mathcal{C}$  where  $C_a$  and  $C_b$  obtained by identifying  $c_1$  and  $c_2$  in the cycle  $c_1, a, c_2, b$  of  $G^*$ .

We call such a  $c$ -disk system a planar  $c$ -disk system with the realizer graph  $S^*$ .

Note that in the above definition, we implicitly use the well known result of Whitney [58] saying that a 3-connected planar graph has a unique planar embedding. We say two vertices  $(v_2, v_1)$  are co-facial in the unique embedding of  $S^*$  if there exists a face  $F \in \mathcal{F}^*$  such that  $v_1, v_2$  belong to the boundary of the face  $F$ . Let  $(G, a, b, c)$  be a rooted multigraph with minimum degree three. Let  $(H, a, b, c)$  be a rooted graph and  $(S, a, b, c)$  be a  $(G, a, b, c)$ -subdivision in  $(H, a, b, c)$  satisfying condition (\*) in Theorem 2.2.1. Let  $\mathcal{C}$  be a planar  $c$ -disk system of  $S$  with realizer  $S^*$ . Let  $\mathcal{T} : (P_1, P_2, P_3)$  be an  $S$ -triad with feet  $x_1, x_2, x_3$ . We say  $\mathcal{T}$  is an *essential*  $S$ -triad if  $c \in \{x_1, x_2, x_3\}$ , say  $c = x_1$ , and there exists a closed curve  $\psi$  passing through  $c_1, c_2, x_2, x_3$  and disjoint from  $S^*$  except at  $c_1, c_2, x_2, x_3$  such that  $\psi$  bounds two closed disks  $D_1, D_2$  in the plane and  $a \in D_1, b \in D_2$ , or equivalently there exists  $k \in \{2, 3\}$  such that each pair of vertices  $(c_1, x_k), (x_k, x_{5-k}), (x_{5-k}, c_2)$  is co-facial in the unique embedding of  $S^*$ .

Note that any  $I, T$ - or  $X$ -rerouting which does not involve the vertex  $c$  of planar  $c$ -disk system  $S$ , naturally extends to its realizer  $S^*$ . Similarly any  $I, T$ - or  $X$ -rerouting of  $S^*$  which does not involve vertices  $c_1$  and  $c_2$  naturally extends to its planar  $c$ -disk system  $S$ .

**Lemma 3.0.16.** *Let  $(G, a, b, c)$  be a rooted multigraph with minimum degree three with two parallel  $ac$  edges and two parallel  $bc$  edges. Let  $(H, a, b, c)$  be an internally 4-connected rooted multigraph and  $(S, a, b, c)$  be a  $(G, a, b, c)$ -subdivision in  $(H, a, b, c)$  satisfying condition  $(*)$  in Theorem 2.2.1. Let  $\mathcal{C}$  be a planar  $c$ -disk system of  $S$  with realizer  $S^*$ . Moreover assume that  $a, b$  are not co-facial in  $\mathcal{C}$ . Suppose  $H$  contains an  $S$ -triad with feet  $x_1, x_2, x_3$ . Then  $H$  has a  $G$ -subdivision  $S'$  obtained from  $S$  by repeated  $I, T$ -reroutings and at most one triad exchange such that  $S'$ , the  $c$ -disk system  $\mathcal{C}'$  induced in  $S'$  satisfy one of the following conditions:*

- (i) *there exists an  $S'$ -jump, or*
- (ii) *there exists an essential  $S'$ -triad, or*
- (iii) *there exists a solid free  $S'$ -cross.*

*Proof.* Let  $\mathcal{T} : (P_1, P_2, P_3)$  be an  $S$ -triad with feet  $x_1, x_2, x_3$  and center  $o$ . Through the proof we use the same labeling for vertices of  $S$  and  $S^*$  except for  $c$  and  $c_1, c_2$  where  $c_1, c_2$  are the two vertices obtained by splitting of the vertex  $c$ . Assume  $C_1, C_2, C_3$  are the three cycles in  $\mathcal{C}$  such that  $x_i, x_j$  belong to  $C_k$  for  $\{i, j, k\} = \{1, 2, 3\}$ . Note that we can assume the  $C_i$ 's are chosen such that  $\{C_a, C_b\} \cap \{C_1, C_2, C_3\} = \emptyset$  where  $C_a : a, c, C_b : b, c$ .

Let  $\mathcal{F}^*$  be the set of facial cycles of  $S^*$  and  $J : c_1, a, c_2, b$  be a facial cycle of  $S^*$  by Definition 3.0.15. Note that there exists a natural one to one correspondence  $\varphi : \mathcal{F}^* \setminus J \rightarrow \mathcal{C} \setminus \{C_a, C_b\}$ , by relabeling  $c_i$  to  $c$ ,  $i = \{1, 2\}$ .

If  $\mathcal{T}$  is essential then (ii) holds. So assume that  $\mathcal{T}$  is not essential. Thus either  $c \notin \{x_1, x_2, x_3\}$  in which case let  $x_1^* = x_1$ , or  $x_1 = c$  and there exists a  $j \in \{1, 2\}$

such that  $c_j, x_2, x_3$  are pairwise co-facial in the unique embedding of  $S^*$  in which case let  $x_1^* = c_j$ . Therefore  $x_1^*, x_2, x_3$  are pairwise co-facial in the unique embedding of  $S^*$ . This implies that there exists a closed curve  $\psi$  passing through  $x_1^*, x_2, x_3$  and disjoint from  $S^*$  except at  $x_1^*, x_2, x_3$ . The curve  $\varphi$  bounds two closed disks  $D_1, D_2$  in the sphere and the set  $\{x_1^*, x_2, x_3\}$  is a cut set in  $S^*$ . By symmetry between  $c_1, c_2$ , we consider two cases:

**Case 1**  $c_1, c_2 \in D_1$ .

Since there are two parallel edges between  $a, c$  and two parallel edges between  $b, c$  in  $S$  and also the graph  $S^*$  is a simple graph,  $ac_1, ac_2, bc_1, bc_2$  are edges in  $S^*$ . Thus  $\{a, b, c_1, c_2\} \subseteq D_1$ . Therefore  $\{x_1, x_2, x_3\}$  is a cut set in  $S$ . Let  $B$  be the set of branch vertices of  $S^*$ . If  $|(D_2 \setminus \{x_1, x_2, x_3\}) \cap B| \geq 2$  then it is easy to see that there exists an internal 3-separation in  $(H, a, b, c)$ , a contradiction. Therefore, we may assume  $|(D_2 \setminus \{x_1, x_2, x_3\}) \cap B| = 1$ . Let  $\{v\} = (D_2 \setminus \{x_1, x_2, x_3\}) \cap B$ . Note that  $v \notin \{a, b, c_1, c_2\}$  and the property (ii) in Definition 2.1.1 implies that  $\deg(v) = 3$  in  $S$  and  $S^*$ .

Let  $Z_1, Z_2, Z_3$  be the three segments of  $(S, a, b, c)$ , or equivalently of  $(S^*, a, b, c)$ , with one end at  $v$  and the other ends at  $v_1, v_2, v_3$ , respectively. Assume  $x_i \in Z_i, i = 1, 2, 3$ . Suppose  $S$  and  $(P_1, P_2, P_3)$  are chosen so that there is no  $(G, a, b, c)$ -subdivision  $(S', a, b, c)$  obtained from  $(S, a, b, c)$  by a sequence of  $I, T$ -reroutings such that there exists an  $(S', a, b, c)$ -triad with feet on  $Z_1, Z_2, Z_3$ , where the sum of the lengths of its legs is strictly smaller than  $|E(x_1Z_1v_1)| + |E(x_2Z_2v_2)| + |E(x_3Z_3v_3)|$ . Let  $X = (V(P_1) \cup V(P_2) \cup V(P_3) \cup V(vZ_1x_1) \cup V(vZ_2x_2) \cup V(vZ_3x_3)) \setminus \{x_1, x_2, x_3\}$  and  $Y = V(S) \setminus (X \cup \{x_1, x_2, x_3\})$ . Since  $(H, a, b, c)$  is internally 4-connected, there exists a path  $P$  from  $x \in X$  to  $y \in Y$  disjoint from  $V(S) \cup V(P_1) \cup V(P_2) \cup V(P_3)$  except at  $x, y$ .

Note that by the triad exchange operation, i.e. replacing  $vZ_1x_1, vZ_2x_2, vZ_3x_3$  by

$P_1, P_2, P_3$ , respectively, there is symmetry between  $vZ_1x_1, vZ_2x_2, vZ_3x_3$  and  $P_1, P_2, P_3$ . So we may assume  $x \in P_1 \cup P_2 \cup P_3$ . By symmetry between  $P_1, P_2, P_3$ , we may assume  $x \in P_1 \setminus \{x_1\}$ . If  $y$  and  $x_2$  do not belong to the same disk then  $P_1 \cup P_2 \cup P$  contains a jump with ends at  $x_2, y$ , so (i) holds. So assume there exists a disk  $C \in \mathcal{C}$  such that  $C$  contains  $x_2, y$ . Note that it is easy to see that  $C \neq C_2$ .

Now, by the triad exchange operation, i.e. replacing  $vZ_1x_1, vZ_2x_2, vZ_3x_3$  by  $P_1, P_2, P_3$ , and by switching the labels of  $vZ_1x_1, vZ_2x_2, vZ_3x_3$  by  $P_1, P_2, P_3$ , respectively, we can assume  $x \in vZ_1x_1 \setminus \{x_1\}$ . Note that if  $y \notin C_2 \cup C_3$ , then  $x$  and  $y$  are not on the same disk, so (i) holds. Thus, by symmetry, we assume that  $y \in C_2$ . Note that  $x_2 = v_2$  since otherwise the fact that  $C \neq C_2, x_2 \in Z_2 \setminus \{v_2\}$  would imply that  $Z_2 \subseteq C_1 \cap C_3 \cap C$ , a contradiction.

Note that the cross  $(P, P_1 \cup P_3)$  is a free solid cross unless one of the following happens: either  $y = c$  and  $Z_1$  is a special segment, or by symmetry between  $x_1, x_3$ ,  $x_1 = c$  and  $Z_3$  is a special segment, or  $x_1 = v_1, x_3 = v_3$  and there is a segment  $Z$  with ends  $v_1, v_3$  such that  $y \in \text{Int}(Z)$ . Below, we study these three cases separately.

**Case 1.1**  $y = c$  and  $Z_1$  is a special segment.

Since  $Z_1$  is a special segment  $\{a, b\} \cap \{v, v_1\} \neq \emptyset$ . Note that in this case  $v \neq a, v \neq b$  since otherwise the fact that  $|(D_2 \setminus \{x_1, x_2, x_3\}) \cap B| = 1$  and  $ac, bc$  are edges implies that  $c \in \{x_1, x_2, x_3\}$  contradicting the fact that  $y = c$ . So by symmetry assume that  $v_1 = a$ .

Since  $Z_1$  is a special segment, there exists  $j \in \{1, 2\}$  such that  $c_j \in \varphi^{-1}(C_2)$  and  $c_{3-j} \in \varphi^{-1}(C_3)$  in the unique embedding of  $S^*$ . By symmetry, we can assume  $j = 1$ .

Note that  $C \neq C_1$  because otherwise  $\varphi^{-1}(C) = \varphi^{-1}(C_1)$  implying that  $V(Z_3) \cup \{c_1\} \subseteq \varphi^{-1}(C_1) \cap \varphi^{-1}(C_2)$  which contradicts the fact that  $\varphi^{-1}(C_1)$  and  $\varphi^{-1}(C_2)$  are facial cycles in  $S^*$ . Similarly,  $C \neq C_3$  because otherwise  $c_1 \in \varphi^{-1}(C_3)$  implying  $V(Z_1) \cup \{c_1\} \subseteq \varphi^{-1}(C_2) \cap \varphi^{-1}(C_3)$ . Therefore  $x_2 = v_2$  since otherwise the path  $x_2Z_2v_2$  belongs to  $C \cap C_1 \cap C_3$ , which contradicts the definition of  $c$ -disk system, see

Definition 2.1.1.

Note that  $v_2 \neq b$  since otherwise the facts that  $c_1ac_2b$  and  $\varphi^{-1}(C_3)$  are facial cycles in  $S^*$  imply that the degree of  $c_2$  is two in  $S^*$ , a contradiction. Therefore,  $C \neq C_a, C \neq C_b$  and this implies that  $\varphi^{-1}(C)$  is well defined.

Note that by definition of  $S$ -triad  $x_3 \notin C_3$ . If  $x_3 \neq v_3$  then the fact that  $c_1ac_2b$  is a facial cycle and  $c_1 \in \varphi^{-1}(C_2), c_2 \in \varphi^{-1}(C_3)$  shows that  $(yPxP_1o, P_2, P_3)$  is an essential triad, so (ii) holds. So we may assume  $x_3 = v_3$ . Note that  $\{v_3, c\} \subseteq V(C_2), \{v_2, c\} \subseteq V(C_3)$  and  $\{v_2, c\} \subseteq V(C)$ . If there exists no  $C^* \in \mathcal{C}$  such that  $\{c, v_3, v_2\} \subseteq V(C^*)$  then the fact that  $c_1ac_2b$  is a facial cycle and  $c_1 \in \varphi^{-1}(C_2), c_2 \in \varphi^{-1}(C_3)$  shows that  $(yPxP_1o, P_2, P_3)$  is an essential triad, so (ii) holds. Therefore, there exists  $C^* \in \mathcal{C}$  such that  $\{c, v_3, v_2\} \subseteq V(C^*)$ .

Since  $a \in \varphi^{-1}(C_3) \cap \varphi^{-1}(C_2) \cap J$  implies that  $\deg(a) = 3$  in  $S$  and  $S^*$ . Therefore  $a$  only belongs to  $C_a, C_2, C_3$ . This immediately shows that either  $x_1 \neq a$  implying  $x_1$  only belongs to  $V(C_2) \cap V(C_3)$ , or  $x_1 = a$  implying  $x_1$  only belongs to  $C_a, C_2, C_3$ . Note that  $c \notin C_1$ , so by property (i) of Definition 2.1.1 and the fact that  $\{v_2, v_3\} \subseteq V(C_1) \cap V(C^*), v_2v_3 \in E(G)$ , or equivalently  $v_2, v_3$  are ends of a segment in  $S$ , called  $Q$ . Now, let  $S'$  be obtained from  $S$  by  $I$ -rerouting  $Q$  through  $v_3P_3oP_2v_2$ . It is easy to see that  $P_1$  is an  $S'$ -jump.

**Case 1.2** By symmetry between  $x_1, x_3, x_1 = c$  and  $Z_3$  is a special segment.

Since  $Z_3$  is a special segment  $c \in C_1 \cap C_2$ . But this immediately shows that  $c, x_2, x_1 \in C_1$ , a contradiction with the definition of triad.

**Case 1.3**  $x_1 = v_1, x_3 = v_3$  and there is a segment  $Z$  with ends  $v_1, v_3$  such that  $y \in \text{Int}(Z)$ .

Note that  $C \neq C_1$  since otherwise  $C$  would contain  $x_1, x_2, x_3$ , a contradiction with definition of an  $S$ -triad. Note that  $x_2 = v_2$ ; otherwise the edges in  $x_2Z_2v_2$  appear in

at least three cycles, a contradiction with Definition 2.1.1. Since  $y$  belongs to  $\text{Int}(Z)$ , the disk  $C$  contains  $v_1, v_3$ . Therefore,  $v_1, v_2, v_3$  belong to  $C$  which equivalently means that  $x_1, x_2, x_3$  are in the same disk, contradiction with definition of an  $S$ -triad.

**Case 2**  $c_1 \in D_1$  and  $c_2 \in D_2$ .

Because  $ac_1, ac_2, bc_1, bc_2$  are edges of  $S^*$ ,  $a, b \subseteq \{x_1^*, x_2, x_3\}$ . Suppose  $x_2 = a, x_3 = b$ . Since  $c_1 \in D_1, c_2 \in D_2$ , we can infer that  $c \neq x_1$ , equivalently  $x_1^* \neq c_1$  and  $x_1^* \neq c_2$ . The choice of  $C_1 \in \mathcal{C}$  shows that  $\varphi^{-1}(C_1)$  is a facial cycle in the unique embedding of  $S^*$ . So there exists a path in  $\varphi^{-1}(C_1)$ , called  $Q$ , connecting  $a$  to  $b$ . But it is easy to see that the cycle  $J$  in  $S^*$  separates  $x_1$  from  $Q$  in the unique embedding of  $S^*$ , contradicting the fact that  $J$  is facial cycle by Definition 3.0.15.  $\square$

**Theorem 3.0.17.** *Let  $(G, a, b, c)$  be a rooted multigraph with minimum degree three with two parallel  $ac$  edges and two parallel  $bc$  edges. Let  $(H, a, b, c)$  be an internally 4-connected rooted multigraph and  $(S, a, b, c)$  be a  $(G, a, b, c)$ -subdivision in  $(H, a, b, c)$  satisfying condition  $(*)$  in Theorem 2.2.1. Let  $\mathcal{C}$  be a planar  $c$ -disk system of  $S$  with realizer  $S^*$ . Moreover assume that  $a, b$  are not co-facial in  $\mathcal{C}$ . Suppose  $H$  contains an  $S$ -triad with feet  $x_1, x_2, x_3$ . Then  $H$  has a  $G$ -subdivision  $S'$  obtained from  $S$  by repeated  $I, T$ -reroutings and at most one triad exchange such that  $S'$  and the  $c$ -disk system  $\mathcal{C}'$  induced in  $S'$  satisfy one of the following conditions:*

- (i) *there exists an  $S'$ -jump, or*
- (ii) *there exists a degenerate  $S'$ -cross, or*
- (iii) *there exists a  $c$ -blocking  $S'$ -cross, or*
- (iv) *there exists a weakly free solid  $S'$ -cross anchored at  $c$  around a vertex of degree four, or*
- (v) *there exists a free solid  $S'$ -cross*

- (vi) *there exists a double facial  $S'$ -cross, or*
- (vii) *there exists a blocking interlaced  $S$ -fork of type I, or*
- (viii) *there exists an essential  $S'$ -triad, or*
- (ix) *there exists an  $S'$ -pyramid, or*
- (x)  *$G$  is isomorphic to  $K_4$ , or*
- (xi)  *$\mathcal{A}(S', H)$  is list colorable.*

*Proof.* Theorem 2.2.10 implies one of the outcomes listed in the statement of Theorem 2.2.10 holds. The fact that  $(H, a, b, c)$  is internally 4-connected, the result of Lemma 3.0.11 and Lemma 3.0.12 imply that that (vi), (ix) and (viii) of Theorem 2.2.10 do not hold. The rest of the proof easily follows from Lemma 3.0.14 and Lemma 3.0.16. □

## CHAPTER IV

### OBSTRUCTIONS FOR $c$ -, $ac$ -, $abc$ -PLANARITY

The proofs of the following two lemmas are trivial.

**Lemma 4.0.18.** *Suppose  $(G, x_1, x_2, x_3)$  is a rooted graph. Then  $(G, x_1, x_2, x_3)$  is  $x_1$ -planar if and only if the graph  $(H, x_1, x_2, x_3)$  obtained from  $G$  by adding the edges  $x_1x_2, x_1x_3$  is  $x_1$ -planar.*

**Lemma 4.0.19.** *Let  $G$  be a 3-connected minor-minimal non-projective planar graph and  $v$  be a vertex of degree three in  $G$ . Let  $v_1, v_2, v_3$  be the neighbors of  $v$  in  $G$ . Then  $\{v_1, v_2, v_3\}$  is an independent set in  $G$ .*

**Corollary 4.0.20.** *Let  $G$  be a 3-connected minor-minimal non-projective planar graph with an internal 3-separation  $(G_1, G_2)$ . Then  $|V(G_1)|, |V(G_2)| \geq 5$ .*

Here we present a proof of Theorem 1.8.2, mentioned in Section 1.8.

*Proof of Theorem 1.8.2.* For seeking a contradiction, assume  $(G_1, a, b, c)$  and  $(G_2, a, b, c)$  are both  $a$ -planar. Let  $a_1, b_1, a_2, b_2$  be four distinct points on the boundary of a disk  $D$ , in the order listed. Let  $c \neq a_1, c \neq a_2$  be a point on the chord connecting  $a_1, a_2$ , called  $\ell$ . We may assume the boundary of  $D$  is a circle and  $x_1, x_2$  are antipodal points, for  $x = a, b$ . Note that the chord  $\ell$  divides the disk  $D$  into two closed disks  $D_1, D_2$  such that  $D_1 \cap D_2 = \ell$  and  $b_1 \in D_1, b_2 \in D_2$ . Since  $(G_1, a, b, c)$  is  $a$ -planar, there exists an embedding  $\Sigma$  of  $(G_1, a, b, c)$  in  $D_1$  by splitting of the vertex  $a$  into  $a'$  and  $a''$  such that  $\Sigma(a') = a_1, \Sigma(a'') = a_2, \Sigma(b) = b_1$  and  $\Sigma(c) = c$ . Similarly since  $(G_2, a, b, c)$  is  $a$ -planar, there exists an embedding  $\Gamma$  of  $(G_2, a, b, c)$  in  $D_2$  by splitting of the vertex  $a$  into  $\dot{a}$  and  $\ddot{a}$  such that  $\Sigma(\dot{a}) = a_1, \Sigma(\ddot{a}) = a_2, \Sigma(b) = b_2$  and  $\Sigma(c) = c$ . Now, by identifying antipodal points on the boundary of  $D$ , we obtain an embedding



of  $G$  in the projective plane. This contradicts the fact that  $G$  is a minor-minimal non-projective planar graph, and it completes the proof of the theorem.  $\square$

The following theorem is due to Robertson and Seymour [45].

**Theorem 4.0.21** ([45]). *Let  $H$  be an internally 3-connected graph and  $a, b, c$  be three distinct vertices of  $H$ . If  $H$  has no planar embedding in which  $a, b, c$  are all incident with the outer face, then  $H$  contains a double fork with feet on  $a, b, c$ .*

**Lemma 4.0.22.** *Let  $G$  be a 3-connected minor-minimal non-projective planar graph with an internal 3-separation  $(G_1, G_2)$  such that  $V(G_1) \cap V(G_2) = \{a, b, c\}$ . Then  $G_1$  and  $G_2$  each contains a double fork with feet on  $a, b, c$ .*

*Proof.* By symmetry, it suffices we show that  $(G_1, a, b, c)$  contains a double fork with feet on  $a, b, c$ . Note that the fact that  $G$  is 3-connected implies that  $(G_1, a, b, c)$  is 3-connected. We claim that  $(G_1, a, b, c)$  does not have a drawing in the disk such that  $a, b, c$  are on the boundary of the disk. For proving the claim assume  $(G_1, a, b, c)$  has such a drawing in the disk, say  $\Sigma$ . Let  $G^*$  be obtained from  $G$  by contracting all edges in  $G_1$  except edges with one end in  $a, b$  or  $c$  and call the new vertex obtained from this contraction operation  $v^*$ . Since  $G$  is 3-connected,  $\deg(v^*) = 3$  in  $G^*$ , and  $\{a, b, c\}$  are the set of neighbors of  $v^*$ . Since  $G$  is minor minimal non-projective planar,  $G^*$  is projective planar. Let  $\Sigma^*$  be such an embedding. Now it is easy to see that by removing the vertex  $v^*$  in  $\Sigma^*$  and identifying the vertices  $a, b, c$  of  $G^*$  in  $\Sigma^*$  with the vertices  $a, b, c$  of  $G_1$  in  $\Sigma$ , we obtain an embedding of  $G$  in the projective plane, a contradiction. Thus the assertion of the lemma follows from this claim and Theorem 4.0.21.  $\square$

The following lemma is trivial, but we use it frequently often in this paper. So for more clarification, we state it here and we omit the proof.

**Lemma 4.0.23.** *Let  $G$  be a double fork with feet on  $a, b, c$ . Then  $(G, a, b, c)$  contains  $\mathcal{O}_i(\alpha, \beta, 3)$  for some  $i \in \{1, 8, 13\}$  and  $\{\alpha, \beta\} = \{1, 2\}$ , (See Figure 1.3, 1.4 and 1.5) as a rooted subdivision, or  $(G, a, b, c)$  has a unique  $c$ -planar embedding.*

From now on we will need to contract edges frequently, and to facilitate the exposition, we introduce the following rules. We label the vertices of a graph  $S$  by a combination of numbers in  $\mathbb{N} \cup \{0\}$  and letters. If  $m, n \in \mathbb{N} \cup \{0\}$  and  $mn \in E(S)$  the graph obtained from  $S$  by contracting the edge  $mn$  has the same labeling as  $S$  except the new vertex will get the label  $i = \min\{m, n\}$ . If  $ix \in E(S)$  where  $x$  is a letter and  $i \in \mathbb{N} \cup \{0\}$  is an edge, the graph obtained from  $S$  has the same labeling as  $S$  except the label of the new vertex will be  $i$ . Finally, if  $xy \in E(S)$ , where  $x, y$  are letters, which holds rarely, we specify the label of the new vertex obtained by contracting the edge  $xy$  by  $x$  where  $x$  has the lower alphabetical order compared to  $y$ .

#### 4.1 Obstructions for $c$ -planarity

Let  $(K_{2,3}, 1, 2, 3)$ ,  $(K'_{2,3}, 1, 2, 3)$ ,  $(K''_{2,3}, 1, 2, 3)$  be the rooted graphs shown in Figure 4.1. If  $(G, a, b, c)$  contains one of  $K_{2,3}(1, 2, 3)$ ,  $K'_{2,3}(1, 2, 3)$ ,  $K''_{2,3}(1, 2, 3)$  as a rooted subdivision, then we say the vertices corresponding to 4, 5 in  $(G, a, b, c)$  are the *centers*.

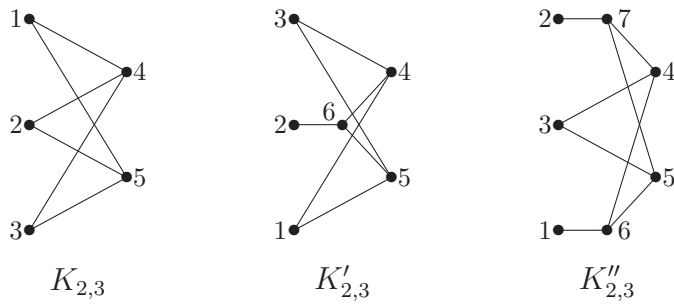


Figure 4.1: Rooted graphs  $(K_{2,3}, 1, 2, 3)$ ,  $(K'_{2,3}, 1, 2, 3)$ ,  $(K''_{2,3}, 1, 2, 3)$  with terminal vertices 1, 2, 3 and centers 4, 5.

