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THE EFFECT OF FOREST DISLOCATIONS ON THE EVOLUTION OF A PHASE-FIELD MODEL FOR PLASTIC SLIP

PATRICK W. DONDL, MATTHIAS W. KURZKE, AND STEPHAN WOJTOWYTSCH

Abstract. We consider the gradient flow evolution of a phase-field model for crystal dislocations in a single slip system in the presence of forest dislocations. The model is based on a Peierls-Nabarro type energy penalizing non-integer slip and elastic stress. Forest dislocations are introduced as a perforation of the domain by small disks where slip is prohibited. The Γ-limit of this energy was deduced by Garroni and Müller (2005 and 2006). Our main result shows that the gradient flows of these Γ-convergent energy functionals do not approach the gradient flow of the limiting energy. Indeed, the gradient flow dynamics remains a physically reasonable model in the case of non-monotone loading. Our proofs rely on the construction of explicit sub- and super-solutions to a fractional Allen-Cahn equation on a flat torus or in the plane, with Dirichlet data on a union of small discs. The presence of these obstacles leads to an additional friction in the viscous evolution which appears as a stored energy in the Γ-limit, but it does not act as a driving force. Extensions to related models with soft pinning and non-viscous evolutions are also discussed. In terms of physics, our results explain how in this phase field model the presence of forest dislocations still allows for plastic as opposed to only elastic deformation.

Contents

1. Introduction

It is well-known that Γ-convergence of functionals is a C^0-type convergence that does not imply convergence of the related dynamics. For example, the 'wiggly' potentials

$$f_\varepsilon : [-1,1] \to \mathbb{R}, \qquad f_\varepsilon(x) = x^2 + 2\varepsilon \sin(x^2/\varepsilon)$$

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converge uniformly, hence also in the sense of Γ-convergence, to the limit $f(x) = x^2$ as $\varepsilon \to 0$ while solutions to the gradient flows of f_ε never travel farther than $\pm\sqrt{\pi\varepsilon}$ from their initial datum into a local minimum and thus do not resemble the gradient flow of the Γ-limit at all.

On the other hand, there are well known conditions under which the gradient flows of Γ-convergent functionals on Hilbert spaces [SS04] and metric spaces [Ser11] approach the gradient flow of a the limiting energy in a suitable sense. In applications, it is not always obvious whether functionals belong to the 'wiggly' or the convergent 'Sandier-Serfaty'-class.

Here, we consider the effect of forest dislocations on the propagation of slip in Peierls-Nabarro-type models following [KCO02]. In [GM05, GM06] it was shown that the corresponding non-local Modica-Mortola type energy functional augmented with the condition that the phase field u_ε vanishes at certain small obstacles Γ-converges to a functional given by the sum of a perimeter and a bulk energy. Essentially, the articles above show (in higher generality) that the energies

$$\mathcal{E}_\varepsilon(u_\varepsilon) = \frac{1}{|\log\varepsilon|}\left([u_\varepsilon]^2_{H^{1/2}(\mathbb{T}^2)} + \int_{\mathbb{T}^2}\frac{1}{\varepsilon}W(u_\varepsilon)\,\mathrm{d}x\right) \tag{1.1}$$

converge to a functional

$$\mathcal{E}(u) = \mathrm{Per}(\{u=1\}) + \Lambda\,\alpha\,\mathcal{H}^2(\{u=1\}), \qquad u\in BV(\mathbb{T}^2,\{0,1\}) \tag{1.2}$$

in the sense of Γ-convergence with respect to the strong L^2-topology when restricted to the spaces

$$\mathcal{E}_\varepsilon : X_\varepsilon \to \mathbb{R}, \qquad X_\varepsilon := \{u_\varepsilon \in H^{1/2}(\mathbb{T}^2) \mid u_\varepsilon \equiv 0 \text{ on } B_\varepsilon(x_{i,\varepsilon}) \text{ for } 1\le i\le N_\varepsilon\}. \tag{1.3}$$

The obstacles $x_{i,\varepsilon}$ have to satisfy certain distribution assumptions and $\frac{\varepsilon}{|\log\varepsilon|}\sum_{i=1}^{N_\varepsilon}\delta_{x_{i,\varepsilon}} \rightharpoonup \Lambda\,\mathcal{H}^2$, where $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure (so $\mathcal{H}^2 = \mathcal{L}^2$ is the Lebesgue measure) and W is a non-negative smooth multi-well potential vanishing quadratically at the integers. The constant α is determined through the solution of a cell-problem. Dislocations in this model are given heuristically by $\mathbb{Z}+1/2$-level sets of the slip u (see Figure 1).

The focus of this article is the evolution that arises as the limit of the L^2-gradient flows of the energies $\mathcal{E}_\varepsilon$. In technical terms, we are interested in the behaviour of solutions to the evolution equation

$$\begin{cases} c_\varepsilon\varepsilon\, u_t &= \frac{1}{|\log\varepsilon|}\left(\mathrm{A}\,u - W'(u)\right) \quad t>0,\ x\in\mathbb{T}^2\setminus\bigcup_{i=1}^{N_\varepsilon}\overline{B_\varepsilon(x_{i,\varepsilon})}\\ u &\equiv 0 \quad t\ge 0,\ x\in\bigcup_{i=1}^{N_\varepsilon}\overline{B_\varepsilon(x_{i,\varepsilon})}\\ u &= u^0 \quad t=0\end{cases} \tag{1.4}$$

as $\varepsilon\to 0$, where $\mathrm{A} := -(-\Delta)^{1/2}$ is the fractional Laplacian or order $s = 1/2$ and the pinning condition is interpreted as $u_\varepsilon(t,\cdot)\in X_\varepsilon$ for all $t>0$ for weak solutions while the differential equation is only tested with functions supported in $\mathbb{T}^2\setminus\bigcup_{i=1}^{N_\varepsilon}\overline{B_\varepsilon(x_{i,\varepsilon})}$. The case $c_\varepsilon\equiv 1$ corresponds to the time-normalised gradient flow dynamics of (1.1) under the pinning constraint (1.3). Depending on the exact problem, we identify the scaling regime c_ε in which the evolution approaches non-trivial limiting dynamics and give results on the limits of solutions of (1.4) for suitable initial conditions.

We show that this problem belongs to the 'wiggly' world, i.e., that the gradient flows of $\mathcal{E}_\varepsilon$ do not approach the dynamics of the limiting problem. The idea behind this is that the non-locality in the energy is too weak to summon a driving force on an otherwise unloaded flat dislocation from the pinning constraint on a relevant time scale. On the other hand, if an external force (or a curvature term) acts to expand the $\{u_\varepsilon\approx 1\}$-phase, we do see a resistance from the energy barrier. Thus the perforation of the domain induces a friction which only resists other forces but does not initiate movement.

Three different terms appear in the dynamics on different time-scales: The curvature driven evolution stemming from the diffuse line-energy which acts on the gradient flow scale, a non-local interaction between interfaces (kink/kink repulsion and kink/anti-kink attraction) stemming from the next order Γ-limit which is $|\log\varepsilon|^{-1}$-small with respect to the curvature flow, and the diffuse bulk term, which acts as a driving force, but only on a time-scale which is roughly $\varepsilon^{1/2}$-slow with respect to the other terms, although it can act against other forces on the fast scale.

In this article, we mostly focus on the situation of infinite parallel straight interfaces to be able to neglect the curvature-driven evolution. We construct explicit sub- and super-solutions to estimate the speed of motion of an interface. At some non-straight interfaces, we can obtain bounds by using sub-solutions as barriers to show non-expansive behaviour (i.e., u non-decreasing) and energy methods to show non-shrinking of an initial condition (i.e., u non-increasing).

Physically, our results provide a justification why the phase-field model is valid beyond the applicability of the Γ-limit, where the bulk term stemming from the forest dislocations induces a dislocation evolution to return to the undeformed state $u = 0$ at macroscopic velocities.

The article is structured as follows. In Section 2, we explain the mathematical setting and the heuristic reasoning behind our results as well as a brief statement of our main theorems. In Section 3 we construct sub-solutions and apply them to a one-dimensional analogue of our problem in order to obtain results in this simpler setting. In Section 4, we state the main results in more precise and general terms and show how the one-dimensional proofs can be adapted to yield the full results. Section 5 is devoted to the discussion of different related models, in particular non-viscous evolution. We conclude the article with a brief summary and some open problems. In an appendix, we briefly discuss parabolic equations with fractional differential operators on bounded domains.

2. Background and Heuristics

2.1. **The Energy Limit.** The energies $\mathcal{E}_\varepsilon$ are obtained as a model for crystal dislocations in [KCO02] and motivated in their current form in [GM05, GM06]. As the characteristic length scale ε of crystal grids is typically very small compared to the behaviour of a crystal on the length scale we are interested in, it is desirable to have a simpler continuum limit $\varepsilon \to 0$ available. This has been formalised by Garroni and Müller as follows.

Theorem 2.1. [GM06] *Let $x_{i,\varepsilon} \in \mathbb{T}^2$ be points such that $1 \le i \le N_\varepsilon$ with $\frac{\varepsilon}{|\log\varepsilon|} N_\varepsilon \to \Lambda$ satisfying the following assumptions:*

(1) (equi-distributed) *For $r_\varepsilon \sim N_\varepsilon^{-1/2}$ there exist constants $c, C > 0$ such that*

$$c\, r_\varepsilon^2\, N_\varepsilon \le N_\varepsilon(Q_\varepsilon) \le C\, r_\varepsilon^2\, N_\varepsilon$$

where $N_\varepsilon(Q_\varepsilon)$ is the number of obstacles in Q_ε and Q_ε is a square of side length r_ε.

(2) (well-separated) *There exists $\beta < 1$ independent of $\varepsilon > 0$ such that $d(x_{i,\varepsilon}, x_{j,\varepsilon}) > 6\,\varepsilon^\beta$ for all $1 \le i \ne j \le N_\varepsilon$.*

(3) (finite capacity density) *The obstacles approach a multiple of the Lebesgue measure through $\frac{\varepsilon}{|\log\varepsilon|}\sum_{i=1}^{N_\varepsilon} \delta_{x_i} \rightharpoonup \Lambda\,\mathcal{L}^2$ for $\Lambda \in (0,\infty)$.*

Take the space

$$X_\varepsilon := \{u_\varepsilon \in H^{1/2}(\mathbb{T}^2) \mid u_\varepsilon \equiv 0 \text{ on } B_\varepsilon(x_{i,\varepsilon}) \text{ for } 1 \le i \le N_\varepsilon\}$$

and the energy functional

$$\mathcal{E}_\varepsilon : X_\varepsilon \to \mathbb{R}, \qquad \mathcal{E}_\varepsilon(u_\varepsilon) = \frac{1}{|\log\varepsilon|}\left([u_\varepsilon]_{1/2}^2 + \int_{\mathbb{T}^2} \frac{1}{\varepsilon} W(u_\varepsilon)\,\mathrm{d}x\right)$$

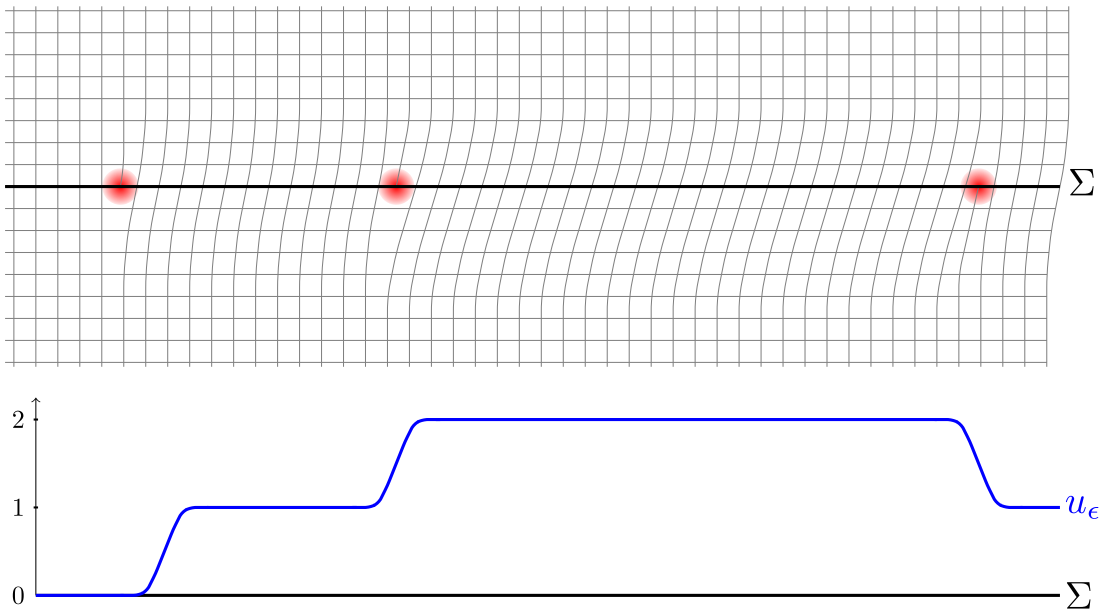

FIGURE 1. The phase-field u_ε counts the number of half-planes wedged into a crystal grid. Dislocations are the level sets of $\mathbb{Z}+1/2$, i.e, the interfaces between the phases. The dislocations we consider all lie in the same plane Σ of the crystal grid and their Burgers vectors $\vec{b}$ are integer multiples of a single vector $\vec{b_0}$.

where W is a periodic multi well potential satisfying $W \geq c\,\mathrm{dist}^2(\cdot,\mathbb{Z})$ for some $c > 0$. Then

$$\left[\Gamma(L^2) - \lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon\right](u) = \int_{\mathbb{T}^2}\alpha(u)\,\mathrm{d}x + 4\int_{J_u}[u]\,\mathrm{d}\mathcal{H}^1 \;\; = \int_{\mathbb{T}^2}\alpha(u)\,\mathrm{d}x + 4\,|Du|(\mathbb{T}^2)$$

where $u \in BV(\mathbb{T}^2,\mathbb{Z})$, $[u] = u^+ - u^-$ denotes the jump of u on the jump set J_u and $\alpha(z)$ is determined as the solution of the cell problem

$$\alpha(z) = \inf\left\{\frac{1}{2}\,[w]^2_{1/2,\mathbb{R}^2} + \int_{\mathbb{R}^2} W(w)\,\mathrm{d}x \;\middle|\; w - z \in H^{1/2}(\mathbb{R}^2),\ w \equiv 0 \text{ on } B_1(0)\right\}. \tag{2.1}$$

In [GM05, GM06] the precise statement is given also for anisotropic kernels, different scalings of the number of obstacles, different obstacle sizes proportional to ε, and finite strength pinning. Furthermore, pre-compactness of finite energy sequences is established.

Let us briefly comment on this result. The Γ-limit is essentially the sum of two terms, the perimeter functional which occurs as the limit of the unconstrained non-local Modica-Mortola functional (see [ABS98, SV12] for double-well potentials and [GM06, Kur06, Kur07] for periodic potentials), and the bulk term which stems from the pinning constraint (see [MK64, MK74, CM97, AB02] for the local case). In the critical scaling $N_\varepsilon \sim \frac{|\log\varepsilon|}{\varepsilon}$ both terms appear on the same order.

Remark 2.2. In one dimension, the critical scaling is $N_\varepsilon \sim |\log\varepsilon|$. The difference arises due to the different scaling of the $H^{1/2}$-semi-norm in different dimensions.

2.2. **Viscous Evolution.** In this article, we compare the gradient-flow dynamics associated to the functionals $\mathcal{E}_\varepsilon$ with those of the continuum limit. If we assume that both halves of the crystal relax on a timescale much faster than the motion of dislocations, we can describe the dynamics by a quasi-static evolution, i.e. only the jump condition between the traces in upper and lower half-space along the slip plane needs to be evolved according to the gradient flow of the energy $\mathcal{E}_\varepsilon$ and the distortion field in upper and lower half space approaches the associated energy minimum instantaneously.

According to [IS09], solutions to the associated evolution equation of $\mathcal{E}_\varepsilon$ *without* the pinning constraint

$$\varepsilon\, u_t = \frac{1}{|\log\varepsilon|}\left(\mathrm{A}\, u - \frac{1}{\varepsilon}\, W'(u)\right)$$

converge to level set mean curvature flow.

Remark 2.3. The ε in front of the time derivative is the correct time scaling for a phase field gradient flow since the interface moves with speed $O(1)$ if the time derivative is $O(1/\varepsilon)$.

In one dimension, the perimeter functional has no interesting dynamics, so the behaviour of the evolution equation (without obstacles) should be governed by the next order Γ-limit. At a simple step function $\chi_{[r_1,r_2]}$ on the real line we can modify arguments from [Kur06, Kur07] for closely related energies to see that

$$\Gamma(L^2) - \lim_{\varepsilon\to 0} |\log\varepsilon|\cdot\left(\mathcal{E}_\varepsilon - \frac{1}{\pi}\right) = \frac{1}{\pi}\,\log|r_2 - r_1| + c_0 \tag{2.2}$$

where $\mathcal{E}_\varepsilon$ is given by the same formula as above, but in dimension one and on a space without pinning constraint. Here $c_0 > 0$ is a constant depending on the potential W. In particular, the next order term in the Γ-expansion vanishes only logarithmically in ε rather than exponentially fast as in the classical local functional. Using non-variational techniques, Gonzalez and Monneau showed in [GM12] that in one dimension (or at straight parallel interfaces), we still expect to see attraction of interfaces on the slower timescale

$$\frac{1}{|\log\varepsilon|}\,(\varepsilon\, u_t) = \frac{1}{|\log\varepsilon|}\left(\mathrm{A}\, u - \frac{1}{\varepsilon}\, W'(u)\right).$$

This motion contrasts with the (local) Allen-Cahn equation in one dimension

$$\varepsilon\, u_t = \varepsilon\,\Delta u - \frac{1}{\varepsilon}\, W'(u),$$

which becomes exponentially slow in ε [CP89]. The stronger attraction here stems from the non-locality of the half Laplacian as compared to the full Laplace operator, occurring analogously in the next order Γ-limit (2.2) at a simple step function. The heavy tails of the singular kernel force slower decay of optimal interfaces for the fractional Allen-Cahn equation, which translates into stronger attraction (see Section 3.2).

Now consider the effect of pinning, just in one dimension. Heuristically, we simply take a function u with one or two interfaces and pinned obstacles on points $d_\varepsilon\mathbb{Z}$ such that the interfaces are $O(d_\varepsilon)$ away from the nearest obstacle, 0 outside and 1 in between the obstacles. The obstacle at md_ε contributes an amount roughly proportional to

$$\frac{1}{|\log\varepsilon|}\int_{m\, d_\varepsilon-\varepsilon}^{m\, d_\varepsilon+\varepsilon}\frac{1}{|x|^2}\,\mathrm{d}x \approx \frac{\varepsilon}{|\log\varepsilon|\, m^2\, d_\varepsilon^2}$$

to the attractive force in the 1/2-Laplacian on an interface at $x = 0$, using the representation of the half-Laplacian as a singular integral operator

$$
\begin{aligned}
-(-\Delta)^{1/2}u(x) &= P.V.\int_{\mathbb{R}^n} \frac{u(y)-u(x)}{|x-y|^{n+1}}\,\mathrm{d}y \\
&= \int_{B_\rho(x)} \frac{u(y)-u(x)-\langle \nabla u(x), y-x\rangle}{|y-x|^{n+1}}\,\mathrm{d}y + \int_{\mathbb{R}^n\setminus B_\rho(x)} \frac{u(y)-u(x)}{|x-y|^{n+1}}\,\mathrm{d}y. \qquad (2.3)
\end{aligned}
$$

The expression $P.V.\int$ denotes that the integral needs to be understood in the principal value sense $P.V.\int_{\mathbb{R}^n} = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus B_\varepsilon(0)}$. This interpretation will be implied in the following. The integrals in the second expression exist and use the symmetry of the integral kernel and the antisymmetry of the linear term for cancellation effects and $\rho \in (0,\infty)$ can be chosen freely. This form will be frequently used for estimates in the following. Note that our normalisation of the fractional Laplacian (and the $H^{1/2}$-semi-norm) differ from the usual one by a dimension-dependent constant.

We can sum over $m \in \mathbb{Z}$ and obtain a term proportional to $\varepsilon/(d_\varepsilon^2\,|\log\varepsilon|)$, which is much smaller than the attractive force between interfaces for $d_\varepsilon \gg \sqrt{\varepsilon}$ and small in the natural gradient flow time scaling for $d_\varepsilon \gg \sqrt{\varepsilon/|\log\varepsilon|}$. Seeing that the interesting amount of obstacles in one dimension would be $N_\varepsilon \sim |\log\varepsilon|$ on a periodic interval, the natural distance between obstacles scales as $d_\varepsilon \sim 1/|\log\varepsilon|$. We are led to the conjecture that the obstacles' contribution to the the contracting force vanishes in the limit $\varepsilon \to 0$, and that the dynamics are independent of the presence of obstacles in this scenario.

