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MASS-GROWTH OF A FINITE TUBE REINFORCED BY A PAIR OF HELICAL FIBRES

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Abstract

Several types of tube-like fibre-reinforced tissue, including the layers of arteries and veins, different kinds of muscle, biological tubes as well as plants and trees, are reinforced by a pair of helical fibres wound symmetrically around the tube axis in opposite directions. In many cases, this kind of biological structures grow in an axially symmetric manner that preserves their own shape as well as the direction and shape of their embedded pair of helical fibres. This study considers and investigates the influence that preservation of fibre direction exerts on pseudo-elastic (elastic-like) mass-growth modelling of the described fibre-reinforced structure. Complete sets of necessary conditions that enable the implied axisymmetric tube mass-growth to take place are sought, found and presented. These hold in addition to, and simultaneously with standard kinematic relations and equilibrium equations met in conventional hyperelasticity. They thus render this mass-growth mathematical model the properties of an apparently overdetermined boundary value problem. However, the additional information they provide leads to identification of admissible classes of strain energy densities for growth that enable realisation of the implied type of tube mass-growth. The analysis is applicable to several different types of mass-growth of tube-like tissue reinforced by a pair of symmetrically wound helical fibres. This is demonstrated with an application which considers that mass-growth of the fibre-reinforced tube takes place in an incompressible manner; namely, in a non-isochoric manner that along with fibre direction, it also preserves the material density of the growing tube.

Keywords: Anisotropic mass-growth, Elastic-like mass-growth, Growth of cylinders and tubes, Helical fibres, Mass-growth modelling, Pseudo-elastic mass-growth.

1. Introduction

Fibres of helical shape are met in several different kinds of plant and bone structures (e.g., [1-3]), as well as in various forms of tube-like soft biological tissue. Arteries and veins (e.g., [4-7]), muscles (e.g., [8]), and even living creatures of tubular shape [9] are referred to as well-known relevant examples. In most cases, these structures grow in time by preserving their tubular shape as well as the shape/direction of their helical fibres. The latter grow with and within the tube that they are embedded in, and are usually met in the form of two fibre families wound symmetrically about the tube axis in opposite directions. The axially symmetric shape of the tube is thus also preserved during the observed mass-growth deformation, along with the shape and direction of either fibre family.

It was recently shown [10] that, if mass-growth preserves the direction and shape of a single family of fibres embedded in, and growing with and within the tube, then the corresponding set of pseudo-elasticity type equations appears overdetermined. However, the extra equations that emerged in the model enabled formation of one or more linear relationships between the strain energy density for growth, W, and its derivatives with respect to the principal deformation invariants of the growing system. These relationships were thus regarded as partial differential equations (PDEs) for the unknown function W, and their solution provided valuable information regarding admissible classes or forms of W that enable the tube to grow without disturbing the shape or the direction of its fibres.

The outlined new development [10] and its concepts are generally expected to apply further to hyperelastic mass-growth problems of several different types of soft or hard biological tissue. Such problems include fibre-reinforced tissue of different geometrical shapes and features, and/or tissue reinforced by more than one family of unidirectional fibres. They thus also include the case of present interest, where a growing hyperelastic tube is reinforced by two families of fibres wound symmetrically about its axis. A proper mathematical formulation of this problem is presented in Section 2, which resembles closely its Reference [10] counterpart. Section 3 derives then the necessary conditions which guarantee that both families of helical fibres embedded in, and growing with the tube preserve their direction during the axially symmetric deformation of interest. These conditions hold regardless of the tube constitution but attention next focuses on cases that the tube mass-growth can be characterised as elastic-like or pseudo-elastic.

It is recalled in this context that when the number of fibre families embedded in a hyperelastic material is bigger than one, the classical deformation invariants involved in the strain energy density of the fibrous composite are not any more independent [11, 12]. It follows that relationships between W and its derivatives, which, in analogy with [10], would reflect mass-growth ability of some relevant fibre-reinforced tissue to preserve fibre direction, can be considered as PDEs for W only after: (i) the exact number of independent invariants is identified; (ii) a complete basis of precisely the same number of independent invariants is formed; and (iii) W is considered and treated as a function of those independent invariants only. Part (i) of this challenge is already dealt with in [11, 12].

As is also shown in [11, 12], only seven invariants can be independent in the particular case of two families of unidirectional fibres and, among those seven, six are strain invariants while the remaining one is a non-strain invariant. Such a set of seven independent invariants have most recently identified in [13].

With the help of an Appendix, this set of independent invariants is also introduced in Section 4, which considers non-isochoric, incompressible tube growth and extends the theoretical analysis presented in [10] towards the present case of principal interest. Comparisons can thus be made against corresponding modelling features and analytical results detailed in Section 7 of [10], which dealt with incompressible mass-growth of a corresponding tube reinforced by a single family of helical fibres.

In a particular application, Section 5 considers purely dilatational mass-growth and describes the influence that the aforementioned newly introduced set of invariants and relevant findings exert on the tube constitutive and equilibrium equations. That Section also demonstrates the manner in which the presented mathematical model identifies specific (admissible) relevant forms of the strain energy density that enable realisation of the assumed pattern of tube mass-growth.

The outlined theoretical developments are similarly applicable to further particular cases of axisymmetric mass-growth of tube-like tissue reinforced by a pair of symmetrically wound helical fibres. Several of those cases are referred to in the concluding Section 6, which summarises the progress made so far in this subject and outlines a number of important, relevant research challenges that need to be dealt with in the near future.

2. Problem formulation

At $t = t_0$, where t denotes time, an un-deformed circular cylindrical tube of finite axial length, $2H$, and constant mass density, ρ_0, occupies the region

$$A \le R \le B, \quad 0 \le \Theta < 2\pi, \quad -H < Z < H, \qquad (2.1)$$

where, R, Θ and Z are standard cylindrical polar coordinate parameters (Figure 1). The non-negative constant parameters A and B ($0 \le A < B$) represent the inner and the outer radii of the tube, respectively. If $A = 0$ this representation corresponds to the particular case of a non-hollow (solid) cylinder.

At $t > t_0$, the tube grows in a manner that is independent of its azimuthal coordinate parameter. The most general form of such an axially symmetric deformation pattern is as follows:

$$r = r(R, Z; t), \quad \theta = \Theta + g(R, Z; t), \quad z = z(R, Z; t), \tag{2.2}$$

where r, θ and z represent cylindrical polar coordinate parameters in the current, continuously deforming configuration. The implied dynamic combination of tube radial and axial expansion with azimuthal and axial shear strains depends on the form of the functions $r(R,\ Z;\ t)$, $g(R,\ Z;\ t) = \hat{g}\,(r,\ z;\ t)$ and $z(R,\ Z;\ t)$, which are to be determined subject to the initial conditions

$$\begin{aligned} &r(R, Z; t_0) = R, \quad g(R, Z; t_0) = \hat{g}(r, z; t_0) = 0, \\ &z(R, Z; t_0) = Z, \end{aligned} \tag{2.3}$$

that preserve consistency between (2.2) and (2.1).