**Lemma 4.1.1.** *Let  $(G, a', b, c)$  be a rooted graph and  $(H, a, b, c)$  obtained from  $(G, a, b, c)$  by adding the edge  $aa'$ .*

- (i) If  $(G, a', b, c)$  contains  $(\mathcal{O}_1, 1, 2, 3)$  as rooted subdivision then  $(H, a, b, c)$  contains  $(\mathcal{O}_8, 1, 2, 3)$  as rooted subdivision;
- (ii) If  $(G, a', b, c)$  contains  $(\mathcal{O}_8, 1, 2, 3)$  as rooted subdivision then  $(H, a, b, c)$  contains  $(\mathcal{O}_{13}, 1, 2, 3)$  as rooted minor;
- (iii) If  $(G, a', b, c)$  contains  $(\mathcal{O}_9, 1, 2, 3)$  as rooted subdivision then  $(H, a, b, c)$  contains  $(\mathcal{O}_{12}, 1, 2, 3)$  as rooted minor;
- (iv) If  $(G, a', b, c)$  contains  $(\mathcal{O}_4, 1, 2, 3)$  as rooted subdivision then  $(H, a, b, c)$  contains  $(\mathcal{O}_{14}, 1, 2, 3)$  as rooted minor;
- (v) If  $(G, a', b, c)$  contains  $(\mathcal{O}_3, 1, 2, 3)$ ,  $(\mathcal{O}_6, 1, 2, 3)$  or  $(\mathcal{O}_{10}, 1, 2, 3)$  as rooted subdivision then  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 1, 2, 3)$  as rooted minor.

**Lemma 4.1.2.** *Let  $(G, a, b', c)$  be a graph isomorphic to  $\mathcal{O}_1(1, 2, 3)$ ,  $(\mathcal{O}_2, 1, 2, 3)$ ,  $\mathcal{O}_6(\alpha, \beta, 3)$ , or  $\mathcal{O}_7(1, 2, 3)$  where  $\{\alpha, \beta\} = \{1, 2\}$ . Let  $(G, a, b, c)$  be obtained from  $(G, a, b', c)$  by adding the edge  $bb'$ , then  $(G, a, b, c)$  contains either  $(\mathcal{O}_2, 1, 2, 3)$  or  $\mathcal{O}_5(1, 2, 3)$  as a rooted minor or  $\mathcal{O}_3(1, 2, 3)$ ,  $\mathcal{O}_4(1, 2, 3)$ ,  $\mathcal{O}_8(1, 2, 3)$  as a rooted subdivision where  $a, b, c$  correspond to  $1, 2, 3$ .*

**Lemma 4.1.3.** *Let  $(G, a, b, c)$  be an internally 4-connected rooted graph. Suppose  $(G, a, b, c)$  is not a  $c$ -planar rooted graph, then  $(G, a, b, c)$  contains  $\mathcal{O}_i(\alpha, \beta, 3)$  for some  $i \in \{1, 3, 4, 6, 7, 8, 9, 10\}$  as a rooted subdivision or  $(G, a, b, c)$  contains  $\mathcal{O}_j(\alpha, \beta, 3)$  for some  $j \in \{2, 5, 12, 13, 14, 15, 18, 25\}$ , or  $\mathcal{O}_5(\alpha, 3, \beta)$ ,  $\mathcal{O}_{25}(3, \alpha, \beta)$  as a rooted minor, where  $\{\alpha, \beta\} = \{1, 2\}$ . See Figure 1.3 and 1.4 and 1.5.*

*Proof.* Let the multigraph  $H$  be obtained from  $(G, a, b, c)$  by adding two parallel  $ac$  edges and two parallel  $bc$  edges and the edge  $ab$ , if they do not exist. Note that there is a symmetry between 1 and 2 so in the whole proof we use this symmetry implicitly for reducing case analysis. Because  $(H, a, b, c)$  is not  $c$ -planar and  $H$  is internally 3-connected, Theorem 4.0.21 implies that  $(H, a, b, c)$  contains either,  $(K_{2,3}, \alpha, \beta, 3)$ ,

$(K'_{2,3}, \alpha, \beta, 3)$ ,  $(K'_{2,3}, 1, 3, 2)$  or  $(K''_{2,3}, \alpha, \beta, 3)$  as a rooted subdivision. Note that if  $(H, a, b, c)$  contains  $(K'_{2,3}, 1, 3, 2)$  then it also contains  $(\mathcal{O}_1, \alpha, \beta, 3)$  as a rooted minor, this immediately implies that  $(G, a, b, c)$  contains  $(\mathcal{O}_1, \alpha, \beta, 3)$  as a rooted minor as stated in the statement of lemma.

Now we consider three major cases:

**Case 1**  $(H, a, b, c)$  contains  $(K_{2,3}, 1, 2, 3)$  as a rooted subdivision.

Let  $J$  be the multigraph obtained from  $(K_{2,3}, 1, 2, 3)$  by adding the two parallel 13 edges and two parallel 23 edges. Let  $\mathcal{C} = \{134, 135, 342, 352, 1425, 313, 323\}$  be a double cycle cover for  $J$ . Note that  $J$  is isomorphic to the graph presented in Figure 2.1 where 1, 2, 3 correspond to  $a, b, c$ , respectively. It is easy to see that  $\mathcal{C}$  is a 3-disk system, and  $(H, a, b, c)$  contains a  $(J, 1, 2, 3)$ -subdivision, called  $S$ , as a subgraph. It is easy to see that  $H, J, S, \mathcal{C}$  satisfy conditions of Theorem 3.0.17. So one of the outcomes of Theorem 3.0.17 holds. Obviously, (xi) of Theorem 3.0.17 does not hold. Since  $J$  is not isomorphic to  $K_4$ , so (x) of Theorem 3.0.17 do not hold. By applying Lemma 2.1.3, we can see that (ix) of Theorem 3.0.17 does not hold. Moreover, since there is no special segment in  $S$ , (ii), (iii), (vi) and (vii) of Theorem 3.0.17 do not hold. Now, we are going to analyze the other possible outcomes, i.e. (i), (iv), (v) and (viii) of Theorem 3.0.17.

**Case 1.1** The outcome (i) of Theorem 3.0.17 holds, i.e. there exists an  $S$ -jump.

By symmetry between 4, 5, we may assume that there exists a path  $P$  from  $w \in \text{seg}(3, 4)$  to  $z \in \text{seg}(1, 5] \cup \text{seg}(2, 5] \cup \text{seg}(3, 5)$ . If  $z = 5$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_1, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (4, 5, w)$ . If  $z \in \text{seg}(1, 5)$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_1, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (4, z, w)$ . If  $z \in \text{seg}(3, 5)$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_1, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (4, 5, z)$ .

**Case 1.2** The outcome (iv) of Theorem 3.0.17 holds, i.e. there exists weakly free

$S$ -cross anchored at  $c$ .

Let  $(P_1, P_2)$  be an  $S$ -cross with feet on  $u_1, 6, 7, v_2$  where  $u_1, 7 \in \text{seg}[2, 5]$  and  $v_2, 6 \in \text{seg}[2, 4]$ . We claim that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 1, 3, 2)$  as a rooted minor. For proving the claim it is enough to show that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 1, 3, 2)$  as a rooted minor, when  $u_1 = 5, v_2 = 4$ . In which case  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 1, 3, 2)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (6, 7, 4, 5)$  as a rooted minor.

**Case 1.3** The outcome (v) of Theorem 3.0.17 holds, i.e. there exists a free solid  $S$ -cross.

We are going to show that either  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$  a rooted minor, or it contains  $(\mathcal{O}_1, 1, 2, 3)$ ,  $(\mathcal{O}_6, 1, 2, 3)$  or  $(\mathcal{O}_7, 1, 2, 3)$  as a rooted subdivision. Let  $(P_1, P_2)$  be the free cross with feet  $u, w, v, z$ .

Note that the feet of the cross are on the disk 1425. We may assume without loss of generality  $u \in \text{seg}[2, 4] \cup \text{seg}[2, 5), v \in \text{seg}[1, 4] \cup \text{seg}[1, 5]$  and  $w \in \text{seg}(1, 5] \cup \text{seg}(2, 5], z \in \text{seg}[1, 4] \cup \text{seg}[2, 4]$ . Now if  $u = 2, v = 1, w = 4, z = 5$  and  $P_1$  is an edge then  $(H, a, b, c)$  contains  $(\mathcal{O}_7, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5) \leftrightarrow (4, 5)$ . If  $P_1$  is not an edge, there exists a path  $P$  from  $x \in \text{Int}(P_1)$  to  $y \in S$ . If  $y$  does not belong to the disk 1425, then  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, x)$ . If  $y$  belongs to the disk 1425 then there exists a path  $P' \subset P_1 \cup P$  such that  $(P', P_2)$  forms a free cross, unless  $y = 4$  in which case it is easy to see that  $(H, a, b, c)$  contains  $(\mathcal{O}_1, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (x, w, 4)$ . Note that the rest of case analysis will investigate possible outcomes of the free cross  $(P', P_2)$

Now by symmetry we consider two main possibilities: either  $u \neq 2$  or  $w \neq 5$ . In the first possibility, assume that  $u \neq 2$ . Without loss of generality, we may assume that  $u \in \text{seg}(2, 4)$ . Now if  $z \in \text{seg}(2, 4]$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_1, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (u, w, z)$ , and if  $z \in \text{seg}[1, 4]$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_1, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (u, w, 4)$ .

In the second possibility, assume that  $w \neq 5$  and  $u = 2$ . Now either  $v = 1$  or  $v \neq 1$ . We start by analyzing the case  $v \neq 1$ . If  $v \in \text{seg}(1, 5)$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_1, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (v, w, 5)$ . If  $v \in \text{seg}(1, 4)$  and  $z \in (2, 4]$ ,  $(H, a, b, c)$  contains  $(\mathcal{O}_1, 1, 2, 3)$  as a rooted subdivision with  $(4, 5, 6) \leftrightarrow (v, w, 4)$ . Finally, if  $v \in \text{seg}(1, 4)$  and  $z \in (1, 4)$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_1, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (v, w, z)$ .

Thus we can assume  $v = 1$ . If  $P_1$  is not a path then similarly as the argument presented above either  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, x)$  or it contains  $(\mathcal{O}_1, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (x, w, 4)$ , or we get a free cross  $(P', P_2)$  where the possible outcomes were already investigated above. Therefore  $P_1$  is an edge. If  $z \in \text{seg}[4, 2)$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_6, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (4, 5, w)$ . Finally, if  $z \in \text{seg}(4, 1)$  then similarly as before  $(H, a, b, c)$  contains  $(\mathcal{O}_6, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (z, 5, w)$ .

**Case 1.4** The outcome (viii) of Theorem 3.0.17 holds, i.e. there exists an essential  $S$ -triad

Let  $(P_1, P_2, P_3)$  be an essential triad with center 0 and feet on  $v_1, v_2, 3$ , respectively. Note that  $v_1, v_2 \in \text{seg}(2, 4) \cup \text{seg}(1, 4) \cup \text{seg}(2, 5) \cup \text{seg}(1, 5)$  and there is a symmetry between  $1, 4, v_1$  and  $2, 5, v_2$ , respectively. By considering these symmetries, we analyze the following cases: in the first possibility,  $v_1 \in \text{seg}[2, 4), v_2 \in \text{seg}[1, 5)$  in which case  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, 0)$ . In the second possibility,  $v_1 \in \text{seg}(2, 4], v_2 \in \text{seg}(2, 5]$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_1, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (v_1, v_2, 0)$ .

**Case 2**  $(H, a, b, c)$  contains  $(K'_{2,3}, 1, 2, 3)$  as a rooted subdivision.

Let  $J$  be the multigraph obtained from  $(K'_{2,3}, 1, 2, 3)$  by adding the two parallel 13 edges and two parallel 23 edges. Let  $\mathcal{C} = \{134, 135, 3462, 3562, 1465, 313, 323\}$  be a double cycle.

It is easy to see that  $\mathcal{C}$  is a 3-disk system, and  $(H, a, b, c)$  contains a  $(J, 1, 2, 3)$ -subdivision, called  $S$ , as a subgraph. It is easy to see that  $H, J, S, \mathcal{C}$  satisfy conditions of Theorem 3.0.17. So one of the outcomes of Theorem 3.0.17 holds. Obviously, (xi) of Theorem 3.0.17 does not hold. Since  $J$  is not isomorphic to  $K_4$ , so (x) of Theorem 3.0.17 does not hold. By applying Lemma 2.1.3, we can see that (ix) of Theorem 3.0.17 does not hold. Now, we are going to analyze the other outcomes, i.e., (i), (iv), (v), (vi), (vii) and (viii) of Theorem 3.0.17.

**Case 2.1** The outcome (i) of Theorem 3.0.17 holds, i.e. there exists an  $S$ -jump.

Let  $P$  be an  $S$ -jump with ends  $w, z$ . If  $w \in \text{seg}(3, 4) \cup \text{seg}(3, 5)$  and  $z \in \text{seg}(1, 5] \cup \text{seg}(1, 4] \cup \text{seg}[4, 6) \cup \text{seg}[5, 6)$  then by the analysis presented in Case 1.1 and Lemma 4.1.1(i),  $(H, a, b, c)$  contains  $(\mathcal{O}_8, 1, 2, 3)$  as a rooted subdivision. So we may assume that  $w \in \text{seg}(2, 6)$  and  $z \in \text{seg}[1, 4) \cup \text{seg}[1, 5)$ . We consider two major possibilities. First, by symmetry assume that  $z \in \text{seg}(1, 5)$ , in which if  $w \in \text{seg}(2, 6)$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_8, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6, 7) \leftrightarrow (z, 6, 5, w)$ , and if  $w = 2$  then it contains  $(\mathcal{O}_{10}, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6, 7) \leftrightarrow (z, 4, 6, 5)$ .

In the second possibility, assume that  $z = 1$ . If  $w \in \text{seg}(2, 6)$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_3, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, 6, w)$ . Thus, assume that  $w = 2$ . If  $P$  is an edge then  $(H, a, b, c)$  contains  $(\mathcal{O}_6, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (4, 5, 6)$ . If  $P$  is not an edge then there exists a path  $P'$  from  $x \in \text{Int}(P)$  to  $y \in S$ . If  $y \in \text{seg}[3, 4) \cup \text{seg}[3, 5)$  then we get  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, x)$ . If  $y \in \text{seg}[5, 6) \cup \text{seg}[4, 6)$ , by symmetry say  $y \in \text{seg}[4, 6)$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_{10}, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6, 7) \leftrightarrow (x, 5, y, 6)$ . Finally, if  $y = 6$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_3, 1, 2, 3)$  as rooted subdivision with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, 6, x)$ .

Thus to summarize,  $(H, a, b, c)$  contains  $(\mathcal{O}_i, 1, 2, 3)$  for some  $i \in \{3, 6, 8, 10\}$  as a rooted subdivision, or it contains  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted minor.

**Case 2.2** The outcome (iv) of Theorem 3.0.17 holds, i.e. there exists weakly free  $S$ -cross anchored at  $c$ .

This case is similar to Case 1.2, so  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 1, 3, 2)$  as a rooted minor.

**Case 2.3** The outcome (ii) of Theorem 3.0.17 holds, i.e. there exists a degenerate  $S$ -cross.

Let  $(P_1, P_2, P)$  be the degenerate  $S$ -cross where  $3, u_2, v_1, v_2$  are feet of the  $S$ -cross  $(P_1, P_2)$ ,  $2, v_2, v_1, 6$  appear on  $\text{seg}[2, 6]$  in the order listed, and  $u_2 \in \text{seg}(3, 4] \cup \text{seg}[4, 6)$ . Assume  $7 \in V(\text{Int}(P_1))$  be the other end of the path  $P$ . If  $v_1 \in \text{seg}(2, 6)$  then we claim that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 1, 2, 3)$  as a rooted minor. For proving the claim it is enough to show that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 1, 2, 3)$  as a rooted minor, when  $u_2 = 4, v_2 = 2$ . In which case it is easy to see that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (6, 7, v_1, 4)$ .

So we assume that  $v_1 = 6$ . Now either  $v_2 = 2$  or  $v_2 \in \text{seg}(2, 6)$ . Then it is easy to see that  $(H, a, b, c)$  contains either  $(\mathcal{O}_{10}, 1, 2, 3)$  or  $(\mathcal{O}_8, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6, 7) \leftrightarrow (u_2, 5, 6, 7)$  or  $(4, 5, 6, 7) \leftrightarrow (u_2, 6, v_2, 7)$ , respectively.

Thus to summarize,  $(H, a, b, c)$  contains either  $(\mathcal{O}_8, 1, 2, 3)$ ,  $(\mathcal{O}_{10}, 1, 2, 3)$  as a rooted subdivision, or it contains  $(\mathcal{O}_5, 1, 2, 3)$  as a rooted minor.

**Case 2.4** The outcome (iii) of Theorem 3.0.17 holds, i.e. there exists a 3-blocking  $S$ -cross.

Let  $(P_1, P_2, P_3)$  be a connected cross with feet on  $3, x, y, z$  and connection  $7, 8$  where  $7 \in \text{Int}(P_1)$  and  $8 \in \text{Int}(P_2)$ . Without loss of generality we may assume,  $x \in \text{seg}[2, 6)$ ,  $y \in \text{seg}(2, 6]$  and  $z \in \text{seg}(3, 5] \cup \text{seg}[5, 6)$ . We claim that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 1, 3, 2)$  as a rooted minor. For proving the claim it is enough to show that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 1, 3, 2)$  as a rooted minor, when  $x = 2$ . In which case it is easy to see that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 1, 3, 2)$  as a rooted minor with signature



$(4, 5, 6, 7) \leftrightarrow (y, 8, 7, z)$ .

**Case 2.5** The outcome (v) of Theorem 3.0.17 holds, i.e. there exists a free solid  $S'$ -cross.

Let  $(P_1, P_2)$  be a free cross with feet  $u, w, v, z$ . By considering the symmetry, we consider two possibilities: either the feet of the free cross are on the disk 1465, or they are on the disk 3562.

In the first possibility, it is easy to see that by applying Lemma 4.1.2 and the result of Case 1.3 above,  $(H, a, b, c)$  contains either  $(\mathcal{O}_2, 1, 2, 3)$  or  $(\mathcal{O}_5, 1, 2, 3)$  as a rooted minor or  $(\mathcal{O}_3, 1, 2, 3)$  or  $(\mathcal{O}_4, 1, 2, 3)$  as a rooted subdivision.

Now we investigate the second possibility, i.e.  $u, w, v, z$  belong to the disk 3562. Since the cross is solid and free, we may assume that  $u \in \text{seg}(3, 5)$ ,  $v \in \text{seg}(5, 6] \cup \text{seg}(2, 6]$ ,  $w \in \text{seg}(3, 5] \cup \text{seg}[5, 6)$  and  $z \in \text{seg}[2, 6)$ .

If  $v \in \text{seg}(2, 6)$  then since we are going to show that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 1, 2, 3)$  as a rooted minor, we can assume that  $w = 5$  and  $z = 2$ . In which case it is easy to see that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (6, u, v, 5)$ . If  $w \in \text{seg}(3, 5)$  then since we are going to show that  $(H, a, b, c)$  contains  $(\mathcal{O}_{18}, 1, 2, 3)$  as a rooted minor, we can assume that  $v = 6$  and  $z = 2$ . In which case it is easy to see that  $(H, a, b, c)$  contains  $(\mathcal{O}_{18}, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6, 7, 8) \leftrightarrow (6, w, 4, 5, u)$ .

Thus, we assume that  $v \in \text{seg}(5, 6]$  and  $w \in \text{seg}[5, 6)$ . First assume that  $v = 6$ , in which case if  $w \in \text{seg}(5, 6)$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_8, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6, 7) \leftrightarrow (5, 6, u, w)$ , and if  $w = 5, z = 2$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_{10}, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, u, 6)$ , and finally, if  $w = 5, z \in \text{seg}(2, 6)$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_8, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6, 7) \leftrightarrow (5, 6, u, z)$

Now assume that  $v \in \text{seg}(5, 6)$ , in which case if  $z \in \text{seg}(2, 6)$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_8, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6, 7) \leftrightarrow (5, 6, u, z)$ ,

and if  $z = 2, w = 5$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_{10}, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, u, 6)$ , and finally, if  $z = 2, w \in \text{seg}(5, 6)$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_8, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6, 7) \leftrightarrow (5, v, u, w)$ .

Thus to summarize the whole case analysis of Case 2.5,  $(H, a, b, c)$  contains  $(\mathcal{O}_i, 1, 2, 3)$  for some  $i \in \{3, 4, 8, 10\}$  as a rooted subdivision, or it contains  $(\mathcal{O}_i, 1, 2, 3)$  for some  $i \in \{2, 5, 18\}$  as a rooted minor.

**Case 2.6** The outcome (vi) of Theorem 3.0.17 holds, i.e. there exists a double facial  $S'$ -cross.

Let  $(P, P_1, P_2)$  be the double facial  $S$ -cross where  $P$  has an end at 3 and the other at  $u \in \text{seg}[2, 6]$ ,  $P_1$  has one end at  $u_1 \in \text{seg}(3, 5] \cup \text{seg}[5, 6)$  and the other at  $v_1 \in \text{seg}[2, 6)$ ,  $P_2$  has one end at  $u_2 \in \text{seg}[4, 3) \cup \text{seg}[4, 6)$  and the other at  $v_2 \in \text{seg}[2, 6)$ . If  $u \in (2, 6)$  then we claim that  $(H, a, b, c)$  contains  $(\mathcal{O}_{15}, 1, 2, 3)$  as a rooted minor. For proving the claim it is enough to show that  $(H, a, b, c)$  contains  $(\mathcal{O}_{15}, 1, 2, 3)$  as a rooted minor, when  $v_1 = v_2 = 2, u_1 = 5, u_2 = 4$ . In which case it is easy to see that  $(H, a, b, c)$  contains  $(\mathcal{O}_{15}, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (5, u, 6, 4)$ .

So we may assume that  $u = 6$ . Now we claim that if  $u_1 \in \text{seg}(3, 5)$  or  $u_2 \in \text{seg}(3, 4)$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted minor. For proving the claim, by symmetry assume that  $u_1 \in \text{seg}(3, 5)$ . Now it is enough to show that  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted minor, when  $v_1 = v_2 = 2, u_2 = 4$ . In which case it is easy to see that  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 6, u_1)$  as a rooted minor.

Thus we assume that  $u_1 \in \text{seg}[5, 6]$  and  $u_2 \in \text{seg}[4, 6]$ .

Now if  $v_1, v_2 \in \text{seg}(2, 6)$  such that  $2, v_1, v_2, 6$  appear on  $\text{seg}[2, 6]$  in the order listed then  $(H, a, b, c)$  contains  $(\mathcal{O}_8, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6, 7) \leftrightarrow (u_2, u_1, 6, v_1)$ . So we may assume by symmetry that  $v_1 = 2$  and  $v_2 \in \text{seg}[2, 6]$ .

Now if  $v_2 \in \text{seg}(2, 6)$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_{10}, 1, 2, 3)$  as a rooted subdivision

with signature  $(4, 5, 6, 7) \leftrightarrow (v_2, u_2, u_1, 6)$ . Thus we can assume  $v_1 = v_2 = 2$ . In which case if  $u_2 = 4$  and  $u_1 = 5$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_9, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (4, 5, 6)$ . So by symmetry assume that  $u_1 \in \text{seg}(5, 6)$ . It is easy to see that  $(H, 1, 2, 3)$  contains  $(\mathcal{O}_{10}, a, b, c)$  as a rooted subdivision with signature  $(4, 5, 6, 7) \leftrightarrow (u_2, 5, u_1, 6)$ .

Thus to summarize,  $(H, 1, 2, 3)$  contains  $(\mathcal{O}_i, 1, 2, 3)$  for some  $i \in \{8, 9, 10\}$  as a rooted subdivision, or it contains  $(\mathcal{O}_2, 1, 2, 3)$  or  $(\mathcal{O}_{15}, 1, 2, 3)$  as a rooted minor.

**Case 2.7** The outcome (vii) of Theorem 3.0.17 holds, i.e. there exists a blocking interlaced  $S$ -fork of type I.

Without loss of generality we may assume that there exist an  $S$ -fork  $(P_1, P_2, P_3)$  with feet on  $3, u, v$  and center  $o$  where  $2, u, v, 6$  appear on  $\text{seg}[2, 6]$  in the order listed, an  $S$ -path  $P$  disjoint from  $P_1, P_2, P_3$  except at  $3$  connecting  $3$  to  $w$  where  $u, w, v$  appear on  $\text{seg}[2, 6]$  in the order listed, and an  $S$ -path  $R$  disjoint from  $P, P_1, P_2, P_3$  except maybe at  $u$  between  $v' \in \text{seg}(3, 5] \cup \text{seg}[5, 6)$  to  $u'$  where  $u', u, 2$  appear on  $\text{seg}(2, 6)$  in the order listed. We claim that  $(H, a, b, c)$  contains  $(\mathcal{O}_{15}, 1, 2, 3)$  as a rooted minor. For proving the claim it is enough to show that  $(H, a, b, c)$  contains  $(\mathcal{O}_{15}, 1, 2, 3)$  as a rooted minor, when  $u = 2, v = 6, v' = 5$  and  $u' = 2$ . In which case it is easy to see that  $(H, a, b, c)$  contains  $(\mathcal{O}_{15}, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (o, w, 6, 5)$  as a rooted minor.

**Case 2.8** The outcome (viii) of Theorem 3.0.17 holds, i.e. there exists an essential  $S'$ -triad with one feet on  $c$ .

Note that this case is similar to case 1.4 except the fact that  $K_{2,3}$  is replaced by  $K'_{2,3}$ . This easily implies that  $(H, a, b, c)$  either contains  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted minor or  $(\mathcal{O}_8, 1, 2, 3)$  as a rooted subdivision.

**Case 3**  $(H, a, b, c)$  contains  $(K''_{2,3}, 1, 2, 3)$  as a rooted subdivision.

Let  $J$  be the multigraph obtained from  $(K''_{2,3}, 1, 2, 3)$  by adding the two parallel

13 edges and two parallel 23 edges. Let  $\mathcal{C} = \{1346, 1356, 3472, 3572, 4567, 313, 323\}$  be a double cycle cover for  $J$ . It is easy to see that  $\mathcal{C}$  is a 3-disk system, and  $(H, a, b, c)$  contains a  $(J, 1, 2, 3)$ -subdivision, called  $S$ , as a subgraph. It is easy to see that  $H, J, S, \mathcal{C}$  satisfy conditions of Theorem 3.0.17. Similarly to Case 2, the only possible outcomes are (i), (v), (vi), (vii) and (viii) of Theorem 3.0.17.

