LINEAR DIFFERENTIAL POLYNOMIALS IN ZERO-FREE MEROMORPHIC FUNCTIONS

James K. Langley

University of Nottingham, School of Mathematical Sciences
NG7 2RD, United Kingdom; james.langley@nottingham.ac.uk

In fondest memory of Günter Frank and Milne Anderson

Abstract. The paper determines all meromorphic functions f in $\mathbf{C}$ such that f and F have finitely many zeros, where $F = f^{(k)} + a_{k-1}f^{(k-1)} + \ldots + a_0 f$ with $k \geq 3$ and the a_j rational functions.

1. Introduction

Let the function f be meromorphic in an annulus $\Omega(r_1) = \{z \in \mathbf{C} \colon r_1 < |z| < \infty\}$, with r_1 positive (not necessarily the same at each occurrence in this paper). Let $k \geq 2$ and let $a_0, \ldots, a_{k-1}$ be functions which are rational at infinity, that is, analytic on some $\Omega(r_1)$ with at most a pole at ∞. Write $D = d/dz$ and

$$(1.1) \qquad F = L[f], \quad L = D^k + a_{k-1}D^{k-1} + \ldots + a_0,$$

in which $L[y]$ denotes the operator L acting on the function y. The central objective of this paper is the classification of all those f for which f and F have no zeros in $\Omega(r_1)$. By a standard change of variables $f = e^P g, F = e^P G$, with P a polynomial, it may be assumed that $a_{k-1}(\infty) = 0$.

This problem, part of which appeared as 1.42 in the collection [1], has a long history going back to Hayman's conjecture in [10], proved in [3, 18], that if $k \geq 2$ then the only meromorphic functions f in the plane for which f and $f^{(k)}$ have no zeros are those of form $f(z) = e^{az+b}$ or $f(z) = (az+b)^{-n}$ with $a, b \in \mathbf{C}$ and $n \in \mathbf{N} = \{1, 2, \ldots\}$: more generally, if f and $f^{(k)}$ have finitely many zeros then $f = Re^P$, with R a rational function and P a polynomial [5, 18], so that f'/f is rational. The problem for $k = 2$ and coefficients which are rational at infinity was fully solved in [18, 19].

Theorem 1.1. [18, 19] *Let the function f be meromorphic in $S \leq |z| < \infty$ for some $S > 0$ and let the functions a_1 and a_0 be analytic there and rational at infinity. Assume that $a_1(\infty) = 0$ and that f and $F = f'' + a_1 f' + a_0 f$ have no zeros in $S \leq |z| < \infty$.*

(a) If

$$\deg_{\infty}(a_0) = \lim_{z \to \infty} \frac{\log |a_0(z)|}{\log |z|}$$

is even then at least one of the following holds.
(i) The function f'/f is rational at infinity.

(ii) *The function f satisfies*

$$(1.2) \qquad \frac{f'}{f} = -\frac{a_1}{2} + \frac{g'}{2g} + \frac{A}{g}, \quad g^2 = \frac{f}{F}, \quad g' = \left(2\frac{f_1'}{f_1} + a_1\right)g + B,$$

where $A, B \in \mathbf{C}$ and g is analytic in $|z| \geq S$, while f_1 is a solution of the homogeneous equation

$$(1.3) \qquad\qquad\qquad\qquad w'' + a_1 w' + a_0 w = 0$$

which admits unrestricted analytic continuation without zeros in $|z| \geq S$.

(iii) *There exist solutions f_1, f_2 of (1.3), such that*

$$(1.4) \qquad f = A f_2 \left(1 + B\left(\frac{f_2}{f_1}\right)^{1/N}\right)^{-N}, \quad A, B \in \mathbf{C}, \quad N \in \mathbf{N}.$$

Here both f_1 and f_2 admit unrestricted analytic continuation without zeros in $|z| > R_1$ for some $R_1 > 0$, and $(f_2/f_1)^{1/N}$ is analytic in $|z| > R_1$.

