Bank-Laine functions, the Liouville transformation and the Eremenko-Lyubich class

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Abstract

The Bank-Laine conjecture concerning the oscillation of solutions of second order homogeneous linear differential equations has recently been disproved by Bergweiler and Eremenko. It is shown here, however, that the conjecture is true if the set of finite critical and asymptotic values of the coefficient function is bounded. It is also shown that if E is a Bank-Laine function of finite order with infinitely many zeros, all real and positive, then its zeros must have exponent of convergence at least $3/2$, and an example is constructed via quasiconformal surgery to demonstrate that this result is sharp. MSC 2000: 30D35.

1 Introduction

If f is a non-constant entire function, let

$$\rho(f) = \limsup_{r \to +\infty} \frac{\log^+ T(r,f)}{\log r}, \quad \lambda(f) = \limsup_{r \to +\infty} \frac{\log^+ N(r,1/f)}{\log r} \leq \rho(f),$$

denote its order of growth and the exponent of convergence of its zeros [11]. In their landmark paper [1], Bank and Laine proved the following results on the oscillation of solutions of

$$y'' + A(z)y = 0. \tag{1}$$

Theorem 1.1 ([1]) *Let A be an entire function, let f_1, f_2 be linearly independent solutions of (1) and let $E = f_1 f_2$, so that $\lambda(E) = \max\{\lambda(f_1), \lambda(f_2)\}$.*
(i) If A is a polynomial of degree $n > 0$ then $\lambda(E) = (n+2)/2$.
(ii) If $\lambda(E) < \rho(A) < +\infty$ then $\rho(A) \in \mathbb{N} = \{1, 2, \ldots\}$.
(iii) If A is transcendental and $\rho(A) < 1/2$ then $\lambda(E) = +\infty$.

The case where $1/2 \leq \rho(A) < 1$ was considered by Rossi [22] and Shen [23].

Theorem 1.2 ([22, 23]) *Let A be an entire function of order $\rho(A)$ and let $E = f_1 f_2$, where f_1, f_2 are linearly independent solutions of (1). If $\rho(A) = 1/2$ then $\lambda(E) = +\infty$, while*

$$\frac{1}{\rho(A)} + \frac{1}{\lambda(E)} \leq 2 \quad \textit{if } 1/2 < \rho(A) < 1. \tag{2}$$

In particular, if $1/2 \leq \rho(A) < 1$ then $\rho(E) > 1$.

The methods of [1] focused on the product $E = f_1 f_2$ of linearly independent solutions f_j of (1), and in particular on the equation

$$4A = \left(\frac{E'}{E}\right)^2 - 2\,\frac{E''}{E} - \frac{c^2}{E^2}, \quad c = W(f_1, f_2), \tag{3}$$

linking E and A, in which the Wronskian $W(f_1, f_2) = f_1 f_2' - f_1' f_2$ is constant by Abel's identity. The paper [1] inspired much subsequent activity concerning the zeros of solutions of (1) and, more generally, linear differential equations with entire coefficients [16], and gave rise to the Bank-Laine conjecture – *let A be a transcendental entire function of finite order $\rho(A)$ and let f_1, f_2 be linearly independent solutions of* (1)*: if $\lambda(f_1 f_2)$ is finite then $\rho(A) \in \mathbb{N}$.* However, two remarkable recent papers of Bergweiler and Eremenko [5, 6] show via quasiconformal constructions not only that the Bank-Laine conjecture is false, but also that the inequality (2) is sharp.

When A is a non-constant polynomial in (1), satisfying $A(z) = a_n z^n(1 + o(1))$ as $z \to \infty$, there are $n + 2$ critical rays given by $\arg z = \theta^*$, where $a_n e^{i(n+2)\theta^*}$ is real and positive, and the Liouville transformation

$$Y(Z) = A(z)^{1/4} y(z), \quad Z = \int_{z_1}^{z} A(t)^{1/2}\, dt, \tag{4}$$

may be applied in sectors symmetric about these rays. This reduces (1) to a sine-type equation

$$\frac{d^2Y}{dZ^2} + \left(1 + \frac{O(1)}{Z^2}\right) Y = 0,$$

for which solutions asymptotic to $e^{\pm iZ}$ on a sectorial region in the Z plane are delivered by Hille's method [14, 15]. On one side of the critical ray, one of the corresponding solutions $A(z)^{-1/4} e^{\pm iZ}(1 + o(1))$ of (1) is large while the other is small, and these roles are reversed as the critical ray is crossed.

In contrast, for transcendental entire A, although a local analogue of Hille's method was developed in [17], applying on small neighbourhoods of maximum modulus points of A, the analytic continuation and estimation of Z in (4) present substantial difficulties. However, it turns out that for a certain class of entire functions A the transformation (4) may be adapted so as to be readily applicable on components where $|A(z)|$ is large.

The Eremenko-Lyubich class $\mathcal{B}$ plays a key role in complex dynamics [3, 9, 25] and consists of those transcendental meromorphic functions A with the following property: there exists a positive real number $M = M(A)$ such that all finite critical and asymptotic values of A have modulus less than M. Now suppose that $A \in \mathcal{B}$ is entire. Then, by standard results from [21, p.287] (see also [4]), all components U_M of the set $\{z \in \mathbb{C} : |A(z)| > M\}$ correspond to logarithmic singularities of A^{-1} over ∞; in particular, $v = \log A(z)$ maps each such U_M conformally onto the half-plane H given by $\operatorname{Re} v > \log M$. Under the change of variables

$$A(z) = e^v, \quad z = \phi(v), \quad \frac{A'(z)}{A(z)} = \frac{dv}{dz} = \frac{1}{\phi'(v)}, \tag{5}$$

in which $z = \phi(v)$ is the inverse mapping from H to U_M, a solution $y(z)$ of (1) on U_M transforms to a solution $w(v) = y(z)$ on H of

$$w''(v) - \frac{\phi''(v)}{\phi'(v)} w'(v) + e^v \phi'(v)^2 w(v) = 0, \tag{6}$$

and the second formula in (4) becomes, for a suitable choice of $z_1 = \phi(v_1)$,

$$Z = \int_{v_1}^{v} e^{u/2} \phi'(u)\, du. \tag{7}$$

The fact that ϕ' varies relatively slowly on H, by classical theorems on conformal mappings [13], makes it possible to prove the following theorem.

Theorem 1.3 *Suppose that A is a transcendental entire function in the Eremenko-Lyubich class $\mathcal{B}$, and let $E = f_1 f_2$, where f_1, f_2 are linearly independent solutions of (1). Then exactly one of the following holds.*
(A) The functions A and E satisfy $\rho(A) = \rho(E) = 1$ and

$$T(r, A) + T(r, E) = O(r) \quad \textit{as} \quad r \to +\infty. \tag{8}$$

(B) There exists $d > 0$ such that the zeros of E satisfy

$$n(r, 1/E) > \exp\left(d r^{1/2}\right) \quad \textit{as} \quad r \to +\infty, \tag{9}$$

and in particular $\rho(E) = \lambda(E) = +\infty$.

It follows from Theorem 1.3 that the Bank-Laine conjecture, despite being false in general [5], is true when the coefficient A is entire and in the class $\mathcal{B}$. An example going back to [1] shows that each of conclusions (A) and (B) can occur: if $A(z) = -e^{2z} - 1/4$ then (1) has solutions

$$f_1(z) = e^{-z/2} \exp\left(-e^z\right), \quad f_2(z) = e^{-z/2} \exp\left(e^z\right), \quad f_1(z) f_2(z) = e^{-z}, \quad \rho(f_1 f_2) = 1,$$

as well as solutions

$$g_1(z) = e^{-z/2} \sinh\left(e^z\right), \quad g_2(z) = e^{-z/2} \cosh\left(e^z\right), \quad \lambda(g_1 g_2) = +\infty.$$

An example will be given in Section 4 to show that the exponent $1/2$ in (9) is sharp.