**Case 3.1** The outcome (i) of Theorem 3.0.17 holds, i.e. there exists an  $S$ -jump.

Let  $P$  be an  $S$ -jump with ends  $w, z$ . If  $w \in \text{seg}(3, 4) \cup \text{seg}(3, 5)$  and  $z \in \text{seg}(1, 5] \cup \text{seg}(1, 4] \cup \text{seg}[4, 6) \cup \text{seg}[5, 6)$  or if  $w \in \text{seg}[4, 6] \cup \text{seg}[6, 5]$  and  $z \in \text{seg}[2, 7) \cup \text{seg}[1, 6)$  then as the summary of Case 3.1 and Lemma 4.1.1 imply  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 1, 2, 3)$ ,  $(\mathcal{O}_{13}, 1, 2, 3)$  as a rooted minor. So by symmetry, we may assume that  $w \in \text{seg}[1, 6]$  and  $z \in \text{seg}[2, 7] \cup \text{seg}[1, 5)$ . We consider two major possibilities. First, by symmetry assume that  $z \in \text{seg}(2, 7)$ , then it is easy to see that by contracting the edge  $7z$ , and  $1w$ , if  $w \neq 1$  the rooted graph  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 1, 3)$  with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, 6, 7)$  as a rooted minor.

In the second possibility, assume  $w = 1, z = 2$ . If  $P$  is an edge then  $(H, a, b, c)$  contains  $(\mathcal{O}_{25}, 3, 2, 1)$  with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, 6, 7)$  as a rooted minor. If  $P$  is not an edge then the fact that  $(H, a, b, c)$  is internally 4-connected implies that there exists a path  $P'$  from  $x \in \text{Int}(P)$  to  $y \in S$ . If  $y \in \text{seg}[3, 4) \cup \text{seg}[3, 5)$  then  $(H, a, b, c)$  contains  $\mathcal{O}_2$  with signature  $(4, 5, 6) \leftrightarrow (4, 5, x)$  as a rooted minor. If  $y \in \text{seg}[4, 6) \cup \text{seg}[5, 6) \cup \text{seg}[4, 7] \cup \text{seg}[5, 7]$ , by symmetry say  $y \in \text{seg}[4, 7]$ . Since we are going to show that  $(H, a, b, c)$  contains either  $(\mathcal{O}_5, 2, 1, 3)$  or  $(\mathcal{O}_{18}, 1, 2, 3)$  as a rooted minor, we may assume either  $y = 4$  or  $y = 7$ . If  $y = 4$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_{18}, 1, 2, 3)$  with signature  $(4, 5, 6, 7, 8) \leftrightarrow (4, 5, x, 6, z)$  as a rooted minor. If  $y = 7$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 1, 3)$  with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, 6, 7)$  as a rooted minor.

**Case 3.2** The outcome (ii) of Theorem 3.0.17 holds, i.e. there exists a degenerate  $S$ -cross.

As the summary of Case 2.3 and Lemma 4.1.1 imply  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 1, 2, 3)$  or  $(\mathcal{O}_{13}, 1, 2, 3)$  as a rooted subdivision.

**Case 3.3** The outcome (iii) of Theorem 3.0.17 holds, i.e. there exists a 3-blocking  $S$ -cross.

As the analysis of Case 2.4 shows  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 1, 3, 2)$  as a rooted minor.

**Case 3.4** The outcome (v) of Theorem 3.0.17 holds, i.e. there exists a free solid  $S'$ -cross.

As the summary of Case 2.5 and Lemma 4.1.1 imply  $(H, a, b, c)$  contains  $(\mathcal{O}_i, 1, 2, 3)$  for some  $i \in \{2, 5, 13, 14, 18\}$  as a rooted minor.

**Case 3.5** The outcome (vi) of Theorem 3.0.17 holds, i.e. there exists a double facial  $S'$ -cross.

As the summary of Case 2.6 and Lemma 4.1.1 imply  $(H, a, b, c)$  contains  $(\mathcal{O}_i, 1, 2, 3)$  for some  $i \in \{2, 5, 12, 13, 15\}$  as a rooted minor.

**Case 3.6** The outcome (vii) of Theorem 3.0.17 holds, i.e. there exists a blocking interlaced  $S$ -fork of type I.

As the analysis of Case 2.7 shows  $(H, a, b, c)$  contains  $(\mathcal{O}_{15}, 1, 2, 3)$  as a rooted minor.

**Case 3.7** The outcome (viii) of Theorem 3.0.17 holds, i.e. there exists an essential  $S'$ -triad with one feet on  $c$ .

As the summary of Case 2.8 and Lemma 4.1.1 imply  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$  or  $(\mathcal{O}_{13}, 1, 2, 3)$  as a rooted minor.

□

**Theorem 4.1.4.** (i) *The rooted graphs  $\mathcal{O}_i(1, 2, 3)$ ,  $1 \leq i \leq 7$  are minor minimal with respect to the non-3-planarity property.*

(ii) *If  $(G, a, b, c)$  is an internally 4-connected not  $c$ -planar rooted graph then  $(G, a, b, c)$  contains  $\mathcal{O}_i(1, 2, 3)$  for some  $i \in \{1, 3, 4, 6, 7\}$  as a rooted subdivision or  $(G, a, b, c)$  contains  $\mathcal{O}_2(1, 2, 3)$ ,  $\mathcal{O}_5(1, 2, 3)$  as a rooted minor. See Figure 1.3.*

*Proof.* For proving (i), note that by the proof of Lemma 4.1.3 and the statement of Lemma 4.0.23, the rooted graphs  $O_i(1, 2, 3)$ ,  $1 \leq i \leq 7$  are not  $c$ -planar. We leave the proof the fact that  $O_i(1, 2, 3)$ ,  $1 \leq i \leq 7$  are minor minimal with respect to not  $c$ -planarity to the reader.

For proving (ii), first we claim that if any of the graphs  $(\mathcal{O}_8, 1, 2, 3)$ ,  $(\mathcal{O}_9, 1, 2, 3)$ ,  $(\mathcal{O}_{10}, 1, 2, 3)$ ,  $(\mathcal{O}_{12}, 1, 2, 3)$ ,  $(\mathcal{O}_{13}, 1, 2, 3)$ ,  $(\mathcal{O}_{14}, 1, 2, 3)$ ,  $(\mathcal{O}_{15}, 1, 2, 3)$ ,  $(\mathcal{O}_{18}, 1, 2, 3)$  or  $(\mathcal{O}_{25}, 1, 2, 3)$  is not  $z$ -planar where  $\{x, y, z\} \in \{1, 2, 3\}$ , then they contain either  $O_i(1, 2, 3)$  for some  $i \in \{1, 3, 4, 6\}$ .

The rooted graph  $(\mathcal{O}_8, 1, 2, 3)$  is not 23-planar and it contains  $(\mathcal{O}_1, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (4, 5, 6)$ . The rooted graph  $(\mathcal{O}_9, 1, 2, 3)$  is not 23-planar and it contains  $(\mathcal{O}_1, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (4, 5, 6)$ . The rooted graph  $(\mathcal{O}_{10}, 1, 2, 3)$  is not 23-planar and it contains  $(\mathcal{O}_1, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (4, 6, 7)$ . The rooted graph  $(\mathcal{O}_{12}, 1, 2, 3)$  is not 123-planar and it contains  $(\mathcal{O}_1, 3, 2, 1)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (4, 5, 6)$  and  $(\mathcal{O}_4, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (4, 5, 6)$ . The rooted graph  $(\mathcal{O}_{13}, 1, 2, 3)$  is not 123-planar and it contains  $(\mathcal{O}_1, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (4, 5, 6)$ . The rooted graph  $(\mathcal{O}_{15}, 1, 2, 3)$  is not 123-planar and it contains  $(\mathcal{O}_1, 3, 2, 1)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (4, 5, 6)$  and  $(\mathcal{O}_3, 2, 1, 3)$  as a rooted subdivision with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, 6, 7)$ . The rooted graph  $(\mathcal{O}_{18}, 1, 2, 3)$  is not 123-planar and it contains  $(\mathcal{O}_1, 3, 2, 1)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (6, 7, 4)$  and  $(\mathcal{O}_1, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (4, 5, 7)$ . The rooted graph  $(\mathcal{O}_{25}, 1, 2, 3)$  is not 123-planar and it contains  $(\mathcal{O}_1, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (4, 5, 6)$  and  $(\mathcal{O}_6, 3, 2, 1)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, 6)$ . This completes the proof of the claim. Now the proof of (ii) follows from Lemma 4.1.3 and the above claim.  $\square$

**Lemma 4.1.5.** *Let  $(G, a, b, c)$  be a 3-connected rooted graph. Let  $G = (G_1, G_2)$  be*

an internal 3-separation such that  $V(G_1) \cap V(G_2) = \{a', b', c\}$ ,  $\{a, b, c\} \subseteq V(G_2)$  and  $E(G_1) = E(G[V(G_1)])$ . Moreover assume that  $G_1$  is isomorphic to  $K_{2,3}$ . If  $(G, a, b, c)$  is not  $c$ -planar then  $(G, a, b, c)$  contains either  $(\mathcal{O}_3, 1, 2, 3)$  or  $(\mathcal{O}_6, 1, 2, 3)$  as a rooted subdivision, see Figure 1.3, or it contains  $(\mathcal{O}_i, \alpha, \beta, 3)$  for some  $i \in \{2, 5, 12, 15\}$ , or  $(\mathcal{O}_5, 3, \alpha, \beta)$ ,  $(\mathcal{O}_5, \alpha, 3, \beta)$  or  $(\mathcal{O}_{25}, 3, \alpha, \beta)$  as a rooted minor, where  $\alpha, \beta = \{1, 2\}$ . See Figure 1.5.

*Proof.* The 3-connectivity of  $(G, a, b, c)$  implies that there exist two disjoint paths  $P_a, P_b$  in  $H$  connecting  $a, b$  to  $a', b'$ , respectively. It is trivial that  $\{a', b', c\}$  is a vertex cut in  $H$ . By symmetry between  $a$  and  $b$ , we may assume that  $b \neq b'$ . Now we consider two major cases,  $a = a'$  and  $a \neq a'$ .

Let the multigraph  $H$  be obtained from  $(G, a, b, c)$  by adding the two parallel  $ac$  edges and two parallel  $bc$  edges, if they do not exist.

**Case 1**  $a = a'$ .

Let  $J$  be the multigraph obtained from  $K'_{2,3}$  by adding the two parallel 13 edges and two parallel 23 edges where 1, 2, 3 are terminal, 4, 5 are centers and 6 is the only neighbor of 2. Let  $\mathcal{C} = \{135, 134, 3562, 1465, 3462, 131, 232\}$  be a double cycle cover for  $J$ . It is easy to see that  $\mathcal{C}$  is a 3-disk system, and  $H$  contains a  $J$ -subdivision, called  $S$ , as a subgraph where  $a, b, c$  correspond to 1, 2, 3, respectively. Note that  $H, J, S, \mathcal{C}$  satisfy the hypothesis of Theorem 2.2.10, so one of the outcomes listed in the statement of Theorem 2.2.10 holds. Note that since  $\{1, 3, 6\}$  is a vertex cut in  $H$  and  $E(G_1) = E(G[V(G_1)])$ , the outcomes (ii), (iv), (v), (vi), (vii) or (x) of Theorem 2.2.10 do not hold. By applying Lemma 2.1.3, we can see that (xi) of Theorem 2.2.10 does not hold. Since  $J$  is not isomorphic to  $K_4$ , (xii) of Theorem 2.2.10 does not hold. Finally since  $G$  is not 3-planar, (xiii) of Theorem 2.2.10 does not hold. So either (i), (iii), (viii) or (ix) holds.

If (i) of Theorem 2.2.10 holds, then  $H$  contains a path  $P$  from 1 to  $z \in \text{seg}[2, 6]$ .

If  $z \in \text{seg}(2, 6]$  then  $H$  contains  $\mathcal{O}_3(1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, 6, 7)$ . If  $z = 2$  in which case either  $P$  is an edge or  $V(\text{Int}(P)) \neq \emptyset$ . If  $P$  is an edge then  $H$  contains  $\mathcal{O}_6(1, 2, 3)$  with signature  $(4, 5, 6) \leftrightarrow (4, 5, 6)$  as a rooted minor, and if  $V(\text{Int}(P)) \neq \emptyset$  then the fact that  $H$  is 3-connected and  $\{1, 3, 6\}$  is a vertex cut in  $G_1$  implies that there exists a path  $R$  from  $u \in V(\text{Int}(P))$  to  $v \in (P_2 \cup \{3\}) \setminus \{1, 2\}$ . So if  $v = 3$  then it is easy to see that  $H$  contains  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, u)$ . Finally, if  $v \in V(P_2) \setminus \{2\}$  then similarly as before  $H$  contains  $(\mathcal{O}_3, 1, 2, 3)$  as a rooted subdivision.

If (iii) of Theorem 2.2.10 holds then there exist  $t_1, w, t_2 \in V(P_b)$ , where  $2, t_1, w, t_2, 6$  appear on  $P_b$  in the order listed, such that there exists a 3-blocking cross  $(P_1, P_2, P_3)$  with feet on  $3, t_1, w, t_2$  and connections  $7, 8$  where  $7 \in P_1$  and  $8 \in P_2$ . We assume that  $t_1 = 2$  and  $t_2 = 6$  since in the rest of the proof, we show that  $H$  contains some rooted graph as a rooted minor. Now, it is easy to see that by contracting the edge  $56$ ,  $H$  contains  $(\mathcal{O}_5, 1, 3, 2)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (8, w, 7, 6)$ .

If (viii) of Theorem 2.2.10 holds then there exist  $t_1, w, t_2 \in V(P_b)$  where  $2, t_1, w, t_2, 6$  appear on  $P_b$  in the order listed, such that there exist a double fork with centers  $7$  and  $8$  and feet  $3, t_1, t_2$  and an  $S$ -path from  $3$  to  $w$ . It is not hard to see that  $H$  contains  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (3, 7, 8)$ .

If (ix) of Theorem 2.2.10 holds then there exist  $t_1, t_2 \in V(P_b)$  such that there exists a double connected fork  $(P_1, P_2, P_3; Q_1, Q_2, Q_3)$  with centers  $7$  and  $8$  and feet  $3, t_1, t_2$  and connections  $w_1, w_2$ . We may assume  $2, t_1, t_2, 6$  appear on  $P_b$  in the order listed. We assume that  $t_1 = 2$  and  $t_2 = 6$  since in the rest of the proof we show that  $H$  contains some rooted graph as a rooted minor. Since the double fork is connected, we consider two cases either  $w_1 = w_2$ , or  $w_1 \neq w_2$ . In the first case, note that  $w_1 \notin \{7, 8\}$ . If  $w_1 \in P_1 \cap Q_1$  then it is easy to see that  $H$  contains  $(\mathcal{O}_5, 1, 3, 2)$  as a rooted minor by contracting the edge  $46$  with signatures  $(4, 5, 6, 7) \leftrightarrow (7, 8, w_1, 4)$ . If  $w_1 \in P_2 \cap Q_2$  then  $H$  contains  $(\mathcal{O}_{15}, 2, 1, 3)$  as a rooted minor by contracting the edge  $46$  with signature



$(4, 5, 6, 7) \leftrightarrow (7, 8, w_1, 4)$ . If  $w_1 \in P_3 \cap Q_3$  then  $H$  contains  $(\mathcal{O}_{15}, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6, 7) \leftrightarrow (7, 8, w_1, 6)$ . In the second case, i.e.  $w_1 \neq w_2$ , if  $w_1 \neq 7$  then contract the edge  $7w_1$  and label the new vertex by 7 and also if  $w_2 \neq 8$  then contract the edge  $8w_2$  and relabel the new vertex by 8. Now by contracting the edge 46, it is not hard to see that  $H$  contains  $\mathcal{O}_{12}(1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (7, 8, 4)$ .

**Case 2**  $a \neq a'$

Let  $J$  be the multigraph obtained from  $K''_{2,3}$  by adding the two parallel 13 edges and two parallel 23 edges where 1, 2, 3 are terminal, 4, 5 are centers, 6 is the only neighbor of 1 and 7 is the only neighbor of 2. Let  $\mathcal{C} = \{1643, 1653, 4657, 3472, 3572, 131, 232\}$  be a double cycle cover for  $J$ . It is easy to see that  $\mathcal{C}$  is a 3-disk system, and  $H$  contains a  $J$ -subdivision, called  $S$ , as a subgraph. Since we are proving (i), we can assume  $1' = 6$  and  $2' = 7$ . Note that  $H, J, S, \mathcal{C}$  satisfy the hypothesis of Theorem 2.2.10, so one of the outcomes listed in the statement of Theorem 2.2.10 holds. Note that similarly as the case where  $1 = 1'$ , since  $\{1, 3, 6\}$  is a vertex in  $H$  and  $E(G'_1) = E(G_1[V(G'_1)])$ , outcomes (ii), (iv), (v), (vi), (vii), (xii), (xiii) or (x) of Theorem 2.2.10 do not hold. So either (i), (iii), (viii), (ix) holds.

If (i) of Theorem 2.2.10 holds then by symmetry between 1 and 2, we may assume there is a path  $P$  from  $w \in \text{seg}[1, 6]$  to  $z \in \text{seg}[2, 7]$ . We assume that  $w = 1$  since in the rest of the proof we show that  $H$  contains some graph as a rooted minor. If  $z \in \text{seg}(2, 7]$  then by contracting the edge  $z7$ , if it exists,  $H$  contains  $(\mathcal{O}_5, 2, 1, 3)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, 6, 7)$ . If  $z = 2$  then either  $P$  is an edge or  $V(\text{Int}(P)) \neq \emptyset$ . If  $P$  is an edge then  $H$  contains  $(\mathcal{O}_{25}, 3, 2, 1)$  with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, 6, 7)$  as a rooted minor, and if  $V(\text{Int}(P)) \neq \emptyset$  then the fact that  $H$  is 3-connected and  $\{1, 3, 6\}$  is a cut set in  $G$  implies that there exists a path  $R$  from  $u \in V(\text{Int}(P))$  to  $v \in (P_2 \cup P_1 \cup \{3\}) \setminus \{1, 2\}$ . So if  $v = 3$  then it is easy to see that  $H$

contains  $\mathcal{O}_2$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, u)$ . If  $v \in V(P_2) \setminus \{2\}$  then similarly as before  $H$  contains  $(\mathcal{O}_5, 1, 2, 3)$  or  $(\mathcal{O}_5, 2, 1, 3)$  as a rooted minor.

By symmetry between  $a$  and  $b$ , if (iii), (viii), (ix) of Theorem 2.2.10 holds then a similar argument presented for the case  $a = a'$  shows that  $H$  contains either  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 1, 3, 2)$ ,  $(\mathcal{O}_{12}, 1, 2, 3)$  or  $(\mathcal{O}_{15}, 2, 1, 3)$  as a rooted minor.

By combining the result of the cases  $a = a'$  and  $a \neq a'$ , we can see that  $(G_1, a, b, c)$  contains either  $\mathcal{O}_3(1, 2, 3)$  or  $\mathcal{O}_6(1, 2, 3)$  as a rooted subdivision or it contains either  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 2, 1, 3)$ ,  $(\mathcal{O}_5, 2, 1, 3)$ ,  $(\mathcal{O}_5, 1, 2, 3)$ ,  $(\mathcal{O}_{12}, 1, 2, 3)$ ,  $(\mathcal{O}_{15}, 2, 1, 3)$ ,  $(\mathcal{O}_{15}, 1, 2, 3)$  or  $(\mathcal{O}_{25}, 3, 2, 1)$  as a rooted minor.

□

**Lemma 4.1.6.** *Let  $(G, a, b, c)$  be a 3-connected rooted graph. If  $(G, a, b, c)$  is minor minimal not  $c$ -planar then  $(G, a, b, c)$  contains  $O_i(\alpha, \beta, 3)$  for some  $i \in \{1, 3, 4, 6, 8\}$  as a rooted subdivision, see Figure 1.3, or  $(G, a, b, c)$  contains  $(\mathcal{O}_j, \alpha, \beta, 3)$  for some  $j \in \{2, 5, 12, 13, 15\}$ , or  $(\mathcal{O}_5, 3, \alpha, \beta)$ ,  $(\mathcal{O}_5, \alpha, 3, \beta)$  or  $(\mathcal{O}_{25}, 3, \alpha, \beta)$  as a rooted minor, where  $\alpha, \beta = \{1, 2\}$ . See Figure 1.5.*

*Proof.* We consider two cases, either  $(G, a, b, c)$  is internally 4-connected or it is not internally 4-connected. If  $(G, a, b, c)$  is internally 4-connected then by Theorem 4.1.4,  $G$  contains either  $O_i(1, 2, 3)$  for some  $i \in \{1, 3, 4, 6, 7\}$  as a rooted subdivision or  $\mathcal{O}_2, \mathcal{O}_5$  as a rooted minor.

So assume that  $(G, a, b, c)$  is not internally 4-connected. Let  $G = (G_1, G_2)$  be an internal 3-separation in  $G$  such that  $V(G_1) \cap V(G_2) = \{a', b', c'\}$  and  $\{a, b, c\} \subset V(G_1)$  and the number of vertices of  $G_1$  is as small as possible. Since  $H$  is 3-connected we may assume there exist three disjoint paths  $P_a, P_b, P_c$  in  $G_1$  connecting  $a, b, c$  to  $a', b', c'$ , respectively. We claim that  $(G_1, a', b', c')$  does not have a drawing on the disk such that  $a', b', c'$  are on the boundary of the disk. For proving the claim assume  $G_1$  has such a drawing on the disk. Let  $G^*$  be obtained from  $G$  by contracting all edges in  $G_1$  except edges with one ends at  $a', b'$  or  $c'$  and calling the new vertex  $v^*$ . It is easy to see

that  $\deg(v^*) = 3$  in  $G^*$ . Since  $(G, a, b, c)$  is minor minimal not  $c$ -planar,  $(G^*, a, b, c)$  has a  $c$ -planar drawing. It is not hard to see that this drawing can be extended to a  $c$ -planar embedding of  $(G, a, b, c)$ . This proves the claim. Now we want to show that  $G_1$  contains a double fork on  $a', b', c'$ . First, note that the graph  $G_1^*$  obtained from  $G_1$  by adding the edges  $a'b', a'c', b'c'$  is 3-connected and it does not have an embedding in the disk such that  $a', b', c'$  are on the boundary of the disk. Thus as Theorem 4.0.21 implies,  $G_1^*$  contains a double fork with feet on  $a', b', c'$ . This immediately shows that  $G_1$  contains a double fork with feet on  $a', b', c'$ .

If  $c \neq c'$  then by symmetry either  $a = a', b = b'$  or  $a \neq a', b = b'$  or  $a \neq a', b \neq b'$  which implies that  $(G, a, b, c)$  contains either  $(\mathcal{O}_1, 1, 2, 3)$  or  $(\mathcal{O}_8, 1, 2, 3)$  as a rooted subdivision or  $(G, a, b, c)$  contains  $(\mathcal{O}_{13}, 1, 2, 3)$  as a rooted minor. So from now on, we may assume  $c = c'$ .

Now we consider two main cases, either  $(G_1, a', b', c')$  is  $c$ -planar, or it is not  $c$ -planar.

- $(G_1, a', b', c)$  is  $c$ -planar.

Note that since  $(G, a, b, c)$  is minor minimal not  $c$ -planar and  $\{a', b', c\}$  is a vertex cut,  $G_1$  is isomorphic to  $K_{2,3}$  where  $a', b', c$  are terminals. Now by applying Lemma 4.1.5,  $(G, a, b, c)$  contains either  $(\mathcal{O}_3, 1, 2, 3)$  or  $(\mathcal{O}_6, 1, 2, 3)$  as a rooted subdivision, or  $(G, a, b, c)$  contains  $(\mathcal{O}_i, \alpha, \beta, 3)$  for some  $i \in \{2, 5, 12, 15\}$ , or  $(\mathcal{O}_5, 3, \alpha, \beta)$ ,  $(\mathcal{O}_5, \alpha, 3, \beta)$   $(\mathcal{O}_{25}, 3, \alpha, \beta)$  as a rooted minor, where  $\alpha, \beta = \{1, 2\}$ .

- $(G_1, a', b', c)$  is not  $c$ -planar.

By symmetry we consider two cases either  $a = a'$  and  $b \neq b'$  or  $a \neq a'$  and  $b \neq b'$ .

If  $a = a'$  and  $b \neq b'$  then since  $(G, a, b, c)$  is minor minimal not  $c$ -planar, edges in the path  $P_b$  are not contractible, i.e. there is an edge between  $b'c$  or  $b'a$  in  $(G_1, a, b', c)$ . By Theorem 4.1.4,  $(G_1, a, b', c)$  contains either  $(\mathcal{O}_6, 1, 2, 3)$  or  $(\mathcal{O}_7, 1, 2, 3)$  as a rooted subdivision where  $a, b, c$  correspond to  $x, y, 3$ , respectively,  $\{x, y\} = \{1, 2\}$ . It is easy

to see that if  $(G_1, a, b', c)$  contains  $(\mathcal{O}_6, 1, 2, 3)$  as a rooted subdivision then  $(G, a, b, c)$  contains  $(\mathcal{O}_3, 1, 2, 3)$  as a rooted subdivision if  $(G_1, a, b', c)$  contains  $(\mathcal{O}_6, 1, 2, 3)$  as a rooted subdivision then  $(G, a, b, c)$  contains  $(\mathcal{O}_5, 1, 2, 3)$  as a rooted minor. If  $(G_1, a, b', c)$  contains  $(\mathcal{O}_7, 1, 2, 3)$  (Note that there is a symmetry between 1, 2) as a rooted subdivision then  $(G, a, b, c)$  contains  $(\mathcal{O}_4, 1, 2, 3)$  as a rooted subdivision.

Now suppose  $a \neq a'$  and  $b \neq b'$ . Note that  $a'b, ab' \notin E(G)$  because  $\{a', b', c\}$  is a vertex cut in  $G$ , and  $(G, a, b, c)$  is minor minimal not  $c$ -planar and  $(G_1, a', b', c)$  is not  $c$ -planar. The fact that  $(G, a, b, c)$  is minor minimal not  $c$ -planar implies that edges in the path  $P_a$  or  $P_b$  are not contractible, i.e. there is an edge between  $a'c$  or  $b'c$  in  $(G_1, a, b', c)$ . By Theorem 4.1.4, this does not hold. This completes the proof of the lemma.  $\square$

Here we present proof of Theorem 1.8.3, mentioned in Section 1.8.