In one dimension, the pinning is expected to have an effect on the evolution if $d_\varepsilon \sim \sqrt{\varepsilon/|\log\varepsilon|}$, which is the natural length scale in two dimensions (since the natural scaling for the number of obstacles is $N_\varepsilon \sim \frac{|\log\varepsilon|}{\varepsilon}$). We would still expect two-dimensional solutions to become slow in this scaling since the one-dimensional case corresponds to solutions constant in one direction or the effect of pinning along whole lines, not just on circles.

A two dimensional version of the argument above gives the contribution

$$
\frac{1}{|\log\varepsilon|}\int_{B_\varepsilon(id_\varepsilon, jd_\varepsilon)} \frac{1}{|x|^3}\,\mathrm{d}x \approx \frac{\varepsilon^2}{|\log\varepsilon|\,d_\varepsilon^3}\,\frac{1}{(i^2+j^2)^{3/2}}
$$

for a single obstacle and thus the scaling proportional to $\varepsilon^2/(d_\varepsilon^3\,|\log\varepsilon|)$ for the contribution of the obstacles to the driving force. Again, inserting $d_\varepsilon = 1/\sqrt{N_\varepsilon} = \sqrt{\varepsilon/|\log\varepsilon|}$, we see that this force should be $O(\varepsilon^{1/2}|\log\varepsilon|^{1/2})$ which is negligible compared to the attraction between interfaces, let alone curvature.

Technically, the relevant consideration is whether this back-of-the-envelope calculation gives the right scaling or whether the pinning induces further non-local effects. In particular, we need to investigate how quickly minimisers of the cell-problem (2.1) approach $z \in \mathbb{Z}$ at infinity.

Another interesting question is how pinning interacts with other terms. Namely, when external forces, the attraction of interfaces, or curvature terms are driving an interface to expand the phase $\{u = 0\}$, a moving interface must create new obstacles during the movement (for example by Orowan loops). This would lead to an increase in the bulk energy term which may dominate the potential energy gain. In this case, the presence of obstacles prevents motion.

These heuristic considerations suggest that the forest dislocations do not act as a driving force on the relevant time-scale, but may act against other driving forces to prevent motion. In this sense, it is more appropriate to think of the obstacles as creating a friction term in the dynamic case rather than a stored energy as it appears in the Γ-limit. Studying the gradient flow of the present phase field model thus provides insight into the treatment of stored energy hardening terms in macroscopic models for plastic evolution, in particular how to include a Bauschinger effect.

2.3. **Main Results and Idea of Proof.** We will always assume that $W \in C^\infty(\mathbb{R})$, that W is 1-periodic and satisfies $W \geq c\,\mathrm{dist}^2(\cdot, \mathbb{Z})$ for some $c > 0$ and $W(0) = 0$. Note that the conditions together imply that $W''(0) > 0$. Additional conditions will be placed on W in Sections 3.2 and 3.4 to ensure the right behaviour of the second derivatives of the optimal transition profile between two neighbouring potential wells in one dimension and of a corrector function for moving interfaces. The prototype of an admissible potential is $W(z) = \sin(\pi z) + 1$. Furthermore, we make the following assumptions on the distribution of obstacles $x_{i,\varepsilon}$:

(1) the assumptions of Theorem 2.1 hold and additionally
(2) the obstacles are arranged as perturbations of a square grid.

The second condition is stated in a precise fashion in Theorem 4.5. Admissible configurations are a perfect square grid with length scale $d_\varepsilon \sim N_\varepsilon^{-1/2}$, small perturbations of the grid on a small fraction of this length scale, or a square grid on a slightly smaller length scale with vacancies and potentially multiple points $x_{i,\varepsilon}$ close to a single node of the grid. For technical reasons a truly random arrangement of $x_{i,\varepsilon}$ as identically and uniformly distributed points on $\mathbb{T}^2$ is admissible neither for our results nor in Theorem 2.1.

Heuristic Statement of the Main Results. *Under the assumptions above, the following holds:*

(1) The gradient-flows of $\mathcal{E}_\varepsilon$ do not converge to the gradient flow of $\mathcal{E}$ in any time-scale, nor to pure mean curvature flow.
(2) For a suitably aligned single straight interface on $\mathbb{R}^2$, a time-rescaling $c_\varepsilon \leq \sqrt{\frac{\varepsilon}{|\log \varepsilon|}}$ is necessary to obtain a moving interface in the limit $\varepsilon \to 0$. In the plane or on a flat torus, two suitably aligned and sufficiently close interfaces attract on a time-scale of $c_\varepsilon = \frac{1}{|\log \varepsilon|}$. This motion is independent of the presence of obstacles $x_{i,\varepsilon}$.
(3) If we apply an external force to increase the amount of slip $|u|$ or mean curvature flow would act in that way, the Garroni-Müller energy barrier has to be overcome and the presence of obstacles can prevent such motion.

Remark 2.4. Point (2) in the main results can be interpreted in the sense that in the unrescaled time-scale, dislocations remain stationary after unloading in contrast to the evolution of the Γ-limit. This extends the validity of the phase-field model to non-monotone loading.

Remark 2.5. Informally speaking, the essence of our results can be stated as follows. In the gradient flow time scaling and in the presence of forest dislocations, straight parallel dislocation lines are stationary. If an exterior force is applied in the direction of increasing the amount of slip, the dislocations remain stationary until a certain threshold is reached, while they offer neither resistance nor help to an exterior force which acts in the direction of decreasing the amount of slip, see Figure 2. In particular, the results derived here are consistent with the mechanical Bauschinger effect observed in [DR10], in the sense that reverting a plastic deformation is associated with a dramatic yield strength drop, but the reversal does of course not take place spontaneously.

To simplify the constructions, we have presented proofs for slip functions taking only the values in $[0, 1]$, but extensions to positive slip are possible. For technical reasons, we focus on *signed* slip and interfaces which are aligned with the forest dislocations. By that we mean that if the forest dislocations are located on a square grid $d_\varepsilon \cdot \mathbb{Z}^2$ and a straight interface in $\mathbb{R}^2$ meets the x-axis at an angle $\phi \in [0, 2\pi)$, then we require $\tan(\phi) \in \mathbb{Q}$. We believe that also this restriction is of a purely technical nature.

In one dimension, the second restriction does not appear and the results are sharp.

The precise statements of these results can be found in the main text, most importantly in Theorems 4.3, 4.5 and Corollary 4.7. The exact time-scaling c_ε for a gradient flow u_ε of $\mathcal{E}_\varepsilon$ for a

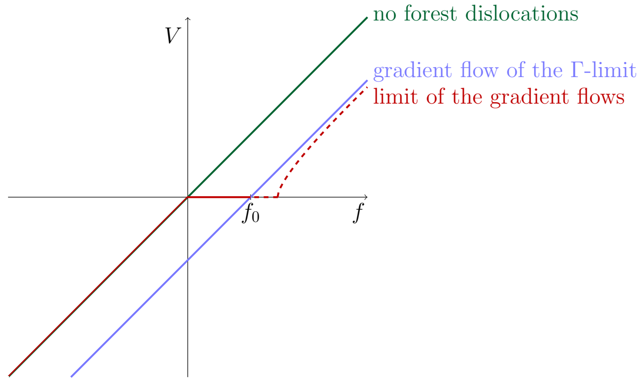

FIGURE 2. We consider the normal velocity $V(f)$ of a single straight dislocation in the sharp interface limit without time rescaling under an applied exterior force f. In the figure above, both the force and the velocity of the interface are chosen to be positive in the direction of increasing slip. The three lines illustrate the kinetic relation derived from a viscous evolution without forest dislocations, the gradient flow of the Garroni-Müller Γ-limit (with $f_0 = \Lambda \cdot \alpha(1)$), and the limit of gradient flows of the phase field model with forest dislocations, respectively. The regime of the dotted line has not been treated here, and is to be taken as a conjecture.

single straight and aligned interface in the plane is not known, but the bounds

$$\sqrt{\frac{\varepsilon}{|\log \varepsilon|^3}} \leq c_\varepsilon \leq \sqrt{\frac{\varepsilon}{|\log \varepsilon|}}$$

hold. Stronger results are available in one dimension, see Theorems 3.8, 3.11, 3.12 and 3.16.

As seen in the heuristic calculations above or in other situations of ‘dynamic meta-stability’ [BK90], the pinning constraint induces no motion on the macroscopic time-scale since its energy dissipation is highly localised at the obstacles. A similar phenomenon is observed in the aforementioned ODE model

$$\dot{x} = -2x\left(1 - \sin\left(\frac{x^2}{\varepsilon}\right)\right)$$

which is the gradient flow of $F(x) = x^2 + \varepsilon\cos(x^2/\varepsilon)$. Short bursts of very fast motion can be observed here before getting trapped in a local energy minimum. Replacing the sin-function by a suitable modification, we can instead observe very fast motion at steep drops alternating with very slow motion on almost flat segments. The overall motion becomes slow as $\varepsilon \to 0$ due to the many flat segments of the potential. An energy dissipation argument for such a system can be found in [Mie12].

Unfortunately, directly using energy dissipation techniques appears impossible in our model. Instead, we construct viscosity sub- and super-solutions of (1.4) which ‘trap’ a solution. If we can establish a certain behaviour for both the sub- and super-solution, it follows that it also holds for the solution. Using suitable estimates, the slowness of sub- and super-solutions (or their behaviour according to kink/anti-kink attraction) can be established through a rigorous version of the heuristic calculation given above. For a one-dimensional problem without pinning, a similar approach has been used in [GM12].

The main difficulty in the proof therefore lies in the construction of suitable sub-solutions for the non-local evolution equation. We first prove analogue statements in one dimension in

Theorems 3.8, 3.11, 3.12 and 3.16 because this is technically easier and then modify the arguments to yield the result in two dimensions.

The construction proceeds in two steps. First we construct a stationary sub-solution at a pinning site by considering a periodic constrained minimisation cell problem and obtaining sharp decay estimates for the solution as $\varepsilon \to 0$. Then we carefully glue the stationary sub-solution to a modified optimal profile for the transition between the potential wells with precise estimates in order to not destroy the sub-solution property.

3. One-dimensional Dynamics

In this section, we construct sub- and super-solutions to the evolution equation (1.4) in one space dimension for various initial conditions. This is significantly simpler than the two-dimensional case even at straight parallel interfaces, so we devote an entire section to demonstrate the techniques that will later be refined for the two-dimensional evolution. The rate we obtain is optimal in one dimension, while there is a difference of order $O(|\log\varepsilon|)$ between the upper and the lower bound in the two-dimensional case.

3.1. Periodic Obstacles.

We consider a rescaled version of the problem where a forest dislocation/obstacle has length scale $O(1)$. The same length scale occurs in the transition of an optimal profile between two potential wells.

Lemma 3.1. *Denote by S^1_l the circle of length $l \gg 0$ and take $R, M > 0$ and an arbitrary point $x_0 \in S^1_l$. Then, if $l \gg 1$ is large enough, there exists a function $\bar u_l \in H^{1/2}(S^1_l)$ such that*

$$\bar u_l \equiv 0 \text{ on } B_R(x_0) \qquad \text{and} \qquad \mathrm{A}\,\bar u_l - W'(\bar u_l) = \frac{M}{l} \text{ on } S^1_l \setminus B_R(x_0)$$

in the weak sense. The function $\bar u_l$ has the following properties:

- *$0 \le \bar u_l < 1$ and $\bar u_l > 0$ outside $\overline{B_R(x_0)}$.*
- *$\bar u_l \in C^{0,1/2}(S^1_l)$ and $\bar u_l \in C^\infty_{loc}(S^1_l \setminus \overline{B_R(x_0)})$.*
- *If $x \in S^1_l$ and $\bar x$ denotes the reflection of x through x_0, then $\bar u_l(\bar x) = \bar u_l(x)$.*
- *If we identify $S^1_l = [-l/2, l/2)$ and $x_0 = 0$, then $\bar u_l$ is monotonically increasing on $[0, l/2)$.*
- *Let $\beta > 0$. Then the set $\{\bar u_l < 1 - \beta\}$ is contained in $[-c_\beta, c_\beta]$ for some $c_\beta > 0$ independently of l.*
- *Let $-x_0$ denote the antipodal point of x_0 Then*
$$u_l(-x_0) \le 1 - \frac{1}{W''(0)}\left(\frac{M}{l} + \frac{R}{l^2}\right).$$
- *There exists a constant $c_2 > 0$ such that*
$$\bar u_l(x) \ge 1 - \frac{M}{W''(0)\,l} - \frac{c_2}{|x|^2}.$$

All constants may depend on R. M may depend on l and the constants are uniform as long as $M_l \le M_0$ is uniformly bounded.

Proof. **Set Up.** For the time being, replace W by a smooth double-well potential on $\mathbb{R}$ which agrees with the original on $[0,1]$ which is monotone and convex outside that interval such that W' has linear and W has quadratic growth at ∞ and such that $W(t) \ge t$ for $t \ge 2$. Once we see that all relevant functions take values only in $[0,1)$, we can pass back to the original multi-well potential. Consider the Hilbert space

$$X_l := \left\{ u \in H^{1/2}(S^1_l) \mid u \equiv 0 \text{ on } B_R(x_0) \right\}.$$

We will show that for every $l \gg 1$, the energy

$$\mathcal{E}_l(u) = [u]^2_{1/2,S^1_l} + \int_{S^1_l} W(u)\,\mathrm{d}x + \frac{M}{l}\int_{S^1_l} |u|\,\mathrm{d}x$$

has a minimiser $\bar u = \bar u_l$ in the open set

$$U_l := \left\{ u \in X_l \;\middle|\; \frac{1}{l}\int_{S^1_l} u\,\mathrm{d}x > \frac{1}{2}\right\}.$$

Subsequently, we will show that $\bar u_l$ has the properties we claim in the Lemma.

Finite Energy. First, we show that there exist functions $u_l \in U_l$ such that

$$\mathcal{E}_l(u_l) \le C$$

for a universal constant C (which depends on R, M). Take a smooth function $\eta : \mathbb{R} \to \mathbb{R}$ such that $\eta(r) = 0$ for $r \le R$, $\eta(r) = 1$ for $r \ge 2R$ and monotone in between. Then set

$$u_l(x) = \eta(d(x, x_0))$$

where d is the usual distance function on the circle. The function u_l is smooth and has energy

$$\begin{aligned}
\mathcal{E}_l(u_l) &= \int_{S^1_l}\int_{S^1_l} K(x,y)\,|u_l(x) - u_l(y)|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{S^1_l} W(u_l)\,\mathrm{d}x + \frac{M}{l}\int_{S^1_l} u_l\,\mathrm{d}x\\
&\le \int_{B_{3R}(x_0)}\int_{B_{3R}(x_0)} K(x,y)\,|u_l(x) - u_l(y)|^2\,\mathrm{d}x\,\mathrm{d}y\\
&\quad + \int_{S^1_l\setminus B_{3R}(x_0)}\int_{B_{2R}(x_0)} K(x,y)\,|1 - u_l(x)|^2\,\mathrm{d}x\,\mathrm{d}y + 2\int_R^{2R} W(\eta)\,\mathrm{d}r + M.
\end{aligned}$$

Recall that the kernel K of the $H^{1/2}$ semi-norm on S^1_l is

$$K(x,y) = \sum_{k\in\mathbb{Z}} \frac{1}{|x - (y + lk)|^2} = \frac{1}{l^2}\sum_{k\in\mathbb{Z}} \frac{1}{|\,(x-y)/l + k\,|^2} = \frac{1}{4\pi^2\, l^2\, \sin^2\left(\frac{x-y}{\pi l}\right)}$$

when we identify $S^1_l = (-l/2, l/2]$. To see this, take the periodic covering of the circle by the real line and use that the half-Laplacian on the circle agrees with the half-Laplacian of the periodically lifted function on $\mathbb{R}$. Clearly, this gives the above kernel for the half-Laplacian and by extension for the $H^{1/2}$-semi-norm. Observe that $|x-y| \le |x| + |y| \le 3R + l/2$ for $x \in B_{3R}(0)$, $y \in S^1_l = [-l/2, l/2]$, so

$$\begin{aligned}
\left| K(x,y) - \frac{1}{|x-y|^2}\right| &= \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{|x - (y + lk)|^2}\\
&= \frac{1}{l^2}\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{|(x-y)/l + k|^2}\\
&\le l^{-2} \max_{z\in B_{1/2+3R/l}(0)} \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{|z-k|^2}\\
&\le C\, l^{-2}
\end{aligned}$$

where C is uniform for all $l \gg 1$. Using $|x-y|^2 \geq |y|^2/3$ for $x \in B_{2R}(x_0)$ and $y \in S^1_l \setminus B_{3R}(x_0)$, the remaining integral is then estimated by

$$\begin{aligned}\int_{S^1_l\setminus B_{3R}(x_0)}\int_{B_{2R}(x_0)} K(x,y)\,|1-u_l(x)|^2\,\mathrm{d}x\,\mathrm{d}y &\leq \int_{S^1_l\setminus B_{3R}(x_0)}\int_{B_{2R}(x_0)} K(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &\leq \int_{S^1_l\setminus B_{3R}(x_0)}\int_{B_{2R}(x_0)} \frac{1}{|x-y|^2}+C\,l^{-2}\,\mathrm{d}x\,\mathrm{d}y\\ &\leq \int_{S^1_l\setminus B_{3R}(0)}\int_{B_{2R}(0)} \frac{3}{|y|^2}+C\,l^{-2}\,\mathrm{d}x\,\mathrm{d}y\\ &= 2R\left(\int_{[-l/2,l/2]\setminus[-3R,3R]}\frac{3}{|y|^2}\,\mathrm{d}y + C\,l^{-1}\right)\\ &\leq C.\end{aligned}$$

Similarly, the local integral is uniformly bounded for large l where the kernel approaches the kernel of the half-Laplacian on the real line.

Minimisers. The direct method of the calculus of variations establishes that $\mathcal{E}_l$ has a minimiser $\bar u_l$ in the closure $\overline{U_l}$ of U_l. We will show that for large enough l, $\bar u_l$ must lie inside U_l.

Assume that $1/2 < \frac{1}{l}\int_{S^1_l}\bar u_l\,\mathrm{d}x < 4/7$. Then there are two possibilities:

- There exists a set $A_l \subset S^1_l$ such that $|A_l| \to \infty$ as $l\to\infty$ and $1/3 \leq \bar u_l \leq 2/3$ on A_l or
- there exists no such set.

In the first case, we observe that

$$\mathcal{E}_l(\bar u_l) \geq \int_{S^1_l} W(\bar u_l)\,\mathrm{d}x \geq |A_l| \min_{t\in[1/3,2/3]} W(t)$$

which goes to infinity when l becomes large. Hence this is not possible for minimisers for large enough l. In the second case, we know that the sets

$$B_l := \{\bar u_l \geq 2/3\}, \qquad D_l := \{\bar u_l \leq 1/3\}$$

satisfy $|B_l|+|D_l| \geq l-c$ for some constant $c>0$. Observe that $\bar u_l \geq 0$, since this cut-off decreases the energy and cannot violate the integral condition. So we deduce that

$$4/7\cdot l > \int_{S^1_l}\bar u_l\,\mathrm{d}x \geq 2/3\,|B_l| \tag{3.1}$$

Conversely, we know that

$$\int_{\{\bar u_l\geq 2\}}\bar u_l\,\mathrm{d}x \leq \int_{S^1_l} W(\bar u_l)\,\mathrm{d}x \leq C$$

since $W(t)\geq t$ for $t\geq 2$. Therefore

$$\frac{l}{2} < \int_{S^1_l}\bar u_l\,\mathrm{d}x \leq 2\,|B_l| + C + \frac{2}{3}\,|S^1_l\setminus(B_l\cup D_l)| + \frac{1}{3}\,|D_l|.$$

We renormalise $b_l := |B_l|/l$ and $d_l := |D_l|/l$ and since in this case also $|S^1_l\setminus(B_l\cup D_l)|\leq C$ for some constant, the inequalities read

$$6/7 > b_l, \qquad 1/2 < 2b_l + C/l + d_l/3.$$

For the same reason, we have $1-b_l \geq d_l \geq 1-b_l-\frac{C}{l}$, whence

$$\frac{1}{2} < 2b_l + \frac{C}{l} + \frac{1-b_l}{3} \qquad\Rightarrow\qquad \frac{1}{2}-\frac{1}{3}-\frac{C}{l} < \frac{5}{3}\,b_l \qquad\Rightarrow\qquad \frac{1}{10}-\frac{C}{l} < b_l.$$

In total, we find that
$$1/11 < b_l < 6/7$$
for all large enough l. Consequently, we obtain that $\delta < b_l, d_l < 1-\delta$ for some $\delta > 0$ and all large enough l. We can roll up the circle to an interval $I_l = [0, l]$ and use $K(x,y) \geq |x-y|^{-2}$. Then the re-arrangement result [ABS98, Proposition 6.1] states that B_l and D_l are ideally distributed as two sub-intervals at opposite ends of I_l. We compute
$$\begin{aligned}
\mathcal{E}_l(\bar u_l) &\geq [\bar u_l]^2_{1/2, S^1_l}\\
&\geq \int_{B_l}\int_{D_l}\left(\frac{1}{3}\right)^2 K(x,y)\,\mathrm{d}x\,\mathrm{d}y\\
&\geq \frac{1}{9}\int_0^{|B_l|}\int_{|B_l|+c}^{l} \frac{1}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y\\
&= \frac{1}{9}\int_0^{|B_l|} \frac{1}{|B_l|+c-y} - \frac{1}{l-y}\,\mathrm{d}y\\
&= \frac{1}{9}\left[\log(|B_l|+c) - \log(c) - \log(l) + \log(l-|B_l|)\right]\\
&= \frac{1}{9}\log\left(\frac{(|B_l|+c)(l-|B_l|)}{c\,l}\right)\\
&\sim \log\left(\frac{|B_l|\,|D_l|}{|B_l|+|D_l|}\right).
\end{aligned}$$
If both $|B_l|$ and $|D_l|$ go to infinity as $l \to \infty$ (as above), then $\mathcal{E}_l(\bar u_l) \to \infty$ as well, which leads to a contradiction. This key estimate is used in [ABS98, Lemma 4.5] to establish Γ-convergence to the perimeter functional for the fractional Modica-Mortola energy with a double well potential and no pinning. Thus indeed $|D_l| \leq C$ so that $|B_l| \geq l - C - c$ and
$$\frac{1}{l}\int_{S^1_l} \bar u_l\,\mathrm{d}x > 4/7,$$
so $\bar u_l \in U_l$. We finally come to establishing the properties we claimed for $\bar u_l$.