The second main result of this paper concerns the location of zeros of Bank-Laine functions, that is, entire functions E such that $E(z) = 0$ implies $E'(z) = \pm 1$. By [2, Lemma C], an entire function E is a Bank-Laine function if and only if $E = f_1 f_2$, where f_1, f_2 are linearly independent solutions of (1) with A entire and $W(f_1, f_2) = 1$. Although a Bank-Laine function with no restriction on its growth may have an arbitrary sequence (a_n) of zeros, subject only to $a_n \to \infty$ without repetition [24], the following result was proved in [7] concerning Bank-Laine functions with real zeros.

Theorem 1.4 ([7]) *Let E be a Bank-Laine function of finite order, with infinitely many zeros, all real, and denote by $n(r)$ the number of zeros of E lying in $[-r, r]$. Then $n(r) \neq o(r)$ as $r \to +\infty$. If, in addition, all zeros of E are positive, then $n(r) \neq O(r)$ as $r \to +\infty$.*

The first assertion of Theorem 1.4 is evidently sharp, because of $\sin z$. The next theorem will establish a sharp lower bound for $\lambda(E)$ when E is a Bank-Laine function of finite order with infinitely many zeros, all real and positive. Here it is sufficient to consider the case where E is real entire, because otherwise it is possible to write $E = \Pi e^{P+iQ}$, where Π is the canonical product over the zeros of E, while P and Q are real polynomials; thus $e^{iQ(z)} = \pm 1$ at every zero of E and $F = \Pi e^P$ is also a Bank-Laine function.

Theorem 1.5 *Let E be a real Bank-Laine function of finite order, with infinitely many zeros, all real and positive. Then the exponent of convergence $\lambda(E)$ of the zeros of E is at least $3/2$. Moreover, if $\lambda(E) = 3/2$ then E and the associated coefficient function A have order $\rho(E) = \rho(A) = 3/2$.*

To demonstrate the sharpness of Theorem 1.5, quasiconformal techniques will be used in Section 6 to construct a real Bank-Laine function E, with only positive zeros, such that E and its associated coefficient function A satisfy $\lambda(E) = \rho(E) = \rho(A) = 3/2$, so that A provides a further counter-example to the Bank-Laine conjecture.

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2 A refinement of Hille's method

The following lemma is an extension of a method from [17], and provides bounds for the error terms in Hille's method [14, 15].

Lemma 2.1 *Let $c > 0$ and $0 < \varepsilon < \pi$. Then there exists $d > 0$, depending only on c and ε, with the following properties. Suppose that the function A is analytic, with $|1 - A(z)| \leq c|z|^{-2}$, on a domain containing*

$$\Omega = \Omega_{R,S} = \{z \in \mathbb{C} : \, 1 \leq R \leq |z| \leq S < +\infty, \, |\arg z| \leq \pi - \varepsilon\}.$$

Then the equation (1) has linearly independent solutions $U(z), V(z)$ satisfying

$$\begin{aligned} U(z) &= e^{-iz}(1 + \delta_1(z)), \quad U'(z) = -ie^{-iz}(1 + \delta_2(z)), \\ V(z) &= e^{iz}(1 + \delta_3(z)), \quad V'(z) = ie^{iz}(1 + \delta_4(z)), \end{aligned} \tag{10}$$

in which

$$|\delta_j(z)| \leq \frac{d}{|z|} \quad \textit{for} \quad z \in \Omega^*_{R,S} = \Omega_{R,S} \setminus \{z \in \mathbb{C} : \, \mathrm{Re}(z) < 0, \, |\mathrm{Im}(z)| < R\}. \tag{11}$$

Proof. Let $X = Se^{i\sigma}$, where $\sigma = \min\{\pi/2, \pi - \varepsilon\}$. Choose an analytic solution v on Ω of

$$v'' + 2iv' - Fv = 0, \quad F = 1 - A, \tag{12}$$

such that $v(X) = 1, v'(X) = 0$, and write

$$L(z) = v(z) - 1 + \frac{1}{2i}\int_X^z (e^{2i(t-z)} - 1)F(t)v(t)\,dt, \quad L'(z) = v'(z) - \int_X^z e^{2i(t-z)}F(t)v(t)\,dt, \tag{13}$$

so that

$$L''(z) = v''(z) + 2i\int_X^z e^{2i(t-z)}F(t)v(t)\,dt - F(z)v(z) = -2iL'(z).$$

Since $L(X) = L'(X) = 0$, the existence-uniqueness theorem implies that $L(z) \equiv 0$ on Ω.

Now let $z \in \Omega^*_{R,S}$ and let γ_z describe the clockwise arc of the circle $|t| = S$ from X to the first point x of intersection with the line $\mathrm{Im}(t) = \mathrm{Im}(z)$, followed by the straight line segment from x to z; then $|e^{2i(t-z)}| \leq 1$ on $\gamma_z \subseteq \Omega$. Since $L(z) = 0$, (13) gives

$$|v(z) - 1| \leq \int_X^z |F(t)v(t)|\,|dt|, \quad |v(z)| \leq 1 + \int_X^z |F(t)v(t)|\,|dt|. \tag{14}$$

Now parametrize γ_z by $t = \zeta(s)$, where s denotes arc length on γ_z. Using (14), write

$$H(s) = 1 + \int_0^s |F(\zeta(\sigma))v(\zeta(\sigma))|\,d\sigma, \quad H'(s) = |F(\zeta(s))v(\zeta(s))| \leq |F(\zeta(s))|H(s),$$

and

$$|v(\zeta(s)) - 1| \leq H(s) - 1 = \exp\left(\int_0^s \frac{H'(\sigma)}{H(\sigma)}\,d\sigma\right) - 1 \leq \exp\left(\int_0^s |F(\zeta(\sigma))|\,d\sigma\right) - 1,$$

which leads to

$$|v(z) - 1| \leq \exp(I_z) - 1, \quad I_z = \int_X^z |F(t)|\,|dt|. \tag{15}$$

Let $d_1, d_2, \ldots$ denote positive constants which depend only on c and ε. The circle $|t| = S$ contributes at most $d_1 S^{-1} \leq d_1|z|^{-1}$ to I_z in (15), while the contribution J_z from the horizontal part of γ_z satisfies:

$$\begin{aligned} J_z &\leq \int_{\mathrm{Re}\,z}^{+\infty} \frac{c}{t^2}\,dt = \frac{c}{\mathrm{Re}\,z} \leq \frac{d_2}{|z|} \quad \text{if } |\arg z| \leq \pi/4; \\ J_z &\leq \int_{\mathbb{R}} \frac{c}{x^2 + (\mathrm{Im}\,z)^2}\,dx \leq \frac{d_3}{|\mathrm{Im}\,z|} \leq \frac{d_4}{|z|} \quad \text{if } \pi/4 \leq |\arg z| \leq \pi - \varepsilon. \end{aligned}$$

Since $R \geq 1$, (13) and (15) now deliver

$$|v(z) - 1| \leq \exp\left(\frac{d_5}{|z|}\right) - 1 \leq \frac{d_6}{|z|} \leq d_6, \quad |v'(z)| \leq \int_X^z |F(t)|(1 + d_6)\,|dt| \leq \frac{d_7}{|z|}.$$

Now set $V(z) = v(z)e^{iz}$; then (12) implies that V solves (1), and the estimates (10) and (11) for V follow at once. To obtain U it is only necessary to apply the above argument to the equation solved by $\overline{y(\bar{z})}$ for every solution $y(z)$ of (1). $\square$

Unbounded sectorial regions may be handled as follows.