*Proof of Theorem 1.8.3.* The proof of Theorem 1.8.3 follows from the proof of Theorem 4.1.4 and Lemma 4.1.6.  $\square$

## 4.2 Obstructions for $ac$ -planarity

**Lemma 4.2.1.** *Let  $(G, a, b, c)$  be a rooted graph and  $(H, a, b, c)$  be obtained from  $(G, a, b, c)$  by adding the edge  $ab$ . If  $(G, a, b, c)$  contains  $(\mathcal{O}_i, 3, 2, 1)$  for some  $i \in \{3, 4, 6, 8, 9, 10\}$  as a rooted subdivision, then  $(H, a, b, c)$  contains  $(\mathcal{O}_{24}, 1, 2, 3)$ ,  $(\mathcal{O}_{23}, 3, 2, 1)$ ,  $(\mathcal{O}_{26}, 2, 1, 3)$ ,  $(\mathcal{O}_{25}, 1, 2, 3)$ ,  $(\mathcal{O}_{28}, 2, 1, 3)$  or  $(\mathcal{O}_{27}, 1, 2, 3)$  as a rooted minor, respectively.*

**Lemma 4.2.2.** *Let  $(G, a, b, c)$  be an internally 4-connected rooted graph. If  $(G, a, b, c)$  is minor minimal not  $ac$ -planar rooted graph then  $G$  contains  $(\mathcal{O}_i, \beta, 1, \gamma)$  for some  $i \in \{3, 4, 6, 8, 9, 10, 11\}$ ,  $\{\beta, \gamma\} = \{2, 3\}$  as a rooted subdivision or  $G$  contains  $(\mathcal{O}_j, \beta, 1, \gamma)$  for some  $j \in \{2, 5, 12, 13, 15, 16, 24, 26, 28\}$ ,  $\{\beta, \gamma\} = \{2, 3\}$  or  $(\mathcal{O}_k, \alpha, 2, \gamma)$  for some  $k \in \{5, 12, 15, 16, 17, 18, 19, 23, 24, 25, 27\}$ ,  $\{\alpha, \gamma\} = \{1, 3\}$ , or  $(\mathcal{O}_5, 2, 3, 1)$  as a rooted minor. See Figure 1.4 and 1.5.*

*Proof.* Our proof strategy is that in the first step, we determine the lists  $\mathcal{L}_S, \mathcal{L}_M$  of rooted graphs such that the rooted  $(G, a, b, c)$  must contain a member of these lists as a rooted subdivision or minor, respectively, for not being  $c$ -planar. Then, by assuming that  $(G, a, b, c)$  contains one of the rooted graph in  $\mathcal{L}_S$  as a rooted subdivision, we find the graphs listed in statement of the theorem such that  $(G, a, b, c)$  must contain as a rooted subdivision or rooted minor for not being  $ac$ -planar. In formally speaking, first we prevent  $c$ -planarity and then  $a$ -planarity.

The analysis presented in Theorem 4.1.4 shows that  $\mathcal{L}_S = \{\mathcal{O}_i(\alpha, \beta, 3) : i = 1, 3, 4, 6, 7 \text{ and } \{\alpha, \beta\} = \{1, 2\}\}$  and  $\mathcal{L}_M = \{\mathcal{O}_i(\alpha, \beta, 3) : i = 2, 5 \text{ and } \{\alpha, \beta\} = \{1, 2\}\}$ . This completes the first step.

Let the multigraph  $H$  be obtained from  $(G, a, b, c)$  by adding two parallel  $ac$  edges and two parallel  $ab$  edges, if they do not exist.

Now we consider the following five cases:

**Case 1**  $(G, a, b, c)$  contains  $(\mathcal{O}_1, 1, 2, 3)$  or  $(\mathcal{O}_1, 2, 1, 3)$  as a rooted subdivision.

Because of the symmetry assume that  $(G, a, b, c)$  contains  $(\mathcal{O}_1, 1, 2, 3)$  as a rooted subdivision. Note that in this case  $(H, a, b, c)$  contains  $(K'_{2,3}, 3, 1, 2)$ . The similar case analysis presented in Case 2 of Lemma 4.1.3 where 1, 2, 3 play roles of 3, 1, 2, respectively, shows that  $(H, a, b, c)$  contains  $\mathcal{O}_i(3, 1, 2)$  for some  $i \in \{3, 4, 6, 8, 9, 10\}$  as a rooted subdivision where  $a, b, c$  correspond to 3, 1, 2, respectively, or  $(H, a, b, c)$  contains  $\mathcal{O}_2(3, 1, 2)$ ,  $(\mathcal{O}_5, 3, 1, 2)$ ,  $(\mathcal{O}_{15}, 3, 1, 2)$  or  $(\mathcal{O}_5, 3, 2, 1)$  as a rooted minor.

**Case 2**  $(G, a, b, c)$  contains  $(\mathcal{O}_3, 1, 2, 3)$  or  $(\mathcal{O}_3, 2, 1, 3)$  as a rooted subdivision.

If  $(G, a, b, c)$  contains  $(\mathcal{O}_3, 2, 1, 3)$  as a rooted subdivision then there is nothing to prove. So we assume that  $(G, a, b, c)$  contains  $(\mathcal{O}_3, 1, 2, 3)$  as a rooted subdivision. Note that in this case  $(H, a, b, c)$  contains  $(K'_{2,3}, 1, 2, 3)$  as a rooted minor. Moreover there exists a path  $Q$  from 1 to vertex  $7 \in \text{seg}(2, 6)$ . the analysis of this case is similar

to Case 2 in Lemma 4.1.3 where roles of 1 and 3 are switched, but because of the existence of the path  $Q$ , we prefer to go through the analysis separately.

Let  $J$  be the multigraph obtained from  $(K'_{2,3}, 1, 2, 3)$  by adding the two parallel 13 edges and two parallel 21 edges. Let  $\mathcal{C} = \{134, 135, 1462, 1562, 3465, 131, 121\}$  be a double cycle cover for  $J$ . It is easy to see that  $\mathcal{C}$  is a 1-disk system, and  $(H, a, b, c)$  contains a  $(J, 1, 2, 3)$ -subdivision, called  $S$ , as a subgraph. It is easy to see that  $H, J, S, \mathcal{C}$  satisfy conditions of Theorem 3.0.17. So one of the outcomes of Theorem 3.0.17 holds. Similarly to Case 2 in Lemma 4.1.3, (xi), (x), (ix) and (iv) of Theorem 3.0.17 do not hold. Now, we are going to analyze the other outcomes, i.e., (i), (ii), (iii), (v), (vi), (vii) and (viii) of Theorem 3.0.17.

**Case 2.1** The outcome (i) of Theorem 3.0.17 holds, i.e. there exists an  $S$ -jump.

Let  $P$  be an  $S$ -jump with ends  $w, z$  where  $w \in \text{seg}(1, 4) \cup \text{seg}(1, 5) \cup \text{seg}[2, 6)$  and  $z \in \text{seg}(3, 5] \cup \text{seg}(3, 4] \cup \text{seg}[4, 6) \cup \text{seg}[5, 6)$ .

We consider two possibilities. First, assume that  $w \in \text{seg}(1, 5)$  and  $z \in \text{seg}(1, 4] \cup \text{seg}(3, 4] \cup \text{seg}[4, 6)$ . If  $P \cap Q \neq \emptyset$ , let  $x \in P \cap Q$  and we claim that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 1, 2, 3)$  as a rooted minor. For proving the claim, it is enough to assume that  $z = 4$ . Now by contracting the edge 27, it is easy to see that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, 6, x)$ . So  $P \cap Q = \emptyset$ . Similarly as before, by assuming  $z = 4$ , it is easy to see that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 1, 3)$  as a minor with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, w, 6)$ .

In the second possibility assume that  $w \in \text{seg}[2, 6)$  and  $z \in \text{seg}[3, 4)$ . If  $P \cap Q \neq \emptyset$  then let  $x \in P \cap Q$ . It is easy to see that  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, x)$ . Therefore, assume that  $P \cap Q = \emptyset$ . If  $w \neq 2$  then because we are going to show that  $(H, a, b, c)$  contains  $(\mathcal{O}_{15}, 2, 1, 3)$  as a rooted minor, we can assume that  $w = 7$  and  $z = 3$ . In which case  $(H, a, b, c)$  contains  $(\mathcal{O}_{15}, 2, 1, 3)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, 6, 7)$ . So assume that  $w = 2$ . If  $z = 3$  and  $P$  is an edge  $(H, a, b, c)$  contains  $(\mathcal{O}_{24}, 3, 1, 2)$

with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, 6, 7)$ . If  $P$  is not an edge then because  $(H, a, b, c)$  is 3-connected, we infer that there exists a path  $P'$  from  $u \in \text{Int}(P)$  to  $v \in S$ . Note that if  $P' \cap Q \neq \emptyset$  and  $t \in P' \cap Q$  then by contracting the edge  $ut$ ,  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, t)$ . Thus, we assume that  $P' \cap Q = \emptyset$ . If  $v \in \text{seg}(2, 6)$  then  $P \cup P'$  contains a jump, which was already analyzed. If  $v = 6$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_{19}, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6, 7, 8) \leftrightarrow (4, 5, 6, 7, u)$ . Now, we assume that  $v = 4$  and by contracting the edge  $46$ ,  $(H, a, b, c)$  contains  $(\mathcal{O}_{16}, 3, 2, 1)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (u, 5, 4, 7)$ .

**Case 2.2** The outcome (ii) of Theorem 3.0.17 holds, i.e. there exists a degenerate  $S$ -cross.

Let  $(P_1, P_2, P)$  be the degenerate  $S$ -cross where  $1, u_2, v_1, v_2$  are feet of the  $S$ -cross  $(P_1, P_2)$ ,  $2, v_2, v_1, 6$  appear on  $\text{seg}[2, 6]$  in the order listed, and  $u_2 \in \text{seg}(1, 4] \cup \text{seg}[4, 6)$ ,  $v_1 \in \text{seg}(2, 6] \cup \text{seg}(4, 6]$ ,  $v_2 \in \text{seg}[2, 6)$ . Let  $8 \in V(\text{Int}(P_1))$  be the other end of the path  $P$ .

If  $v_1 \in \text{seg}(2, 6)$  then we are going to show that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 3, 1)$  as a rooted minor. To show this, let  $v_2 = 2, u_2 = 4$ , then it is easy to see that by contracting the edge  $56$ ,  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 3, 1)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (4, v_1, 8, 5)$ . Therefore, we can assume that  $v_1 \in \text{seg}(4, 6]$ . Now, we are going to show that  $(H, a, b, c)$  contains  $\mathcal{O}_5$  or  $(\mathcal{O}_{17}, 1, 2, 3)$  as a rooted minor. For showing this, we assume that  $v_1 = 6, v_2 = 2, u_2 = 4$ . If  $(P_1 \cup P_2 \cup P) \cap Q = \emptyset$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_{17}, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6, 7, 8) \leftrightarrow (7, 8, 5, 4, 6)$ . If  $P_1 \cap P_2 \cap Q \neq \emptyset$ , then  $P_1 \cup P_2 \cup Q$  contains a connected cross, thus as Case 2.3 in the proof of this lemma implies,  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 3, 1)$  as a rooted minor. If  $Q \cap P_2 \neq \emptyset$ , let  $9 \in P_2 \cap Q$  then by contracting the edge  $79$ ,  $(H, a, b, c)$  contains  $(\mathcal{O}_{17}, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6, 7, 8) \leftrightarrow (7, 8, 5, 4, 6)$ . If  $Q \cap P_1 \neq \emptyset$  or  $Q \cap P \neq \emptyset$ , we can assume that  $8 \in Q \cap (P_1 \cup P)$  then by contracting the edge  $56$ ,  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 3, 1)$  as a rooted minor with signature

$(4, 5, 6, 7) \leftrightarrow (4, 7, 8, 5)$ .

**Case 2.3** The outcome (iii) of Theorem 3.0.17 holds, i.e. there exists a 1-blocking  $S$ -cross.

This case is similar to Case 2.4 in the proof of Lemma 4.1.3 by switching the role of  $a$  and  $c$ , so  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 3, 1)$  as a rooted minor.

**Case 2.4** The outcome (v) of Theorem 3.0.17 holds, i.e. there exists a free solid  $S'$ -cross.

Let  $(P_1, P_2)$  be the free cross with feet  $u, w, v, z$ . By considering the symmetry, we consider two possibilities: either the feet of the free cross are on the disk 3465, or they are on the disk 1562.

In the first possibility, i.e.  $u, w, v, z$  belong to the disk 3465, Since we are going to show that  $(H, a, b, c)$  contains either  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_{12}, 2, 1, 3)$  or  $(\mathcal{O}_{13}, 1, 2, 3)$  as a rooted minor, we assume that  $u = 5, w = 3, v = 4, z = 6$ . If  $(P_1 \cup P_2) \cap Q = \emptyset$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_{12}, 2, 1, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, 6)$ . If  $P_2 \cap Q \neq \emptyset$  then let  $x \in V(P_2 \cap Q)$ , in which case  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, x)$ . So finally assume that  $P_2 \cap Q = \emptyset$ ,  $P_1 \cap Q \neq \emptyset$  then let  $x \in V(P_1 \cap Q)$ , in which case  $(H, a, b, c)$  contains  $(\mathcal{O}_{13}, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6, 7, 8) \leftrightarrow (6, x, 5, 7, 4)$ .

In the second possibility, i.e.  $u, w, v, z$  belong to the disk 1562, since the cross is solid and free, we may assume that  $u \in \text{seg}(1, 5)$ ,  $v \in \text{seg}(5, 6] \cup \text{seg}(2, 6]$ ,  $w \in \text{seg}(1, 5] \cup \text{seg}[5, 6)$  and  $z \in \text{seg}[2, 6)$ . Let we assume  $8 = u \in \text{seg}(1, 5)$ . If  $v \in \text{seg}(2, 6)$ , or  $w \in \text{seg}(1, 5)$  then the same argument presented in Case 2.5 in the proof of Lemma 4.1.3 where the roles of  $a$  and  $c$  are switched shows that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 3, 2, 1)$  or  $(\mathcal{O}_{18}, 3, 2, 1)$  as a rooted minor, respectively. So we may assume that  $v \in \text{seg}(5, 6]$  and  $w \in \text{seg}[5, 6)$ . Because we are going to show that  $(H, a, b, c)$  contains either  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 2, 3, 1)$  or  $(\mathcal{O}_{17}, 1, 2, 3)$  as a rooted minor, we can assume that  $v = 6, w = 5$  and  $z = 2$ . If  $(P_1 \cup P_2) \cap Q = \emptyset$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_{17}, 1, 2, 3)$



as a rooted minor with signature  $(4, 5, 6, 7, 8) \leftrightarrow (7, 8, 4, 5, 6)$ . If  $P_1 \cap Q \neq \emptyset$  then let  $x \in P_1 \cap Q$ , it is easy to see that by contracting the edges 58 and 46,  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 3, 1)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (x, 5, 8, 4)$ . If  $P_2 \cap Q \neq \emptyset$  then let  $x \in P_2 \cap Q$ , it is easy to see that by contracting the edges 28 and 26,  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (8, 4, x)$ . Finally, if  $P_1 \cap P_2 \cap Q \neq \emptyset$  then  $P_1 \cup P_2 \cup Q$  contains a 1-blocking connected cross, so as Case 2.3 in this lemma implies,  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 3, 1)$  as a rooted minor.

**Case 2.5** The outcome (vi) of Theorem 3.0.17 holds, i.e. there exists a double facial  $S'$ -cross.

Let  $(P, P_1, P_2)$  be the double facial  $S$ -cross where  $P$  has an end at 1 and the other end at  $u \in \text{seg}[2, 6]$ ,  $P_1$  has one end at  $u_1 \in \text{seg}(1, 5] \cup \text{seg}[5, 6)$  and the other end at  $v_1 \in \text{seg}[2, 6)$ ,  $P_2$  has one end at  $u_2 \in \text{seg}(1, 4] \cup \text{seg}[4, 6)$  and the other end at  $v_2 \in \text{seg}[2, 6)$ . Because we are going to show that  $(H, a, b, c)$  contains either  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 3, 1, 2)$  or  $(\mathcal{O}_{15}, 3, 2, 1)$  as a rooted minor, by possibly switching the role of  $P$  and  $Q$ , we can assume that  $u = 6$ ,  $v_1 = v_2 = 2$ ,  $u_1 = 5$  and  $u_2 = 4$ . If  $(P_1 \cup P_2 \cup P) \cap Q = \emptyset$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_{15}, 3, 2, 1)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (4, 7, 6, 5)$ . If  $(P_1 \cup P_2) \cap Q \neq \emptyset$  and  $P \cap Q \neq \emptyset$  then by symmetry between  $P_1, P_2$ , assume that  $P_1 \cap Q \neq \emptyset$  and  $P \cap Q \neq \emptyset$  in which case  $P_1 \cup P \cup Q$  contains a 1-blocking cross, so by Case 2.3,  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 3, 1)$  as a rooted minor. Therefore by symmetry between  $P_1, P_2$ , either  $P_1 \cap Q = \emptyset$  or  $P \cap Q = \emptyset$ . If  $P \cap Q = \emptyset$  and  $x \in P_1 \cap Q$  then by contracting the edge 27,  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (5, 6, x)$ . Finally, if  $P_1 \cap Q = \emptyset$  and  $x \in P \cap Q$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_{15}, 3, 2, 1)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (4, 7, 6, 5)$ .

**Case 2.6** The outcome (vii) of Theorem 3.0.17 holds, i.e. there exists a blocking interlaced  $S$ -fork of type I.

This case is similar to Case 2.7 in the proof of Lemma 4.1.3 where roles of  $a$  and  $c$

are switched, so  $(H, a, b, c)$  contains  $(\mathcal{O}_{15}, 3, 2, 1)$  as a rooted minor.

**Case 2.7** The outcome (viii) of Theorem 3.0.17 holds, i.e. there exists an essential  $S'$ -triad with one feet on  $c$ .

Let  $(P_1, P_2, P_3)$  be an essential triad with center 8 and feet on  $v_1, v_2, 1$ , respectively. Note that  $v_1, v_2 \in \text{seg}(2, 4) \cup \text{seg}(3, 4) \cup \text{seg}(2, 5) \cup \text{seg}(3, 5)$  and there is a symmetry between  $1, 4, v_1$  and  $2, 5, v_2$ , respectively. By considering these symmetries, we analyze the following two possible outcomes: First, if  $v_1 \in \text{seg}[2, 4), v_2 \in \text{seg}[3, 4)$  in which case  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, 8)$ .

Second, if  $v_1 \in \text{seg}(1, 4], v_2 \in \text{seg}(1, 5]$  since we are going to show that either  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 1, 3)$  or  $(\mathcal{O}_{13}, 1, 2, 3)$ , we assume that  $v_1 = 4, v_2 = 5$ . If  $(P_1 \cup P_2 \cup P_3) \cap Q = \emptyset$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 1, 3)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, 6, 7)$ . If  $(P_1 \cup P_2 \cup P_3) \cap Q \neq \emptyset$  then since we are going to show that  $(H, a, b, c)$  contains  $(\mathcal{O}_{13}, 1, 2, 3)$  as a rooted minor, we can assume that  $8 \in V(Q)$ , in which case the signature  $(4, 5, 6, 7, 8) \leftrightarrow (6, 8, 5, 7, 4)$  is the proof of the claim.

**Case 3**  $(G, a, b, c)$  contains  $(\mathcal{O}_4, 1, 2, 3)$  or  $(\mathcal{O}_4, 2, 1, 3)$  as a rooted subdivision.

If  $(G, a, b, c)$  contains  $(\mathcal{O}_4, 2, 1, 3)$  as a rooted subdivision then there is nothing to prove. So we assume that  $(G, a, b, c)$  contains  $(\mathcal{O}_4, 1, 2, 3)$  as a rooted subdivision. Let  $(K_{2,3}^*, 1, 2, 3)$  be a rooted graph obtained from  $(K'_{2,3}, 1, 2, 3)$  by adding the edge 45. Note that in this case  $(H, a, b, c)$  contains  $(K_{2,3}^*, 1, 2, 3)$  as a rooted minor. Moreover there exists a path  $Q$  from 1 to vertex 6 in  $(H, a, b, c)$ .

Let  $J$  be the multigraph obtained from  $(K_{2,3}^*, 1, 2, 3)$  by adding the two parallel 13 edges and two parallel 21 edges. Let  $\mathcal{C} = \{134, 135, 1462, 1562, 345, 456, 131, 121\}$  be a double cycle cover for  $J$ . It is easy to see that  $\mathcal{C}$  is a 1-disk system, and  $(H, a, b, c)$  contains a  $(J, 1, 2, 3)$ -subdivision, called  $S$ , as a subgraph. It is easy to see that  $H, J, S, \mathcal{C}$  satisfy conditions of Theorem 3.0.17. So one of the outcomes of

Theorem 3.0.17 holds. Similarly to Case 2 in Lemma 4.1.3, (xi), (x), (ix) and (iv) of Theorem 3.0.17 do not hold. Moreover (viii) of Theorem 3.0.17 does not hold. Now, we are going to analyze the other outcomes, i.e., (i), (ii), (iii), (v), (vi) and (vii) of Theorem 3.0.17.

**Case 3.1** The outcome (i) of Theorem 3.0.17 holds, i.e. there exists an  $S$ -jump.

Let  $P$  be an  $S$ -jump with ends  $w, z$ , where  $w \in \text{seg}(1, 4) \cup \text{seg}(1, 5) \cup \text{seg}[2, 6]$  and  $z \in \text{seg}(3, 5] \cup \text{seg}(3, 4] \cup \text{seg}[4, 6) \cup \text{seg}[5, 6) \cup \text{seg}(5, 4)$ . We consider two possibilities.

First assume that  $w \in \text{seg}(1, 5)$  and  $z \in \text{seg}(1, 4] \cup \text{seg}(3, 4] \cup \text{seg}[4, 6) \cup \text{seg}(4, 5)$ . If  $w = 1$  and  $z \in \text{seg}(4, 5)$ , in which case either  $P \cap Q = \emptyset$  implying  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 1, 3)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, z, 6)$ , or  $P \cap Q \neq \emptyset$  implying  $(H, a, b, c)$  contains  $(\mathcal{O}_{15}, 3, 1, 2)$  as a rooted minor by contracting the edge 26 and with signature  $(4, 5, 6, 7) \leftrightarrow (5, x, z, 4)$  where  $x \in P \cap Q$ . If  $w \in \text{seg}(1, 5)$  then since we are going to show that  $(H, a, b, c)$  contains either  $(\mathcal{O}_5, 2, 1, 3)$  or  $(\mathcal{O}_{13}, 1, 2, 3)$  as a rooted minor, we may assume that  $z = 4$ . Now, either  $P \cap Q = \emptyset$  implying  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 1, 3)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, w, 6)$ , or  $P \cap Q \neq \emptyset$  implying  $(H, a, b, c)$  contains  $(\mathcal{O}_{13}, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6, 7, 8) \leftrightarrow (5, x, w, 6, 4)$  where  $x \in P \cap Q$ .

In the second possibility assume that  $w \in \text{seg}[2, 6]$  and  $z \in \text{seg}[3, 4) \cup \text{seg}(4, 5)$ . If  $z \in \text{seg}(4, 5)$  then since we are going to show that  $(H, a, b, c)$  contains either  $(\mathcal{O}_5, 1, 2, 3)$  or  $(\mathcal{O}_{13}, 1, 2, 3)$  as a rooted minor, we may assume that  $w = 2$ . Now, either  $P \cap Q = \emptyset$  implying  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, z, 6)$ , or  $P \cap Q \neq \emptyset$  implying  $(H, a, b, c)$  contains  $(\mathcal{O}_{13}, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6, 7, 8) \leftrightarrow (6, z, 4, x, 5)$ , where  $x \in P \cap Q$ .

Thus by symmetry between 4, 5 assume that  $z \in \text{seg}[3, 4)$ . If  $z \in \text{seg}(3, 4)$  then since we are going to show that  $(H, a, b, c)$  contains either  $(\mathcal{O}_2, 1, 2, 3)$  or  $(\mathcal{O}_{16}, 3, 1, 2)$  as a rooted minor, we may assume that  $w = 2$ . Now, either  $P \cap Q = \emptyset$ , implying  $(H, a, b, c)$  contains  $(\mathcal{O}_{16}, 3, 1, 2)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (z, 6, 4, 5)$ ,

or  $P \cap Q \neq \emptyset$ , implying  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, x)$  where  $x \in P \cap Q$ .

So assume that  $z = 3$  and,  $w = 6$  or  $2$ . If  $w = 6$  then either  $P \cap Q = \emptyset$  implying  $(H, a, b, c)$  contains  $(\mathcal{O}_{12}, 2, 1, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, 6)$ , or  $P \cap Q \neq \emptyset$  implying  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, x)$  where  $x \in P \cap Q$ . If  $w = 2$  and  $P$  is an edge then  $(H, a, b, c)$  contains  $(\mathcal{O}_{23}, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, 6)$ . If  $P$  is not an edge then either  $P \cap Q \neq \emptyset$  implying  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, x)$  where  $x \in P \cap Q$ , there exists a path  $P'$  from  $x \in \text{Int}(P)$  to  $y \in S \cup Q$ . Note that if  $y \in Q \cup \text{seg}[2, 6] \cup \text{seg}[4, 6] \cup \text{seg}[5, 6] \cup \text{seg}[4, 5] \cup \text{seg}[3, 4] \cup \text{seg}[3, 5]$  then in the above case analysis possible outcomes have already been investigated, and if  $y \in \text{seg}[1, 4] \cup \text{seg}[1, 5]$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, x)$ .

**Case 3.2** The outcome (ii) of Theorem 3.0.17 holds, i.e. there exists a degenerate  $S$ -cross.

Let  $(P_1, P_2, P)$  be the degenerate  $S$ -cross where  $1, u_2, v_1, v_2$  are feet of the  $S$ -cross and  $2, v_2, v_1, 6$  appear on  $\text{seg}[2, 6]$  in the order listed. Assume  $u_2 \in \text{seg}(1, 4] \cup \text{seg}[4, 6)$ ,  $v_1 \in \text{seg}(2, 6] \cup \text{seg}(4, 6]$ ,  $v_2 \in \text{seg}[2, 6)$ . Assume  $8 \in V(\text{Int}(P_1))$  is the other end of the path  $P$ .

Because we are going to show that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 3, 1)$  as a rooted minor, we assume that  $v_1 = 6, u_2 = 4, v_2 = 2$ . It is easy to see that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 3, 1)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (4, 6, 7, 5)$ .