2.1 *Kinematics*

The basic features of the deformation pattern (2.2) are captured by the relevant deformation gradient tensor, customary denoted with $\boldsymbol{F}$. In the particular cylindrical polar co-ordinate system implied in (2.1), this will be over-signed with a so-called "hat" and, for the axially symmetric deformation (2.2), will accordingly by given as follows:

$$\hat{\boldsymbol{F}} = \begin{bmatrix} \dfrac{\partial r}{\partial R} & \dfrac{\partial r}{R\partial\Theta} & \dfrac{\partial r}{\partial Z} \\ \dfrac{r\partial\theta}{\partial R} & \dfrac{r\partial\theta}{R\partial\Theta} & \dfrac{r\partial\theta}{\partial Z} \\ \dfrac{\partial z}{\partial R} & \dfrac{\partial z}{R\partial\Theta} & \dfrac{\partial z}{\partial Z} \end{bmatrix} = \begin{bmatrix} \chi_r & 0 & \gamma_{rz} \\ \gamma_{\theta r} & \chi_\theta & \gamma_{\theta z} \\ \gamma_{zr} & 0 & \chi_z \end{bmatrix}, \tag{2.4}$$

where, the appearing amounts of radial, azimuthal and axial stretch are as follows:

$$\chi_r = r_{,R}, \quad \chi_\theta = r/R, \quad \chi_z = z_{,Z}, \tag{2.5}$$

and the associated amounts of shear are

$$\gamma_{rz} = r_{,Z}, \quad \gamma_{\theta r} = rg_{,R}, \quad \gamma_{\theta z} = rg_{,Z}, \quad \gamma_{zr} = z_{,R}. \tag{2.6}$$

Here, as well as in what follows, a comma denotes partial differentiation with respect to the indicated suffice(s).

The radial, azimuthal and axial components of the velocity vector, $\boldsymbol{v}$, are

$$v_r = \dot{r}, \quad v_\theta = r\dot{\theta} = r\dot{g}, \quad v_z = \dot{z}, \tag{2.7}$$

respectively, where a dot denotes differentiation with respect to time. The assumed axial symmetry implies that the rate of deformation tensor obtains the simplified form

$$\hat{\boldsymbol{d}} = \begin{bmatrix} & & \\ \frac{1}{2} r & & \\ \frac{1}{2} & & \end{bmatrix}, \tag{2.8}$$

which yields the volumetric rate of deformation as

$$\nabla \cdot \boldsymbol{v} = \mathrm{tr}\hat{\boldsymbol{d}} = \quad . \tag{2.9}$$

The deformation and equilibrium concepts outlined in this Section, as well as the fibre deformation conditions detailed afterwards in Section 4 hold regardless of the cause of the assumed axisymmetric deformation. However, attention will next focus on dynamic deformation which is solely due to action of the rate of growth, r_g. This scalar quantity is here considered as known function of space and time, and maintains mass-growth through its non-zero contribution in the continuity equation with growing mass,

$$\dot{\rho} + \rho\nabla \cdot \boldsymbol{v} = r_g, \tag{2.10}$$

where ρ represents the mass density of the growing continuum. In the present case of interest, admissible forms of the function r_g are evidently only those ones that enable the tube to attain and maintain axially symmetric deformation patterns of the form (2.2).

2.2 *Equilibrium and boundary conditions*

It is assumed that stress equilibrium prevails at all times and, in the absence of body forces, is governed by the quasi-static equations of motion,

$$\nabla \cdot \boldsymbol{\sigma} = 0, \tag{2.11}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor. Due to the prevailing axial symmetry, the relevant cylindrical polar coordinate version of the corresponding radial, azimuthal and axial equations of motion are respectively simplified as follows:

$$\begin{aligned} &\sigma_{rr,r} + \sigma_{zr,z} + \frac{1}{r}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) = 0, \\ &\frac{1}{r^2}\left(r^2\sigma_{r\theta}\right)_{,r} + \sigma_{z\theta,z} = 0, \\ &\sigma_{zz,z} + \frac{1}{r}\ \left(r\sigma_{rz}\right)_{,r} = 0. \end{aligned} \tag{2.12}$$

The number of equations (2.12) matches the number of the principal unknown functions

$r(R,\ Z;\ t)$, $g(R,\ Z;\ t)$ and $z(R,\ Z;\ t)$ appearing in (2.2). Solution of these PDEs may be attempted only after material constitution is specified, and a relevant constitutive equations is thus determined. However, solution of s specific boundary value problem requires further specification of an associated set of boundary conditions.

A deformation pattern of the form (2.2) may generally be unsustainable if not some or all of the curved and flat boundaries of the tube (denoted at $t > t_0$ with $r = a$, $r = b$ and $z = \pm h$) are supported by some set of externally applied normal tractions,

$$\sigma_{rr}\big|_{r=a} = q_a\left(z;t\right), \quad \sigma_{rr}\big|_{r=b} = q_b\left(z;t\right), \quad \sigma_{zz}\big|_{z=\pm h} = q_{\pm h}, \tag{2.13}$$

which may be determined in an *a-posteriori* manner. In the same context, the presupposed axisymmetric deformation may not be sustainable without appropriate external support of the type

$$\begin{gathered} \theta - \Theta = g\left(A,Z;t\right) = \hat{g}\left(a,z;t\right) = \alpha\left(z;t\right), \\ \sigma_{r\theta}\big|_{r=b} = \tau_{b\theta}\left(z;t\right), \\ \sigma_{rz}\big|_{r=a} = \tau_{az}\left(z;t\right), \quad \sigma_{rz}\big|_{r=b} = \tau_{bz}\left(z;t\right), \\ \sigma_{zr}\big|_{z=\pm h} = \tau_{\pm hr}\left(r;t\right), \quad \sigma_{z\theta}\big|_{z=\pm h} = \tau_{\pm h\theta}\left(r;t\right), \end{gathered} \tag{2.14}$$

which accommodates possible shear type of boundary growth.

Satisfaction of several of the non-homogeneous traction boundary conditions (2.13) and (2.14) is often required when deformation is due to externally applied loading. However, unless necessary, the homogeneous version of any of those traction boundary conditions (where the corresponding q- and τ-quantity is zero) should be given priority in problems that the deformation pattern (2.2) is totally due to mass-growth activity.

3. Growth/deformation patterns that preserve the direction of two families of helical fibres

Consider that the material of the tube is reinforced by two families of continuously distributed helical fibres wound about the tube axis symmetrically in opposite directions. In the reference configuration ($t = t_0$), the fibre directions are determined by the unit vectors

$$\hat{\boldsymbol{A}}^{(1)} = \left\{0, \hat{A}_\Theta, \hat{A}_Z\right\}^T, \quad \hat{\boldsymbol{A}}^{(2)} = \left\{0, -\hat{A}_\Theta, \hat{A}_Z\right\}^T, \tag{3.1}$$

where

$$\hat{A}_\Theta = \cos\Phi, \quad \hat{A}_Z = \sin\Phi, \tag{3.2}$$

and Φ denotes the angle that either family forms with the Z-axis of the adopted cylindrical polar co-ordinate system.

Axially symmetric deformations of the form (2.2) require from the directions of both fibre families and, therefore, from Φ to be independent of the azimuthal coordinate parameter, Θ. Both families are also required to have the same fibre density and, except for their directions, to be essentially indistinguishable. For simplicity, it is also assumed that Φ is constant. Hence, 2Φ is the constant angle that the regular helices of those families form at each material point of the tube.

Standard rules of vector transformation in continuum mechanics reveal that, at $t > t_0$, the unit vectors (3.1) transform into the following:

$$\begin{aligned} \boldsymbol{b}^{(1)} &= \left\{b_r^{(1)}, b_\theta^{(1)}, b_z^{(1)}\right\}^T = \hat{\boldsymbol{F}}\hat{\boldsymbol{A}}^{(1)} \\ &= \left\{\gamma_{rz}\hat{A}_Z,\ \ \chi_\theta\hat{A}_\Theta + \gamma_{\theta z}\hat{A}_Z,\ \ \chi_z\hat{A}_Z\right\}^T, \\ \boldsymbol{b}^{(2)} &= \left\{b_r^{(2)}, b_\theta^{(2)}, b_z^{(2)}\right\}^T = \hat{\boldsymbol{F}}\hat{\boldsymbol{A}}^{(2)} \\ &= \left\{\gamma_{rz}\hat{A}_Z,\ -\chi_\theta\hat{A}_\Theta + \gamma_{\theta z}\hat{A}_Z,\ \ \chi_z\hat{A}_Z\right\}^T, \end{aligned} \tag{3.3}$$

which define the direction of the deformed fibre families. Interest here focuses into the case that fibre directions and, therefore, the angles they form with any of the cylindrical polar coordinate directions remain unchanged during deformation.