Lemma 2.2 *Suppose that $c > 0$ and $0 < \varepsilon < \pi$, and that the function A is analytic, with $|1 - A(z)| \leq c|z|^{-2}$, on $\Omega' = \{z \in \mathbb{C} : 1 \leq R \leq |z| < +\infty, |\arg z| \leq \pi - \varepsilon\}$. Then there exist $d > 0$, depending only on c and ε, and solutions U, V of (1) on*

$$\Omega'' = \{z \in \mathbb{C} : R < |z| < +\infty, |\arg z| < \pi - \varepsilon\} \setminus \{z : \mathrm{Re}(z) \leq 0, |\mathrm{Im}(z)| \leq R\},$$

such that U and V satisfy $W(U, V) = 2i$ and (10), with $|\delta_j(z)| \leq d/|z|$, on Ω''.

Proof. Taking a sequence $S_n \to +\infty$ yields solutions U_n, V_n of (1) on Ω^*_{R,S_n}, with corresponding error terms $\delta_{j,n}(z), j = 1, 2, 3, 4$. Here the functions $z\delta_{j,n}(z)$ are uniformly bounded, since the constant d is independent of S in (11). Thus, by normal families, it may be assumed that the $U_n, V_n, \delta_{j,n}$ converge locally uniformly on Ω''. The limit functions U, V satisfy (10), with $|\delta_j(z)| \leq d/|z|$ on Ω''. Since $W(U, V)$ is constant, by Abel's identity, it follows that $W(U, V) = 2i$. □

Finally, a change of variables $z \to -z$ shows that Lemmas 2.1 and 2.2 hold if $\Omega_{R,S}$ and $\Omega^*_{R,S}$, and correspondingly Ω' and Ω'', are replaced by their reflections across the imaginary axis.

3 Estimates in a half-plane

Throughout this section let $H = \{v \in \mathbb{C} : \mathrm{Re}\, v > 0\}$ and let $\phi : H \to \mathbb{C} \setminus \{0\}$ be analytic and univalent. For $v, v_1 \in H$, define $Z = Z(v, v_1)$ as in (7) by

$$Z(v, v_1) = \int_{v_1}^{v} e^{u/2}\phi'(u)\, du = 2e^{v/2}\phi'(v) - 2e^{v_1/2}\phi'(v_1) - 2\int_{v_1}^{v} e^{u/2}\phi''(u)\, du. \tag{16}$$

Since $0 \notin \phi(H)$ the image of H under $\log \phi$ contains no disc of radius greater than π; thus applying Bieberbach's theorem and Koebe's one quarter theorem [13, Theorems 1.1 and 1.2] to ϕ and $\log \phi$ respectively gives, for $u \in H$,

$$\left|\frac{\phi''(u)}{\phi'(u)}\right| \leq \frac{4}{\mathrm{Re}\, u}, \quad \left|\frac{\phi'(u)}{\phi(u)}\right| \leq \frac{4\pi}{\mathrm{Re}\, u}. \tag{17}$$

The fact that the estimates (17) are independent of ϕ is the key to the results of this section and the proof of Theorem 1.3.

Lemma 3.1 *Let ε be a small positive real number. Then there exists a large positive real number N_0, depending on ε but not on ϕ, with the following property.*

Let $v_0 \in H$ be such that $S_0 = \mathrm{Re}\, v_0 \geq N_0$, and define v_1, v_2, v_3, K_2 and K_3 by

$$v_j = \frac{2^j S_0}{128} + iT_0, \quad T_0 = \mathrm{Im}\, v_0, \quad K_j = \left\{v_j + re^{i\theta} : r \geq 0, -\frac{\pi}{2^j} \leq \theta \leq \frac{\pi}{2^j}\right\}. \tag{18}$$

Then the following three conclusions all hold:
(i) $Z = Z(v, v_1)$ satisfies, for $v \in K_2$,

$$Z = Z(v, v_1) = \int_{v_1}^{v} e^{u/2}\phi'(u)\, du = 2e^{v/2}\phi'(v)(1 + \delta(v)), \quad |\delta(v)| < \varepsilon. \tag{19}$$

(ii) $\psi = \psi(v, v_1) = \log Z(v, v_1)$ *is univalent on a domain containing* K_3.
(iii) There exists a domain D*, with* $v_0 \in D \subseteq K_3$*, mapped univalently by* Z *onto a sectorial region* M_3 *satisfying*

$$Z_0 = Z(v_0, v_1) \in M_3 = \{Z \in \mathbb{C} : \, |Z_0|/8 < |Z| < +\infty, \, |\arg(\eta Z)| < 3\pi/4\}, \qquad (20)$$

where $\eta = 1$ *if* $\operatorname{Re} Z_0 \geq 0$ *and* $\eta = -1$ *if* $\operatorname{Re} Z_0 < 0$.

Proof. To prove (i) assume that $S_0 = \operatorname{Re} v_0$ is large and let $v \in K_2$, so that

$$S = \operatorname{Re} v \geq \frac{S_0}{32} = 2 \operatorname{Re} v_1. \qquad (21)$$

Now v_1 may be joined to v by a straight line segment L_v which is parametrised with respect to $s = \operatorname{Re} u$, and an elementary arc length estimate $|du| \leq (\sec \pi/4) ds \leq 2\, ds$ holds on L_v. Thus (17) delivers, for $u \in L_v$,

$$|\phi'(u)| \leq \left(\frac{S}{s}\right)^8 |\phi'(v)|, \quad |\phi''(u)| \leq \frac{4}{s} |\phi'(u)| \leq \frac{4}{s} \left(\frac{S}{s}\right)^8 |\phi'(v)|, \qquad (22)$$

which implies by (21) that

$$\left| \frac{e^{v_1/2}\phi'(v_1)}{e^{v/2}\phi'(v)} \right| \leq \left(\frac{S}{\operatorname{Re} v_1} \right)^8 \exp\left(\frac{1}{2} \operatorname{Re}\,(v_1 - v) \right) \leq S^8 \exp\left(-S/4\right) < \frac{\varepsilon}{4} \qquad (23)$$

provided S_0 is large enough. Moreover, (22) leads to

$$\left| \frac{1}{e^{v/2}\phi'(v)} \int_{v_1}^{v} e^{u/2} \phi''(u)\, du \right| \leq \Psi(S) = \frac{8S^8}{e^{S/2}} \int_1^S e^{s/2} s^{-9}\, ds. \qquad (24)$$

Since $\lim_{S \to +\infty} \Psi(S) = 0$ by L'Hôpital's rule, (21) implies that $\Psi(S) < \varepsilon/4$ if S_0 is large enough. Thus (19) follows from (16), (23) and (24), which proves (i).

Next, (19) gives, on K_2,

$$\psi(v) = \psi(v, v_1) = \log Z(v, v_1) = \frac{v}{2} + \log 2 + \log \phi'(v) + \delta_1(v), \quad |\delta_1(v)| \leq 2|\delta(v)| < 2\varepsilon.$$

Since ε is small and S_0 is large, (17), (18) and Cauchy's estimate for derivatives now deliver

$$\left| \psi'(v) - \frac{1}{2} \right| \leq \frac{8}{\operatorname{Re} v} \leq \frac{1}{4}, \qquad (25)$$

and hence $\operatorname{Re} \psi'(v) > 0$, on a convex domain containing K_3, which proves (ii).