Symmetry and Monotonicity. Due to [BI94, Theorem 3] $\bar u$ agrees with its monotonically decreasing rearrangement around $-x_0$ since rearranging decreases the non-local term in the energy (strictly, if the function was not symmetric decreasing from a point before) while leaving the local ones invariant and preserving the integral constraint. So, when we identify $S^1_l = (-l/2, l/2]$ and $x_0 = 0$ by the usual covering map, we see that $\bar u_l(x) = \bar u_l(-x)$ and $\bar u_l$ is monotonically increasing on $[0, l/2)$.

Growth. When showing that $\bar u_l \in U_l$, we showed that one of the sequences of sets
$$B_l := \{x \in S^1_l \mid \bar u_l \geq 2/3\}, \qquad D_l := \{x \in S^1_l \mid \bar u_l \leq 1/3\}$$
must have uniformly bounded measures. Since $\frac{1}{l}\int_{S^1_l} \bar u_l\,\mathrm{d}x > 1/2$, this can only be D_l. Since D_l has uniformly bounded measure and for $\beta < 1/2$ also the measure of the set $\{1/2 \leq \bar u_l < 1-\beta\}$ is bounded from the double-well part of the energy, it follows that the sequence of sets
$$D^\beta_l := \{x \in S^1_l \mid \bar u_l \leq 1-\beta\}$$
must have uniformly bounded measures depending on $\beta \in (0,1)$. Since $\bar u_l$ is monotone growing away from $x_0 = 0$, the sets D^β_l are intervals and there exists a constant $c_\beta > 0$ such that
$$D^\beta_l \subset (-c_\beta, c_\beta) \subset (-l/2, l/2] = S^1_l$$
for all sufficiently large l.

Boundedness. We have seen that $\bar u_l \geq 0$ and that

$$\frac{1}{l}\int_{S^1_l} \bar u_l \,\mathrm{d}x > \frac{4}{7}, \qquad \frac{1}{l}\int_{\{\bar u_l\geq 2\}} \bar u_l\,\mathrm{d}x \leq \frac{C}{l}.$$

So clearly, $\min\{\bar u_l, 2\}$ satisfies the integral constraint for large enough l, vanishes on $B_1(x_0)$ and has strictly lower energy than $\bar u_l$ unless $\bar u_l \leq 2$ since we modified $W(t)$ to be monotonically increasing for $t\geq 1$. Thus $\bar u_l \leq 2$ for all sufficiently large l.

Regularity. Since $\bar u_l$ lies in the interior of U_l, we can calculate the change of energy in $\mathcal{E}_l$ under small variations of $\bar u_l$. $\mathcal{E}_l$ is smooth when we vary only where $\bar u_l > 0$. Thus, $\bar u$ satisfies the Euler-Lagrange equation

$$\begin{cases} \bar u_l = 0 & \text{on } [-a_l, a_l]\\ \mathrm{A}\,\bar u_l = W'(\bar u_l) + M/l & \text{in } S^1_l\setminus[-a_l,a_l]\end{cases}$$

in a weak sense. The right hand side of the equation lies in $L^\infty(S^1_l)$ since $\bar u_l$ is bounded. Due to [SV14], $\bar u_l$ is a viscosity solution of the same equation and thus continuous. In fact,

$$\bar u_l \in C^{0,1/2}(S^1_l)\cap C^\infty\left(S^1_l\setminus[-a_l,a_l]\right)$$

due to [ROS14, Propositions 1.1 and 1.4, Theorem 1.2]. The proofs in the literature are usually presented for bounded domains on Euclidean space, but also work in the periodic case.

Determining the Vanishing Set. It remains to show that $a_l = R$. Assume that $a_l > R$ and take $\phi\in C^\infty_c(R,a_l)$ with $\phi\geq 0$. Then we know that

$$\begin{aligned} 0 &\leq \frac{\mathcal{E}_l(\bar u_l + t\phi) - \mathcal{E}_l(\bar u_l)}{t}\\ &= \langle \bar u_l, \phi\rangle_{H^{1/2}} + \frac{t}{2}\,[\phi]^2_{H^{1/2}} + \frac{1}{t}\int_{S^1_l} W(t\phi)\,\mathrm{d}x + \frac{M}{l}\int_{S^1_l}\phi\,\mathrm{d}x\\ &\to -\langle \mathrm{A}\,\bar u_l, \phi\rangle_{L^2} + \frac{M}{l}\int_{S^1_l}\phi\,\mathrm{d}x\end{aligned}$$

as $t\to 0$ since W vanishes quadratically at zero and $W(\bar u_l + t\phi) = W(\bar u_l)$ where $\bar u_l\neq 0$. We can replace the $H^{1/2}$-inner product with an L^2-inner product since $\bar u_l\equiv 0$ is smooth on the support of ϕ. It follows that $\mathrm{A}\,u_l \leq \frac{M}{l}$ on (R, a_l), but this can easily be seen to be false for all large l since

$$\mathrm{A}\,\bar u_l(x) \geq \int_{c_\eta}^{l/2}\frac{\eta}{|y-x|^2}\,\mathrm{d}y \geq \frac{\eta}{c_\eta} - \frac{2\eta}{l} \not\to 0$$

for all $\eta\in(0,1)$. Thus $a_l \equiv R$ for $l\gg 1$.

Improved Boundedness. By boundedness, symmetry and monotonicity, $\bar u_l$ is maximal at the antipodal point $-x_0$ of x_0. Due to smoothness, we can easily argue that $\mathrm{A}\,\bar u_l$ is defined pointwise around $-x_0$, and thus

$$\begin{aligned}\mathrm{A}\,\bar u_l(-x_0) &= \int_{S^1_l} K(x,y)\left[\bar u_l(x) - \bar u_l(x_0)\right]\mathrm{d}x\\ &\leq \int_{-l/2}^{l/2}\frac{u(x) - u(-x_0)}{\min\{|x-l/2|, |x+l/2|\}^2}\,\mathrm{d}x\\ &\leq \int_{-R}^{R}\frac{-\bar u_l(-x_0)}{\min\{|x-l/2|,|x+l/2|\}^2}\,\mathrm{d}x \qquad \leq (2R)\cdot\frac{-1}{(l/2)^2}\end{aligned}$$

since $\bar u_l$ is maximal at $-x_0$, hence

$$-8R/l^2 \geq \mathrm{A}\,\bar u_l(-x_0) = W'(\bar u_l(-x_0)) + M/l$$

holds pointwise. Since $W' > 0$ outside $[0,1]$, this directly shows that

$$\bar u_l \leq \bar u_l(-x_0) \leq 1 - \frac{1}{W''(1)}\left(\frac{M}{l} + \frac{R}{l^2}\right) < 1$$

for large enough l. From now on, we can use the original potential W.

Improved Growth I. Take $0 < \beta < 1/8$ such that $-W'(1-t) \geq \frac{W''(1)\,t}{2}$ for $t \in [0,\beta]$ and that W' is monotonically increasing on $[1-\beta, 1]$. Assume that $S^1_l = J \cup J^c$ and $w_l \in C^0(S^1_l) \cap H^{1/2}(S^1_l)$ such that $\bar u_l \geq w_l$ on J and

$$\mathrm{A}\, w_l - W'(w_l) \geq \frac{M}{l} \quad \text{as well as} \quad \bar u_l \geq 1-\beta \text{ on } J^c.$$

Then by a simple application of the maximum principle we have $\bar u_l \geq w_l$ on S^1_l. Assume the contrary. Then $w_l - \bar u_l$ has a positive maximum somewhere in J^c. We find that

$$0 \geq \mathrm{A}(w_l - \bar u_l) \geq W'(w_l) + M/l - W'(\bar u_l) - M/l = W'(w_l) - W'(\bar u_l) > 0$$

at this point since $1 \geq w_l > \bar u_l \geq 1-\beta$ are both in the area where W' is monotonically increasing and obtain a contradiction. We will now construct comparison functions w_l.

By monotonicity and growth beyond $1-\beta$ on uniformly finite intervals, there exists an interval J_β around x_0 such that $\bar u_l \geq 1-\beta$ outside of J_β independently of l. Take $l \gg 1$ and define

$$w_l : S^1_l = (-l/2, l/2] \to \mathbb{R}, \qquad w_l(x) = \left(1 - \frac{\gamma M}{l} - \frac{\tilde c_2}{|x|^2}\right)_+.$$

We easily calculate

$$w_l(x) = 0 \quad \Leftrightarrow \quad |x| \leq \sqrt{\frac{\tilde c_2}{1-\gamma M/l}}, \qquad w_l(x) \geq 1-\beta \quad \Leftrightarrow \quad |x| \geq \sqrt{\frac{\tilde c_2}{\beta - \gamma M/l}} \geq 2\sqrt{2\,\tilde c_2},$$

so in particular $\overline{w}_l$ vanishes on an interval

$$[-\sqrt{\tilde c_2}, \sqrt{\tilde c_2}] \subset J \subset [-\sqrt{2\tilde c_2}, \sqrt{2\tilde c_2}].$$

In particular we can choose $\tilde c_2$ so large that $\bar u_l(x) \leq 1-\beta$ implies $w_l(x) = 0$. So we observe that

$$\begin{aligned}
\mathrm{A}\, w_l(x) - W'(w_l(x)) &= \mathrm{A}\, w_l(x) - W'\left(1 - \frac{\gamma M}{l} - \frac{\tilde c_2}{|x|^2}\right)\\
&\geq \mathrm{A}\, w_l(x) + \frac{W''(1)}{2}\left(\frac{\gamma M}{l} + \frac{\tilde c_2}{|x|^2}\right)\\
&\geq \frac{W''(1)\,\gamma}{2}\,\frac{M}{l} + \left(\mathrm{A}\, w_l(x) + \frac{W''(1)\,\tilde c_2}{2\,|x|^2}\right)
\end{aligned}$$

whenever $w_l \geq 1-\beta$. Therefore we can choose $\gamma \geq 2/W''(1)$ and only need to show that the second term is non-negative for $|x| \geq 2\sqrt{2\tilde c_2}$. Without loss of generality, take $x > 0$. Now we observe that $w_l(x) = 0 \Rightarrow |x| \leq \sqrt{2\,\tilde c_2}$ and recall that $\frac{c}{|x|^2} \leq K_l(x) \leq \frac{C}{|x|^2}$ where K_l is the kernel of the half-Laplacian on $S^1_l = (-l/2, l/2]$ and the constants c, C are uniform in $l \gg 1$.

First, let us assume that $0 < x < l/2 - 1$. We do not use the integrals in our estimates which have the correct sign (i.e. over the domain $x < y < l/2$ and its mirror image) except for the one over $[x, x+1)$ and we replace the ones with a negative sign which are on the far half $(-l/2, 0)$ of the circle by their counterpart on the near side $(0, l/2)$. Since the kernel K_l is monotone, this is admissible. Pick $2/3 < \alpha < 1$ and compute

$$\begin{aligned}
\mathrm{A}\, w_l(x) &= P.V. \int_{-l/2}^{l/2} [w_l(y) - w_l(x)]\, K_l(x-y)\, \mathrm{d}y \\
&\geq -\int_{\sqrt{2c_2}}^{\sqrt{2c_2}} K_l(x-y)\, \mathrm{d}y + 2\int_{\sqrt{2c_2}}^{x^\alpha} K_l(x-y)\, [w_l(y) - w_l(x)]\, \mathrm{d}y \\
&\quad + 2\int_{x^\alpha}^{x-1} K_l(x-y) \int_x^y w_l'(t)\, \mathrm{d}t\, \mathrm{d}y + \int_{x-1}^{x+1} K_l(x-y) \int_x^y (y-t)\, w_l''(t)\, \mathrm{d}t\, \mathrm{d}y \\
&\geq -C\left(\frac{\sqrt{2c_2}}{|x-\sqrt{2c_2}|^2} + \frac{1}{|x-x^\alpha|^2}\int_{c_2}^{x^\alpha} \frac{c_2}{y^2}\, \mathrm{d}y + \frac{c_2}{|x^\alpha|^3}\int_{x^\alpha}^{x-1} \frac{1}{|y-x|}\, \mathrm{d}y + \frac{c_2}{|x|^4}\right)
\end{aligned}$$

In the first term, we used that the jump is at most 1, in the second we pulled out the integral kernel and used only the negative part of the difference. If $x^\alpha \leq \sqrt{2\tilde{c}_2}$, the second term simply disappears. In the third term, we kept the integral kernel and used the largest possible value of the derivative, and the fourth term is estimated solely by the second derivative of $|x|^{-2}$. Constants were absorbed into C. Thus

$$\mathrm{A}\, w_l(x) \geq -C\left(\frac{\sqrt{2c_2}}{|x-\sqrt{c_2}|^2} + \frac{1}{|x-x^\alpha|^2} + \frac{c_2 \log(x)}{|x^\alpha|^3} + \frac{c_2}{|x|^4}\right)$$

When we choose $\tilde{c}_2$ large enough, we can see that

$$\mathrm{A}\, w_l(x) \geq -\frac{W''(1)\, \tilde{c}_2}{2\, |x|^2}$$

for all $|x| \geq \sqrt{2\, \tilde{c}_2}$. Finally, we observe that the argument can also applied for $|x| > l/2 - 1$ when we replace $1/x^2$ by a function $f_l \in C^2(S_l^1)$ satisfying

$$\frac{c}{|x|^2} \leq f_l(x) \leq \frac{C}{|x|^2}, \qquad |f_l'(x)| \leq \frac{C}{|x|^3}, \qquad |f_l''(x)| \leq \frac{C}{|x|^3}.$$

Improved Growth II. Finally, we use the comparison function

$$w_l(x) = \left(1 - \frac{\gamma M}{l} - \frac{c_2}{|x|^2}\right)_+$$

with the sharper growth rate $\gamma = \frac{1}{W''(1)}$ in the leading order term. Using a more precise Taylor expansion of W at the well, we observe that

$$\begin{aligned}
\mathrm{A}\, w_l(x) - W'(w_l(x)) &= \mathrm{A}\, w_l(x) - W'\left(1 - \frac{\gamma M}{l} - \frac{c_2}{|x|^2}\right) \\
&\geq \mathrm{A}\, w_l(x) + W''(1)\left(\frac{\gamma M}{l} + \frac{c_2}{|x|^2}\right) - C_{W'''}\left(\frac{\gamma M}{l} + \frac{c_2}{|x|^2}\right)^2 \\
&\geq W''(1)\, \gamma\, \frac{M}{l} + \left(\mathrm{A}\, w_l(x) + \frac{W''(1)\, c_2}{|x|^2}\right) - C_{W'''}\left(\frac{\gamma M}{l} + \frac{c_2}{|x|^2}\right)^2
\end{aligned}$$

whenever $w_l \geq 1 - \beta$ for a constant $C_{W'''}$ which depends only on the third derivatives of W. Choosing c_2 large enough, we see that $w_l(x) \leq 1 - \frac{2}{W''(0) l} - \frac{\tilde{c}_2}{|x|^2} \leq \bar{u}_l(x)$ for all $x \leq \sqrt{l}$, and by the same argument as above, we see that w_l is a sub-solution on $S_l^1 \setminus [-\sqrt{l}, \sqrt{l}]$, such that we find also in this ansatz that $w_l(x) \leq \bar{u}_l(x)$ for all $x \in S_l^1$ and $l \gg 1$. The important argument in this second improved growth estimate is that $|x|^{-2}$ is small (in fact, comparable to l^{-1}) outside of the relevant interval, so that the third derivative correction term can be estimated to be small. $\square$

Remark 3.2. The same method can be applied on tori $T_l^d = (S_l^1)^d$ in any dimension $d \geq 1$, but only yields solutions to

$$\mathrm{A}\, u - W'(u) = \frac{M}{l^d}$$

with faster decaying constant on the right hand side. That decay rate is not sufficient for later applications, since an interface along a straight line is essentially one-dimensional and exerts a force of order $1/l$.

The case $d = 1$ is special in the proof above since $S_l^d = T_l^d$. On tori in higher dimensions, the re-arrangement is significantly more involved, since for example the monotonically increasing and monotonically decreasing rearrangements do not agree.

Remark 3.3. Assume that we are instead constructing an obstacle with boundary conditions at $a \in \mathbb{Z}$, say $a \geq 2$. Then we modify W outside $[0, a]$ instead and obtain the same results as before under the integral side condition $\frac{1}{l}\int_{S_l^1} u \,\mathrm{d}x > a - 1/2$. The proof is only slightly more involved.

For technical purposes, it will be helpful to continue the solutions onto a larger set.

Lemma 3.4. *Let $m \in \mathbb{N}, m \geq 2$ and $u \in C^2(S_l^1)$. Identify $S_r^1 = (-r/2, r/2]$ and define*

$$u_{l,ml} : S_{ml}^1 \to \mathbb{R}, \quad u_{l,ml}(x) = \begin{cases} u(x) & x \in S_l^1 \\ u(l/2) & x \in S_{ml}^1 \setminus S_l^1 \end{cases}.$$

If u is maximal at $l/2$, then

$$\mathrm{A}^{S_{ml}^1}\, u_{l,ml}(x) \geq \mathrm{A}^{S_l^1}\, u(x)$$

for $x \in S_l^1$ and

$$\mathrm{A}^{S_{ml}^1}\, u_{l,ml}(x) \geq \mathrm{A}^{S_l^1}\, u(l/2)$$

for $x \in S_{ml}^1 \setminus S_l^1$. The same holds true for $m = \infty$ (i.e. $S_{ml}^1 = \mathbb{R}$).

Proof. A function on S_l^1 or S_{ml}^1 can be interpreted as a function on $\mathbb{R}$ with period l or ml respectively. In this way, every function on S_l^1 can also be interpreted as a function on S_{ml}^1, since the periods are compatible. Now let $x \in S_l^1$, then

$$\mathrm{A}^{S_{ml}^1}\, u_{l,ml}(x) = \int_{-\infty}^{\infty} \frac{u_{l,ml}(y) - u_{l,ml}(x)}{|x-y|^2}\,\mathrm{d}y \geq \int_{-\infty}^{\infty} \frac{u(y) - u(x)}{|x-y|^2}\,\mathrm{d}y = \mathrm{A}^{S_l^1}\, u(x)$$

since $u_{l,ml}(x) = u(x)$ and $u_{l,ml}(y) \geq u(y)$ for all $y \in \mathbb{R}$. If on the other hand $x \in S_{ml}^1 \setminus S_l^1$, then we may assume up to translation that $\frac{l}{2} < x < \frac{ml}{2}$, thus

$$\begin{aligned}
\mathrm{A}^{S_{ml}^1}\, u_{l,ml}(x) &= \int_{-\infty}^{\infty} \frac{u_{l,ml}(y) - u_{l,ml}(x)}{|x-y|^2}\,\mathrm{d}y \\
&= \int_{\mathbb{R}\setminus[l/2,ml/2]} \frac{u_{l,ml}(y) - u(l/2)}{|x-y|^2}\,\mathrm{d}y \\
&\geq \int_{\mathbb{R}\setminus[l/2,ml/2]} \frac{u(y) - u(l/2)}{|x-y|^2}\,\mathrm{d}y \\
&\geq \int_{\mathbb{R}\setminus[l/2,ml/2]} \frac{u(y) - u(l/2)}{|l/2-y|^2}\,\mathrm{d}y \\
&\geq \int_{\mathbb{R}} \frac{u(y) - u(l/2)}{|l/2-y|^2}\,\mathrm{d}y \\
&= \mathrm{A}^{S_l^1}\, u(l/2)
\end{aligned}$$

since again $u_{l,ml}(y) \geq u(y)$ and since the missing part of the integral over the real line which we include in the last step is non-positive. The fact that choosing $x = l/2$ minimises the function

$$x \mapsto \int_{\mathbb{R}\setminus[l/2,ml/2]} \frac{u(y) - u(l/2)}{|x-y|^2}\,\mathrm{d}y$$

can be seen easily since the function is by design symmetric under reflection around the midpoint between $l/2$ and $ml/2$ and concave as

$$\frac{d}{dx}\int_{\mathbb{R}\setminus[l/2,ml/2]} \frac{u(y) - u(l/2)}{|x-y|^2}\,\mathrm{d}y = (-2)\int_{\mathbb{R}\setminus[l/2,ml/2]} [u(y) - u(l/2)]\frac{1}{(x-y)^3}\,\mathrm{d}y$$
$$\frac{d^2}{dx^2}\int_{\mathbb{R}\setminus[l/2,ml/2]} \frac{u(y) - u(l/2)}{|x-y|^2}\,\mathrm{d}y = (-2)(-3)\int_{\mathbb{R}\setminus[l/2,ml/2]} [u(y) - u(l/2)]\frac{1}{(x-y)^4}\,\mathrm{d}y$$

which is clearly negative since u is maximal at $l/2$. □

We formulated the Lemma in the setting of smooth functions to compute the singular integral directly, but by density it also holds in the distributional sense for $u \in H^{1/2}(S^1_l)$, thus in particular

$$\mathrm{A}^{S^1_{ml}}\,\bar{u}_{l,ml} - W'(\bar{u}_{l,ml}) \geq \frac{M}{l}$$

on $\{\bar{u}_{l,ml} > 0\} = S^1_{ml} \setminus B_R(x_0)$.