This requirement imposes on fibre directions the following restrictions:

$$\boldsymbol{b}^{(1)} = \lambda_1\left(t\right)\hat{A}^{(1)}, \quad \boldsymbol{b}^{(2)} = \lambda_2\left(t\right)\hat{A}^{(2)}, \tag{3.4}$$

where, λ_1 and λ_2 represent fibre growth-stretch in the first and second family, respectively. Upon assuming that

$$\hat{A}_\Theta\hat{A}_Z \neq 0, \tag{3.5}$$

(3.4) may alternatively be written as follows:

$$\begin{gathered} b_r^{(1)} = b_r^{(2)} = 0, \\ \frac{b_\theta^{(1)}}{\hat{A}_\Theta} = \frac{b_z^{(1)}}{\hat{A}_Z} = \lambda_1\left(t\right), \quad -\frac{b_\theta^{(2)}}{\hat{A}_\Theta} = \frac{b_z^{(2)}}{\hat{A}_Z} = \lambda_2\left(t\right). \end{gathered} \tag{3.6}$$

Nevertheless, (3.4) make also clear that

$$b_z^{(1)} / \hat{A}_Z = b_z^{(2)} / \hat{A}_Z = \chi_z, \tag{3.7}$$

which, compared with (3.6b, c), lead to the following results:

$$\chi_z = \lambda_1(t) = \lambda_2(t) \equiv \lambda(t), \quad \gamma_{\theta z} = rg_{,Z} = 0,$$
$$b_\theta^{(1)} / \hat{A}_\Theta = -b_\theta^{(2)} / \hat{A}_\Theta = \chi_\theta = \lambda(t). \qquad (3.8)$$

The first of these relationships reveals that both fibre families experience identical fibre growth-stretch, $\lambda(t)$. Hence, connection of (3.8a) with (2.5c) yields axial placement in the form

$$z = \lambda(t)Z + \hat{z}(R;t); \quad \hat{z}(R,t_0) = 0, \quad \lambda(t_0) = 1, \qquad (3.9)$$

which also satisfy the initial conditions (2.3a, c).

Similarly, connection of (3.8b) with (2.5b) yields the radial placement as follows:

$$r = \lambda(t)R, \qquad (3.10)$$

and, as $\gamma_{rz} = r_{,Z} = 0$, the restrictions (3.6a) are satisfied identically. Finally, (3.8c) reveals that the most general form of the azimuthal placement is

$$g = g(R;t); \quad g = g(R;t_0) = 0, \qquad (3.11)$$

where the initial condition (2.3b) is also taken into consideration.

It is pointed out that (3.10) provides already the form of the radial placement for growth while (3.9) is on course to do the same for the axial placement. Despite that the functions $\lambda(t)$, $g(R;t)$ and $\hat{z}(R;t)$ are still to be determined, it is anticipated that, after constitutive equations for growth are introduced, the corresponding complete set of governing equations will appear overdetermined. It will be shown though in Sections 4 and 5 below that the extra information provided through the use of the equilibrium equations (2.12) can be used to feedback and, hence, elucidate particular features of the adopted type of mass-growth constitution.

The deformation features of the growing tube can now be updated and captured by the following simplified form of the deformation gradient and Cauchy-Green deformations tensors:

$$\hat{\boldsymbol{F}} = \begin{bmatrix} \chi_r & 0 & 0 \\ \gamma_{\theta r} & \chi_\theta & 0 \\ \gamma_{zr} & 0 & \chi_z \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ \lambda R g_{,R} & \lambda & 0 \\ \hat{z}_{,R} & 0 & \lambda \end{bmatrix},$$
$$\hat{\boldsymbol{C}} = \hat{\boldsymbol{F}}^{\boldsymbol{T}}\hat{\boldsymbol{F}} = \begin{bmatrix} \chi_r^2 + \gamma_{\theta r}^2 + \gamma_{zr}^2 & \chi_\theta\gamma_{\theta r} & \chi_z\gamma_{zr} \\ \chi_\theta\gamma_{\theta r} & \chi_\theta^2 & 0 \\ \chi_z\gamma_{zr} & 0 & \chi_z^2 \end{bmatrix}, \qquad (3.12)$$

where (2.5) and (2.6) still hold. It is thus noted that, regardless of the final form of the unknown functions $g(R;t)$ and $\hat{z}(R;t)$, the components of both $\hat{\boldsymbol{F}}$ and $\hat{\boldsymbol{C}}$ are at most functions of the radial co-ordinate parameter, R, and the time, t.

Finally, the rate of deformation tensor (2.8) obtains the simplified form

$$\hat{\boldsymbol{d}} = \begin{bmatrix} & \frac{}{2} & \\ \frac{1}{2}R\dot{g} & & \\ \frac{1}{2\lambda} & & \end{bmatrix}, \qquad (3.13)$$

and yields the volumetric rate of deformation as

$$\nabla \cdot \boldsymbol{v} = \mathrm{tr}\hat{\boldsymbol{d}} = 3\dot{\lambda} \quad . \qquad (3.14)$$

4. Incompressible mass-growth

In [10, 13. 14], the term incompressible mass-growth is associated with a particular class of non-isochoric, processes that preserve material density ($\rho/\rho_0 = 1 \neq \det\boldsymbol{F}$) and enable (2.10) to split into

$$\rho = \rho_0, \quad \nabla \cdot \boldsymbol{v} = r_g / \rho_0 \neq 0. \qquad (4.1)$$

Hence, nonnection of (4.1b) with (3.14) yields

$$r_g = 3\rho_0 \dot{\lambda} / \lambda. \qquad (4.2)$$

By virtue of (3.8), such incompressible mass-growth patterns of the fibre-reinforced tube are possible if r_g depends only on time. Integration of (4.2) with respect to time yields thus the fibre growth-stretch parameter as follows:

$$\lambda(t) = \exp\left(\frac{1}{3\rho_0}\int_{t_0}^{t} r_g \, dt\right), \qquad (4.3)$$

where use is made of the initial condition (3.9c).

4.1 *Material constitution*

The relevant constitutive equation [10, 13, 14],

$$\boldsymbol{\sigma} = \boldsymbol{F}\left(\frac{\partial W}{\partial \boldsymbol{C}} + \frac{\partial W}{\partial \boldsymbol{C}^{\boldsymbol{T}}}\right)\boldsymbol{F}^{\boldsymbol{T}} + W\boldsymbol{I}, \qquad (4.4)$$

describes a kind of purely elastic mass-growth behaviour. Here, W represents the strain energy density for growth, while the deformation gradient tensor, $\boldsymbol{F}$, and the right Cauchy-Green deformation tensor, $\boldsymbol{C}$, are measured in some arbitrary orthogonal co-ordinate system.

The constitutive equation (4.4) may thus be regarded as a superposition of the term $W\boldsymbol{I}$ on the standard constitutive equation met it conventional hyperelasticity. It is, however, recalled [10] that the strain energy density for growth is not necessarily identical with its conventional hyperelasticity counterpart.

