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RESEARCH ARTICLE

Fano 3-folds in $\mathbb{P}^2 \times \mathbb{P}^2$ format, Tom and Jerry

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Abstract We study $\mathbb{Q}$-factorial terminal Fano 3-folds whose equations are modelled on those of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$. These lie in codimension 4 in their total anticanonical embedding and have Picard rank 2. They fit into the current state of classification in three different ways. Some families arise as unprojections of degenerations of complete intersections, where the generic unprojection is a known prime Fano 3-fold in codimension 3; these are new, and an analysis of their Gorenstein projections reveals yet other new families. Others represent the "second Tom" unprojection families already known in codimension 4, and we show that every such family contains one of our models. Yet others have no easy Gorenstein projection analysis at all, so prove the existence of Fano components on their Hilbert scheme.

Keywords Fano 3-fold · Segre embedding · Gorenstein format

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1 Introduction

1.1 Fano 3-folds, Gorenstein rings and $\mathbb{P}^2 \times \mathbb{P}^2$

A *Fano 3-fold* is a complex projective variety X of dimension 3 with $\mathbb{Q}$-factorial terminal singularities and $-K_X$ ample. We construct several new Fano 3-folds, and others which explain known phenomena. The anticanonical ring

$$R(X) = \bigoplus_{m\in\mathbb{N}} H^0(X, -mK_X)$$

of a Fano 3-fold X is Gorenstein, and provides an embedding $X \subset \mathrm{w}\mathbb{P}$ in weighted projective space (wps) that we exploit here, focusing on the case $X \subset \mathrm{w}\mathbb{P}^7$ of codimension 4.

According to folklore, when seeking Gorenstein rings in codimension 4 one should look to $\mathbb{P}^2 \times \mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Each embeds by the Segre embedding as a projectively normal variety in codimension 4 with Gorenstein coordinate ring (by [16, Section 5] since their hyperplane sections are subcanonical). We consider $W = \mathbb{P}^2 \times \mathbb{P}^2$, expressed as

$$W \xrightarrow{\cong} V : \left(\bigwedge^2 \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} = 0 \right) \subset \mathbb{P}^8, \tag{1}$$

or, in words, as the locus where a generic 3×3 matrix of forms drops rank. As part of a more general theory of weighted homogeneous varieties, the case of $\mathbb{P}^2 \times \mathbb{P}^2$ was worked out by Szendrői [32], which was the inspiration for our study here.

The number of deformation families of Fano 3-folds is finite [20,21], and the Graded Ring Database (GRDB) [4,6] has a list of rational functions $P(t)$ that includes all Hilbert series $P_X(t) = \sum_{m\in\mathbb{N}} h^0(-mK_X)t^m$ of Fano 3-folds with $\mathrm{Pic}(X) = \mathbb{Z}\cdot(-K_X)$. (In fact, we do not know of any Fano 3-fold whose Hilbert series is not on that list, even without this additional condition.) An attempt at an explicit classification, outlined in [2], aims to describe all deformation families of Fano 3-folds for each such Hilbert series. All families whose general member lies in codimension $\leqslant 2$ are known [12], and almost certainly those in codimension 3 are too [2,6]. An analysis of (Gorenstein) projections [8,24,34] provides much of the classification in codimension 4, but it is not complete, and codimension 4 remains at the cutting edge.

We use the methods of [8] freely, although we work through an example in detail in Sect. 3 and explain any novelties as they arise.

1.2 The aims of this paper

We describe families of Fano 3-folds $X \subset \mathrm{w}\mathbb{P}^7$ whose equations are a specialisation of the format (1); that is, they are regular pullbacks, as in Sect. 2. It is usually hard to describe the equations of varieties in codimension 4—see papers from Kustin and Miller [22] to Reid [31]—but if we decree the format in advance, then the equations come almost for free, and the question becomes how to put a grading on them to give

Fano 3-folds. Our results come in three broad flavours, which we explain in Sects. 4–6 and summarise here.

Section 4: **Unprojecting Pfaffian degenerations.** We find new varieties in $\mathbb{P}^2 \times \mathbb{P}^2$ format that have the same Hilbert series as known Fano 3-folds but lie in different deformation families. From another point of view, we understand this as the unprojection analysis of degenerations of complete intersections, and this treatment provides yet more families not exhibited by [8]. (The key point is that the unprojection divisor $D \subset Y$ does not persist throughout the degeneration $Y \rightsquigarrow Y_0$, and so the resulting unprojection is not a degeneration in a known family.)

For example, No. 1.4 in Takagi's analysis [33] exhibits a single family of Fano 3-folds with Hilbert series

$$P_{26989}(t) = \frac{1 - 3t^2 - 4t^3 + 12t^4 - 4t^5 - 3t^6 + t^8}{(1-t)^7(1-t^2)} = 1 + 7t + 26t^2 + 66t^3 + \cdots ;$$

this is number 26989 in the GRDB. Our $\mathbb{P}^2 \times \mathbb{P}^2$ analysis finds another family with $\rho_X = 2$, and a subsequent degeneration–unprojection analysis of the situation finds a third family.

Theorem 1.1 *There are three deformation families of Fano 3-folds X with Hilbert series $P_X = P_{26989}$. Their respective general members $X \subset \mathbb{P}(1^7, 2)$ all lie in codimension 4 with degree $-K_X^3 = 17/2$ and a single orbifold singularity $\frac{1}{2}(1,1,1)$, and with invariants:*

	ρ_X	$e(X)$	$h^{2,1}(X)$	Construction	N
Family 1 [33, 1.4]	1	− 14	9	Sect. 3.1: c.i. unprojection	6
Family 2	2	− 16	11	Sect. 3.2: Tom_3	5
Family 3	2*	− 12	9*	Sect. 3.3: $\text{Jer}_{1,3}$	7

(The superscript * in Family 3 indicates a computer algebra calculation.)

We prove this particular result in Sect. 3; the last two columns of the table refer to the unprojection calculation (N is the number of nodes, as described in Sect. 3), which is explained in the indicated sections. The Euler characteristic $e(X)$ is calculated during the unprojection following [8, Section 7] and the other invariants follow. We do not know whether there are any other deformation families realising the same Hilbert series $P_X = P_{26989}(t)$.

We calculate the Hodge number $h^{2,1}(X)$ in Family 3 using Ilten's computer package [19] for the computer algebra system Macaulay2 [17] following [15]: denoting the affine cone over X by A_X, [15, Theorem 2.5] gives

$$H^{2,1}(X) \cong \left(T^1_{A_X}\right)_{-1},$$

and this is exactly what [19] calculates (compare [5, Section 4.1.3]).

In this case, all three families lie in codimension 4. It is more common that the known family lies in codimension 3 and we find new families in codimension 4.

Thus the corresponding Hilbert scheme contains different components whose general members are Fano 3-folds in different codimensions, a phenomenon we had not seen before.

Further analysis of degenerations finds yet more new Fano 3-folds even where there is no $\mathbb{P}^2 \times \mathbb{P}^2$ model; the following result is proved in Sect. 4.2.

Theorem 1.2 *There are two deformation families of Fano 3-folds X with Hilbert series $P_X = P_{548}$. Their respective general members X have degree $-K_X^3 = 1/15$, and are distinguished by their embedding in wps and Euler characteristics as follows:*

	$X \subset \mathrm{w}\mathbb{P}$	$e(X)$	# nodes
Family 1	$X \subset \mathbb{P}(1, 3, 4, 5, 6, 7, 10)$	-42	8
Family 2	$X \subset \mathbb{P}(1, 3, 4, 5, 6, 7, 9, 10)$	-40	9

In this case there is no $\mathbb{P}^2 \times \mathbb{P}^2$ model: such a model would come from a specialised Tom unprojection, but the Tom and Jerry analysis outlined in Sect. 4.2 rules this out.

