Particle detectors and the zero mode of a quantum field

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(Dated: April 2014)

We study the impact of the zero mode of a quantum field on the evolution of a particle detector. For a massless scalar field in a periodic cavity, we show that the impact of the zero mode on the Unruh-DeWitt detector and its derivative-coupling generalisation is necessarily nonvanishing but can be made negligible in some limits, including those commonly occurring in non-relativistic quantum optics. For the derivative-coupling detector this can be accomplished by just tuning the zero mode's initial state, but the standard Unruh-DeWitt detector requires a more subtle and careful tuning. Applications include an inertial detector with arbitrary velocity, where we demonstrate the regularity of the ultrarelativistic limit, and a detector with uniform acceleration.

arXiv:1404.5621v2 [quant-ph] 10 Jul 2014

I. INTRODUCTION

The response of a particle detector coupled to a quantum field has been subject of extensive research since the 70s to the present. While protocols that would directly measure the state of a quantum field are difficult to envisage, it is conceptually straightforward to make a measurement on a detector that has been allowed to interact with a quantum field. A spatially pointlike detector has a particular advantage in that it can be identified with an 'observer' who is moving through the spacetime: such detectors have been used to quantify the particle content in a given state of a quantum field as seen by a local observer [1–5], to analyze the entanglement contained in the vacuum state of a quantum field [6], to study metrology settings [7], to analyze the decoherence effects of relativistic trajectories [8, 9], to propose schemes of universal quantum computing via relativistic motion [10] and to set up scenarios of quantum communication in the relativistic limit [11, 12]. The question 'How many times does a particle detector click for a given field state and a given trajectory in spacetime?' is relevant from quantum optics [13] to the study of very fundamental problems as the quantum effects associated with the presence of horizons [14], or to serve as a witness of primordial quantum fluctuations which may give information about the nature of the gravitational interaction [15].

To model the field-detector interaction it is commonplace to use the so-called Unruh-DeWitt (UDW) detector [1, 2], which is a two-level quantum system that couples in a pointlike manner to a scalar field along its worldline. This model encompasses all the fundamental features of the light-matter interaction when there is no orbital angular momentum exchange involved [16, 17], and it is a useful tool for addressing a range of issues, from fundamental studies of the particle content in a given field state to quantum gravity, and from fundamental quantum optics to relativistic quantum information. A limit of the UDW detector model yields the Jaynes-Cummings model, which is ubiquitous in quantum optics as a phenomenological model of light-matter interaction [13]. Additionally, the UDW detector is a powerful effective model to describe the way in which superconducting qubits couple to microwave guides [18, 19].

In this paper we address the response of an UDW detector when the scalar field has a mode of vanishing frequency, known as a zero mode. Zero modes occur for a massless field in static cavities in Minkowski spacetime with Neumann or periodic boundary conditions, and they also occur in spatially compact cosmological spacetimes when the field is massless and couples conformally to the curvature [15]. Since the zero mode has a vanishing frequency, it cannot be treated as a harmonic oscillator, and it has no distinguished Fock vacuum. In this paper we tackle the following question: How does the zero mode, and particularly the inherent ambiguity in its quantum state, affect the response of an UDW detector?

Eliminating the zero mode by hand from the field mode expansion would lead to an ill-defined quantum field theory, with nonvanishing commutators for the field operator at spacelike-separated events. Dropping the zero mode by hand from the coupling between the detector and the field would give a detector model that is as such mathematically consistent; however, as seen in [17], the full linear coupling is necessary to model the $\boldsymbol{p}\cdot\boldsymbol{A}$ term by which an atomic electron couples to the quantized electromagnetic field. In quantum optics, these zero-mode issues arise with the common UDW and Jaynes-Cummings models with periodic and Neumann boundary conditions, and there it is usual to assume at the outset that any zero modes will have negligible effect and to drop them by hand. Our aim is to examine under which circumstances such dropping can be justified.

We will see that a non-careful treatment of the zero mode is dangerous: the zero mode is able to produce important effects on the detector's dynamics (both its click counting statistics and its quantum coherence). We will quantitatively study when these effects are maximized and minimized. In particular, we find that, by suitably choosing the initial state of the quantum field, it is indeed possible to minimize zero-mode effects on the detector dynamics under the UDW interaction. We will analyze both the direct effects on the detector coming from the choice of the zero mode and the possible cross-talk effects coming from the effective coupling of the zero mode and the rest of the field modes through their backreaction on the detector.

We work in a periodic cavity in $(1+1)$-dimensional Minkowski spacetime. We consider both the usual UDW detector [1, 2], which couples linearly to the scalar field, and its modification that couples linearly to the proper time derivative of the field [20–23].

We start in Section II by reviewing the quantisation of a massless scalar field in a $(1+1)$-dimensional periodic cavity, paying special attention to the zero mode and to its contribution to the two-point function. Section III analyses the density matrix of a conventional UDW detector and Section IV the transition probability of UDW-type detector with a derivative coupling. The results are summarised and discussed in Section V.

Throughout this paper, we use units in which $\hbar = c = 1$. The spacetime signature is $(-+)$ where the minus direction is timelike.

II. PRELIMINARIES: MASSLESS SCALAR FIELD IN A PERIODIC CAVITY

In this section we review the quantisation of a real, massless scalar field on a $(1+1)$-dimensonal flat, static spacetime with the spatial topology of a circle. In quantum optics terminology, this may be described as quantisation in a periodic cavity, or as quantisation in $(1+1)$-dimensonal Minkowski spacetime with periodic boundary conditions. We shall establish the notation in a way that includes explicitly the zero-mode contributions to the mode expansion of the field and its two-point function.

A. Spacetime and quantum field

The spacetime is a flat static cylinder with spatial circumference $L > 0$. We work in standard Minkowski coordinates (t, x) in which the metric reads

$$ds^2 = -dt^2 + dx^2 \ , \tag{II.1}$$

with the periodic identification $(t, x) \sim (t, x + L)$.

The quantum field is a free real massless scalar field ϕ, with the action

$$S = \frac{1}{2} \int \mathrm{d}t \, \mathrm{d}x \left[(\partial_t \phi)^2 - (\partial_x \phi)^2 \right] \ , \tag{II.2}$$

and the field equation is $\Box\phi = 0$. The (indefinite) Klein-Gordon inner product reads

$$(\phi, \phi') = \mathrm{i} \int \mathrm{d}x \, (\phi^* \, \partial_t \phi' - \phi' \, \partial_t \phi^* \,) \ , \tag{II.3}$$

where the star denotes complex conjugation.

We expand the Heisenberg picture field operator in the spatial Fourier modes in the usual fashion. We split the expansion as

$$\phi(t, x) = \phi_{\mathrm{osc}}(t, x) + \phi_{\mathrm{zm}}(t) \ , \tag{II.4}$$

where the oscillator-mode contribution $\phi_{\mathrm{osc}}(t, x)$ contains the Fourier components that are not spatially constant and the zero-mode contribution $\phi_{\mathrm{zm}}(t)$ contains the Fourier component that is spatially constant. We consider each contribution in turn.

B. Oscillator modes

The positive frequency oscillator modes of the classical field are

$$\phi_n(t, x) = \frac{1}{\sqrt{4\pi|n|}} \exp\!\left(-\mathrm{i}\frac{2\pi|n|}{L} t + \mathrm{i}\frac{2\pi n}{L} x \right) \ , \tag{II.5}$$

where $n \in \mathbb{Z} \setminus \{0\}$. The normalisation is such that $(\phi_m, \phi_n) = \delta_{mn}$. We thus have

$$\phi_{\mathrm{osc}}(t, x) = \sum_{n \neq 0} \left[a_n \phi_n(t, x) + a_n^\dagger \phi_n^*(t, x) \right] \ , \tag{II.6}$$

where the nonvanishing commutators of the annihilation and creation operators are

$$[a_n, a_m^\dagger] = \delta_{nm} \ . \tag{II.7}$$

The oscillator modes have a Fock vacuum $|0\rangle$, satisfying $a_n|0\rangle = 0$. The oscillator-mode contribution to the Fock vacuum Wightman function is

$$\begin{aligned} \langle 0|\phi_{\mathrm{osc}}(t, x)\phi_{\mathrm{osc}}(t', x')|0\rangle &= \sum_{n \neq 0} \phi_n(t, x)\phi_n^*(t', x') \\ = \sum_{n=1}^{\infty} \frac{1}{4\pi n} \Bigg\{ &\exp\!\left[-\mathrm{i}\frac{2\pi n}{L}(\Delta u - \mathrm{i}\epsilon) \right] \\ &+ \exp\!\left[-\mathrm{i}\frac{2\pi n}{L}(\Delta v - \mathrm{i}\epsilon) \right] \Bigg\} \ , \end{aligned} \tag{II.8}$$

where $u = t - x$, $v = t + x$, $\Delta u = u - u'$, $\Delta v = v - v'$, and $\epsilon \to 0_+$. The sum in (II.8) can be evaluated in closed form, with the result

$$\begin{aligned} &\langle 0|\phi_{\mathrm{osc}}(t, x)\phi_{\mathrm{osc}}(t', x')|0\rangle \\ &\quad = -\frac{1}{4\pi} \ln\left\{ 1 - \exp\!\left[-\mathrm{i}\frac{2\pi}{L}(\Delta u - \mathrm{i}\epsilon) \right] \right\} \\ &\qquad - \frac{1}{4\pi} \ln\left\{ 1 - \exp\!\left[-\mathrm{i}\frac{2\pi}{L}(\Delta v - \mathrm{i}\epsilon) \right] \right\} \ . \end{aligned} \tag{II.9}$$

C. Zero mode

We denote the spatially constant Fourier component of the classical field by Q. From (II.2), the Lagrangian for Q is

$$L_{\mathrm{zm}} = \frac{L}{2}\dot{Q}^2 \ . \tag{II.10}$$

Q has hence the dynamics of a nonrelativistic free particle on the real line, with L taking the role of the mass. The Hamiltonian reads

$$H_{\mathrm{zm}} = \frac{1}{2L}P^2 \ , \tag{II.11}$$

where P is the momentum conjugate to Q.