The form (4.4) of the constitutive equation implies that the initial values that W and $\boldsymbol{\sigma}$ acquire at $t = t_0$ are interconnected, and potentially influenced by earlier mass-growth stages [14, 15]. Hence, (4.4) is associated with the initial conditions

$$W\Big|_{t=t_0} = W_0,$$
$$\boldsymbol{\sigma}\Big|_{\substack{t=t_0\\ \boldsymbol{F}=\boldsymbol{I}}} = \left(\frac{\partial W}{\partial \boldsymbol{C}} + \frac{\partial W}{\partial \boldsymbol{C}^{\boldsymbol{T}}}\right)\Bigg|_{\substack{t=t_0\\ \boldsymbol{F}=\boldsymbol{I}}} + W_0\boldsymbol{I} = \boldsymbol{T}_0, \tag{4.5}$$

where W_0 and $\boldsymbol{T}_0$ denote the initial value of W and the prestress tensor, respectively. This set of initial conditions is of importance in mass-growth problems that require from the initial configuration to be updated at regular time intervals. However, the influence that potential prestress may exert on the present analysis is disregarded in what follows, where, for simplicity, it is assumed that

$$\boldsymbol{T}_0 = \boldsymbol{0}. \tag{4.6}$$

The form of W must be an isotropic invariant of $\boldsymbol{C}$ and two unit non-parallel vectors, $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$. Such a form of W is expressible in terms of an appropriate set of deformation invariants, such as the classical set

$$\begin{gathered}
I_1 = \mathrm{tr}\boldsymbol{C},\quad I_2 = \frac{1}{2}\left\{(\mathrm{tr}\boldsymbol{C})^2 - \mathrm{tr}\boldsymbol{C}^2\right\},\quad I_3 = \det\boldsymbol{C},\\
I_4 = \boldsymbol{A}^{(1)T}\boldsymbol{C}\boldsymbol{A}^{(1)},\quad I_5 = \boldsymbol{A}^{(1)T}\boldsymbol{C}^2\boldsymbol{A}^{(1)},\\
I_6 = \boldsymbol{A}^{(2)T}\boldsymbol{C}\boldsymbol{A}^{(2)},\quad I_7 = \boldsymbol{A}^{(2)T}\boldsymbol{C}^2\boldsymbol{A}^{(2)},\\
I_8 = \boldsymbol{A}^{(1)T}\boldsymbol{C}^2\boldsymbol{A}^{(2)},\quad I_9 = \boldsymbol{A}^{(1)T}\boldsymbol{A}^{(2)} = \cos 2\Phi,\\
I_{10} = \boldsymbol{A}^{(1)T}\boldsymbol{C}^2\boldsymbol{A}^{(2)}.
\end{gathered} \tag{4.7}$$

If the sense of the fibres is not significant, then I_8 and I_{10} are customarily multiplied by I_9 and, hence, become even in both $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ (e.g., [11, 12, 16, 17]). It will be seen later though (see Appendix) that this modification is not necessary, at least in cases of a tube reinforced by a pair of equivalent fibre families wound symmetrically about its axis.

Due to certain syzygies that exist among the invariants listed in (4.7) (e.g., (30) in [12]), I_{10} is long known and/or claimed redundant (e.g., [16-19]). In this context, [11] and [12] show further that only seven of the deformation invariants listed in (4.7) can be independent, while only one among those seven, notably I_9, is a non-strain invariant.

The latter observations create a mathematical conflict in the present mass-growth problem. This conflict emerges from the fact that, in analogy with [10], the extra information involved in the partially known from (3.9) and (3.10) of the placement functions is expected to transform some of the equilibrium equations into relationships between W and its derivatives with respect to the employed deformation invariants. Such relationships need then to be regarded as PDEs, and solved for the unknown function W.

However, the implied relationships can be regarded as PDEs for W only if a set of precisely seven independent invariants is identified and used in the description of W, instead of the full classical set of non-independent invariants listed in (4.7).

4.2 *Constitutive equations in terms of independent invariants*

Such a set of precisely seven independent invariants has been identified in [13] (see also Appendix), and is as follows

$$\begin{aligned}
J_1 &= I_1 + 2\left(I_8 I_9 - I_4 - I_6\right)^2\left(1 - I_9^2\right)^{-1},\\
J_2 &= \left(I_4 + I_6 - I_8\right)\left(1 - I_9\right)^{-1},\\
J_3 &= \left(I_4 + I_6 + I_8\right)\left(1 + I_9\right)^{-1},\\
J_4 &= 2\left(I_4 - I_6\right)\left(1 - I_9^2\right)^{-1/2},\\
J_5 &= \pm\left\{\left(1 + I_9\right)^{-1}\left[\left(I_5 + I_7\right)/2 + I_{10}\right] - \left(J_3^2 + J_4^2\right)\right\}^{1/2},\\
J_6 &= \pm\left\{\left(1 - I_9\right)^{-1}\left[\left(I_5 + I_7\right)/2 - I_{10}\right] - \left(J_2^2 + J_4^2\right)\right\}^{1/2},\\
J_7 &= I_9.
\end{aligned} \tag{4.8}$$

In spite their cumbersome and unattractive form, when measured in the cylindrical polar co-ordinate system implied in (2.1), these invariants acquire the following simple values:

$$\begin{gathered}
J_1 = \hat{C}_{RR} \equiv \hat{C}_{11},\quad J_2 = \hat{C}_{\Theta\Theta} \equiv \hat{C}_{22},\\
J_3 = \hat{C}_{ZZ} \equiv \hat{C}_{33},\quad J_4 = \hat{C}_{\Theta Z} \equiv \hat{C}_{23},\\
J_5 = \hat{C}_{ZR} \equiv \hat{C}_{31},\quad J_6 = \hat{C}_{R\Theta} \equiv \hat{C}_{12},\\
J_7 = \cos 2\Phi,
\end{gathered} \tag{4.9}$$

where the suffices R, Θ and Z are meant to represent their Cartesian parameter counterparts 1, 2 and 3, respectively.

Unlike the known weakness of their classical counterparts listed in (4.7), these new invariants acquire simple physical meaning. Indeed, each of $J_1 - J_3$ represents amount of stretch along one of a corresponding cylindrical polar co-ordinate direction, while each of $J_4 - J_6$ measures amount of shear encountered on a relevant cylindrical polar co-ordinate surface. The independence of $J_1 - J_6$ is underpinned by the fact that each of these six strain invariants can be activated/controlled independently from the others by means of appropriately chosen homogeneous deformations (e.g., [16, 19]).

The standard non-stain invariant J_7, also met in the classical set (4.5), cannot enter the form of W on its own, but only through the influence it exerts on the remaining strain invariants, $J_1 - J_6$. It is accordingly evident that the strain energy density for growth may be represented in the following dual form:

$$W\left(I_1, I_2, \ldots, I_{10}\right) = \hat{W}\left(J_1, J_2, J_3, J_4, J_5\ J_6\right). \quad (4.10)$$

Upon replacing W with $\hat{W}$ in (4.4) and, then, developing the latter in the usual manner, one obtains the components of the Cauchy stress tensor as follows:

$$\begin{aligned}\sigma_{ij} = 2F_{iM}F_{jN}\left(\hat{W}_1\frac{\partial J_1}{\partial C_{MN}} + \hat{W}_2\frac{\partial J_2}{\partial C_{MN}} + \hat{W}_3\frac{\partial J_3}{\partial C_{MN}}\right) + \\ F_{iM}F_{jN}\left\{\hat{W}_4\left(\frac{\partial J_4}{\partial C_{MN}} + \frac{\partial J_4}{\partial C_{NM}}\right) + \hat{W}_5\left(\frac{\partial J_5}{\partial C_{MN}} + \frac{\partial J_5}{\partial C_{NM}}\right) + \right. \\ \left. \hat{W}_6\left(\frac{\partial J_6}{\partial C_{MN}} + \frac{\partial J_6}{\partial C_{NM}}\right)\right\} + \hat{W}\boldsymbol{I},\end{aligned} \quad (4.11)$$

where

$$\hat{W}_\ell \quad \frac{\partial \hat{W}}{\partial J_\ell}, \quad (\ell \qquad 6), \quad (4.12)$$

and all remaining suffices take the values 1, 2 and 3; the standard summation convention of repeated indices also applies.