Now let

$$L_3 = \{v \in K_3 : \, \operatorname{Re} v \geq S_0/8\}.$$

Then, for $v \in L_3$, integration along the line segment from v_0 to $\operatorname{Re} v + iT_0$ followed by that from $\operatorname{Re} v + iT_0$ to v yields, in view of (25),

$$\psi(v) - \psi(v_0) = \frac{v - v_0}{2} + \eta(v), \quad |\eta(v)| \leq 8 \left(\left| \log \frac{\operatorname{Re} v}{S_0} \right| + \tan \frac{\pi}{8} \right). \qquad (26)$$

Since S_0 is large this implies that, for $v \in \partial L_3$ with $\operatorname{Re} v = S_0/8$,

$$\operatorname{Re}(\psi(v) - \psi(v_0)) \leq -\frac{7S_0}{16} + 8\left(\log 8 + \tan\frac{\pi}{8}\right) \leq \log\frac{1}{16}.$$

On the other hand, all other $v \in \partial L_3$ satisfy, by (18) and (26),

$$\begin{aligned} |\operatorname{Im}(v - v_0)| &\geq \left(\operatorname{Re} v - \frac{S_0}{16}\right)\tan\frac{\pi}{8} \geq \frac{\operatorname{Re} v}{2}\tan\frac{\pi}{8}, \\ |\operatorname{Im}(\psi(v) - \psi(v_0))| &\geq \frac{\operatorname{Re} v}{4}\tan\frac{\pi}{8} - 8\left(\left|\log\frac{\operatorname{Re} v}{S_0}\right| + \tan\frac{\pi}{8}\right) \geq 4\pi. \end{aligned}$$

Moreover, $\operatorname{Re}(\psi(v) - \psi(v_0)) \to +\infty$ as $v \to \infty$ in K_3, again by (26). Thus the strip

$$\left\{\psi(v_0) + \sigma + i\tau : \sigma \geq \log\frac{1}{8}, -2\pi \leq \tau \leq 2\pi\right\}$$

lies in the interior of $\psi(L_3)$, which completes the proof of (iii) and the lemma. □

Proposition 3.1 *There exists a positive real number N_1, independent of ϕ, with the following property. If $v_0 \in H$ satisfies*

$$\min\{S_0, |e^{v_0/2}\phi'(v_0)|\} > N_1, \quad S_0 = \operatorname{Re} v_0,$$

and if w_1, w_2 are linearly independent solutions of (6) with

$$W(w_1, w_2) = \pm\phi', \quad |w_1(v_0)w_2(v_0)| \geq 1, \tag{27}$$

then w_1w_2 has a sequence of distinct zeros $\zeta_m \to \infty$ in H which satisfy

$$|\phi(\zeta_m)| = O(\log m)^2 \quad \textit{as} \quad m \to +\infty. \tag{28}$$

Proof. Observe first that, by Abel's identity, the Wronskian of any two local solutions of (6) is a constant multiple of ϕ'. Fix a small positive ε and assume that $v_0 \in H$, that w_1, w_2 are linearly independent solutions of (6) which satisfy (27), and finally that S_0 and $|e^{v_0/2}\phi'(v_0)|$ are both large. Let v_1, v_2, v_3, K_2 and K_3 be as in (18), and define Z by (16). By Lemma 3.1, $Z_0 = Z(v_0, v_1)$ is large and there exist $\eta \in \{-1, 1\}$ and a domain $D \subseteq K_3$, both as in conclusion (iii), so that $M_3 = Z(D)$ satisfies (20). The change of variables

$$w(v) = e^{-v/4}W(Z), \quad w_j(v) = e^{-v/4}W_j(Z), \tag{29}$$

transforms (6) on D to the equation on M_3 given by

$$W''(Z) + (1 + G(Z))W(Z) = 0, \quad G(Z) = \frac{1}{16e^v\phi'(v)^2}\left(1 + 4\frac{\phi''(v)}{\phi'(v)}\right). \tag{30}$$

Here the derivatives in the first equation are with respect to Z, and

$$|G(Z)| \leq \frac{1}{|Z|^2} \tag{31}$$

on $M_3 = Z(D)$, by (17), (19) and the fact that $S_0 = \mathrm{Re}\, v_0$ is large. Now apply Lemma 2.2 with

$$\Omega' = \{Z \in \mathbb{C}: \, |Z_0|/4 \leq |Z| < +\infty, \, |\arg(\eta Z)| \leq 5\pi/8\} \subseteq M_3,$$

and let $M_4 = \Omega''$, so that $Z_0 = Z(v_0, v_1) \in M_4 \subseteq \Omega' \subseteq M_3$, by the choice of η. Since $|Z_0|$ is large, there exist solutions $U_1(Z), U_2(Z)$ of (30) on M_4, which satisfy $W(U_1, U_2) = 2i$ and

$$|U_1(Z)e^{iZ} - 1| + |U_2(Z)e^{-iZ} - 1| \leq \frac{d}{|Z|}, \tag{32}$$

in which the positive constant d is independent of v_0 and Z_0, by (31).

Suppose first that, on M_4,

$$W_1(Z) = \sigma_1 U_1(Z), \quad W_2(Z) = \sigma_2 U_2(Z), \quad \sigma_j \in \mathbb{C} \setminus \{0\}.$$

Then (19), (27) and (29) give

$$\pm\phi' = W(w_1, w_2) = e^{-v/2} W(W_1, W_2) \frac{dZ}{dv} = W(W_1, W_2)\phi' = 2i\sigma_1\sigma_2\phi',$$

so that $|\sigma_1\sigma_2| = 1/2$. But $\mathrm{Re}\, v_0$ and $|Z_0|$ are large, which implies, in view of (29) and (32), that

$$w_1(v_0)w_2(v_0) = e^{-v_0/2} W_1(Z_0)W_2(Z_0) = e^{-v_0/2}\sigma_1\sigma_2 U_1(Z_0)U_2(Z_0)$$

is small, a contradiction.

Because w_1 and w_2 are interchangeable, it now follows that at least one of W_1 and W_2, without loss of generality W_1, is a non-trivial linear combination

$$W_1(Z) = A_1 U_1(Z) - A_2 U_2(Z), \quad A_1, A_2 \in \mathbb{C} \setminus \{0\}, \tag{33}$$

of U_1, U_2 on M_4. Fix a small positive κ and suppose that

$$Z^* = \frac{1}{2i} \log \frac{A_1}{A_2} + \pi n,$$

where n is an integer of large modulus and appropriate sign, depending on η. Then $Z^* \in M_4$ and (32) implies that, on $|Z - Z^*| = \kappa$,

$$\frac{1}{2i} \log \frac{A_2 U_2(Z)}{A_1 U_1(Z)} - \pi n = Z - Z^* + J(Z), \quad |J(Z)| < \kappa.$$

Hence W_1 has a zero Z^{**} with $|Z^{**} - Z^*| < \kappa$, by Rouché's theorem and (33).

It follows that $W_1(Z)$ has distinct zeros $X_1, X_2, \ldots$, which tend to infinity in M_4 and satisfy $|X_m| \leq c_0 + c_1 m$, where $c_0, c_1, \ldots$ denote positive constants which may depend on v_0 and ϕ, but not on m. By (19), these zeros X_m satisfy, with $\zeta_m \in K_3$ and ε small,

$$X_m = Z(\zeta_m) = 2e^{\zeta_m/2}\phi'(\zeta_m)(1 + \delta(\zeta_m)), \quad |\delta(\zeta_m))| < \varepsilon.$$

Using (17) to estimate $|\log|\phi'(\zeta_m)||$ then gives, in view of (18),

$$\begin{aligned} |\zeta_m| &\leq c_2 + c_3 \mathrm{Re}\, \zeta_m \leq c_4 + c_5 \log^+ |X_m| + c_6 \log |\zeta_m|, \\ |\zeta_m| &\leq c_7 + c_8 \log^+ |X_m| \leq c_9 + c_{10} \log m. \end{aligned}$$

Now (28) is obtained by applying the Koebe distortion theorem [13, Theorem 1.3] with

$$\mu(\lambda) = \phi(v), \quad v = \frac{1+\lambda}{1-\lambda}, \quad |\lambda| < 1, \quad \zeta_m = \frac{1+\lambda_m}{1-\lambda_m}.$$