(iv) *There exist solutions f_1, f_2 of (1.3), each admitting unrestricted analytic continuation without zeros in $|z| > R_1$ for some $R_1 > 0$, a function M which is rational at infinity, and non-constant polynomials Q, Q_1 such that*

$$\frac{f'}{f} = \frac{f_2'}{f_2} + \frac{Q(M)M'}{e^M + 1}, \quad \text{where} \quad Q(M)M' = \frac{f_1'}{f_1} - \frac{f_2'}{f_2} \quad \text{or} \quad Q_1(M)e^{-M} = \frac{f_1}{f_2}.$$

(b) *If $\deg_\infty(a_0)$ is odd then f may be determined by applying part (a) to*

$$\phi(z) = f(z^2), \quad \Phi(z) = 4z^2 F(z^2) = \phi''(z) + (2za_1(z^2) - 1/z)\phi'(z) + 4z^2 a_0(z^2)\phi(z).$$

A refinement of this theorem for meromorphic functions in the plane may be found in [21, Theorem 1.3]. For $k \geq 3$ and f, F zero-free in the whole plane, the case of constant coefficients was solved in full by Steinmetz in [23], while polynomial coefficients were treated in [4] for entire f, and for meromorphic f by Brüggemann in [2].

Theorem 1.2. [2, 4] *Let the function f be meromorphic in the plane, such that f and $F = L[f]$ have no zeros, where $k \geq 3$ and $a_0, \ldots, a_{k-2}$ are polynomials, not all constant, with $a_{k-1} \equiv 0$. Then $f = (H')^{-(k-1)/2}e^H$ or $f = (H')^{-(k-1)/2}H^{-m}$ for some $m \in \mathbf{N}$, where H''/H' is a polynomial.*

The following theorem, which settles all cases, will be proved.

Theorem 1.3. *Let $k \geq 3$ and let the function f be meromorphic in some annulus $\Omega(r_1)$, with f'/f not rational at infinity. Assume that f and $F = L[f]$ have no zeros in $\Omega(r_1)$, where L is as in (1.1) with the a_j analytic in $\Omega(r_1)$ and rational at infinity, and with $a_{k-1}(\infty) = 0$. Then f satisfies at least one of the following.*

(i) *The logarithmic derivative f'/f has a representation*

$$(1.5) \quad \frac{f'}{f} = -\frac{a_{k-1}}{k} - \left(\frac{k-1}{2}\right)\frac{H''}{H'} + H' \quad \text{or} \quad \frac{f'}{f} = -\frac{a_{k-1}}{k} - \left(\frac{k-1}{2}\right)\frac{H''}{H'} - m\frac{H'}{H},$$

where $m \in \mathbf{N}$ and $H_0 = H''/H'$ is rational at infinity, with $H_0(\infty) \neq 0$, while the equation $L[y] = 0$ has linearly independent local solutions y_j satisfying

$$(1.6) \qquad \frac{y_j'}{y_j} = -\frac{a_{k-1}}{k} - \left(\frac{k-1}{2}\right)\frac{H''}{H'} + (j-1)\frac{H'}{H}, \quad j = 1, \ldots, k,$$

and f is given locally by either $f = cy_1 \exp(y_2/y_1)$ or $f = cy_1^{m+1} y_2^{-m}$, where $c \in \mathbf{C} \setminus \{0\}$.

(ii) *There exist a polynomial Q and functions ν_1, ν_0, both rational at infinity, such that f'/f has a representation*

(1.7)
$$\frac{f'}{f} = \frac{Q(T)T'}{1 - e^{-T}} + \frac{y_1'}{y_1}, \quad T = \log\left(\frac{v}{u}\right),$$

where y_1 is a solution of $L[y] = 0$, while v and u are linearly independent solutions of

(1.8)
$$y'' + \nu_1 y' + \nu_0 y = 0$$

which continue without zeros in some annulus $\Omega(r_2)$. Here $Q(T)$ is rational at infinity, and u, v, y_1'/y_1 and $a_0, \ldots, a_{k-2}$ all have representations in terms of $Q(T)$, T, a_{k-1} and their derivatives. Moreover, if T' is not rational at infinity then k is even and $z^{-1/2} T'(z)$ is rational at infinity.

