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A COMPLETE SET OF INDEPENDENT AND PHYSICALLY MEANINGFUL INVARIANTS IN THE MECHANICS OF SOLIDS REINFORCED BY TWO FAMILIES OF FIBRES

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Abstract

It has recently been shown [2, 3] that only seven of the classical deformation invariants employed in hyperelasticity of solids reinforced by two families of unidirectional fibres are independent. This short communication demonstrates a manner in which such a set of seven invariants is conveniently identified without much deviation from well-known features that characterise their classical counterparts. It also shows that, unlike several of their classical counterparts, these newly identified invariants have all their own physical meaning. This new development is immediate applicable on mass-growth problems of tissue that preserve fibre direction [1] and, notably, on problems involving mass-growth of a circular tube reinforced by two families of helices wound symmetrically around the tube in opposite directions.

Keywords: Deformation invariants, Complete set of invariants, Independent invariants, Hyperelastic deformations, Hyperelastic mass-growth, Two families of fibres.

1. Introduction

Mass-growth of a fibre-reinforced circular cylindrical tube can naturally be modelled with the use of known principles employed in the theory of hyperelastcity. It is thus recently shown [1] that, if mass-growth preserves the direction and shape of a single family of fibres embedded in, and growing with and within the tube, then the corresponding set of hyperelasticity type equations appears overdetermined. However, the extra equations emerging in the model enable formation of one or more linear relationships between the strain energy density for growth, *W*, and its derivatives with respect to the principal deformation invariants of the growing system. These relationships are thus regarded as partial differential equations for the unknown function *W*, and their solution provides valuable information regarding admissible classes or forms of *W* that enable the tube to grow without disturbing the shape and direction of its fibres.

The outlined new development and its concepts are naturally expected to apply on, and, hence, become of interest and importance in hyperelastic mass-growth problems of several types of soft or hard biological tissue. These include fibre-reinforced tissue of different geometrical shapes and features, and/or tissue reinforced by more than one unidirectional family of fibres.

However, if the number, *n* say, of fibre families embedded in a hyperelastic material is bigger than one, then the classical deformation invariants involved in the strain energy density are not any more independent (e.g., [2, 3]). It follows that relationships between *W* and its derivatives that, in analogy with [1] would reflect mass-growth ability of the material to preserve fibre direction, can be considered as partial differential equations for *W* only after: (i) the exact number, *m* say, of independent invariants is identified; (ii) a manner is found for a complete basis of precisely *m* independent invariants to be formed; and (iii) *W* is considered as function of those m independent invariants only.

Part (i) of this challenge is already dealt with by Shariff [2] who showed further that, for $n \geq 2$, the number of independent invariants is given by the formula $m = 2n + 3$. Moreover, Shariff [2] showed that six of those $2n + 3$ independent invariants are strain, while the remaining are $2n - 3$ non-strain invariants. However, parts (ii) and (iii) of the outlined challenge are still issues of ongoing debate, and their resolution may well depend on special features of a particular elasticity problem of interest.

This short communication is motivated by its author's interest to extend the analysis detailed in [1] towards hyperelastic mass-growth modelling of tissue reinforced by two or more families of fibres. It accordingly considers the particular but practically important case of tissue reinforced by two unidirectional families of fibres ($n = 2$) and, based on the aforementioned useful result [2, 3], demonstrates a manner in which a complete set of six independent strain invariants can conveniently be identified without much deviation from well-known features that characterise their classical counterparts. The analysis is demonstrated in Cartesian co-ordinates (Section 2), and shows that each one of these newly identified independent invariants has its own physical meaning. Section 3 then shows that a suitable extension to cylindrical polar co-ordinates makes these findings directly applicable to hyperelasticity problems (of either mechanical or mass-growth nature) of a tube reinforced by two families of helical fibres wound around the tube symmetrically in opposite directions.

