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Can quantum probability provide a new direction

for cognitive modeling?

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Short abstract (100 words):

One of the dominant traditions in cognitive modeling is classic (Bayesian) probability (CP) theory. Yet considerable evidence has accumulated that human judgment often goes against classical principles. We discuss quantum probability (QP) theory as an alternative formal probabilistic framework for understanding cognition. In QP theory, probabilistic assessment is often strongly context and order dependent, individual states can be superposition states (which are indefinite with respect to some specific judgment), and composite systems can be entangled (they cannot be decomposed into simpler subsystems). We present several fundamental empirical findings which indicate that cognitive processes often obey quantum rather than classical probability principles.

Long abstract (250 words):

Classical (Bayesian) probability (CP) theory has led to an influential research tradition for modeling cognitive processes. Cognitive scientists have been trained to work with CP principles for so long that it is hard to even imagine alternative ways to formalize probabilities. Yet, in physics, quantum probability (QP) theory has been the dominant probabilistic approach for nearly 100 years. Could QP theory provide us with any advantages in cognitive modeling as well? Note first that both CP and QP theory share the fundamental assumption that it is possible to model cognition on the basis of formal, probabilistic principles. But why consider a QP approach? The answers are that (a) there are many well established empirical findings (e.g., from the influential Tversky, Kahneman research tradition) which are hard to reconcile with CP principles; and (b) these same findings have natural and straightforward accounts with quantum principles. In QP theory, probabilistic assessment is often strongly context and order dependent, individual states can be superposition states (which are impossible to associate with specific values), and composite systems can be entangled (they cannot be decomposed into their subsystems). All these characteristics appear perplexing from a classical perspective. Yet our thesis is that they provide a more accurate and powerful account of certain cognitive processes. We first introduce QP theory and illustrate its application with psychological examples. We then review empirical findings which motivate the use of quantum theory in cognitive theory, but also discuss ways in which QP and CP theories converge. Finally, we consider the implications of a QP theory approach to cognition for human rationality.

1. Preliminary issues

1.1 Why move towards Quantum Probability theory?

In this paper we evaluate the potential of quantum probability (QP) theory for modeling cognitive processes. What is the motivation for employing QP theory in cognitive modeling? Does the use of QP theory offer the promise of any unique insights or predictions regarding cognition? Also, what do quantum models imply regarding the nature of human rationality? In other words, is there anything to be *gained*, by seeking to develop cognitive models based on QP theory? Especially over the last decade, there has been growing interest in such models, encompassing publications in major journals, special issues, dedicated workshops, and a comprehensive book (Busemeyer & Bruza, 2011). Our strategy in this paper is to briefly introduce QP theory, summarize progress with selected, QP models, and motivate answers to the above questions. We note that this paper is *not* about the application of quantum physics to brain physiology. This is a controversial issue (Litt et al., 2006; Hammeroff, 2007), about which we are agnostic. Rather, we are interested in QP theory as a mathematical framework for cognitive modeling. QP theory is potentially relevant in any behavioral situation which involves uncertainty. For example, Moore (2002) reported that the likelihood of a 'yes' response to the questions 'Is Gore honest?' and 'Is Clinton honest?' depends on the relative order of the questions. We will later discuss how QP principles can provide a simple and intuitive account for this and a range of other findings.

QP theory is a formal framework for assigning probabilities to events (Hughes, 1989, Isham, 1989). QP theory can be distinguished from quantum mechanics, the latter being a theory of physical phenomena . For the present purposes it is sufficient to consider QP theory as the abstract foundation of quantum mechanics not specifically tied to physics (for more refined characterizations see e.g. Aerts & Gabora, 2005; Atmanspacher, Romer, & Wallach, 2002, Khrennikov, 2010; Redei & Summers, 2007). The development of quantum theory has been the result of intense effort from some of the greatest scientists of all time, over a period of more than 30 years. The idea of 'quantum' was first proposed by Planck in the early 1900s and advanced by Einstein. Contributions from Bohr, Born, Heisenberg, and Schrödinger all led to the eventual formalization of QP theory by von Neumann and Dirac in the 1930s. Part of the appeal of using QP theory in cognition relates to confidence in the robustness of its mathematics. Few other theoretical frameworks in any science have been scrutinized so intensely, led to such surprising predictions, and also changed human

existence as much as QP theory (when applied to the physical world; quantum mechanics has enabled the development of, e.g., the transistor, and so the microchip, and the laser).

QP theory is, in principle, applicable not just in physics, but in any science where there is a need to formalize uncertainty. For example, researchers have been pursuing applications in areas as diverse as economics (Baaquie, 2004) and information theory (e.g., Grover, 1997; Nielsen & Chuang, 2000). The idea of using quantum theory in psychology has been around for nearly 100 years: Bohr, one of the founding fathers of quantum theory, was known to believe that aspects of quantum theory could provide insight about cognitive process (Busemeyer, Wang, Pothos, & Atmanspacher, in preparation). However, Bohr never made any attempt to provide a formal cognitive model based on QP theory and such models have started appearing only fairly recently (Aerts & Aerts, 1995; Aerts & Gabora, 2005; Atmanspacher, Filk, & Romer, 2004; Blutner, 2009; Bordley, 1998; Bruza, Kitto, Nelson, & McEvoy, 2009; Busemeyer, Wang, & Townsend, 2006; Busemeyer et al., 2011; Conte, et al., 2009; Khrennikov, 2010; Lambert-Mogiliansky, Zamir, & Zwirn, 2009; Pothos & Busemeyer, 2009; Yukalov & Sornette, 2010). But what are the features of quantum theory which make it a promising framework for understanding cognition? It seems essential to address this question, before expecting readers to invest the time for understanding the (relatively) new mathematics of QP theory.

Superposition, entanglement, incompatibility, and interference are all related aspects of QP theory, which endow it with a unique character. Consider a cognitive system, which concerns the cognitive representation of some information about the world (e.g., the story about the hypothetical Linda, used in Tversky & Kahneman's, 1983, famous experiment; Section 3.1). Questions posed to such systems ('Is Linda feminist?') can have different outcomes ('Yes, Linda is feminist'). Superposition has to do with the nature of uncertainty about question outcomes. The classical notion of uncertainty concerns our lack of knowledge about the state of the system that determines question outcomes. In QP theory, there is a deeper notion of uncertainty that arises when a cognitive system is in a superposition among different possible outcomes. Such a state is *not* consistent with *any* single possible outcome (that this is the case is not obvious; this remarkable property follows from the Kochen-Specker theorem). Rather, there is a potentiality (Isham, 1989, p.153) for different possible outcomes, and if the cognitive system evolves in time, so does the potentiality for each possibility. In quantum physics, superposition appears puzzling: what does it mean for a particle to have a potentiality for different positions, without it actually existing at any particular position? By contrast, in psychology superposition appears an intuitive way to characterize the fuzziness (the conflict, ambiguity, and ambivalence) of everyday thought.

Entanglement concerns the compositionality of complex cognitive systems. QP theory allows the specification of entangled systems for which it is not possible to specify a joint probability distribution from the probability distributions of the constituent parts. In other words, in entangled composite systems, a change in one constituent part of the system necessitates changes in another part. This can lead to inter-dependencies between the constituent parts not possible in classical theory and surprising predictions, especially when the parts are spatially or temporally separated.