**Case 3.3** The outcome (iii) of Theorem 3.0.17 holds, i.e. there exists a 1-blocking  $S$ -cross.

This case is similar to Case 2.4 in the proof of Lemma 4.1.3 where the roles of  $a$  and  $c$  are switched, so  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 3, 1)$  as a rooted minor.

**Case 3.4** The outcome (v) of Theorem 3.0.17 holds, i.e. there exists a free solid

$S'$ -cross.

Let  $(P_1, P_2)$  be the free solid  $S$ -cross with feet  $u, w, v, z$ . We consider three possibilities: either the feet of the free cross are on the disk 1426, 345 or 456.

In the first possibility, i.e.  $u, w, v, z$  belong to the disk 1462, since  $(P_1, P_2)$  is a free solid  $S$ -cross and we are going to show that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 3, 1)$  as a rooted minor. We may assume that  $u \in \text{seg}(1, 4)$ ,  $v = 6$ ,  $w = 4$  and  $z = 2$ . It is easy to see that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 3, 1)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (4, 6, u, 5)$ .

In the second possibility, i.e.  $u, w, v, z$  belong to the disk 345. We are going to show that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 1, 3)$  as a rooted minor. We assume either  $u = 3, v \in \text{seg}(4, 5), w = 5, z \in \text{seg}(3, 4)$ , or  $u = 4, v \in \text{seg}(3, 5), w = 5, z \in \text{seg}(3, 4)$  which both imply that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 1, 3)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (v, z, 5, 4)$

In the third possibility, i.e.  $u, w, v, z$  belong to the disk 456. We are going to show that  $(H, a, b, c)$  contains  $(\mathcal{O}_{13}, 1, 2, 3)$  as a rooted minor. We assume either  $u = 4, v \in \text{seg}(5, 6), w = 6, z \in \text{seg}(4, 5)$ , or  $u = 5, v \in \text{seg}(4, 6), w = 4, z \in \text{seg}(5, 6)$  which implies that  $(H, a, b, c)$  contains  $(\mathcal{O}_{13}, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6, 7, 8) \leftrightarrow (v, z, 5, 6, 4)$

**Case 3.5** The outcome (vi) of Theorem 3.0.17 holds, i.e. there exists a double facial  $S'$ -cross.

Let  $(P, P_1, P_2)$  be the double facial  $S$ -cross, where  $P$  has an end at 1 and the other end at  $u \in \text{seg}[2, 6]$ ,  $P_1$  has one end at  $u_1 \in \text{seg}(1, 5] \cup \text{seg}[5, 6)$  and the other end at  $v_1 \in \text{seg}[2, 6)$ ,  $P_2$  has one end at  $u_2 \in \text{seg}(1, 4] \cup \text{seg}[4, 6)$  and the other end at  $v_2 \in \text{seg}[2, 6)$ . Because we are going to show that  $(H, a, b, c)$  contains  $(\mathcal{O}_{12}, 3, 2, 1)$  as a rooted minor, by possibly switching the role of  $P$  and  $Q$ , we can assume that  $u = 6$ ,  $v_1 = v_2 = 2$ ,  $u_1 = 5$  and  $u_2 = 4$ . It is easy to see that  $(H, a, b, c)$  contains  $(\mathcal{O}_{12}, 3, 2, 1)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 6, 5)$ .

**Case 3.6** The outcome (vii) of Theorem 3.0.17 holds, i.e. there exists a blocking interlaced  $S$ -fork of type I.

This case is similar to Case 2.7 in the proof of Lemma 4.1.3 where roles of  $a$  and  $c$  are switched, so  $(H, a, b, c)$  contains  $(\mathcal{O}_{15}, 3, 2, 1)$  as a rooted minor.

**Case 4**  $(G, a, b, c)$  contains  $(\mathcal{O}_6, 1, 2, 3)$  or  $(\mathcal{O}_6, 2, 1, 3)$  as a rooted subdivision.

If  $(G, a, b, c)$  contains  $(\mathcal{O}_6, 2, 1, 3)$  as a rooted subdivision then the assertion of the lemma hold. So assume that  $(G, a, b, c)$  contains  $(\mathcal{O}_6, 1, 2, 3)$  as a rooted subdivision. Note that in this case  $(H, a, b, c)$  contains  $(K'_{2,3}, 1, 2, 3)$ . The case analysis presented in Case 2 of Lemma 4.1.3 where roles of  $a$  and  $c$  are switched shows that  $(H, a, b, c)$  contains  $\mathcal{O}_i(3, 2, 1)$  for some  $i = \{3, 4, 6, 8, 9, 10\}$  as a rooted subdivision, or  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 2, 3, 1)$ ,  $(\mathcal{O}_5, 3, 2, 1)$ ,  $(\mathcal{O}_{15}, 3, 2, 1)$  or  $(\mathcal{O}_{18}, 3, 2, 1)$  as a rooted minor. Because  $ab$  is an edge in  $(G, a, b, c)$ , by applying Lemma 4.2.1, we can say that  $(H, a, b, c)$  contains  $(\mathcal{O}_{24}, 1, 2, 3)$ ,  $(\mathcal{O}_{23}, 3, 2, 1)$ ,  $(\mathcal{O}_{26}, 2, 1, 3)$ ,  $(\mathcal{O}_{25}, 1, 2, 3)$ ,  $(\mathcal{O}_{28}, 2, 1, 3)$  or  $(\mathcal{O}_{27}, 1, 2, 3)$  as a rooted minor. Thus to summarize this case  $(H, a, b, c)$  contains either  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 3, 1, 2)$ ,  $(\mathcal{O}_5, 3, 2, 1)$ ,  $(\mathcal{O}_{15}, 3, 2, 1)$ ,  $(\mathcal{O}_{18}, 3, 2, 1)$ ,  $(\mathcal{O}_{24}, 1, 2, 3)$ ,  $(\mathcal{O}_{23}, 3, 2, 1)$ ,  $(\mathcal{O}_{26}, 2, 1, 3)$ ,  $(\mathcal{O}_{25}, 1, 2, 3)$ ,  $(\mathcal{O}_{28}, 2, 1, 3)$  or  $(\mathcal{O}_{27}, 1, 2, 3)$  as a rooted minor.

**Case 5**  $(G, a, b, c)$  contains  $(\mathcal{O}_7, 1, 2, 3)$  as a rooted subdivision.

Let  $(\tilde{K}_{2,3}, 1, 2, 3)$  be a rooted graph obtained from  $(K_{2,3}, 1, 2, 3)$  by adding the edge 45. Note that in this case  $(H, a, b, c)$  contains  $(\tilde{K}_{2,3}, 1, 2, 3)$  as a rooted minor. Moreover there exists an edge from  $a$  to vertex  $b$  in  $(H, a, b, c)$ .

Let  $J$  be the multigraph obtained from  $(\tilde{K}_{2,3}, 1, 2, 3)$  by adding the two parallel 13 edges and two parallel 21 edges. Let  $\mathcal{C} = \{134, 135, 142, 152, 345, 452, 131, 121\}$  be a double cycle cover for  $J$ . It is easy to see that  $\mathcal{C}$  is a 1-disk system, and  $(H, a, b, c)$  contains a  $(J, 1, 2, 3)$ -subdivision, called  $(S, 1, 2, 3)$  as a subgraph. It is easy to see that  $H, J, S, \mathcal{C}$  satisfy conditions of Theorem 3.0.17. So one of the outcomes of Theorem 3.0.17 holds. Similarly as Case 1 in Lemma 4.1.3, (ii), (iii), (vi), (vii), (xi), (x) and (ix) of Theorem 3.0.17 do not hold. Moreover (viii) of Theorem 3.0.17 does

not hold. Now, we are going to analyze the other outcomes, i.e., (i), (iv) and (v) of Theorem 3.0.17.

**Case 5.1** The outcome (i) of Theorem 3.0.17 holds, i.e. there exists an  $S$ -jump.

Let  $P$  be an  $S$ -jump with ends  $w, z$ , where  $w \in \text{seg}[1, 4] \cup \text{seg}[1, 5) \cup \text{seg}[2, 4)$  and  $z \in \text{seg}[3, 5] \cup \text{seg}(3, 4] \cup \text{seg}(2, 4] \cup \text{seg}(2, 5] \cup \text{seg}(5, 4)$ . By symmetry, we consider two possibilities.

In the first possibility assume,  $w \in \text{seg}[1, 4)$  and  $z \in \text{seg}(1, 4] \cup \text{seg}(3, 4] \cup \text{seg}[4, 6) \cup \text{seg}(4, 5)$ . If  $w = 1$  and  $z \in \text{seg}(4, 5)$ , in which case  $(H, a, b, c)$  contains  $(\mathcal{O}_6, 2, 1, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (4, 5, z)$ . So assume that  $w \in \text{seg}(1, 4)$ . If  $z \in \text{seg}(2, 5) \cup \text{seg}(3, 5)$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 3, 2, 1)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (5, w, z, 4)$ . If  $z = 5$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_6, 2, 1, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (4, 5, 6)$ . If  $z \in \text{seg}(4, 5)$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_6, 2, 1, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (4, 5, z)$ . Finally if  $z \in \text{seg}(1, 5)$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_6, 2, 1, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (4, 5, w)$ .

In the second possibility, by symmetry assume that  $w \in \text{seg}[2, 4)$  and  $z \in \text{seg}[3, 5)$ . We start by assuming that  $z \in \text{seg}(3, 5)$ . If  $w \in \text{seg}(2, 4)$  in which case  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (4, z, w, 5)$ . If  $w = 2$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_6, 2, 1, 3)$  as a rooted subdivision. Thus assume  $z = 3$ . If  $w = 2$  and  $P$  is an edge then  $(H, a, b, c)$  contains  $(\mathcal{O}_{11}, 2, 1, 3)$  as a rooted subdivision with signature  $(4, 5) \leftrightarrow (4, 5)$ . If  $P$  is not an edge then there exists a path  $P'$  from  $x \in \text{Int}(P)$  to  $y \in S$ . If  $y \in \text{seg}[1, 4) \cup \text{seg}[1, 5)$ , then it is easy to see that  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, x)$ . If  $y = 4$  or  $y = 5$ , say by symmetry  $y = 4$ , then  $(H, a, b, c)$  contains  $(\mathcal{O}_6, 2, 1, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (5, x, 4)$ . If  $y \in \text{seg}(4, 5)$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_{15}, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (4, x, y, 5)$ . Finally, if  $y \in S \setminus (\text{seg}[1, 4] \cup \text{seg}[1, 5] \cup \text{seg}[4, 5])$ , the above case analysis has already investigated

the other possible outcomes.

**Case 5.2** The outcome (iv) of Theorem 3.0.17 holds, i.e. there exists weakly free  $S$ -cross anchored at  $c$ .

The same argument as presented in Case 1.2 in the proof of Lemma 4.1.3, where the roles of  $a$  and  $c$  are switched implies that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 3, 1)$  as a rooted minor.

**Case 5.3** The outcome (v) of Theorem 3.0.17 holds, i.e. there exists a free solid  $S'$ -cross.

Let  $(P_1, P_2)$  be the free cross with feet  $u, w, v, z$ . Since in the following case analysis, appearance of the edge 12 does not change the results, by symmetry we consider only one possibility, i.e. the feet of the cross belong to the disk 245.

Since we are going to show that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 3, 1)$  as a rooted minor, we assume either  $u = 4, v \in \text{seg}(2, 5), w = 2, z \in \text{seg}(4, 5)$ , or  $u = 5, v \in \text{seg}(2, 4), w = 4, z \in \text{seg}(2, 5)$  which both imply that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 2, 3, 1)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (v, z, 4, 5)$ . Note that in proving the existence of  $(\mathcal{O}_5, 2, 3, 1)$  as a rooted minor, we did not use the fact that there are edges connecting terminal vertices this gives us an opportunity to use this case analysis in the future.

□

**Lemma 4.2.3.** *Let  $(G, a, b, c)$  be a 3-connected rooted graph. Let  $G = (G_1, G_2)$  be an internal separation such that  $V(G_1) \cap V(G_2) = \{a, b', c\}$ ,  $\{a, b, c\} \subseteq V(G_2)$  and  $E(G_1) = E(G[V(G_1)])$ . Moreover assume that  $G_1$  is isomorphic to  $K_{2,3}$ . If  $(G, a, b, c)$  is not  $ac$ -planar then  $(G, a, b, c)$  contains  $(\mathcal{O}_i, \alpha, 2, \gamma)$  for some  $i \in \{2, 12, 15, 24\}$ ,  $\{\alpha, \gamma\} = \{1, 3\}$ , or  $(\mathcal{O}_5, 2, 3, 1)$ ,  $(\mathcal{O}_5, 1, 3, 2)$ ,  $(\mathcal{O}_{26}, 2, 1, 3)$  or  $(\mathcal{O}_{26}, 3, 1, 2)$  as a rooted minor.*

*Proof.* Our proof strategy is that in the first step, we determine the lists  $\mathcal{L}_S, \mathcal{L}_M$  of rooted graphs such that  $(G, a, b, c)$  must contain a member of these lists as a rooted subdivision or minor, respectively, for not being  $c$ -planar. Then, by assuming that



$(G, a, b, c)$  contains one of the rooted graph in  $\mathcal{L}_S$  as a rooted subdivision, we show that  $(G, a, b, c)$  contains one of the graph listed in the statement of the Lemma as a rooted subdivision or rooted minor.

The analysis presented in Case 1 of Lemma 4.1.5 shows that  $\mathcal{L}_S = \{\mathcal{O}_3(1, 2, 3), \mathcal{O}_6(1, 2, 3)\}$  and  $\mathcal{L}_M = \{(O_2, 1, 2, 3), (O_5, 1, 3, 2), (O_{12}, 1, 2, 3), (O_{15}, 1, 2, 3), (O_{15}, 2, 1, 3)\}$ .

Let  $H$  be the multigraph obtained from  $(G, a, b, c)$  by adding two parallel  $ac$  edges and two parallel  $ab$  edges.

Now we consider two cases:

**Case 1**  $(G, a, b, c)$  contains  $(\mathcal{O}_3, 1, 2, 3)$  as a rooted subdivision.

Let  $J$  be the multigraph obtained from  $K'_{2,3}$  by adding the two parallel 13 edges and two parallel 12 edges where 1, 2, 3 are terminal, 4, 5 are centers and 6 is the only neighbor of 2. Let  $\mathcal{C} = \{135, 134, 1562, 3465, 1462, 313, 212\}$  be a double cycle cover for  $J$ . It is easy to see that  $\mathcal{C}$  is a 1-disk system, and  $H$  contains a  $J$ -subdivision, called  $S$ , as a subgraph where  $a, b, c$  correspond to 1, 2, 3, respectively, where  $b'$  correspond to 6. Note that  $H, J, S, \mathcal{C}$  satisfy the hypothesis of Theorem 2.2.10, so one of the outcomes listed in the statement of Theorem 2.2.10 holds. Note that since  $(G, a, b, c)$  actually contains  $(\mathcal{O}_3, 1, 2, 3)$  as a rooted subdivision, there is a path, called  $Q$ , from 1 to vertex  $7 \in \text{seg}(2, 6)$  in  $H$ .

Note that since  $\{1, 3, 6\}$  is a vertex cut in  $H$  and  $E(G_1) = E(G[V(G_1)])$ , the outcomes (ii), (iv), (v), (vi), (vii) and (x) of Theorem 2.2.10 do not hold. By applying Lemma 2.1.3, we can see that (xi) of Theorem 2.2.10 does not hold. Since  $J$  is not isomorphic to  $K_4$ , (xii) of Theorem 2.2.10 does not hold. Finally since  $G$  is not 1-planar, (xiii) of Theorem 2.2.10 does not hold. So either (i), (iii), (viii) or (ix) holds.

Note that we are going to apply a similar case analysis that we did in Case 1 of Lemma 4.1.5, where the roles of  $a$  and  $c$  are switched. Note that as Case 1 of Lemma 4.1.5 shows, the occurrence of (iii), (viii), (ix) of Theorem 2.2.10 imply

that  $(H, a, b, c)$  contains either  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 2, 3, 1)$ ,  $(\mathcal{O}_{12}, 3, 2, 1)$ ,  $(\mathcal{O}_{15}, 3, 2, 1)$  or  $(\mathcal{O}_{15}, 3, 1, 2)$  as a rooted minor. Thus we may assume that (i) of Theorem 2.2.10 holds. Note that since  $\{1, 3, 6\}$  is a vertex cut, the only possible  $S$ -jump in  $H$  is a path  $P$  from 3 to  $z \in \text{seg}[2, 6)$ . If there exists a vertex  $x \in P \cap \text{Int}(Q)$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (4, 5, x)$ . So we assume that  $P$  and  $Q$  are internally disjoint. If  $z = 2$  then either  $P$  is an edge, in which case  $(H, a, b, c)$  contains  $(\mathcal{O}_{24}, 3, 2, 1)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, 6, 7)$ , and if  $P$  is not an edge the same analysis presented in Case 1 of Lemma 4.1.5 shows that  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted minor.

**Case 2**  $(G, a, b, c)$  contains  $(\mathcal{O}_6, 1, 2, 3)$  as a rooted subdivision.

Let  $H, J, S, \mathcal{C}$  be as defined in the previous case. Note that in this case  $(G, a, b, c)$  contains an edge from  $a$  to  $b$ , we use this fact when we are identifying some rooted minors in  $(G, a, b, c)$ . Similarly as the previous case only (i), (iii), (viii) and (ix) of Theorem 2.2.10 can occur where the occurrence of the last three imply that  $(H, a, b, c)$  contains either  $(\mathcal{O}_2, 1, 2, 3)$ ,  $\mathcal{O}_5(2, 3, 1)$ ,  $(\mathcal{O}_{12}, 3, 2, 1)$ ,  $(\mathcal{O}_{15}, 1, 2, 3)$  or  $(\mathcal{O}_{15}, 3, 2, 1)$  as a rooted minor.

Thus we may assume that (i) of Theorem 2.2.10 holds. Note that since  $\{1, 3, 6\}$  is a vertex cut, the only possible  $S$ -jump in  $H$  is a path  $P$  from 3 to  $z \in \text{seg}[2, 6)$ . If  $z \in \text{seg}(2, 6)$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_{24}, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, 6, z)$ . If  $z = 2$  then either  $P$  is an edge, in which case  $(H, a, b, c)$  contains  $(\mathcal{O}_{26}, 2, 1, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, 6)$ , and if  $P$  is not an edge the same analysis presented in Case 1 of Lemma 4.1.5 shows that  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$  as a rooted minor.  $\square$

**Lemma 4.2.4.** *Let  $(G, a, b, c)$  be a 3-connected rooted graph. Let  $G = (G_1, G_2)$  be an internal separation such that  $V(G_1) \cap V(G_2) = \{a', b', c\}$ ,  $\{a, b, c\} \subseteq V(G_2)$  and  $E(G_1) = E(G[V(G_1)])$ . Moreover assume that  $G_1$  is isomorphic to  $K_{2,3}$ . If  $(G, a, b, c)$  is not  $c$ -planar then  $(G, a, b, c)$  contains either  $(\mathcal{O}_3, \alpha, \beta, 3)$  or  $(\mathcal{O}_6, \alpha, \beta)$ , as a rooted*

subdivision or it contains  $(\mathcal{O}_i, \alpha, \beta, 3)$  for some  $i \in \{2, 5, 12, 15\}$ , or  $(\mathcal{O}_{25}, 3, \alpha, \beta)$ ,  $(\mathcal{O}_5, \alpha, 3, \beta)$  as a rooted minor where  $\{\alpha, \beta\} = \{1, 2\}$ .

*Proof.* Proof of this lemma is the same as Case 2 in Lemma 4.1.4.  $\square$

**Lemma 4.2.5.** *Suppose  $(G, a, b, c)$  is not ac-planar. Let  $G = (G_1, G_2)$  be an internal 3-separation such that  $V(G_1) \cap V(G_2) = \{a, b', c'\}$ ,  $\{a, b, c\} \subseteq V(G_2)$  and  $E(G_1) = E(G[V(G_1)])$ . Assume  $(G_1, a, b', c')$  is internally 4-connected rooted not  $c'$ -planar but  $a$ -planar rooted graph. Assume  $b, c \notin V(G_1)$  and  $(G, a, b, c)$  is a rooted graph such that  $V(G) = V(G_1) \cup \{b, c\}$ , and  $(G, a, b, c)$  is obtained from  $(G, a, b', c')$  by adding the edges  $bb'$  and  $cc'$ . Then  $(G, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 2, 1, 3)$ ,  $(\mathcal{O}_5, 1, 2, 3)$ ,  $\mathcal{O}_5, 1, 3, 2)$ ,  $(\mathcal{O}_2, 1, 3, 2)$ ,  $(\mathcal{O}_{12}, 1, 2, 3)$ ,  $(\mathcal{O}_{15}, 2, 1, 3)$ ,  $(\mathcal{O}_{15}, 1, 2, 3)$  or  $(\mathcal{O}_{25}, 3, 2, 1)$  as a rooted minor.*

*Proof.* Note that there exist paths  $P_b, P_c$  which is internally disjoint from  $G_1$  and they connect  $b'$  to  $b$  and  $c'$  to  $c$  in  $G$ , respectively. The proof of this lemma is similar to the proof of Lemma 4.2.9 and Case 2 in Lemma 4.1.5. Since  $(G_1, a, b', c')$  is internally 4-connected not  $c$ -planar, by applying Lemma 4.1.5, and the fact that  $(G_1, a, b, c')$  is  $a$ -planar, someone can conclude that the rooted graph  $(G_1, a, b', c')$  contains  $(\mathcal{O}_i, 1, 2, 3)$  for some  $i \in \{1, 3, 4, 6, 7\}$  as a rooted subdivision.

So we consider two cases regarding whether  $(G_1, a, b', c')$  contains  $(\mathcal{O}_i, 1, 2, 3)$  for some  $i \in \{1, 3, 4, 6, 7\}$  as a rooted subdivision.

**Case 1**  $(G_1, a, b', c')$  contains  $(\mathcal{O}_1, 1, 2, 3)$ ,  $(\mathcal{O}_3, 1, 2, 3)$ ,  $(\mathcal{O}_4, 1, 2, 3)$ ,  $(\mathcal{O}_6, 1, 2, 3)$  as a rooted subdivision.

By switching the roles of  $a$  and  $c$ , a similar case analysis presented in Case 2 of Lemma 4.1.5 shows that  $(H, a, b, c)$  contains either  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 2, 1, 3)$ ,  $(\mathcal{O}_5, 1, 2, 3)$ ,  $(\mathcal{O}_{12}, 1, 2, 3)$ ,  $(\mathcal{O}_{15}, 2, 1, 3)$  or  $(\mathcal{O}_{25}, 3, 2, 1)$  as a rooted minor.

**Case 2**  $(G_1, a, b', c')$  contains  $(\mathcal{O}_7, 1, 2, 3)$  as a rooted subdivision.

Let  $(G^*, 1, 6, 7)$  be isomorphic to  $(\mathcal{O}_7, 1, 6, 7)$ , i.e. we just rename the vertex 3 in  $(\mathcal{O}_7, 1, 2, 3)$  by 6 in  $(G^*, 1, 6, 7)$ . Let  $J$  be the multigraph obtained from  $(G^*, 1, 6, 7)$  by adding the edge 36 and two parallel 12 edges and two parallel 13 edges where 1, 2, 3 are terminal. Let  $\mathcal{C} = \{1375, 1374, 475, 465, 1462, 1562, 121, 131\}$  be a double cycle cover for  $J$ . It is easy to see that  $\mathcal{C}$  is 1-disk system, and  $H$  contains a  $J$ -subdivision, called  $S$ , as a subgraph where  $a, b, c$  correspond to 1, 2, 3, respectively. Note that  $H, J, S, \mathcal{C}$  satisfy the hypothesis of Theorem 2.2.10, so one of the outcomes listed in the statement of Theorem 2.2.10 holds. Similarly as Case 1, we can see the only possible outcomes are (i), (iii), (viii) or (ix) of Theorem 2.2.10. Similarly as the argument presented in Case 1, occurrence of (iii) or (viii) or (ix) of Theorem 2.2.10 lead to the fact that  $(H, a, b, c)$  contains either  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 1, 3, 2)$ ,  $(\mathcal{O}_{12}, 1, 2, 3)$ ,  $(\mathcal{O}_{15}, 1, 2, 3)$  or  $(\mathcal{O}_{15}, 2, 1, 3)$  as a rooted minor. So we only focus on analyzing on occurrence of (i) in Theorem 2.2.10.

Thus, let  $P$  be a path from  $w \in \text{seg}[2, 6]$  to  $z \in \text{seg}[3, 7]$ . If  $w = 6, z = 7$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_{14}, 1, 2, 3)$  as a rooted minor. The other possibility for  $w$  and  $z$  lead to the same case analysis presented in Case 2 of Lemma 4.1.5 (by switching the roles of  $a$  and  $c$ ) implying that  $(H, a, b, c)$  contains either  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 2, 1, 3)$ ,  $(\mathcal{O}_5, 1, 2, 3)$ ,  $(\mathcal{O}_2, 1, 3, 2)$ ,  $(\mathcal{O}_{12}, 1, 2, 3)$ ,  $(\mathcal{O}_{15}, 2, 1, 3)$  or  $(\mathcal{O}_{25}, 3, 2, 1)$  as a rooted minor.  $\square$

**Lemma 4.2.6.** *Let  $(G_1, a', b, c)$  be an internally 4-connected rooted not  $c$ -planar but  $a'$ -planar rooted graph. Assume  $a \notin V(G_1)$  and  $(G, a, b, c)$  is a rooted graph such that  $V(G) = V(G_1) \cup \{a\}$ , and  $(G, a, b, c)$  is obtained from  $(G, a', b, c)$  by adding the edge  $aa'$ . Then  $(G, a, b, c)$  contains  $(\mathcal{O}_4, 2, 1, 3)$  or  $(\mathcal{O}_8, 2, 1, 3)$  as a rooted subdivision, or it contains either  $(\mathcal{O}_5, 1, 2, 3)$  or  $(\mathcal{O}_{14}, 3, 1, 2)$  as a rooted minor.*

**Lemma 4.2.7.** *Let  $(G_1, a', b', c)$  be an internally 4-connected rooted not  $c$ -planar. Assume  $a, b \notin V(G_1)$  and  $(G, a, b, c)$  is a rooted graph such that  $V(G) = V(G_1) \cup \{a, b\}$ , and  $(G, a, b, c)$  is obtained from  $(G, a', b', c)$  by adding the edge  $aa'$  and  $bb'$ . Then*

$(G, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 1, 2, 3)$ ,  $(\mathcal{O}_{13}, 1, 2, 3)$  or  $(\mathcal{O}_{14}, 3, 2, 1)$  which are listed in Figure 1.5 as a rooted minor.