To quantise, let Q_S and P_S be the Schrödinger picture position and momentum operators corresponding to Q and P. Q_S and P_S are time-independent, they satisfy $[Q_S, P_S] = \mathrm{i}$, and they commute with all a_n and $a_n^\dagger$. The Heisenberg picture position operator reads

$$Q_H(t) = Q_S + L^{-1}P_S t \ . \tag{II.12}$$

It follows that the zero-mode contribution to the Heisenberg picture field operator (II.4) is given by

$$\phi_{\mathrm{zm}}(t) = Q_H(t) \ , \tag{II.13}$$

and ϕ_{zm} commutes with ϕ_{osc}. While the zero mode does not have a Fock vacuum, its contribution to the Wightman function is nevertheless well defined: denoting the Heisenberg picture quantum state of the zero mode by $|\psi\rangle$, the zero-mode contribution to the Wightman function is

$$\begin{aligned}\langle\psi|\phi_{\mathrm{zm}}(t)\phi_{\mathrm{zm}}(t')|\psi\rangle &= \langle\psi|Q_S^2|\psi\rangle + \langle\psi|P_S Q_S|\psi\rangle L^{-1}t \\ &+ \langle\psi|Q_S P_S|\psi\rangle L^{-1}t' + \langle\psi|P_S^2|\psi\rangle L^{-2}tt' \ . \end{aligned} \tag{II.14}$$

We emphasise that while the oscillator-mode contribution (II.8) to the Wightman function depends on t and t' only though the combination $t-t'$, the same does not hold for the zero-mode contribution (II.14): the Fock vacuum for the oscillator modes is time translation invariant, but the zero mode has no time translation invariant states. This will be significant in Sections III and IV below.

D. Stress-energy tensor

The renormalised stress-energy tensor of the quantum field may be computed from the Wightman function by point-splitting [3], using as the short distance subtraction term the Minkowski vacuum Wightman function [3, 24],

$$\langle\phi(t,x)\phi(t',x')\rangle_{\mathrm{Mink}} = -\frac{1}{4\pi}\ln\left[(\epsilon + \mathrm{i}\Delta u)(\epsilon + \mathrm{i}\Delta v)\right] \ . \tag{II.15}$$

We assume the field to be minimally coupled to curvature. When the oscillator modes are in the Fock vacuum, their contribution is [3]

$$T_{tt}^{\mathrm{osc}} = T_{xx}^{\mathrm{osc}} = -\frac{\pi}{6L^2} \ , \quad T_{tx}^{\mathrm{osc}} = 0 \ . \tag{II.16}$$

The zero-mode contribution is

$$T_{tt}^{\mathrm{zm}} = T_{xx}^{\mathrm{zm}} = \frac{\langle\psi|P_S^2|\psi\rangle}{2L^2} \ , \quad T_{tx}^{\mathrm{zm}} = 0 \ . \tag{II.17}$$

Note that the zero-mode contribution (II.17) is time translation invariant, even though the Wightman function (II.14) is not. Note also that both T_{tt}^{zm} and T_{xx}^{zm} are strictly positive.

III. DENSITY MATRIX OF THE UDW DETECTOR IN SECOND ORDER PERTURBATION THEORY

In this section we consider the evolution of the UDW detector's reduced density matrix in second order perturbation theory in the coupling constant, identifying explicitly the contributions from the zero mode of the field. The full reduced density matrix, rather than just the transition probabilities, is required for examining for example how the detector suffers decoherence when interacting with an arbitrary state of the field.

We would in particular like to identify situations where the effects of the zero mode on the detector's time evolution is negligible. This task consists of two steps:

1. Identify those zero mode initial states ('safe' initial states) for which the zero mode has a negligible effect on the detector's evolution, assuming that back-reaction of the detector on the zero mode is neglected.

2. Identify conditions under which a 'safe' zero mode initial state remains 'safe' under back-reaction from the detector.

We will see that guaranteeing the first condition is a very difficult endeavour. It will not be enough to demand that the zero mode initial state have vanishing energy, as one could have naively suspected.

On the other hand, we will see that the second condition can be satisfied by imposing constraints exclusively on the oscillator modes. If the oscillator modes are initialized in any ensemble of Fock states (Fock states or any diagonal density matrix in the Fock basis, such as a thermal state), a 'safe' zero-mode state does remain 'safe'.

A. Coupled dynamics

We consider a pointlike detector whose worldline $\mathsf{x}(\tau)$ is parametrised by the proper time τ. The detector is a two-level quantum system, with a Hilbert space spanned by the orthonormal energy eigenstates $|g\rangle$ and $|e\rangle$ whose respective energies are 0 and Ω. If $\Omega > 0$, $|g\rangle$ is the ground state and $|e\rangle$ is the excited state; if $\Omega < 0$, the roles are reversed. We employ a notation that is adapted to the case $\Omega > 0$, but all the formulas remain valid also for $\Omega < 0$, except in subsection III E where we take $\Omega > 0$.

The standard UDW interaction Hamiltonian is [1–4]

$$H = \lambda\chi(\tau)\mu(\tau)\phi\big(\mathsf{x}(\tau)\big) \ , \tag{III.1}$$

where $\mu(\tau)$ is the monopole moment operator, given by

$$\mu(\tau) = \sigma^+ e^{\mathrm{i}\Omega\tau} + \sigma^- e^{-\mathrm{i}\Omega\tau} \ , \tag{III.2}$$

and $\sigma^\pm$ are the usual raising and lowering operators, with the nonvanishing matrix elements $\langle e|\,\sigma^+\,|g\rangle = \langle g|\,\sigma^-\,|e\rangle = 1$. The switching function χ specifies how the interaction is switched on and off. We assume that χ is smooth. We also assume either that χ has compact support, in which case the system is strictly uncoupled both before and after the interaction, or that χ has sufficiently strong falloff properties for the system to be treated as asymptotically uncoupled in the distant past and future.

Working perturbatively to second order in λ, the interaction picture time evolution operator U for the full system is

$$U = U^{(0)} + U^{(1)} + U^{(2)} + \mathcal{O}(\lambda^3) \ , \tag{III.3}$$

where

$$U^{(0)} = \mathbb{1} \ , \tag{III.4a}$$
$$U^{(1)} = -\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}\tau\, H(\tau) \ , \tag{III.4b}$$
$$U^{(2)} = -\int_{-\infty}^{\infty}\mathrm{d}\tau\int_{-\infty}^{\tau}\mathrm{d}\tau'\, H(\tau)H(\tau') \ . \tag{III.4c}$$

Given an initial density matrix ρ_0, the final density matrix ρ_T is hence given by

$$\rho_T = [\mathbb{1}+U^{(1)}+U^{(2)}+\mathcal{O}(\lambda^3)]\rho_0[\mathbb{1}+U^{(1)}+U^{(2)}+\mathcal{O}(\lambda^3)]^\dagger \ . \tag{III.5}$$

Writing $\rho_T = \rho_0 + \rho_T^{(1)} + \rho_T^{(2)} + \mathcal{O}(\lambda^3)$, this gives

$$\rho_T^{(1)} = U^{(1)}\rho_0 + \rho_0 {U^{(1)}}^\dagger \ , \tag{III.6a}$$
$$\rho_T^{(2)} = U^{(1)}\rho_0 {U^{(1)}}^\dagger + U^{(2)}\rho_0 + \rho_0 {U^{(2)}}^\dagger \ . \tag{III.6b}$$

B. Detector state: Evolution equations

We take the detector's initial state to be an arbitrary density matrix, denoted by $\rho_{\mathrm{d},0}$. For the initial state of the field, we assume that the zero mode is in a pure state $|\psi\rangle$, and the oscillator modes are in a Fock state $|F\rangle$, that is, in a pure state in which each of the modes is in a number operator eigenstate. The initial density matrix for the full coupled system is hence

$$\rho_0 = \rho_{\mathrm{d},0} \otimes |F\rangle\langle F| \otimes |\psi\rangle\langle\psi| \ . \tag{III.7}$$

The time evolution of the detector's density matrix is given by

$$\rho_{d,T} = \mathrm{Tr}_{\mathrm{osc,zm}}\left(U\rho_0 U^\dagger\right) \ , \tag{III.8}$$

where the subscripts osc and zm indicate that the trace is over all the field degrees of freedom, including both the oscillator modes and the zero mode. We wish to analyse under what circumstances the contributions to $\rho_{d,T}$ can be separated into two decoupled parts, one accounting for the oscillator mode effects and the other for the zero mode effects.

Recall first that the interaction Hamiltonian (III.1) splits into the oscillator-mode contribution H_{osc} and the zero-mode contribution H_{zm} as

$$H = H_{\mathrm{osc}} + H_{\mathrm{zm}} \ , \tag{III.9a}$$
$$H_{\mathrm{osc}} = \lambda\chi(\tau)\mu(\tau)\sum_{n\neq 0}\left[a_n\phi_n(x,t) + a_n^\dagger\phi_n^*(x,t)\right] \ , \tag{III.9b}$$
$$H_{\mathrm{zm}} = \lambda\chi(\tau)\mu(\tau)\left(Q_S + \frac{P_S}{L}t\right) \ , \tag{III.9c}$$

ϕ_n is given by (II.5), and t and x are understood as functions of τ since the field couples to the detector at the detector's location. It follows that

$$U^{(1)} = U^{(1)}_{\mathrm{osc}} + U^{(1)}_{\mathrm{zm}} \ , \tag{III.10a}$$
$$U^{(1)}_{\mathrm{osc}} = -\mathrm{i}\lambda\int_{-\infty}^{\infty}\mathrm{d}\tau\, H_{\mathrm{osc}}(\tau) \ , \tag{III.10b}$$
$$U^{(1)}_{\mathrm{zm}} = -\mathrm{i}\lambda\int_{-\infty}^{\infty}\mathrm{d}\tau\, H_{\mathrm{zm}}(\tau) \ . \tag{III.10c}$$

Writing

$$\rho_{0,\mathrm{zm}} = \mathrm{Tr}_{\mathrm{osc}}\,\rho_0 \ , \qquad \rho_{0,\mathrm{d,osc}} = \mathrm{Tr}_{\mathrm{zm}}\,\rho_0 \ , \tag{III.11}$$

we thus have

$$\begin{aligned}\mathrm{Tr}_{\mathrm{osc,zm}}\big(U^{(1)}\rho_0\big) &= \mathrm{Tr}_{\mathrm{osc}}\left(U^{(1)}_{\mathrm{osc}}\rho_{0,\mathrm{d,osc}}\right) \\ &\quad + \mathrm{Tr}_{\mathrm{zm}}\left(U^{(1)}_{\mathrm{zm}}\rho_{0,\mathrm{d,zm}}\right) \\ &= \mathrm{Tr}_{\mathrm{zm}}\left(U^{(1)}_{\mathrm{zm}}\rho_{0,\mathrm{d,zm}}\right) \ ,\end{aligned} \tag{III.12}$$

where the last equality holds because H_{osc} is off-diagonal in the Fock basis. From (III.6a) and (III.12) we see that the order λ contribution to $\rho_{d,T}$ comes entirely from the zero mode of the field.