In quantum theory there is a fundamental distinction between compatible and incompatible questions for a cognitive system. Note that the terms compatible and incompatible have a specific, technical meaning in QP theory, which should not be confused with their lay use in language. If two questions A, B about a system are compatible, it is always possible to define the conjunction between A and B. In classical systems, it is assumed by default that all questions are compatible. Therefore, for example, the conjunctive question 'is A and B true' always has a yes or no answer and the order between questions A, B in the conjunction does not matter. By contrast, in QP theory, if two questions A, B are incompatible, it is impossible to define a single question regarding their conjunction. This is because an answer to question A implies a superposition state regarding question B (e.g., if A is true at a time point, then B can be neither true nor false at the same time point). Instead, QP defines conjunction between incompatible questions in a sequential way, such as 'A and then B'. Crucially, the outcome of question A can affect the consideration of question B, so that interference and order effects can arise. This is a novel way to think of probability, and one which is key to some of the most puzzling predictions of quantum physics. For example, knowledge of the position of a particle imposes uncertainty on its momentum. However, incompatibility may make more sense when considering cognitive systems and, in fact, it was first introduced in psychology. The physicist Niels Bohr borrowed the notion of incompatibility from the work of William James. For example, answering one attitude question can interfere with answers to subsequent questions (if they are incompatible), so that their relative order becomes important. Human judgment and preference often display order and context effects and we shall argue that in such cases quantum theory provides a natural explanation of cognitive process.

1.2 Why move away from existing formalisms?

By now, we have hopefully convinced readers that QP theory has certain unique properties, whose potential for cognitive modeling appears, at the very least, intriguing. For many researchers, the inspiration for applying quantum theory in cognitive modeling has been the widespread interest in cognitive models based on classical probability (CP) theory (Anderson, 1991; Griffiths et al., 2010;

Oaksford & Chater, 2007; Tenenbaum et al., 2011). Both CP and QP theories are formal probabilistic frameworks. They are founded on different axioms (the Kolmogorov and Dirac/ von Neumann axioms respectively) and so often produce divergent predictions regarding the assignment of probabilities to events. However, they share profound commonalities as well, such as the central objective of quantifying uncertainty and similar mechanisms for manipulating probabilities. Regarding cognitive modeling, quantum and classical theorists share the fundamental assumption that human cognition is best understood within a formal probabilistic framework.

As Griffiths et al. (2010, p.357) note "probabilistic models of cognition pursue a top-down or 'function-first' strategy, beginning with abstract principles that allow agents to solve problems posed by the world … and then attempting to reduce these principles to psychological and neural processes." That is, the application of CP theory to cognition requires a scientist to create hypotheses regarding cognitive representations and inductive biases and so elucidate the fundamental questions of *how* and *why* a cognitive problem is successfully addressed. In terms of Marr's (1982) analysis, CP models are typically aimed at the computational and algorithmic levels, though perhaps it is more accurate to characterize them as top-down or function-first (as Griffiths et al., 2010, did).

We can recognize the advantage of CP cognitive models in at least two ways. First, in a CP cognitive model, the principles which are invoked (the axioms of CP theory) work as a logical "team" and always deductively constrain each other. By contrast, alternative cognitive modeling approaches (e.g., based on heuristics) work "alone" and therefore are more likely to fall foul of arbitrariness problems, whereby it is possible to manipulate each principle in the model independently of other principles. Second, neuroscience methods and computational bottom-up approaches are typically unable to provide much insight into the fundamental why and how questions of cognitive process (Griffiths et al., 2010). Overall, there are compelling reasons for seeking to understand the mind with CP theory. The intention of QP cognitive models is aligned with that of CP models. Thus, it makes sense to present QP theory side-by-side with CP theory, so readers can appreciate their commonalities and differences.

A related key issue is this: if CP theory is so successful and elegant (at least, in cognitive applications), why seek an alternative? Moreover, part of the motivation for using CP theory in cognitive modeling is the strong intuition supporting many CP principles. For example, the probability of *A* and *B* is the same as the probability of *B* and *A* (

conceptual difficulties (in the 1960s, Feynman famously said "I think I can safely say that nobody understands quantum mechanics."). A classical theorist might argue that, when it comes to modeling psychological *intuition*, we should seek to apply a computational framework which is as intuitive as possible (CP theory) and avoid the one which can lead to puzzling and, superficially at least, counterintuitive predictions (QP theory).

But human judgment often goes directly against CP principles. A large body of evidence has accumulated to this effect, mostly associated with the influential research program of Tversky and Kahneman (Tversky & Kahneman, 1973, 1974; Kahneman et al., 1982; Tversky & Shafir, 1992). Many of these findings relate to order/ context effects, violations of the law of total probability (which is fundamental to Bayesian modeling), and failures of compositionality. Thus, if we are to understand the intuition behind human judgment in such situations, we have to look for an alternative probabilistic framework. Quantum theory was originally developed so as to model analogous effects in the physical world and so, perhaps, it can offer insight into these aspects of human judgment which seem paradoxical from a classical perspective. This situation is entirely analogous to that faced by physicists early in the last century. On the one hand, there was the strong intuition from classical models (e.g., Newtonian physics, classic electromagnetism). On the other hand, there were compelling empirical findings which were resisting explanation on the basis of classical formalisms. Thus, physicists had to turn to quantum theory, and so paved the way for some of the most impressive scientific achievements.

It is important to note that other cognitive theories embody order/ context effects or interference effects or other quantum-like components. For example, a central aspect of the Gestalt theory of perception concerns how the dynamic relationships between the parts of a distal layout together determine the conscious experience corresponding to the image. Query theory (Johnson et al., 2007) is a proposal for how value is constructed through a series of (internal) queries and has been used to explain the endowment effect in economic choice. In query theory, value is constructed, rather than read off, and also different queries can interfere with each other, so that query order matters. In configural weight models (e.g., Birnbaum, 2008) we also encounter the idea that, in evaluating gambles, the context of a particular probability-consequence branch (e.g., its rank order) will affect its weight. The theory also allows weight changes depending on the observer perspective (e.g., buyer vs. seller). Anderson's (1978) integration theory is a family of models for how a person integrates information from several sources, and also incorporates a dependence on order. Fuzzy Trace Theory (Reyna, 2008; Reyna & Brainerd, 1995) is based on a distinction between verbatim and gist information, the latter corresponding to the general semantic qualities of an

event. Gist information can be strongly context and observer dependent and this has led Fuzzy Trace Theory to some surprising predictions (e.g., Brainerd, Reyna, & Ceci, 2008).

This brief overview shows that there is a diverse range of cognitive models that include a role for context or order and a comprehensive comparison is not practical here. However, when comparisons have been made, the results favored quantum theory (e.g., averaging theory was shown to be inferior to a matched quantum model, Trueblood & Busemeyer, 2011). In some other cases, we can view QP theory as a way to formalize previously informal conceptualizations (e.g., for query theory and the Fuzzy Trace Theory).

Overall, there is a fair degree of flexibility in the particular specification of computational frameworks in cognitive modeling. When it comes to CP and QP models, this flexibility is tempered by the requirement of adherence to the axioms in each theory: all specific models have to be consistent with these axioms. This is exactly what makes CP (and QP) models appealing to many theorists and why, as noted, in seeking to understand the unique features of QP theory, it is most natural to compare it with CP theory.

In sum, a central aspect of this paper is the debate of whether psychologists should explore the utility of quantum theory in cognitive theory; or whether the existing formalisms are (mostly) adequate and that a different paradigm is not necessary. Note, we do not develop an argument that CP theory is unsuitable for cognitive modeling; it clearly is, in many cases. And, moreover, as we shall see, CP and QP processes sometimes converge in their predictions. Rather, what is at stake is whether there are situations where the distinctive features of QP theory provide a more accurate and elegant explanation for empirical data. In the next section we provide a brief consideration of the basic mechanisms in QP theory. Perhaps contrary to common expectation, the relevant mathematics is simple and mostly based on geometry and linear algebra. We next consider empirical results which appear puzzling from the perspective of CP theory, but can naturally be accommodated within QP models. Finally, we discuss the implications of QP theory for understanding rationality.

2. Basic assumptions in QP theory and psychological motivation

2.1 The outcome space

CP theory is a set-theoretic way to assign probabilities to the possible outcomes of a question. First, a sample space is defined, in which specific outcomes about a question are subsets of this sample space. Then, a probability measure is postulated, which assigns probabilities to disjoint outcomes in

an additive manner (Kolmogorov, 1932). The formulation is different in QP theory, which is a geometric theory of assigning probabilities to outcomes (Isham, 1989). A vector space (called a Hilbert space) is defined, in which possible outcomes are represented as subspaces of this vector space. Note that our use of the terms questions and outcomes are meant to imply the technical QP terms observables and propositions.