The constitutive equation (4.11) holds in any orthogonal co-ordinate system, including the cylindrical polar co-ordinate system implied in (2.1) and (4.9). Hence, use of the notation adopted in (4.7), along with the intermediate results:

$$\frac{\partial J_1}{\partial \hat{C}_{MN}} = \frac{\partial \hat{C}_{11}}{\partial \hat{C}_{MN}} = \delta_{1M}\delta_{1N},$$
$$\frac{\partial J_2}{\partial \hat{C}_{MN}} = \frac{\partial \hat{C}_{22}}{\partial \hat{C}_{MN}} = \delta_{2M}\delta_{2N}, \ldots$$
$$\frac{\partial J_4}{\partial \hat{C}_{MN}} = \frac{\partial \hat{C}_{23}}{\partial \hat{C}_{MN}} = \delta_{2M}\delta_{3N}, \ldots$$

one transforms (4.11) into its following cylindrical polar coordinate version:

$$\begin{aligned}\sigma_{ij} = 2\left(\hat{W}_1\hat{F}_{i1}\hat{F}_{j1} + \hat{W}_2\hat{F}_{i2}\hat{F}_{j2} + \hat{W}_3\hat{F}_{i3}\hat{F}_{j3}\right) + \\ \hat{W}_4\left(\hat{F}_{i2}\hat{F}_{j3} + \hat{F}_{i3}\hat{F}_{j2}\right) + \hat{W}_5\left(\hat{F}_{i1}\hat{F}_{j3} + \hat{F}_{i3}\hat{F}_{j1}\right) + \\ \hat{W}_6\left(\hat{F}_{i1}\hat{F}_{j2} + \hat{F}_{i2}\hat{F}_{j1}\right) + \hat{W}\boldsymbol{I}.\end{aligned} \quad (4.13)$$

When regarded in association with the components of $\boldsymbol{\sigma}$, suffices 1, 2 and 3 are meant to represent r, θ and z, respectively, whereas, if associated with the components of $\hat{\boldsymbol{C}}$, they are meant to represent R, Θ and Z, respectively.

In the present case of interest, $\hat{\boldsymbol{F}}$ and $\hat{\boldsymbol{C}}$ obtain the forms shown in (3.12), and, hence, (4.11) produces the normal stress components:

$$\begin{aligned}\sigma_{rr} &= 2\lambda^2\hat{W}_1 + \hat{W}, \\ \sigma_{\theta\theta} &= 2\lambda^2\left[\left(\hat{W}_1 Rg_{,R} + \hat{W}_6\right)Rg_{,R} + \hat{W}_2\right] + \hat{W}, \\ \sigma_{zz} &= 2\left[\left(\hat{W}_1\hat{z}_{,R} + \lambda\hat{W}_5\right)\hat{z}_{,R} + \lambda^2\hat{W}_3\right] + \hat{W},\end{aligned} \quad (4.14)$$

along with the following set shear stresses:

$$\begin{aligned}\sigma_{\theta z} = \sigma_{z\theta} &= \lambda\left[\left(2\hat{W}_1\hat{z}_{,R} + \lambda\hat{W}_5\right)Rg_{,R} + \lambda\hat{W}_4 + \hat{W}_6\hat{z}_{,R}\right], \\ \sigma_{zr} = \sigma_{rz} &= \lambda\left(2\hat{W}_1\hat{z}_{,R} + \lambda\hat{W}_5\right), \\ \sigma_{r\theta} = \sigma_{\theta r} &= \lambda^2\left(2\hat{W}_1 Rg_{,R} + \hat{W}_6\right).\end{aligned} \quad (4.15)$$

4.3 *Equilibrium*

The components of $\hat{\boldsymbol{F}}$ and $\hat{\boldsymbol{C}}$ defined in (3.12) are at most functions of R and t. Hence, expressions (4.9) imply that $\hat{W}$ and, therefore, all stresses defined in (4.14) and (4.15) are at most functions of R and t as well. It follows that, while (3.10) also implies that R is proportional to r, the equilibrium equations (2.12) simplify as follows:

$$\sigma_{rr,r} + \frac{1}{r}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) = 0,$$
$$\left(r^2\sigma_{r\theta}\right)_{,r} = 0, \quad \frac{1}{r}\left(r\sigma_{rz}\right)_{,r} = 0. \tag{4.16}$$

Direct integration of (4.16b, c) yields

$$\sigma_{r\theta}\left(r;t\right) = \frac{b^2\tau_{b\theta}(t)}{r^2}, \quad \sigma_{rz}\left(r;t\right) = \frac{h\tau_{\pm h\theta}(t)}{r}, \tag{4.17}$$

where unique specification of the appearing arbitrary functions of time, $\tau_{b\theta}$ and $\tau_{\pm h\theta}$, is linked with satisfaction of the corresponding pair of boundary condition listed in (2.14).

However, the expressed priority on the homogeneous version of those boundary condition, namely

$$\tau_{b\theta}\left(t\right) = \tau_{\pm h\theta}\left(t\right) = 0, \tag{4.18}$$

is found compatible with the assumed incompressible mass-growth if

$$\sigma_{r\theta}\left(r;t\right) = \sigma_{rz}\left(r;t\right) = 0 \tag{4.19}$$

throughout the body of the growing tube.

Connection of (4.19) with the last pair of (4.15) requires

$$2\hat{W}_1\hat{z}_{,R} + \lambda\hat{W}_5 = 0, \quad 2R\hat{W}_1 g_{,R} + \hat{W}_6 = 0, \tag{4.20}$$

and these conditions are satisfied if

$$\hat{z}_{,R} = -\frac{\lambda\hat{W}_5}{2\hat{W}_1}, \quad g_{,R} = -\frac{\hat{W}_6}{2R\hat{W}_1}, \quad \hat{W}_1 \neq 0, \tag{4.21}$$

or

$$\hat{W}_1 = \hat{W}_5 = \hat{W}_6 = 0, \quad \hat{z}_{,R} g_{,R} \neq 0, \tag{4.22}$$

or

$$\hat{W}_1 = \hat{W}_5 = \hat{W}_6 = \hat{z}_{,R} g_{,R} = 0. \tag{4.23}$$

In any of these three cases, use of the ultimate form of the normal stresses (4.14a, b) will convert the last remaining equilibrium equation (4.16a) into a PDE for the unknown strain energy density for growth, $\hat{W}$. Potential solutions of that PDE are expected to provide admissible forms of $\hat{W}$ that enable realisation of the incompressible mass-growth implied by (4.1).

The most general form which that PDE may attain is associated with certain mass-growth patterns that enable validity of (4.21) in association with $\hat{z}_{,R} g_{,R} \neq 0$. That form of a PDE is highly non-linear and cumbersome and, since it is also particularly lengthy, is not cited here explicitly. Instead, a relatively simple example application detailed next, and this demonstrates the manner in which admissible forms of $\hat{W}$ are meant to be determined.