Section 5: **Second Tom.** The Big Table [9] lists all (general) Fano 3-folds in codimension 4 that have a Type I projection. Such projections can be of Tom type or Jerry type (see [8, 2.3]). The result of that paper is that every Fano 3-fold admitting a Type I projection has at least one Tom family and one Jerry family. However in some cases there is a second Tom or second Jerry (or both). Two of these cases were already known to Szendrői [32], even before the Tom and Jerry analysis was developed.

Euler characteristic is of course constant in families, but whenever there is a second Tom, the Euler characteristics of members of the two Tom families differ by 2. Theorem 5.1 below says that in this case the Tom family with smaller Euler characteristic always contains special members in $\mathbb{P}^2 \times \mathbb{P}^2$ format.

Section 6: **No Type I projection.** Finally, we find some Fano 3-folds that are harder to describe, including some that currently have no construction by Gorenstein unprojection. Such Fano 3-folds were expected to exist, but this is the first construction of them in the literature we are aware of. It may be the case that there are other families of such Fano 3-folds having Picard rank 1, but our methods here cannot answer that question.

1.3 Summary of results

Our approach starts with a systematic enumeration of all possible $\mathbb{P}^2 \times \mathbb{P}^2$ formats that could realise the Hilbert series of a Fano 3-fold after appropriate specialisation. In Sect. 7, following [7,27], we find 53 varieties in $\mathbb{P}^2 \times \mathbb{P}^2$ format that have the Hilbert series of a Fano 3-fold. We summarise the fate of each of these 53 cases in Table 1; the final column summarises our results, as we describe below, and the rest of the paper explains the calculations that provide the proof.

The columns of Table 1 are as follows. Column k is an adjunction index, described in Sect. 7.1, and columns a and b refer to the vectors in Sect. 2 that determine the weights on the weighted $\mathbb{P}^2 \times \mathbb{P}^2$. Column GRDB lists the number of the Hilbert series

Table 1 53 Fano 3-fold Hilbert series in $\mathbb{P}^2\times\mathbb{P}^2$ format (number of nodes is given as a superscript to Tom/Jer)

k	a	b	GRDB	c	T/J	w$\mathbb{P}$ in GRDB	codim 4 models in this paper
4	000	112	26989	4		$\mathbb{P}(1^7,2)$	Tom^5, Jer^7 in $\mathbb{P}(1^7,2)$
5	000	122	20652	4	TTJ	$\mathbb{P}(1^5,2^3)$	Second Tom
5	001	112	20543	3	n/a	$\mathbb{P}(1^5,2^2)$	Tom^7, Jer^9 in $\mathbb{P}(1^5,2^3)$
5	001	112	24078	4	TTJ	$\mathbb{P}(1^6,2,3)$	Second Tom
6	000	222	12960	4	TJ	$\mathbb{P}(1^3,2^5)$	Subfamily of Tom
6	001	122	16339	4	TTJJ	$\mathbb{P}(1^4,2^3,3)$	Second Tom
7	001	123	11436	3	n/a	$\mathbb{P}(1^3,2^3,3)$	Tom^{13} in $\mathbb{P}(1^3,2^3,3^2)$
7	001	123	16228	4	TTJJ	$\mathbb{P}(1^4,2^2,3,4)$	Second Tom
7	011	122	11455	4	TTJJ	$\mathbb{P}(1^3,2^3,3^2)$	Second Tom
8	001	223	11157	5	n/a	$\mathbb{P}(1^3,2^2,3^2,4^2)$	Bad 1/4 point
8	001	223	6878	4	TTJJ	$\mathbb{P}(1^2,2^3,3^3)$	Second Tom
8	011	123	11125	4	TTJJ	$\mathbb{P}(1^3,2^2,3^2,4)$	Second Tom
9	001	233	5970	4	TTJJ	$\mathbb{P}(1^2,2^2,3^3,4)$	Second Tom
9	012	123	11106	4	TTJJ	$\mathbb{P}(1^3,2^2,3,4,5)$	Second Tom
9	012	123	11021	4	TTJJ	$\mathbb{P}(1^3,2,3^2,4^2)$	Second Tom
9	012	123	5962	3	n/a	$\mathbb{P}(1^2,2^2,3^3)$	Tom^{11}, Jer^{13} in $\mathbb{P}(1^2,2^2,3^3,4)$
9	012	123	6860	4	TTJ	$\mathbb{P}(1^2,2^3,3^2,5)$	Second Tom
10	001	234	5870	4	TTJJ	$\mathbb{P}(1^2,2^2,3^2,4,5)$	Second Tom
10	011	233	5530	4	TTJJ	$\mathbb{P}(1^2,2,3^3,4^2)$	Second Tom
10	012	124	10984	3	n/a	$\mathbb{P}(1^3,2,3,4,5)$	Bad 1/4 point
10	012	124	5858	3	n/a	$\mathbb{P}(1^2,2^2,3^2,5)$	Tom^{13}, Jer^{14} in $\mathbb{P}(1^2,2^2,3^2,4,5)$
11	011	234	5306	4	TTJJ	$\mathbb{P}(1^2,2,3^2,4^2,5)$	Second Tom
11	012	134	5302	3	n/a	$\mathbb{P}(1^2,2,3^2,4^2)$	Tom^{16} in $\mathbb{P}(1^2,2,3^2,4^2,5)$
11	012	134	5844	3	n/a	$\mathbb{P}(1^2,2^2,3,4,5)$	Bad 1/6 point and no 1/5
11	012	134	10985	4	TTJJ	$\mathbb{P}(1^3,2,3,4,5,6)$	Second Tom
12	012	234	1766	4	no I	$\mathbb{P}(1,2,3^3,4^2,5)$	Quasismooth $\mathbb{P}^2\times\mathbb{P}^2$ model
12	012	234	5215	4	TTJJ	$\mathbb{P}(1^2,2,3,4^2,5^2)$	Second Tom
12	012	234	2427	4	TTJJ	$\mathbb{P}(1,2^2,3^2,4,5^2)$	Second Tom
12	012	234	5268	4	TTJJ	$\mathbb{P}(1^2,2,3^2,4,5,6)$	Second Tom
13	001	345	1413	4	TTJJ	$\mathbb{P}(1,2,3^2,4^2,5^2)$	Second Tom
13	012	235	5177	4	TJ	$\mathbb{P}(1^2,2,3,4,5^2,6)$	Bad 1/5 point
13	012	235	2422	4	TTJJ	$\mathbb{P}(1,2^2,3^2,4,5,7)$	Second Tom
14	011	345	5002	4	TTJJ	$\mathbb{P}(1^2,3,4^2,5^2,6)$	Second Tom
14	012	245	5163	4	TTJJ	$\mathbb{P}(1^2,2,3,4,5,6,7)$	Second Tom
14	012	245	1410	4	TJJ	$\mathbb{P}(1,2,3^2,4^2,5,7)$	Bad 1/4 point
14	013	235	4999	3	n/a	$\mathbb{P}(1^2,3,4^2,5^2,6)$	Bad 1/4 point
14	013	235	1396	3	n/a	$\mathbb{P}(1,2,3^2,4,5^2)$	Tom^9, Jer^{11} in $\mathbb{P}(1,2,3^2,4,5^2,6)$
15	012	345	878	4	no I	$\mathbb{P}(1,3^2,4^2,5^2,6)$	Quasismooth $\mathbb{P}^2\times\mathbb{P}^2$ model