2. Identification of a suitable set of independent invariants

It is initially observed that the aforementioned formula [2], namely $m = 2n + 3$, is applicable not only for $n \geq 2$, but also in the case of hyperelastic materials reinforced by a single family of fibres. Indeed, for $n = 1$, that formula gives the right number of independent deformation invariants that W is dependent on in the case of transversely isotropic hyperelasticity. This classical set of invariants is

$$\begin{aligned} &I_1 = \mathrm{tr}\boldsymbol{C}, \quad I_2 = \frac{1}{2}\left\{\left(\mathrm{tr}\boldsymbol{C}\right)^2 - \mathrm{tr}\boldsymbol{C}^2\right\}, \\ &I_3 = \det\boldsymbol{C}, \quad I_4 = \boldsymbol{A}^T\boldsymbol{C}\boldsymbol{A}, \quad I_5 = \boldsymbol{A}^T\boldsymbol{C}^2\boldsymbol{A}, \end{aligned} \tag{1}$$

where $\boldsymbol{C}$ represents the right Cauchy-Green deformation tensor, and the unit vector $\boldsymbol{A}$ defines the fibre direction. Not all of the invariants listed in (1) have a clear physical meaning but their number ($m = 2\times1 + 3 = 5$) is smaller than six. These are all strain invariants. Hence, as is correctly implied in [2], the remaining, and certainly more important part of Shariff's claim applies only for $n \geq 2$.

The set of classical invariants employed in the present case of interest ($n = 2$) consists of the set listed in (1) augmented by the following:

$$\begin{aligned} &I_6 = \boldsymbol{B}^T\boldsymbol{C}\boldsymbol{B}, \quad I_7 = \boldsymbol{B}^T\boldsymbol{C}^2\boldsymbol{B}, \quad I_8 = \boldsymbol{A}^T\boldsymbol{C}\boldsymbol{B}, \\ &I_9 = \boldsymbol{A}\boldsymbol{B}, \quad I_{10} = \boldsymbol{A}^T\boldsymbol{C}^2\boldsymbol{B}, \end{aligned} \tag{2}$$

where the unit vector $\boldsymbol{B}$ ($\neq \pm\boldsymbol{A}$) defines the fibre direction of the second fibre family. It is noted in passing that, if the sense of the fibres is not significant, then I_8 and I_{10} are customarily multiplied by I_9 and, hence, become even in $\boldsymbol{A}$ and $\boldsymbol{B}$ (e.g., [3, 4]). It will be seen later though (see (6) below) that the present development does not require such a refinement of I_8 and I_{10}.

I_{10} is already known and/or claimed redundant (e.g., [2-6]), and will be treated as such in what follows. Suffice it here to mention the existence of the following syzygy:

$$\begin{aligned} 2I_9I_{10} = \left(1 - I_9^2\right)I_2 + 2I_1I_9I_8 + I_4I_6 \\ - I_1\left(I_4 + I_6\right) + I_5 + I_7 - I_8^2. \end{aligned} \tag{3}$$

This is a slightly modified version of the syzygy (30) of [3], obtained there through a detailed cumbersome algebraic process. The implied slight modification applied on that syzygy [3] restored its consistency with the presently used forms of I_8 and I_{10}. It is also pointed out for later use that, with the exception of I_3, (3) involves all the remaining invariants listed in (1) and (2).

When expressed explicitly in a generic Cartesian co-ordinate system Ox_i, each of the ten classical invariants (1) and (2) forms an expression of the six strain components of $\boldsymbol{C}$, and the six non-strain components of $\boldsymbol{A}$ and $\boldsymbol{B}$. Each of the latter is a unit vector and, hence, only two of its three components are independent. With I_{10} known to be redundant, a possible total number of ten independent components of $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ still exceeds the number (nine) of the invariants remaining in (1) and (2). However, the arguments detailed in [2, 3] make it clear that only seven of the invariants listed in (1) and (2) can be independent, while only one of those seven can be a non-strain invariant; notably I_9.

A question then arises whether a particular Cartesian co-ordinate system, $O\hat{x}_i$ say, can be identified, in which all the components of $\boldsymbol{A}$ and $\boldsymbol{B}$ are expressible in terms of the single non-strain invariant, I_9. In such a particular co-ordinate system, each of the ten classical invariants listed in (1) and (2) will naturally become expressible in terms of seven parameters, namely I_9 and the six strain components of the particular form attained by the symmetric Cauchy-Green deformation tensor, $\hat{\boldsymbol{C}}$ say.