In order λ^2, the first term in (III.6b) gives

$$\begin{aligned}&\mathrm{Tr}_{\mathrm{osc,zm}}\left(U^{(1)}\rho_0{U^{(1)}}^\dagger\right) \\ &\quad = \mathrm{Tr}_{\mathrm{osc,zm}}\left(U^{(1)}_{\mathrm{osc}}\rho_0{U^{(1)}_{\mathrm{osc}}}^\dagger\right) + \mathrm{Tr}_{\mathrm{osc,zm}}\left(U^{(1)}_{\mathrm{zm}}\rho_0{U^{(1)}_{\mathrm{zm}}}^\dagger\right)\end{aligned}$$

$$= \mathrm{Tr}_{\mathrm{osc}}\left(U^{(1)}_{\mathrm{osc}}\rho_{0,\mathrm{d,osc}}U^{(1)\dagger}_{\mathrm{osc}}\right) + \mathrm{Tr}_{\mathrm{zm}}\left(U^{(1)}_{\mathrm{zm}}\rho_{0,\mathrm{d,zm}}U^{(1)\dagger}_{\mathrm{zm}}\right) , \tag{III.13}$$

where the cross terms that would involve both $U^{(1)}_{\mathrm{osc}}$ and $U^{(1)}_{\mathrm{zm}}$ are absent because H_{osc} is off-diagonal in the Fock basis. For the last two terms in (III.6b), using (III.4c), (III.9) and (III.10) gives

$$\mathrm{Tr}_{\mathrm{osc,zm}}\big(U^{(2)}\rho_0\big) = \mathrm{Tr}_{\mathrm{osc}}\big(U^{(2)}_{\mathrm{osc}}\rho_{0,\mathrm{d,osc}}\big) + \mathrm{Tr}_{\mathrm{zm}}\big(U^{(2)}_{\mathrm{zm}}\rho_{0,\mathrm{d,zm}}\big) , \tag{III.14}$$

where

$$U^{(2)}_{\mathrm{osc}} = -\int_{-\infty}^{\infty}\!\mathrm{d}\tau\int_{-\infty}^{\tau}\!\mathrm{d}\tau'\, H_{\mathrm{osc}}(\tau)H_{\mathrm{osc}}(\tau') , \tag{III.15a}$$

$$U^{(2)}_{\mathrm{zm}} = -\int_{-\infty}^{\infty}\!\mathrm{d}\tau\int_{-\infty}^{\tau}\!\mathrm{d}\tau'\, H_{\mathrm{zm}}(\tau)H_{\mathrm{zm}}(\tau') , \tag{III.15b}$$

and terms involving $H_{\mathrm{osc}}(\tau)H_{\mathrm{zm}}(\tau')$ and $H_{\mathrm{zm}}(\tau)H_{\mathrm{osc}}(\tau')$ have vanished because H_{osc} is off-diagonal in the Fock basis.

Collecting, we see from (III.12), (III.13) and (III.14) that the detector's density matrix after the time evolution is given to order λ^2 by

$$\rho_{T,\mathrm{d}} = \rho_{0,\mathrm{d}} + \rho^{\mathrm{osc}\,(2)}_{T,\mathrm{d}} + \rho^{\mathrm{zm}\,(1)}_{T,\mathrm{d}} + \rho^{\mathrm{zm}\,(2)}_{T,\mathrm{d}} + \mathcal{O}(\lambda^3) , \tag{III.16a}$$

$$\rho^{\mathrm{osc}\,(2)}_{T,\mathrm{d}} = \mathrm{Tr}_{\mathrm{osc}}\left(U^{(1)}_{\mathrm{osc}}\rho_{0,\mathrm{d,osc}}U^{(1)\dagger}_{\mathrm{osc}}\right) + \mathrm{Tr}_{\mathrm{osc}}\big(U^{(2)}_{\mathrm{osc}}\rho_{0,\mathrm{d,osc}} + \mathrm{H.c.}\big) , \tag{III.16b}$$

$$\rho^{\mathrm{zm}\,(1)}_{T,\mathrm{d}} = \mathrm{Tr}_{\mathrm{zm}}\left(U^{(1)}_{\mathrm{zm}}\rho_{0,\mathrm{d,zm}} + \mathrm{H.c.}\right) , \tag{III.16c}$$

$$\rho^{\mathrm{zm}\,(2)}_{T,\mathrm{d}} = \mathrm{Tr}_{\mathrm{zm}}\left(U^{(1)}_{\mathrm{zm}}\rho_{0,\mathrm{d,zm}}U^{(1)\dagger}_{\mathrm{zm}}\right) + \mathrm{Tr}_{\mathrm{zm}}\big(U^{(2)}_{\mathrm{zm}}\rho_{0,\mathrm{d,zm}} + \mathrm{H.c.}\big) . \tag{III.16d}$$

The oscillator-mode contribution and the zero mode contribution to the detector's evolution are hence decoupled to order λ^2: there is no 'back-reaction' of the zero mode on the oscillator modes and vice versa through the interaction with the detector (the oscillator modes depositing energy in the detector and the detector transferring that energy into the zero mode).

We emphasise that this decoupling relies on the assumption that the oscillator modes are initially in a Fock state. The decoupling would continue to hold if the oscillator mode initial state were generalised to a diagonal density matrix in the Fock basis, such as for example a thermal state, but it would not hold for arbitrary initial states, such as, for instance, a coherent state.

C. Detector state: Solution

We now specialise to the case where the oscillator-mode initial state $|F\rangle$ is the Fock vacuum $|0\rangle$.

We employ a matrix representation in which

$$|g\rangle = \begin{pmatrix}1\\0\end{pmatrix} , \qquad |e\rangle = \begin{pmatrix}0\\1\end{pmatrix} , \tag{III.17}$$

so that the monopole moment operator (III.2) has the form

$$\mu(\tau) = \begin{pmatrix}0 & e^{-\mathrm{i}\Omega\tau}\\ e^{\mathrm{i}\Omega\tau} & 0\end{pmatrix} . \tag{III.18}$$

We parametrise the detector's initial density matrix as

$$\rho_{0,\mathrm{d}} = a\,|g\rangle\langle g| + b\,|g\rangle\langle e| + b^*\,|e\rangle\langle g| + (1-a)\,|e\rangle\langle e| , \tag{III.19}$$

where $a\in\mathbb{R}$, $b\in\mathbb{C}$, and $(a-\frac{1}{2})^2 + |b|^2 \le \frac{1}{4}$. The matrix representation is

$$\rho_{0,\mathrm{d}} = \begin{pmatrix}a & b\\ b^* & 1-a\end{pmatrix} . \tag{III.20}$$

We address the oscillator-mode contribution and zero-mode contribution in (III.16) in turn.

1. *Oscillator-mode contribution* (III.16b)

To analyse the oscillator-mode contribution (III.16b), we introduce the quantities

$$u_n(\tau) = \phi_n\big(t(\tau), x(\tau)\big) , \tag{III.21a}$$

$$I_{n,\pm} = -\mathrm{i}\lambda\int_{-\infty}^{\infty}\mathrm{d}\tau\,\chi(\tau)\,e^{\pm\mathrm{i}\Omega\tau}u^*_n(\tau) , \tag{III.21b}$$

$$G_{n,\pm} = -\lambda^2\int_{-\infty}^{\infty}\!\mathrm{d}\tau\int_{-\infty}^{\tau}\!\mathrm{d}\tau'\,\chi(\tau)\chi(\tau')\,e^{\pm\mathrm{i}\Omega(\tau-\tau')} \times u_n(\tau)u^*_n(\tau') , \tag{III.21c}$$

where $n\in\mathbb{Z}\setminus\{0\}$ and ϕ_n is the oscillator mode function (II.5). In words, u_n is the pull-back of ϕ_n to the detector's worldline.

Consider the first term in (III.16b). With the notation (III.21), we have

$$\begin{aligned} U^{(1)}_{\mathrm{osc}}\rho_{0,\mathrm{d,osc}} &= \sum_{n\neq0}\big(I_{n,+}a^\dagger_n\sigma^+ + I_{n,-}a^\dagger_n\sigma^-\big)\,\rho_{0,\mathrm{d,osc}} \\ &= \sum_{n\neq0}|1_n\rangle\langle0|\otimes\Big[I_{n,+}\big(a\,|e\rangle\langle g| + b\,|e\rangle\langle e|\big) \\ &\qquad + I_{n,-}\big(b^*\,|g\rangle\langle g| + (1-a)\,|g\rangle\langle e|\big)\Big] . \end{aligned} \tag{III.22}$$

Multiplying from the right with $U^{(1)\dagger}_{\mathrm{osc}}$ and tracing over all the field modes, we find

$$\mathrm{Tr}_{\mathrm{osc}}\left(U^{(1)}_{\mathrm{osc}}\rho_{0,\mathrm{d,osc}}U^{(1)\dagger}_{\mathrm{osc}}\right)$$

$$= \sum_{n\neq 0} \begin{pmatrix} (1-a)|I_{n,-}|^2 & b^* I_{n,-} I^*_{n,+} \\ b\, I^*_{n,-} I_{n,+} & a\,|I_{n,+}|^2 \end{pmatrix} . \qquad \text{(III.23)}$$

Consider then the second term in (III.16b). We observe that

$$U^{(2)}_{\mathrm{osc}} \rho_{0,\mathrm{d},\mathrm{osc}} = -\lambda^2 \sum_{n\neq 0} \int_{-\infty}^{\infty} \mathrm{d}\tau \int_{-\infty}^{\tau} \mathrm{d}\tau'\, \chi(\tau)\chi(\tau') \times u_n(\tau) u_n^*(\tau')\, \mu(\tau)\mu(\tau')\, a_n a_n^\dagger\, \rho_{0,\mathrm{d},\mathrm{osc}} + (\text{traceless under } \mathrm{Tr}_{\mathrm{osc}}) . \qquad \text{(III.24)}$$

Writing out the product $\mu(\tau)\mu(\tau')$ using (III.18), adding the Hermitian conjugate, and taking $\mathrm{Tr}_{\mathrm{osc}}$, we obtain

$$\mathrm{Tr}_{\mathrm{osc}}\big(U^{(2)}_{\mathrm{osc}} \rho_{0,\mathrm{d},\mathrm{osc}} + \mathrm{H.c.}\big)$$

$$= \sum_{n\neq 0} \begin{pmatrix} 2a\,\mathrm{Re}\, G_{n,-} & b\big(G_{n,-} + G^*_{n,+}\big) \\ b^*\big(G_{n,+} + G^*_{n,-}\big) & 2(1-a)\,\mathrm{Re}\, G_{n,+} \end{pmatrix} , \qquad \text{(III.25)}$$

where $G_{n,\pm}$ is given in (III.21).