A vector space represents all possible outcomes for questions we could ask about a system of interest. For example, consider a hypothetical person and the general question of her emotional state. Then, one-dimensional subspaces (called rays) in the vector space would correspond to the most elementary emotions possible. The number of unique elementary emotions and their relation to each other determines the overall dimensionality of the vector space. Also, more general emotions, such as happiness, would be represented by subspaces of higher dimensionality. In Figure 1a, we consider the question of whether a hypothetical person is happy or not. But, because it is hard to picture high multidimensional subspaces, for practical reasons we assume that the outcomes of the happiness question are one-dimensional subspaces. Therefore, one ray corresponds to the person definitely being happy and another one to her definitely being unhappy.

Our initial knowledge of the hypothetical person is indicated by the state vector, a unit length vector, denoted as

employed in psychology to model the match between representations has been explored before (Sloman, 1993) and the QP cognitive program can be seen as a way to generalize these early ideas. Also, note that a remarkable mathematical result, Gleason's theorem, shows that the QP way for assigning probabilities to subspaces is unique (e.g., Isham, 1989, p.210). It is not possible to devise another scheme for assigning numbers to subspaces that satisfy the basic requirements for an additive probability measure (i.e., that the probabilities assigned to a set of mutually exclusive and exhaustive outcomes are individually between zero and one, and sum to one).

An important feature of QP theory is the distinction between superposition and basis states. In the above example, after the person has decided that she is happy, then the state vector is

Suppose that we are interested in two questions, whether the person is happy or not, and also whether she is employed or not. In this example, there are two outcomes with respect to the question about happiness, and two outcomes regarding employment. In CP theory, it is always possible to specify a single joint probability distribution over all four possible conjunctions of outcomes for happiness and employment, in a particular situation. (Griffiths, 2003, calls this the unicity principle, and it is fundamental in CP theory). By contrast, in QP theory, there is a key distinction between compatible and incompatible questions. For compatible questions, one can specify a joint probability function for all outcome combinations and in such cases the predictions of CP and QP theories converge (ignoring dynamics). For incompatible questions, it is impossible to determine the outcomes of all questions concurrently. Being certain about the outcome of one question, induces an indefinite state regarding the outcomes of other, incompatible questions.

This absolutely crucial property of incompatibility is one of the characteristics of QP theory which differentiates it from CP theory. Psychologically, incompatibility between questions means that a cognitive agent cannot formulate a single thought for combinations of the corresponding outcomes. This is perhaps because he is not used to thinking about these outcomes together, for example, as in the case of asking whether Linda (Tversky & Kahneman, 1983) can be both a bank teller and a feminist. Incompatible questions need to be assessed one after the other. A heuristic guide of whether some questions should be considered compatible or not is whether clarifying one is expected to interfere with the evaluation of the other. Psychologically, the intuition is that considering one question alters our state of mind (the context), which in turn affects consideration of the second question. Thus, probability assessment in QP theory can be (when we have incompatible questions) order and context dependent, which contrasts sharply with CP theory.

Whether some questions are considered compatible or incompatible is part of the analysis which specifies the corresponding cognitive model. Regarding the questions for happiness and employment for the hypothetical person, the modeler would need to commit a priori as to whether these are compatible or incompatible. We consider in turn the implications of each approach.

2.2.1 Incompatible questions

For outcomes corresponding to one-dimensional subspaces, incompatibility means that subspaces exist at non-orthogonal angles to each other, as in, for example, for the happy and employed subspaces in Figure 1b. Because of the simple relation we assume between happiness and employment, all subspaces can be coplanar, so that the overall vector space is only two-dimensional. Also, recall that certainty about a possible outcome in QP theory means that the state vector is

contained within the subspace for the outcome. For example, if we are certain that the person is happy, then the state vector is aligned with the happy subspace. However, if this is the case, we can immediately see that we have to be somewhat uncertain about the person's employment (perhaps thinking about being happy makes the person a bit anxious about her job). Conversely, certainty about employment aligns the state vector with the subspace for employed, which makes the person somewhat uncertain about her happiness (perhaps her job is sometimes stressful). This is a manifestation of the famous Heisenberg uncertainty principle – being clear on one question forces one to be unclear on another incompatible question.

Since it is impossible to evaluate incompatible questions concurrently, quantum conjunction has to be defined in a sequential way, and so order effects may arise in the overall judgment. For example, suppose that the person is asked first whether she is employed, and then whether she is happy, that is, we have
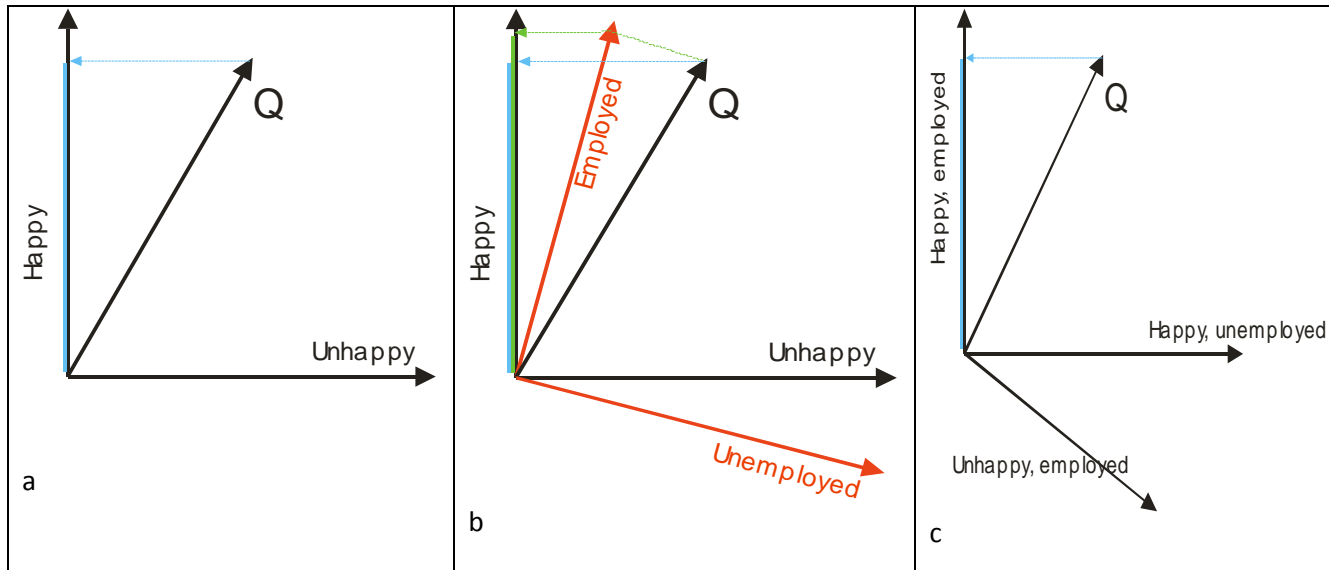
Figure 1. An illustration of basic processes in QP theory. In Figure 1b, all vectors are co-planar, and the figure is a two-dimensional one. In Figure 1c, the three vectors 'Happy, employed', 'Happy, unemployed', and 'Unhappy, employed' are all orthogonal to each other, so that the figure is a three-dimensional one. (The fourth dimension, unhappy, unemployed is not shown).

The magnitude of a projection depends on the angle between the corresponding subspaces. For example, when the angle is large, a lot of amplitude is lost between successive projections. As can be seen in Figure 1b,

defined in exactly the same way. But could order dependence in quantum theory arise as probability dependence in classical theory? The answer is no because

inexorably affects the other (clarifying employment). In other words, in such an entangled state, the possibilities of being happy and employed are strongly dependent on each other. The significance of entanglement is that it can lead to an extreme form of dependency between the outcomes for a pair of questions, which goes *beyond* what is possible in CP theory. In classical theory, one can always construct a joint probability *Prob*(*A*,*B*,*C*) out of pairwise ones, and *Prob*(*A*,*B*), *Prob*(*A*,*C*), and *Prob*(*B*,*C*) are all constrained by this joint. However, in QP theory, for entangled systems, it is not possible to construct a complete joint, because the pairwise probabilities can be stronger than what is allowed classically (Fine, 1982).