5. Application: Dilatational mass-growth

Consider an incompressible process of mass-growth which is purely dilatational, in the sense that $\gamma_{\theta r} = \gamma_{zr} = 0$ or, equivalently,

$$g_{,R} = \hat{z}_{,R} = 0. \tag{5.1}$$

In this particular case, the deformation gradient and Cauchy-Green deformation tensors acquire diagonal form, namely

$$\hat{\boldsymbol{F}} = diag\left\{\lambda(t), \lambda(t), \lambda(t)\right\},$$
$$\hat{\boldsymbol{C}} = diag\left\{\lambda^2, \lambda^2, \lambda^2\right\}. \tag{5.2}$$

By virtue of (5.1), (4.14) and (4.15) enable the nonzero stresses to simplify as follows:

$$\sigma_{rr} = 2\lambda^2\hat{W}_1 + \hat{W}, \quad \sigma_{\theta\theta} = 2\lambda^2\hat{W}_2 + \hat{W},$$
$$\sigma_{zz} = 2\lambda^2\hat{W}_3 + \hat{W}, \quad \sigma_{\theta z} = \sigma_{z\theta} = \lambda^2\hat{W}_4. \tag{5.3}$$

Moreover, (5.2) enables the six independent strain invariants (4.9) to becomae

$$J_1 = J_2 = J_3 = \left[\lambda(t)\right]^2, \ J_4 = J_5 = J_6 = 0, \tag{5.4}$$

and, as a result, $\hat{W}$ and its derivatives, as well as all nonzero stresses, are functions of the time only.

As (5.1) implies that $\hat{z}_{,R} g_{,R} = 0$ the present application fall into a category of problems that require satisfaction of the (4.21) or (4.23). These two cases are accordingly considered separately in what follows. It is however noted that, in either case, it is

$$\hat{W}_5 = \hat{W}_6 = 0. \tag{5.5a}$$

Moreover, priority in the homogeneous form of traction boundary conditions requires that, in either case, $\tau_{\pm h\theta}\left(r;t\right) = 0$ in the last of (2.14). It follows that

$$\sigma_{\theta z} = \hat{W}_4 = 0, \tag{5.5b}$$

and, hence, (4.10) obtains the simplified form

$$W\left(I_1, I_2, ..., I_{10}\right) = \hat{W}\left(J_1, J_2, J_3\right). \tag{5.6}$$

5.1 *Forms of the strain energy density consistent with (4.23)*

In the case that (4.23) holds, its combination with (5.5b) reveals that (5.6) simplifies further into

$$W\left(I_1, I_2, \ldots, I_{10}\right) = \hat{W}\left(J_2, J_3\right), \tag{5.7}$$

and the nonzero stress components become

$$\begin{aligned} \sigma_{rr} &= \hat{W}, \quad \sigma_{\theta\theta} = 2\lambda^2 \hat{W}_2 + \hat{W}, \\ \sigma_{zz} &= 2\lambda^2 \hat{W}_3 + \hat{W}. \end{aligned} \tag{5.8}$$

The last remaining equilibrium equation (4.16a) requires though that $\sigma_{rr} = \sigma_{\theta\theta}$. This is possible only if $\hat{W}_2 = 0$, and (5.6) reduces to

$$\begin{aligned} W\left(I_1, I_2, \ldots, I_{10}\right) &= \hat{W}\left(J_3\right) \\ &= W\left(I_1 + 2\left(I_8 I_9 - I_4 - I_6\right)^2 \left(1 - I_9^2\right)^{-1}\right), \end{aligned} \tag{5.9}$$

which also shows that, when expressed in terms of the classical deformation invariants, the admissible form sought of the strain energy density becomes unconventionally complicated.

As σ_{rr} are σ_{zz} are generally non-zero, the assumed mass-growth process becomes possible only if supported by appropriate sets of externally applied normal tractions. However, by setting $q_{\pm h} = 0$, (2.13c) reveals that homogeneous boundary conditions on the tube flat ends are still applicable if

$$\sigma_{zz} = 0. \tag{5.10}$$

Connection of this condition with (5.8c) requires

$$2 J_3 \hat{W}_3 + \hat{W} = 0. \tag{5.11}$$

Hence, solution of this differential equation specifies the strain energy density for growth (5.9) as follows:

$$\begin{aligned} W\left(I_1, I_2, \ldots, I_{10}\right) &= \hat{W}\left(J_3\right) = \hat{c} J_3^{1/2} \\ &= \hat{c}\left[I_1 + 2\left(I_8 I_9 - I_4 - I_6\right)^2 \left(1 - I_9^2\right)^{-1}\right]^{1/2}, \end{aligned} \tag{5.12}$$

where $\hat{c}$ is an arbitrary constant.

5.2 *Forms of the strain energy density consistent with (4.21)*

In the alternative case where (4.21) is valid, (5.6) holds still, and (5.8) are replaced by

$$\begin{aligned} \sigma_{rr} &= 2\lambda^2 \hat{W}_1 + \hat{W}, \\ \sigma_{\theta\theta} &= 2\lambda^2 \hat{W}_2 + \hat{W}, \quad \sigma_{zz} = 2\lambda^2 \hat{W}_3 + \hat{W}. \end{aligned} \tag{5.13}$$

Satisfaction of the last remaining equilibrium equations (4.16a) requires again that $\sigma_{rr} = \sigma_{\theta\theta}$ or, equivalently,

$$\hat{W}_1 = \hat{W}_2. \tag{5.14}$$

It follows that $\hat{W}\left(J_1, J_2, J_3\right)$ should also be symmetric in J_1 and J_2.

As σ_{rr} are σ_{zz} are generally non-zero, this type of mass-growth is again possible only if supported by appropriate sets of externally applied normal tractions. However, by requiring from one or both of σ_{rr} are σ_{zz} to be zero, one identifies several particular cases in which homogeneous boundary conditions are applicable in the flat ends or on the curved boundaries of the growing tube.

By requiring, for instance, that $\sigma_{zz} = 0$, someone enables the flat ends of the tube to be traction free. Use of (5.13c) leads then again to (5.11) which, however, is now regarded as a PDE rather than as an ordinary differential equation. Its solution thus yields the following class of admissible strain energy densities:

$$\hat{W}\left(J_1, J_2, J_3\right) = \tilde{c}\,[illegible]\,J_3^{1/2}, \tag{5.15}$$

where $\tilde{c}$ [illegible] represents a symmetric but, otherwise, arbitrary function of its arguments.

In a similar manner, by requiring $\sigma_{rr} = 0$, one obtains the PDE

$$2 J_1 \hat{W}_1 + \hat{W} = 0. \tag{5.16}$$

A solution of this equation, which is also symmetric in J_1 and J_2, is as follows:

$$\hat{W}\left(J_1, J_2, J_3\right) = [illegible]\,J_2^{1/2}, \tag{5.17}$$

where [illegible] is an arbitrary function of J_3. Such a form of a strain energy density for growth implies that, while it is also $\sigma_{\theta\theta} = 0$, the curved tube boundaries are free from traction. Hence, σ_{zz} is the only non-zero stress acting throughout the growing tube.

Same features of tube mass-growth are observed with use of the alternative class of strain energy densities

$$\hat{W}\left(J_1, J_2, J_3\right) = \hat{c}\left(J_3\right)\exp\left(-\frac{J_1 + J_2}{2J_3}\right), \qquad (5.18)$$

which are symmetric in J_1 and J_2; here, $\hat{c}\left(J_3\right)$ is an arbitrary function of J_3. This class satisfies simultaneously the differential equations

$$2J_3\hat{W}_1 + \hat{W} = 0, \quad 2J_3\hat{W}_2 + \hat{W} = 0, \qquad (5.19)$$

and, by virtue of (5.13a, c), returns

$$\sigma_{rr} = \sigma_{\theta\theta} = 0\,. \qquad (5.20)$$

Finally, stress-free mass-growth is also possible ($\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zz} = 0$) in cases that $\hat{W}$ satisfies either the set of simultaneous PDEs

$$\begin{aligned} &2J_1\hat{W}_1 + \hat{W} = 0, \\ &2J_2\hat{W}_2 + \hat{W} = 0, \qquad (5.21) \\ &2J_3\hat{W}_3 + \hat{W} = 0, \end{aligned}$$

or the alternative set

$$\begin{aligned} &\left(J_1 + J_2\right)\hat{W}_1 + \hat{W} = 0, \\ &\left(J_1 + J_2\right)\hat{W}_2 + \hat{W} = 0, \qquad (5.22) \\ &2J_3\hat{W}_3 + \hat{W} = 0. \end{aligned}$$

It can readily be verified that forms of $\hat{W}$ which are symmetric in J_1 and J_2 and consistent with the latter alternative sets of PDEs are

$$\hat{W}\left(J_1, J_2, J_3\right) = \breve{c}\left(J_1^{-1/2} + J_2^{-1/2} + J_3^{-1/2}\right), \qquad (5.23)$$

and

$$\hat{W}\left(J_1, J_2, J_3\right) = \breve{c}\left(J_1 + J_2\right)^{-1} J_3^{-1/2}, \qquad (5.24)$$

respectively, where $\breve{c}$ is an arbitrary constant.