This yields, for large m, since ζ_m tends to infinity in K_3,

$$|\phi(\zeta_m)| = |\mu(\lambda_m)| \leq \frac{c_{11}}{(1-|\lambda_m|^2)^2} = c_{11} \left| \frac{|\zeta_m|^2 + 2\mathrm{Re}\,\zeta_m + 1}{4\mathrm{Re}\,\zeta_m} \right|^2 \leq c_{12}|\zeta_m|^2 \leq c_{13}(\log m)^2.$$

$\square$

4 Proof of Theorem 1.3

Let A, f_1 and f_2 be as in the hypotheses, without loss of generality satisfying $W(f_1, f_2) = \pm 1$, and set $E = f_1 f_2$. Choose $M > 0$ such that $|A(0)|$ and all finite critical and asymptotic values of A have modulus at most $M/2$. It may be assumed that $M \leq 1$, because otherwise the f_j may be replaced by the functions $g_j(z) = M^{1/2} f_j(z/M)$, which solve

$$y'' + B(z)y = 0, \quad B(z) = M^{-2}A(z/M).$$

If $T(r, E) = O(r)$ as $r \to +\infty$, then (3) delivers (8), and Theorems 1.1 and 1.2 force $\rho(A) = \rho(E) = 1$.

Assume henceforth that $T(r, E) \neq O(r)$ as $r \to +\infty$ and, following standard notation of the Wiman-Valiron theory [12], denote by $\mu(r, E)$ the maximum term of the Maclaurin series of E, and by $\nu(r, E)$ the central index. Then inequalities from [12] give

$$T(r, E) \leq \log^+ M(r, E) \leq \log^+ \mu(2r, E) + \log 2 \leq \int_1^{2r} \frac{\nu(t, E)}{t}\, dt + O(1), \tag{34}$$

and so it may be assumed that $\nu(r) = \nu(r, E) \neq O(r)$ as $r \to +\infty$.

Let $1/2 < \tau < 1$. It follows from the Wiman-Valiron theory [12] that there exists a sequence (z_n) satisfying

$$|z_n| = r_n \to +\infty, \quad |E(z_n)| = M(r_n, E), \quad \lim_{n\to+\infty} \frac{\nu(r_n)}{r_n} = +\infty, \tag{35}$$

such that, if $z = z_n e^\sigma$, $|\sigma| < \nu(r_n)^{-\tau}$, then

$$E(z) \sim \left(\frac{z}{z_n}\right)^{\nu(r_n)} E(z_n), \quad \frac{E'(z)}{E(z)} \sim \frac{\nu(r_n)}{z}, \quad \frac{E''(z)}{E(z)} \sim \frac{\nu(r_n)^2}{z^2},$$

as well as, in view of (3),

$$A(z) \sim -\frac{\nu(r_n)^2}{4z^2}, \quad A(z)^{-1/2} \sim \pm \frac{2iz}{\nu(r_n)}.$$

Thus (35) delivers $\min\{|E(z_n)|, |A(z_n)|\} \to +\infty$ as $n \to +\infty$, while applying Cauchy's estimate for derivatives to $A^{-1/2}$ yields

$$A(z_n)^{-3/2}A'(z_n) = O\left(\frac{r_n}{\nu(r_n)} \cdot \frac{\nu(r_n)^\tau}{r_n}\right) \to 0.$$

Take n so large that

$$|E(z_n)| > 2, \quad \log|A(z_n)| > N_1, \quad \left|\frac{A(z_n)^{3/2}}{A'(z_n)}\right| > N_1, \tag{36}$$

where N_1 is the positive constant from Proposition 3.1. Then z_n lies in a component C of $\{z \in \mathbb{C} : |A(z)| > 1\}$, and $0 \notin C$ since $|A(0)| < 1$. Because all finite critical and asymptotic values of A have modulus at most $1/2$, a change of variables (5) gives a conformal equivalence between C and the right half-plane $\mathrm{Re}\, v > 0$. Choose σ_n, with $\mathrm{Re}\,\sigma_n > 0$, such that $z_n = \phi(\sigma_n)$. Then $e^{\sigma_n} = A(z_n)$ and (5) and (36) imply that

$$\mathrm{Re}\,\sigma_n > N_1, \quad \left|e^{\sigma_n/2}\phi'(\sigma_n)\right| = \left|\frac{A(z_n)^{3/2}}{A'(z_n)}\right| > N_1. \tag{37}$$

A solution $y(z)$ of (1) transforms under (5) to a solution $w(v) = y(z)$ of (6), and $\{f_1, f_2\}$ to a pair of solutions $\{w_1, w_2\}$ of (6) with $W(w_1, w_2) = \pm\phi'$. Use (37) to apply Proposition 3.1, with $v_0 = \sigma_n = \phi^{-1}(z_n)$. Since $|w_1(v_0)w_2(v_0)| = |E(z_n)| > 2$, by (36), the function E has a sequence of distinct zeros $\phi(\zeta_m)$ satisfying (28). But this gives $c > 0$ such that, for all large $m \in \mathbb{N}$, and for r satisfying $c(\log m)^2 \leq r < c(\log(m+1))^2$,

$$n(c(\log m)^2, 1/E) \geq \frac{m}{2}, \quad n(r, 1/E) \geq \frac{m+1}{3} > \frac{1}{3}\exp\left((r/c)^{1/2}\right),$$

which establishes (9) and completes the proof of Theorem 1.3. □

The following example shows that the exponent $1/2$ in (9) is sharp. Let $A(z) = \cos\sqrt{z}$, which belongs to the Eremenko-Lyubich class $\mathcal{B}$, and let f be a non-trivial solution of (1). Let $\nu(r, f)$ be the central index of f and apply to f the same results from the Wiman-Valiron theory [12] as used in (34) and subsequently. If r is large and lies outside an exceptional set of finite logarithmic measure, and if $|z| = r$ and $|f(z)| = M(r, f)$, then

$$\frac{\nu(r, f)^2}{z^2} \sim \frac{f''(z)}{f(z)} = -A(z), \quad \nu(r, f) \leq \exp(c\sqrt{r}),$$

for some positive constant c. This upper bound for the non-decreasing function $\nu(r, f)$ then holds for all large r, possibly with a larger c, and so applying to f the inequalities of (34) gives $d_1 > 0$ with

$$n(r, 1/f) \leq N(3r, 1/f) \leq T(3r, f) + O(1) \leq \exp(d_1\sqrt{r})$$

as $r \to +\infty$. Because $\rho(A) = 1/2$, conclusion (A) of Theorem 1.3 cannot hold in this case, and so the exponent $1/2$ in (9) is sharp. □

5 Proof of Theorem 1.5

Let E be as in the hypotheses, and assume that the zeros of E have exponent of convergence $\lambda \leq 3/2$. Then the canonical product Π_0 over these zeros has order λ, and $E = \Pi_0 \exp(P_0)$, with P_0 a real polynomial. If P_0 has degree at least 2, then the zeros of E have Nevanlinna deficiency $\delta(0, E) = 1$, which contradicts [7, Theorem 4.1] (see also [18, Theorem 2.1]). It may therefore be assumed that E has order $\lambda \leq 3/2$.