It follows that, after the non-strain invariant I_9 and the redundant invariant I_{10} are temporarily excluded, inversion of a suitably chosen set of six, out of the eight remaining equations, may yield the components of $\hat{\boldsymbol{C}}$ in terms of the invariants listed in (1) and (2). The six strain components of $\hat{\boldsymbol{C}}$ will then also be identified as invariants of the deformation and, along with the non-strain invariant I_9, will naturally form a set of seven independent strain invariants, analogous to those suggested by in [2, 3]. Moreover, with $\hat{\boldsymbol{C}}$ being itself a Cauchy-Green deformation tensor, every member of such a newly formed basis of independent invariants will naturally acquire a straightforward physical meaning.

It will be thus shown next that such a suitable Cartesian frame is the co-ordinate system having its $\hat{x}_1$-axis normal to the plane of $\boldsymbol{A}$ and $\boldsymbol{B}$, and its $\hat{x}_3$-axis parallel to the bisector of the acute angle, 2Φ, formed by this pair of unit vectors (so that $0 < \Phi < \pi/4$). In this particular co-ordinate system, the unit vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ acquire the form

$$\hat{\boldsymbol{A}} = \left(0, \hat{A}_2, \hat{A}_3\right)^T = \left(0, \sin\Phi, \cos\Phi\right)^T,$$
$$\hat{\boldsymbol{B}} = \left(0, -\sin\Phi, \cos\Phi\right)^T = \left(0, -\hat{A}_2, \hat{A}_3\right)^T, \tag{4}$$

and their non-zero components are given in terms of the non-strain invariant, $I_9 = \cos 2\Phi$, as follows:

$$\hat{A}_2 = \frac{1}{\sqrt{2}}\left(1 - I_9\right)^{1/2} > 0,$$
$$\hat{A}_3 = \frac{1}{\sqrt{2}}\left(1 + I_9\right)^{1/2} > 0. \tag{5}$$

By temporarily excluding from the discussion the invariants I_2, I_3 and I_9, one can then express the remaining invariants as follows:

$$\begin{aligned}
I_1 &= \hat{C}_{11} + \hat{C}_{22} + \hat{C}_{33},\\
I_4 &= \hat{A}_2^2\hat{C}_{22} + 2\hat{A}_2\hat{A}_3\hat{C}_{23} + \hat{A}_3^2\hat{C}_{33},\\
I_5 &= \hat{A}_2^2\left(\hat{C}_{21}^2 + \hat{C}_{22}^2 + \hat{C}_{23}^2\right)\\
&\quad + 2\hat{A}_2\hat{A}_3\left(\hat{C}_{21}\hat{C}_{31} + \hat{C}_{22}\hat{C}_{32} + \hat{C}_{23}\hat{C}_{33}\right) + \hat{A}_3^2\left(\hat{C}_{13}^2 + \hat{C}_{23}^2 + \hat{C}_{33}^2\right),\\
I_6 &= \hat{A}_2^2\hat{C}_{22} - 2\hat{A}_2\hat{A}_3\hat{C}_{23} + \hat{A}_3^2\hat{C}_{33},\\
I_7 &= \hat{A}_2^2\left(\hat{C}_{21}^2 + \hat{C}_{22}^2 + \hat{C}_{23}^2\right)\\
&\quad - 2\hat{A}_2\hat{A}_3\left(\hat{C}_{21}\hat{C}_{31} + \hat{C}_{22}\hat{C}_{32} + \hat{C}_{23}\hat{C}_{33}\right) + \hat{A}_3^2\left(\hat{C}_{13}^2 + \hat{C}_{23}^2 + \hat{C}_{33}^2\right),\\
I_8 &= \hat{A}_3^2\hat{C}_{33} - \hat{A}_2^2\hat{C}_{22},\\
I_{10} &= \hat{A}_3^2\left(\hat{C}_{13}^2 + \hat{C}_{23}^2 + \hat{C}_{33}^2\right) - \hat{A}_2^2\left(\hat{C}_{21}^2 + \hat{C}_{22}^2 + \hat{C}_{23}^2\right).
\end{aligned} \tag{6}$$

It is noted that I_8 and I_{10} are even in the components of $\hat{\boldsymbol{A}}$ and/or $\hat{\boldsymbol{B}}$. The aforementioned refinement of I_8 and I_{10} (e.g., [3, 4]) is thus not required by the present development.