Table 1 continued

k	a	b	GRDB	c	T/J	w$\mathbb{P}$ in GRDB	codim 4 models in this paper
15	012	345	4949	4	TTJJ	$\mathbb{P}(1^2, 3, 4, 5^2, 6^2)$	Second Tom
15	012	345	1253	4	TTJ	$\mathbb{P}(1, 2, 3, 4^2, 5^2, 7)$	Second Tom
15	012	345	1218	4	TTJJ	$\mathbb{P}(1, 2, 3, 4, 5^3, 6)$	Second Tom
15	012	345	4989	4	TTJJ	$\mathbb{P}(1^2, 3, 4^2, 5, 6, 7)$	Second Tom
16	012	346	1186	4	TJJ	$\mathbb{P}(1, 2, 3, 4, 5^2, 6, 7)$	Bad 1/5 point
17	012	356	648	4	no I	$\mathbb{P}(1, 3, 4^2, 5^2, 6, 7)$	Bad 1/5 point
17	012	356	4915	4	TTJJ	$\mathbb{P}(1^2, 3, 4, 5, 6, 7, 8)$	Second Tom
18	012	456	577	4	no I	$\mathbb{P}(1, 3, 4, 5^2, 6^2, 7)$	Quasismooth but not terminal
18	012	456	645	4	TJ	$\mathbb{P}(1, 3, 4^2, 5, 6, 7^2)$	Bad $1/4$ point
18	012	456	4860	4	TTJJ	$\mathbb{P}(1^2, 4, 5, 6^2, 7^2)$	Second Tom
19	012	457	570	4	TJJ	$\mathbb{P}(1, 3, 4, 5^2, 6, 7, 8)$	Bad 1/5 point
20	012	467	4839	4	TTJJ	$\mathbb{P}(1^2, 4, 5, 6, 7, 8, 9)$	Second Tom
22	012	568	1091	4	TJJ	$\mathbb{P}(1, 2, 5, 6, 7^2, 8, 9)$	Bad 1/7 point
22	012	568	393	4	TJ	$\mathbb{P}(1, 4, 5^2, 6, 7, 8, 9)$	Bad 1/4, 1/5 points
23	012	578	360	4	no I	$\mathbb{P}(1, 4, 5, 6, 7^2, 8, 9)$	Bad 1/7 point

in [6], column c indicates the codimension of the usual model suggested there, and w$\mathbb{P}$ its ambient space. Column T/J shows the number of distinct Tom and Jerry components according to [8]. For example, TTJ indicates there are two Tom unprojections and one Jerry unprojection in the Big Table [9]. We write 'no I' when the Hilbert series does not admit a numerical Type I projection, and so the Tom and Jerry analysis does not apply, and 'n/a' if the usual model is in codimension 3 rather than 4.

The final column describes the results of this paper; it is an abbreviation of more detailed results. For example, Theorem 1.1 expands out the first line of the table, $k = 4$, and other lines of the table that are not indicated as failing have analogous theorems that the final column summarises. If the $\mathbb{P}^2 \times \mathbb{P}^2$ model fails to realise a Fano 3-fold at all, it is usually because the general member does not have terminal singularities; we say, for example, 'bad 1/4 point' if the format forces a non-quasismooth, non-terminal index 4 point onto the variety.

When the GRDB model is in codimension 3, we list which Tom and Jerry unprojections of a degeneration work to give alternative varieties in codimension 4, indicating the number of nodes as a superscript and the codimension 4 ambient space. (We do not say which Tom or Jerry since that depends on a choice of rows and columns.) In each case the Tom unprojection gives the $\mathbb{P}^2 \times \mathbb{P}^2$ model determined by the parameters a and b. The usual codimension 3 model arises by Type I unprojection with number of nodes being one more that that of the $\mathbb{P}^2 \times \mathbb{P}^2$ Tom model.

When the GRDB model is in codimension 4 with two Tom unprojections, the $\mathbb{P}^2 \times \mathbb{P}^2$ always works to give the second of the Tom families. The further Tom and Jerry analysis of the unprojection is carried out in [8] and we do not repeat the result here. When the GRDB model is in codimension 4 with only a single Tom unprojection, the model usually fails. The exception is family 12,960, which does work as a $\mathbb{P}^2 \times \mathbb{P}^2$ model.

There is also a case of a Hilbert series, number 11,157, where the GRDB offers a prediction of a variety in codimension 5, but this fails as a $\mathbb{P}^2 \times \mathbb{P}^2$ model.

In Sect. 7.1, we outline a computer search that provides the a, b parameters of Table 1 which are the starting point of the analysis here. In Sect. 7.2, we summarise the results of [32] that provide the most general form of the Hilbert series of a variety in $\mathbb{P}^2 \times \mathbb{P}^2$ format; that paper also discovered cases 11,106 and 11,021 of Table 1 that inspired our approach here. First we introduce the key varieties of the $\mathbb{P}^2 \times \mathbb{P}^2$ format in Sect. 2.

2 The key varieties and weighted $\mathbb{P}^2 \times \mathbb{P}^2$ formats

The affine cone $C(\mathbb{P}^2 \times \mathbb{P}^2)$ on $\mathbb{P}^2 \times \mathbb{P}^2$ is defined by the equations (1) on $\mathbb{C}^9$. It admits a 6-dimensional family of $\mathbb{C}^*$ actions, or equivalently six degrees of freedom in assigning positive integer gradings to its (affine) coordinate ring. We express this as follows.

Let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ be two vectors of integers that satisfy $a_1 \leqslant a_2 \leqslant a_3$, and similarly for the b_i, and that $a_1 + b_1 \geqslant 1$. We define a *weighted* $\mathbb{P}^2 \times \mathbb{P}^2$ as

$$V = V(a,b) = \left(\bigwedge^2 \begin{pmatrix} x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \\ x_6 & x_7 & x_8 \end{pmatrix} = 0 \right) \subset \mathbb{P}^8(a_1+b_1, \ldots, a_3+b_3), \qquad (2)$$

where the variables have weights

$$\operatorname{wt} \begin{pmatrix} x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \\ x_6 & x_7 & x_8 \end{pmatrix} = \begin{pmatrix} a_1+b_1 & a_1+b_2 & a_1+b_3 \\ a_2+b_1 & a_2+b_2 & a_2+b_3 \\ a_3+b_1 & a_3+b_2 & a_3+b_3 \end{pmatrix} =: a^T + b. \qquad (3)$$

Thus $V(a,b) = C(\mathbb{P}^2 \times \mathbb{P}^2) /\!\!/ \mathbb{C}^*$, where the $\mathbb{C}^*$ action is determined by the grading. We treat $V(a,b)$ as a key variety for each different pair a, b. (Note that the entries of a and b may also all lie in $\frac{1}{2} + \mathbb{Z}$, without any change to our treatment here).

Proposition 2.1 *$V(a,b)$ is a 4-dimensional, $\mathbb{Q}$-factorial projective toric variety of Picard rank $\rho_V = 2$.*

Proof First we describe a toric variety $W(a,b)$ by its Cox ring. The input data is the weight matrix (3), which is weakly increasing along rows and down columns. The key is to understand the freedom one has to choose alternative vectors $a^{(i)}, b^{(i)}$, for $i = 1, 2$, to give the same matrix. For example, if we choose $a_1^{(1)} = 0$, then $b^{(1)}$ is determined by the top row, and then $a_2^{(1)}$ and $a_3^{(1)}$ are determined by the first column. Alternatively, choosing $b_1^{(2)} = 0$ determines different vectors $a^{(2)}$ and $b^{(2)}$. Concatenating the a and b vectors to give $v^{(i)} = \left(a_1^{(i)}, \ldots, b_3^{(i)}\right) \in \mathbb{Q}^6$ determines a 2-dimensional $\mathbb{Q}$-subspace $U = U_{a,b} \subset \mathbb{Q}^6$ together with a chosen integral basis $\langle v^{(1)}, v^{(2)} \rangle$.