2.3 Time evolution

So far we have seen static QP models, whereby we assess the probability for various outcomes for a state at a single point in time. We next examine how the state can change in time. Time evolution in QP theory involves a rotation (technically, a unitary) operator (the solution to Schrödinger's equation). This dynamic operator evolves the initial state vector, without changing its magnitude. Recall, the state vector is a superposition of components along different basis vectors. So, what evolves is the amplitudes along the different basis vectors. For example, a rotation operator might move the state

As an example, suppose the hypothetical person is due a major professional review and she is a bit anxious about continued employment (so that she is unsure about her being employed or not). Prior to the review, she contemplates whether she is happy to be employed or not. In this example, we assume that the employment and happiness questions are compatible (Figure 1c). In CP theory, the initial probabilities satisfy

happiness after a professional review). The resolution regarding employment eliminates any possible interference effects from her judgment, and the quantum prediction converges to the classical one (Appendix). Thus, in QP theory there is a crucial difference between (just) uncertainty and superposition and it is only the latter which can lead to violations of the law of total probability. In quantum theory, just the knowledge that an uncertain situation has been resolved (without necessarily knowing the outcome of the resolution) can have a profound influence on predictions.

3. The empirical case for QP theory in psychology

In this section we explore whether the main characteristics of QP theory (order/context effects, interference, superposition, entanglement) provide us with any advantage in understanding psychological processes. Many of these situations concern Kahneman and Tversky's hugely influential research program on heuristics and biases (Kahneman, Slovic, & Tversky, 1982; Tversky & Kahneman, 1973, 1974, 1983), one of the few psychology research programs to have been associated with a Nobel prize (in economics, for Kahneman in 2002). This research program was built around compelling demonstrations that key aspects of CP theory are often violated in decision making and judgment. Thus, this is a natural place to start looking for whether QP theory may have an advantage over CP theory.

Our strategy is to first discuss how the empirical finding in question is inconsistent with CP theory axioms. This is not to say that some model broadly based on classical principles cannot be formulated. Rather, that the basic empirical finding is clearly inconsistent with classical principles and that a classical formalism, when it exists, may be contrived. We then present an illustration for how a QP approach can offer the required empirical coverage. Such illustrations will be simplifications of the corresponding quantum models.

3.1 Conjunction fallacy

In a famous demonstration, Tversky and Kahneman (1983) presented participants with a story about a hypothetical person, Linda, who sounded very much like a feminist. Participants were then asked to evaluate the probability of statements about Linda. The important comparison concerned the statements 'Linda is a bank teller' (extremely unlikely given Linda's description) and 'Linda is a bank teller and a feminist'. Most participants chose the second statement as more likely than the first, thus effectively judging that

empirical finding is obtained with different kinds of stories or dependent measures (including betting

procedures that do not rely on the concept of probability; Gavanski & Roskos-Ewoldsen, 1991; Sides,

Osherson, Bonini, & Viale, 2002; Stolarz-Fantino, et al., 2003; Tentori & Crupi, 2012; Wedell & Moro,

2008). But, according to CP theory this is impossible, since the conjunction of two statements can

never be more probable than either statement individually (this finding is referred to as the

conjunction fallacy). The CP intuition can be readily appreciated in frequentist terms: in a sample

space of all possible Linda's, of the ones who are bank tellers, only a subset will be both bank tellers

and feminists. Tversky and Kahneman's explanation was that (classical) probability theory is not

appropriate for understanding such judgments. Rather, such processes are driven by a similarity

mechanism, specifically a representativeness heuristic, according to which participants prefer the

statement 'Linda is a bank teller and a feminist' because Linda is more representative of a

stereotypical feminist. A related explanation, based on the availability heuristic, is that the

conjunctive statement activates memory instances similar to Linda (Tversky & Koehler, 1994).

QP theory provides an alternative way to understand the conjunction fallacy. In Figure 2, we

specify

Psychologically, the QP model explains the conjunction fallacy in terms of the context dependence of probability assessment. Given the information participants receive about Linda, it is extremely unlikely that she is a bank teller. However, once participants think of Linda in more general terms as a feminist, they are more able to appreciate that feminists can have all sorts of professions, including being bank tellers. The projection acts as a kind of abstraction process, so that the projection on to the feminist subspace loses some of the details about Linda, which previously made it impossible to think of her as a bank teller. From the more abstract feminist point of view, it becomes a bit more likely that Linda could be a bank teller, so that while the probability of the conjunction remains low, it is still more likely than the probability for just the bank teller property. Of course, from a QP theory perspective, the conjunctive fallacy is no longer a fallacy, it arises naturally from basic QP axioms.

Busemeyer et al. (2011) presented a quantum model based on this idea and examined in detail the requirements for the model to predict an overestimation of conjunction. In general, QP theory does not always predict an overestimation of conjunction. However, given the details of the Linda problem, an overestimation of conjunction necessarily follows. Moreover, the same model was able to account for several related empirical findings, such as the disjunction fallacy, event dependencies, order effects, and unpacking effects (e.g., Bar-Hillel & Neter, 1993; Carlson & Yates, 1989; Gavanski & Roskos-Ewoldsen, 1991; Stolarz-Fantino, et al., 2003). Also, the QP model is compatible with the representativeness and availability heuristics. The projection operations used to compute probabilities measure the degree of overlap between two vectors (or subspaces), and overlap is a measure of similarity (Sloman, 1993). Thus, perceiving Linda as a feminist allows the cognitive system to establish similarities between the initial representation (the initial information about Linda) and the representation for bank tellers. If we consider representativeness to be a similarity process, as we can do with the QP model, it is not surprising that it is subject to chaining and context effects. Moreover, regarding the availability heuristic (Tversky & Koehler, 1994), the perspective from the QP model is that considering Linda to be a feminist increases availability for other related information about feminism, such as possible professions.
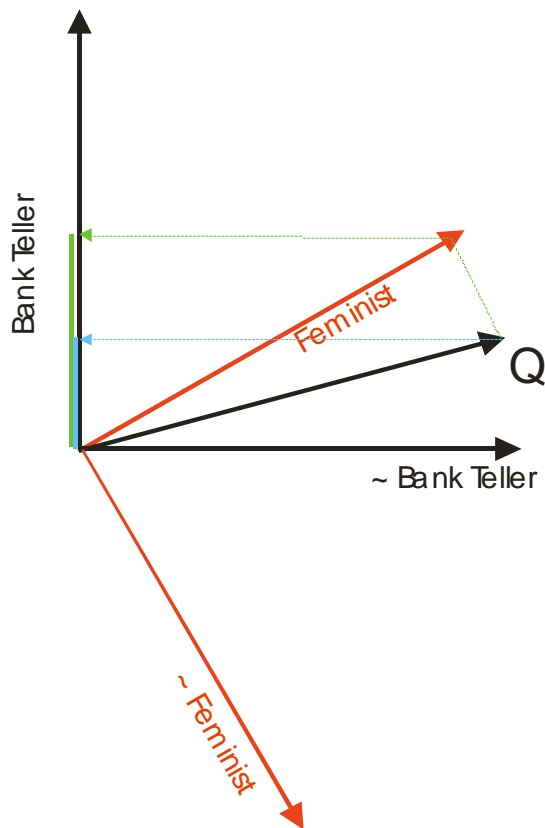
Figure 2. An illustration of the QP explanation for the conjunction fallacy.