The same methods as in Lemma 3.1 can be used to establish the following result for a global minimiser without the negative forcing term M.

Lemma 3.5. *Let $u \in 1 + H^{1/2}(\mathbb{R})$ be a minimiser of*

$$\mathcal{E}(u) = [u]^2_{1/2} + \int_{\mathbb{R}} W(u)\,\mathrm{d}$$

under the constraint $u \equiv 0$ on $[-R, R]$. Then u satisfies $1 - \frac{c}{|x|^2} \leq u(x)$ for all $x \in \mathbb{R}$ and some $c \geq 1$.

3.2. **The Interface.** We recall the following results for transitions between potential wells.

Lemma 3.6. [CSM05] *There exists a function $\phi \in C^{2,\alpha}(\mathbb{R})$ such that ϕ is monotonically increasing,*

$$\lim_{x\to\infty}\phi(x) = 1, \qquad \lim_{x\to-\infty}\phi(x) = 0, \qquad \mathrm{A}\,\phi - W'(\phi) = 0.$$

The function satisfies

$$\frac{c}{1+x^2} < \phi'(x) \leq \frac{C}{1+x^2}$$

for some $C \geq c > 0$.

The estimate on the derivative further implies that

$$\phi(x) = 1 - \int_x^\infty \phi'(t)\,\mathrm{d}t \leq 1 - \int_x^\infty \frac{C}{1+t^2}\,\mathrm{d}t \leq 1 - \frac{C'}{x}$$

for any $C' > C$ and sufficiently large x, as well as $\phi(x) \geq 1 - \frac{c}{x}$ for all sufficiently large x. This has been sharpened to the estimate

$$\left|1 - \frac{1}{W''(0)\,x} - \phi(x)\right| = O(x^{-2})$$

in [GM12, Theorem 3.1]. The same decay holds for large negative x:

$$\left|\phi(x) - \frac{1}{W''(0)\,|x|}\right| = O(|x|^{-2}).$$

Note that the constant in [GM12] is slightly different since the operator used there is the half-Laplacian in its usual normalisation, while we neglected a dimensional constant for easier notation. Under additional conditions, we can also control the second derivative of ϕ. Note that for the popular choice

$$W(u) = \frac{1}{\pi}\,[\cos(\pi u) + 1]$$

(with wells on $\mathbb{Z} + 1/2$) we have the transition function

$$\phi(x) = \frac{2}{\pi}\,\arctan(x), \qquad \phi'(x) = \frac{2/\pi}{1+x^2}, \qquad \phi''(x) = \frac{4x}{\pi\,(1+x^2)^2}$$

so that also

$$|\phi''(x)| \leq C/(1+x^2)^{3/2}. \tag{3.2}$$

In the following, we assume that W is chosen such that the optimal transition function $\phi = \phi_W$ satisfies (3.2).

Lemma 3.7. *Let $L \gg 1$, β as in Lemma 3.1. Then there exists function $\widetilde{\phi} = \widetilde{\phi}_L \in C^{2,\alpha}(\mathbb{R})$ such that*

(1) $\widetilde{\phi}$ is monotone increasing,
(2) $\widetilde{\phi} \equiv 0$ on $(-\infty, -L/W''(0) + \tilde{C})$,
(3) $\widetilde{\phi} \equiv 1 - \frac{1}{L}$ on $[L/W''(0) + \tilde{C}, \infty)$,
(4) whenever $0 < \widetilde{\phi}(x) < \beta$ or $1 - \beta < \widetilde{\phi}(x) \leq 1$ we have

$$\left(\mathrm{A}\,\widetilde{\phi} - W'(\widetilde{\phi})\right)(x) \geq \frac{\bar{c}}{L^2}$$

(5) whenever $\beta \leq \widetilde{\phi}(x) \leq 1 - \beta$, we have

$$\left|\mathrm{A}\,\widetilde{\phi} - W'(\widetilde{\phi})\right|(x) \leq \frac{C}{L^2}.$$

The constants $\bar{c}, C, \tilde{C}$ depend on W, but not on L.

Proof. Take $f_L : \mathbb{R} \to \mathbb{R}$ to be a smooth function such that

$$f_L(z) = \begin{cases} 0 & z \leq \frac{1}{L} \\ z - \frac{C_1}{L^2} & \frac{2}{L} \leq z \leq 1 - \frac{2}{L} \\ 1 - \frac{1}{L} & z \geq 1 - \frac{1}{L} \end{cases}, \qquad f_l(z) \leq z - \frac{C_1}{L^2} \qquad \forall\ z \geq \frac{C_1}{L^2}$$

and $0 \leq f_L' \leq 3$, $|f_L''| \leq \frac{10}{L}$ and define

$$\widetilde{\phi} = f_L \circ \phi.$$

We see that

$$\widetilde{\phi}' = (f_L' \circ \phi)\,\phi' \geq 0,$$

so $\widetilde{\phi}$ is monotone increasing. Furthermore, we obtain that

$$\phi(x) \geq 1 - \frac{1}{W''(0)\,x} - \frac{C}{x^2} \geq 1 - \frac{1}{L} \qquad \forall\ x \geq L/W''(0) + C$$

and thus $\widetilde{\phi}(x) \equiv 1-\frac{1}{L}$ for all $x \geq L/W''(0)+C$. Analogously, $\widetilde{\phi}(x) \equiv 0$ for all $x \leq -(L/W''(0)+C)$. Now compute

$$(\widetilde{\phi}-\phi)' = [(f_L'\circ\phi)-1]\,\phi'$$
$$(\widetilde{\phi}-\phi)'' = [(f_L'\circ\phi)-1]\,\phi'' + (f_L''\circ\phi)\,(\phi')^2.$$

Thus it is easy to see that

$$\left|\widetilde{\phi}-\phi\right| \leq \frac{2}{L}, \qquad \left|\widetilde{\phi}'-\phi'\right| \leq \frac{C}{L^2}, \qquad \left|\widetilde{\phi}''-\phi''\right| \leq \frac{C}{L^3}.$$

When we abbreviate $w=\widetilde{\phi}-\phi$, we can therefore use a representation of A like (2.3) to compute

$$\begin{aligned}\left|\,\mathrm{A}(\widetilde{\phi}-\phi)(x)\right| &= \left|\int_{x-L}^{x+L}\frac{\int_x^y (y-t)w''(t)\,\mathrm{d}t}{|y-x|^2}\,\mathrm{d}y + \int_{\mathbb{R}\setminus[x-L,x+L]}\frac{w(y)-w(x)}{|y-x|^2}\,\mathrm{d}y\right|\\ &\leq \int_{x-L}^{x+L}||w''||_{L^\infty}\,\mathrm{d}y + 2\int_L^\infty\frac{||w||_\infty}{y^2}\,\mathrm{d}y\\ &\leq \frac{C}{L^2}.\end{aligned}$$

Finally, take x such that $\phi(x)\geq 1-\beta$ or $0<\phi_L(x)\leq\beta$. Then we can use that $\tilde{\phi}\leq\phi-\frac{C_1}{L^2}$ or $\widetilde{\phi}=0$ to estimate

$$\begin{aligned}-W'(\widetilde{\phi}(x)) &= -W'(\widetilde{\phi}(x)-\phi(x)+\phi(x))\\ &\geq -W'(\phi(x)) - \frac{2\,W''(\phi(x))}{3}\,(\widetilde{\phi}(x)-\phi(x))\\ &\geq -W'(\phi(x)) - \frac{W''(1)}{2}\,\frac{C_1}{L^2}.\end{aligned}$$

Thus in total

$$\mathrm{A}\,\widetilde{\phi} - W'(\widetilde{\phi}) \geq \frac{C}{L^2}$$

if $\widetilde{\phi}(x)\notin[\beta,1-\beta]\cup\{0\}$ and C_1 is chosen large enough (since the constants involving derivatives do not depend on C_1) and

$$\left|\mathrm{A}\,\widetilde{\phi} - W'(\widetilde{\phi})\right| \leq \frac{C}{L^2}.$$

□

3.3. **Dynamics on the Real Line I.** We can apply the sub-solutions constructed above to the one-dimensional problem by glueing a sub-solution for periodic obstacles to that for an interface. First we describe our results on the real line for a single step and then for a kink/anti-kink pair. The first case cannot be achieved with finite energy, but it helps us identify the time scale on which the pinning constraint induces motion. Here, we need to use solutions to (1.4) in the viscosity sense since a single transition layer does not have finite energy.

Theorem 3.8 (A single step)**.** *Let $x_{i,\varepsilon}=i\,d_\varepsilon$ for $d_\varepsilon\gg\varepsilon$. Then there exist $\underline{u}_\varepsilon\leq\overline{u}_\varepsilon$ which are a viscosity sub- and super-solution of*

$$\begin{cases} c_\varepsilon\varepsilon\,u_t &= \frac{1}{|\log\varepsilon|}\left(\mathrm{A}\,u-\frac{1}{\varepsilon}W'(u)\right) \quad \text{in } \mathbb{R}\setminus\bigcup_{i\in\mathbb{Z}}\overline{B_\varepsilon(id_\varepsilon)}\\ u &= 0 \quad \text{on } \bigcup_{i\in\mathbb{Z}}\overline{B_\varepsilon(id_\varepsilon)}\end{cases} \tag{3.3}$$

respectively with the following property: When we choose $c_\varepsilon=\frac{\varepsilon}{d_\varepsilon^2\,|\log\varepsilon|}$, there are constants $c,C>0$ such that

$$\lim_{\varepsilon\to0}\underline{u}_\varepsilon(t,\cdot)=\chi_{[ct,\infty)}, \qquad \lim_{\varepsilon\to0}\overline{u}_\varepsilon(t,\cdot)=\chi_{[Ct,\infty)}$$

in $L^2_{loc}(\mathbb{R})$ *for all* $t > 0$.

In one dimension, the natural obstacle scale is $d_\varepsilon \sim 1/|\log\varepsilon|$, so the gradient flow equation is slow on a scale of $O(\varepsilon\,|\log\varepsilon|)$. The interface moves with speed $O(1)$ if $d_\varepsilon \sim \sqrt{\varepsilon/|\log\varepsilon|}$, which is the natural distance between obstacles in two space dimensions; it is slow for larger distances and fast for smaller ones. It moves on the same time-scale as an interface would due to the kink/anti-kink attraction for $d_\varepsilon \sim \sqrt{\varepsilon}$. The proof also goes through for $d_\varepsilon = N\varepsilon$ for large enough N.

The assumption that the obstacles are distributed on a lattice can of course be weakened significantly, and we believe that solutions u_ε should in fact converge to a characteristic function with a linearly propagating front $\chi_{[vt,\infty)}$ for periodic obstacles. We do not pursue these questions further.

Proof of Theorem 3.8. **Construction of a sub-solution.** For convenience, we build the sub-solution in the blow-up scale. Choose $l = l_\varepsilon = d_\varepsilon/(\varepsilon N)$ for some sufficiently large $N \in \mathbb{N}$ (to be specified later), $R = 1$ and $M = 1$. We can extend the sub-solution for an obstacle $\bar u_l$ on the circle S^1_l from Lemma 3.1 to the circle of length Nl as in Lemma 3.4.

Define L by $1 - \frac{1}{L} = \bar u_l(-x_0)$ and take $\widetilde\phi$ associated to L. By growth estimates for the optimal profile and the obstacle cell solution, $L = l + O(1)$. Furthermore, by this we know that $\widetilde\phi$ is constant on intervals $(-\infty, l/W''(0) + \tilde C]$ and $[l/W''(0) + \tilde C, \infty)$ for potentially larger constants $\tilde C$. While l and L depend on N, the constants are uniform (at least for $l \gg 1$ bounded from below), and we can take N such that $N > 4\tilde C$. We will place an additional condition on N later.

Our sub-solutions are given by a modified transition $\widetilde\phi$ which moves with speed α in the space between two obstacles. Once we get too close to an obstacle, we jump over it instantaneously. In formulas

$$\tilde u(t,x) = \begin{cases} \widetilde\phi\left(x - \frac{N}{4}l - \alpha t\right) & x \le \frac{3N}{4}l \\ \bar u_{l,Nl}(x) & x \ge \frac{3N}{4l} \end{cases}$$

for $t \in [0, \frac{Nl}{4\alpha}]$ and

$$\tilde u(t,x) = \tilde u\left(t - m\frac{Nl}{4\alpha},\, x - mNl\right) \qquad \text{for } t \in \left(m\frac{Nl}{4\alpha}, (m+1)\frac{Nl}{4\alpha}\right], \quad m \in \mathbb{N}.$$

By construction, u is continuous in space for all times t, jointly upper semi-continuous and non-increasing in time for a fixed point $x \in \mathbb{R}$. Since for fixed x we only jump down as time increases, u is clearly a sub-solution at the points of discontinuity in time. It remains to find α such that u is a sub-solution also where it evolves smoothly.

At smooth points of $\tilde u$ away from the pinning set $\bigcup_{i\in Z} B_1(i\,Nl)$, it is sufficient to verify the inequality $u_t \le \mathrm{A}\,u - W'(u)$ pointwise to obtain that $\tilde u$ is a viscosity sub-solution.

At points where $\tilde u = \widetilde\phi = 0$, $\tilde u$ is clearly a sub-solution as $\partial_t \tilde u = 0$, $W'(\tilde u) = 0$ and $\mathrm{A}\,\tilde u \ge 0$ since 0 is an absolute minimum of $\tilde u$. At points where $\tilde u = u_{l,Nl}$, $\tilde u$ is a sub-solution since

$$\mathrm{A}\,\tilde u(x) - W'(\tilde u(x)) = \mathrm{A}\,\tilde u_{l,Nl}(x) - W'(\tilde u_{l,Nl}(x)) + \mathrm{A}(\tilde u - \tilde u_{l,Nl})(x) \ge \frac{1}{l} - \frac{c}{\left|x - \frac{N}{4}l - \alpha t\right|} \ge 0$$

as the left half-space where $\tilde u < \tilde u_{l,Nl}$ exerts a force proportional to the inverse distance of the interface to the point x. As this is larger than Nl, the first term dominates and the sub-solution property is established for large enough N.

Finally, take (t,x) such that $\tilde u(t,x) = \widetilde\phi(t,x)$ and compute

$$\begin{aligned}
\mathrm{A}\,\tilde u(t,x) - W'(\tilde u(t,x)) &= \mathrm{A}(\tilde u - \widetilde\phi)(t,x) + \mathrm{A}(\widetilde\phi(t,x)) - W'(\widetilde\phi(t,x))\\
&= \mathrm{A}(\tilde u - \widetilde\phi)(t,x) + O(l^{-2})\\
&= \int_{x+\frac{N}{8}l}^{\infty} \frac{(\tilde u - \widetilde\phi)(y)}{|y-x|^2}\,\mathrm{d}y + O(l^{-2})\\
&\geq -\sum_{i=1}^{\infty} \int_{i\,Nl-1}^{i\,Nl+1} \frac{1}{|y-x|^2}\,\mathrm{d}y + \int_{(i-1)\,Nl}^{i\,Nl-1} \frac{c\,|y-i\,Nl|^{-2}}{|y-x|^2}\,\mathrm{d}y\\
&\qquad\qquad + \int_{i\,Nl+1}^{(i+1)Nl} \frac{c\,|y-i\,Nl|^{-2}}{|y-x|^2}\,\mathrm{d}y + O(l^{-2})\\
&\geq -\sum_{i=1}^{\infty} \frac{2}{|i\,Nl - Nl/4|^2} + \frac{2}{|(i-1-1/4)\,Nl|^2} \int_1^l \frac{1}{y^2}\,\mathrm{d}y + O(l^{-2})\\
&= O(N^{-2}l^{-2}) + O(l^{-2})\\
&= O(l^{-2}).
\end{aligned}$$

Now we can finally use that the second $O(l^{-2})$ term is positive where $\widetilde\phi \in (0,\beta]$ or $[1-\beta,1]$, and it compensates the first term for large enough N (which can be chosen independently of l). At the interface we have

$$\bar c := \max_{z\in[\beta,1-\beta]} \widetilde\phi'(z) > 0,$$

so we can choose $\alpha = O(l^{-2})$ such that $\tilde u$ is a sub-solution.

Rescaling. Let us pass back to the original length scale:

$$\underline{u}_\varepsilon(t,x) = \tilde u\left(\frac{t}{c_\varepsilon\varepsilon^2\,|\log\varepsilon|}, \frac{x}{\varepsilon}\right).$$

By construction, $\underline{u}_\varepsilon$ is a sub-solution of (3.3) since (at smooth points)

$$\begin{aligned}
c_\varepsilon\varepsilon\,\partial_t\underline{u}_\varepsilon &= \frac{c_\varepsilon\,\varepsilon}{c_\varepsilon\,\varepsilon^2\,|\log\varepsilon|}(\partial_t\tilde u)\\
&\leq \frac{1}{\varepsilon\,|\log\varepsilon|}\left((\mathrm{A}\,\tilde u) - W'(\tilde u)\right)\\
&= \frac{1}{|\log\varepsilon|}\left(\mathrm{A}\,\underline{u}_\varepsilon - \frac{1}{\varepsilon}\,W'(\underline{u}_\varepsilon)\right).
\end{aligned}$$

We know that the interface in the blow up scale moves by exactly Nl_ε in time $Nl_\varepsilon/(4\alpha) \sim N\,l_\varepsilon^3$, so the rescaled interface moves by $d_\varepsilon = N\varepsilon l_\varepsilon$ at the time t_ε such that

$$\frac{t_\varepsilon}{c_\varepsilon\,\varepsilon^2\,|\log\varepsilon|} \approx N\left(\frac{d_\varepsilon}{\varepsilon}\right)^3 \qquad\Leftrightarrow\qquad \frac{t_\varepsilon}{c_\varepsilon} = \frac{N\,d_\varepsilon^3\,|\log\varepsilon|}{\varepsilon}.$$

To obtain a speed of $O(1)$, we need $t_\varepsilon \sim d_\varepsilon$, so we choose the acceleration factor

$$c_\varepsilon = \frac{\varepsilon}{d_\varepsilon^2\,|\log\varepsilon|}.$$

Limiting behaviour. In the limit $\varepsilon \to 0$, the jumps over shorter and shorter spatial intervals disappear and $\underline{u}_\varepsilon(t,\cdot)$ converges locally in L^1 to the characteristic function of an interval $I(t)$ moving with uniform speed. If c_ε is chosen too small, then $I(t) = \emptyset$ for all positive times, whereas too large c_ε implies $I(t) \equiv [0,\infty)$. In the scaling regime identified above, we have $I(t) = [ct,\infty)$ for some $c > 0$.