We define $W(a,b)$ as a quotient of $\mathbb{C}^6$ by $\mathbb{C}^* \times \mathbb{C}^*$ as follows. In terms of Cox coordinates, it is determined by the polynomial ring R in variables $u_1, u_2, u_3, v_1, v_2, v_3$, bi-graded by the columns of the matrix (giving the two $\mathbb{C}^*$ actions)

$$\begin{pmatrix} a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & b_1^{(1)} & b_2^{(1)} & b_3^{(1)} \\ a_1^{(2)} & a_2^{(2)} & a_3^{(2)} & b_1^{(2)} & b_2^{(2)} & b_3^{(2)} \end{pmatrix}. \tag{4}$$

The irrelevant ideal is $B(a, b) = \langle u_1, u_2.u_3\rangle \cap \langle v_1, v_2.v_3\rangle$, and

$$W(a, b) = (\mathbb{C}^6 \setminus V(B(a, b))) / \mathbb{C}^* \times \mathbb{C}^*.$$

If $W(a, b)$ is well formed, then it is a toric variety determined by a fan (the image of all non-irrelevant cones of the fan of $\mathbb{C}^6$ under projection to a complement of U). The bilinear map

$$\begin{aligned} \Phi_{a,b}\colon \qquad W(a, b) &\to \mathbb{P}(a_1 + b_1, a_1 + b_2, \dots, a_3 + b_3) \\ (u_1, \dots, v_3) &\mapsto (u_1 v_1, u_1 v_2, \dots, u_3 v_3) \end{aligned} \tag{5}$$

is an isomorphism onto its image $V(a, b)$, and the conclusions of the proposition all follow at once. ($\mathbb{Q}$-factoriality holds since the Cox coordinates correspond to the 1-skeleton of the fan, and so any maximal cone with at least five rays must contain all u_i or all v_j, contradicting the choice of irrelevant ideal.)

If $W(a, b)$ is not well formed, then, just as for wps, there is a different weight matrix that is well formed and determines a toric variety W' isomorphic to $W(a, b)$. (See Iano-Fletcher [18, 6.9–20] for wps and Ahmadinezhad [1, 2.3] for the general case.) The proposition follows using W'. ◻

The well forming process used in the proof is easy to use. For example, if an integer $n > 1$ divides every entry of some row of the weight matrix (4), then we may divide that row through by n; the subspace $U \subset \mathbb{Q}^6$ is unchanged by this. Or if an integer $n > 1$ divides all columns except one, then the corresponding Cox coordinate u appears only as u^n in the coordinate rings of standard affine patches and we may truncate R by replacing the generator u by u^n; this does not change the coordinate rings of the affine patches, and so the scheme it defines is isomorphic to the original (c.f. [1, Lemma 2.9] for the more general statement). This multiplies the u column of (4) by n, changing the subspace $U \subset \mathbb{Q}^6$, and then we may divide the whole matrix by n as before. See [1, 2.3] for the complete process.

Having said that, in practice we will work with non-well-formed quotients if they arise, since they still admit regular pullbacks that are well formed, and the grading on the target wps is something we fix in advance. More importantly for us here is that well forming step $u \rightsquigarrow u^n$ destroys the $\mathbb{P}^2 \times \mathbb{P}^2$ structure, so we avoid it.

Example 2.2 Consider $V(a, b) \subset \mathbb{P}(2^6, 3^3)$ for $a = (1, 1, 1), b = (1, 1, 2)$. Selecting $a^{(i)}$ and $b^{(i)}$ as above gives bi-grading matrix

$$\left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 2 & 2 & 3 \\ 2 & 2 & 2 & 0 & 0 & 1 \end{array}\right)$$

on variables $u_1, u_2, u_3, v_1, v_2, v_3$. (We use the vertical line in the bi-grading matrix to indicate the irrelevant ideal $B(a,b)$.) The map Φ of (5) is then

$$\begin{aligned} W(a,b) &\to \mathbb{P}(2,2,3,2,2,3,2,2,3) = \mathbb{P}(2^6,3^3) \\ (u_1,\dots,v_3) &\mapsto (u_1v_1, u_1v_2, \dots, u_3v_3), \end{aligned}$$

since the monomials having gradings $\binom{2}{2}$ and $\binom{3}{3}$, as necessary. The image $V(a,b)$ is defined by (2), and we often write the target weights of Φ in matching array:

$$\begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}.$$

In this case $V(a,b)$ is not well formed: the locus $V(a,b)\cap\mathbb{P}(2^6)$ has dimension 3 (by Hilbert–Burch), so has codimension 1 in $V(a,b)$ but nontrivial stabiliser $\mathbb{Z}/2$ in the wps. Well forming the gradings using v_3^2, as above, gives a new bi-grading

$$\left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 1 & 3 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right).$$

That process is well established, but has a problem: for this presentation W' of W, the Segre map is not bi-linear: u_1v_1 has bidegree $\binom{1}{1}$, but u_1v_3 has an independent bidegree $\binom{3}{2}$. We could use $u_1^2v_3$ instead, which has proportional bidegree $\binom{3}{3}$. Taking $V' = \operatorname{Proj} R$, where R is the graded ring of forms of degrees $\binom{m}{m}$ for $m \geqslant 0$, gives $W' \to V' \subset \mathbb{P}(1^6, 3^6)$, which is now well formed, but we have lost the codimension 4 property of V we want to exploit. In a case like this, we work directly with the non-well-formed $W(a,b)$ and its non-well-formed image $V \subset \mathbb{P}(2^6,3^3)$.

We use the varieties $V(a,b)$ as key varieties to produce new varieties from by regular pullback; see [30, Section 1.5] or [7, Section 2]. In practical terms, that means writing equations in the form of (1) inside a wps $\mathrm{w}\mathbb{P}^7$ where the x_i are homogeneous forms of positive degrees, and the resulting loci $X \subset \mathrm{w}\mathbb{P}^7$ are the Fano 3-folds we seek.

Alternatively, we may treat X as a complete intersection in a projective cone over $V(a,b)$, as in Sect. 3.2 below, where the additional cone vertex variables may have any positive degrees; this point of view is taken by Corti-Reid and Szendrői in [14, 26,29,32]. It follows from this description that the Picard rank of X is 2.

3 Unprojection and the proof of Theorem 1.1

The Hilbert series number 26989 in the Graded Ring Database (GRDB) [6] is

$$P = \frac{1-3t^2-4t^3+12t^4-4t^5-3t^6+t^8}{(1-t)^7(1-t^2)}.$$

In Sect. 3.1 we describe the known family of Fano 3-folds $X^{(1)} \subset \mathbb{P}(1^7,2)$ that realise this Hilbert series, $P_{X^{(1)}} = P$. These 3-folds are not smooth: the general member of the family has a single $\frac{1}{2}(1, 1, 1)$ quotient singularity. We exhibit a different family in Sect. 3.2 with the same Hilbert series in $\mathbb{P}^2 \times \mathbb{P}^2$ format, and the subsequent "Tom and Jerry" analysis yields a third distinct family in Sect. 3.3.