6. Conclusions

This study built up on an earlier, pioneering step [10] of modelling mass-growth of a tube reinforced by embedded fibres that grow, with and within the tube, without changing shape/direction. While [10] dealt with the general problem of a growing tube reinforced by a single fibre family, the present investigation advanced further and considered that, as is very often observed in nature, there are embedded in the tube two families of helical fibres which are wound symmetrically about the tube axis.

The applications detailed in Section 5, dealt with relatively simple mass-growth patterns of incompressible and, also, dilatational tube mass-growth. However, those pilot examples demonstrated clearly the manner in which the presented analysis enables determination of admissible strain energies that render the tube ability to grow without disturbing the shape and/or direction of the embedded pair of helical fibres.

When compared with the results described in (3.12) and (3.13) of [10] for a single family of helical fibres, their counterparts (3.9) and (3.13) obtained in Section 3 reveal that presence of two families of helical fibres stabilises considerably the tube mass-growth behaviour, in the sense that (i) restricts growth into a radially proportional pattern, and (ii) decouples axial growth from its azimuthal counterpart. Moreover, through extra freedom offered by the presence of the arbitrary function $\hat{z}\left(R;t\right)$, (3.9) reveals that (iii) a pair of helical fibres enhances substantially the tube ability to grow in a non-dilatational manner (e.g., arteries, veins, plants and trees).

It is pointed out though that the kind of incompressible mass-growth considered in Section 4 refers to a relatively confined set of problems that the present model, detailed previously in Section 3, can be applied to. There exists, in fact, a vast area of relevant applications which is still completely unexplored. This refers to compressible mass-growth of hard and/or soft biological tissue, namely to non-isochoric mass-growth that allows mass density to vary with time. Investigation and proper study of mass-growth compressibility is thus regarded as a subject of immediate research interest.

The principal mathematical difficulty observed in that case stems from the fact that, in general, and regardless of the number of helical or other fibre families embedded in the tube, the continuity equation with growing mass (2.10) may need to be solved simultaneously with the equilibrium equations (2.12). In principle, such a solution should also be sought for general functional forms of the rate of growth, r_g, which, however, may depend not only on the time and/or the spatial parametrisation, but also on the observed mass-growth strain and deformation. Nevertheless, search for this kind of complicated solutions may be simplified by initially considering and studying cases of compressible mass-growth which is either nearly incompressible or associated with evidence (theoretical or experimental) that reveals *a-priori* the form of the volumetric rate of deformation (2.9).

It is accordingly fitting at this point to mention the additional mathematical hurdle which, as is mentioned in the Introduction, stems from the fact that the classical deformation invariants involved in the strain energy density are not independent when the number fibre families is bigger than one. This well-known fact suggests that corresponding relationships between W and its derivatives that reflect mass-growth ability of tissue to preserve fibre direction cannot be considered as partial differential equations for W.

The mathematical difficulty caused by the latter observation has been confronted successfully in [13] (see also Appendix) when the growing tissue is reinforced by two families of fibres. However, there are recently reported examples of arterial wall layers which are reinforced not only by a pair of symmetrically wound helices, but also by a third or even a fourth fibre family [5]. Mass-growth examples of tissue that preserves the shape and direction of more than two families of fibres are accordingly emerging in the literature. Their potential/future consideration and study will thus naturally require appropriate extension of the results reported in the Appendix or, more generally, in [13].

Appendix

The non-zero components (3.2) of the non-parallel unit vectors (3.1) are given in terms of the non-strain invariant $I_9 = \cos 2\Phi$ as follows:

$$\begin{aligned} \hat{A}_\Theta &= \frac{1}{\sqrt{2}}\left(1-I_9\right)^{1/2} > 0, \\ \hat{A}_Z &= \frac{1}{\sqrt{2}}\left(1+I_9\right)^{1/2} > 0. \end{aligned} \tag{A.1}$$

After I_2, I_3 and I_9 are temporarily excluded from the discussion, the remaining of the invariants listed in (4.7) are evaluated in the cylindrical polar co-ordinate system (2.1) and are found to be

$$\begin{aligned} I_1 &= \hat{C}_{RR} + \hat{C}_{\Theta\Theta} + \hat{C}_{ZZ}, \\ I_4 &= \hat{A}_\Theta^2 \hat{C}_{\Theta\Theta} + 2\hat{A}_\Theta \hat{A}_Z \hat{C}_{\Theta Z} + \hat{A}_3^2 \hat{C}_{ZZ}, \\ I_5 &= \hat{A}_\Theta^2\left(\hat{C}_{\Theta R}^2 + \hat{C}_{\Theta\Theta}^2 + \hat{C}_{\Theta Z}^2\right) \\ &\quad +2\hat{A}_\Theta \hat{A}_Z\left(\hat{C}_{\Theta R}\hat{C}_{ZR} + \hat{C}_{\Theta\Theta}\hat{C}_{Z\Theta} + \hat{C}_{\Theta Z}\hat{C}_{ZZ}\right) \\ &\quad + \hat{A}_Z^2\left(\hat{C}_{RZ}^2 + \hat{C}_{\Theta Z}^2 + \hat{C}_{ZZ}^2\right), \\ I_6 &= \hat{A}_\Theta^2 \hat{C}_{\Theta\Theta} - 2\hat{A}_\Theta \hat{A}_Z \hat{C}_{\Theta Z} + \hat{A}_Z^2 \hat{C}_{ZZ}, \\ I_7 &= \hat{A}_\Theta^2\left(\hat{C}_{\Theta R}^2 + \hat{C}_{\Theta\Theta}^2 + \hat{C}_{\Theta Z}^2\right) \\ &\quad -2\hat{A}_\Theta \hat{A}_Z\left(\hat{C}_{\Theta R}\hat{C}_{ZR} + \hat{C}_{\Theta\Theta}\hat{C}_{Z\Theta} + \hat{C}_{\Theta Z}\hat{C}_{ZZ}\right) \\ &\quad + \hat{A}_Z^2\left(\hat{C}_{RZ}^2 + \hat{C}_{\Theta Z}^2 + \hat{C}_{ZZ}^2\right), \\ I_8 &= \hat{A}_Z^2 \hat{C}_{ZZ} - \hat{A}_\Theta^2 \hat{C}_{\Theta\Theta}, \\ I_{10} &= \hat{A}_Z^2\left(\hat{C}_{RZ}^2 + \hat{C}_{\Theta Z}^2 + \hat{C}_{ZZ}^2\right) - \hat{A}_\Theta^2\left(\hat{C}_{\Theta R}^2 + \hat{C}_{\Theta\Theta}^2 + \hat{C}_{\Theta Z}^2\right). \end{aligned} \tag{A.2}$$

It is initially observed that these classical invariants are all quadratic in, and, therefore, even functions of the non-zero components of the fibre direction vectors, $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$. The sense of the fibres is thus not significant by default, and (see Section 4 above) I_8 and I_{10} need not be multiplied by I_9.