By virtue of (4), (6) may initially be thought of as a set of seven simultaneous algebraic equations for the six strain components of the symmetric tensor $\hat{\boldsymbol{C}}$. However, the syzygy (3) reveals that I_{10} depends on the remaining invariants listed in (1) and (2), with I_3 excluded. Hence, (6) is essentially equivalent to a set of six simultaneous algebraic equations for the six independent components of $\hat{\boldsymbol{C}}$.

Inversion of (6) leads thus to the following set of new invariants:

$$
\begin{aligned}
J_1 &\equiv \hat{C}_{11} = I_1 + 2\left(I_8 I_9 - I_4 - I_6\right)^2 \left(1 - I_9^2\right)^{-1}, \\
J_2 &\equiv \hat{C}_{22} = \left(I_4 + I_6 - I_8\right)\left(1 - I_9\right)^{-1}, \\
J_3 &\equiv \hat{C}_{33} = \left(I_4 + I_6 + I_8\right)\left(1 + I_9\right)^{-1}, \\
J_4 &\equiv \hat{C}_{23} = 2\left(I_4 - I_6\right)\left(1 - I_9^2\right)^{-1/2}, \\
J_5 &\equiv \hat{C}_{31}^2 = \left(1 + I_9\right)^{-1}\left[I_{10} + \left(I_5 + I_7\right)/2\right] \\
&\qquad - \left(J_3^2 + J_4^2\right), \\
J_6 &\equiv \hat{C}_{12}^2 = \left(1 - I_9\right)^{-1}\left[-I_{10} + \left(I_5 + I_7\right)/2\right] \\
&\qquad - \left(J_2^2 + J_4^2\right), \\
J_7 &= I_9,
\end{aligned}
\tag{7}
$$

which, in agreement with [2, 3], consists six strain invariants and one non-strain invariant. Moreover, in agreement with [7, 8], the number of these invariants reduces naturally to six in the particular case that ***A*** and ***B*** are orthogonal ($J_7 = I_9 = 0$).

Evidently, each of J_1 – J_3 represents amount of stretch along the direction of the corresponding $\hat{x}_i$-axis. On the other hand, each of the remaining strain invariants, namely $J_4 – J_6$, is a measure of the amount of shear encountered on the co-ordinate plane noted by the indices of the corresponding non-diagonal component of $\hat{\boldsymbol{C}}$. Their independence is thus underpinned by the fact that each of these six strain invariants can be activated or controlled independently from the others, by means of appropriately chosen homogeneous deformation(s) (e.g., [5, 6, 9]).

It is pointed out that non-strain invariants, like J_7, never enter on their own any form of a strain energy density, W. Hence, the fact that all six of J_1 – J_6 depend on J_7 is a rule rather than an exception.

It is now recalled that neither I_2 nor I_3 were included in (6), while, by virtue of (1), the following relationships still hold:

$$
I_2 = \frac{1}{2}\left\{\left(\mathrm{tr}\hat{\boldsymbol{C}}\right)^2 - \mathrm{tr}\hat{\boldsymbol{C}}^2\right\}, \quad I_3 = \det\hat{\boldsymbol{C}}. \tag{8}
$$

Moreover, I_3 is absent in (3) and is thus also absent in all new invariants listed in (7).

It follows that by inserting (7) into the right hand side of (8.b) one more syzygy can be obtained and, along with (3), relate the classical invariants listed in (1) and (2). That syzygy will be analogous to its counterpart implied in [3], through lengthy intermediate relevant calculations detailed there; see Section 3 of [3], after (37).