**Lemma 4.2.8.** *Suppose  $(G, a, b, c)$  is not ac-planar. Let  $G = (G_1, G_2)$  be an internal 3-separation such that  $V(G_1) \cap V(G_2) = \{a, b', c\}$ ,  $\{a, b, c\} \subseteq V(G_2)$  and  $E(G_1) = E(G[V(G_1)])$ . Assume  $(G_1, a, b', c)$  is internally 4-connected not  $c$ -planar but  $a$ -planar. Then  $(G, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 2, 3, 1)$ ,  $(\mathcal{O}_5, 3, 1, 2)$ ,  $(\mathcal{O}_{12}, 3, 2, 1)$ ,  $(\mathcal{O}_{12}, 2, 1, 3)$ ,  $(\mathcal{O}_{15}, 3, 1, 2)$ ,  $(\mathcal{O}_{15}, 2, 1, 3)$ ,  $(\mathcal{O}_{23}, 1, 2, 3)$ ,  $(\mathcal{O}_{24}, 3, 2, 1)$ ,  $(\mathcal{O}_{25}, 1, 2, 3)$  as a rooted minor.*

*Proof.* Note that there exists a path  $P_b$  which is internally disjoint from  $G_1$  and it connects  $b'$  to  $b$  in  $G$ . Since  $(G_1, a, b', c)$  is internally 4-connected not  $c$ -planar, then by Lemma 4.1.5, and the fact that  $(G_1, a, b', c)$  is  $a$ -planar, someone can conclude that the rooted graph  $(G_1, a, b', c)$  contains  $(\mathcal{O}_i, 1, 2, 3)$  for some  $i = 1, 3, 4, 6, 7$  as a rooted subdivision.

So we consider five cases regarding whether  $(G_1, a, b, c')$  contains  $(\mathcal{O}_i, 1, 2, 3)$  for some  $i \in \{1, 3, 4, 6, 7\}$  as a rooted subdivision.

**Case 1**  $(G_1, a, b, c')$  contains  $(\mathcal{O}_1, 1, 2, 3)$  as a rooted subdivisions.

Let  $(G^*, 1, 7, 3)$  be isomorphic to  $(\mathcal{O}_1, 1, 2, 3)$ , i.e. we just rename the vertex 2 in  $(\mathcal{O}_1, 1, 2, 3)$  by 7 in  $(G^*, 1, 7, 3)$ . Let  $J$  be the multigraph obtained from  $(G^*, 1, 7, 3)$  by adding the edge 27 and two parallel 12 edges and two parallel 13 edges where 1, 2, 3 are terminal. Let  $\mathcal{C} = \{3651, 3641, 4657, 1572, 1472, 121, 131\}$  be a double cycle cover for  $J$ . It is easy to see that  $\mathcal{C}$  is 1-disk system, and  $H$  contains a  $J$ -subdivision, called  $S$ , as a subgraph where  $a, b, c$  correspond to 1, 2, 3, respectively. Note that  $H, J, S, \mathcal{C}$  satisfy the hypothesis of Theorem 2.2.10, so one of the outcomes listed in the statement of Theorem 2.2.10 holds. Note that since  $\{1, 3, 7\}$  is a vertex cut in  $H$  and  $E(G_1) = E(G[V(G_1)])$ , the outcomes (ii), (iv), (v), (vi), (vii) and (x) of Theorem 2.2.10 do not hold. By applying Lemma 2.1.3, we can see that (xi) of

Theorem 2.2.10 does not hold. Since  $J$  is not isomorphic to  $K_4$ , (xii) of Theorem 2.2.10 does not hold. Finally since  $G$  is not 1-planar, (xiii) of Theorem 2.2.10 does not hold. So either (i), (iii), (viii) or (ix) holds.

If (i) of Theorem 2.2.10 holds, i.e, there exists a path  $P$  from  $w \in \text{seg}[2, 7]$  to 3. If  $w \in \text{seg}(2, 7]$  then since we are going to show that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 3, 1, 2)$  as a rooted minor, we may assume that  $w = 7$ . Then it is easy to see that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 3, 1, 2)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, 6, 7)$ . If  $w = 2$  and  $P$  is an edge then  $(H, a, b, c)$  contains  $(\mathcal{O}_{25}, 1, 3, 2)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, 6, 7)$ , and if  $P$  is not an edge then there exists a path  $P'$  from  $x \in \text{Int}(P)$  to  $y \in S$ . Since  $\{1, 3, 7\}$  is a cut set,  $y \in \text{seg}(2, 7)$  which implies that  $(H, a, b, c)$  contains  $(\mathcal{O}_5, 3, 1, 2)$  as a rooted minor as before.

If (iii) or (viii) or (ix) of Theorem 2.2.10 holds, by contracting the edge 67,  $H$  contains either  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 2, 3, 1)$ ,  $(\mathcal{O}_{12}, 3, 2, 1)$  or  $(\mathcal{O}_{15}, 3, 1, 2)$  as a rooted minor, the case analysis is exactly the same as Case 2 in Lemma 4.1.5 by switching the roles of  $a$  and  $c$ .

**Case 2**  $(G_1, a, b', c)$  contains  $(\mathcal{O}_3, 1, 2, 3)$  as a rooted subdivision.

Let  $(G^*, 1, 8, 3)$  be isomorphic to  $(K'_{2,3}, 1, 2, 3)$ , i.e. we just rename the vertex 2 in  $(K'_{2,3}, 1, 2, 3)$  by 8 in  $(G^*, 1, 8, 3)$ . Let  $J$  be the multigraph obtained from  $(G^*, 1, 8, 3)$  by adding the edge 28 and two parallel 12 edges and two parallel 13 edges where 1, 2, 3 are terminal. Let  $\mathcal{C} = \{351, 341, 4653, 15682, 14682, 121, 131\}$  be a double cycle cover for  $J$ . It is easy to see that  $\mathcal{C}$  is 1-disk system, and  $H$  contains a  $J$ -subdivision, called  $S$ , as a subgraph where  $a, b, c$  correspond to 1, 2, 3, respectively. Note that there exists an edge in  $(H, a, b, c)$  from vertex 1 to a vertex in  $\text{seg}(6, 8)$ , called 7. It is easy to see that  $H, J, S, \mathcal{C}$  satisfy the hypothesis of Theorem 2.2.10, so one of the outcomes listed in the statement of Theorem 2.2.10 holds. Similarly as Case 1, we just analyzed the occurrence of (i) of Theorem 2.2.10, since either the other outcomes do not hold

or they lead to existence of  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 2, 3, 1)$ ,  $(\mathcal{O}_{12}, 3, 2, 1)$  or  $(\mathcal{O}_{15}, 3, 1, 2)$  as a rooted minor in  $(H, a, b, c)$

If (i) of Theorem 2.2.10 holds, i.e, there exists a path  $P$  from  $w \in \text{seg}[2, 8]$  to 3. If  $w \in \text{seg}(2, 8]$  then since we are going to show that  $(H, a, b, c)$  contains  $(\mathcal{O}_{15}, 2, 1, 3)$  as a rooted minor, we may assume that  $w = 8$ . Then it is easy to see that by contracting the edge 78,  $(H, a, b, c)$  contains  $(\mathcal{O}_{15}, 2, 1, 3)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, 6, 7)$ . If  $w = 2$  and  $P$  is an edge then by contracting the edge 78,  $(H, a, b, c)$  contains  $(\mathcal{O}_{24}, 3, 2, 1)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, 6, 7)$ , and if  $P$  is not an edge then there exists a path  $P'$  from  $x \in \text{Int}(P)$  to  $y \in S$ . Since  $\{1, 3, 8\}$  is a cut set,  $y \in \text{seg}(2, 8)$  which implies that  $(H, a, b, c)$  contains  $(\mathcal{O}_{15}, 2, 1, 3)$  as a rooted minor as before.

**Case 3**  $(G_1, a, b', c)$  contains  $(\mathcal{O}_4, 1, 2, 3)$  as a rooted subdivision.

Let  $(G^*, 1, 7, 3)$  be isomorphic to  $(\mathcal{O}_4, 1, 2, 3)$ , i.e. we just rename the vertex 2 in  $(\mathcal{O}_4, 1, 2, 3)$  by 7 in  $(G^*, 1, 7, 3)$ . Let  $J$  be the multigraph obtained from  $(G^*, 1, 7, 3)$  by removing the edge 16 and adding the edge 27 and two parallel 12 edges and two parallel 13 edges. Let  $\mathcal{C} = \{351, 341, 345, 456, 15672, 14672, 121, 131\}$  be a double cycle cover for  $J$ . It is easy to see that  $\mathcal{C}$  is 1-disk system, and  $H$  contains a  $J$ -subdivision, called  $S$ , as a subgraph where  $a, b, c$  correspond to 1, 2, 3, respectively. Note that there exists an edge in  $(H, a, b, c)$  from vertex 1 to a vertex in 6. Remember, we remove the edge 16 from  $J$ , just to make  $H, J, S, \mathcal{C}$  satisfying the hypothesis of Theorem 2.2.10. So one of the outcomes listed in the statement of Theorem 2.2.10 holds. Similarly as Case 1, we just analyzed the occurrence of (i) of Theorem 2.2.10, since the other outcomes either do not hold or they lead to existence of  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 2, 3, 1)$ ,  $(\mathcal{O}_{12}, 3, 2, 1)$  or  $(\mathcal{O}_{15}, 3, 1, 2)$  as a rooted minor in  $(H, a, b, c)$

If (i) of Theorem 2.2.10 holds, i.e, there exists a path  $P$  from  $w \in \text{seg}[2, 7]$  to 3. If  $w \in \text{seg}(2, 7]$  then since we are going to show that  $(H, a, b, c)$  contains  $(\mathcal{O}_{12}, 2, 1, 3)$

as a rooted minor, we may assume that  $w = 7$ . Then it is easy to see that by contracting the edge 67,  $(H, a, b, c)$  contains  $(\mathcal{O}_{12}, 2, 1, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, 6)$ . If  $w = 2$  and  $P$  is an edge then by contracting the edge 67,  $(H, a, b, c)$  contains  $(\mathcal{O}_{23}, 1, 2, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, 6)$ , and if  $P$  is not an edge then there exists a path  $P'$  from  $x \in \text{Int}(P)$  to  $y \in S$ . Since  $\{1, 3, 7\}$  is a cut set,  $y \in \text{seg}(2, 7)$  which implies that  $(H, a, b, c)$  contains  $(\mathcal{O}_{12}, 2, 1, 3)$  as a rooted minor as before.

**Case 4**  $(G_1, a, b', c)$  contains  $(\mathcal{O}_6, 1, 2, 3)$  as a rooted subdivision.

Let  $(G^*, 1, 7, 3)$  be isomorphic to  $(K'_{2,3}, 1, 2, 3)$ , i.e. we just rename the vertex 2 in  $(K'_{2,3}, 1, 2, 3)$  by 7 in  $(G^*, 1, 7, 3)$ . Let  $J$  be the multigraph obtained from  $(G^*, 1, 7, 3)$  by adding the edge 27 and two parallel 12 edges and two parallel 13 edges where 1, 2, 3 are terminal. Let  $\mathcal{C} = \{351, 341, 3456, 15672, 14672, 121, 131\}$  be a double cycle cover for  $J$ . It is easy to see that  $\mathcal{C}$  is 1-disk system, and  $H$  contains a  $J$ -subdivision, called  $S$ , as a subgraph where  $a, b, c$  correspond to 1, 2, 3, respectively. Note that there exists an edge in  $(H, a, b, c)$  from vertex 1 to a vertex in 7. It is easy to see that  $H, J, S, \mathcal{C}$  satisfying the hypothesis of Theorem 2.2.10. So one of the outcomes listed in the statement of Theorem 2.2.10 holds. Similarly as Case 1, we just analyzed the occurrence of (i) of Theorem 2.2.10, since the other outcomes either do not hold or they lead to existence of  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 2, 3, 1)$ ,  $(\mathcal{O}_{12}, 3, 2, 1)$  or  $(\mathcal{O}_{15}, 3, 1, 2)$  as a rooted minor in  $(H, a, b, c)$ .

If (i) of Theorem 2.2.10 holds, i.e, there exists a path  $P$  from  $w \in \text{seg}[2, 7]$  to 3. If  $w \in \text{seg}(2, 7]$  then since we are going to show that  $(H, a, b, c)$  contains  $(\mathcal{O}_{15}, 2, 1, 3)$  as a rooted minor, we may assume that  $w = 7$ . Then it is easy to see that  $(H, a, b, c)$  contains  $(\mathcal{O}_{15}, 2, 1, 3)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, 6, 7)$ . If  $w = 2$  and  $P$  is an edge then  $(H, a, b, c)$  contains  $(\mathcal{O}_{24}, 3, 2, 1)$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, 6, 7)$ , and if  $P$  is not an edge then there exists a path  $P'$



from  $x \in \text{Int}(P)$  to  $y \in S$ . Since  $\{1, 3, 7\}$  is a cut set,  $y \in \text{seg}(2, 7)$  which implies that  $(H, a, b, c)$  contains  $(\mathcal{O}_{15}, 2, 1, 3)$  as a rooted minor as before.

**Case 5**  $(G_1, a, b', c)$  contains  $(\mathcal{O}_7, 1, 2, 3)$  as a rooted subdivision.

The analysis of this case is the same as Case 3, so  $(H, a, b, c)$  contains  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 2, 3, 1)$ ,  $(\mathcal{O}_{12}, 3, 2, 1)$ ,  $(\mathcal{O}_{15}, 3, 1, 2)$ ,  $(\mathcal{O}_{12}, 2, 1, 3)$   $(\mathcal{O}_{23}, 1, 2, 3)$  as a rooted minor.

□

**Lemma 4.2.9.** *Suppose  $(G, a, b, c)$  is not ac-planar. Let  $G = (G_1, G_2)$  be an internal 3-separation such that  $V(G_1) \cap V(G_2) = \{a, b, c'\}$ ,  $\{a, b, c\} \subseteq V(G_2)$  and  $E(G_1) = E(G[V(G_1)])$ . Assume  $(G_1, a, b, c')$  is internally 4-connected not  $c'$ -planar but  $a$ -planar. Then  $(G, a, b, c)$  contains  $(\mathcal{O}_3, 2, 1, 3)$ ,  $(\mathcal{O}_4, 2, 1, 3)$ ,  $(\mathcal{O}_6, 2, 1, 3)$  as a rooted subdivision or it contains either  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 2, 1, 3)$ ,  $(\mathcal{O}_5, 3, 2, 1)$ ,  $(\mathcal{O}_5, 3, 1, 2)$ ,  $(\mathcal{O}_{12}, 3, 1, 2)$ ,  $(\mathcal{O}_{15}, 3, 2, 1)$ ,  $(\mathcal{O}_{15}, 3, 1, 2)$ , or  $(\mathcal{O}_{25}, 1, 3, 2)$  as a rooted minor.*

*Proof.* Note that there exists a path  $P_c$  which is internally disjoint from  $(G_1, a, b, c')$  and it connects  $c'$  to  $c$  in  $G$ . Since  $(G_1, a, b, c')$  is internally 4-connected not  $c$ -planar, then by Lemma 4.1.5, and the fact that  $(G_1, a, b, c')$  is  $a$ -planar, we can conclude that the rooted graph  $(G_1, a, b, c')$  contains  $(\mathcal{O}_i, 1, 2, 3)$  for some  $i = \{1, 3, 4, 6, 7\}$  as a rooted subdivision.

So we consider five cases regarding whether  $(G_1, a, b, c')$  contains  $(\mathcal{O}_i, 1, 2, 3)$  for some  $i = \{1, 3, 4, 6, 7\}$  as a rooted subdivision.

**Case 1**  $(G_1, a, b, c')$  contains  $(\mathcal{O}_1, 1, 2, 3)$  as a rooted subdivision.

Let  $(G^*, 1, 2, 7)$  be isomorphic to  $(\mathcal{O}_1, 1, 2, 3)$ , i.e. we just rename the vertex 3 in  $(\mathcal{O}_1, 1, 2, 3)$  by 7 in  $(G^*, 1, 2, 7)$ . Let  $J$  be the multigraph obtained from  $(G^*, 1, 2, 7)$  by adding the edge 37 and two parallel 12 edges and two parallel 13 edges where 1, 2, 3 are terminal. Let  $\mathcal{C} = \{241, 251, 14673, 15673, 2465, 131, 121\}$  be a double

cycle cover for  $J$ . It is easy to see that  $\mathcal{C}$  is 1-disk system, and  $H$  contains a  $J$ -subdivision, called  $S$ , as a subgraph where  $a, b, c$  correspond to 1, 2, 3, respectively. Note that  $H, J, S, \mathcal{C}$  satisfy the hypothesis of Theorem 2.2.10, so one of the outcomes listed in the statement of Theorem 2.2.10 holds. Note that since  $\{1, 2, 7\}$  is a vertex cut in  $H$  and  $E(G_1) = E(G[V(G_1)])$ , the outcomes (ii), (iv), (v), (vi), (vii) and (x) of Theorem 2.2.10 do not hold. By applying Lemma 2.1.3, we can see that (xi) of Theorem 2.2.10 does not hold. Since  $J$  is not isomorphic to  $K_4$ , (xii) of Theorem 2.2.10 does not hold. Finally since  $G$  is not 1-planar, (xiii) of Theorem 2.2.10 does not hold. So either (i), (iii), (viii) or (ix) holds.

If (i) of Theorem 2.2.10 holds, i.e, there exists a path  $P$  from  $w \in \text{seg}[3, 7]$  to 2. If  $w \in \text{seg}(3, 7)$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_3, 3, 1, 2)$  as a rooted subdivision. If  $w = 3$  and  $P$  is an edge then  $(H, a, b, c)$  contains  $(\mathcal{O}_6, 3, 1, 2)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (4, 5, 6)$ , and if  $P$  is not an edge then there exists a path  $P'$  from  $x \in \text{Int}(P)$  to  $y \in S$ . Since  $\{1, 2, 7\}$  is a cut set,  $y \in \text{seg}(3, 7)$  which implies that  $(H, a, b, c)$  contains  $(\mathcal{O}_3, 3, 1, 2)$  as a rooted subdivision as before.

If (iii) or (viii) or (ix) of Theorem 2.2.10 holds, by contracting the edge 67,  $H$  contains either  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 2, 1, 3)$ ,  $(\mathcal{O}_{15}, 3, 2, 1)$ ,  $(\mathcal{O}_{15}, 3, 1, 2)$  or  $(\mathcal{O}_{12}, 3, 1, 2)$  as a rooted minor. The case analysis is exactly the same as Case 1 in Lemma 4.1.5 where  $a, b, c$  play the roles of  $c, a, b$ , respectively, in Case 1 in Lemma 4.1.5.

**Case 2**  $(G_1, a, b, c')$  contains  $(\mathcal{O}_3, 1, 2, 3)$ ,  $(\mathcal{O}_4, 1, 2, 3)$  or  $(\mathcal{O}_6, 1, 2, 3)$  as a rooted subdivision.

Similarly as Case 2 in Lemma 4.1.5,  $(H, a, b, c)$  contains either  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 3, 2, 1)$ ,  $(\mathcal{O}_5, 3, 1, 2)$ ,  $(\mathcal{O}_5, 2, 1, 3)$ ,  $(\mathcal{O}_{12}, 3, 1, 2)$ ,  $(\mathcal{O}_{15}, 3, 2, 1)$  or  $(\mathcal{O}_{25}, 1, 3, 2)$ , as a rooted minor. Note that  $a, b, c$  play the roles of  $c, a, b$ , respectively, in Case 2 in Lemma 4.1.5.

**Case 3**  $(G_1, a, b, c')$  contains  $(\mathcal{O}_7, 1, 2, 3)$  as a rooted subdivision.

Let  $(G^*, 1, 2, 6)$  be isomorphic to  $(\mathcal{O}_7, 1, 2, 3)$ , i.e. we just rename the vertex 3 in

$(\mathcal{O}_7, 1, 2, 3)$  by 6 in  $(G^*, 1, 2, 6)$ . Let  $J$  be the multigraph obtained from  $(G^*, 1, 2, 7)$  by adding the edge 36 and two parallel 12 edges and two parallel 13 edges where 1, 2, 3 are terminal. Let  $\mathcal{C} = \{125, 142, 245, 456, 1463, 1563\}$  be a double cycle cover for  $J$ . It is easy to see that  $\mathcal{C}$  is 1-disk system, and  $H$  contains a  $J$ -subdivision, called  $S$ , as a subgraph where  $a, b, c$  correspond to 1, 2, 3, respectively. Note that  $H, J, S, \mathcal{C}$  satisfy the hypothesis of Theorem 2.2.10, so one of the outcomes listed in the statement of Theorem 2.2.10 holds. Similarly as Case 1, we can see the only possible outcomes are (i), (iii), (viii) or (ix) of Theorem 2.2.10. Similarly as the argument presented in Case 1 occurrence of (iii) or (viii) or (ix) of Theorem 2.2.10 lead to the fact that  $(H, a, b, c)$  contains either  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 2, 1, 3)$ ,  $(\mathcal{O}_{15}, 3, 2, 1)$ ,  $(\mathcal{O}_{15}, 3, 1, 2)$  or  $(\mathcal{O}_{12}, 3, 1, 2)$  as a rooted minor. So we only focus on analyzing on occurrence of (i) in Theorem 2.2.10.

Thus, let  $P$  be a path from  $w \in \text{seg}[3, 6]$  to 2. If  $w \in \text{seg}(3, 6)$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_3, 3, 1, 2)$  as a rooted subdivision. If  $w = 6$  then  $(H, a, b, c)$  contains  $(\mathcal{O}_4, 3, 1, 2)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (4, 5, 6)$ . If  $w = 3$  and  $P$  is an edge then  $(H, a, b, c)$  contains  $(\mathcal{O}_6, 3, 1, 2)$  as a rooted subdivision with signature  $(4, 5, 6) \leftrightarrow (4, 5, 6)$ , and if  $P$  is not an edge then there exists a path  $P'$  from  $x \in \text{Int}(P)$  to  $y \in S$ . Since  $\{1, 2, 6\}$  is a cut set,  $y \in \text{seg}(3, 6]$  which implies that  $(H, a, b, c)$  contains either  $(\mathcal{O}_3, 3, 1, 2)$  or  $(\mathcal{O}_4, 3, 1, 2)$  as a rooted subdivision as before.  $\square$

**Theorem 4.2.10.** *Let  $(G, a, b, c)$  be a 3-connected rooted graph. If  $(G, a, b, c)$  is minor minimal not ac-planar rooted graph then  $(G, a, b, c)$  contains contains  $(\mathcal{O}_i, \beta, 1, \gamma)$  for some  $i \in \{3, 4, 6, 8, 9, 10, 11\}$ ,  $\{\beta, \gamma\} = \{2, 3\}$  as a rooted subdivision or  $G$  contains  $(\mathcal{O}_j, \beta, 1, \gamma)$  for some  $j \in \{2, 5, 12, 13, 15, 16, 24, 26, 28\}$ ,  $\{\beta, \gamma\} = \{2, 3\}$  or  $(\mathcal{O}_k, \alpha, 2, \gamma)$  for some  $k \in \{5, 12, 14, 15, 16, 17, 18, 19, 23, 24, 25, 27\}$ ,  $\{\alpha, \gamma\} = \{1, 3\}$ , or  $(\mathcal{O}_\ell, \alpha, 3, \beta)$  for some  $\ell \in \{5, 25\}$ ,  $\{\alpha, \beta\} = \{1, 2\}$  as a rooted minor. These are listed in Figure 1.4 and 1.5 as a rooted minor.*

*Proof.* The proof strategy is similar to the one presented in the proof of Lemma 4.1.6. We consider two cases, either  $(G, a, b, c)$  is internally 4-connected or it is not internally

4-connected. If  $(G, a, b, c)$  is internally 4-connected then by Lemma 4.2.2,  $(G, a, b, c)$  contains  $(\mathcal{O}_i, \beta, 1, \gamma)$  for some  $i \in \{3, 4, 6, 8, 9, 10, 11\}$ ,  $\{\beta, \gamma\} = \{2, 3\}$  as a rooted subdivision or  $G$  contains  $(\mathcal{O}_j, \beta, 1, \gamma)$  for some  $j \in \{2, 5, 12, 13, 15, 16, 24, 26, 28\}$ ,  $\{\beta, \gamma\} = \{2, 3\}$  or  $(\mathcal{O}_k, \alpha, 2, \gamma)$  for some  $k \in \{5, 12, 15, 16, 17, 18, 19, 23, 24, 25, 27\}$ ,  $\{\alpha, \gamma\} = \{1, 3\}$ , or  $(\mathcal{O}_5, 2, 3, 1)$  as a rooted minor.

So assume that  $(G, a, b, c)$  is not internally 4-connected. Let  $G = (G_1, G_2)$  be an internal 3-separation in  $G$  such that  $V(G_1) \cap V(G_2) = \{a', b', c'\}$  and  $\{a, b, c\} \subset V(G_1)$  and the number of vertices of  $G_1$  is as small as possible. Since  $H$  is 3-connected we may assume there exist three disjoint paths  $P_a, P_b, P_c$  in  $G_1$  connecting  $a, b, c$  to  $a', b', c'$ , respectively. The same argument presented in the proof of Lemma 4.1.6 shows that  $G_1$  contains a double fork with feet on  $a', b', c'$ . If  $c \neq c'$  and  $a \neq a'$  then it is easy to see that  $(G, a, b, c)$  contains either  $(\mathcal{O}_8, 2, 1, 3)$  as a rooted subdivision or  $(G, a, b, c)$  contains  $(\mathcal{O}_{13}, 1, 2, 3)$  as a rooted minor. So from now on, we may assume either  $c = c'$  or  $a = a'$ , but for preserving symmetry, we do not specify which one until we need them.

By symmetry, we consider three main cases, either  $(G_1, a', b', c')$  is not  $a'c'$ -planar,  $(G_1, a', b', c')$  is not  $c'$ -planar but it is  $a'$ -planar, or  $(G_1, a', b', c')$  is both  $a'$ -planar and  $c'$ -planar.