3.2 Failures of commutativity in decision making

We next consider failures of commutativity in decision making, whereby asking the same two questions in different orders can lead to changes in response (Feldman & Lynch, 1988; Schuman & Presser, 1981; Tourangeau, Rasinski, & Bradburn, 1991). Consider the questions 'Is Clinton honest?' and 'Is Gore honest?' and the same questions in a reverse order. When the first two questions were asked in a Gallup poll, the probabilities of answering yes for Clinton and Gore were 50% and 68% respectively. The corresponding probabilities for asking the questions in the reverse order were, by contrast, 57% and 60% (Moore, 2002). Such order effects are puzzling according to CP theory, since, as noted, the probability of saying yes to question *A* and then yes to question *B* equals

In Figure 3, there are two sets of basis vectors, one for evaluating whether Clinton is honest or not and another for Gore. The two sets of basis vectors are not entirely orthogonal; we assume that if a person considers Clinton honest, then she is a little bit more likely to consider Gore as honest as well, and vice versa (since they ran together). The initial state vector is fairly close to the

participants (all medical practitioners) had to make a decision about a disease based on two types of clinical information. The order of presenting this information influenced the decision, with results suggesting that the information presented last was weighted more heavily (a recency effect). Trueblood and Busemeyer's (2011) model involved considering a tensor product space for the state vector, with one space corresponding to the presence or not of the disease (this is the event we are ultimately interested in) and the other space to positive or negative evidence, evaluated with respect to the two different sources of information (one source of information implies positive evidence for the disease and the other negative evidence). Considering each source of clinical information involved a rotation of the state vector, in a way reflecting the impact of the information on the disease hypothesis. The exact degree of rotation was determined by free parameters. Using the same number of parameters, the QP theory model produced better fits to empirical results than the anchoring and adjustment model of Hogarth and Einhorn (1992), for the medical diagnosis problem and for the related jury decision one.

3.3 Violations of the sure thing principle

The model Trueblood and Busemeyer (2011) developed is an example of a dynamic QP model, whereby the inference process requires evolution of the state vector. This same kind of model has been employed by Pothos and Busemeyer (2009) and Busemeyer, Wang, and Lambert-Mogiliansky (2009) to account for violations of the sure thing principle. The sure thing principle is the expectation that human behavior ought to conform to the law of total probability. For example, in a famous demonstration, Shafir and Tversky (1992) reported that participants violated the sure thing principle in a one-shot prisoner's dilemma task. This is a task whereby the participant receives different payoffs depending on whether he decides to cooperate or defect, relative to another (often hypothetical) opponent. Usually the player does not know the opponents' move, but in some conditions Shafir and Tversky told participants what the opponent had decided to do. When participants were told that the opponent was going to cooperate, they decided to defect; and when they were told that the opponent was defecting, they decided to defect as well. The payoffs were specified in such a way so that defection was the optimal strategy. The expectation from the sure thing principle is that, when no information was provided about the action of the opponent, participants should also decide to defect (it is a 'sure thing' that defection is the best strategy, since it is the best strategy in all particular cases of opponent's actions). However, surprisingly, in the no knowledge case, many participants reversed their judgment and decided to cooperate (Busemeyer, Matthew, & Wang, 2005; Croson, 1999; Li & Taplan, 2002). Similar results have been reported for

the two-stage gambling task (Tversky & Shafir, 1992) and a novel categorization – decision making paradigm (Busemeyer, Wang, Lambert-Mogiliansky, 2009; Townsend et al., 2000). Therefore, violations of the sure thing principle in decision making, though relatively infrequent, are not exactly rare either. Note this research has established violations of the sure thing principle using within participants designs.

Shafir and Tversky (1992) suggested that participants perhaps adjust their beliefs for the other player's action, depending on what they are intending to do (this principle was called wishful thinking and follows from cognitive dissonance theory and related hypotheses, e.g., Festinger, 1957, Krueger, DiDonato, & Freestone, 2011). Thus, if there is a slight bias for cooperative behavior, in the unknown condition participants might be deciding to cooperate because they imagine that the opponent would cooperate as well. Tversky and Shafir (1992) described such violations of the sure thing principle as failures of consequential reasoning. When participants are told that the opponent is going to defect, they have a good reason to defect as well, and likewise when they are told that the opponent is going to cooperate. However, in the unknown condition, it is as if these (separate) good reasons for defecting under each known condition cancel out (Busemeyer & Bruza, 2011, Chapter 9).

This situation is similar to the generic example for violations of the law of total probability we considered in Section 2. Pothos and Busemeyer (2009) developed a quantum model for the two stage gambling task and prisoner's dilemma embodying these simple ideas. A state vector was defined in a tensor product space of two spaces, one corresponding to the participant's intention to cooperate or defect and one for the belief of whether the opponent is cooperating or defecting. A unitary operator was then specified to rotate the state vector depending on the payoffs, increasing the amplitudes for those combinations of action and belief maximizing payoff. The same unitary operator also embodied the idea of wishful thinking, rotating the state vector so that the amplitudes for the 'cooperate – cooperate' and 'defect – defect' combinations for participant and opponent actions increased. Thus, the state vector developed as a result of two influences. The final probabilities for whether the participant is expected to cooperate or defect were computed from the evolved state vector, by squaring the magnitudes of the relevant amplitudes.

Specifically, the probability of defecting when the opponent is known to defect is based on the projection

operator

it is neutral with respect to the *A* and *B* subspaces (i.e., prior to the similarity comparison, a participant would not be thinking more about *A* than *B*, or vice versa).

Let us consider one of Tversky's (1977) main findings, that the similarity of Korea to China was judged greater than the similarity of China to Korea (actually, North Korea and Red China; similar asymmetries were reported for other countries). Tversky's proposal was that symmetry is violated, because we have more extensive knowledge for China than for Korea, and so China has more distinctive features relative to Korea. He was able to describe empirical results with a similarity model based on a differential weighting of the common and distinctive features of Korea and China. But, the only way to specify these weights was with free parameters and alternative values for the weights could lead to either no violation of symmetry or a violation in a way opposite to the empirically observed one.

By contrast, using QP theory, if one simply assumes that the dimensionality of the China subspace is greater than the dimensionality of the Korea one, then a violation of symmetry in the required direction readily emerges, without the need for parameter manipulation. As shown in Figure 4, in the Korea to China comparison (4a), the last projection is to a higher dimensionality subspace, than the last projection in the China to Korea comparison (4b). Therefore, in the Korea to China case (4a), more of the amplitude of the original state vector is retained, which leads to a prediction for a higher similarity judgment. This intuition was validated with computational simulations by Pothos and Busemeyer (2011), whose results indicate that, as long as one subspace has a greater dimensionality than another, on average the transition from the lower dimensionality subspace to the higher dimensionality one, would retain more amplitude than the converse transition (it has not been proved that this is always the case, but note that participant results with such tasks are not uniform).
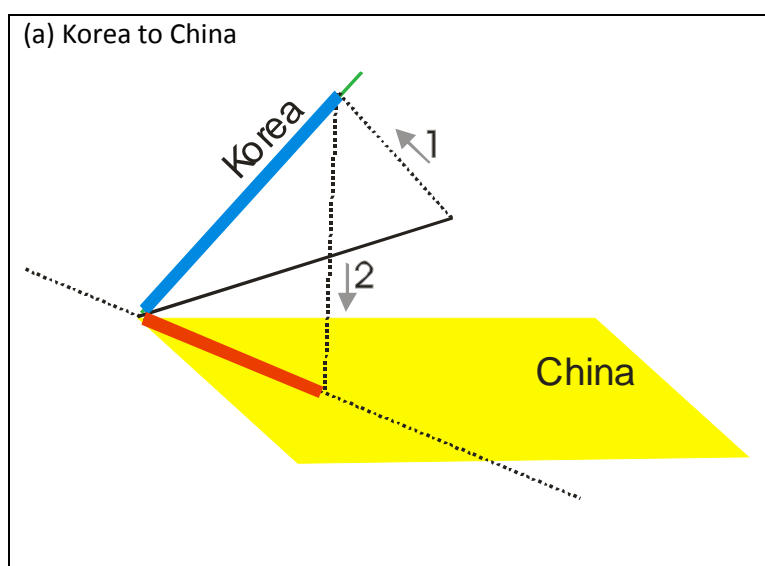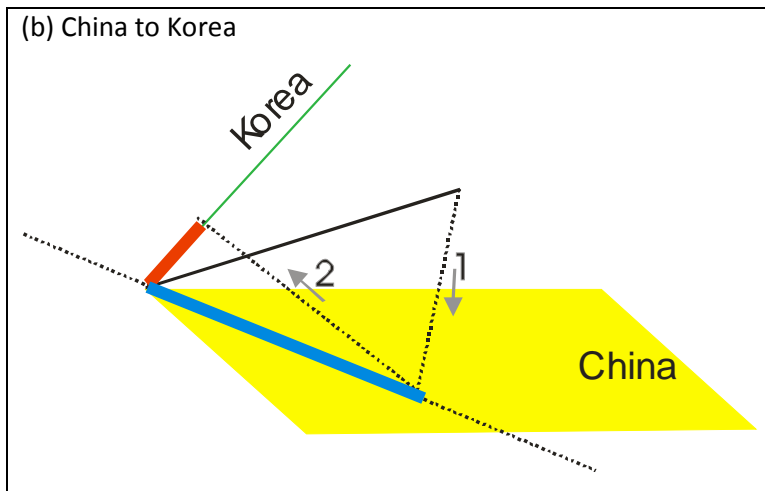
(a) Korea to China