$\rho^{\mathrm{osc}\,(2)}_{T,\mathrm{d}}$ is hence given by the sum of (III.23) and (III.25). As $\rho^{\mathrm{osc}\,(2)}_{T,\mathrm{d}}$ must by construction be traceless and a is a continuous parameter, setting the trace of this sum to zero yields the identities

$$0 = \sum_{n\neq 0} \Big(|I_{n,\pm}|^2 + 2\,\mathrm{Re}\, G_{n,\mp} \Big) , \qquad \text{(III.26)}$$

which may be used to simplify $\rho^{\mathrm{osc}\,(2)}_{T,\mathrm{d}}$ to the final form

$$\rho^{\mathrm{osc}\,(2)}_{T,\mathrm{d}} = \sum_{n\neq 0} \begin{pmatrix} (1-a)|I_{n,-}|^2 - a\,|I_{n,+}|^2 & b^* I_{n,-} I^*_{n,+} + b\big(G_{n,-} + G^*_{n,+}\big) \\ b\, I^*_{n,-} I_{n,+} + b^*\big(G_{n,+} + G^*_{n,-}\big) & a\,|I_{n,+}|^2 - (1-a)|I_{n,-}|^2 \end{pmatrix} . \qquad \text{(III.27)}$$

2. *Zero-mode contributions* (III.16c) *and* (III.16d)

To evaluate the order λ zero-mode contribution (III.16c), we note from (III.9c) that

$$U^{(1)}_{\mathrm{zm}} \rho_{0,\mathrm{d},\mathrm{zm}} = -\mathrm{i}\lambda \int_{-\infty}^{\infty} \mathrm{d}\tau\, \chi(\tau) \Big(Q_S + \frac{P_S}{L} t(\tau) \Big) |\psi\rangle\langle\psi| \begin{pmatrix} 0 & e^{-\mathrm{i}\Omega\tau} \\ e^{\mathrm{i}\Omega\tau} & 0 \end{pmatrix} \begin{pmatrix} a & b \\ b^* & 1-a \end{pmatrix} . \qquad \text{(III.28)}$$

Adding the Hermitian conjugate and taking $\mathrm{Tr}_{\mathrm{zm}}$, we find

$$\rho^{\mathrm{zm}\,(1)}_{T,\mathrm{d}} = -\lambda \int_{-\infty}^{\infty} \mathrm{d}\tau\, \chi(\tau) \left(\langle Q_S \rangle_\psi + \frac{\langle P_S \rangle_\psi}{L} t(\tau) \right) \begin{pmatrix} 2\,\mathrm{Re}(\mathrm{i}b^* e^{-\mathrm{i}\Omega\tau}) & \mathrm{i}(1-2a) e^{-\mathrm{i}\Omega\tau} \\ \mathrm{i}(2a-1) e^{\mathrm{i}\Omega\tau} & 2\,\mathrm{Re}(\mathrm{i}b e^{\mathrm{i}\Omega\tau}) \end{pmatrix} . \qquad \text{(III.29)}$$

The order λ^2 zero-mode contribution (III.16d) may be evaluated similarly, with the result

$$\begin{aligned}
\rho^{\mathrm{zm}\,(2)}_{T,\mathrm{d}} = &-\lambda^2 \int_{-\infty}^{\infty} \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}\tau'\, \chi(\tau)\chi(\tau') \Big(\langle Q_S^2 \rangle_\psi + \frac{\langle P_S^2 \rangle_\psi}{L^2} t(\tau) t(\tau') + \frac{\langle Q_S P_S \rangle_\psi}{L} t(\tau') + \frac{\langle P_S Q_S \rangle_\psi}{L} t(\tau) \Big) \\
&\qquad \times \begin{pmatrix} (1-a) e^{-\mathrm{i}\Omega(\tau-\tau')} & b^* e^{-\mathrm{i}\Omega(\tau+\tau')} \\ b\, e^{\mathrm{i}\Omega(\tau+\tau')} & a\, e^{\mathrm{i}\Omega(\tau-\tau')} \end{pmatrix} \\
&- \lambda^2 \int_{-\infty}^{\infty} \mathrm{d}\tau \int_{-\infty}^{\tau} \mathrm{d}\tau'\, \chi(\tau)\chi(\tau') \Big(\langle Q_S^2 \rangle_\psi + \frac{\langle P_S^2 \rangle_\psi}{L^2} t(\tau) t(\tau') + \frac{\langle Q_S P_S \rangle_\psi}{L} t(\tau') + \frac{\langle P_S Q_S \rangle_\psi}{L} t(\tau) \Big) \\
&\qquad \times \begin{pmatrix} a\, e^{-\mathrm{i}\Omega(\tau-\tau')} & b\, e^{-\mathrm{i}\Omega(\tau-\tau')} \\ b^* e^{\mathrm{i}\Omega(\tau-\tau')} & (1-a) e^{\mathrm{i}\Omega(\tau-\tau')} \end{pmatrix} \\
&- \lambda^2 \int_{-\infty}^{\infty} \mathrm{d}\tau \int_{-\infty}^{\tau} \mathrm{d}\tau'\, \chi(\tau)\chi(\tau') \Big(\langle Q_S^2 \rangle_\psi + \frac{\langle P_S^2 \rangle_\psi}{L^2} t(\tau) t(\tau') + \frac{\langle P_S Q_S \rangle_\psi}{L} t(\tau') + \frac{\langle Q_S P_S \rangle_\psi}{L} t(\tau) \Big) \\
&\qquad \times \begin{pmatrix} a\, e^{\mathrm{i}\Omega(\tau-\tau')} & b\, e^{-\mathrm{i}\Omega(\tau-\tau')} \\ b^* e^{\mathrm{i}\Omega(\tau-\tau')} & (1-a) e^{-\mathrm{i}\Omega(\tau-\tau')} \end{pmatrix} . \qquad \text{(III.30)}
\end{aligned}$$

D. When is the zero-mode contribution small?

With the explicit formulas at hand, we are ready to address the central question: under what conditions are the zero-mode contributions to the detector's reduced density matrix small compared with the oscillator-mode contributions?

We can make the following observations.

(i). Suppose the zero mode initial state $|\psi\rangle$ is 'safe', in the sense of negligible zero-mode contributions to the detector's evolution. As the oscillator-mode contributions and the zero-mode contributions to the detector's density matrix are decoupled, the total state remains 'safe' throughout its evolution, to the order λ^2 in which we are working: the total state cannot become 'unsafe' via interaction of the zero mode and the oscillator modes through their mutual interaction with the detector. We recall again that this conclusion relies on the oscillator modes having been prepared in a Fock state, and the conclusion would continue to hold also when the oscillator modes are prepared in a noncoherent ensemble of Fock states, such as a thermal state.

(ii). Suppose the zero mode initial state $|\psi\rangle$ has a nonvanishing expectation value of Q_S or P_S. Then the contribution of the zero mode to the detector's density matrix is not only non-negligible but in fact dominant: the zero mode gives a nonzero contribution already in order λ while the oscillator-mode contributions appear only in order λ^2.

(iii). Suppose the zero mode initial state $|\psi\rangle$ has vanishing expectation values of Q_S and P_S. This does not suffice to guarantee that the zero-mode contribution to the detector's evolution would be negligible: the zero-mode contributions occur then in the same λ^2 order as the oscillator-mode contributions, and they cannot be identically vanishing for any $|\psi\rangle$ as they are linear combinations of the expectation values of Q_S^2, P_S^2, Q_SP_S and P_SQ_S.

(iv). Suppose the detector is initially in an incoherent superposition of the ground state and the excited state, so that $b = 0$ in (III.20). Then (III.29) shows that the evolution of the diagonal elements in the detector's reduced density matrix is of order λ^2, regardless of the initial state of the zero mode. The zero mode can therefore not give the dominant perturbative contribution to the transition probabilities between the two eigenstates of the detector, although it can give the dominant contribution when the initial state is a coherent superposition of the two energy eigenstates.

(v). The explicit appearance of t in (III.29) and (III.30) shows that the zero mode contribution to the detector's evolution is explicitly time-dependent. This is a consequence of the fact, noted in subsection II C, that the zero mode does not have time translation invariant states.