Another syzygy that relates the classical invariants may also be obtained by inserting the components of $\hat{\boldsymbol{C}}$ listed in (7) into (8.a). That syzygy will evidently be also independent of I_3. The considerable amount of algebra required for derivation of the explicit form of such lengthy syzygy makes the latter impractical, and certainly unnecessary for the purposes of the present short communication. However, its potential length suggests that, rather than (3), that syzygy is analogous to the third syzygy implied in [3], through the intermediate formulas (34)-(36), (29) and (28.a) presented there along with other necessary relevant calculations.

3. Cylindrical polar co-ordinates: Tube reinforced by a pair of helical families of fibres

As is mentioned in the Introduction, this short communication stemmed from the author’s interest to extend the hyperelastic mass-growth modelling presented in [1] towards mass-growth of tube-like tissue that preserves the shape and direction of two families of unidirectional fibres. In particular, tubes reinforced by two families of continuously distributed helical fibres wound symmetrically in opposing directions are met very often in several different kinds of plant and bone structures (e.g., [10, 11]), as well as in various forms of tube-like soft or hard biological tissue. Arteries and veins (e.g., [12-15]), muscles (e.g., [16]), and even living creatures of tubular shape [17] are referred to as well-known relevant examples.

Moreover, the use of this particular kind of tube fibre-reinforcement is long known in the construction of a variety of common man made articles, such as tyres and fire hose. Tube reinforcement with helical fibres has, therefore, been of paramount importance in both the foundation and development of the non-linear theory of fibre-reinforced materials (e.g. [18, 19]). As tubes reinforced in this particular manner are thus so commonly met in nature and practice, it is almost not surprising that the aforementioned analysis, results and conclusions are directly applicable to their case.

Consider in this context that, in its undeformed configuration, the circular cylindrical tube of interest has axial length $2H$ and occupies the region

$$A \le R \le B, \;\; 0 \le \Theta < 2\pi, \;\; -H < Z < H, \tag{9}$$

where, R, Θ and Z are standard cylindrical polar coordinates, and the non-negative constants A and B ($0 \le A < B$) represent its inner and the outer radii (e.g., [1]). In that cylindrical polar co-ordinate system the unit vectors implied in (4) acquire the form

$$\begin{aligned} \hat{\boldsymbol{A}} &= \left(0, \hat{A}_\Theta, \hat{A}_Z\right)^T = \left(0, \sin\Phi, \cos\Phi\right)^T, \\ \hat{\boldsymbol{B}} &= \left(0, -\sin\Phi, \cos\Phi\right)^T = \left(0, -\hat{A}_\Theta, \hat{A}_Z\right)^T, \end{aligned} \tag{10}$$

thus showing that 2Φ represents here the angle between a pair of helical fibres wound symmetrically in opposing directions.

If Φ is an acute angle, then the remaining of the analysis detailed in the preceding section holds still, provided that all indices 1, 2 and 3 appearing there in the components of $\hat{\boldsymbol{A}}$, $\hat{\boldsymbol{B}}$ and $\hat{\boldsymbol{C}}$ are replaced by R, Θ and Z, respectively. If Φ is an obtuse angle, then (10) should be replaced by the following

$$\begin{aligned} \hat{\boldsymbol{A}} &= \left(0, \hat{A}_\Theta, \hat{A}_Z\right)^T = \left(0, \cos\Phi, \sin\Phi\right)^T, \\ \hat{\boldsymbol{B}} &= \left(0, -\cos\Phi, \sin\Phi\right)^T = \left(0, -\hat{A}_\Theta, \hat{A}_Z\right)^T, \end{aligned} \tag{11}$$

thus implying that the sign of I_9 should further be reversed everywhere in (5) and (7).

The cylindrical polar co-ordinate system (R, Θ, Z) is thus seen to be particularly privileged in modelling and solving hyperelasticity problems concerning tubes reinforced by two families of helical fibres wound symmetrically in opposing directions. This is because, when deformation (or mass-growth) of such a tube is modelled in cylindrical polar co-ordinates, each of the components of the Cauchy-Green deformation tensor represents one of the six independent strain invariants, J_1 – J_6, defined in (7). The seventh independent, non-strain invariant still represents the cosine of the angle formed by the two families of helical fibres.

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