**Case 4**  $(G_1, a', b', c')$  is not  $a'c'$ -planar.

Here, by symmetry we assume that  $c = c'$ . By Lemma 4.2.2, we know that  $(G_1, a', b', c)$  contains  $(\mathcal{O}_i, \beta, 1, \gamma)$  for some  $i \in \{3, 4, 6, 8, 9, 10, 11\}$ ,  $\{\beta, \gamma\} = \{2, 3\}$  as a rooted subdivision or  $G$  contains  $(\mathcal{O}_j, \beta, 1, \gamma)$  for some  $j \in \{2, 5, 12, 13, 15, 16, 24, 26, 28\}$ ,  $\{\beta, \gamma\} = \{2, 3\}$  or  $(\mathcal{O}_k, \alpha, 2, \gamma)$  for some  $k \in \{5, 12, 15, 16, 17, 18, 19, 23, 24, 25, 27\}$ ,  $\{\alpha, \gamma\} = \{1, 3\}$ , or  $(\mathcal{O}_5, 2, 3, 1)$  as a rooted minor.

Note that the if  $a \neq a'$  then the fact that  $(G, a, b, c)$  is minor minimal not  $ac$ -planar implies that the edges in the path  $P_a$  is contractible if and only if there is no

edge between  $a'$  and  $c$  and no edge between  $a'$  and  $b$  in  $(G_1, a', b', c)$ . It is easy to see that none of the graphs listed above have an edge between 2 and 3 or 2 and 1 except  $(\mathcal{O}_6, 2, 1, 3)$  and  $(\mathcal{O}_{11}, 2, 1, 3)$  which implies that  $(G, a, b, c)$  contains either  $(\mathcal{O}_3, 2, 1, 3)$  or  $(\mathcal{O}_4, 2, 1, 3)$  as a rooted subdivision.

Thus assume  $a = a'$  and the fact that  $G \neq G_1$  implies that  $b \neq b'$ . the fact that  $(G, a, b, c)$  is minor minimal not  $ac$ -planar implies that the edges in the path  $P_b$  is contractible if and only if there is no edge between  $b'$  and  $c$  and no edge between  $b'$  and  $a$  in  $(G_1, a, b', c)$ . It is easy to see that none of the graphs listed above have an edge between 1 and 3 or 1 and 2 except  $(\mathcal{O}_6, 2, 1, 3), (\mathcal{O}_{11}, 2, 1, 3)$  which implies that  $(G, a, b, c)$  contains either  $(\mathcal{O}_5, 1, 2, 3)$  or  $(\mathcal{O}_{12}, 2, 1, 3)$  as a rooted minor.

**Case 5**  $(G_1, a', b', c')$  is not  $c'$ -planar but it is  $a'$ -planar.

If  $a \neq a'$  then by Lemma 4.2.6 and Lemma 4.2.7,  $(G_1, a, b, c)$  contains either  $(\mathcal{O}_8, 2, 1, 3), (\mathcal{O}_4, 2, 1, 3)$  as a rooted subdivision, or it contains either  $(\mathcal{O}_5, 1, 2, 3), (\mathcal{O}_{13}, 1, 2, 3)$  or  $(\mathcal{O}_{14}, 3, 2, 1)$  as a rooted minor.

So we assumed that  $a = a'$ . Note that by applying Lemma 4.2.8, 4.2.9 or 4.2.5 to the case  $b \neq b', c = c'$  or  $b \neq b', c = c'$ , or  $b \neq b', c \neq c'$ , respectively, we can see that  $(G, a, b, c)$  contains either  $(\mathcal{O}_3, 2, 1, 3), (\mathcal{O}_4, 2, 1, 3)$  or  $(\mathcal{O}_6, 2, 1, 3)$  as a rooted subdivision or it contains either  $(\mathcal{O}_i, 1, 2, 3), (\mathcal{O}_j, 1, 3, 2), (\mathcal{O}_k, 2, 1, 3), (\mathcal{O}_\ell, 2, 3, 1), (\mathcal{O}_m, 3, 1, 2), (\mathcal{O}_n, 3, 2, 1)$  for some  $i \in \{2, 5, 12, 15, 23, 25\}, j \in \{2, 5, 25\}, k \in \{5, 12, 15\}, \ell \in \{2, 5\}, m \in \{2, 5, 12, 15\}, n \in \{2, 5, 12, 15, 24, 25\}$  as a rooted minor.

**Case 6**  $(G_1, a', b', c')$  is  $c'$ -planar and  $a'$ -planar.

Since  $(G, a, b, c)$  is minor minimal not  $ac$ -planar, then  $(G_1, a', b', c')$  is isomorphic to  $(K_{2,3}, 1, 2, 3)$  where  $a, b, c$  correspond to 1, 2, 3. Without loss of generality, assume  $c = c'$ . If  $a \neq a'$  then by switching the roles of  $a$  and  $b$  in the proof of Lemma 4.1.5 (note that in analyzing both Case 1 and Case 2 in the proof of Lemma 4.1.5, we have

the assumption of  $b \neq b'$ ), we infer  $(G_1, a, b, c)$  contains  $(\mathcal{O}_3, 2, 1, 3)$  or  $(\mathcal{O}_6, 2, 1, 3)$  as a rooted subdivision or it contains either  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 1, 2, 3)$ ,  $(\mathcal{O}_5, 2, 1, 3)$ ,  $(\mathcal{O}_5, 3, 1, 2)$ ,  $(\mathcal{O}_{12}, 2, 1, 3)$ ,  $(\mathcal{O}_{15}, 1, 2, 3)$ ,  $(\mathcal{O}_{15}, 2, 1, 3)$  or  $(\mathcal{O}_{25}, 2, 3, 1)$  as a rooted minor.

So we may assume that  $a = a'$ . Because  $\{a, b, c\} \neq \{a', b', c'\}$ , we have  $b \neq b'$ . Now, the rest of the proof is similar to the strategy, we had in Lemma 4.2.2. In the first step, we determine the lists  $\mathcal{L}_S, \mathcal{L}_M$  of rooted graphs such that  $(G, a, b, c)$  must contain a member of these lists as a rooted subdivision or minor, respectively, for not being  $c$ -planar. Then, by assuming that  $(G, a, b, c)$  contains one of the rooted graph in  $\mathcal{L}_S$  as a rooted subdivision, we find a list such that  $(G, a, b, c)$  must contains a member of the list as a rooted minor for not being  $ac$ -planar. Note that the proof presented here is slightly different than the proof of Lemma 4.2.2, and difference comes from the fact that  $\{a, b', c\}$  is a cut set in  $(G, a, b, c)$ .

Now by Lemma 4.1.5,  $\mathcal{L}_S = \{(\mathcal{O}_3, 1, 2, 3), (\mathcal{O}_6, 1, 2, 3)\}$  and  $\mathcal{L}_M = \{(\mathcal{O}_i, \alpha, \beta, 3) : i = 2, 5, 12, 15, \{\alpha, \beta\} = \{1, 2\}\} \cup \{(\mathcal{O}_5, 3, 1, 2), (\mathcal{O}_5, 3, 2, 1), (\mathcal{O}_5, 1, 3, 2), (\mathcal{O}_5, 2, 3, 1)\} \cup \{(\mathcal{O}_{25}, 3, 1, 2), (\mathcal{O}_{25}, 3, 2, 1)\}$ , and this finishes the first step. In the second step, we know that  $(G, a, b, c)$  contains either  $(\mathcal{O}_3, 1, 2, 3)$  or  $(\mathcal{O}_6, 1, 2, 3)$  as a subdivision, where  $\{1, 3, 6\}$  is a cut set in  $(G, a, b, c)$ . Note that the vertex 6 in  $(\mathcal{O}_3, 1, 2, 3)$  or  $(\mathcal{O}_6, 1, 2, 3)$  corresponds to vertex  $b'$  in  $(G, a, b, c)$ .

Assume  $(G, a, b, c)$  contains  $(\mathcal{O}_3, 1, 2, 3)$  as a rooted subdivision. Let  $J$  be the multigraph obtained from  $(K'_{2,3}, 1, 2, 3)$  by adding the two parallel 13 edges and two parallel 21 edges . Let  $\mathcal{C} = \{134, 135, 1562, 1462, 3564, 131, 121\}$  be a double cycle cover for  $J$ . It is easy to see that  $\mathcal{C}$  is a 1-disk system, and  $(H, a, b, c)$  contains a  $(J, 1, 2, 3)$ -subdivision, called  $S$ , as a subgraph. It is easy to see that  $H, J, S, \mathcal{C}$  satisfy conditions of Theorem 2.2.10. So one of the outcomes listed in the statement of Theorem 2.2.10 holds. Moreover,  $(G, a, b, c)$  contains an edge  $Q$  from 1 to vertex  $7 \in \text{seg}(2, 6)$ . Since  $\{1, 3, 6\}$  is a cut set, similarly as Lemma 4.1.5, we can see the only possible outcomes are (i), (iii), (viii) or (ix) of Theorem 2.2.10. Note that by

switching the role of  $a$  and  $c$ , as Case 2 of Lemma 4.1.5 shows, the occurrence of (iii), (viii) or (ix) of Theorem 2.2.10 lead to the fact that  $(H, a, b, c)$  contains either  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 3, 1, 2)$ ,  $(\mathcal{O}_5, 3, 2, 1)$ ,  $(\mathcal{O}_5, 2, 3, 1)$ ,  $(\mathcal{O}_{12}, 3, 2, 1)$  or  $(\mathcal{O}_{15}, 3, 1, 2)$  as a rooted minor. So we only focus on analyzing on occurrence of (i) in Theorem 2.2.10.

For analyzing on occurrence of (i) in Theorem 2.2.10, we can follow the same analysis presented in the in the second possibility of Case 2.1 in Lemma 4.2.2 to show that  $(G, a, b, c)$  contains either  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_{15}, 2, 1, 3)$  or  $(\mathcal{O}_{24}, 3, 1, 2)$ . Note that the fact that  $\{1, 3, 6\}$  is a cut set in  $(G, a, b, c)$  makes the case analysis presented in the second possibility of Case 2.1 in Lemma 4.2.2 shorter.

Now assume  $(G, a, b, c)$  contains  $(\mathcal{O}_6, 1, 2, 3)$  as a rooted subdivision. Let  $H, J, S, \mathcal{C}$  be as defined above. Moreover,  $(G, a, b, c)$  contains an edge with one ends at  $a$  and another one at  $b$ . So one of the outcomes listed in the statement of Theorem 2.2.10 holds. Since  $\{1, 3, 6\}$  is a cut set, similarly as Lemma 4.1.5, we can see the only possible outcomes are (i), (iii), (viii) or (ix) of Theorem 2.2.10. Note that by switching the role of  $a$  and  $c$ , as Case 2 of Lemma 4.1.5 shows, the occurrence of (iii), (viii) or (ix) of Theorem 2.2.10 lead to the fact that  $(H, a, b, c)$  contains either  $(\mathcal{O}_2, 1, 2, 3)$ ,  $(\mathcal{O}_5, 3, 1, 2)$ ,  $(\mathcal{O}_5, 3, 2, 1)$ ,  $(\mathcal{O}_5, 2, 3, 1)$ ,  $(\mathcal{O}_{12}, 3, 2, 1)$  or  $(\mathcal{O}_{15}, 3, 1, 2)$  as a rooted minor. Thus we only focus on analyzing on the occurrence of (i) in Theorem 2.2.10.

Thus, let  $P$  be a path from  $w \in \text{seg}[2, 6)$  to  $z = 3$ . If  $w \in \text{seg}(2, 6)$  then  $(G, a, b, c)$  contains  $(\mathcal{O}_{24}, 3, 1, 2)$  as a rooted minor. If  $w = 2$  and  $P$  is an edge then  $(G, a, b, c)$  contains  $(\mathcal{O}_{26}, 2, 1, 3)$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, 6)$ , and if  $P$  is not an edge then there exists a path  $P'$  from  $x \in \text{Int}(P)$  to  $y \in S$ . Since  $\{1, 3, 6\}$  is a cut set,  $y \in \text{seg}(2, 6]$  which implies that  $(H, a, b, c)$  contains  $(\mathcal{O}_{24}, 3, 1, 2)$  as a rooted minor as before.  $\square$

Now the proof of Theorem 1.8.4 follows from Lemma 4.2.2 and Theorem 4.2.10.

### 4.3 Obstructions for $abc$ -planarity

**Lemma 4.3.1.** *Let  $(G_1, a, b, c)$  be an internally 4-connected rooted not  $ac$ -planar graph. Assume  $b \notin V(G_1)$  and  $(G, a, b, c)$  is a rooted graph such that  $V(G) = V(G_1) \cup \{b\}$ , and  $(G, a, b, c)$  is obtained from  $(G, a, b', c)$  by adding the edge  $bb'$ . Then  $(G, a, b, c)$  contains  $(\mathcal{O}_i, \alpha, \beta, \gamma)$  for some  $i \in \{2, 5, 12, 13, 14, 22\}$  where  $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$ , as a rooted minor.*

**Theorem 4.3.2.** *Let  $(G, a, b, c)$  be an internally 4-connected. If  $(G, a, b, c)$  is minor minimal not  $ac$ -planar rooted graph then  $(G, a, b, c)$  contains  $(\mathcal{O}_i, \alpha, \beta, \gamma)$ ,  $i \in \{2, 5, 12, 13, 14, 15, 16, 17, 18, 19, 20, 23, 24, 25, 26, 27, 28\}$  where  $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$ , as a rooted minor.*

*Proof.* In the proof of this theorem because of the symmetry between  $a, b, c$ , we do not list the order of the root when we are going to show that  $(G, a, b, c)$  contains one of the rooted graph listed in the statement of the theorem as a rooted minor.

Our proof strategy is that in the first step, we determine the lists  $\mathcal{L}_S, \mathcal{L}_M$  of rooted graphs such that  $(G, a, b, c)$  must contain a member of these lists as a rooted subdivision or minor, respectively, for not being  $ac$ -planar. Then, by assuming that  $(G, a, b, c)$  contains one of the rooted graph in  $\mathcal{L}_S$  as a rooted subdivision, we find the list of graphs mentioned in the statement of the theorem.

The analysis presented in Lemma 4.2.2 shows that  $\mathcal{L}_S = \{(\mathcal{O}_i, 2, 1, 3) : i = 3, 4, 6, 8, 9, 10, 11\}$  as a rooted subdivision. This completes the first step.

Now we consider the following seven cases:

**Case 1**  $(G, a, b, c)$  contains  $(\mathcal{O}_3, 2, 1, 3)$  as a rooted subdivision.

The analysis of this case is exactly the same as Case 2 in Lemma 4.2.2 which shows that  $(G, a, b, c)$  contains either  $\mathcal{O}_2, \mathcal{O}_5, \mathcal{O}_{12}, \mathcal{O}_{13}, \mathcal{O}_{15}, \mathcal{O}_{16}, \mathcal{O}_{17}, \mathcal{O}_{18}$  or  $\mathcal{O}_{19}$  as a rooted minor.

**Case 2**  $(G, a, b, c)$  contains  $(\mathcal{O}_4, 2, 1, 3)$  as a rooted subdivision.



The analysis of this case is exactly the same as Case 3 in Lemma 4.2.2 which shows that  $(G, a, b, c)$  contains either  $\mathcal{O}_2, \mathcal{O}_5, \mathcal{O}_{12}, \mathcal{O}_{13}, \mathcal{O}_{15}, \mathcal{O}_{16}$  or  $\mathcal{O}_{23}$  as a rooted minor.

**Case 3**  $(G, a, b, c)$  contains  $(\mathcal{O}_6, 2, 1, 3)$  as a rooted subdivision.

The analysis of this case is exactly the same as Case 4 in Lemma 4.2.2 which shows that  $(G, a, b, c)$  contains either  $\mathcal{O}_2, \mathcal{O}_5, \mathcal{O}_{15}, \mathcal{O}_{23}, \mathcal{O}_{24}, \mathcal{O}_{25}, \mathcal{O}_{26}, \mathcal{O}_{27}$  or  $\mathcal{O}_{28}$  as a rooted minor.

**Case 4**  $(G, a, b, c)$  contains  $(\mathcal{O}_8, 1, 2, 3)$  as a rooted subdivision.

The analysis of this case is exactly the same as Case 3 in Lemma 4.1.3 which shows that  $(G, a, b, c)$  contains either  $\mathcal{O}_2, \mathcal{O}_5, \mathcal{O}_{13}, \mathcal{O}_{14}, \mathcal{O}_{15}, \mathcal{O}_{25}$  or  $\mathcal{O}_{21}$  as a rooted minor.

Let the multigraph  $H$  be obtained from  $(G, a, b, c)$  by adding two parallel  $ab$  edges and two parallel  $bc$  edges, if they do not exist.

**Case 5**  $(G, a, b, c)$  contains  $(\mathcal{O}_9, 2, 1, 3)$  as a rooted subdivision.

Let  $J$  be the multigraph obtained from  $(\mathcal{O}_9, 1, 2, 3)$  by adding the two parallel 12 edges and two parallel 23 edges. Let  $\mathcal{C} = \{142, 152, 342, 352, 346, 365, 156, 146, 212, 232\}$  be a double cycle cover for  $J$ . It is easy to see that  $\mathcal{C}$  is a 2-disk system, and  $(H, a, b, c)$  contains a  $(J, 1, 2, 3)$ -subdivision, called  $S$ , as a subgraph. It is easy to see that  $H, J, S, \mathcal{C}$  satisfy conditions of Theorem 3.0.17. So one of the outcomes of Theorem 3.0.17 holds. Similarly as Case 1 in Lemma 4.1.3, (ii), (iii), (vi), (vii), (ix), (x), (xi) of Theorem 3.0.17 do not hold. Moreover, (iv) and (viii) of Theorem 3.0.17 do not hold. Now, we are going to analysis of other possible outcomes, i.e. (i) and (v) of Theorem 3.0.17.

**Case 5.1** The outcome (i) of Theorem 3.0.17 holds, i.e. there exists an  $S$ -jump.

Let  $P$  be a path with ends  $w, z \in S$ . Since we are going to show that  $(H, a, b, c)$  contains either  $\mathcal{O}_2, \mathcal{O}_{12}, \mathcal{O}_{15}$  or  $\mathcal{O}_{28}$  as a minor, by using symmetry we just have four possibilities.

In the first possibility, if  $w = 4, z = 5$  then  $(H, a, b, c)$  contains  $\mathcal{O}_{12}$  as a rooted

minor with signature  $(4, 5, 6) \leftrightarrow (4, 6, 5)$ .

In the second possibility, if  $w = 2, z = 6$  then  $(H, a, b, c)$  contains  $\mathcal{O}_2$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, 6)$ .

In the third possibility, if  $w = 1, z \in \text{seg}(3, 4)$  then  $(H, a, b, c)$  contains  $\mathcal{O}_2$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (5, 6, z)$ .

In the fourth possibility, assume  $w = 1, z = 3$ . If  $P$  is an edge then  $(H, a, b, c)$  contains  $\mathcal{O}_{28}$  as a rooted minor, and if  $P$  is not an edge then there is a path  $P'$  from  $x \in \text{Int}(P)$  to  $y \in S$ . Since we are going to show that  $(H, a, b, c)$  contains  $\mathcal{O}_2$  or  $\mathcal{O}_{15}$  as a rooted minor, we can assume either  $y = 4, 2$  or  $6$ . If  $y = 4$  or  $y = 2$  then  $(H, a, b, c)$  contains  $\mathcal{O}_2$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (x, 6, 5)$ , and if  $y = 6$  then  $(H, a, b, c)$  contains  $\mathcal{O}_{15}$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (x, 4, y, 5)$ .

**Case 5.2** The outcome (v) of Theorem 3.0.17 holds, i.e. there exists a free solid  $S$ -cross.

Let  $(P_1, P_2)$  be the free cross with feet  $u, w, v, z$ . By symmetry, we may assume that  $u, w, v, z$  belong to the disk 146. Now by contracting the edge 56 and applying the same analysis as presented in Case 5.3 in Lemma 4.2.2 shows that  $(H, a, b, c)$  contains  $\mathcal{O}_5$  as a rooted minor.

**Case 6**  $(G, a, b, c)$  contains  $(\mathcal{O}_{10}, 2, 1, 3)$  as a rooted subdivision.

Let  $J$  be the multigraph obtained from  $(\mathcal{O}_{10}, 1, 2, 3)$  by adding the two parallel 12 edges and two parallel 23 edges. Let  $\mathcal{C} = \{2412, 2561, 1476, 3765, 2473, 253, 212, 232\}$  be a double cycle cover for  $J$ . It is easy to see that  $\mathcal{C}$  is a 2-disk system, and  $(H, a, b, c)$  contains a  $(J, 1, 2, 3)$ -subdivision, called  $S$ , as a subgraph. It is easy to see that  $H, J, S, \mathcal{C}$  satisfy conditions of Theorem 3.0.17. So one of the outcomes of Theorem 3.0.17 holds. Similarly to Case 1 in Lemma 4.1.3, (ii), (iii), (vi), (vii), (ix), (x), (xi) of Theorem 3.0.17 do not hold. Now, we are going to analysis of other possible outcomes, i.e. (i), (iv), (v) and (viii) of Theorem 3.0.17.

**Case 6.1** The outcome (i) of Theorem 3.0.17 holds, i.e. there exists an  $S$ -jump.

Let  $P$  be a path with ends  $w, z \in S$ . Since we are going to show that  $(H, a, b, c)$  contains either  $\mathcal{O}_2, \mathcal{O}_5, \mathcal{O}_{17}, \mathcal{O}_{18}$  or  $\mathcal{O}_{27}$  as a minor, by using symmetry we just have six possibilities.

In the first possibility, if  $w \in \text{seg}(1, 4), z = 5$  then by contracting 67,  $(H, a, b, c)$  contains  $\mathcal{O}_5$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (4, 5, w, 6)$ .

In the second possibility, if  $w \in \text{seg}(1, 4), z = 3$  then  $(H, a, b, c)$  contains  $\mathcal{O}_2$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (5, 7, w)$ .

In the third possibility, if  $w = 4, z = 5$  then  $(H, a, b, c)$  contains  $\mathcal{O}_5$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (5, 7, 6, 4)$ .

In the fourth possibility, if  $w = 2, z \in \text{seg}(6, 7)$  then  $(H, a, b, c)$  contains  $\mathcal{O}_2$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, z)$ .

In the fifth possibility, if  $w \in \text{seg}(2, 4), z = 6$  then  $(H, a, b, c)$  contains  $\mathcal{O}_{18}$  as a rooted minor with signature  $(4, 5, 6, 7, 8) \leftrightarrow (6, 4, 5, w, 7)$ .

In the sixth possibility, assume  $w = 1, z = 3$ . If  $P$  is an edge then  $(H, a, b, c)$  contains  $\mathcal{O}_{27}$  as a rooted minor, and if  $P$  is not an edge then there is a path  $P'$  from  $x \in \text{Int}(P)$  to  $y \in S$ . Since we are going to show that  $(H, a, b, c)$  contains  $\mathcal{O}_2$  or  $\mathcal{O}_{16}$  as a rooted minor, we can assume either  $y = 2, 4$  or  $6$ . If  $y = 2$  then  $(H, a, b, c)$  contains  $\mathcal{O}_2$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, x)$ , and if  $y = 4$  then by contracting 24, 67 the rooted graph  $(H, a, b, c)$  contains  $\mathcal{O}_2$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 6, x)$ . Finally, if  $y = 6$  then  $(H, a, b, c)$  contains  $\mathcal{O}_{17}$  as a rooted minor with signature  $(4, 5, 6, 7, 8) \leftrightarrow (8, 7, 5, 4, 6)$ .

**Case 6.2** The outcome (iv) of Theorem 3.0.17 holds, i.e. there exists weakly free  $S$ -cross anchored at  $c$ .

The analysis of this case is exactly the same as Case 1.2 in Lemma 4.1.3 which shows that  $(G, a, b, c)$  contains  $\mathcal{O}_5$  as a rooted minor.

**Case 6.3** The outcome (v) of Theorem 3.0.17 holds, i.e. there exists a free solid

$S$ -cross.

Let  $(P_1, P_2)$  be the free cross with feet  $u, w, v, z$ . By symmetry either  $u, w, v, z$  belong to the disk 1476, or to the disk 1652.

First, assume that  $u, w, v, z$  belong to the disk 1476. Since we are going to show that  $(H, a, b, c)$  contains  $\mathcal{O}_5$  as a rooted minor, we can assume either  $u = 4, w \in \text{seg}(1, 6), v = 6, z = 7$  implying  $(H, a, b, c)$  contains  $\mathcal{O}_5$  as a rooted minor by contracting 56 and with signature  $(4, 5, 6, 7) \leftrightarrow (w, 4, 7, 5)$ , or  $u = 1, w \in \text{seg}(1, 6), v \in \text{seg}(6, 7), z = 7$  implying  $(H, a, b, c)$  contains  $\mathcal{O}_5$  as a rooted minor by contracting 47, 56 and with signature  $(4, 5, 6, 7) \leftrightarrow (v, w, 5, 4)$ , or  $u = 1, w = 6, v \in \text{seg}(6, 7), z = 4$  implying  $(H, a, b, c)$  contains  $\mathcal{O}_5$  as a rooted minor with signature  $(4, 5, 6, 7) \leftrightarrow (6, 7, v, 4)$ .

Second, assume that  $u, w, v, z$  belong to the disk 1652. Since we are going to show that  $(H, a, b, c)$  contains  $\mathcal{O}_2$  as a rooted minor, we can assume  $u = 1, w \in \text{seg}(1, 5), v = 5, z = 6$  implying  $(H, a, b, c)$  contains  $\mathcal{O}_2$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, 6)$ .

**Case 6.4** The outcome (viii) of Theorem 3.0.17 holds, i.e. there exists an essential  $S$ -triad with one feet on  $c$ .

Let  $(P_1, P_2, P_3)$  be an essential triad with center 0 and feet on  $v_1, v_2, 1$ , respectively. By considering symmetry and the fact that we are going to show  $(H, a, b, c)$  contains  $\mathcal{O}_2, \mathcal{O}_{18}$  as a rooted minor, we analyze the following possibilities: If  $v_1 = 1, v_2 = 7$  in which case  $(H, a, b, c)$  contains  $\mathcal{O}_2$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, 0)$ . If  $v_1 = 7, v_2 = 6$  then  $(H, a, b, c)$  contains  $\mathcal{O}_2$  as a rooted minor with signature  $(4, 5, 6) \leftrightarrow (4, 5, 0)$ . If  $v_1 = 4, v_2 = 6$  then  $(H, a, b, c)$  contains  $\mathcal{O}_{18}$  as a rooted minor with signature  $(4, 5, 6, 7, 8) \leftrightarrow (6, 4, 5, 0, 7)$ .