Recall (from [8, Section 4], for example) that if $X \dashrightarrow Y \supset D$ is a Gorenstein unprojection and Y is quasismooth away from N nodes, all of which lie on D, then

$$e(X) = e(Y) + 2N - 2. \tag{6}$$

3.1 The classical 7 x 12 family

A general member of the first family can be constructed as the unprojection of a coordinate $D = \mathbb{P}^2$ inside a c.i. $Y_{2,2,2} \subset \mathbb{P}^6$ (see, for example, Papadakis [23]). In general, Y has six nodes that lie on D: in coordinates x, y, z, u, v, w, t of $\mathbb{P}^6$, setting $D = (u = v = w = t = 0)$, the general Y has equations defined by

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,4} \\ A_{2,1} & \cdots & A_{2,4} \\ A_{3,1} & \cdots & A_{3,4} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

for general linear forms $A_{i,j}$; singularities occur when the 3×4 matrix drops rank, which is calculated by evaluating the numerator of the Hilbert series of that locus at 1:

$$P_{\text{sings}} = \frac{1 - 4t^3 + 3t^4}{(1-t)^3} = \frac{1 + 2t + 3t^2}{1-t}, \quad \text{so there are } 1 + 2 + 3 = 6 \text{ nodes.}$$

The coordinate ring of X has a 7×12 free resolution. If Y_{gen} is a nonsingular small deformation of Y, then $e(Y_{\text{gen}}) = -24$ (by the usual Chern class calculation, since Y_{gen} is a smooth 2, 2, 2 complete intersection) so, by (6),

$$e(X) = -24 + 12 - 2 = -14.$$

This family is described by Takagi [33]; it is no. 1.4 in the tables there of Fano 3-folds of Picard rank 1.

3.2 A $\mathbb{P}^2 \times \mathbb{P}^2$ family with Tom projection

Consider the $\mathbb{P}^2 \times \mathbb{P}^2$ key variety $V_{a,b} \subset \mathbb{P}(1^6, 2^3)$, where $a = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $b = \left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right)$. We define a quasismooth variety $X^{(2)} \subset \mathbb{P}(1^7, 2)$ in codimension 4 as a regular pullback.

In explicit terms, in coordinates x, y, z, t, u, v, w, s on $\mathbb{P}(1^7, 2)$, a 3×3 matrix M of forms of degrees

$$a^T + b = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$$

gives a quasismooth $X^{(2)} = (\wedge^2 M = 0) \subset \mathbb{P}(1^7, 2^2)$; for example,

$$M = \begin{pmatrix} x & t & s \\ y & u & x^2 - z^2 + t^2 + v^2 \\ z & v & xt + yu + w^2 \end{pmatrix}$$

works. Alternatively, note that $X^{(2)}$ may be viewed as a complete intersection

$$X^{(2)} = C_1 V_{a,b} \cap Q_1 \cap Q_2 \subset \mathbb{P}(1^7, 2^3),$$

where $C_1 V_{a,b} \subset \mathbb{P}(1^7, 2^3)$ is the projective cone over $V_{a,b}$ on a vertex of degree 1 (by introducing a new variable of degree 1), and Q_i are general quadrics (which are quasilinear, and so may be used to eliminate two variables of degree 2). The general such $X^{(2)}$ is quasismooth (since in particular the intersection misses the vertex). Described in these terms, $C_1 V_{a,b}$ has Picard rank 2, and so $\rho_{X^{(2)}} = 2$.

Any such $X^{(2)}$ has a single quotient singularity $\frac{1}{2}(1, 1, 1)$, at the coordinate point $P_s \in X^{(2)}$ as the explicit equations make clear, since y, z, u, v are implicit functions in a neighbourhood of $P_s \in X^{(2)}$. The Gorenstein projection from this point P_s has image $Y = (\operatorname{Pf} N = 0) \subset \mathbb{P}^6$, where

$$N = \begin{pmatrix} 0 & x & y & z \\ & t & u & v \\ & & x^2 - z^2 + t^2 + v^2 & xt + yu + w^2 \\ & & & 0 \end{pmatrix}$$

is an antisymmetric 5×5 matrix, and $\operatorname{Pf} N$ denotes the sequence of five maximal Pfaffians of N. (The nonzero entries of N are those of M^T with the entry s deleted.)

This Y contains the projection divisor $D = (y = z = u = v = 0)$ and has five nodes on D (either by direct calculation, or by the formula of [8, Section 7]). The divisor $D \subset Y$ is in Tom_3 configuration: entries $n_{i,j}$ of the skew 5×5 matrix N defining Y lie in the ideal $I_D = (y, z, u, v)$ if both $i \neq 3$ and $j \neq 3$; that is, all entries off row 3 and column 3 of N are in I_D. Thus, in particular, we can reconstruct $X^{(2)}$ from $D \subset Y$ as the Tom_3 unprojection. It follows from Papadakis–Reid [25, Section 2.4] that $\omega_{X^{(2)}} = \mathcal{O}_{X^{(2)}}(-1)$ and so $X^{(2)}$ is a Fano 3-fold.

It remains to show that $e(X^{(2)}) = -16$, so that this Fano 3-fold must lie in a different deformation family from the classical one constructed in Sect. 3.1.

The degree of the $(1, 2)$ entry $f_{1,2}$ of N is in fact zero while the degree of $f_{4,5}$ is 2, although each entry is of course the zero polynomial in this case; we denote this by indicating the degrees of the entries with brackets around those that are zero in this case:

$$\begin{pmatrix} (0) & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 2 & 2 \\ & & & (2) \end{pmatrix}.$$

We may deform Y by varying these two entries to $f_{1,2} = \varepsilon$ and $f_{4,5} = \varepsilon f$, where $\varepsilon \neq 0$ and f is a general quadric on $\mathbb{P}^6$ (and, of course, the skew symmetric entries in $f_{2,1}$ and $f_{5,4}$). Denoting the deformed matrix by N_ε, and $Y_\varepsilon = (\mathrm{Pf}\ N_\varepsilon = 0)$, we see a small deformation of Y to a smooth Fano 3-fold $Y_\varepsilon \subset \mathbb{P}^6$ that is a 2, 2, 2 complete intersection. (The nonzero constant entries of N_ε provide two syzygies that eliminate two of the five Pfaffians.) As in Sect. 3.1, the smoothing Y_ε has Euler characteristic -24, so by (6) we have that $e_{X^{(2)}} = -24 + 10 - 2 = -16$.

Note that the Pfaffian smoothing Y_ε of Y destroys the unprojection divisor $D \subset Y$: for D to lie inside Y_ε the entries $f_{3,4}$ and $f_{3,5}$ of N_ε would have to lie in I_D (so N_ε would be in $\mathrm{Jer}_{4,5}$ format with the extra constraint $f_{4,5} = 0$), but then Y would be singular along D since three of the five Pfaffians would lie in I_D^2.

3.3 A third family by Jerry unprojection

A Tom and Jerry analysis following [8] shows that varieties $D \subset Y \subset \mathbb{P}^6$ defined by Pfaffians as in Sect. 3.2 by the maximal Pfaffians of a syzygy matrix N with weights

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 2 & 2 \\ & & & 2 \end{pmatrix}.$$

can also be constructed in $\mathrm{Jer}_{1,3}$ format: that is, with all entries $f_{i,j}$ of N lying in I_D whenever i or j lie in $\{1, 3\}$. The general such $D \subset Y$ has seven nodes on D. Unprojecting $D \subset Y$ gives a general member $X^{(3)}$ of a third family with $e(X^{(3)}) = -24 + 2 \times 7 - 2 = -12$.

This completes the proof of Theorem 1.1.

4 Unprojecting Pfaffian degenerations

4.1 $\mathbb{P}^2 \times \mathbb{P}^2$ models with a codimension 3 Pfaffian component

Each of the Fano Hilbert series 1396, 5302, 5858, 5962, 11436, 20543 is realised by a codimension 3 Pfaffian model, which is the simple default model presented in the GRDB. (So too are 4999, 5844 and 10,984, but we do not find new models for these.) We show that they can also be realised by a $\mathbb{P}^2 \times \mathbb{P}^2$ model in a different deformation family (and sometimes a third model too). The key point is that a projection of the usual model admits alternative degenerations in higher codimension that also contain a divisor that can be unprojected.