Construction of super-solutions. Here we work directly on the macroscopic scale. Let us make the ansatz

$$\overline{u}_\varepsilon(t,x) = \min\left\{\phi\left(\frac{x-\alpha t}{\varepsilon}\right), \bar{u}_{l_\varepsilon}\left(\frac{x}{\varepsilon}\right)\right\}.$$

In the stationary case $\alpha = 0$ this minimum of two solutions is clearly a super-solution. Still for small positive α, it suffices to consider (t,x) such that $\overline{u}_\varepsilon(t,x) = \phi\left(\frac{x-\alpha t}{\varepsilon}\right)$ since at other points (including the non-smooth points where ϕ and $\bar{u}_{l_\varepsilon}$ meet) the super-solution property is still easily established. The function is continuous by construction and satisfies the pinning constraint. Finally, compute

$$\begin{aligned}
c_\varepsilon\varepsilon\,\partial_t\overline{u}_\varepsilon(t,x) &= -\alpha\,\frac{\varepsilon}{d_\varepsilon^2\,|\log\varepsilon|}\,\phi'\left(\frac{x-\alpha t}{\varepsilon}\right)\\
&\geq \frac{-c\alpha\varepsilon}{d_\varepsilon^2\,|\log\varepsilon|}\,\frac{1}{\left(\frac{x-\alpha t}{\varepsilon}\right)^2+1}\\
\frac{1}{|\log\varepsilon|}\left(\mathrm{A}\,\overline{u}_\varepsilon - \frac{1}{\varepsilon}\,W'(\overline{u}_\varepsilon)\right) &= \frac{1}{\varepsilon\,|\log\varepsilon|}\,((\mathrm{A}\,\phi) - W'(\phi)) + \frac{1}{|\log\varepsilon|}\left(\mathrm{A}\,\overline{u}_\varepsilon - \frac{1}{\varepsilon}(\mathrm{A}\,\phi)\right)\\
&= \frac{1}{|\log\varepsilon|}\,\mathrm{A}\left(\overline{u}_\varepsilon - \phi\left(\frac{\cdot-\alpha t}{\varepsilon}\right)\right)\\
&\leq \frac{1}{|\log\varepsilon|}\sum_{i\in\mathbb{Z}}\int_{[id_\varepsilon-\varepsilon,id_\varepsilon+\varepsilon]}\frac{-\phi\left(\frac{y-\alpha t}{\varepsilon}\right)}{|y-x|^2}\,\mathrm{d}y\\
&\leq -\frac{2\varepsilon\,\phi\left(\frac{x-\alpha t-d_\varepsilon}{\varepsilon}\right)}{|\log\varepsilon|\,d_\varepsilon^2}
\end{aligned}$$

by just considering the index $i \in \mathbb{Z}$ such that $x - d_\varepsilon \leq id_\varepsilon \leq x$. Since $\phi(z) \geq c\,\min\{1, -1/z\}$ vanishes more slowly than ϕ', this shows that

$$c_\varepsilon\varepsilon\,\partial_t\overline{u}_\varepsilon(t,x) \geq \frac{1}{|\log\varepsilon|}\left(\mathrm{A}\,\overline{u}_\varepsilon - \frac{1}{\varepsilon}\,W'(\overline{u}_\varepsilon)\right)$$

for suitably small α which is independent of $\varepsilon > 0$. □

Remark 3.9. It is possible to prove a comparison principle for the evolution equation (1.4) in the viscosity sense. This has been done for equations on the whole space and operators of the type $(-\Delta)^s$ for $s > 1/2$ in [Imb05], but the methods go through for $s \leq 1/2$ and equations on domains with only minor modifications. Thus, the existence of a viscosity solution u_ε with $\underline{u}_\varepsilon \leq u_\varepsilon \leq \overline{u}_\varepsilon$ follows directly by Perron's method. For a viscosity solution with given initial data, additional barriers have to be constructed. It is well known that u solves

$$\begin{cases} c_\varepsilon\varepsilon\,u_t &= \frac{1}{|\log\varepsilon|}\left(\mathrm{A}\,u - \frac{W'(u)}{\varepsilon}\right) & t>0, x\in\Omega_\varepsilon\\ u &\equiv 0 & t\geq 0, x\in\Omega_\varepsilon^c\\ u &= u^0 & t=0\end{cases}$$

in the viscosity sense if and only if $v := e^{-\lambda t}u$ solves an equation of the same form with the non-linearity

$$f_\lambda(t,v) = -\frac{e^{-\lambda t}W\left(e^{\lambda t}v\right)}{\varepsilon} - \lambda v.$$

in place of $W'(u)/\varepsilon$. When we choose λ large enough (depending on W and ε), the function f_λ is monotone in u uniformly in t. For an initial condition u^0 smooth enough to establish

$$\mathrm{A}\,u^0 - \frac{W'(u^0)}{\varepsilon} \in L^\infty(\Omega_\varepsilon)$$

pointwise, we can then construct sub- and supersolutions by

$$\overline{v}(t,x) = u_0(x) + Ct\cdot\chi_{\Omega_\varepsilon}, \qquad \overline{v}(t,x) = u_0(x) - Ct\cdot\chi_{\Omega_\varepsilon}$$

for some large constant $C > 0$. This includes all initial conditions in $C_b^2(\mathbb{R})$ and all initial conditions that we are interested in. Since v attains the initial condition, also u does. The domain Ω_ε can be chosen to be periodic or the perforated real line $\mathbb{R}\setminus B_\varepsilon(d_\varepsilon\cdot\mathbb{Z})$.

3.4. **The Corrector.** We have seen that the pinning constraint induces motion on a time-scale which is strictly slower than the $\log\varepsilon$-timescale on which the next-order term in a Γ-expansion (2.2) acts as a kink/anti-kink attraction. We want to show that the pinning does not affect the attraction and annihilation of a single kink/anti-kink pair. For this purpose, we need a more refined construction to obtain the exact speed of an interface rather than just the order in ε. Therefore, we need to know the behaviour of a moving interface to the next order.

Set

$$\eta := \frac{1}{W''(0)}\int_{-\infty}^{\infty}(\phi')^2\,\mathrm{d}x.$$

Lemma 3.10. *There exists a function $\psi\in H^{1/2}(\mathbb{R})\cap C^{1,\alpha}(\mathbb{R})\cap L^\infty(\mathbb{R})$ for some $\alpha>0$ which solves*

$$\mathrm{A}\,\psi - W''(\phi)\psi = \phi' + \eta\,\left(W''(\phi)-W''(0)\right).$$

The solution ψ satisfies the estimate

$$|\psi'(x)| \leq \frac{C}{1+x^2}$$

and if $W\in C^{3,1}(\mathbb{R})$ then also $\psi\in C^{2,\alpha}(\mathbb{R})$ for some $\alpha>0$ and

$$|\psi''(x)| \leq \frac{C}{1+x^2}.$$

The Lemma is proved in [GM12, Theorem 3.2] without the decay estimate, see also [PV17, Lemma 2.2]. A simple proof can be obtained in a similar way as the one of the decay estimate on ϕ'.

Idea of Proof: We only sketch the proof of the decay estimates. Consider the case $W_\alpha(z) = \frac{1-\sin(\pi z)}{\pi^2\alpha}$ for $\alpha>0$ which has the explicit solution $\phi_\alpha(t) = \frac{1}{\pi}\arctan(\frac{t}{\alpha})$ (with potential wells at ±1 instead of 0 and 1). We note that the derivative of any optimal transition ϕ_α satisfies

$$\mathrm{A}\,\phi_\alpha' - W''(\phi_\alpha)\phi_\alpha' = 0$$

so in particular

$$\mathrm{A}(\phi_\alpha') + \frac{1}{\alpha}\,\phi_\alpha' \geq 0$$

for all large (positive and negative) x. Given another potential W, we split $W''(\phi) = f_+(x) + f_-(x)$ with $\inf f_+ > 0$ and f_- compactly supported. This splitting allows us to use a comparison with the solutions ϕ_α for a suitable α, taking the compactly supported 'bad' term to the other side. We calculate formally

$$\begin{aligned}
\mathrm{A}\,\psi - W''(\phi)\psi &= \phi' + \eta\,\left(W''(\phi)-W''(0)\right)\\
\mathrm{A}\,\psi' - W''(\phi)\psi' &= W^{(3)}(\phi)\phi'\psi + \phi'' + \eta\,W^{(3)}(\phi)\phi'\\
\mathrm{A}\,\psi'' - W''(\phi)\psi'' &= W^{(3)}(\phi)\left\{2\phi'\psi' + \phi''\psi + \eta\phi''\right\} + W^{(4)}(\phi)\left\{(\phi')^2\psi + \eta(\phi')^2\right\}.
\end{aligned}$$

If $W\in C^{3,1}(\mathbb{R})$, then the last equation makes sense with a right hand side in $L^\infty(\mathbb{R})\cap L^1(\mathbb{R})$ and the regularity of ψ can be improved to $C^{2,\alpha}(\mathbb{R})$. The decay now follows as in the proof of [CSM05, Theorem 1.6]. □

Note that due to the decay estimate on the derivative, $\phi \pm \varepsilon\psi$ is still monotone increasing for all small enough $\varepsilon > 0$. As before, this is needed when a moving interface comes close to an obstacle and jumps instantaneously to ensure that the jump is pointwise down in time and thus to preserve the sub-solution property at jump points.

3.5. **Dynamics on the Real Line II.** We can now use the slowness of the obstacle-driven evolution in comparison to the kink/anti-kink attraction to show that the pinning constraint has no influence on the motion of a single kink/anti-kink pair.

Theorem 3.11 (Asymptotically flat crystal). *Let $x_{i,\varepsilon} = i\,d_\varepsilon$ for $d_\varepsilon \gg \sqrt{\varepsilon}$. Then there exist $\underline{u}_\varepsilon \leq \overline{u}_\varepsilon$ which are a viscosity sub- and super-solution of*

$$\begin{cases} \frac{1}{|\log\varepsilon|}\varepsilon\,u_t &= \frac{1}{|\log\varepsilon|}\left(\mathrm{A}\,u - \frac{1}{\varepsilon}\,W'(u)\right) \quad \text{in } \mathbb{R}\setminus\bigcup_{i\in\mathbb{Z}}\overline{B_\varepsilon(id_\varepsilon)} \\ \qquad u &= 0 \qquad\qquad\qquad\qquad\quad \text{on } \bigcup_{i\in\mathbb{Z}}\overline{B_\varepsilon(id_\varepsilon)} \end{cases}$$

respectively such that

$$\lim_{\varepsilon\to 0}\underline{u}_\varepsilon(t,\cdot) = \lim_{\varepsilon\to 0}\overline{u}_\varepsilon(t,\cdot) = \chi_{[-r(t),r(t)]}$$

in $L^2(\mathbb{R})$ for all $t > 0$ with

$$r(t) = \sqrt{r(0)^2 - \frac{t}{\int_{-\infty}^{\infty}(\phi')^2\,\mathrm{d}t}}\,.$$

Proof. Choose $\delta > 0$. Following [PV17] we know that for all small enough $\varepsilon > 0$

$$\left[\phi\left(\frac{x+\overline{x}_\delta(t)}{\varepsilon}\right) - \varepsilon\psi\left(\frac{x+\overline{x}_\delta(t)}{\varepsilon}\right)\right] + \left[\phi\left(\frac{\overline{x}_\delta(t)-x}{\varepsilon}\right) - \varepsilon\psi\left(\frac{\overline{x}_\delta(t)-x}{\varepsilon}\right)\right] - 1$$

is a sub-solution of the unpinned equation

$$\varepsilon\,u_t = \mathrm{A}\,u - \frac{1}{\varepsilon}\,W'(u)$$

when we choose $\overline{x}_\delta$ as the solution of the ordinary differential equation

$$\dot{\overline{x}}_\delta = \frac{1}{\int_{-\infty}^{\infty}(\phi')^2\,\mathrm{d}x}\,\frac{-1}{2\overline{x}_\delta} - \delta, \qquad \overline{x}_\delta(0) = r(0) - \delta.$$

We just sketch the modifications which we need to make in the previous proof to apply it in this situation. Again, we can modify the interface choosing

$$\varepsilon^{-1/2} = \varepsilon^{-1}\,\varepsilon^{1/2} \ll L_\varepsilon \ll \varepsilon^{-1}\,d_\varepsilon.$$

This time, we need to modify the function $\phi_\varepsilon = \phi - \varepsilon\psi$. Abbreviate again $w_\varepsilon = (f_L\circ\phi_\varepsilon) - \phi_\varepsilon$ and compute

$$w_\varepsilon'' = [(f_L'\circ\phi_\varepsilon) - 1]\,\phi_\varepsilon'' + (f_L''\circ\phi_\varepsilon)\,(\phi_\varepsilon')^2 \leq C\left(\frac{1}{L^3} + \frac{\varepsilon}{L^2}\right)$$

so

$$\mathrm{A}\,w_\varepsilon(x) \leq 2L\,||\,w_\varepsilon''||_{L^\infty(x-L,x+L)} + \frac{2}{L}\,||\,w_\varepsilon||_{L^\infty(\mathbb{R})} \leq C\left(\frac{1}{L^2} + \frac{\varepsilon}{L}\right).$$

The contribution of the "bending modification" to the attraction thus is $O(L^{-2} + \varepsilon\,L^{-1}) = O(L_\varepsilon^{-2})$, which was seen to be slow compared to the kink/anti-kink attraction in the previous proof where we also saw that the insertion of obstacles contributes to the force on the same order. When constructing sub-solutions in this setting, we only have to jump over obstacles when we come L_ε-close (as before), but the obstacles are $d_\varepsilon/\varepsilon$-far apart, which is significantly further by our choice of L_ε. Thus both the additional attraction and the fast motion close to obstacles disappear in the limit $\varepsilon \to 0$. Thus the sub-solution converges to

$$\chi_{[-\overline{x}_\delta, \overline{x}_\delta]}$$

strongly in $L^1(\mathbb{R})$. Now it suffices to take $\delta \to 0$. Super-solutions are obtained similarly. □

It is expected that the Theorem results can be extended to the case where several up and down steps occur by combining our methods with those of [PV15, PV17].

We see that motion becomes slow also in this time scale as the compact step becomes wider and wider. If we take a limit such that one transition remains fixed at the origin and let the other one go to $\pm\infty$, we partially recover the statement of the previous Theorem as we see that in this time-scaling, the evolution of a single step is stationary. To recover the optimal time-scale, we could couple the initial width $r(0) = r_\varepsilon(0)$ of the step to ε.

3.6. **Periodic dynamics.** On a circle of finite radius, there is no analogue of a single step. Instead, we can consider the situation in which $\{u_\varepsilon \approx 1\}$ is the majority phase. Without pinning, the majority phase takes over the minority phase in logarithmic time in a gradient flow. This happens precisely as it would if $\{u_\varepsilon \approx 1\}$ is the minority phase and the pinning has no effect, just as on the real line. If $\{u_\varepsilon \approx 1\}$ however is the majority phase, this would increase the energy, and the evolution becomes stationary on all timescales.

The use of energy methods relies on an analogue of Theorem 2.1 being valid in one dimension. We formulate it at the end of this section in Proposition 3.15 and assume its validity throughout.

Theorem 3.12. *Denote by $S^1_R = [-R/2, R/2]$ the circle of radius R and $x_{i,\varepsilon}$ be points on S^1_R such that*

$$\min_{i\neq j} |x_{i,\varepsilon} - x_{j,\varepsilon}| \geq d_\varepsilon \gg \sqrt{\varepsilon}$$

The number of points is denoted by N_ε.

(1) *Let $r < R/2$. There exists a weak solution u_ε of*

$$\begin{cases} \frac{1}{|\log\varepsilon|}\varepsilon\, u_t &= \frac{1}{|\log\varepsilon|}\left(\mathrm{A}\, u - \frac{1}{\varepsilon}\, W'(u)\right) \quad \text{in } S^1_R \setminus \bigcup_{i=1}^{N_\varepsilon} \overline{B_\varepsilon(x_{i,\varepsilon})} \\ u &= 0 \quad \text{on } \bigcup_{i=1}^{N_\varepsilon} \overline{B_\varepsilon(x_{i,\varepsilon})} \end{cases} \tag{3.4}$$

such that $u_\varepsilon(0,\cdot) \to \chi_{[-r/2,r/2]}$ which satisfies

$$u_\varepsilon(t,\cdot) \to \chi_{[-r(t),r(t)]}$$

for all $t > 0$ independently of the distribution of points $x_{i,\varepsilon}$. Here $r(t)$ solves

$$\dot{r} = -\frac{1}{2r} + 2\sum_{n=1}^{\infty} \frac{2r}{(nR)^2 - 4r^2}, \qquad r(0) = r/2.$$

If the points $x_{i,\varepsilon}$ satisfy the conditions of Theorem 2.1/Proposition 3.15, then also $\mathcal{E}_\varepsilon(u_\varepsilon(0,\cdot)) \to \mathcal{E}(\chi_{[-r/2,r/2]})$.

(2) *Let $r > R/2$ and assume that the points $x_{i,\varepsilon}$ satisfy the conditions of Proposition 3.15. Then there exists a weak solution u_ε of*

$$\begin{cases} c_\varepsilon \varepsilon\, u_t = \frac{1}{|\log\varepsilon|}\left(\mathrm{A}\, u - \frac{1}{\varepsilon}\, W'(u)\right) & \text{in } S^1_R \setminus \bigcup_{i=1}^{N_\varepsilon} \overline{B_\varepsilon(x_{i,\varepsilon})} \\ u = 0 & \text{on } \bigcup_{i=1}^{N_\varepsilon} \overline{B_\varepsilon(x_{i,\varepsilon})} \end{cases}$$

such that $u_\varepsilon(0,\cdot) \to \chi_{[-r/2,r/2]}$ and $\mathcal{E}_\varepsilon(u_\varepsilon(0,\cdot)) \to \mathcal{E}(\chi_{[-r/2,r/2]})$ which satisfies

$$u_\varepsilon(t,\cdot) \to \chi_{[-r/2,r/2]}$$

for all $t > 0$, independently of $c_\varepsilon \to 0$.

Proof. **Proof of (1).** Note that the results of [PV15] also hold in this setting and that an unpinned solution to the time-rescaled gradient flow is governed by this ODE. In the proof, the discussion showing that the constants in front of ε-powers are finite is slightly more involved than in the case of finitely many layers. One has to use periodicity in an essential way always combining the force exerted by a kink/anti-kink pair to obtain cancellations between otherwise infinite forces.

We construct sub- and super-solutions periodically on $\mathbb{R}$ and then take them as functions on the circle. Note that the series

$$U(t,x) := \sum_{k\in\mathbb{Z}} \left[(\phi - \varepsilon\,\psi)\left(\frac{x-kR-r(t)/2}{\varepsilon}\right) - (\phi-\varepsilon\psi)\left(\frac{x-kR+r(t)/2}{\varepsilon}\right)\right]$$

converges absolutely and uniformly for all $x\in\mathbb{R}$ by comparison with $\sum_{k=1}^\infty k^{-2}$ since for $kR > x + r/2$ we have

$$\begin{aligned}\phi\left(\frac{x-kR-r/2}{\varepsilon}\right) - \phi\left(\frac{x-kR+r/2}{\varepsilon}\right) &= 1 - \frac{1}{W''(0)\,\frac{x-kR-r/2}{\varepsilon}} + O\left(\frac{\varepsilon^2}{(x-kR-r/2)^2}\right) \\ &\quad - \left(1 - \frac{1}{W''(0)\,\frac{x-kR-r/2}{\varepsilon}} + O\left(\frac{\varepsilon^2}{(x-kR-r/2)^2}\right)\right) \\ &= \frac{1}{W''(0)}\left(\frac{\varepsilon}{x-kR-r/2} - \frac{\varepsilon}{x-kR+r/2}\right) + O(\varepsilon^2\,(kR)^{-2}) \\ &= \frac{\varepsilon}{W''(0)}\,\frac{r}{(x-kR)^2-(r/2)^2} + O(\varepsilon^2\,(kR)^{-2}).\end{aligned}$$

A similar estimate holds for $kR < x - r/2$ and for ψ. Hence, the partial sums of the series converge and a continuous limit exists. Since super-solutions for a finite number of kink/anti-kink pairs have precisely this form (with a function r which starts out slightly wider than the width of the limiting step and moves slightly slower than by the limiting ODE, compare the role $\overline{x}_\delta$ plays in the proof of Theorem 3.11), we can construct a limiting (viscosity) super-solution to the unpinned problem using this series. Now, it is easy to see that again

$$\overline{u}(t,x) = \min\{U(t,x), \overline{u}_l(x)\}$$

is a super-solution to the pinned equation where $\overline{u}_l$ is constructed as in Lemma 3.1 for $M = 0$.

For weak sub-solutions, both the optimal transitions after the bending modification and the extended obstacle sub-solutions are constant for large arguments, so they can be glued together rather than added up. The proof proceeds like that of Theorem 3.11 with an additional term in the calculations from periodicity. The resulting ODE is computed by periodicity:

$$\begin{aligned}\dot r &= \left(\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{r-(r+nR)} - \frac{1}{r-(-r+nR)}\right) - \frac{1}{2r} \\ &= -\frac{1}{2r} + 2\sum_{n=1}^\infty \frac{2r}{(nR)^2-4r^2}.\end{aligned}$$

The initial condition with converging energies which lies above the sub-solution is given by

$$u_0 = \min\left\{\sum_{k\in\mathbb{Z}}\left[\phi\left(\frac{x-kl-r/2-\delta_\varepsilon}{\varepsilon}\right) - \phi\left(\frac{x-kl+r/2+\delta_\varepsilon}{\varepsilon}\right)+1\right], \bar u\left(\frac{x}{\varepsilon}\right)\right\}$$

where $\bar u$ is the periodic solution of Lemma 3.1 with $M\equiv 0$. Furthermore, δ_ε is a small parameter which ensures that the initial condition lies above the sub-solution.

Proof of (2). Fix $t > 0$. For any sequence $c_\varepsilon > 0$ and solutions u_ε to the evolution equation, we observe that $u_\varepsilon(t, \cdot) \to \chi_{E(t)}$ up to a subsequence since the initial energies are bounded, and the energy decreases along the gradient flow.