In both cases (i) and (ii) there exist $r_3 > 0$ and functions $\widetilde{a}_1$, $\widetilde{a}_0$, each rational at infinity, such that $f'' + \widetilde{a}_1 f' + \widetilde{a}_0 f$ has no zeros in $\Omega(r_3)$.

The conclusions of Theorems 1.1 and 1.3 are closely related, and the last assertion of Theorem 1.3 makes it clear that this is no coincidence. If Q is a constant d in (1.7) then integration shows that f is a constant multiple of $y_1 (v/u - 1)^d$. Conclusion (1.5) may be compared with that of Theorem 1.2, and links closely to (1.2) of Theorem 1.1 and [21, Theorem 1.3(II)]. Examples II and III in Section 2 demonstrate that in (1.7) the multiplicities of poles of f may be unbounded, in sharp contrast to the situation in Theorem 1.2, where any poles of f must all have the same multiplicity m. Example III also shows that T' need not be rational at infinity in (1.7).

Some previous partial results for rational coefficients may be found in [13, 17]. Methods from [2, 3, 4, 23] are essential to the proof of Theorem 1.3; these are supplemented by a result (Lemma 3.1) on integer-valued analytic functions, facilitating the analytic continuation of several asymptotic representations. A decisive role is played by a criterion (Lemma 13.1) for certain auxiliary functions to satisfy a second order differential equation, which simplifies the subsequent analysis considerably.

The author acknowledges extensive discussions and correspondence on this problem with the late Günter Frank; these took place over many years and have contributed substantially to the methodology of this paper. Indeed, the Wronskian-based method invented by Frank [3, 5] underpins much of the successful work on these and related problems. Thanks are also due to the referees for their valuable comments.

2. Examples

Throughout the paper c will be used to denote non-zero constants, not always the same at each occurrence, and $\mathbf{C}^*$ will denote $\mathbf{C} \setminus \{0\}$.

2.1. Example I. This example goes back to [4], and may be compared with conclusion (i) of Theorem 1.3 and that of Theorem 1.2. Let H be such that $\delta = H''/H' \not\equiv 0$ is a polynomial, and write

$$g = (H')^{-k} e^H, \quad h = (H')^{-k} H^{-m}, \quad D = \frac{d}{dz}, \quad m \in \mathbf{N}.$$

Then it is easy to check (see the remark following (6.5) below) that

$$(D + \delta) \ldots (D + k\delta)[g] = e^H, \quad (D + \delta) \ldots (D + k\delta)[h] = cH^{-m-k}.$$

Taking f to be $e^P g$ or $e^P h$ for a suitably chosen polynomial P gives polynomial coefficients a_j with $a_{k-1} = 0$ such that f and $F = L[f]$ have no zeros.

2.2. Example II. Let P be a non-constant polynomial which takes positive integer values at all zeros of $1 - e^z$, and write

$$(2.1) \quad \frac{f'(z)}{f(z)} = \frac{P(z)}{1 - e^z}, \quad \frac{f''(z)}{f(z)} = \frac{Q_1(z)e^z + Q_0(z)}{(1 - e^z)^2}, \quad Q_1 = P - P', \quad Q_0 = P' + P^2.$$

Then f is meromorphic and zero-free in the plane, with a pole of multiplicity $P(z)$ at a zero z of $1 - e^z$. A standard calculation yields polynomials R_j such that

$$\frac{f'''(z)}{f(z)} = \frac{R_2(z)e^{2z} + R_1(z)e^z + R_0(z)}{(1 - e^z)^3}.$$

If $F = f''' + b_2 f'' + b_1 f'$, where the b_j are rational functions, then

$$\frac{F(z)}{f(z)} = \frac{B_2(z)e^{2z} + B_1(z)e^z + B_0(z)}{(1 - e^z)^3},$$
$$B_2 = R_2 - b_2 Q_1 + b_1 P,$$
$$B_1 = R_1 + b_2(Q_1 - Q_0) - 2b_1 P,$$
$$B_0 = R_0 + b_2 Q_0 + b_1 P.$$