For example, consider series number 20543

$$P_{20543}(t) = \frac{1-4t^3+4t^5-t^8}{(1-t)^5(1-t^2)^2}.$$

There is a well-known family that realises this as $X = (\operatorname{Pf} M = 0) \subset \mathbb{P}(1^5, 2^2)$ in codimension 3, where M has degrees

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ & 2 & 2 & 2 \\ & & 2 & 2 \\ & & & 2 \end{pmatrix}.$$

A typical member of this family has a two $\frac{1}{2}(1,1,1)$ quotient singularities, and making the Gorenstein projection from either of them presents X as a Type I unprojection of

$$\mathbb{P}^2 = D \subset Y_{3,3} \subset \mathbb{P}(1^5, 2).$$

In general, Y has eight nodes lying on D, and it smooths to a nonsingular Fano 3-fold Y_{gen} with Euler characteristic $e(Y_{\text{gen}}) = -40$. Thus a general X has Euler characteristic $e(X) = -40 + 2\times 8 - 2 = -26$.

A quasismooth $\mathbb{P}^2\times\mathbb{P}^2$ family. We can write another (quasismooth) model $X \subset \mathbb{P}(1^5, 2^3)$ in codimension 4 in $\mathbb{P}^2\times\mathbb{P}^2$ format with weights

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}.$$

Projecting from $\frac{1}{2}(1,1,1)$ has image $Y = (\operatorname{Pf} M = 0) \subset \mathbb{P}(1^5, 2^2)$ where M has degrees

$$\begin{pmatrix} (0) & 1 & 1 & 2 \\ & 1 & 1 & 2 \\ & & 2 & 3 \\ & & & (3) \end{pmatrix}, \tag{7}$$

and Y has seven nodes lying on D; in coordinates x, y, z, t, u, w, v, we may take $D = \mathbb{P}^2$ to be $(t = u = v = w = 0)$. By varying the $(1,2)$ entry from zero to a unit, Y has a deformation to a quasismooth 3, 3 complete intersection Y_{gen} as before, and so, $e(X) = e(Y_{\text{gen}}) + 2\times 7 - 2 = -40 + 14 - 2 = -28$. Thus these $\mathbb{P}^2\times\mathbb{P}^2$ models are members of a different deformation family from the original one.

More is true in this case: the general member of this new deformation family is in $\mathbb{P}^2\times\mathbb{P}^2$ format. Starting with matrix (7) and $D = \mathbb{P}^2$ as above, the $(1,2)$ entry of the general Tom_3 matrix is necessarily the zero polynomial. In general, the four entries $(1,4)$, $(1,5)$, $(2,4)$ and $(2,5)$ of the matrix are in the ideal $\langle t, u, v, w\rangle$, and for the general member these four variables are dependent on those entries. Thus the $(4,5)$ entry can be arranged to be zero by row-and-column operations.

Another family in codimension 4. There is a third deformation family in this case. The codimension 3 format (7) also admits a Jerry$_{15}$ unprojection with nine nodes on D, giving $X \subset \mathbb{P}(1^5, 2^3)$ in codimension 4 with $e(X) = -24$.

4.2 Pfaffian degenerations of codimension 2 Fano 3-folds

The key to the cases in Sect. 4.1 that the $\mathbb{P}^2 \times \mathbb{P}^2$ model exposes is the degeneration of a codimension 2 Fano 3-fold. More generally, Table 3 of [3] lists 13 cases of Fano 3-fold degenerations where the generic fibre is a codimension 2 complete intersection and the special fibre is a codimension 3 Pfaffian. In each case, the anti-symmetric 5×5 syzygy matrix of the special fibre has an entry of degree 0, which is the zero polynomial in the degeneration, but when nonzero serves to eliminate a single variable. (In fact [3] describes the graded rings of K3 surfaces, but these extend to Fano 3-folds by the usual extension–deformation method introducing a new variable of degree 1.)

For example, $Y_{12,13} \subset \mathbb{P}(1, 3, 4, 5, 6, 7)$ degenerates to codimension 3

$$Y^0 \subset \mathbb{P}(1, 3, 4, 5, 6, 7, 9) \quad \text{with syzygy degrees} \quad \begin{pmatrix} 0 & 3 & 4 & 7 \\ & 5 & 6 & 9 \\ & & 9 & 12 \\ & & & 13 \end{pmatrix}.$$

Both of these realise Fano Hilbert series number 547, and the Euler characteristic of a general member is $e(Y) = -56$.

The codimension 2 family has a subfamily whose members contain a Type I unprojection divisor,

$$D = \mathbb{P}(1, 3, 7) \subset Y = Y_{12,13} \subset \mathbb{P}(1, 3, 4, 5, 6, 7)$$

on which Y has eight nodes; the unprojection of $D \subset Y$ gives the codimension 3 Pfaffian family

$$\text{Hilbert Series No. 548:} \qquad X_{12,13,14,15,16} \subset \mathbb{P}(1, 3, 4, 5, 6, 7, 10). \tag{8}$$

Imposing the same unprojection divisor $D \subset Y^0$ can be done in two distinct ways, coming from different Tom and Jerry arrangments. In one way, there are degenerations $Y^t_{12,13} \rightsquigarrow Y^0$ which contain the same D in every fibre Y_t. These unproject to a degeneration of the family (8) by the following lemma: indeed unprojection commutes with regular sequences by [10, Lemma 5.6], and so unprojection commutes with flat deformation, if one fixes the unprojection divisor; so the lemma is a particular case of [10, Lemma 5.6].

Lemma 4.1 *Let $\mathbb{P} = \mathbb{P}(a_0, \ldots, a_s)$ be any wps and fix $D = \mathbb{P}(a_0, \ldots, a_d) \subset \mathbb{P}$, for some $d \leqslant s - 2$. Suppose $Y_t \subset \mathcal{Y} \to \mathcal{T}$ is a flat 1-dimensional family of projectively Gorenstein subschemes of $\mathbb{P}$ over smooth base $0 \in \mathcal{T}$, each one containing D and with* $\dim Y_t = \dim D + 1 = d + 1$, *and with $\omega_Y = \mathcal{O}_Y(k_Y)$. Let $\mathcal{X} \ni X_t \subset \mathbb{P}(a_0, \ldots, a_s, b)$ be the unprojection of $D \times \mathcal{T} \subset \mathcal{Y}$, where $b = k_Y - k_D = a_0 + \cdots + a_d - 1$. Then $\mathcal{X}$*

is flat over $\mathcal{T}$, and for each closed point $t \in \mathcal{T}$ the fibre $X_t \in \mathcal{X}$ is the unprojection of $D \subset Y_t$.

But the Jer_{24} unprojection is different: small deformations of Y^0 do not contain D. Indeed, in this $D \subset Y^0$ model, Y^0 has nine nodes on D, which is a numerical obstruction to any such deformation. This $D \subset Y^0$ unprojects to a codimension 4 Fano 3-fold

$$X^0 \subset \mathbb{P}(1, 3, 4, 5, 6, 7, 9, 10)$$

with the same Hilbert Series No. 548 as (8) but lying in a different component: it has Euler characteristic $-56 + 2 \times 9 - 2 = -40$. This proves Theorem 1.2.

5 $\mathbb{P}^2 \times \mathbb{P}^2$ and the second Tom

The Big Table [9], which contains the results of [8], lists deformation families of Fano 3-folds in codimension 4 that have a Type I projection to a Pfaffian 3-fold in codimension 3. The components are listed according to the Tom or Jerry type of the projection: the type of projection is invariant for sufficiently general members of each component. The result of this section gives an interpretation of the Big Table of [8], but does not describe any new families of Fano 3-folds.