Figures 4a and 4b. Figure 4a corresponds to the similarity of Korea to China and 4b of China to Korea. Projecting to a higher dimensionality subspace last (as in 4a) retains more of the original amplitude than projecting onto a lower dimensionality subspace last (as in 4b).

3.5 Other related empirical evidence

Tversky and Kahneman are perhaps the researchers who most vocally pointed out a disconnect between CP models and cognitive process and, accordingly, we have emphasized QP theory models for some of their most influential findings (and related findings). A skeptical reader may ask, is the applicability of QP theory to cognition mostly restricted to decision making and judgment? Empirical findings which indicate an inconsistency with CP principles are widespread across most areas of cognition. Such findings are perhaps not as well established as the ones reviewed above, but they do provide encouragement regarding the potential of QP theory in psychology. We have just considered a QP theory model for asymmetries in similarity judgment. Relatedly, Hampton (1988a, see also Hampton, 1988b) reported an overextension effect for category membership. Participants rated the strength of category membership of a particular instance to different categories. For example, the rated membership of 'cuckoo' to the pet and bird categories were 0.575 and 1 respectively. However, the corresponding rating for the conjunctive category pet bird was 0.842, a finding analogous to the conjunction fallacy. This paradigm also produces violations of disjunction. Aerts and Gabora (2005) and Aerts (2009) provided a QP theory account of such findings. Relatedly, Aerts and Sozzo (2011) examined membership judgments for pairs of concept combinations, and they empirically found extreme forms of dependencies between concept combination pairs, which indicated that it would be impossible to specify a complete joint distribution over all combinations. These results could be predicted by a QP model using entangled states to represent concept pairs.

In memory research, Brainerd and Reyna (2008) discovered an episodic over-distribution effect. In a training part, participants were asked to study a set of items T. In test, the training items T were presented together with related new ones, R (and some additional foil items). Two sets of instructions were employed. With the verbatim instructions (V), participants were asked to identify only items from the set T. With the gist instructions (G), participants were required to select only R items. In some cases, the instructions (denoted as V or G) prompted participants to select test items from the T or R sets. From a classical perspective, since a test item comes from either the T set or the R one, but not both, it has to be the case that

results, in this case, perhaps depended too much on an arbitrary bias parameter. Another example from memory research is Bruza et. al.'s (2009) application of quantum entanglement (which implies a kind of holism inconsistent with classical notions of causality) to explain associative memory findings, which cannot be accommodated within the popular theory of spreading activation.

Finally, in perception, Conte et al. (2009) employed a paradigm involving the sequential presentation of two ambiguous figures (each figure could be perceived in two different ways) or the presentation of just one of the figures. It is possible that seeing one figure first may result in some bias in perceiving the second figure. Nonetheless, from a classical perspective, one still expects the law of total probability to be obeyed, so that

2008). But such approaches are often unsatisfactory. Arbitrary interpretations of the relevant probabilistic mechanism are unlikely to generalize to related empirical situations (e.g., disjunction fallacies). Also, the introduction of post hoc parameters will lead to models which are descriptive and limited in insight. Thus, employing a formal framework in arbitrarily flexible ways to cover problematic findings is possible, but of arguable explanatory value, and also inevitably leads to criticism (Jones & Love, 2011). But are the findings we considered particularly problematic for CP theory?

CP theory is a formal framework, that is, a set of inter-dependent axioms which can be productively employed to lead to new relations. Thus, when obtaining psychological evidence for a formal framework, we do not just support the particular principles under scrutiny. Rather, such evidence corroborates the psychological relevance of *all* possible relations which can be derived from the formal framework. For example, one cannot claim that one postulate from a formal framework is psychologically relevant, but another is not, and still maintain the integrity of the theory.

The ingenuity of Tversky, Kahneman, and their collaborators (Kahneman et al., 1982; Shafir & Tversky, 1992; Tversky & Kahneman, 1973) was exactly that they provided empirical tests of principles which are at the heart of CP theory, such as the law of total probability and the relation between conjunction and individual probabilities. Thus, it is extremely difficult to specify any *reasonable* CP model consistent with their results, as such models simply lack the necessary flexibility. There is a clear sense in which if one wishes to pursue a formal, probabilistic approach for the Tversky, Kahneman type of findings, then CP theory is not the right choice, even if it is not actually possible to disprove the applicability of CP theory for such findings.

4.2 Heuristics vs. formal probabilistic modeling

The critique of CP theory by Tversky, Kahneman and collaborators can be interpreted in a more general way, as a statement that the attempt to model cognition with any axiomatic set of principles is misguided. These researchers thus motivated their influential program involving heuristics and biases. Many of these proposals sought to relate generic memory or similarity processes to performance in decision making (e.g., the availability and representativeness heuristics; Tversky & Kahneman, 1983). Other researchers have developed heuristics as individual computational rules. For example, Gigerenzer and Todd's (1999) 'take the best' heuristic offers a powerful explanation of behavior in a particular class of problem-solving situations.

Heuristics, however well motivated, are typically isolated: confidence in one heuristic does not extend to other heuristics. Thus, cognitive explanations based on heuristics are markedly different from ones based on a formal axiomatic framework. Theoretical advantages of heuristic models are that individual principles can be examined independently from each other and that no commitment has to be made regarding the overall alignment of cognitive process with the principles of a formal framework. Some theorists would argue that we can only understand cognition through heuristics. However, it is also often the case that heuristics can be re-expressed in a formal way or re-interpreted within CP or QP theory. For example, the heuristics from the Tversky and Kahneman research program, which were developed specifically as an alternative to CP models, often invoke similarity or memory processes, which can be related to order/context effects in QP theory. Likewise, failures of consequential reasoning in Prisoner's Dilemma (Tversky & Shafir, 1992) can be formalized with quantum interference effects.

The contrast between heuristic and formal probabilistic approaches to cognition is a crucial one for psychology. The challenge for advocates of the former is to specify heuristics which cannot be reconciled with formal probability theory (CP or QP). The challenge for advocates of the latter is to show that human cognition is overall aligned with the principles of (classical or quantum) formal theory.

4.3 Is QP theory more complex than CP theory?

We have discussed the features of QP theory, which distinguish it from CP theory. These distinctive features typically emerge when considering incompatible questions. We have also stated that QP theory can behave like CP theory for compatible questions (Section 2.2.2). Accordingly, there might be a concern that QP theory is basically all of CP theory (for compatible questions) and a bit more too (for incompatible ones), so that it provides a more successful coverage of human behavior simply because it is more flexible.