E. Example: Zero mode in a harmonic oscillator ground state

As an explicit example, we consider the zero mode state $|\psi\rangle$ whose wave function in the Q-representation is the Gaussian $\langle Q\,|\psi\,\rangle = (\gamma/\pi)^{1/4}\exp\!\left(-\frac{1}{2}\gamma Q^2\right)$ with $\gamma > 0$. In words, $|\psi\rangle$ is a harmonic oscillator ground state with a frequency proportional to γ. We emphasise that as $|\psi\rangle$ is in the Heisenberg picture and the zero mode Hamiltonian H_{zm} (II.11) is not that of a harmonic oscillator, the Schrödinger picture time evolution of $|\psi\rangle$ is not a pure phase but a spreading Gaussian. Nevertheless, in $|\psi\rangle$ we have

$$\langle Q_S\rangle = \langle P_S\rangle = 0\ , \tag{III.31a}$$

$$\langle Q_S^2\rangle = \frac{1}{2\gamma}\ ,\quad \langle P_S^2\rangle = \frac{\gamma}{2}\ , \tag{III.31b}$$

$$\langle Q_SP_S\rangle = -\langle P_SQ_S\rangle = \frac{\mathrm{i}}{2}\ , \tag{III.31c}$$

and from (II.11) we further see that $\langle H_{\mathrm{zm}}\rangle = \gamma/(4L)$. It follows from (III.29) that ${\rho_{T,\mathrm{d}}^{\mathrm{zm}}}^{(1)} = 0$, and we can use (III.30) to estimate ${\rho_{T,\mathrm{d}}^{\mathrm{zm}}}^{(2)}$. We can in particular ask whether the limit of small γ, in which the zero mode contribution to the stress-energy tensor (II.17) is small, suffices to make also ${\rho_{T,\mathrm{d}}^{\mathrm{zm}}}^{(2)}$ small.

We specialise to a detector that is static in the rest frame of the cylinder, therefore $t(\tau) = \tau$.

To estimate the strength of the zero-mode contribution to the detector's dynamics, we introduce the following estimator modelled on the $U^{(1)}\rho_0U^{(1)}$ terms in (III.30):

$$\begin{aligned}E_{\mathrm{zm}}^{\pm} =& \frac{\lambda^2}{2}\int_{-\infty}^{\infty}\mathrm{d}\tau\int_{-\infty}^{\infty}\mathrm{d}\tau'\,\chi(\tau)\chi(\tau')\left[\frac{1}{\gamma}+\frac{\gamma}{2L^2}\tau\tau'\right.\\ &\left.+\frac{\mathrm{i}}{L}(\tau'-\tau)\right]e^{\pm\mathrm{i}\Omega(\tau-\tau')}\ .\end{aligned} \tag{III.32}$$

For the Gaussian switching function

$$\chi(\tau) = \frac{1}{\pi^{1/4}\sigma^{1/2}}e^{-\tau^2/(2\sigma^2)}\ , \tag{III.33}$$

where the positive parameter σ is the effective duration of the interaction, the integrals can be evaluated analytically, yielding

$$E_{\mathrm{zm}}^{\pm} = \lambda^2\sqrt{\pi}e^{-\sigma^2\Omega^2}\left(\frac{\gamma}{2L^2}\sigma^5\Omega^2+\frac{1}{\gamma}\sigma\mp\frac{2\sigma^3\Omega}{L}\right). \tag{III.34}$$

Here, inspired by the usual quantum optics convention [13], we call E_{zm}^{+} and E_{zm}^{-} the counterrotating-wave and rotating-wave contributions respectively. Note that the contributions from the zero mode of the counter-rotating wave terms E_{zm}^{+} can be exactly cancelled by choosing γ suitably. Specifically, for a given interaction time, length of the cavity and detector gap, we have two values of γ,

$$\gamma_{\pm} = \frac{(2\pm\sqrt{2})L}{\sigma^2\Omega}, \tag{III.35}$$

which cancel the contribution to the detector dynamics coming from the zero-mode counter-rotating wave terms. The contribution of the zero-mode rotating-wave terms E_{zm}^{-} cannot be cancelled for any value of γ, but we will discuss later that in this case its contribution to the detector dynamics is not relevant for the long interaction time regime.

To estimate the strength of the oscillator-mode contribution to the detector's dynamics, we introduce similar estimators

$$E^{\pm}_{\text{osc}} = \sum_{n=1}^{\infty} \frac{\lambda^2}{2n\pi} \left| \int_{-\infty}^{\infty} \mathrm{d}\tau \, \chi(\tau) \, e^{\mathrm{i}(2\pi n/L \pm \Omega)\tau} \right|^2 , \quad \text{(III.36)}$$

modelled on the diagonal components of $\rho^{\text{osc}\,(2)}_{T,\text{d}}$ (III.27). For the Gaussian switching function (III.33), we obtain

$$E^{\pm}_{\text{osc}} = \sum_{n=1}^{\infty} \frac{\lambda^2 \sigma}{n\sqrt{\pi}} \exp\left[-\frac{\sigma^2 (2\pi n \pm L\Omega)^2}{L^2} \right] . \quad \text{(III.37)}$$

Let us study the relative strength of the zero-mode contribution as compared to the oscillator-mode contribution as a function of the zero mode natural frequency parameter γ and the effective total detection time σ. The relevant relative strength estimator is

$$\begin{aligned} S^{\pm}_{\text{zm}} &\equiv \frac{|E^{\pm}_{\text{zm}}|}{|E^{\pm}_{\text{osc}}|} \\ &= \pi e^{-\sigma^2\Omega^2} \left| \frac{\frac{\gamma}{2L^2}\sigma^5\Omega^2 + \frac{1}{\gamma}\sigma \mp \frac{2\sigma^3\Omega}{L}}{\sum_{n=1}^{\infty} \frac{\sigma}{n} \exp\left[-\frac{\sigma^2(2\pi n \pm L\Omega)^2}{L^2} \right]} \right| . \end{aligned} \quad \text{(III.38)}$$

We may assume without loss of generality that Ω is positive, so that $|g\rangle$ is the ground state and $|e\rangle$ is the excited state of the detector, as is the standard convention in quantum optics. Then S^{-}_{zm} corresponds to the rotating-wave terms and S^{+}_{zm} corresponds to the counter-rotating wave terms [13]. The behaviour for the two is markedly different.

Let us begin with the rotating-wave terms S^{-}_{zm}. We can see that if these terms contribute to the detector dynamics, and in the case where the detector is resonant with one of the field modes (i.e., if the gap of the atom coincides with an integer multiple of $2\pi/L$), the impact of the resonant oscillator mode on the detector (III.36) will not be exponentially suppressed with the effective interaction duration σ. Instead, the contribution of the resonant mode $2\pi n/L = \Omega$ will grow linearly with σ, whereas the contributions from the zero mode and from all the non-resonant oscillator modes are exponentially suppressed with σ.

Looking at (III.21b), one can see that the terms E^{-}_{osc} are nonexponentially vanishing only when we consider a detector which is excited and on resonance with one of the field modes, this is, when $\omega_n = \Omega$. It is very easy to see that similar terms would appear if we consider a detector initially in its ground state but it is resonant with an excited field mode (see for instance [16]).

This is a relieving result which rescues the usual quantum optical intuition behind the single-mode approximation [13]: In standard quantum optical setups, where we consider a detector's emission to or absorption from a resonant field mode, the zero mode-dynamics can be safely neglected for large interaction times since it is exponentially suppressed with the duration of the interaction, whereas the contribution of the resonant mode increases with time. By the same token, the contribution of the non-resonant modes can also be neglected for the stationary detector at rest. On the other hand, from (III.34), we also conclude that for a gapless detector $\Omega = 0$, the zero-mode contribution will not be exponentially suppressed with the duration of the interaction. In this case, however, the effect of the zero mode can be made arbitrarily small by choosing γ such that σ/γ is small, as shown in (III.34).

The counter-rotating contribution S^{+}_{zm}, however, is a whole different story, as illustrated in Fig. 1. Notice that in this case the impact estimators of the zero mode and the oscillator modes on the detector state are both small when the effective interaction time σ is long, but as shown in Fig. 1, the contribution of the zero mode decays with σ much slower than the contribution of the oscillator modes. Although it is indeed possible to pick a value of γ that cancels the relative strength of the zero mode in the detector's response, this value would need to be large if the detector's energy gap is small. So the zero-mode contribution can be exactly cancelled, provided the value of γ is adapted to σ. As mentioned above, from (III.34), we see that in the gapless detector limit $\Omega \to 0$, the contribution of the zero mode to the dynamics can be made arbitrarily small by taking σ/γ sufficiently small, but this will not be the case for non-zero detector gaps, where the value of γ such that the contribution of the zero mode exactly cancels is a function of σ.

Nevertheless, if the gap of the detector increases, we see that the relative strength of the zero mode as compared with the impact of the oscillator modes becomes non-negligible if the interaction timescale is large enough, and even gives the leading contribution in the limit of very long interaction times.

This is hinting that the impact of the zero mode on the detector dynamics, while minimizable by choosing appropriate initial states for the zero mode, may be highly non-negligible for large gap detectors and for long interaction times, provided that there are no resonance-effects of the atom capturing or emitting quanta into a resonant mode, as it is for example the case of a detector in the ground state coupled to the field vacuum.