By virtue of (A.1), (A.2) are regarded as a set of seven simultaneous algebraic equations for the six strain components of the symmetric tensor $\hat{\boldsymbol{C}}$. However, as is explained in Section 4 (see also [13]), I_{10} is not independent of the remaining invariants and, therefore, (A.2) is essentially equivalent to a set of six algebraic equations for the six components of $\hat{\boldsymbol{C}}$. By solving this set of equations, one obtains (4.9) and (4.8), which reveal that the components of $\hat{\boldsymbol{C}}$ are also invariants of the deformation.

References

1. J.R. Barnett, V.A. Bonham, Cellulose microfibril angle in the cell wall of wood fibres. Biol Rev **79** (2004) 461-472.
2. A.G. Reisinger, D.H. Pahr, P.K. Zysset, Elastic anisotropy of bone lamellae as a

function of fibril orientation pattern. Biomech Model Mechanobiol **10** (2011) 67–77.

3. M. Fournier, J. Dlouhá, G. Jaouen, T. Almeras, Integrative biomechanics of tree ecology: beyond wood density and strength. J. Experim. Botany **64** (2013) 4793-4815.
4. C.J. Cyron, J.D. Humphrey, Preferred fiber orientations in healthy arteries and veins understood from netting analysis. Math. Mech. Solids **20** (2015) 680-696.
5. G.A, Holzapfel, J.A. Niestrawska, R.W. Ogden, A.J. Reinisch, A.J Schriefl, Modelling non-symmetric collagen fibre dispersion in arterial walls. J. R. Soc. Interface **12** (2015) 20150188. (doi.org/10.1098/rsif.2015.0188).
6. N. Qi, H. Gao, R.W. Ogden, N.A. Hill, G.A. Holzapfel, H.-C. Han, X. Luo, Investigation of the optimal collagen fibre orientation in human iliac arteries. J. Mech. Behav. Biomed. Mater. **52** (2015) 108-119.
7. A.J. Schriefl, G. Zeindlinger, D.M. Pierce, P. Regitnig, G.A, Holzapfel, Determination of the layer-specific distributed collagen fibre orientations in human thoracic and abdominal aortas and common iliac arteries. J. R. Soc. Interface **9** (2012) 1275-1286.
8. M. De Eguileor, R, Valvassori, G. Lanzavecchia, A. Grimaldi, Morphogenesis of helical fibres in haplotaxids. Hydrobiologia **334** (1996) 207-217.
9. M. De Eguileor, A. Grimaldi, G, Lanzavecchia, G, Tettamanti, R. Valvassori, Dimensional and numerical growth of helical fibers in leeches: An unusual pattern. J Exper Zoology A: Ecol Genet Physiol. **281** (1998) 171-187.
10. K.P. Soldatos, On the preservation of fibre direction during axisymmetric hyperelastic mass-growth of a finite fibre-reinforced tube. J. Eng. Math. **109** (2018) 173-210.
11. M.H.B.M. Shariff, The number of independent invariants of an *n*-preferred direction anisotropic solid. Math. Mech. Solids **22** (2017) 1989-1996.
12. M.H.B.M. Shariff, R. Bustamante, On the independence of strain invariants of two preferred direction nonlinear elasticity. Int. J. Eng. Sci. **97** (2015) 18-25.
13. K.P. Soldatos, A complete set on independent and physically meaningful invariants in the mechanics of solids reinforced by two families of fibres. Mech. Res. Comm. **94** (2018) 49-52.
14. K.P Soldatos, Modelling framework for mass-growth. Mech. Res. Comm. **50** (2013) 50-57.
15. K.P. Soldatos, Modelling framework for mass-growth II: the general case. Mech. Res. Comm. **65** (2015) 35-42.
16. A.J.M. Spencer, Deformations of fibre-reinforced materials, Clarendon Press, Oxford, 1972.
17. A.J.M. Spencer, Constitutive theory for strongly anisotropic solids. In: Spencer, A.J.M. (ed.) Continuum theory of the mechanics of fibre-reinforced composites. CISM courses and lectures, Springer, Wien, 1984, pp. 1–32.
18. A.J.M. Spencer, Theory of invariants. In: Eringen, A.C. (ed.) Continuum Physics, Vol. I – Mathematics, Academic Press, New York, 1971, pp 240-353.
19. R.W. Ogden, Non-linear Elastic Deformations, Wiley, New York, 1984.

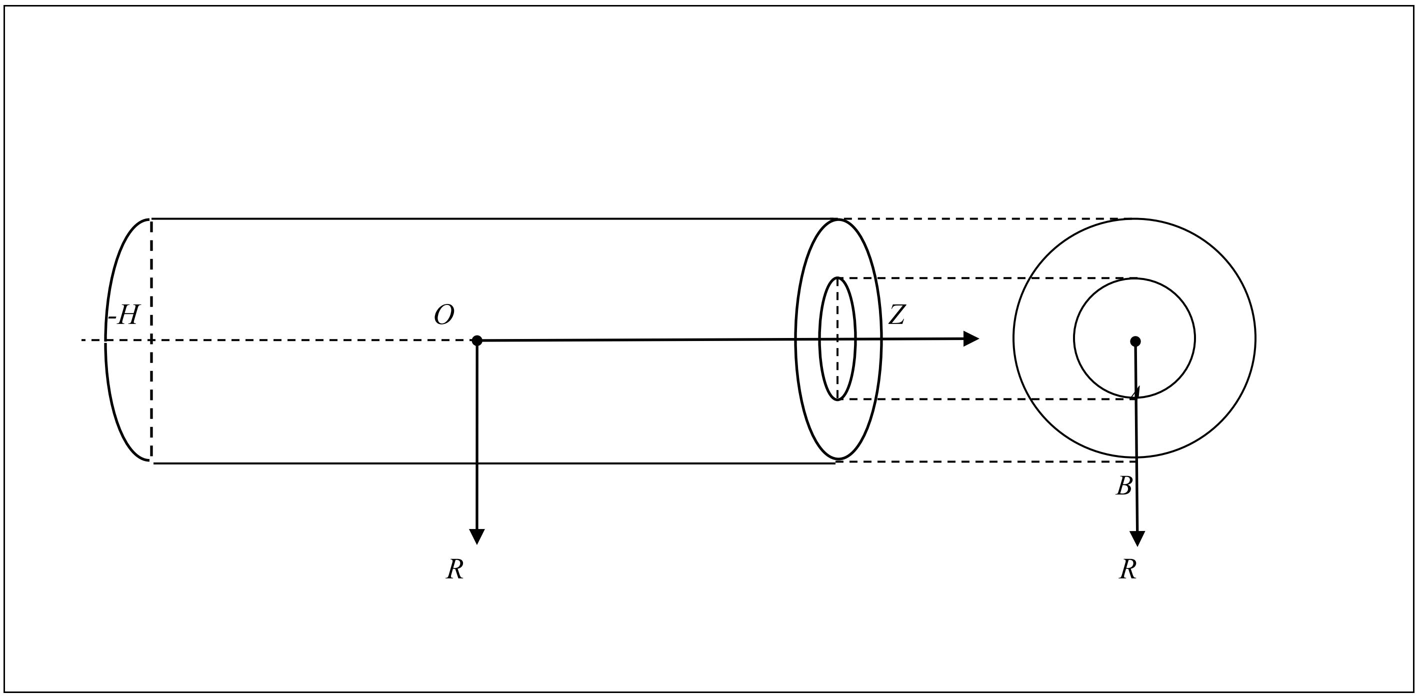

FIGURE 1: Nomenclature of a finite hollow circular cylindrical tube and its cross-section in the reference configuration. Nomenclature in the current configurations is obtained by replacing capital letters with their low case version.