Assume there is $c_\varepsilon \to 0$ such that $u_\varepsilon(t, \cdot) \to \chi_{E(t)}$ with $E(t) \neq [-r/2, r/2]$. Since the unpinned evolution equation wants to expand the $\{u = 1\}$ phase under these initial conditions, we can easily construct a stationary sub-solution to the initial condition: the kink/anti-kink attraction acts at the interface in the direction of expanding the $\{u \approx 1\}$ phase and the contracting force of the obstacles becomes negligible. Thus $[-r, r] \subset E(t)$ for all $t > 0$.

We assume that an analogue of Theorem 2.1 holds in one dimension. For a contradiction, assume that $[-r, r] \subsetneq E(t) \subsetneq S^1_R$. Then $\mathcal{E}(\chi_{E(t)}) \geq 2 + \Lambda\,\mathcal{L}^1(E(t)) > \mathcal{E}(\chi_{[-r/2,r/2]})$, which is a contradiction. The inequality holds since any set which is neither empty nor the whole circle has a perimeter ≥ 2 in one dimension and since the $\{u = 1\}$ phase was assumed to be expanding.

If $E(t) = S^1_R$, then we use the fact that $u_\varepsilon \in C^0([0, T], L^2)$ evolves continuously when considered as an L^2-valued function. This allows us to choose a different sequence $\tilde{c}_\varepsilon$ such that the integral of u_ε at time t is always strictly bounded away from both r and R. Thus we have reduced this case to the previous one and obtain a contradiction like before. □

Remark 3.13. It is an open question whether the statement above is stable in the sense that all solutions to (3.4) with initial conditions u^0_ε satisfying

$$u^0_\varepsilon \to \chi_{[-r,r]}, \qquad \mathcal{E}_\varepsilon(u^0_\varepsilon) \to \mathcal{E}(\chi_{[-r,r]})$$

behave in the same way.

Remark 3.14. Since we employ energy methods for the derivation of this result, it is not clear whether also dilute obstacles whose capacity vanishes asymptotically in the energy can impede interface motion.

Finally, let us state the result needed for the use of energy methods above.

Proposition 3.15. *Let $x_{i,\varepsilon} \in S^1$ be points such that $1 \leq i \leq N_\varepsilon$ with $N_\varepsilon/|\log\varepsilon| \to \Lambda$ satisfying the following assumptions:*

(1) (well-seperated) *There exists $\beta < 1$ independent of $\varepsilon > 0$ such that $d(x_{i,\varepsilon}, x_{j,\varepsilon}) > \varepsilon^\beta$ for all $1 \leq i \neq j \leq N_\varepsilon$.*

(2) (finite capacity density) *The obstacles approach a multiple of the Lebesgue measure through $\frac{1}{|\log\varepsilon|}\sum_{i=1}^{N_\varepsilon}\delta_{x_i} \rightharpoonup \Lambda\,\mathcal{L}^2$ for $\Lambda \in (0, \infty)$.*

Take the space

$$X_\varepsilon := \{u_\varepsilon \in H^{1/2}(S^1) \mid u_\varepsilon \equiv 0 \text{ on } B_\varepsilon(x_{i,\varepsilon}) \text{ for } 1 \leq i \leq N_\varepsilon\}$$

and the energy functional

$$\mathcal{E}_\varepsilon : X_\varepsilon \to \mathbb{R}, \qquad \mathcal{E}_\varepsilon(u_\varepsilon) = \frac{1}{|\log\varepsilon|}\left([u_\varepsilon]^2_{1/2} + \int_{S^1}\frac{1}{\varepsilon}W(u_\varepsilon)\,\mathrm{d}x\right)$$

where W is a periodic multi-well potential and $W \geq c\,\mathrm{dist}^2(\cdot, \mathbb{Z})$ for some $c > 0$. Then

$$\left[\Gamma(L^2) - \lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon\right](u) = \int_{S^1}\alpha(u)\,\mathrm{d}x + 4\int_{J_u}[u]\,\mathrm{d}\mathcal{H}^1$$

where $u \in BV(S^1, \mathbb{Z})$, $[u] = u^+ - u^-$ denotes the jump of u on the jump set J_u and $\alpha(z)$ is determined as the solution of the cell problem

$$\alpha(z) = \inf\left\{\frac{1}{2}[w]^2_{1/2,\mathbb{R}^2} + \int_{\mathbb{R}} W(w)\,\mathrm{d}x \;\middle|\; w - z \in H^{1/2}(\mathbb{R}),\ w \equiv 0 \text{ on } B_1(0)\right\}.$$

We will not prove the proposition in this article, the sceptical reader may also take it as a conjecture. For a useful compactness property, either a more restrictive distribution of obstacles or a potential growing at ∞ should be imposed.

3.7. **External Driving Forces.** Let us assume that an external sheer force is applied to the crystal. On the scale where the crystal can be assumed to be periodic, an applied force is constant in space and thus enters the evolution equation as an additive constant.

Theorem 3.16. *Denote by $S^1_R = [-R/2, R/2]$ the circle of radius R and let $x_{i,\varepsilon}$ be points on S^1_R satisfying the conditions of Proposition 3.15. Let $r < R$, $f \in \mathbb{R}$. There exists a weak solution u_ε of*

$$\begin{cases} \varepsilon\, u_t = \frac{1}{|\log\varepsilon|}\left(\mathrm{A}\, u - \frac{1}{\varepsilon}\, W'(u)\right) + f & \text{in } \mathbb{R}\setminus \bigcup_{i=1}^{N_\varepsilon} \overline{B_\varepsilon(x_{i,\varepsilon})} \\ u = 0 & \text{on } \bigcup_{i=1}^{N_\varepsilon} \overline{B_\varepsilon(x_{i,\varepsilon})} \end{cases} \tag{3.5}$$

with an initial condition satisfying

$$u_\varepsilon(0,\cdot) \to \chi_{[-r/2,r/2]} \quad \text{in } L^2(S^1), \qquad \mathcal{E}_\varepsilon(u_\varepsilon(0)) \to \mathcal{E}(\chi_{[-r/2,r/2]})$$

such that the following hold:

(1) If $f < 0$, then $u_\varepsilon(t,\cdot) \to \chi_{[-r/2+|f|t, r/2-|f|t]}$ in $L^2(S^1)$ for all $t > 0$ (the characteristic function of the empty set being zero).

(2) If $f = f_\varepsilon = \overline{f}\,|\log\varepsilon|^{-1}$ for $\overline{f} < 0$ and $c_\varepsilon = \frac{1}{|\log\varepsilon|}$, then $u_\varepsilon(t,\cdot) \to \chi_{[-r(t),r(t)]}$ for all $t > 0$ where

$$\dot r = -\frac{1}{2r} + 2\sum_{n=1}^{\infty} \frac{2r}{(nR)^2 - 4r^2} + \overline{f}, \qquad r(0) = r/2.$$

(3) There exists $f_0 > 0$ such that for $0 < f < f_0$, we have $u_\varepsilon(t,\cdot) \to \chi_{[-r/2,r/2]}$ in $L^2(S^1)$ for all $t > 0$. This also holds if we accelerate the solutions to any faster time-scale.

The proof proceeds exactly as before, but the applied force now acts on the fast time-scale when it is contracting so that the kink/anti-kink attraction disappears in the limit. In the other direction, we note that for small forces, an energy barrier still needs to be overcome. For the case of small negative forcing or order $|\log\varepsilon|^{-1}$ compare [GM12] where periodic forcing is considered.

4. Two-dimensional Dynamics

4.1. **Technical Points.** The one-dimensional evolution reaches the macroscopic time-scale at an obstacle distance of $d_\varepsilon \sim \sqrt{\varepsilon/|\log\varepsilon|}$ which is the natural distance of obstacles in the setting of Theorem 2.1. In two dimensions, this is still expected to be slow.

Let us for the moment assume that the obstacles are distributed on a perfect grid $d_\varepsilon \cdot \mathbb{Z}^2$ in the plane. Then we can use the moving interface sub-solutions constructed in one dimension as a sub-solution by extending them as constant in the second direction. They are still pinned sub-solutions since at $\{u = 0\}$, the sub-solution property holds trivially on the non-pinned set. However, these extended sub-solutions can be considered as sub-solutions in the situation when the obstacles are ε-tubes around lines rather than unions of ε-balls. The volume of the pinning set is $N_\varepsilon \cdot \varepsilon^2 \sim \varepsilon\,|\log\varepsilon|$, while the volume of the ε-tubes is proportional to $\sqrt{N_\varepsilon}\cdot\varepsilon \sim \sqrt{\varepsilon\,|\log\varepsilon|}$, so considerably larger. While the volume of the pinning set is a bad proxy for estimating its influence, this simple observation suggests that using a one-dimensional construction may well over-estimate the influence of pinning. Indeed, a back-of-the-envelope calculation like in Section 2.2 gives much slower speed for super-solutions.

The modification of the interface needs to be done more carefully in this setting, since the flattening out of the interface to facilitate glueing induces motion on the macroscopic time-scale in this scaling. The refined modification is presented below.

Lemma 4.1. *Let $1/2 < \zeta \leq 1$, $F > 0$, $l \gg 1$. Then there exists*

$$\bar u \, \in \, C^{1/2}\left(\mathbb{R}^2\right) \, \cap \, C^{1,1/2}_{loc}\left(B_2 \setminus \overline{B_1}\right) \, \cap \, C^{1,1/2}\left(\mathbb{R}^2 \setminus B_{3/2}\right)$$

with the following properties:

(1) *We have*

$$\bar u \equiv 0 \text{ on } B_1 \qquad \text{and} \quad \mathrm{A}\, \bar u - W'(\bar u) \geq \frac{1}{l} + \frac{F}{l^{1+\zeta}} \text{ on } \mathbb{R}^2 \setminus B_1.$$

(2) *The function $\bar u$ is constant on $\mathbb{R}^2 \setminus B_{l^\zeta/3}$ and*

$$\left| \lim_{|x|\to\infty} u(x) - \left(1 - \frac{1}{W''(0)\, l}\right) \right| \leq \frac{C}{l^{2\zeta}}$$

(3) *The growth estimate*

$$\bar u(x) \geq 1 - \frac{1}{W''(0)\, l} - \frac{c}{|x|^2}$$

holds for some $c > 0$.

All constants are independent of l.

Proof. Take the sub-solution $\bar u$ constructed in Lemma 3.1 on the circle of length l^ζ with

$$M = M_l = \frac{1}{l^{1-\zeta}} + \frac{F}{l}$$

and extend it to the whole real line as in Lemma 3.4. Now set

$$u_{2D}(x) = \bar u(|x|).$$

Since the norm is 1-Lipschitz, the function u_{2D} is $C^{0,1/2}$-Hölder continuous and since the norm is C^∞-smooth away from the origin, u_{2D} is exactly as smooth as $\bar u$ outside $\overline{B_1(0)}$.

The function is constant on $\mathbb{R}^2 \setminus B_{l^\zeta/2}(0)$ and satisfies the well-known growth estimate

$$\bar u(|x|) \geq 1 - \frac{l^{\zeta-1} + F\, l^{-1}}{W''(1)\, l^\zeta} - \frac{c}{|x|^2}$$

Note that F vanishes in the error estimate to leading order since $l^{1+\gamma} \gg l^{2\gamma}$. The sub-solution property is established by comparing the rotationally symmetric extension to a non-radial extension. Assume that $x = |x| \cdot e_1$ and observe that

$$\begin{aligned} \mathrm{A}\, u_{2D}(x) &= \int_{\mathbb{R}^2} \frac{\bar u(|y|) - \bar u(|x|)}{|y-x|^3}\, \mathrm{d}y \\ &\geq \int_{\mathbb{R}^2} \frac{\bar u(y_1) - \bar u(x_1)}{|y-x|^3}\, \mathrm{d}y \\ &= c_{2,1} \int_{\mathbb{R}} \frac{u(y_1) - u(x_1)}{|y_1 - x_1|^2}\, \mathrm{d}y_1 \\ &= c_{2,1}\, \mathrm{A}\, \bar u(|x|). \end{aligned}$$

since $\bar u$ is monotone growing away from the origin and $|x| = x_1$. The same holds after rotation for any point $x \in \mathbb{R}^2 \setminus B_1(0)$.

We used that for a function $f : \mathbb{R}^n \to \mathbb{R}$, $f(\hat x, x_n) = g(\hat x)$ for some $g : \mathbb{R}^{n-1} \to \mathbb{R}$ we have $\mathrm{A}^{\mathbb{R}^n} f(\hat x, x_n) = \mathrm{A}^{R^{n-1}} g(\hat x)$ when the fractional Laplacian is computed as a singular integral. The normalising constant $c_{2,1} \neq 1$ appears here because we neglected normalising the fractional Laplacian before. Since the same re-normalisation affects the half-Laplacian acting on the interface and the obstacle sub-solution in the same way, we will not make a difference here and remark only that the Lemma holds for the properly normalised operator. □

In two dimensions, the downward force exerted by the pinning constraint decays faster and the optimal transition approaches +1 more quickly. This is connected to the fact that small balls shrink faster in two dimensions, or equivalently, that the boundary condition at infinity has a stronger upwards pull since large circles have increasing measure while two points in one dimension always have the same mass.

Lemma 4.2. *Let $u \in 1 + H^{1/2}(\mathbb{R}^2)$ be a minimiser of*

$$\mathcal{E}(u) = [u]_{1/2}^2 + \int_{\mathbb{R}^2} W(u)\,\mathrm{d}x$$

under the constraint $u \equiv 0$ on $B_R(0)$. Then $u \in C^{1/2}(\mathbb{R}^2)$ is radially symmetric, smooth away from the pinning set, and satisfies $1 - \frac{c}{|x|^3} \leq u(x) < 1$ for all $x \in \mathbb{R}^2$ and some $c \geq 1$.

The proof is a slight variation of that of Lemma 3.1 or Lemma 3.5.

4.2. **Dynamics in the Plane.** We will now prove that the pinning constraint acts on a much slower time-scale than the kink/anti-kink attraction by considering the model problem of a single infinitely long straight interface on $\mathbb{R}^2$ perfectly aligned with the grid.

Theorem 4.3. *Let $\Gamma_\varepsilon = d_\varepsilon \cdot \mathbb{Z}^2$ for $d_\varepsilon \gg \varepsilon$. Then there exists $\underline{u}_\varepsilon \leq \overline{u}_\varepsilon$ which are a viscosity sub- and super-solution of*

$$\begin{cases} c_\varepsilon^\pm \varepsilon\, u_t &= \frac{1}{|\log\varepsilon|}\left(\mathrm{A}\, u - \frac{1}{\varepsilon} W'(u)\right) \quad \text{in } \mathbb{R}^2 \setminus \bigcup_{i\in\mathbb{Z}} \overline{B_\varepsilon(id_\varepsilon)} \\ \quad u &= 0 \qquad\qquad\qquad\qquad\qquad \text{on } \bigcup_{i\in\mathbb{Z}} \overline{B_\varepsilon(id_\varepsilon)} \end{cases}$$

respectively. When we choose $c_\varepsilon^+ = \frac{\varepsilon^2}{d_\varepsilon^3\,|\log\varepsilon|}$ for the time scaling of the super-solution and $c_\varepsilon^- = |\log\varepsilon|\, c_\varepsilon^+$, there are constants $c, C > 0$ such that

$$\lim_{\varepsilon\to 0} \underline{u}_\varepsilon(t,\cdot) = \chi_{[ct,\infty)\times\mathbb{R}}, \qquad \lim_{\varepsilon\to 0} \overline{u}_\varepsilon(t,\cdot) = \chi_{[Ct,\infty)\times\mathbb{R}}$$

in $L^2_{loc}(\mathbb{R})$ for all $t > 0$. In particular, the gradient flow is slow of some order c_ε between

$$\varepsilon^{1/2}\,|\log\varepsilon|^{1/2} \leq c_\varepsilon \leq \varepsilon^{1/2}\,|\log\varepsilon|^{3/2}$$

in the line-tension scaling $d_\varepsilon \sim \varepsilon^{1/2}|\log\varepsilon|^{-1/2}$.

Here, we miss the optimal order by a logarithmic term as the sub-solution moves on a faster time-scale than the super-solution. This discrepancy is due to our use of a radially extended function rather than a fully two-dimensional construction. The two-dimensional growth rate is observed in Lemma 4.2, and we expect the super-solution to give the right order of movement rather than the sub-solution – see Remark 3.2 for the difficulties related to constructing sub-solutions directly in two dimensions.

Proof of Theorem 4.3. Like in the proof of Theorem 3.8, we begin by constructing sub-solutions in a blow-up scale.

First modification. Like in the one-dimensional case, denote $l = l_\varepsilon = d_\varepsilon/(\varepsilon N)$ for suitably large $N \in \mathbb{N}$. Take f_L like in Lemma 3.7, but this time for $L = l^{3/2}$. Choose $g \in C^\infty(\mathbb{R})$ such that

$$0 \leq g \leq 1, \qquad g(t) = \begin{cases} 1 & |t| \geq 2l \\ 0 & |t| \leq l \end{cases}, \qquad |g'| \leq \frac{2}{l}, \qquad |g''| \leq \frac{4}{l^2}$$

and $c_1, c_2 > 0$ to be specified later. Set

$$\widetilde{\phi}(x) = f_L \circ \phi(x_1) - \frac{c_1 \log(l)}{l^2}\, g(x_1) - \frac{c_2 \log(l)}{l^3}$$

and compute

$$
\begin{aligned}
\left|\mathrm{A}(\widetilde{\phi}-\phi)(x)\right| &\leq |\mathrm{A}(f_L\circ\phi-\phi)(x)| + \left|\frac{c_1\,\log l}{l^2}\ \mathrm{A}\,g(x)\right| \\
&\leq \frac{C}{L^2} + \frac{c_1\log(l)}{l^2}\left(\int_{B_l(x)}\frac{||\,D^2 g||}{|y-x|}\,\mathrm{d}y + \int_{\mathbb{R}^2\setminus B_l(x)}\frac{2\,||g||}{|x-y|^3}\,\mathrm{d}y\right) \\
&\leq \frac{C}{l^3} + \frac{c_1\log(l)}{l^2}\left(\frac{4}{l^2}\int_0^l \frac{1}{r}\,(2\pi r)\,\mathrm{d}r + 2\int_l^\infty \frac{1}{r^3}\,(2\pi r)\,\mathrm{d}r\right) \\
&\leq \frac{C\,\log(l)}{l^3}.
\end{aligned}
$$

like in Lemma 3.7 because $L^2 = l^3$ and again using the property that the fractional Laplacian of a function of n variable which does not depend on the last variable reduces reduces to the fractional Laplacian of its profile one dimension below. Thus $\widetilde{\phi}$ satisfies

$$
\begin{aligned}
\mathrm{A}\,\widetilde{\phi} - W'(\widetilde{\phi}) &= \mathrm{A}(\widetilde{\phi}-\phi) + \mathrm{A}\,\phi - W'(\phi) + \left[W'(\phi) - W'(\widetilde{\phi})\right] \\
&= \mathrm{A}(\widetilde{\phi}-\phi) + \left[W'(\phi) - W'(\widetilde{\phi})\right] \\
&\geq \begin{cases} \frac{-c_W\,c_2\,\log l}{l^3} & \widetilde{\phi}(x)\in[\beta,1-\beta] \\ \frac{\tilde{c}_W\,c_2\,\log l}{l^3} & \widetilde{\phi}(x)\in(0,\beta] \text{ or } \widetilde{\phi}(x)\in[1-\beta,1] \\ \frac{\tilde{c}_W\,c_1\,\log l}{l^2} & |x|\geq 2l. \end{cases}
\end{aligned}
$$

All constants are positive and depend only on W''.

Second modification. Let $y_{i,j,\varepsilon} = Nl\,(i,j)$ be an enumeration of the pinning sites after rescaling with $i,j\in\mathbb{Z}$. For pinning sites with $Nli \leq l^{3/2}+l$, we need an additional modification to flatten the interface before we can insert obstacles.