Theorem 5.1 *For every Hilbert series listed in the Big Table* [9] *that is realised by two distinct Tom projections, there is a Fano 3-fold in $\mathbb{P}^2 \times \mathbb{P}^2$ format that lies on the family containing 3-folds with the smaller* (*more negative*) *Euler characteristic.*

The theorem is proved simply by constructing each case. There are 29 Hilbert series that have two Tom families. Using 'TTJ' to indicate a series realised by two Tom components and one Jerry component and 'TTJJ' to indicate two of each, they are (Table 2).

For example, for Hilbert Series No. 4839,

$$P_{4839}(t) = \frac{1 - t^{11} - 2t^{12} - 2t^{13} - 2t^{14} - t^{15} - t^{16} + \cdots - t^{40}}{\prod_{a \in [1,1,4,5,6,7,8,9]} (1 - t^a)},$$

[9] describes four deformation families of Fano 3-folds

$$X \subset \mathbb{P}(1, 1, 4, 5, 6, 7, 8, 9).$$

A general such X has Type I projections from both $\frac{1}{5}(1, 1, 4)$ and $\frac{1}{9}(1, 1, 8)$. (It is enough to consider just one of these centres of projection, but [8] calculates both, drawing the same conclusion twice.)

We construct a $\mathbb{P}^2 \times \mathbb{P}^2$ model for P_{4839}. Consider $\mathbb{P} = \mathbb{P}^7(1, 1, 4, 5, 6, 7, 8, 9)$ with coordinates x, y, z, t, u, v, w, s. The 2×2 minors of the matrix

$$\begin{pmatrix} z & u & v \\ t & v + x^7 - y^7 & w + z^2 + x^8 \\ u + x^6 + y^6 & w & s \end{pmatrix} \quad \text{of weights} \quad \begin{pmatrix} 4 & 6 & 7 \\ 5 & 7 & 8 \\ 6 & 8 & 9 \end{pmatrix}$$

Table 2 Hilbert series in $\mathbb{P}^2 \times \mathbb{P}^2$ format that admit a second Tom unprojection

GRDB	$\mathbb{P}^2 \times \mathbb{P}^2$ weights	T/J families	Centre: # nodes
1253	$\begin{pmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{pmatrix}$	TTJ	$\frac{1}{7}:6$
1218	$\begin{pmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{pmatrix}$	TTJJ	$\frac{1}{5}:9$
1413	$\begin{pmatrix} 3 & 4 & 5 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{pmatrix}$	TTJJ	$\frac{1}{5}:7$
2422	$\begin{pmatrix} 2 & 3 & 5 \\ 3 & 4 & 6 \\ 4 & 5 & 7 \end{pmatrix}$	TTJJ	$\frac{1}{7}:5$
2427	$\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{pmatrix}$	TTJJ	$\frac{1}{5}:6$
4839	$\begin{pmatrix} 4 & 6 & 7 \\ 5 & 7 & 8 \\ 6 & 8 & 9 \end{pmatrix}$	TTJJ	$\frac{1}{5}:20;\ \frac{1}{9}:13$
4860	$\begin{pmatrix} 4 & 5 & 6 \\ 5 & 6 & 7 \\ 6 & 7 & 8 \end{pmatrix}$	TTJJ	$\frac{1}{7}:13$
4915	$\begin{pmatrix} 3 & 5 & 6 \\ 4 & 6 & 7 \\ 5 & 7 & 8 \end{pmatrix}$	TTJJ	$\frac{1}{4}:19;\ \frac{1}{8}:11$
4949	$\begin{pmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{pmatrix}$	TTJJ	$\frac{1}{6}:11$
4989	$\begin{pmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{pmatrix}$	TTJJ	$\frac{1}{4}:15;\ \frac{1}{7}:10$
5002	$\begin{pmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 4 & 5 & 6 \end{pmatrix}$	TTJJ	$\frac{1}{4}:14;\ \frac{1}{5}:11;\ \frac{1}{6}:10$
5163	$\begin{pmatrix} 2 & 4 & 5 \\ 3 & 5 & 6 \\ 4 & 6 & 7 \end{pmatrix}$	TTJJ	$\frac{1}{3}:19;\ \frac{1}{7}:9$
5215	$\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{pmatrix}$	TTJJ	$\frac{1}{5}:9$
5268	$\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{pmatrix}$	TTJJ	$\frac{1}{3}:14;\ \frac{1}{5}:8$
5306	$\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 3 & 4 & 5 \end{pmatrix}$	TTJJ	$\frac{1}{3}:13;\ \frac{1}{4}:9;\ \frac{1}{5}:8$
5530	$\begin{pmatrix} 2 & 3 & 3 \\ 3 & 4 & 4 \\ 3 & 4 & 4 \end{pmatrix}$	TTJJ	$\frac{1}{3}:11;\ \frac{1}{4}:8$

Table 2 continued

GRDB	$\mathbb{P}^2 \times \mathbb{P}^2$ weights	T/J families	Centre: # nodes
5870	$\begin{pmatrix} 2 & 3 & 4 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$	TTJJ	$\frac{1}{3}:10;\ \frac{1}{5}:7$
5970	$\begin{pmatrix} 2 & 3 & 3 \\ 2 & 3 & 3 \\ 3 & 4 & 4 \end{pmatrix}$	TTJJ	$\frac{1}{3}:9;\ \frac{1}{4}:7$
6860	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$	TTJ	$\frac{1}{5}:4$
6878	$\begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 4 \end{pmatrix}$	TTJJ	$\frac{1}{3}:8$
10985	$\begin{pmatrix} 1 & 3 & 4 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$	TTJJ	$\frac{1}{2}:23;\ \frac{1}{6}:7$
11021	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$	TTJJ	$\frac{1}{4}:7$
11106	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$	TTJJ	$\frac{1}{2}:15;\ \frac{1}{5}:6$
11125	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 3 & 4 \end{pmatrix}$	TTJJ	$\frac{1}{2}:14;\ \frac{1}{3}:7;\ \frac{1}{4}:6$
11455	$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{pmatrix}$	TTJJ	$\frac{1}{2}:11;\ \frac{1}{3}:6$
16228	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$	TTJJ	$\frac{1}{2}:9;\ \frac{1}{4}:5$
16339	$\begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 2 & 3 & 3 \end{pmatrix}$	TTJJ	$\frac{1}{2}:8;\ \frac{1}{3}:5$
20652	$\begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$	TTJ	$\frac{1}{2}:6$
24078	$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$	TTJ	$\frac{1}{3}:4$

define quasismooth $X \subset \mathbb{P}$ with quotient singularities $\frac{1}{2}(1,1,1)$, $\frac{1}{5}(1,1,4)$ and $\frac{1}{9}(1,1,8)$.

Eliminating either the variable t of degree 5 or s of degree 9 computes the two possible Type I projections, with image a nodal codimension 3 Fano 3-fold Y containing $D = \mathbb{P}(1,1,4)$ or $D = \mathbb{P}(1,1,8)$ with 20 or 13 nodes lying on D respectively. (Both t and s appear only once in the matrix, so eliminating them simply involves omitting that entry and mounting the rest of the matrix in a skew matrix, as usual.)

6 Cases with no numerical Type I projection

The five Hilbert series 360, 577, 648, 878 and 1766 do not admit a Type I projection, and so the analysis of [8] does not apply. Nevertheless each is realised by a variety in $\mathbb{P}^2 \times \mathbb{P}^2$ format exist, although only two of these are Fano 3-folds.

In the two cases 360 and 648 the general $\mathbb{P}^2 \times \mathbb{P}^2$ model is not quasismooth and has a non-terminal singularity, so there is no $\mathbb{P}^2 \times \mathbb{P}^2$ Fano model. (Each of these admit Type II_1 projections, so are instead subject to the analysis of [24]; this is carried out by Taylor [34].) In the case 577, the $\mathbb{P}^2 \times \mathbb{P}^2$ model is quasismooth, but it has a $\frac{1}{4}(1, 1, 1)$ quotient singularity and so is not a terminal Fano 3-fold and again there is no $\mathbb{P}^2 \times \mathbb{P}^2$ Fano model.