This view is incorrect. First, it is true that QP theory for compatible questions behaves a lot like CP theory. For example, for compatible questions, conjunction is commutative, Lüder's law becomes effectively identical to Bayes's law, and no overestimation of conjunction can be predicted. However, CP and QP theories can diverge, even for compatible questions. For example, quantum time-dependent models involving compatible questions can still lead to interference effects, not possible in classical theory (Section 2.3). Though CP and QP theories share the key commonality of being formal frameworks for probabilistic inference, they are founded on different axioms and their

structure (set theoretic vs. geometric) is fundamentally different. QP theory is subject to several, restrictive constraints, but these are different from the ones in CP theory.

For example, CP Markov models must obey the law of total probability, while dynamic QP models can violate this law. However, dynamic QP models must obey the law of double stochasticity, while CP Markov models can violate this law. Double stochasticity is a property of transition matrices that describes the probabilistic changes from an input to an output over time. Markov models require each column of a transition matrix to sum to unity (so that they are stochastic), but QP models require both each row and each column to sum to unity (so they are doubly stochastic). Double stochasticity sometimes fails and this rules out QP models (Busemeyer, Wang, & Lambert-Mogiliansky, 2009; Khrennikov, 2010).

Moreover, QP models have to obey the restrictive law of reciprocity, for outcomes defined by one dimensional subspaces. According to the law of reciprocity, the probability of transiting from one vector to another is the same as the probability of transiting from the second vector to the first, so that the corresponding conditional probabilities have to be the same. Wang and Busemeyer (under review) directly tested this axiom, using data on question order, and found that it was upheld with surprisingly high accuracy.

More generally, a fundamental constraint of QP theory concerns Gleason's theorem, namely that probabilities have to be associated with subspaces via the equation

to their complexity. Accordingly, Shiffrin and Busemeyer (2011) adopted a Bayesian procedure for model comparison, which evaluates models both on the basis of their accuracy and complexity . As Bayesian comparisons depend on priors over model parameters, different priors were examined, including uniform and normal priors. For both priors, the Bayes's factor favored the QP model over the traditional model (on average, by a factor of 2.07 for normal priors, and by a factor of 2.47 for uniform priors).

Overall, yes, QP theory does generalize CP theory in certain ways. For example, it allows both for situations which are consistent with commutativity in conjunction (compatible questions) and situations which are not (incompatible questions). However, QP theory is also subject to constraints which do not have an equivalent in CP theory, such as double stochasticity and reciprocity, and there is currently no evidence that specific QP models are more flexible than CP ones. The empirical question then becomes which set of general constraints is more psychologically relevant. We have argued that QP theory is ideally suited for modeling empirical results which depend on order/context or appear to involve some kind of extreme dependence that rules out classical composition. QP theory was designed by physicists to capture analogous phenomena in the physical world. Having said that, QP theory does not always succeed, and there have been situations when the assumptions of CP models are more in tune with empirical results (Busemeyer, Wang, & Townsend, 2006). Moreover, in some situations the predictions from QP and CP models converge, and in such cases it is perhaps easier to employ CP models.

5. The rational mind

Beginning with Aristotle and up until recently, scholars have believed that humans are rational because they are capable of reasoning on the basis of logic. First, logic is associated with an abstract elegance and a strong sense of mathematical correctness. Second, logic was the only system for formal reasoning and so scholars could not conceive of the possibility that reasoning could be guided by an alternative system. Logic is exactly this, logical, so how could there be an alternative system for rational reasoning? But this view turned out to be problematic. Considerable evidence accumulated that naïve observers do not typically reason with classical logic (Wason, 1960), so classical logic could not be maintained as a theory of thinking.

Oaksford and Chater (2007, 2009) made a compelling case against the psychological relevance of classical logic. The main problem is that classical logic is deductive, so that once a particular conclusion is reached from a set of premises, this conclusion is certain and cannot be altered by the addition of further premises. Of course, this is rarely true for everyday reasoning. The

key aspect of everyday reasoning is its nonmonotonicity, since it is always possible to alter an existing conclusion, with new evidence. Oaksford and Chater (2007, 2009) advocated a perspective of Bayesian rationality, which was partly justified using Anderson's (1990) rational analysis approach. According to rational analysis, psychologists should look for the behavior function which is optimal, given the goals of the cognitive agent and its environment. Oaksford and Chater's Bayesian rationality view has been a major contribution to the recent prominence of cognitive theories based on CP theory. For example, CP theories are often partly justified as *rational* theories of the corresponding cognitive problems, which makes them easier to promote, than alternatives. For example, in categorization, the rational model of categorization (e.g., Sanborn et al., 2011) has been called, well, rational. By contrast, the more successful Generalized Context Model (Nosofsky, 1984) has received less corresponding justification (Wills & Pothos, 2012).

There has been considerable theoretical effort to justify the rational status of CP theory. We can summarize the relevant arguments under three headings, Dutch book, long term convergence, and optimality. The Dutch book argument concerns the long term consistency of accepting bets. If probabilities are assigned to bets in a way which goes against the principles of CP theory, then this guarantees a net loss (or gain) across time. In other words, probabilistic assignment inconsistent with CP theory leads to unfair bets (de Finetti, Machi, & Smith, 1993). Long term convergence refers to the fact that if the true hypothesis has any degree of non-zero prior probability, then, in the long run, Bayesian inference will allow its identification. Finally, optimality is a key aspect of Anderson's (1990) rational analysis and concerns the accuracy of probabilistic inference. According to advocates of CP theory, this is the optimal way to harness the uncertainty in our environment and make accurate predictions regarding future events and relevant hypotheses.

These justifications are not without problems. Avoiding a Dutch book requires expected value maximization, rather than expected utility maximization, i.e., the decision maker is constrained to use objective values rather than personal utilities, when choosing between bets. However, decision theorists generally reject the assumption of objective value maximization and instead allow for subjective utility functions (Savage, 1954). This is essential, for example, in order to take into account the observed risk aversion in human decisions (Kahneman & Tversky, 1979). When maximizing subjective expected utility, CP reasoning can fall prey to Dutch book problems (Wakker, 2010). Long term convergence is also problematic, because if the true hypothesis has a prior probability of zero, then it can never be identified. This is invariably the case in Bayesian models, since it is not possible to assign a non-zero probability to all candidate hypotheses. Overall, a priori arguments, such as the Dutch book or long term convergence, are perhaps appealing under simple,

idealized conditions. However, as soon as one starts taking into account the complexity of human cognition, such arguments break down.

Perhaps the most significant a priori justification for the rationality of CP theory concerns optimality of predictions. If reasoning on the basis of CP theory is optimal, in the sense of predictive accuracy, then this seems to settle the case in favor of CP theory. For example, is it more accurate to consider Linda as just a bank teller, compared to a bank teller and a feminist? By contrast, QP theory embodies a format for probabilistic inference which is strongly perspective and context dependent. For example, Linda may not look like a bank teller initially, but from the perspective of feminism such a property becomes more plausible. But, equally, optimality must be evaluated under the constraints and limited resources of the cognitive system (Simon 1955).

The main problem with classical optimality is that it assumes a measurable, objective reality and an omniscient observer. Our cognitive systems face the problem of making predictions for a vast number of variables that can take on a wide variety of values. For the cognitive agent to take advantage of classical optimality, it would have to construct an extremely large joint probability distribution to represent all these variables (this is the principle of unicity). But for complex possibilities, it is unclear as to where such information would come from. For example, in Tversky and Kahneman's (1983) experiment we are told about Linda, a person we have never heard of before. Classical theory would assume that this story generates a sample space for all possible characteristic combinations for Linda, including unfamiliar ones such as feminist bank teller. This just seems implausible, let alone practical, considering that for the bulk of available knowledge, we have no relevant experience. It is worth noting that Kolmogorov understood this limitation of CP theory (Busemeyer & Bruza, 2011, chapter 12). He pointed out that his axioms apply to a sample space from a single experiment and different experiments require new sample spaces. But his admonitions were not formalized and CP modelers do not take them into account.