There exist an entire function A and solutions f_1, f_2 of (1) such that $W(f_1, f_2) = 1$ and $E = f_1 f_2$. Then $f_j(z) = 0$ gives $E'(z) = (-1)^j$ and each f_j has infinitely many zeros, as may be seen by considering the graph of E on the real axis. Define U by

$$U = \frac{f_2}{f_1}, \quad \frac{U'}{U} = \frac{W(f_1, f_2)}{f_1 f_2} = \frac{1}{E}.$$

Lemma 5.1 *The coefficient function A in (1) has order at most λ but is transcendental.*

Proof. The first assertion is an immediate consequence of the Bank-Laine equation (3). The second may be deduced from a theorem of Steinmetz [26], or from a combination of Theorem 1.4 with the result of Edrei, Fuchs and Hellerstein [8] that if E is an entire function of finite order and genus at least 1, all of whose zeros are positive, then 0 is a Nevanlinna deficient value of E, from which the transcendence of A follows using (3). It may also be proved using Hille's method as follows. Suppose that A is a polynomial. Since the f_j have infinitely many positive zeros, the positive real axis must be one of the $2 + \deg(A)$ critical rays for the equation, and each f_j must be large in both adjacent sectors. Let L be the first other critical ray encountered when moving counter-clockwise from the positive real axis. Since the f_j have only positive zeros, both f_j must change from large to small as this critical ray L is crossed. A contradiction then arises from the fact that linearly independent solutions cannot be small in the same sector, because the Wronskian is a non-zero constant. $\square$

Because U'/U has order at most $3/2$ and is never 0, while all zeros and poles of U are simple, U has no critical values and finitely many asymptotic values [19]. Since U'/U is real, there exists $\theta \in \mathbb{R}$ such that $U = f_2/f_1 = e^{2i\theta} U_0$, with U_0 real meromorphic. But replacing f_1 by $f_1 e^{i\theta}$ and f_2 by $f_2 e^{-i\theta}$ leaves E unchanged, and so it may be assumed that $\theta = 0$ and U is real meromorphic.

Take zeros $x_0, x_1, x_2 \in \mathbb{R}$ of f_2, with $x_0 < x_1 < x_2$, and let R be the supremum of all $r > 0$ such that the branch of U^{-1} mapping 0 to x_1 admits unrestricted analytic continuation in the open disc $B(0, r)$ of centre 0 and radius r. Then R is finite, and U maps a simply connected domain Ω_1, with $x_1 \in \Omega_1$, univalently onto $B(0, R)$. Moreover, U^{-1} has a singularity over some α with $|\alpha| = R$, and Ω_1 contains a path γ which tends to infinity and is mapped by U onto the half-open line segment $[0, \alpha)$. If α is real then, because U is real meromorphic and univalent on Ω_1, the path γ must coincide with $(-\infty, x_1]$ or $[x_1, +\infty)$, contradicting the fact that $x_0, x_2 \notin \gamma$. Hence $\alpha \notin \mathbb{R}$ and, since U has finitely many critical and asymptotic values, U^{-1} has logarithmic singularities over α and $\overline{\alpha}$.

Lemma 5.2 *Let $F(z) = (E(z) - E(0))/z$. Then there exist $M_0 > 0$ and disjoint non-empty components Σ_1, Σ_2 of the set $\{z \in \mathbb{C} : |F(z)| > M_0\}$.*

Proof. There exists $M > 0$ such that for each $\beta \in \{\alpha, \overline{\alpha}\}$ there is a component Ω_β of the set $\{z \in \mathbb{C} : |U(z) - \beta| < 1/M\}$ mapped univalently by $v = \log 1/(U(z) - \beta)$ onto the half-plane H_0 given by $\mathrm{Re}\, v > \log M$. It may be assumed that M is so large that $\Omega_\beta \cap B(0,1) = \emptyset$ and $\Omega_\alpha \cap \Omega_{\overline{\alpha}} = \emptyset$. Let $\phi : H_0 \to \Omega_\beta$ be the inverse function and write

$$U(z) = \beta + e^{-v}, \quad z = \phi(v) \in \Omega_\beta, \quad v \in H_0. \tag{38}$$

Then

$$E(z) = \frac{U(z)}{U'(z)} = \frac{\beta + e^{-v}}{-e^{-v}} \cdot \phi'(v) = -(1 + \beta e^v)\phi'(v), \tag{39}$$

and ϕ satisfies, on H_0, as in (17),

$$\left| \frac{\phi''(v)}{\phi'(v)} \right| \leq \frac{4}{\mathrm{Re}\, v - \log M}, \quad \left| \frac{\phi'(v)}{\phi(v)} \right| \leq \frac{4\pi}{\mathrm{Re}\, v - \log M}. \tag{40}$$

It follows from (38), (39) and (40) that there exists $c_1 > 0$ such that, as $v \to +\infty$ on $\mathbb{R}$,

$$|z| = |\phi(v)| = o(v^{c_1}) = o(e^v |\phi'(v)|) = o(|E(z)|), \quad F(z) \to \infty,$$

whereas if $\mathrm{Re}\, v = 1 + \log M$ then

$$|E(z)| \leq |z|(1 + |\beta| Me) \left| \frac{\phi'(v)}{\phi(v)} \right| \leq |z|(1 + |\beta| Me)4\pi, \quad |F(z)| \leq (1 + |\beta| Me)4\pi + |E(0)|.$$

Hence there exist $M_0 > 0$ such that the set $\{v \in H_0 : |F(\phi(v))| > M_0\}$ has a component whose closure with respect to the finite plane lies in H_0. □

The remainder of the proof follows lines fairly similar to [22, 23]. By (3) and well known estimates for logarithmic derivatives [10], there exist positive integers M_1, M_2 such that

$$\left| \frac{E'(z)}{E(z)} \right| + \left| \frac{E''(z)}{E(z)} \right| \leq |z|^{M_1}, \quad |A(z)| = \frac{1}{4|E(z)|^2} + O\left(|z|^{M_2}\right), \tag{41}$$

for all z outside a union U_1 of countably many open discs, whose centres tend to infinity and whose radii have finite sum. Choose a polynomial P, of degree at most M_2, such that

$$B(z) = \frac{A(z) - P(z)}{z^{M_2+1}} \tag{42}$$

is entire. For $j = 1, 2$ define a subharmonic function $u_j(z)$ on $\mathbb{C}$ by $u_j(z) = \log |F(z)/M_0|$ on Σ_j, with $u_j(z) = 0$ on $\mathbb{C} \setminus \Sigma_j$, where M_0 and Σ_1, Σ_2 are as in Lemma 5.2. Similarly, let Σ_3 be a component of the set $\{z \in \mathbb{C} : |B(z)| > 1\}$, and set $u_3(z) = \log |B(z)|$ on Σ_3, with $u_3(z) = 0$ on $\mathbb{C} \setminus \Sigma_3$. These u_j have orders satisfying $\rho(u_j) \leq \rho(F) = \rho(E) = \lambda$, for $j = 1, 2$, while $\rho(u_3) \leq \rho(B) = \rho(A) \leq \lambda$.

For $j = 1, 2, 3$ and $t > 0$ let $\theta_j(t)$ be the angular measure of the intersection of Σ_j with the circle $S(0, t)$ of centre 0 and radius t. If $j = 1, 2$ and $|z|$ is large and $z \in \Sigma_j \cap \Sigma_3$, then (42) implies that z lies in the exceptional set U_1 of (41). Hence there exists a set $F_0 \subseteq [1, +\infty)$, of finite linear measure, such that if $r \in [1, +\infty) \setminus F_0$ then the following all hold: (a) $S(0, r)$ does not meet U_1; (b) $\Sigma_j \cap \Sigma_{j'} \cap S(0, r) = \emptyset$ for $j \neq j'$; (c) no Σ_j contains $S(0, r)$.