Thus F may be made zero-free in some $\Omega(r_1)$ by setting

$$0 = R_1 + b_2(Q_1 - Q_0) - 2b_1 P = R_0 + b_2 Q_0 + b_1 P,$$
$$(2.2) \qquad \frac{F(z)}{f(z)} = \frac{B_2(z)e^{2z}}{(1 - e^z)^3},$$

these equations being solvable for b_1 and b_2, since $(Q_1 - Q_0)P + 2Q_0 P = (Q_1 + Q_0)P \not\equiv 0$ by (2.1). Similar calculations show that it is possible to achieve each of

$$(2.3) \qquad \frac{F(z)}{f(z)} = \frac{B_1(z)e^z}{(1 - e^z)^3}; \quad \frac{F(z)}{f(z)} = \frac{B_0(z)}{(1 - e^z)^3}.$$

Finally, should it be the case that $b_2(\infty) \neq 0$, there exist a polynomial Q_2 and rational functions a_j, with $a_2(\infty) = 0$, such that writing $h = e^{Q_2} f$ gives

$$\frac{F(z)}{f(z)} = \frac{f''' + b_2 f'' + b_1 f'}{f} = \frac{h''' + a_2 h'' + a_1 h' + a_0 h}{h}.$$

2.3. Example III. This is adapted from [19]. Let $Y(z) = z^{m/2}$, where $m \in \mathbf{N}$, and set $h = \cosh Y$. Then h is entire with only simple zeros. Let P_1 be an even polynomial which takes negative integer values at all odd integer multiples of $\pi i/2$, and set $P = P_1(Y)$. Then P is a polynomial and setting

$$\frac{f'}{f} = P \cdot \frac{h'}{h} = \frac{PY' \sinh Y}{\cosh Y} = \frac{-2P_1(Y)Y'}{1 + e^{2Y}} + P_1(Y)Y'$$

defines f as a meromorphic function in the plane, with no zeros. Next, set $R = f'' + b_1 f' + b_0 f$, where $b_1 = -P'/P - Y''/Y'$ and $b_0 = -(PY')^2$. This gives, since $h'' = (Y''/Y')h' + (Y')^2 h$,

$$\frac{R}{f} = (P - P^2)\left((Y')^2 - \left(\frac{h'}{h}\right)^2\right) = \frac{(P - P^2)(Y')^2}{h^2},$$

and so R is zero-free in some $\Omega(r_1)$. Moreover, $S = R(P - P^2)^{-1}(Y')^{-2}$ satisfies $S/f = h^{-2}$ and $S'/S = (P - 2)h'/h$. Hence the same construction, with P replaced by $P - 2$, gives rational functions c_j, d_j and e_j such that

$$\frac{S'' + c_1 S' + c_0 S}{S} = \frac{R'' + d_1 R' + d_0 R}{R} = \frac{f^{(4)} + e_3 f^{(3)} + \ldots + e_0 f}{R} = \frac{F}{R}$$

is free of zeros in some $\Omega(r_2)$, as is F.

3. Preliminaries

Lemma 3.1. *Let the function g be analytic on the half-plane H^+ given by $\operatorname{Re} z \geq 0$, such that $g(n) \in \mathbf{Z}$ for all $n \in \mathbf{Z} \cap H^+$ and $|g(z)| = o\left(2^{|z|}\right)$ as $z \to \infty$ in H^+. Then g is a polynomial. Next, let $h(z) = e^{2\pi i \alpha z} u(z)$, where $\alpha \in \mathbf{R}$ and u is analytic on H^+, with $\log^+ |u(z)| = o(|z|)$ as $z \to \infty$ in H^+, and assume that $h(n) = 1$ for all large $n \in \mathbf{N}$. Then $u(z) \equiv 1$ and $\alpha \in \mathbf{Z}$.*