Additionally, one may argue that it could be challenging to ever experimentally acknowledge these effects by probing the state of the detector: the zero mode becomes more relevant for long times, but in absence of spontaneous emission or absorption, the vacuum response of the detector for both the zero mode and the oscillator modes is exponentially suppressed with time. Indeed, an inertial UDW detector of positive gap in the ground state, which remained on forever while interacting with the vacuum, has a vanishing probability of excitation, as it can be checked from (III.27) considering $\Omega > 0$. This means

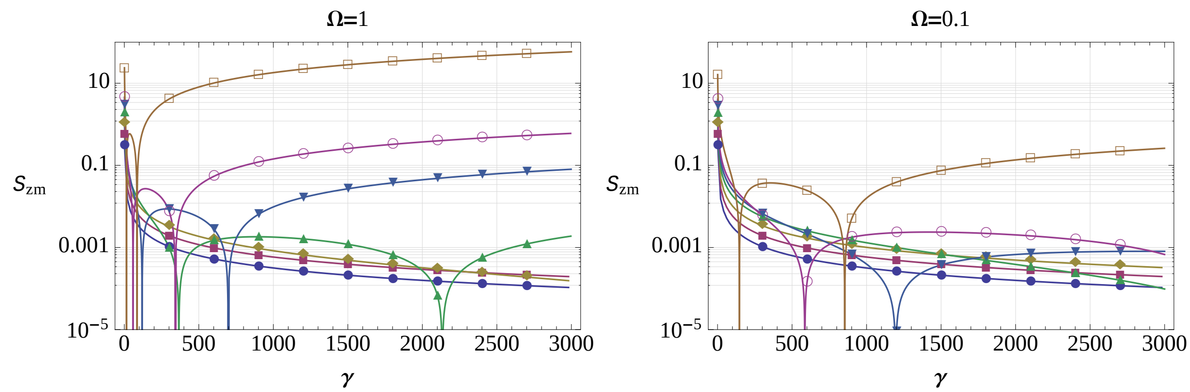

FIG. 1. (Color online): The plots show the quantity $S^+_{\rm zm}$ (III.38), which characterises the relative strength of the zero mode and oscillator mode contributions to the detector's density matrix in the counterrotating case, as a function of γ for $\Omega = 1$ and $\Omega = 0.1$ with selected values of the effective interaction duration σ. Solid circle is $\sigma = 10^{-5}$, solid square is $\sigma = 10^{-3}$, rhombus is $\sigma = 10^{-2}$, triangle is $\sigma = 0.04$, inverted triangle is $\sigma = 0.07$, hollow circle is $\sigma = 0.1$, and hollow square is $\sigma = 0.2$. $S^+_{\rm zm}$ has exact zeros at $\gamma_\pm = (2 \pm \sqrt{2})L/(\sigma^2\Omega)$, which is within the plotted range of γ only for the higher values of σ.

that even though the zero-mode contribution is relatively large as compared to the oscillator-mode contribution, in the regimes where this happens the overall action on the detector would be negligible. Notice that this may be amplified by choosing a more sudden switching than a Gaussian smearing.

We also make the following hypothesis: Another scenario where these effects may be non-trivial is in the interaction of more than one detector with the same quantum field, and in the extraction of correlations from the background field [6, 25].

IV. TRANSITION PROBABILITY OF A DERIVATIVE-COUPLING DETECTOR

A. Derivative-coupling detector

We have seen that the zero-mode contribution to the evolution of the UDW detector is well defined, and in certain circumstances it can be arranged to be small compared with the oscillator-mode contribution. One unappealing property of the zero-mode contribution however is that it is not invariant under time translations in any state of the zero mode, not even when the detector is stationary. Further, the terms involving $t(\tau)$ and $t(\tau')$ in (III.29) and (III.30) show that the time dependence may be significant, with potentially polynomial growth at late times.

In this section we consider a modified UDW detector for which the zero-mode contribution to the detector's transition probabilities is time translation invariant. In the notation of Section III, the new interaction Hamiltonian is

$$H = \lambda\chi(\tau)\mu(\tau)\dot{\phi}\big(\mathsf{x}(\tau)\big) \ , \tag{IV.1}$$

where the overdot denotes derivative with respect to τ, and the assumptions on the switching function χ are as in Section III. In words, the detector couples to the proper time derivative of the field at the detector's location, rather than to the field itself. The derivative ameliorates the effects that stem from the infrared behaviour of a massles field in $(1 + 1)$ dimensions in a number of contexts [20–23], and in our context it will restore time translation invariance.

A price to pay for the derivative in (IV.1) is, however, that the detector's ultraviolet properties become similar to those of the non-derivative UDW detector (III.1) in $(3+1)$ dimensions. If we try to proceed with (IV.1) as in Section III, we find that the off-diagonal components in the counterpart of ${\rho^{\rm osc}_{T,\rm d}}^{(2)}$ (III.27) are ill-defined, due to a divergence of the sums at large $|n|$. This ultraviolet problem occurs even in the simpler setting of a non-derivative UDW detector (III.1) in $(3 + 1)$ dimensional Minkowski spacetime, as can be seen by expressing the time evolution of the detector's density matrix in terms of the two-point function of the field as in [11] and considering the short-distance Hadamard form of the two-point function [24, 26].

We shall therefore not consider the detector's full density matrix but just the diagonal elements, which give the transition probabilities. Adapting the standard treatment of the non-derivative UDW detector [1–4] and working to leading order in λ, the probability of a transition from $|g\rangle$ to $|e\rangle$ is equal to

$$P(\Omega) = \lambda^2 F(\Omega) \ , \tag{IV.2}$$

where the response function F is given by

$$F(\Omega) = \int \mathrm{d}\tau\, \mathrm{d}\tau'\, \chi(\tau)\chi(\tau')\, e^{-\mathrm{i}\Omega(\tau-\tau')}\partial_\tau\partial_{\tau'} W(\tau, \tau') \ , \tag{IV.3}$$

and the correlation function W is obtained by pulling back the field's Wightman function, in the unperturbed state of the field, to the detector's worldline. Note that

W is a distribution and must be given an appropriate iϵ prescription. For us this prescription will be straightforward, but it would require more attention if one wished to address the limit in which the interaction is switched on and off sharply [23, 27–33].

From now on we drop the factor λ^2 in (IV.2) and refer to the response function (IV.3) as the probability.

We specialise to the situation in which the field is initially in the state discussed in Section II: in the free field Heisenberg picture, the oscillator modes are in the Fock vacuum $|0\rangle$ and the zero mode is in a state $|\psi\rangle$. The Wightman function is then given by the sum of the oscillator-mode contribution (II.8) and the zero-mode contribution (II.14).

B. Zero-mode contribution to the response function

By (II.14) and (IV.3), the zero-mode contribution to the response function reads

$$F_{\mathrm{zm}}(\Omega) = \frac{\langle\psi|P_S^2|\psi\rangle}{L^2}\left|\int_{-\infty}^{\infty} \mathrm{d}\tau\,\chi(\tau)\,e^{-\mathrm{i}\Omega\tau}\frac{dt(\tau)}{d\tau}\right|^2 . \tag{IV.4}$$

Two crucial observations are immediate.

First, since F_{zm} depends on $t(\tau)$ only through its derivative, any additive constant in $t(\tau)$ will drop out. F_{zm} is manifestly invariant under time translations.

Second, F_{zm} depends on $|\psi\rangle$ only in that it is proportional to the single expectation value $\langle\psi|P_S^2|\psi\rangle$. As P_S^2 is a positive definite operator with a continuous spectrum, $\langle\psi|P_S^2|\psi\rangle$ is strictly positive for any $|\psi\rangle$; however, $\langle\psi|P_S^2|\psi\rangle$ can be made arbitrarily small by a suitable choice of $|\psi\rangle$. In this sense, F_{zm} can always be made as small as desired by a suitable choice of the state of the zero mode.

We emphasise that both of these properties are in a marked contrast with those of the non-derivative detector of Section III. In (III.30), additive constants in $t(\tau)$ do not drop out, and the matrix elements involving $|\psi\rangle$ cannot all be simultaneously made arbitrarily small for any choice of $|\psi\rangle$. We also note that the $|\psi\rangle$-dependent overall coefficient $\langle\psi|P_S^2|\psi\rangle/L^2$ in (IV.4) is, up to a numerical factor, equal to the tt and xx components of the zero-mode contribution to the field's stress-energy tensor (II.17).

C. Stationary detector

1. Response of a stationary detector

As a first example, we consider a detector on the inertial worldline

$$t = \tau\cosh\beta\ , \qquad x = \tau\sinh\beta\ , \tag{IV.5}$$

where $\beta\in\mathbb{R}$ is the rapidity with respect to the cylinder's rest frame, so that the detector's velocity with respect to the cylinder's rest frame is $\tanh\beta$. This is the most general stationary trajectory on the cylinder.

Pulling back (II.8) and (II.14) to the worldline (IV.5), we find that the correlation function is given by

$$W(\tau,\tau') = W_{\mathrm{osc}}(\tau,\tau') + W_{\mathrm{zm}}(\tau,\tau'), \tag{IV.6}$$

where

$$W_{\mathrm{osc}}(\tau,\tau') = \langle 0|\phi_{\mathrm{osc}}\big(t(\tau),x(\tau)\big)\phi_{\mathrm{osc}}\big(t(\tau'),x(\tau')\big)|0\rangle = \sum_{n=1}^{\infty}\frac{1}{4\pi n}\left\{\exp\left[\frac{2\pi n e^{-\beta}}{\mathrm{i}L}(\Delta\tau-\mathrm{i}\epsilon)\right] + \exp\left[\frac{2\pi n e^{\beta}}{\mathrm{i}L}(\Delta\tau-\mathrm{i}\epsilon)\right]\right\}\ , \tag{IV.7a}$$

$$\begin{aligned} W_{\mathrm{zm}}(\tau,\tau') &= \langle\psi|Q_H\big(t(\tau)\big)Q_H\big(t(\tau')\big)|\psi\rangle \\ &= \langle\psi|Q_S^2|\psi\rangle + \langle\psi|P_SQ_S|\psi\rangle L^{-1}\tau\cosh\beta + \langle\psi|Q_SP_S|\psi\rangle L^{-1}\tau'\cosh\beta + \langle\psi|P_S^2|\psi\rangle L^{-2}\tau\tau'\cosh^2\beta\ , \end{aligned} \tag{IV.7b}$$

where $\Delta\tau = \tau-\tau'$. Hence

$$\partial_\tau\partial_{\tau'}W_{\mathrm{osc}}(\tau,\tau') = \frac{\pi}{L^2}\sum_{\eta=\pm1}\sum_{n=1}^{\infty} n e^{-2\eta\beta}\exp\left[-\mathrm{i}\frac{2\pi n e^{-\eta\beta}}{L}(\Delta\tau-\mathrm{i}\epsilon)\right]\ , \tag{IV.8a}$$

$$\partial_\tau\partial_{\tau'}W_{\mathrm{zm}}(\tau,\tau') = \frac{\cosh^2\beta}{L^2}\langle\psi|P_S^2|\psi\rangle\ , \tag{IV.8b}$$

where $\eta=1$ indicates terms that come from the right-movers (field modes with positive momentum) and $\eta=-1$ indicates terms that come from the left-movers (field modes with negative momentum). From (IV.3) we then obtain

$$F(\Omega) = F_{\mathrm{osc}}(\Omega) + F_{\mathrm{zm}}(\Omega)\ , \tag{IV.9a}$$

$$F_{\rm osc}(\Omega) = \sum_{\eta=\pm1} \sum_{n=1}^{\infty} |I_n^\eta|^2 \ , \tag{IV.9b}$$

$$I_n^\eta = \frac{\sqrt{\pi n}}{L} e^{-\eta\beta} \hat{\chi}\left(\Omega + \frac{2\pi n e^{-\eta\beta}}{L}\right) \ , \tag{IV.9c}$$

$$F_{\rm zm}(\Omega) = \frac{\cosh^2\beta}{L^2} \langle\psi|P_S^2|\psi\rangle \, |\hat{\chi}(\Omega)|^2 \ , \tag{IV.9d}$$

where $\hat{\chi}$ is the Fourier transform of χ,

$$\hat{\chi}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}\tau \, e^{-\mathrm{i}\omega\tau} \chi(\tau) \ . \tag{IV.10}$$

We note that the detector's velocity enters $F_{\rm osc}$ through a Doppler shift: the field modes with momentum in (respectively opposite to) the detector's velocity contribute with a redshift (blueshift).