Quantum theory assumes no measurable objective reality, rather judgment depends on context and perspective. The same predicate (e.g., that Linda is a bank teller) may appear plausible or not, depending on the point of view (e.g., depending on whether we accept Linda as a feminist or not). Note that QP theory does assume systematic relations between different aspects of our knowledge, in terms of the angle (and relative dimensionality) between different subspaces. But, each inference changes the state vector and so the perspective from which all other outcomes can be evaluated. Note also that context effects in QP theory are very different from conditional probabilities in CP theory. The latter are still assessed against a common sample space. With the former, the sample space for a set of incompatible outcomes changes every time an incompatible question is evaluated (since this changes the basis for evaluating the state).

If we cannot assume an objective reality and an omniscient cognitive agent, then perhaps the perspective-driven probabilistic evaluation in quantum theory is the best *practical* rational scheme. In other words, quantum inference is optimal, for when it is impossible to assign probabilities to all relevant possibilities and combinations concurrently. This conclusion resonates with Simon's (1955) influential idea of bounded rationality, according to which cognitive theory needs to incorporate assumptions about the computational burden which can be supported by the human brain. For example, classically, the problem of assessing whether Linda is a feminist and a bank teller requires the construction of a bivariate joint probability space, which assigns a probability density for each outcome regarding these questions. By contrast, a QP representation is simpler: it requires a univariate amplitude distribution for each question, and the two distributions can be related through a rotation. As additional questions are considered (e.g., whether Linda might be tall or short) the efficiency of the QP representation becomes more pronounced. Note that classical schemes could be simplified, by assuming independence between particular outcomes. However, independence assumptions are not appropriate for many practical situations and will introduce errors in inference.

Note that the perspective dependence of probabilistic assessment in QP theory may seem to go against an intuition that 'objective' (classical) probabilities are somehow more valid or correct. However, this same probabilistic scheme does lead to more accurate predictions in the physical world, in the context of quantum physics. If the physical world is not 'objective' enough for CP theory to be used, there is a strong expectation that the mental world, with its qualities of flux and interdependence of thoughts, would not be as well.

The application of QP theory to cognition implies a strong interdependence between thoughts, such that it is typically not possible to have one thought, without repercussions for other thoughts. These intuitions were extensively elaborated in the work of Fodor (1983), with his proposals that thought is isotropic and Quinean, so that revising or introducing one piece of information can in principle impact on most other information in our knowledge base. Oaksford and Chater (2007, 2009) argued that it is exactly such characteristics of thought which make CP theory preferable to classical logic for cognitive modeling. However, Fodor's (1983) arguments also seem to go against the neat reductionism in CP theory, required by the principle of unicity and the law of total probability, according to which individual thoughts can be isolated from other, independent ones, and the degree of interdependence is moderated by the requirement to always have a joint probability between all possibilities. QP theory is not subject to these constraints.

Overall, accepting a view of rationality inconsistent with classical logic was a major achievement accomplished by CP researchers (e.g., Oaksford & Chater, 2007, 2009). For example,

how can it be that in the Wason selection task the 'falsificationist' card choices are not the best ones? Likewise, accepting a view of rationality at odds with CP theory is the corresponding challenge for QP researchers. For example, how could it not be that

cognitively feasible to apply the unicity principle?), the greater the psychological relevance of CP theory.

Another challenge concerns further understanding the rational properties of quantum inference. The discussion in Section 5 focused on the issue of accuracy, assuming that the requirements from the principle of unicity have to be relaxed. However, there is a further, potentially relevant literature on quantum information theory (Nielsen & Chuang, 2010), which concerns the processing advantages of probabilistic inference based on QP theory. For example, a famous result by Grover (1997) shows how a quantum search algorithm will outperform any classical algorithm. The potential psychological relevance of such results (e.g., in categorization theory) is an issue for much further work (e.g., is it possible to approximate quantum information algorithms in the brain?). These are exciting possibilities regarding both the rational basis of quantum cognitive models and the general applicability of quantum theory to cognitive theory.

6.2 Empirical challenges

So far, the quantum program has involved employing quantum computational principles to explain certain, prominent empirical findings. Such quantum models do not simply provide re-descriptions of results which have already had (some) compelling explanation. Rather, we discussed results which have presented on-going challenges and have resisted explanation based on classical principles. One objective for future work is to continue identifying empirical situations which are problematic from a classical perspective.

Another objective is to look for new, surprising predictions, which take advantage of the unique properties of quantum theory, such as superposition, incompatibility, and entanglement. For example, Trueblood and Busemeyer (2011) developed a model to accommodate order effects in the assessment of evidence in McKenzie et al.'s (2002) task. The model successfully described data from both the original conditions and a series of relevant extensions. Moreover, Wang and Busemeyer (under review) identified several types of order effects which can occur in questionnaires, such as consistency and contrast (Moore, 2002). Their quantum model was able to make quantitative, parameter free predictions for these order effects. In perception, Atmanspacher and Filk (2010) proposed an experimental paradigm for bistable perception, so as to test the predictions from their quantum model regarding violations of the temporal Bell inequality (such violations are tests of the existence of superposition states).

Overall, understanding the quantum formalism to the extent that surprising, novel predictions for cognition can be generated is no simple task (in physics, this was a process which took several decades). The current encouraging results are a source of optimism.

6.3 Implications for brain neurophysiology

An unresolved issue is how QP computations are implemented in the brain. We have avoided a detailed discussion of this research area because, though exciting, is still in its infancy. One perspective is that the brain does not instantiate any quantum computation at all. Rather, interference effects in the brain can occur if neuronal membrane potentials have wave-like properties, a view which has been supported in terms of the characteristics of EEG signals (Barros & Suppes, 2009). Relatedly, Ricciardi and Umezawa (1967; Jibu & Yasue, 1995; Vitiello, 1995) developed a quantum field theory model of human memory, which still allows a classical description of brain activity. The most controversial (Atmanspacher, 2004; Litt et al., 2006) perspective is that the brain directly supports quantum computations. For quantum computation to occur, a system must be isolated from the environment, since environmental interactions cause quantum superposition states to rapidly decohere into classical states. Penrose (1989) and Hammeroff (1998) suggested that microtubules prevent decoherence for periods of time long enough to enable meaningful quantum computation; in this view, the collapse of superposition states is associated with experiences of consciousness.

Overall, in cognitive science it has been standard to initially focus on identifying the mathematical principles underlying cognition, and later address the issue of how the brain can support the postulated computations. However, researchers have been increasingly seeking bridges between computational and neuroscience models. Regarding the QP cognitive program, this is clearly an important direction for future research.

6.4 The future of QP theory in psychology

There is little doubt that extensive further work is essential before all aspects of QP theory can acquire psychological meaning. But this does not imply that current QP models are not satisfactory. In fact, we argue that the quantum approach to cognition embodies all the characteristics of good cognitive theory: it is based on a coherent set of formal principles, the formal principles are related to specific assumptions about psychological process (e.g., the existence of order/context effects in judgment), and it leads to quantitative computational models which can parsimoniously account for

both old and new empirical data. The form of quantum cognitive theories is very much like that of CP ones, and the latter have been hugely influential in recent cognitive science. The purpose of this article is to argue that researchers attracted to probabilistic cognitive models need not be restricted to classical theory. Rather, quantum theory provides many theoretical and practical advantages and its applicability to psychological explanation should be further considered.

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Appendix. An elaboration of some of the basic definitions in QP theory.

(See Busemeyer and Bruza, 2011, for an extensive introduction)

Projectors (or projection operators).

Projectors are idempotent linear operators. For a one dimensional subspace, corresponding, for

example, to the

Then the direct sum space is formed by all possible pairs of vectors, one from *E* and another from ~*E*.

Time dependence.

The quantum state vector changes over time according to Schrödinger's equation,

Now, suppose that the person is determined to find out whether she will be employed or not, before having this inner reflection about happiness (perhaps she intends to delay thinking about her happiness, until after her professional review). Then, the state after learning about her employment will be either