Let S be large and positive: then a well known consequence of Carleman's estimate for harmonic measure [27, pp.116-7] gives, as $r \to +\infty$,

$$\begin{aligned} 9 \log \frac{r}{S} &\leq \int_{[S,r]\setminus F_0} \left(\sum_{j=1}^{3} 1 \right)^2 \frac{dt}{t} + O(1) \leq \int_{[S,r]\setminus F_0} \left(\sum_{j=1}^{3} \theta_j(t) \right) \left(\sum_{j=1}^{3} \frac{1}{\theta_j(t)} \right) \frac{dt}{t} + O(1) \\ &\leq 2 \sum_{j=1}^{3} \int_{[S,r]\setminus F_0} \frac{\pi}{t\theta_j(t)} \, dt + O(1) \leq 2 \sum_{j=1}^{3} \log(\max\{u_j(z) : \, |z| = 2r\}) + O(1) \\ &\leq 2 \sum_{j=1}^{3} (\rho(u_j) + o(1)) \log r \leq (6\lambda + o(1)) \log r \leq (9 + o(1)) \log r. \end{aligned}$$

It follows at once that $\rho(u_j) = \lambda = 3/2$ for each j. □

6 A Bank-Laine function with positive zeros

The construction of an example demonstrating the sharpness of Theorem 1.5 will involve domains D_0, D_1, D_2 and D_3 defined by

$$\begin{aligned} D_0 &= \{u \in \mathbb{C} : \, 0 < |u| < +\infty, \, 0 < \arg u < 3\pi/2\}, \\ D_1 &= E_1 \cup E_2, \\ E_1 &= \{s + it : \, -\pi/2 < s < 0, \, -\infty < t < +\infty\}, \\ E_2 &= \{s + it : \, -\pi/2 < s < \pi/2, \, 0 < t < +\infty\}, \\ D_2 &= \{v \in \mathbb{C} : \, 0 < |v| < +\infty, \, -\pi/2 < \arg v < 0\}, \\ D_3 &= D_2 \cup \{\zeta \in \mathbb{C} : \, |\zeta| < 1, \, \mathrm{Re}\, \zeta > 0\}. \end{aligned} \tag{43}$$

Lemma 6.1 *Let $h : (-\infty, 1] \to (-\infty, 0]$ be a continuous bijection, such that $h(1) = 0$ while h' is continuous and has positive upper and lower bounds for $-\infty < y < 1$ (that is, there exists $\varepsilon > 0$ such that $\varepsilon < h'(y) < 1/\varepsilon$ for $-\infty < y < 1$). Then there exists a homeomorphism ψ from the closure of D_3 to that of D_2, such that: (A) ψ maps D_3 quasiconformally onto D_2, with $\psi(z) \to \infty$ and $\psi(z) = O(|z|)$ as $z \to \infty$ in D_3; (B) $\psi(iy) = ih(y)$ for $-\infty < y \leq 1$; (C) $\psi(z)$ is real and strictly increasing as z describes the boundary of D_3 clockwise from i to infinity.*

Proof. Let $\phi : D_3 \to D_2$ be a conformal bijection such that $\phi(i) = 0$ and $\phi(z) \to \infty$ as $z \to \infty$ in D_3. Then $\phi(z)$ is real and strictly increasing as z describes the boundary of D_3 clockwise from i to infinity. Moreover, there exists a continuous bijection $k : (-\infty, 1] \to (-\infty, 0]$ with $k(1) = 0$ and $\phi(iy) = ik(y)$. It is clear from the reflection principle that $k'(y)$ is continuous and positive for $-\infty < y < 1$, and it will be shown that $k'(y)$ has positive upper and lower bounds for $-\infty < y < 1$.

Take the restriction of ϕ to $\{z \in D_3 : \, |z| > r_1\}$, for some large positive r_1, and reflect twice, first across the imaginary axis and then across the real axis. This shows that

$$L_1 = \lim_{y \to -\infty} k'(y) = \lim_{z \to \infty} \phi'(z)$$

exists and is finite and positive, and gives $\phi(z) = O(|z|)$ as $z \to \infty$ in D_3.

Next, extend $\phi : D_3 \to D_2$ by reflection across the imaginary axis to a conformal mapping onto the lower half-plane, and apply the reflection principle to $\phi_1(u) = \phi(e^{iu})$ on the half-disc $\{u \in \mathbb{C} : |u - \pi/2| < r_2, \operatorname{Im} u > 0\}$, for some small positive r_2. This extended function has $\phi_1'(\pi/2) \neq 0$, which shows that $L_2 = \lim_{y \to 1-} k'(y)$ exists and is finite and non-zero, and hence positive by continuity.

The function $H = h \circ k^{-1}$ is a continuous bijection from $(-\infty, 0]$ to itself, and so there exists a homeomorphism η from the closure of D_2 to itself given by $\eta(x + iy) = x + iH(y)$ for $x \geq 0$ and $y \leq 0$. Furthermore, the chain rule shows that $H'(y)$ is continuous, with positive upper and lower bounds, for $-\infty < y < 0$. Hence η is C^1 on D_2 with

$$2\frac{\partial \eta}{\partial \bar{z}} = \eta_x + i\eta_y = 1 - H'(y), \quad 2\frac{\partial \eta}{\partial z} = \eta_x - i\eta_y = 1 + H'(y),$$

which ensures that η is quasiconformal on D_2, and that $\eta(z) = O(|z|)$ as $z \to \infty$ in D_2.

It now follows that $\psi = \eta \circ \phi$ is a homeomorphism from the closure of D_3 to that of D_2, quasiconformal on D_3 itself, and satisfies

$$\psi(iy) = \eta(\phi(iy)) = \eta(ik(y)) = iH(k(y)) = ih(y) \quad \text{for } -\infty < y \leq 1.$$

Finally, $\psi(z) = O(|\phi(z)|) = O(|z|)$ as $z \to \infty$ in D_3. □

Lemma 6.2 *Let E_0 be the closure of the domain D_0 in (43), and define F on $E_0 \setminus D_1$ by*

$$\begin{aligned}
F(s+it) &= f_1(s+it) \quad \text{for } -\infty < s \leq -\pi/2,\ t \in \mathbb{R}, \\
F(s+it) &= f_2(s+it) \quad \text{for } \pi/2 \leq s < +\infty,\ 0 \leq t < +\infty, \\
f_1(u) &= -i\exp(2e^{iu}), \\
f_2(u) &= \cot(u/2) = -i\left(\frac{1+e^{iu}}{1-e^{iu}}\right).
\end{aligned} \tag{44}$$

Then F extends to a mapping from E_0 into the extended plane, continuous with respect to the spherical metric, with the following properties.
(i) $H = \log F$ maps D_1 quasiconformally onto the quadrant D_2, with $H(\pi/2) = 0$.
(ii) Let L_0 be the path consisting of the line segment from π to 0 followed by the negative imaginary axis in the direction of $-i\infty$. Then $F(u)$ is real and strictly increasing as u describes L_0, and each $u_0 \in L_0$ has $s_0 > 0$ such that $\operatorname{Im} F(u) < 0$ on $D_0 \cap B(u_0, s_0)$.
(iii) F is locally injective on E_0;
(iv) There exists $c > 0$ such that $|F(u)| \leq \exp\exp(c|u|)$ for $u \in D_0$ lying on the circles $|u| = (4n+1)\pi/2$, $n \in \mathbb{N}$.