Choose $1/2<\gamma<1$ and a bump-function η such that

$$
0\leq\eta\leq 1, \qquad \eta(x) = \begin{cases} 1 & |x|\leq l^\gamma \\ 0 & |x|\geq 2l^\gamma \end{cases}, \qquad |\nabla\eta|\leq\frac{2}{l^\gamma}, \qquad |D^2\eta|\leq\frac{4}{l^{2\gamma}}
$$

and set

$$
\tilde{u}_\lambda(x) = \left(1 - \sum_{|i|\leq\frac{2\,l^{1/2}}{N}}\ \sum_{j\in\mathbb{Z}}\eta(x-y_{i,j,\varepsilon})\right)\widetilde{\phi}(x_1+\lambda l) + \sum_{|i|\leq 2\sqrt{l}/N,\ j\in\mathbb{Z}} U_{i,\varepsilon}\,\eta(x-y_{i,j,\varepsilon})
$$

with

$$
U_{i,\varepsilon} = \widetilde{\phi}(\,(Ni-\lambda)l) + O(l^{\gamma-2})
$$

for a small term $O(l^{\gamma-2})$ to be chosen later. This is a function which mostly looks like (a translated version of) $\widetilde{\phi}$, but is flattened at the pinning sites. The parameter λ will later be used for the time-evolution. Note that

$$
|\widetilde{\phi}(x) - U_{i,\varepsilon}| = \frac{1}{W''(0)}\left|\frac{1}{x} - \frac{1}{Nil-\lambda}\right| + O(l^{-2}) + O(l^{\gamma-2}) \leq C\,l^{\gamma-2}
$$

where $\eta(x - y_{i,j,\varepsilon}) \neq 0$, so when we set

$$\begin{aligned}\eta_{i,j}(x) &= \eta(x - y_{i,j,\varepsilon}),\\ w &:= \tilde{u}_\lambda - \widetilde{\phi}\\ &= \sum_{i,j} \Big(U_{i,\varepsilon} - \widetilde{\phi}(x_1 + \lambda l)\Big)\ \eta_{i,j}\end{aligned}$$

we observe that

$$\begin{aligned}\nabla w(x) &= \sum_{i,j} \Big(U_{i,\varepsilon} - \widetilde{\phi}\Big)\ \nabla\eta_{i,j} - \widetilde{\phi}'\,\eta_{i,j}\,e_1,\\ D^2 w(x) &= \sum_{i,j} \Big(U_{i,\varepsilon} - \widetilde{\phi}\Big)\ D^2\eta_{i,j} - \widetilde{\phi}'\,(e_1 \otimes \nabla\eta_{i,j} + \nabla\eta_{i,j} \otimes e_1) - \widetilde{\phi}''\,\eta_{i,j}\,e_1 \otimes e_1\\ &= O\left(l^{-(2+\gamma)}\right).\end{aligned}$$

Therefore, the usual argument shows that

$$|\mathrm{A}\,w(x)| \leq \int_{B_{l^\gamma}(x)} \frac{||D^2 w||_{L^\infty}}{|x-y|}\,\mathrm{d}y + \int_{\mathbb{R}^2\setminus B_{l^\gamma}(x)} \frac{||w||_{L^\infty}}{|x-y|^3}\,\mathrm{d}y = O(l^{-2})$$

on $\mathbb{R}^2$ where the matrix norm for $D^2 w$ is the pointwise norm for symmetric bilinear forms

$$||D^2 w|| = \max_{v\in S^1} \sqrt{|v^T\,D^2 w\,v|}$$

as needed for radial integration. This estimate can be improved if we are far from the next pinning site. Namely, assume that $\min_{i,j}|x - y_{i,j,\varepsilon}| \geq 2l$, then the first term vanishes and the sharper estimate

$$\begin{aligned}|\mathrm{A}\,w(x)| &\leq \sum_{i,j} \frac{C}{|x - y_{i,j,\varepsilon}|^3} \int_{B_{l^\gamma}(y_{i,j,\varepsilon})} 2\,||w||\,\mathrm{d}y\\ &\leq \sum_{i,j} \frac{C\,l^{2\gamma}\,l^{\gamma-2}}{|x - y_{i,j,\varepsilon}|^3}\\ &= O(l^{3\gamma-5})\end{aligned}$$

holds. From now on, take $\gamma = 2/3$, so that $\mathrm{A}\,w = O(l^{-3})$. Note that this holds true for all x with $|x| \leq 2l$.

Inserting Obstacles: Right Half-Space. We now insert the obstacle sub-solutions from Lemma 4.1 into the flattened out sites and on the half-spaces that are flattened out by f_L. First we deal with the right hand side of the interface where ϕ is close to 1.

We concentrate on the obstacles in the flattened discs since the flattened half-plane can be treated similarly. The height of the obstacle at $y_{i,j,\varepsilon}$ is $1 - \frac{1}{W''(0)\,(Nil-\lambda)} - O((Nil)^{\gamma-2})$ for $j \leq 2l^{1/2}/N$ which is

$$1 - \frac{1}{W''(0)\,l} - O(l^{-4/3}) = 1 - \frac{1}{W''(0)\,(Nil)} - \frac{F}{(Nil)^{4/3}} + o(l^{-4/3})$$

for some bounded sequence $F = F_j \in \mathbb{R}$, so we choose $\zeta = 1/3$ in Lemma 4.1. A lower order perturbation in either term gives a matching height between the interface and the inserted obstacle so that we can glue the obstacle into the modified domain. The sub-solution for an obstacle on $\mathbb{R}^2$ is constant for arguments $|x| \geq l^\zeta = l^{1/3}$ and the flattened out disc in the obstacle has a radius of $l^{2/3}$, so the glueing does not cause more problems. The resulting function is denoted by u.

Let us check that the sub-solution condition is still satisfied at the obstacles we just inserted. An ideal interface would exert a non-local force of

$$\begin{aligned}
\mathrm{A}\,\phi(Nil) &= W'(\phi(Nil)) \\
&= W''(0)\,\big(1-\phi(Nil)\big) + O\big(\,(1-\phi(Nil))^{-2}\big) \\
&= W''(0)\cdot\frac{1}{W''(0)\,Nil} + O((Nil)^{-2}) \\
&= \frac{1}{Nil} + O((Nil)^{-2})
\end{aligned}$$

at $y_{i,j,\varepsilon}$ which is compensated by the obstacle by construction. The same is true up to order $O(l^{\gamma-2}l^{-\gamma}) = O(l^{-2})$ for the modified interface $\tilde{u}_\lambda$, which is also compensated. The obstacle $y_{i',j',\varepsilon}$ has a distance of $Nl\,\sqrt{(i-i')^2+(j-j')^2}$ to $y_{i,j,\varepsilon}$, so their contribution to the non-local force is negligible. Namely, the force created by the obstacles at a point x is

$$\begin{aligned}
\sum_{i,j}\int_{B_l^{1/3}(y_{i,j,\varepsilon})}\frac{u(y)-\tilde{u}_\lambda(y)}{|y-x|^3}\,\mathrm{d}y &\le \sum_{i,j}\left[\int_{B_1(y_{i,j,\varepsilon}}\frac{1}{|y-x|^3}\,\mathrm{d}y + \int_{B_{l^{1/3}}(y_{i,j,\varepsilon})\setminus B_1}\frac{\frac{1}{|y-y_{i,j,\varepsilon}|^2}}{|y-x|^3}\,\mathrm{d}y\right] \\
&= O\left(\frac{1+\log(l)}{\operatorname{dist}(x,Nl\mathbb{Z})^3}\right) \\
&= \begin{cases} O(l^{-2}\log(l)) & \text{if } \operatorname{dist}(x,Nl\mathbb{Z})\ge l^{2/3} \\ O(l^{-3}\log(l)) & \text{if } \operatorname{dist}(x,Nl\mathbb{Z})\ge l \end{cases}
\end{aligned}$$

which is compensated either by a sufficiently large constant c_1 close to the obstacles or by c_2 away from the obstacles. At the interface, it can be compensated by a speed $O(l^{-3}\log(l))$.

Inserting Obstacles: Left Half-Space. Since the interface could only be modified for $|x| \ge L \ge l^{3/2}$, we also need to insert obstacles into the flattened out discs in the left half space in two dimensions.

Being close to phase $\{u\approx 0\}$, the construction of a stationary obstacle sub-solution does not go through. Instead we can take a sub-solution of the periodic obstacle problem at phase $\{u\approx 1\}$ for $F=0$, an auxiliary double-well potential $\widetilde{W}$ and multiply it by a factor

$$h_l \sim \frac{1}{W''(0)\,l}$$

which allows for continuous glueing. Here, the interface pulls upwards with force $\sim \frac{1}{l}$, while the self-interaction force of the obstacle is

$$\begin{aligned}
\mathrm{A}(h_l u) - \widetilde{W}'(h_l u) &= h_l\ \mathrm{A}\,u - \widetilde{W}'(h_l u) \\
&= h_l\widetilde{W}'(u) - \widetilde{W}'(h_l u) \\
&\le \frac{C_{\widetilde{W}}}{l}.
\end{aligned}$$

When we choose $\widetilde{W}$ suitably, $C_{\widetilde{W}}$ can be made as small as we need by choosing any initial $\widetilde{W}$ and then multiplying by a suitably small constant. Thus the upwards pull of the interface compensates the self-interaction.

Conclusion and Rescaling. Overall, the calculations show that a function as constructed above is a sub-solution if the interface moves with a speed $\lambda = O(\log(l)\,l^{-3})$. Again, the suitable monotonicity of $\widetilde{\phi}$ and the precise construction ensure the sub-solution property at non-smooth times. When passing to the macroscopic scale as

$$\underline{u}_\varepsilon(t,x) = u\left(\frac{t}{\varepsilon^2|\log\varepsilon|\,c_\varepsilon}, \frac{x}{\varepsilon}\right)$$

we observe that the interface moves a distance $d_\varepsilon \sim \varepsilon l_\varepsilon$ in a time t_ε proportional to the product of the re-scaling factor with the quotient of the travelled distance l_ε in the blow-up scale and the speed $l_\varepsilon^{-3}\,\log(l_\varepsilon)$ in the blow-up scale, i.e.

$$t_\varepsilon \sim \varepsilon^2|\log\varepsilon|\,c_\varepsilon \cdot \frac{d_\varepsilon}{\varepsilon}\left(\frac{\varepsilon^3\,|\log\varepsilon|}{d_\varepsilon^3}\right)^{-1} = \frac{c_\varepsilon\,d_\varepsilon^4}{\varepsilon^2}.$$

To obtain a uniform speed on the order $O(1)$ we choose $t_\varepsilon \sim d_\varepsilon$ or equivalently

$$c_\varepsilon^- = \frac{\varepsilon^2}{d_\varepsilon^3}.$$

Super-solutions. Super-solutions are constructed in analogy to the one-dimensional case. The growth rate $1 - \frac{c}{|x|^3}$ from Lemma 4.2 leads to the fact that the logarithmic term in the integral is not present in the super-solution. A simple calculation shows that super-solutions move on the slower time-scale

$$c_\varepsilon^+ = \frac{\varepsilon^2}{d_\varepsilon^3\,|\log\varepsilon|}.$$

□

Remark 4.4. A similar argument can be made when the interface is not perfectly aligned with the grid. Take the grid $\Gamma_\varepsilon = S(\phi)\cdot\left(d_\varepsilon\cdot\mathbb{Z}^2\right)$ where $S(\phi)$ is the rotation matrix

$$S(\phi) = \begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}.$$

If we take ϕ such that $\tan(\phi)\in\mathbb{Q}$, then the first component z_1 of $z\in\Gamma_\varepsilon$ is

$$z_1 = d_\varepsilon\,(n\cos\phi + m\sin\phi) = d_\varepsilon\cos\phi\,(n + m\tan\phi)$$

for some $n,m\in\mathbb{Z}$. Since we assumed $\tan\phi$ to be rational, this is a discrete periodic subset of the real line and the distance between any two points is proportional to d_ε. Also the fractional Laplacian can be estimated as before, so Theorem 4.3 also holds for rotated square grids.

Equally well, we could rotate the interface instead of the grid. This resembles the settings of [DY06, DKY08] where a straight front in a periodic medium is considered for sharp interface mean curvature flow and for a local Allen-Cahn equation. Our setting differs in the use of a non-local differential operator and in that the obstacles are comparable to the size of the interface, but the distances between them lie on a much larger scale.

4.3. **Dynamics on a Torus.** Finally we state the main result as applied to the case of [GM06]. Note that the inclusion of a constant force f in the energy is a compact perturbation, so

$$\widetilde{\mathcal{E}}_\varepsilon(u) := \mathcal{E}_\varepsilon(u) - \int_{\mathbb{T}^2} fu\,\mathrm{d}x \qquad \xrightarrow{\Gamma(L^2)} \qquad \mathcal{E}(u) - \int_{\mathbb{T}^2} fu\,\mathrm{d}x$$

for any constant $f\in\mathbb{R}$.

Theorem 4.5. *Denote by $\mathbb{T}_a^2 = \mathbb{R}^2/\left(a\cdot\mathbb{Z}^2\right)$ the flat square torus with volume $A = a^2$. Consider the evolution equation*

$$\begin{cases} c_\varepsilon\,\varepsilon u_t &= \frac{1}{|\log\varepsilon|}\left(\mathrm{A}\,u - \frac{1}{\varepsilon}W'(u)\right) + f \quad \text{in } [0,\infty)\times\left[\mathbb{T}_a^2\setminus\bigcup_{k=1}^{N_\varepsilon}\overline{B_\varepsilon(x_{i,\varepsilon})}\right]\\ u &= 0 \quad \text{on } (0,\infty)\times\bigcup_{k=1}^{N_\varepsilon}\overline{B_\varepsilon(x_{i,\varepsilon})}\\ u &= u_\varepsilon^0 \quad \text{at } t=0\end{cases}$$

where the pinning sites $x_{i,\varepsilon}$ satisfy the assumptions of Theorem 2.1 and additionally

(1) *the distribution assumption*

$$\{x_{1,\varepsilon}, \dots, x_{N_\varepsilon,\varepsilon}\} \subset \bigcup_{j,k=1}^{\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}} B_{r_\varepsilon}\left(d_\varepsilon \cdot (j,k)\right).$$

for $r_\varepsilon \ll d_\varepsilon = \sqrt{\frac{\varepsilon}{|\log\varepsilon|}}$ (i.e. the pinning sites lie in small discs around grid points) and

(2) *the number of obstacles per disk is uniformly bounded:*

$$\#\left(\{x_{1,\varepsilon}, \dots, x_{N_\varepsilon,\varepsilon}\} \cap B_{r_\varepsilon}\left(d_\varepsilon \cdot (j,k)\right)\right) \le M$$

for some $M \in \mathbb{N}$ independently of ε, j, k.

Then we find $u_\varepsilon^0 \to \chi_{[-r/2,r/2]\times[0,a]} =: u$ in $L^2(\mathbb{T}_a^2)$ (a strip of width r around the torus) such that $\mathcal{E}_\varepsilon(u_\varepsilon^0) \to \mathcal{E}(u)$ and the following hold:

(i) *If $f = 0$ and $r > a/2$, then $u_\varepsilon(t,\cdot) \to u$ in $L^2(\mathbb{T}_a^2)$ independently of $c_\varepsilon \to 0$.*

(ii) *If $f = 0$ and $r < a/2$, then $u_\varepsilon(t,\cdot) \to \chi_{[-r(t),r(t)]\times[0,a]}$ in $L^2(\mathbb{T}_a^2)$ for $c_\varepsilon = |\log\varepsilon|^{-1}$ where*

$$\dot r = -\frac{1}{2r} + 2\sum_{n=1}^\infty \frac{2r}{(nR)^2 - 4r^2}, \qquad r(0) = r/2.$$

(iii) *If $0 < f < f_0$ for some f_0 depending only on the capacity α of dislocations (i.e. the potential W) and the limiting density $\Lambda = \lim_{\varepsilon\to 0} \frac{\varepsilon}{|\log\varepsilon|} N_\varepsilon \in (0,\infty)$, then $u_\varepsilon(t,\cdot) \to u$ in $L^2(\mathbb{T}_a^2)$ independently of $c_\varepsilon \to 0$. (This is valid also for $f_\varepsilon > 0$ if $f_\varepsilon \gg |\log\varepsilon|^{-1}$.)*

(iv) *If $f < 0$, then $u_\varepsilon(t,\cdot) \to \chi_{[-r/2+|f|t, r/2-|f|t]\times[0,a]}$ for $c_\varepsilon \equiv 1$.*

(v) *If $f = f_\varepsilon = \overline{f}\,|\log\varepsilon|^{-1}$ for $\overline{f} < 0$, then $u_\varepsilon(t,\cdot) \to \chi_{[-r(t),r(t)]\times[0,a]}$ for all $t > 0$ where*

$$\dot r = -\frac{1}{2r} + 2\sum_{n=1}^\infty \frac{2r}{(nR)^2 - 4r^2} + \overline{f}, \qquad r(0) = r/2$$

if $c_\varepsilon = \frac{1}{|\log\varepsilon|}$.

If $W \in C^4(\mathbb{R})$ and $W^{(3)}(0) = 0$ (e.g. if $W' = \sin$), then we can generalise the distribution assumptions as follows.

(1') *There exist $d'_\varepsilon \gg \varepsilon^{2/3}|\log\varepsilon|^{1/3}$ and $r_\varepsilon \ll d'_\varepsilon$ such that*

$$\{x_{1,\varepsilon}, \dots, x_{N_\varepsilon,\varepsilon}\} \subset \bigcup_{j,k=1}^{1/d'_\varepsilon} B_{r_\varepsilon}\left(d'_\varepsilon \cdot (j,k)\right).$$

(2') *the number of obstacles per disk is uniformly bounded:*

$$\#\left(\{x_{1,\varepsilon}, \dots, x_{N_\varepsilon,\varepsilon}\} \cap B_{r_\varepsilon}\left(d'_\varepsilon \cdot (j,k)\right)\right) \le M$$

for some $M \in \mathbb{N}$ independently of ε, j, k.

The proof is a combination of the analogous statement in one dimension and the more subtle modification of the interface in two dimensions described above with a few additional facets:

(1) Note that

$$|\psi'(x)| \le \frac{C}{1+|x|^2} \qquad \Rightarrow \qquad |\psi(x)| \le \frac{C}{1+|x|},$$

so at the nearest obstacle where we need to modify we use $l_\varepsilon \sim d_\varepsilon/\varepsilon \sim (\varepsilon\,|\log\varepsilon|)^{-1/2}$ to calculate

$$\varepsilon\psi\left(l_\varepsilon\right) = O(\varepsilon^{3/2}\,|\log\varepsilon|^{1/2}) \ll \left(\varepsilon\,|\log\varepsilon|\right)^{3/2} = l_\varepsilon^3.$$

This allows us to carry out the same modifications as before without paying much attention to the corrector, which is a lower order perturbation only at the closest pinning site. We have enough 'wiggle room' to come closer to the pinning sites and jump shorter by a logarithmic term, so again we can argue that neither the contracting force nor the jumps matter in the limit.

If $W^{(3)}(0) = 0$, then we can show that ψ decays as x^{-2} at $\pm\infty$, not just as x^{-1}, thus we can come closer to the corrected interface with the obstacles without having to take care of bigger complications in the modification process.

If we could improve the order at which the sub-solution moves to the order of the super-solution, it would suffice to require $d_\varepsilon \gg \varepsilon^{2/3}$, which is the order at which the bulk term induces logarithmically fast motion.

(2) When we denote by u the solution to the cell-problem from Lemma 4.2 and by $x_{i,\varepsilon}$ the pinning sites, we see that the initial condition

$$u_\varepsilon^0(x) = \min\left\{\phi\left(\frac{x_1 + r/2}{\varepsilon}\right), \phi\left(\frac{-x_1 - r/2}{\varepsilon}\right), u\left(\frac{x - x_{i,\varepsilon}}{\varepsilon}\right)\right\}_{1\leq i\leq N_\varepsilon}$$

is trapped between the sub- and super-solution constructed before. Since all three components have converging energies, also $\mathcal{E}_\varepsilon(u_\varepsilon^0) \to \mathcal{E}(u^0)$.

The first statement covers the case of obstacles located on a square grid, the second case allows for relatively general arrangements in a denser grid with many vacancies.

In particular, we see that in none of the four cases above we obtain the gradient flow of the the limiting energy as limit of the evolutions, which behaves as follows:

(1) If $f = 0$, the $\{u = 1\}$-phase contracts with constant velocity stemming from the bulk energy term.
(2) If $f < 0$, the $\{u = 1\}$-phase contracts with constant velocity stemming from both the bulk-energy term and the external force. Here, the behaviour is correct, but the velocity is governed only by the external force.
(3) If $0 < f < f_0$, the $\{u = 1\}$-phase contracts with constant velocity stemming from the bulk-energy which dominates the small external force in the opposite direction. Here, in fact a small force f_ε already suffices to cause qualitatively wrong behaviour.

Remark 4.6. Similarly as for the propagation of a front on the whole space, we can also have tilted grids. On a torus, we obtain a tilted grid by labelling equidistant points on the edges of a square and then connecting id_ε on the bottom with $(i + k)d_\varepsilon$ on the top (and periodically extended). The same result holds then, up to slight technical complications.

Finally, we apply Theorem 4.5 to show that the limit of the pure gradient flows without external force in the usual fast time scale is not mean curvature flow.