However, there is a quasismooth Fano 3-fold $X \subset \mathbb{P}(1, 3^2, 4^2, 5^2, 6)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ format with weights

$$\begin{pmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{pmatrix}$$

realising Hilbert series 878. It has $4 \times \frac{1}{3}(1, 1, 2)$, $2 \times \frac{1}{4}(1, 1, 3)$ quotient singularities. There is also a quasismooth Fano 3-fold $X \subset \mathbb{P}(1, 2, 3^3, 4^2, 5)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ format with weights

$$\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{pmatrix}$$

realising Hilbert series 1799. It has $2 \times \frac{1}{2}(1, 1, 1)$, $5 \times \frac{1}{3}(1, 1, 2)$ quotient singularities. Each of these two admit only Type II_2 projections, and an analysis by Gorenstein projection has not yet been attempted. Presumably such an analysis can in principle work, once we have much better understanding of Type II unprojection, but until then our models are the only Fano 3-folds known to realise these two Hilbert series.

7 Enumerating $\mathbb{P}^2 \times \mathbb{P}^2$ formats

7.1 Enumerating $\mathbb{P}^2 \times \mathbb{P}^2$ formats and cases that fail

The Hilbert series $P_X(t) = \sum_{m \in \mathbb{N}} h^0(-mK_X)t^m$ of such Gorenstein rings $R(X)$ satisfy the orbifold integral plurigenus formula [11, Theorem 1.3]

$$P_X(t) = P_{\mathrm{ini}}(t) + \sum_{Q \in \mathcal{B}} P_{\mathrm{orb}}(Q)(t), \tag{9}$$

where P_{ini} is a function only of the genus g of X, where $g + 2 = h^0(-K_X)$, and P_{orb} is a function of a quotient singularity $Q = \frac{1}{r}(1, a, -a)$, the collection of which form the basket $\mathcal{B}$ of X (see [13, Section 9]). When $X \subset \mathrm{w}\mathbb{P}$ is quasismooth, and so is an orbifold, the basket $\mathcal{B}$ is exactly the collection of quotient singularities of X. Thus the numerical data $g, \mathcal{B}$ gives the basis for a systematic search of Hilbert series with given properties, which we develop further here.

We may enumerate all $\mathbb{P}^2 \times \mathbb{P}^2$ formats $V(a, b)$ and then list all genus–basket pairs $g, \mathcal{B}$ whose corresponding series (9) has matching numerator. This algorithm is explained in [7, Section 4]. It works systematically through increasing $k \in \mathbb{N}$, where $k = 3\left(\sum a_i + \sum b_i\right)$, the sum of the weights of the ambient space of the image of Φ in (5).

The enumeration does not have a termination condition, even though there can only be finitely many solutions for Fano 3-folds, so this does not directly give a classification. Nevertheless, we search for $\mathbb{P}^2 \times \mathbb{P}^2$ formats for each $k = 1, \ldots, 31$ to start the investigation. This reveals 53 cases whose numerical data (basket and genus) match those of a Fano 3-fold. The number # of cases found per value of k is:

k	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24–31
#	1	3	2	3	3	5	4	4	4	3	5	5	1	2	3	1	1	0	2	1	0

This hints that we may have found all Fano Hilbert series that match some $\mathbb{P}^2 \times \mathbb{P}^2$ format, since the algorithm stops producing results after $k = 23$. Of course that is not a proof that there are no other cases, and we do not claim that; the results here only use the outcome of this search as their starting point, so how that outcome arises is not relevant.

7.2 Weighted GL(3, $\mathbb{C}$) × GL(3, $\mathbb{C}$) varieties according to Szendrői

The elementary considerations we deploy for the key varieties $V(a, b)$ are part of a more general approach to weighted homogeneous spaces by Grojnowski and Corti–Reid [14], with other cases developed by Qureshi and Szendrői [28,29]. The particular case of $\mathbb{P}^2 \times \mathbb{P}^2$ was worked out detail by Szendrői [32], which we sketch here.

In the treatment of [32], $G = \mathrm{GL}(3, \mathbb{C}) \times \mathrm{GL}(3, \mathbb{C})$ has weight lattice $M = \mathrm{Hom}(T, \mathbb{C}^*) \cong \mathbb{Z}^6$, for the maximal torus $T \subset G$. The construction of a weighted $\mathbb{P}^2 \times \mathbb{P}^2$, denoted $\mathrm{w}\Sigma(\mu, u)$, is determined by the choice of a coweight vector $\mu \in \mathrm{Hom}(M, \mathbb{Z})$, in coordinates say $\mu = (a_1, a_2, a_3, b_1, b_2, b_3) \in \mathrm{Hom}(M, \mathbb{Z})$, and an integer $u \in \mathbb{Z}$. These data are subject to the positivity conditions that all $a_i + b_j + u > 0$. The construction of $\mathrm{w}\Sigma(\mu, u)$ is described in [28, Section 2.2]. It embeds in wps

$$\mathrm{w}\Sigma(\mu, u) \hookrightarrow \mathrm{w}\mathbb{P}^8(a_1 + b_1 + u, \ldots, a_3 + b_3 + u), \tag{10}$$

with image defined by 2×2 minors

$$\mathrm{w}\Sigma = \left\{ \bigwedge^2 \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} = 0 \right\} \subset \mathrm{w}\mathbb{P}^8 \tag{11}$$

with respect to the weights

$$\deg\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} = \begin{pmatrix} a_1+b_1+u & a_1+b_2+u & a_1+b_3+u \\ a_2+b_1+u & a_2+b_2+u & a_2+b_3+u \\ a_3+b_1+u & a_3+b_2+u & a_3+b_3+u \end{pmatrix}.$$

The following theorem then follows from the general Hilbert series formula of [28, Theorem 3.1].

Theorem 7.1 (Szendrői [32]) *The Hilbert series of* $\mathrm{w}\Sigma(\mu, u)$ *in the embedding* (10) *is*

$$P(t) = \frac{P_{\mathrm{num}}(t)}{\prod_{i,j}(1 - t^{a_i+b_j+u})},$$

where the Hilbert numerator $P_{\mathrm{num}}(t)$ *is*

$$1 - \left(\sum_{i,j} t^{-a_i-b_j}\right)t^{2u+s} + \left(4 + \sum_{i\neq j} t^{-a_i+a_j} + \sum_{i\neq j} t^{-b_i+b_j}\right)t^{3u+s} - \left(\sum_{i,j} t^{a_i+b_j}\right)t^{4u+s} + t^{6u+2s},$$

with $s = a_1 + a_2 + a_3 + b_1 + b_2 + b_3$.

This numerator exposes the 9×16 resolution. The 2×2 minors in (11) are visible in the first parentheses; for example $t^{-a_1-b_1}t^{2u+s} = t^{(a_2+b_2+u)+(a_3+b_3+u)}$ carries the degree of $x_5x_9 = x_6x_8$. First syzygies appear in the second parentheses; for example, the syzygy

$$\det\begin{pmatrix} x_4 & x_5 & x_6 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} \equiv 0$$

has degree $\deg(x_4x_5x_9) = (a_2 + b_1 + u) + (a_2 + b_2 + u) + (a_3 + b_3 + u) = (a_2 - a_1) + 3u + s$. The additional parameter $u \in \mathbb{Z}$ in this treatment is absorbed into the a_i in our naive treatment of Sect. 2, so the key varieties we enumerate are the same.

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