Proof. Using the principal argument and logarithm, set

$$\begin{aligned}
h(y) &= -\frac{\pi}{2} + 2y \quad \text{for } -\infty < y \leq 0, \\
h(y) &= -\frac{\pi}{2} + \arg\left(\frac{1+iy}{1-iy}\right) = -\frac{\pi}{2} - i\log\left(\frac{1+iy}{1-iy}\right) \quad \text{for } 0 < y \leq 1.
\end{aligned} \tag{45}$$

For $0 < y < 1$ this gives

$$h(y) = -\frac{\pi}{2} + 2\arctan y, \quad h'(y) = \frac{2}{1+y^2} \to 2 \quad \text{as } y \to 0+,$$

and so $h'(0) = 2$. Thus h is a continuous bijection from $(-\infty, 1]$ to $(-\infty, 0]$ and h' exists and is continuous for $-\infty < y < 1$, with $1 \leq h'(y) \leq 2$ there. Lemma 6.1 gives a homeomorphism ψ from the closure of D_3 to that of D_2, such that ψ maps D_3 quasiconformally onto D_2, with $\psi(z) = O(|z|)$ as $z \to \infty$ in D_3 and $\psi(iy) = ih(y)$ for $-\infty < y \leq 1$. Furthermore, $\psi(z)$ is real and strictly increasing as z describes the boundary of D_3 clockwise from i to infinity, and so is $G = \exp \circ \psi$, which is continuous on the closure of D_3 and satisfies, by (45),

$$\begin{aligned} G(v) &= \exp(ih(y)) = -i\exp(2iy) = -i\exp(2v) \quad \text{for } v = iy,\ -\infty < y \leq 0, \\ G(v) &= \exp(ih(y)) = -i\left(\frac{1+iy}{1-iy}\right) = -i\left(\frac{1+v}{1-v}\right) \quad \text{for } v = iy,\ 0 < y \leq 1. \end{aligned} \tag{46}$$

The next step is to set $F(u) = G(e^{iu})$ on the closure of D_1. Now $v = e^{iu}$ maps D_1 conformally onto D_3, with $v \to 0$ as $\operatorname{Im} u \to +\infty$ and $v \to \infty$ as $\operatorname{Im} u \to -\infty$. Furthermore, the boundary of D_1 is mapped by $v = e^{iu}$ as follows: the line $\operatorname{Re} u = -\pi/2$ to the negative imaginary axis; the half-line $\operatorname{Re} u = \pi/2$, $0 \leq \operatorname{Im} u < +\infty$, to the segment $v = iy$, $0 < y \leq 1$; the real interval $[0, \pi/2]$ to the arc of the unit circle from 1 to i; the negative imaginary axis to the real interval $(1, +\infty)$. Hence (44) and (46) imply that F is well-defined and continuous on E_0, and that (i) and (ii) hold. Moreover, because ψ is injective on D_3 and (44) implies that each $\log|f_j(u)|$ is positive for $-\pi/2 < \operatorname{Re} u < \pi/2$ and negative for $\pi/2 < |\operatorname{Re} u| < 3\pi/2$, the function F is locally injective on E_0. Thus it remains only to prove (iv). By (44), it is enough to bound the growth of $F(u)$ for $u \in D_1$, and hence it suffices to consider the continuous function $G(v)$ on the closure of D_3. But, as $v = e^{iu} \to \infty$ in D_3,

$$|F(u)| = |G(v)| \leq \exp(|\psi(v)|) \leq \exp(O(|v|)) = \exp\left(O(|e^{iu}|)\right) \leq \exp\exp(2|u|).$$

$\square$

Now define $V(z)$ on the open upper half-plane H^+ by $V(z) = F(z^{3/2})$, in which $z^{3/2}$ is the principal branch and F is as in Lemma 6.2. Then V extends first to a (spherically) continuous function from the closed upper half-plane into the extended plane, mapping $\mathbb{R}$ into $\mathbb{R} \cup \{\infty\}$, and then to the whole plane via $V(\overline{z}) = \overline{V(z)}$. The extended function V is locally injective on $\mathbb{C}$, in view of Lemma 6.2, and quasimeromorphic, by [20, Ch. I, Theorem 8.3]. Lemma 6.2(iv) delivers, as $n \to +\infty$,

$$\log^+ \log^+ |V(z)| = O(|z|^{3/2}) \quad \text{for} \quad |z| = r_n = \left(\frac{(4n+1)\pi}{2}\right)^{2/3}. \tag{47}$$

The remainder of the construction proceeds much as in [5]. Let D_4 be the pre-image in H^+ of the domain D_1 under $u = z^{3/2}$, let E_4 be its closure, and F_4 the union of E_4 and its reflection across the real axis. Then V is meromorphic off F_4 and writing $z = re^{i\theta}$ and $u = se^{i\eta}$ shows that the complex dilatation μ_V of V satisfies, for some $C_1, C_2 > 0$,

$$\begin{aligned}\int_{1 \leq |z| < +\infty} \left| \frac{\mu_V(z)}{z^2} \right| dxdy &\leq 2 \int_{1 \leq |z| < +\infty, z \in D_4} \frac{1}{r} \, dr \, d\theta \\ &= C_1 \int_{1 \leq |u| < +\infty, u \in D_1} \frac{1}{s} \, d\eta \, ds \\ &\leq C_2 \int_1^{+\infty} \frac{1}{s^2} \, ds = C_2. \qquad (48)\end{aligned}$$

Let ϕ be the (unique) quasiconformal homeomorphism of the extended plane which solves the Beltrami equation $\dfrac{\partial \phi}{\partial \overline{z}} = \mu_V(z) \dfrac{\partial \phi}{\partial z}$ a.e. and fixes each of 0, 1 and ∞ [20]. In view of (48) and the Teichmüller-Belinskii theorem [20, Ch. V, Theorem 6.1], there exists $\alpha \in \mathbb{C} \setminus \{0\}$ with

$$\phi(z) \sim \alpha z \qquad (49)$$

as $z \to \infty$. Moreover, there exists a locally univalent meromorphic function U such that $V = U \circ \phi$ on $\mathbb{C}$. Let $U_1(z) = \overline{U(\overline{z})}$: then

$$U(\phi(z)) = V(z) = \overline{V(\overline{z})} = \overline{U(\phi(\overline{z}))} = U_1\left(\overline{\phi(\overline{z})}\right).$$

Hence $\phi(z)$ and $\overline{\phi(\overline{z})}$ have the same complex dilation a.e. and both fix 0, 1 and ∞, which implies, by uniqueness, that ϕ is real on $\mathbb{R}$ and U is real meromorphic. Furthermore, $\phi([0, +\infty)) = [0, +\infty)$ and all zeros and poles of U are real and positive, while $E = U/U'$ is a real Bank-Laine function with positive zeros.

Now U satisfies, by Lemma 6.2 and (49),

$$n(r, 1/U) + n(r, U) = O(r^{3/2}) \quad \text{as } r \to +\infty. \qquad (50)$$

Let Π_1 and Π_2 be the canonical products over the zeros and poles of U respectively, which have order at most $3/2$, by (50), and write

$$U = \frac{\Pi_1}{\Pi_2} e^h, \quad \frac{1}{E} = \frac{U'}{U} = \frac{\Pi_1'}{\Pi_1} - \frac{\Pi_2'}{\Pi_2} + h', \qquad (51)$$

where h is an entire function. For $|z| = r_n$, where n is large, (47) and (49) yield

$$\log^+ \log^+ |U(\phi(z))| = O(|z|^{3/2}) = O(|\phi(z)|^{3/2}).$$

Thus the maximum principle delivers

$$\log^+ \log^+ |\Pi_2(\zeta) U(\zeta)| = O(|\zeta|^{3/2})$$

as $\zeta \to \infty$ and hence

$$\log T(r, \Pi_2 U) = O(r^{3/2}) \quad \text{as } r \to +\infty. \qquad (52)$$

Combining (52) with (51) and the lemma of the logarithmic derivative leads to

$$m(r, h') \leq m\left(r, \frac{(\Pi_2 U)'}{\Pi_2 U}\right) + m\left(r, \frac{\Pi_1'}{\Pi_1}\right) + O(1) = O(r^{3/2}) \quad \text{as } r \to +\infty.$$

Hence h' and E have order of growth at most $3/2$. Applying Theorem 1.5 then shows that E is a real Bank-Laine function whose zeros are all real and positive and have exponent of convergence $3/2$, and that E itself has order $3/2$, as has the associated coefficient function A. □

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