Corollary 4.7. *Under the same assumptions as Theorem 4.5 and the assumption that $\Lambda > \Lambda_0 > 0$ for a suitable Λ_0, there exists a sequence of initial conditions $u_\varepsilon^0 \to u = 1 - \chi_{B_r(0)}$ for some small $r > 0$ such that $\mathcal{E}_\varepsilon(u_\varepsilon) \to \mathcal{E}(u)$ and of solutions u_ε of*

$$\begin{cases} \epsilon\,\partial_t u_\varepsilon = \frac{1}{|\log\varepsilon|}\left(\mathrm{A}\,u_\varepsilon - \frac{1}{\varepsilon}\,W'(u_\varepsilon)\right) & \textit{in } [0,\infty)\times\left[\mathbb{T}^2\setminus\overline{\bigcup_{k=1}^{N_\varepsilon} B_\varepsilon(x_{i,\varepsilon})}\right] \\ u = 0 & \textit{on } (0,\infty)\times\overline{\bigcup_{k=1}^{N_\varepsilon} B_\varepsilon(x_{i,\varepsilon})} \\ u = u_\varepsilon^0 & \textit{at } t = 0 \end{cases}$$

such that $u_\varepsilon(t,\cdot) \to 1 - \chi_{E(t)}$ for some set $E(t)$ for all times t, but the boundaries $\partial E(t)$ are not moving by either mean curvature flow or the gradient flow of $\mathcal{E}$.

Proof. At small circles, the line energy dominates the bulk term in both the energy $\mathcal{E}$ and its gradient flow, while circles of radius $r > \frac{1}{\Lambda\alpha}$ are expanding. Assume that Λ_0 is so large that

$\partial B_{2/(\Lambda\alpha)}(0)$ is a round circle on the torus. Then choose $r \in (\,(\Lambda\alpha)^{-1}, 2(\Lambda\alpha)^{-1})$. Both the gradient flow of $\mathcal{E}$ and mean curvature flow of $\partial B_r(0)$ exist smoothly up to some small positive time.

For energetic reasons, the initial set cannot shrink, so $B_r(0) \subset E(t)$ and E does not evolve by mean curvature flow. Using straight interfaces as barriers, we use Theorem 4.5 to show that E cannot leave $[-r, r] \times [0, 1]$ nor $[0, 1] \times [-r, r]$. Thus E is trapped in $[-r, r]^2$ and does not evolve by the gradient flow of $\mathcal{E}$ which is given by circles of increasing radius from $E(0)$. □

On the whole space, we could use the previous results to show that the circle is in fact non-expanding since the angles ϕ with $\tan\phi \in \mathbb{Q}$ are dense in $(-\pi/2, \pi/2)$.

5. Related Models

Let us briefly discuss the validity of our results for similar models concerning the same phenomenon. The first two extensions we discuss concern the dissipation mechanism, while the third one discusses a modification of the energy functional.

5.1. Non-viscous Evolution. It can be argued that the use of a quadratic dissipation is unphysical for the dynamics of dislocations and a rate independent evolution

$$-\,\delta\mathcal{E}_\varepsilon(u_\varepsilon) \in c_\varepsilon\,\mathrm{sign}(\dot u_\varepsilon), \qquad c_\varepsilon > 0 \tag{5.1}$$

associated to a linear dissipation would be physically more sensible. Here

$$\mathrm{sign}(z) = \begin{cases} \{1\} & z > 0\\ [-1, 1] & z = 0\\ \{-1\} & z < 0\end{cases}$$

is the usual set-valued sign function. We believe that the emergence of an asymmetric, stick-slip type motion law from a viscous dissipation is the more interesting observation, in particular as rate-independent dynamics of this problem appear to be stationary in many cases. Namely, the sub-solutions constructed above satisfy

$$-c_\varepsilon \leq \frac{1}{|\log\varepsilon|}\left(\mathrm{A}\,u_\varepsilon - \frac{1}{\varepsilon}\,W'(u_\varepsilon)\right) \leq c_\varepsilon$$

for $c_\varepsilon \geq C\,\frac{\varepsilon}{|\log\varepsilon|\,d_\varepsilon^2}$ for suitable $C > 0$ in the case of Theorem 3.8 and $c_\varepsilon \geq C/|\log\varepsilon|$ in the case of Theorems 3.11 and 3.12. The same is true for many similar initial conditions even without the sign condition. Thus for such a choice of c_ε, the rate-independent evolution can be taken as stationary. The same holds for an evolution law associated to a mixed dissipation

$$-\delta\mathcal{E}_\varepsilon(u_\varepsilon) \in c_\varepsilon\,\mathrm{sign}(\dot u_\varepsilon) + \tilde c_\varepsilon\,\dot u_\varepsilon$$

if c_ε is chosen in the corresponding parameter regime as above, since the last term vanishes identically for stationary solutions of the differential inclusion (5.1).

5.2. Finite Relaxation Speed. We considered the L^2-gradient flow of the energy

$$\mathcal{E}(u) = \frac{1}{|\log\varepsilon|}\left(\frac{1}{2}\,[u]_{1/2}^2 + \frac{1}{\varepsilon}\int_{\mathbb{T}^2} W(u)\,\mathrm{d}x\right)$$

which arose from as an equilibrium localisation on a plane of the crystal grid of the energy

$$\mathcal{F}_\varepsilon(u) = \frac{1}{|\log\varepsilon|}\left(\frac{1}{2}\int_{\mathbb{T}^2\times\mathbb{R}_+} |\nabla u|^2\,\mathrm{d}x + \frac{1}{\varepsilon}\int_{\mathbb{T}^2} W(u)\,\mathrm{d}x\right).$$

The modelling assumption behind this mechanism is that for given dislocations in a plane, the rest of the crystal has relaxed to the minimal Dirichlet energy, which is not quite true in the dynamic case. When we consider the first variation

$$\begin{aligned}\delta F_\varepsilon(u;\phi) &= \frac{1}{|\log\varepsilon|}\left(\int_{\mathbb{T}^2\times\mathbb{R}_+}\langle\nabla u,\nabla\phi\rangle\,\mathrm{d}x + \frac{1}{\varepsilon}\int_{\mathbb{T}^2} W'(u)\phi\,\mathrm{d}x\right)\\ &= \frac{1}{|\log\varepsilon|}\left(\int_{\mathbb{T}^2\times\mathbb{R}_+}\langle\nabla\cdot(\phi\,\nabla u) - (\Delta u)\phi\,\mathrm{d}x + \frac{1}{\varepsilon}\int_{\mathbb{T}^2} W'(u)\phi\,\mathrm{d}x\right)\\ &= \frac{1}{|\log\varepsilon|}\left(\int_{\mathbb{T}^2\times\mathbb{R}_+} -(\Delta u)\phi\,\mathrm{d}x + \int_{\mathbb{T}^2}\partial_\nu u + W'(u)\phi\,\mathrm{d}x\right)\end{aligned}$$

and the inner product

$$\langle v,\phi\rangle := \frac{1}{m_\varepsilon}\int_{\mathbb{R}_+\times\mathbb{T}^2} v\phi\,\mathrm{d}x + \frac{1}{\varepsilon}\int_{\mathbb{T}^2} v\phi,$$

we obtain an evolution equation

$$\tag{5.2}\begin{cases} m_\varepsilon\, u_t &= \frac{1}{|\log\varepsilon|}\Delta u & \text{in } \mathbb{T}^2\times\mathbb{R}_+\\ \varepsilon\, u_t &= \frac{1}{|\log\varepsilon|}\left(\mathrm{A}\,u - \frac{1}{\varepsilon}\,W'(u_\varepsilon)\right) & \text{on } \mathbb{T}^2\setminus\bigcup_{k=1}^{N_\varepsilon}\overline{B_\varepsilon(x_{i,\varepsilon})}\\ u &= 0 & \text{on } \mathbb{T}^2\setminus\bigcup_{k=1}^{N_\varepsilon}\overline{B_\varepsilon(x_{i,\varepsilon})}.\end{cases}$$

The case we considered above corresponds to an infinitely fast relaxation speed in the half-space, i.e. the formal limit $m_\varepsilon\equiv 0$. In that case we could forget about the analytic continuation and only had to track the evolution of the boundary values. We can connect this to the case of positive $m_\varepsilon > 0$ as follows:

All the sub-solutions, super-solutions and solutions constructed above for the gradient flow equation on $\Omega=\mathbb{R}^2$ or $\Omega=\mathbb{T}^2$ were pointwise non-increasing, so their harmonic extensions to $\Omega\times\mathbb{R}_+$ have this property as well. Thus the analytic continuation $\hat u_\varepsilon$ satisfies

$$m_\varepsilon\partial_t\hat u_\varepsilon \le 0 = \frac{1}{|\log\varepsilon|}\Delta\hat u_\varepsilon \quad \text{in } \Omega\times\mathbb{R}_+$$

both in the viscosity or the distributional sense, which means that the analytic solution $\hat u_\varepsilon$ is a sub-solution for (5.2). In this sense, we can at least say that an evolution with a finite relaxation speed can in no case be faster than the limiting case we considered.

5.3. **Finite-Strength Pinning.** The hard constraint $u\equiv 0$ on $\bigcup_{k=1}^{N_\varepsilon}\overline{B_\varepsilon(x_{i,\varepsilon})}$ can be see as the limiting case of the following soft obstacle problems. We consider a version of $\mathcal{E}_\varepsilon$ on the whole space with an additional term in the energy

$$\mathcal{F}_\varepsilon(u) = \frac{1}{|\log\varepsilon}\left([u]_{1/2}^2 + \int_{\mathbb{T}^2} W(u)\,\mathrm{d}x\right) + \sum_{i=1}^{N_\varepsilon}\frac{1}{\varepsilon^2}\int_{B_\varepsilon(x_{i,\varepsilon})} g\left(\frac{x-x_{i,\varepsilon}}{\varepsilon}\right)|u|^q\,\mathrm{d}x.$$

Here $1\le q<\infty$ is a parameter we could choose freely and $g\in C_c(B_1(0))$ is a non-negative function. The hard obstacle arises as the formal limit $g\to+\infty\cdot\chi_{\overline{B_1(0)}}$ for any q. Physically, the case $q=1$ seems the most relevant. This extension has been discussed in [GM05, GM06], and the same Γ-limit statement still holds with a different capacity function $\alpha:\mathbb{Z}\to[0,\infty)$.

Our results apply also here by the following considerations. Observe that

$$\underline{u}_t = 0,\quad W'(\underline{u}) = 0,\quad \mathrm{A}\,\underline{u}\ge 0$$

at $x\in\bigcup_{k=1}^{N_\varepsilon}\overline{B_\varepsilon(x_{i,\varepsilon})}$, so $\underline{u}$ is a sub-solution also to $u_t = \mathrm{A}\,u - \frac{1}{\varepsilon}W'(u) - q\,g(x)\,|u|^{q-2}u$ which is the gradient flow equation of $\mathcal{F}_\varepsilon$. Thus we can use the same sub-solutions to obtain upper bounds on the velocity of interfaces, even in the case $q=1$ since the sub-solutions $\underline{u}_\varepsilon$ do not change sign. For super-solutions, we have to solve a minimisation problem with the soft pinning instead of the hard one instead:

$$\text{Minimise } \frac{1}{2}\,[u]_{1/2}^2 + \int_{S_l^1} W(u)\,\mathrm{d}x + \int_{B_1(x_0)} g\,|u|^q\,\mathrm{d}x \qquad \text{subject to } \frac{1}{l}\int_{S_l^1} u\,\mathrm{d}x > \frac{1}{2}.$$

When we establish that the solution to this problem satisfies $u \not\equiv 1$, $0 \leq u \leq 1$, we obtain matching bounds on the scaling of the velocity of interfaces at a single step on the real line and up to a logarithmic factor in the plane. We observe that also here, the contracting effect of the obstacles vanishes compared to the kink/anti-kink attraction.

6. Conclusion

We have identified the time-scale on which a pinning constraint would naturally act by considering a whole space problem (up to a factor of $O(|\log\varepsilon|)$). We have shown that the gradient flows of the pinned problems do not converge to the gradient flow of the limiting problem under certain assumptions on the distribution of obstacles and given estimates on the behaviour for certain initial conditions. A number of questions remain open.

(1) Is there an explicit law that describes the limit of the evolutions of the ε-problem at curved initial conditions?
(2) How dependent is the limiting motion on the exact (well-prepared) initial condition?
(3) Do the same results hold for more general distributions of obstacles, or can other phenomena occur for less regularly distributed (or moving) obstacles?

Furthermore, our methods used the rotational symmetry of the fractional Laplacian and the fact that all functions we constructed were non-negative. We expect that these constraints could be eliminated. We also believe that a more explicit characterisation of admissible potentials W should be available.

Appendix A. Fractional Evolution Equations

The gradient flow of the energy $\mathcal{E}_\varepsilon$ is given by the fractional parabolic equation

$$\text{(A.1)} \qquad \begin{cases} c_\varepsilon \varepsilon\, u_t &= \frac{1}{|\log\varepsilon|}\left(\mathrm{A}\,u - W'(u)\right) & t>0,\ x \in \mathbb{T}^2 \setminus \bigcup_{i=1}^{N_\varepsilon} \overline{B_\varepsilon(x_{i,\varepsilon})} \\ u &\equiv 0 & t \geq 0,\ x \in \bigcup_{i=1}^{N_\varepsilon} \overline{B_\varepsilon(x_{i,\varepsilon})} \\ u &= u^0 & t = 0 \end{cases}$$

with $c_\varepsilon \equiv 1$ which formally has the structure

$$\text{(A.2)} \qquad \begin{cases} u_t &= \mathrm{A}\,u + f(u) & t>0,\ x\in\Omega \\ u &\equiv 0 & t\geq 0,\ x \in \mathbb{T}^2\setminus\Omega \\ u &= u^0 & t=0,\ x\in\Omega \end{cases}$$

where $\mathrm{A} = -(-\Delta)^{1/2}$ is the fractional Laplacian of order $s = 1/2$ and f is a bounded Lipschitz function. Choosing constants $c_\varepsilon \ll 1$ corresponds to accelerating time to rescale slow motion of the gradient flows to the macroscopic time-scale. Since we derived the equation as the gradient flow of the energy $\mathcal{E}_\varepsilon$, the most natural concept of a solution is that of a weak solution in a Bochner space

$$W_\varepsilon = \left\{ u \in L^2\left([0,T], X_\varepsilon\right) \,\middle|\, \frac{\mathrm{d}u}{\mathrm{d}t} \in L^2\left([0,T], X_\varepsilon^*\right) \right\}$$

where again

$$X_\varepsilon := \{u_\varepsilon \in H^{1/2}(\mathbb{T}^2) \mid u_\varepsilon \equiv 0 \text{ on } B_\varepsilon(x_{i,\varepsilon}) \text{ for } 1 \leq i \leq N_\varepsilon\}.$$

It is well-known that the operator $A : H^{1/2}(\mathbb{T}^2) \to H^{-1/2}(\mathbb{T}^2)$ (and also $A : X_\varepsilon \to X_\varepsilon^*$) is monotone. Furthermore, if u solves (A.2), we note that $v(t,\cdot) = e^{-\lambda t}$ solves the same equation with f replaced by

$$f_\lambda(v) = e^{-\lambda t} f\left(e^{\lambda t} v\right) + \lambda v$$

which can be made monotone increasing for large enough λ as f is Lipschitz. The existence of a weak solution v follows by the theory of monotone operators. Reversing the modification, we obtain a weak solution $u = e^{\lambda t} v$ in the Bochner space W_ε.

If u is smooth, $0 \leq u_0 \leq 1$, the comparison principle implies that $u \leq 1$ for all times and the non-linearity $f = -W'$. We show that remains true for weak solutions.

Since f is bounded, we have $f(u) \in L^\infty(\Omega)$ and [FRRO17] implies that u is Hölder continuous up to the boundary (note that the concept of a weak solution used in that article is weaker than ours). Hence $f(u)$ is also Hölder continuous, which means that u is locally $C^{1,\alpha}$-smooth in Ω and $C^{0,1/2}$-continuous on $\mathbb{T}^2$. While the results of [FRRO17] are given in domains in $\mathbb{R}^n$, the proofs also apply in the periodic case.

In particular, the weak solutions are classical and thus justify the usual calculations that imply a decrease of energy along the time evolution. The solutions are in particular viscosity solutions and we may construct viscosity sub- and super-solutions to understand their behaviour.

Interestingly, the regularity results apply more easily if f is bounded a priori. However, if $W(z) = (z^2-1)^2$ is the usual double-well potential and the initial condition lies between -1 and 1, we may modify W outside $[-1,1]$ to become bounded Lipschitz. In a second step, we may apply the maximum principle to deduce that solutions remain between ± 1, which means that the solution to the modified problem is actually also a solution to the original problem – compare the proof of Lemma 3.1.

We have also investigated solutions of the evolution equation on the whole real line or in the plane. By the same arguments as above, weak solutions exist if the initial condition happens to lie in the space

$$\left\{ u \in L^2(\mathbb{R}^n) \;\middle|\; u \equiv 0 \qquad \text{on } \bigcup_{i=1}^\infty B_\varepsilon(x_{i,\varepsilon}) \right\}.$$

Here we use that we could modify f to become monotone and use a monotone Nemickij operator rather than having to pass to the theory of pseudomonotone operators, where the lack of compactness in the embedding for the whole space problem causes additional challenges.

When we consider solutions to the evolution equation (A.1) on the real line with initial conditions approximating a single step function, on the other hand, we need to understand solutions in the viscosity sense. On the whole space, the theory of viscosity solutions for fractional evolution equations is developed [Imb05, DI06, BI08] and the pinning constraint could be included in the proof of the maximum principle by the doubling of variables in the standard way – see e.g. [Imb05, Theorem 2]. Existence can then be proved using Perron's method.

Consider the Bochner space W over $\Omega \subset \mathbb{T}^2$ and the particular case of a non-linearity f which satisfies $f(1) = 0$, $f < 0$ on $(1,\infty)$ and f is constant close to ∞. Assume further that we have an initial condition $u_0 \leq 1$. Now consider $u_+ := \max\{u, 1\}$. Since $u \in C^0([0,T], L^2(\mathbb{T}^2))$ by embedding theorems $u_+(t,\cdot)$ is well-defined in $L^2(\mathbb{T}^2)$ and we can calculate the integral

$$\left(\int_{\mathbb{T}^2} (u_+)^2 \,\mathrm{d}x\right)(t)$$

pointwise in time. Due to Bochner-space theory, smooth functions are dense in W and we can consider a sequence of functions $u_n \in C^0([0,T) \times \mathbb{T}^2) \cap C^2((0,T) \times \Omega)$ such that $u_n \equiv 0$ on $\mathbb{T}^2 \setminus \Omega$

such that $u_n \to u$ in W. In particular this convergence implies

$$\left(\int_{\mathbb{T}^2} (u_n)_+^2 \,\mathrm{d}x\right)(0) \to 0.$$

Take (t,x) such that $u_n(t,x) > 1$. By continuity, $u_n > 1$ in a neighbourhood of the point, and the Laplacian can be calculated pointwise as a singular integral

$$[\mathrm{A}\, u_{n,+}]\,(t,x) = \int_\Omega \frac{u_{n,+}(t,y) - u_{n,+}(t,x)}{|x-y|^3}\,\mathrm{d}y \geq \int_\Omega \frac{u_n(t,y) - u_n(t,x)}{|x-y|^3}\,\mathrm{d}y = [\mathrm{A}\, u_n]\,(t,x)$$

since $u_{n,+}(t,y) \geq u(t,y)$ and $u_{n,+}(t,x) = u_{n,+}(t,x)$. On the other hand, if $u_{n,+}(t,x) = 1$, then $u_{n,+}$ is minimal at (t,x) and thus $\mathrm{A}\, u_{n,+} \geq 0$ at (t,x) in the distributional sense. It follows that

$$(\partial_t - \mathrm{A})\, u_{n,+} \geq \chi_{\{u_n > 1\}} \cdot (\partial_t - \mathrm{A})\, u_n.$$

Since u_+ is smooth enough to be a Sobolev function, it also lies in W and we compute

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} (u_n)_+^2 \,\mathrm{d}x\right)(t) &= \left(\int_{\mathbb{R}^n} (u_n)_+^2 \,\mathrm{d}x\right)(0) + 2\int_0^t \langle [\partial_t - \mathrm{A} + \mathrm{A}]\,(u_n)_+, (u_n)_+\rangle_{X^*,X}\,\mathrm{d}s \\
&\leq \left(\int_{\mathbb{R}^n} (u_n)_+^2 \,\mathrm{d}x\right)(0) + 2\int_0^t \langle [\partial_t - \mathrm{A}]\,(u_n)_+, (u_n)_+\rangle_{X^*,X}\,\mathrm{d}s \\
&\leq \left(\int_{\mathbb{R}^n} (u_n)_+^2 \,\mathrm{d}x\right)(0) + 2\int_0^t \langle f(u_{n,+}), (u_n)_+\rangle_{L^2,L^2} + \langle \eta_n, u_{n,+}\rangle_{X^*,X}\,\mathrm{d}s \\
&\leq \left(\int_{\mathbb{R}^n} (u_n)_+^2 \,\mathrm{d}x\right)(0) + 2\int_0^t \langle \eta_n, u_{n,+}\rangle_{X^*,X}\,\mathrm{d}s \\
&\to 0
\end{aligned}$$

since $u_n \to u$ strongly in W, thus in particular $f(u_n) \to f(u)$ strongly as well. This implies that $u \leq 1$ for all times and thus the solutions are classical for positive times. The same argument shows $u \geq 0$ and a slight modification implies comparison with a stationary sub- or super-solution.

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Ethical Statement

The authors declare that they have no conflict of interest.

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