2. Limits of long detection and ultrarelativistic velocity

We wish to examine the response in the limit of long detection and in the limit of ultrarelativistic velocity. We choose the switching to be the Gaussian (III.33), normalised so that $\int_{-\infty}^{\infty} \chi^2(\tau) \, \mathrm{d}\tau = 1$. Then

$$\hat{\chi}(\omega) = \pi^{1/4} (2\sigma)^{1/2} \, e^{-\sigma^2\omega^2/2} \ , \tag{IV.11}$$

and from (IV.9) we have

$$F_{\rm osc}(\Omega) = \frac{2\pi^{3/2}\,\sigma}{L^2} \sum_{\eta=\pm1} \sum_{n=1}^{\infty} e^{-2\eta\beta} n \, e^{-\sigma^2(\Omega + 2\pi n e^{-\eta\beta}/L)^2} \ , \tag{IV.12a}$$

$$F_{\rm zm}(\Omega) = \frac{2\pi^{1/2}\cosh^2\beta}{L^2} \langle\psi|P_S^2|\psi\rangle \, \sigma e^{-\sigma^2\Omega^2} \ . \tag{IV.12b}$$

Consider first the limit of long detection, $\sigma \to \infty$, with β fixed. In this limit (IV.12) reduces to

$$\begin{aligned} F_{\rm osc}(\Omega) &= \frac{2\pi^2}{L^2} \sum_{\eta=\pm1} \sum_{n=1}^{\infty} n e^{-2\eta\beta} \, \delta\!\left(\Omega + \frac{2\pi n e^{-\eta\beta}}{L}\right) \\ &= -\frac{\pi}{L} \sum_{\eta=\pm1} \sum_{n=1}^{\infty} e^{-\eta\beta} \Omega \, \delta\!\left(\Omega + \frac{2\pi n e^{-\eta\beta}}{L}\right) \ , \end{aligned} \tag{IV.13a}$$

$$F_{\rm zm}(\Omega) = \frac{2\pi\cosh^2\beta}{L^2} \langle\psi|P_S^2|\psi\rangle \, \delta(\Omega) \ , \tag{IV.13b}$$

where δ is Dirac's delta-function. $F_{\rm osc}$ consists of strict delta-peaks of de-excitation, at the Doppler-shifted frequencies of the oscillator modes. $F_{\rm zm}$ is a strict delta-peak at zero energy. Given that the detector's energy gap is assumed nonvanishing, the zero mode does not contribute to the response in the long detection limit.

It can be verified that formulas (IV.13) also ensue if, instead of working with a switching function, we appeal to stationarity at the outset and consider the transition rate,

$$\dot{F}(\Omega) = \int_{-\infty}^{\infty} \mathrm{d}\tau \, e^{-\mathrm{i}\Omega(\tau-\tau')} \partial_\tau \partial_{\tau'} W(\tau,\tau') \ , \tag{IV.14}$$

obtained from (IV.3) by setting $\chi(\tau) = 1$ and formally factoring out the infinite total detection time [3]. This shows that the normalisation of our switching function (III.33) is well adapted for recovering a transition rate per unit time in the $\sigma \to \infty$ limit.

Consider then the limit of ultrarelativistic velocity, $|\beta| \to \infty$, with σ fixed. $F_{\rm zm}$ (IV.12b) diverges proportionally to $e^{2|\beta|}$. In $F_{\rm osc}$ (IV.12a), the contribution from the blueshifted modes goes to zero, but an integral estimate shows that the contribution from the redshifted modes has a finite limit, given by

$$\begin{aligned} F_{\rm osc}^{|\beta|\to\infty}(\Omega) &= \frac{2\pi^{3/2}\,\sigma}{L^2} \int_0^{\infty} \mathrm{d}x \, x \exp\!\left[-\sigma^2\left(\Omega + \frac{2\pi x}{L}\right)^2\right] \\ &= \frac{1}{4\sigma}\left[\frac{e^{-\sigma^2\Omega^2}}{\pi^{1/2}} - \sigma\Omega \, \mathrm{erfc}(\sigma\Omega)\right] \ . \end{aligned} \tag{IV.15}$$

It can be verified, using the Minkowski vacuum Wightman function (II.15) and (IV.3), that (IV.15) is exactly half of the response of an inertial detector in the Minkowski vacuum in full Minkowski space. The physical picture of the ultrarelativistic motion is hence that the redshifted oscillator modes contribute to the response as if the spacetime were not periodic, while the contribution from the blueshifted oscillator modes is blueshifted beyond the energies accessible to the detector.

We shall not attempt to take the limits of long detection and ultrarelativistic velocity simultaneously. However, if the zero-mode contribution is considered negligible, we note that taking the two limits in succession commutes: both the $|\beta| \to \infty$ limit of (IV.13a) and the $\sigma \to \infty$ limit of (IV.15) give

$$F(\Omega) = -\tfrac{1}{2}\Omega\Theta(-\Omega) \ , \tag{IV.16}$$

where Θ is the Heaviside function. This is exactly half of the transition rate in inertial motion in Minkowski space in Minkowski vacuum, obtained from (IV.14) with (II.15).

It is remarkable, and perhaps surprising, that the oscillator-mode contribution to the response exhibits no pathology in the ultrarelativistic limit. We see this as evidence that the periodic cavity provides a useful arena for analysing detector-field interaction even at relativistic velocities [34–36].

D. Uniformly accelerated detector

As a second example, we consider a detector on the uniformly accelerated worldline

$$t = a^{-1}\sinh(a\tau) \ , \quad x = a^{-1}\cosh(a\tau) \ , \tag{IV.17}$$

where τ is the proper time and the positive parameter a is the proper acceleration. The detector accelerates towards increasing x, and the detector is momentarily at rest in the cylinder's rest frame at the moment $\tau = 0$.

The trajectory is locally stationary with respect to the local Minkowski geometry in its neighbourhood, but it is not stationary with respect to the global time translations that leave the cylinder invariant. In geometric terms, the trajectory is invariant under the local boost-generating Killing vector $x\partial_t + t\partial_x$, but this Killing vector is not globally defined on the cylinder because of the spatial periodicity.

Pulling back (II.8) and (II.14) to the detector's worldline, we find

$$\partial_\tau \partial_{\tau'} W_{\rm osc}(\tau,\tau') = \frac{\pi}{L^2} \sum_{\eta=\pm 1} \sum_{n=1}^{\infty} n\, e^{-\eta a(\tau+\tau')} \exp\left[{\rm i}\eta \frac{2\pi n}{aL} (e^{-\eta a\tau} - e^{-\eta a\tau'} + {\rm i}\eta\epsilon) \right] , \quad \text{(IV.18a)}$$

$$\partial_\tau \partial_{\tau'} W_{\rm zm}(\tau,\tau') = \frac{1}{L^2} \langle\psi|P_S^2|\psi\rangle \cosh(a\tau)\cosh(a\tau') , \quad \text{(IV.18b)}$$

where again $\eta = 1$ comes from the right-movers and $\eta = -1$ comes from the left-movers. From (IV.3) we then obtain

$$F_{\rm osc}(\Omega) = \sum_{\eta=\pm 1} \sum_{n=1}^{\infty} |J_n^\eta|^2 , \quad \text{(IV.19a)}$$

$$J_n^\eta = \frac{\sqrt{\pi n}}{L} \int_{-\infty}^{\infty} {\rm d}\tau\, \chi(\tau)\, e^{-{\rm i}\Omega\tau - \eta a\tau} \times \exp\left({\rm i}\eta \frac{2\pi n}{aL} e^{-\eta a\tau} \right) , \quad \text{(IV.19b)}$$

$$F_{\rm zm}(\Omega) = \frac{\langle\psi|P_S^2|\psi\rangle}{L^2} \left| \int_{-\infty}^{\infty} {\rm d}\tau\, \chi(\tau)\, e^{-{\rm i}\Omega\tau} \cosh(a\tau) \right|^2 . \quad \text{(IV.19c)}$$

As the trajectory is not stationary on the cylinder, we now consider the Gaussian switching function

$$\chi_{\tau_0}(\tau) = \frac{1}{\pi^{1/4}\sigma^{1/2}} e^{-(\tau-\tau_0)^2/(2\sigma^2)} , \quad \text{(IV.20)}$$

where $\sigma > 0$ as before but the new real-valued parameter τ_0 specifies the instant about which χ_{τ_0} is peaked. From (IV.19), we then have

$$F_{\rm osc}(\Omega) = \sum_{\eta=\pm 1} \sum_{n=1}^{\infty} |J_n^\eta|^2 , \quad \text{(IV.21a)}$$

$$J_n^\eta = \frac{\pi^{1/4} n^{1/2}}{La\, \sigma^{1/2}} e^{-(\eta a + {\rm i}\Omega)\tau_0} \int_0^\infty {\rm d}x\, x^{{\rm i}\eta\Omega/a} \times \exp\left[-\frac{(\ln x)^2}{2a^2\sigma^2} + {\rm i}\eta e^{-\eta a\tau_0} \frac{2\pi n}{aL} x \right] , \quad \text{(IV.21b)}$$

$$F_{\rm zm}(\Omega) = \frac{2\pi^{1/2}\sigma}{L^2} \langle\psi|P_S^2|\psi\rangle\, e^{-\sigma^2\Omega^2 + \sigma^2 a^2} \times \left[\cos^2(\sigma^2 a\Omega) + \sinh^2(a\tau_0) \right] . \quad \text{(IV.21c)}$$

Both the oscillator-mode contribution and the zero-mode contribution depend on τ_0.