**Case 7**  $(G, a, b, c)$  contains  $(\mathcal{O}_{11}, 2, 1, 3)$  as a rooted subdivision.

Note that  $(G, a, b, c)$  contains  $(\mathcal{O}_7, 2, 1, 3)$  as a subdivision, therefore by applying

the same case analysis presented in Case 5 of Lemma 4.2.2, we conclude that  $(G, a, b, c)$  contains either  $(\mathcal{O}_6, 1, 2, 3)$  or  $(\mathcal{O}_{11}, 1, 2, 3)$  as a rooted subdivision or it contains  $\mathcal{O}_2, \mathcal{O}_5$  as a rooted minor, using the fact that  $(G, a, b, c)$  also contains the edge  $bc$  (Note that adding the edge  $bc$  is equivalent to adding the edge 13 to  $(\mathcal{O}_{11}, 1, 2, 3)$  and also adding the edge 13 to  $(\mathcal{O}_6, 1, 2, 3)$ ), we can say that  $(G, a, b, c)$  contains  $\mathcal{O}_{26}, \mathcal{O}_{20}, \mathcal{O}_2$  or  $\mathcal{O}_5$  as a rooted minor. Note that  $\mathcal{O}_{26}, \mathcal{O}_{20}$  are obtained from  $(\mathcal{O}_6, 2, 1, 3), (\mathcal{O}_{11}, 2, 1, 3)$  by adding the edge  $bc$ , respectively.  $\square$

**Theorem 4.3.3.** *Let  $(G, a, b, c)$  be a 3-connected graph. If  $(G, a, b, c)$  is minor minimal not ac-planar rooted graph then  $(G, a, b, c)$  contains  $(\mathcal{O}_i, \alpha, \beta, \gamma)$  for some  $i \in \{2, 5, 12, 13, 14, 15, 16, 17, 18, 19, 20, 23, 24, 25, 26, 27, 28\}$  where  $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$ , as a rooted minor. These rooted graphs are listed in Figure 1.5.*

*Proof.* In the proof of this theorem because of the symmetry between  $a, b, c$ , we do not list the order of the root when we are going to show that  $(G, a, b, c)$  contains one of the rooted graph listed in the statement of the theorem as a rooted minor.

The proof strategy is similar to the one presented in the proof of Lemma 4.1.6 and Theorem 4.2.10. We consider two cases, either  $(G, a, b, c)$  is internally 4-connected or it is not internally 4-connected. If  $(G, a, b, c)$  is internally 4-connected then by Theorem 4.3.2,  $G$  contains either  $\mathcal{O}_2, \mathcal{O}_5, \mathcal{O}_{12}, \mathcal{O}_{13}, \mathcal{O}_{14}, \mathcal{O}_{15}, \mathcal{O}_{16}, \mathcal{O}_{17}, \mathcal{O}_{18}, \mathcal{O}_{19}, \mathcal{O}_{20}, \mathcal{O}_{23}, \mathcal{O}_{24}, \mathcal{O}_{25}, \mathcal{O}_{26}, \mathcal{O}_{27}$  or  $\mathcal{O}_{28}$  as a rooted minor.

So assume that  $(G, a, b, c)$  is not internally 4-connected. Let  $G = (G_1, G_2)$  be an internal 3-separation in  $G$  such that  $V(G_1) \cap V(G_2) = \{a', b', c'\}$  and  $\{a, b, c\} \subset V(G_1)$  and the number of vertices of  $G_1$  is as small as possible. Since  $H$  is 3-connected we may assume there exist three disjoint paths  $P_a, P_b, P_c$  in  $G_1$  connecting  $a, b, c$  to  $a', b', c'$ , respectively. The same argument presented in the proof of Lemma 4.1.6 shows that  $G_1$  contains a double fork with feet on  $a', b', c'$ . If  $a \neq a', b \neq b'$  and  $c \neq c'$  then it is easy to see that  $(G, a, b, c)$  contains  $\mathcal{O}_{13}$  as a rooted minor. So from now on, we may assume  $b = b'$ . By symmetry between  $a, c$  and  $a', c'$ , respectively, we consider

the following cases:

- $c = c', a \neq a'$  and  $b \neq b'$

If  $(G_1, a', b', c)$  is not  $c$ -planar, then by applying Lemma 4.2.7, we can see that  $(G, a, b, c)$  contains either  $\mathcal{O}_2, \mathcal{O}_5, \mathcal{O}_{13}$  or  $\mathcal{O}_{14}$ , as a rooted minor. If  $(G_1, a', b', c)$  is  $c$ -planar, then by applying a similar argument as presented in Case 2 in the proof of Lemma 4.1.5, we can infer that  $(G, a, b, c)$  contains either  $\mathcal{O}_2, \mathcal{O}_5, \mathcal{O}_{12}, \mathcal{O}_{15}$  or  $\mathcal{O}_{25}$  as a rooted minor.

- $c = c', a = a'$  and  $b \neq b'$

If  $(G_1, a, b', c)$  is not  $ac$ -planar, then by applying Lemma 4.3.1, we can see that  $(G, a, b, c)$  contains either  $\mathcal{O}_2, \mathcal{O}_5, \mathcal{O}_{13}, \mathcal{O}_{14}, \mathcal{O}_{12}$  or  $\mathcal{O}_{22}$ , as a rooted minor. Now, we consider the possibility that  $(G_1, a, b', c)$  is not  $c$ -planar but it is  $a$ -planar. Then by applying Lemma 4.2.8  $(G, a, b, c)$  contains either  $\mathcal{O}_2, \mathcal{O}_5, \mathcal{O}_{12}, \mathcal{O}_{15}, \mathcal{O}_{23}, \mathcal{O}_{24}$  or  $\mathcal{O}_{25}$  as a rooted minor. Finally, we assume that  $(G_1, a, b', c)$  is  $c$ -planar and  $a$ -planar in which case the same argument presented in the second paragraph of Case 6 in Theorem 4.2.10 is applicable and it shows that  $(G, a, b, c)$  contains  $\mathcal{O}_5, \mathcal{O}_2, \mathcal{O}_{12}, \mathcal{O}_{15}, \mathcal{O}_{24}, \mathcal{O}_{25}, \mathcal{O}_{26}$  as a rooted minor.  $\square$

Now the proof of Theorem 1.8.5 follows from Theorem 4.3.2 and 4.3.3.

## CHAPTER V

### MINOR MINIMAL NON-PROJECTIVE PLANAR GRAPHS WITH AN INTERNAL 3-SEPARATION

**Definition 5.0.4.** Let  $(G_1, a, b, c)$  and  $(G_2, a, b, c)$  be two rooted graphs, where the graph induced by the set  $\{a, b, c\}$  in  $G_1$  is equal to the graph induced by the set  $\{a, b, c\}$  in  $G_2$ . We define  $G_1(a, b, c) + G_2(a, b, c)$  to be the graph obtained from  $G_1(a, b, c)$  and  $G_2(a, b, c)$  by identifying the vertices  $a, b, c$  in  $(G_1, a, b, c)$  with the vertices  $a, b, c$  in  $(G_2, a, b, c)$ . Note that  $|V(G_1(a, b, c) + G_2(a, b, c))| = |V(G_1(a, b, c))| + |V(G_2(a, b, c))| - 3$

**Lemma 5.0.5.** Suppose  $(G, a, b, c)$  is minor minimal either not  $c$ -, not  $bc$ -, or not  $abc$ -planar. Then  $ab \in E(G)$  if and only if  $G \setminus \{ab\}$  is  $c$ -planar.

**Definition 5.0.6.** Let  $(G, a, b, c)$  be a rooted graph such that  $\{a, b, c\}$  is an independent set in  $G$ . We say  $(G, a, b, c)$  is  $a$ -persistent if it is not  $a$ -planar and it is not minor minimal with respect to not the  $a$ -planarity property, i.e. there exists an edge  $e \in E(G, a, b, c)$  such that by deleting or contracting  $e$ , the rooted graph remains not  $a$ -planar.

Definition 5.0.6 will be useful for analyzing the interaction of not  $ac$ -planar graphs with not  $b$ -planar graphs. Therefore in the following lemma, we just focus on the list of not  $ac$ -planar graphs.

**Lemma 5.0.7.** (i) If  $(G, a, b, c)$  is isomorphic to one of the graphs  $(\mathcal{O}_i, 2, 1, 3)$  for  $i \in \{8, 9, 10, 12, 15, 26, 28, 23, 24\}$ , then  $(G, a, b, c)$  is  $a$ -persistent and it is  $c$ -persistent.

(ii) If  $(G, a, b, c)$  is isomorphic to one of the graphs  $(\mathcal{O}_4, 2, 1, 3)$ ,  $(\mathcal{O}_3, 2, 1, 3)$  then  $(G, a, b, c)$  is  $a$ -persistent and it is not  $c$ -persistent.

(iii) If  $(G, a, b, c)$  is isomorphic to  $(\mathcal{O}_5, 1, 2, 3)$  or  $(\mathcal{O}_5, 2, 1, 3)$  then  $(G, a, b, c)$  is  $a$ -persistent but it is not  $c$ -persistent.

*Proof.* (i) Assume  $(G, a, b, c)$  is isomorphic to one of the graphs  $(\mathcal{O}_i, 2, 1, 3)$  for  $i \in \{8, 9, 10, 12, 15, 26\}$ .

By symmetry between  $a, c$ , it is enough to show that  $(G, a, b, c)$  is  $a$ -persistent. It is easy to show that If  $(G, a, b, c)$  is isomorphic to one of the graphs  $(\mathcal{O}_i, 2, 1, 3)$  for  $i = 8, 9, 10$  then it contains  $(\mathcal{O}_1, 3, 2, 1)$  as a proper subgraph, if  $(G, a, b, c)$  is isomorphic to one of the graphs  $(\mathcal{O}_i, 2, 1, 3)$  for  $i = 12, 15, 26$  then it contains  $(\mathcal{O}_4, 3, 2, 1)$ ,  $(\mathcal{O}_3, 3, 2, 1)$ , or  $(\mathcal{O}_6, 3, 2, 1)$  as a proper subgraph, respectively. Thus  $(G, a, b, c)$  is  $a$ -persistent.

Assume  $(G, a, b, c)$  is isomorphic to  $(\mathcal{O}_{28}, 2, 1, 3)$ . It is easy to see that  $(G, a, b, c)$  contains  $(\mathcal{O}_6, 3, 2, 1)$  and  $(\mathcal{O}_9, 1, 2, 3)$ , so  $(G, a, b, c)$  is  $c$ - and  $a$ -persistent.

Assume  $(G, a, b, c)$  is isomorphic to  $(\mathcal{O}_{23}, 2, 1, 3)$ . It is easy to see that  $(G, a, b, c)$  contains  $(\mathcal{O}_4, 3, 2, 1)$  and  $(\mathcal{O}_6, 1, 2, 3)$ , so  $(G, a, b, c)$  is  $a$ - and  $c$ -persistent.

Assume  $(G, a, b, c)$  is isomorphic to  $(\mathcal{O}_{24}, 2, 1, 3)$ . It is easy to see that  $(G, a, b, c)$  contains  $(\mathcal{O}_3, 3, 2, 1)$  and  $(\mathcal{O}_6, 1, 2, 3)$ , so  $(G, a, b, c)$  is  $a$ - and  $c$ -persistent.

(ii) Assume  $(G, a, b, c)$  is isomorphic to either  $(\mathcal{O}_3, 2, 1, 3)$  or  $(\mathcal{O}_4, 2, 1, 3)$ .

Since  $(\mathcal{O}_4, 2, 1, 3)$ ,  $(\mathcal{O}_3, 2, 1, 3)$  are minor minimal not 3-planar and they are minor minimal with respect to this property,  $(G, a, b, c)$  is not  $c$ -persistent. Similarly to part(i), it is easy to see that  $(G, a, b, c)$  contains  $(\mathcal{O}_1, 3, 2, 1)$  as a proper subgraph, so  $(G, a, b, c)$  is  $a$ -persistent.

(iii) If  $(G, a, b, c)$  is isomorphic to  $(\mathcal{O}_5, 1, 2, 3)$  or  $(\mathcal{O}_5, 2, 1, 3)$ , then we proceed as follows.



Since  $(G, a, b, c)$  is minor minimal  $c$ -planar,  $(G, a, b, c)$  is not  $c$ -persistent. It is easy to see that  $(G, a, b, c)$  contains  $(\mathcal{O}_8, 2, 3, 1)$  as a proper subgraph so it is  $a$ -persistent but it is not  $c$ -persistent.

□

Now we are ready to prove our main theorem, i.e. Theorem 1.8.1.

*Proof of Theorem 1.8.1.* Let  $G = (G_1, G_2)$  and  $G_1 \cap G_2 = \{a, b, c\}$  be an internal 3-separation in  $G$ . By Theorem 1.8.2 there exists no  $x \in \{a, b, c\}$  such that  $(G_1, a, b, c)$  and  $(G_2, a, b, c)$  are both  $x$ -planar. By Lemma 4.0.22, we know that  $G_1$  and  $G_2$  each contains a double fork with feet on  $a, b, c$ . Now based on the fact that either  $\{a, b, c\}$  is an independent set or it is not independent, and by symmetry between  $G_1, G_2$  and  $a, b, c$ , without loss of generality, we consider the following cases:

**Case 1** The set  $\{a, b, c\}$  is an independent set.

**Case 1.1**  $(G_1, a, b, c)$  is not  $abc$ -planar and  $(G_2, a, b, c)$  is isomorphic to  $K_{2,3}(1, 2, 3)$ . By Theorem 4.3.3 and Theorem 1.8.5 and the fact that  $\{a, b, c\}$  is an independent set,  $(G_1, a, b, c)$  is isomorphic one of the rooted graphs  $(\mathcal{O}_i, 1, 2, 3)$ , for  $i \in \{2, 5, 12, 13, 14, 15, 16, 17, 18, 19\}$  which implies that  $G$  is isomorphic to  $K_{3,5}, \mathcal{F}_5, \mathcal{D}_3, \mathcal{G}_1, \mathcal{E}_{19}, \mathcal{E}_5, \mathcal{D}_{12}, \mathcal{E}_{11}, \mathcal{E}_{27}, \mathcal{D}_9$ .

**Case 1.2**  $(G_1, a, b, c)$  is not  $ac$ -planar and  $(G_2, a, b, c)$  is not  $b$ -planar.

Since  $\{a, b, c\}$  is an independent set, we may assume  $(G_1, a, b, c)$  is  $b$ -planar, otherwise the possible outcomes are analyzed in Case 1.1. Let  $\{x, y\} = \{a, c\}$  and we assume that  $(G_2, a, b, c)$  is  $x$ -planar; otherwise, the possible outcomes have been already analyzed in 1.1. Note that since  $(G, a, b, c)$  is minor minimal  $(G_2, a, b, c)$  is not  $b$ -persistent. Moreover if  $(G_2, a, b, c)$  is not  $y$ -planar then  $(G_1, a, b, c)$  should not be  $x$ -persistent. Thus, we consider two possibilities. In the first possibility, assume that  $(G_2, a, b, c)$  is  $y$ -planar. So by Theorem 1.8.3,  $(G_2, a, b, c)$  is isomorphic to  $(\mathcal{O}_1, t, 3, s)$

where  $\{t, s\} = \{1, 2\}$  and by Theorem 1.8.4  $(G_1, a, b, c)$  is isomorphic to  $\mathcal{O}_3, \mathcal{O}_4, \mathcal{O}_6, \mathcal{O}_8, \mathcal{O}_9, \mathcal{O}_{10}$ , where  $a, b, c$  correspond to  $z, 1, w$  where  $\{z, w\} = \{2, 3\}$ . Note that  $\mathcal{O}_3, \mathcal{O}_4, \mathcal{O}_6, \mathcal{O}_8, \mathcal{O}_9, \mathcal{O}_{10}$  are the only graphs in the statement of Theorem 1.8.4 which are not  $ac$ -planar and  $\{a, b, c\}$  is an independent set. Now by gluing  $(G_1, a, b, c)$  to  $(G_2, a, b, c)$ , we can imply that  $G$  is isomorphic to either  $\mathcal{F}_1, \mathcal{E}_{19}, \mathcal{F}_1, \mathcal{G}_1, \mathcal{D}_3, \mathcal{F}_1$ , respectively.

In the second possibility, assume that  $(G_2, a, b, c)$  is not  $y$ -planar. Thus,  $(G_1, a, b, c)$  is not  $x$ -persistent. Without loss of generality, assume  $y = a$ . Thus by Theorem 1.8.4 and Lemma 5.0.7,  $(G_1, a, b, c)$  is isomorphic to  $(\mathcal{O}_3, 2, 1, 3)$  or  $(\mathcal{O}_4, 2, 1, 3)$  and also  $(G_2, a, b, c)$  is isomorphic to  $(\mathcal{O}_3, 2, 3, 1)$  or  $(\mathcal{O}_4, 2, 3, 1)$ . If  $(G_1, a, b, c)$  is isomorphic to  $(\mathcal{O}_3, 2, 1, 3)$  and  $(G_2, a, b, c)$  is isomorphic to  $(\mathcal{O}_3, 2, 3, 1)$  then it is easy to see that  $(G, a, b, c)$  must contain a degree two vertex, a contradiction. Similarly, if  $(G_1, a, b, c)$  is isomorphic to  $(\mathcal{O}_4, 2, 1, 3)$  and  $(G_2, a, b, c)$  is isomorphic to  $(\mathcal{O}_4, 2, 3, 1)$  then  $G$  must contain a degree two vertex, a contradiction. If  $(G_1, a, b, c)$  is isomorphic to  $(\mathcal{O}_3, 2, 1, 3)$  and  $(G_2, a, b, c)$  is isomorphic to  $(\mathcal{O}_4, 2, 3, 1)$  then it is easy to see that  $G$  contains  $\mathcal{C}_7$ . If  $(G_1, a, b, c)$  is isomorphic to  $(\mathcal{O}_4, 2, 1, 3)$  and  $(G_2, a, b, c)$  is isomorphic to  $(\mathcal{O}_3, 2, 3, 1)$  then it is easy to see that  $G$  contains  $\mathcal{C}_7$ .

**Case 2** The set  $\{a, b, c\}$  is not an independent set.

**Case 2.1**  $|E(G[a, b, c])| = 1$  and  $E(G[a, b, c]) = \{ab\}$ .

By our assumptions, the edge  $ab \in E(G_1, a, b, c)$ . Note that the fact that  $G$  is minor minimal non-projective planar implies that  $(G_1, a, b, c)$  must be not  $c$ -planar and  $(G_1, a, b, c) \setminus ab$  must be  $c$ -planar. Let  $G'_2(a, b, c)$  be the rooted graph obtained from  $G'_1(a, b, c)$  by adding the edge  $ab$ . It is easy to see that  $(G'_2, a, b, c)$  must be not  $c$ -planar and  $(G_2, a, b, c)$  must be  $c$ -planar.

By symmetry, we consider two possibilities. In the first one, we assume that

$(G_1, a, b, c)$  is not  $abc$ -planar and  $G'_2(a, b, c)$  is not  $c$ -planar. By applying Theorem 1.8.3,  $G'_2(a, b, c)$  is isomorphic to  $(\mathcal{O}_6, 1, 2, 3)$  or  $(\mathcal{O}_7, 1, 2, 3)$ . By applying Theorem 1.8.5,  $G_1(a, b, c)$  is isomorphic to  $\mathcal{O}_{23}, \mathcal{O}_{24}, \mathcal{O}_{25}, \mathcal{O}_{27}, \mathcal{O}_{28}$  where  $a, b, c$  correspond to  $x, y, z$  where  $\{x, y, z\} \in \{1, 2, 3\}$ . It is not hard to see that by gluing  $(G_1, a, b, c)$  and  $(G'_2, a, b, c)$ , we can infer that either  $G = \mathcal{O}_{23} + (\mathcal{O}_6, 1, 2, 3)$ ,  $G = \mathcal{O}_{24} + (\mathcal{O}_7, 1, 2, 3)$ ,  $G = \mathcal{O}_{27} + (\mathcal{O}_7, 1, 2, 3)$  or  $G = \mathcal{O}_{28} + (\mathcal{O}_6, 1, 2, 3)$ , therefore  $G$  contains  $\mathcal{D}_3$  as a minor, or,  $G = \mathcal{O}_{23} + (\mathcal{O}_7, 1, 2, 3)$ , so  $G$  is isomorphic  $\mathcal{C}_7$ , or  $G = \mathcal{O}_{24} + (\mathcal{O}_6, 1, 2, 3)$ ,  $G = \mathcal{O}_{25} + (\mathcal{O}_6, 1, 2, 3)$ ,  $G = \mathcal{O}_{27} + (\mathcal{O}_6, 1, 2, 3)$ , so  $G$  contains  $\mathcal{F}_1$  as a minor, or  $G = \mathcal{O}_{25} + (\mathcal{O}_7, 1, 2, 3)$ , thus  $G$  contains  $\mathcal{E}_{19}$  as a minor, or finally  $G = \mathcal{O}_{28} + (\mathcal{O}_7, 1, 2, 3)$ , therefore  $G$  contains  $K_7 - C_4$  as a minor.

In the second possibility, we assume that  $(G_1, a, b, c)$  is not  $ac$ -planar and  $G'_2(a, b, c)$  is not  $bc$ -planar. Note that we assume that  $(G_1, a, b, c)$  is  $b$ -planar and  $G'_2(a, b, c)$  is  $a$ -planar. Moreover  $(G_1, a, b, c)$  is not  $a$ -persistent and  $G'_2(a, b, c)$  is not  $b$ -persistent. Thus by applying Theorem 1.8.4 and Lemma 5.0.7,  $(G_1, a, b, c)$  is isomorphic to  $(\mathcal{O}_6, 3, 1, 2)$  and  $(G'_2, a, b, c)$  is isomorphic to  $(\mathcal{O}_6, 1, 3, 2)$ . This implies that  $bc \in E(G_2)$  and  $ac \in E(G'_2)$ , a contradiction.

**Case 2.2**  $|E(G[a, b, c])| = 2$  and  $E(G[a, b, c]) = \{ab, ac\}$ .

Let  $G'_2(a, b, c)$  be the rooted graph obtained from  $G_2(a, b, c)$  by adding the edge  $ab$  and  $ac$ . Since  $G$  is minor minimal not projective planar, by applying lemma 5.0.5 on  $(G_1, a, b, c)$  and  $(G'_2, a, b, c)$ , we can infer that  $(G_1, a, b, c)$  and  $(G'_2, a, b, c)$  is not  $bc$ -planar. Without loss of generality, assume that  $(G_1, a, b, c)$  is not  $abc$ -planar.

By applying Theorem 1.8.4  $G'_2(a, b, c)$  is isomorphic to  $(\mathcal{O}_{11}, 1, 2, 3)$ . By applying Theorem 1.8.5,  $G_1(a, b, c)$  is isomorphic to  $\mathcal{O}_{26}$  where  $a, b, c$  correspond to  $x, y, z$  where  $\{x, y, z\} \in \{1, 2, 3\}$ . Note that there is a symmetry between  $b, c$ . It is not hard to see that by gluing  $(G_1, a, b, c)$  and  $(G'_2, a, b, c)$ , we can infer that  $G$  is isomorphic to  $\mathcal{D}_3$ .

**Case 2.3**  $|E(G[a, b, c])| = 3$  and  $E(G[a, b, c]) = \{ab, ac, bc\}$ .

Let  $G'_2(a, b, c)$  be rooted graph obtained from  $G_2(a, b, c)$  by adding the edge  $ab, ac$

and  $bc$ . Since  $G$  is minor minimal not projective planar, by applying lemma 5.0.5 to  $(G_1, a, b, c)$  and  $(G'_2, a, b, c)$ , we can infer that  $(G_1, a, b, c)$  and  $(G'_2, a, b, c)$  is not  $abc$ -planar.

By applying Theorem 1.8.5,  $G_1(a, b, c)$  and  $G'_2(a, b, c)$  are isomorphic to  $(\mathcal{O}_{20}, 1, 2, 3)$ . It is not hard to see that by gluing  $(G_1, a, b, c)$  and  $(G'_2, a, b, c)$ , we can infer that  $G$  is isomorphic to  $K_7 - C_4$ . □

## CHAPTER VI

### CONCLUSION

As we mentioned in Chapter 1, we developed a new technique and modern approach for finding the set  $\Omega$ , the set of minor minimal non-projective planar graphs. In particular, we settled the case where  $G \in \Omega$  and  $G$  is 3-connected with an internal 3-separation. Our main results are as follows:

- (i) We found the set of 3-connected minor minimal non-projective planar graphs with an internal 3-separation.
- (ii) We developed a theory for finding non- $c$ -planar extension of  $c$ -planar graphs.
- (iii) We found the set of minor minimal non- $c$ -planar graphs.
- (iv) We found the set of minor minimal non- $ac$ -planar graphs.
- (v) We found the set of minor minimal non- $abc$ -planar graphs.

Here we briefly highlight our plan for settling the case where  $G \in \Omega$  and  $G$  is internally 4-connected. We break this case into two cases: either  $G$  contains  $V_8$  as a minor or  $G$  does not contain  $V_8$  as a minor, where  $V_8$  is the graph obtained from a cycle of length eight by adding edges joining every pair of diagonally opposite vertices.

In the case that  $G$  does not have  $V_8$  as a minor, we use an unpublished result of Robertson.

**Theorem 6.0.8.** *Let  $G$  be an internally 4-connected graph which does not contain  $V_8$  as a minor. Then either*

- (i)  $G$  is a planar graph, or,

- (ii)  $G$  is isomorphic to the line graph of  $K_{3,3}$ , or,
- (iii)  $G$  has two vertices  $u, v$  such that  $G \setminus \{u, v\}$  is a cycle, or
- (iv)  $G$  has at most seven vertices, or,
- (v) there exists a set  $S \in V(G)$  of size at most four such that  $G \setminus S$  has no edge.

Note that if  $G$  is isomorphic to the line graph of  $K_{3,3}$  or double wheel then  $G$  is projective planar. So the only technical parts for dealing with the case that  $G$  does not contain  $V_8$  as a minor are the last two outcomes of Theorem 6.0.8 which can be done by a simple case analysis since internally 4-connected graphs satisfying (iv) and (v) in Theorem 6.0.8 are few in number. Now, we may assume  $G$  contains  $V_8$  as a minor. Since  $V_8$  is a cubic graph, we may assume  $G$  contains  $V_8$  as a subdivision. For this part we should develop a new theory similar to the theory developed in Chapters 2 and 3.

In this thesis we did not have the chance to explore this idea; however the above strategy is a promising one for finding the set of internally 4-connected minor minimal non-projective planar graphs. Combining the above strategy and our results in this thesis would give a new proof of Theorem 1.3.5.

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