For the zero-mode contribution, we recall that $F_{\rm zm}$ can be always made as small as desired by choosing $|\psi\rangle$ so that the overall coefficient $\langle\psi|P_S^2|\psi\rangle$ is small. A more interesting question however is how the relative magnitudes of $F_{\rm zm}$ and $F_{\rm osc}$ depend on the other parameters when $\langle\psi|P_S^2|\psi\rangle$ is fixed. We illustrate this in Fig. 2, showing a parameter range in which $F_{\rm zm}/F_{\rm osc}$ tends to grow with increasing interaction time and with increasing acceleration.

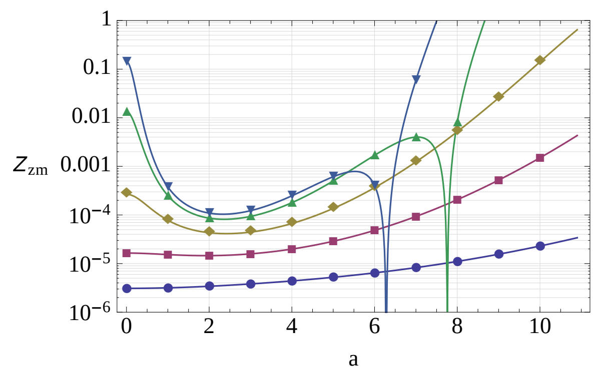

FIG. 2. (Color online): The plot shows $Z_{\rm zm} := F_{\rm zm}/F_{\rm osc}$ evaluated from (IV.21) with $L = 1$, $\Omega = 1$, $\tau_0 = 0$ and $\langle\psi|P_S^2|\psi\rangle = 10^{-6}$, as a function of a, with selected values of σ: solid circle is $\sigma = 0.15$, solid square is $\sigma = 0.25$, rhombus is $\sigma = 0.35$, triangle is $\sigma = 0.45$, and inverted triangle is $\sigma = 0.5$. $Z_{\rm zm}$ has exact zeroes at $a = (\sigma^2\Omega)^{-1}(\pi/2 + k\pi)$, $k = 0, 1, \ldots$, of which only the first zero ($k = 0$) for the largest two values of σ is in the range covered by the plot.

For the oscillator-mode contribution, we expect that $F_{\rm osc}$ should reduce to the response of the accelerating detector (IV.17) in Minkowski space in the Minkowski vacuum. The response in the Minkowski vacuum is given by

$$F_{\rm Mink}(\Omega) = \frac{ae^{-\sigma^2\Omega^2}}{4\pi} \int_{-\infty}^{\infty} \frac{{\rm d}r}{\cosh^2 r}$$

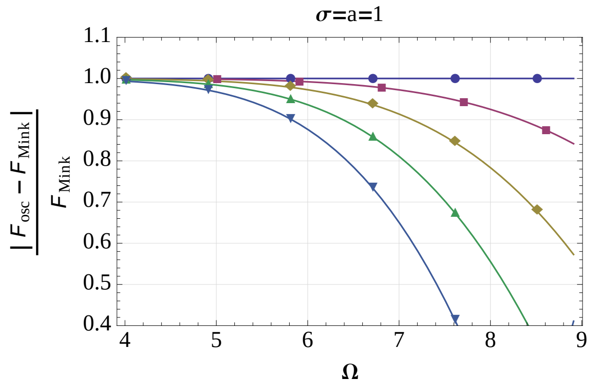

FIG. 3. (Color online): The plot shows $|F_{\rm Mink} - F_{\rm osc}|/F_{\rm Mink}$, evaluated from (IV.21) and (IV.22), with $a = \sigma = 1$ and $\tau_0 = 0$, as a function of Ω, with selected values of L: solid circle is $L = 0.01$, solid square is $L = 0.15$, rhombus is $L = 0.2$, triangle is $L = 0.25$, and inverted triangle is $L = 0.3$. $F_{\rm osc}$ and $F_{\rm Mink}$ get closer to each other when L increases. The slow convergence of the sum in (IV.21a) limits our ability to go to higher values of L.

$$\times \exp\left\{ -\frac{1}{a^2\sigma^2}\left[r + \mathrm{i}\left(\sigma^2 a\Omega - \frac{\pi}{2}\right)\right]^2 \right\} , \tag{IV.22}$$

as can be verified using (II.15), (IV.3) and (IV.17), and in the long detection time limit $F_{\rm Mink}$ reduces to the familiar Planckian result in the Unruh temperature $a/(2\pi)$,

$$F^{\sigma\to\infty}_{\rm Mink}(\Omega) = \frac{\Omega}{e^{2\pi\Omega/a} - 1} , \tag{IV.23}$$

as can be verified using formula 3.985.1 in [37]. Numerical evidence for closeness of $F_{\rm osc}$ and $F_{\rm Mink}$ with increasing L is given in Fig. 3. The slow convergence of the sum in (IV.21a) has prevented us from seeking evidence in a more extensive range of the parameter space.

A limit of particular interest for the oscillator modes would be that of large $|\tau_0|$ with all other parameters fixed. In this limit the detector is moving through the cavity with an ultrarelativistic velocity throughout the vast majority of the effective detection time. Our ultrarelativistic inertial detector result (IV.15) suggests that in this limit $F_{\rm osc}$ should tend to half of the Minkowski vacuum response $F_{\rm Mink}$ (IV.22). If correct, this would be further evidence that the periodic cavity provides a useful arena for analysing detector-field interaction even at relativistic velocities [34–36]. The slow convergence of the sum in (IV.21a) has however not enabled us to obtain conclusive evidence about this question.

$F_{\rm osc}$ can be written in a form that avoids a discrete sum by using for the Fock vacuum Wightman function the summed expression (II.9), and the ϵ-regulator can then be eliminated by the techniques of [23, 29–33]. We have not investigated whether this integral representation improves the stability of numerical evaluation in the limits of interest.

V. CONCLUSIONS

We have assessed the impact of the zero mode of a quantum field on the dynamics of particle detectors. This mode is often neglected when considering the light-matter interaction under periodic or Neumann boundary conditions.

We worked with a massless scalar field in a periodic cavity in $(1+1)$-dimensional Minkowski space. We considered the traditional Unruh-DeWitt (UDW) detector, coupled linearly to the field, as well as a modified UDW detector that couples linearly to the proper time derivative of the field. We treated the interaction perturbatively, to quadratic order in the coupling constant for the detector's transition probabilities.

For the UDW detector, we first showed that when the oscillator modes of the field are initially in a Fock state, or in an ensemble of Fock states, the zero mode of the field is not affected by the backreaction of the non-zero modes on the detector, or vice versa. To quadratic order in the coupling constant, there is hence no danger that energy in the oscillator modes of the field would get transferred to the zero mode via the interaction through the detector, regardless of the state of the detector. This conclusion does however not need to hold if the oscillator modes of the field are initially in a state with a non-diagonal density matrix in the Fock basis, such as a coherent state.

We then showed that the zero mode does have a nonvanishing direct effect on the evolution of the detector's density matrix. This effect can be made negligible in standard quantum optical settings, but situations in which the effect can be significant, and even dominant, can arise in other settings, including the Unruh effect [1] or the harvesting of quantum entanglement from a field [6].

For the derivative-coupling detector, we found that the zero mode has again a nonvanishing direct effect on the detector's transition probabilities, but this effect can be made as small as desired by just tuning the detector's initial state. The effect is invariant under time translations in the cavity, and it is directly proportional to the contribution of the zero mode to the field's renormalised stress-energy tensor. As examples, we considered a detector moving inertially but with an arbitrary velocity, including the ultrarelativistic limit, and a detector in uniformly accelerated motion.

Our analysis provides the basic tools for studying the Unruh effect in a periodic cavity for the derivative-coupling detector. Exploiting these tools for a systematic survey of the parameter space, whether by analytic or numerical techniques, is left to future work.

Systems where a zero mode arises as a consequence of Neumann boundary conditions will have some quantitative differences because of the absence of spatial homogeneity, but we anticipate the qualitative conclusions about the zero mode to be largely similar. We also anticipate that both our analysis and our conclusions can be generalised to cosmological spacetimes in which zero modes arise [15].

It should be interesting to study in detail possible zero-mode effects on relevant well-known results in relativistic quantum information such as, for instance, vacuum entanglement harvesting and farming [6, 25] and relativistic quantum communication [11, 12], when considered in the context of periodic and Neumann cavities. Since we showed that in many relevant cases there is no backreaction of the zero-mode on the oscillatory modes through their interaction with the detector, it is doubtful that the zero mode dynamics alone could destabilize the process of entanglement harvesting or hinder relativistic communication protocols. Nevertheless, as shown by the full density matrix analysis carried over in Section III C, there would be an impact on these phenomena coming from the dynamics of the zero mode that might become relevant in some regimes. Although outside of the scope of this paper, it may be relevant to study the role of the zero-mode dynamics in those scenarios in future work.

As a final comment, we note that while the focus of this paper is theoretical, the conclusions are applicable in all detector-field interaction settings where there are periodic or Neumann boundary conditions. Among laboratory systems this includes closed optical cavities, such as optical-fibre loops [38], and superconducting circuits coupled to periodic [39] or Neumann microwave guides [40].

ACKNOWLEDGMENTS

We thank William Donnelly, Robert Jonsson, Benito Juárez-Aubry, Achim Kempf, Don Marolf and Adrian Ottewill for helpful discussions and correspondence. Both authors thank Achim Kempf, Robert Mann and Gerard Milburn for hospitality at the Banff International Research Centre workshop 13w5153 Entanglement in Curved Spacetime, where this work was begun. J.L. thanks Achim Kempf and Eduardo Martín-Martínez for hospitality at the University of Waterloo. E.M-M. acknowledges the support of the Banting Postdoctoral Fellowship Programme. J.L. was supported in part by STFC (Theory Consolidated Grant ST/J000388/1).

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