Proof. The first assertion is proved in [20]. To prove the second part let $\delta_1 \in (0, \infty)$ be small: then there exist $p \in \mathbf{Z}$ and $q \in \mathbf{N}$ such that

$$\alpha = s + t, \quad s = \frac{p}{q}, \quad |t| < \frac{\delta_1}{2\pi q}.$$

Here $t = 0$ if α is rational, while if $\alpha \notin \mathbf{Q}$ then suitable p and q exist by Dirichlet's approximation theorem [9, p. 155]. Write

$$z = qw, \quad F(w) = e^{2\pi i t q w} u(qw).$$

If $n \in \mathbf{N}$ is large then

$$1 = h(qn) = e^{2\pi i \alpha q n} u(qn) = e^{2\pi i (pn + tqn)} u(qn) = F(n).$$

Thus F is a polynomial, by the first part, and so $F(w) \equiv 1$. Moreover, $t = 0$, because otherwise there exists $\theta \in \{-\pi/4, \pi/4\}$ such that $F(re^{i\theta}) \to 0$ as $r \to +\infty$, and so $u \equiv 1$. Finally, $\alpha = p/q$ must be an integer, since $1 = h(qn + 1) = \exp(2\pi i p/q)$ for large $n \in \mathbf{N}$. $\qquad \square$

Lemma 3.2. *Let d_1, d_2 and λ be positive constants and let g be a zero-free analytic function on the half-plane $\operatorname{Re}(w) > 0$, with $\log^+ |g(w)| \leq d_1 + d_2 |w|^\lambda$ there. Then for $0 < \alpha < \pi/2$ there exists $\mu = \mu_\alpha > 0$ such that $\log^+ |1/g(w)| \leq \mu_\alpha |w|^{1+\lambda}$ as $w \to \infty$ with $|\arg w| \leq \alpha$.*

Proof. This is standard: set $w = (1 + z)/(1 - z)$ and $g(w) = G(z)$ for $|z| < 1$. With $\rho = (1 + r)/2$ this leads to

$$\log M(r, 1/G) \leq \left(\frac{\rho + r}{\rho - r}\right) T(\rho, 1/G) \leq \left(\frac{\rho + r}{\rho - r}\right) (\log M(\rho, G) + O(1))$$
$$= O(1 - r)^{-1-\lambda}$$

as $r \to 1-$. It remains only to observe that there exists $c_1 = c_1(\alpha) > 0$ such that if $|w|$ is large and $|\arg w| \leq \alpha < \pi/2$ then $(1 - |z|^2)^{-1} \leq c_1 |w|$. $\qquad \square$

Lemma 3.3. *Suppose that p and q are (both formal or both locally analytic) solutions of the equations*

(3.1) $$\frac{p'}{p} = d_0 \frac{q'}{q} + d_1, \quad q'' + \nu_1 q' + \nu_0 q = 0,$$

where the d_j and ν_j are rational at infinity, and let L be as in (1.1). Then there exist coefficients b_j, each rational at infinity, such that p and q satisfy

$$(3.2) \qquad \frac{L[p]}{p} = \sum_{j=0}^{k} b_j \left(\frac{q'}{q}\right)^j, \quad b_k = d_0(d_0 - 1)\dots(d_0 - k + 1).$$

Moreover, if $d_0 \not\equiv 0, 1$ and e_1 and e_0 are rational at infinity, then there exist coefficients E_μ, each rational at infinity and depending only on the d_j, e_j and ν_j, such that $E_2 \not\equiv 0$ and

$$(3.3) \qquad E_2\frac{p''}{p} + E_1\frac{p'}{p} + E_0 = \left(\frac{p'}{p}\right)^2 + e_1\frac{p'}{p} + e_0.$$

Proof. Formula (3.2) follows from (3.1) and a simple induction argument, which deliver

$$\left(\frac{q'}{q}\right)' = -\left(\frac{q'}{q}\right)^2 - \nu_1\left(\frac{q'}{q}\right) - \nu_0, \quad \frac{p^{(m)}}{p} = \sum_{j=0}^{m} b_{j,m}\left(\frac{q'}{q}\right)^j, \quad m \in \mathbf{N},$$

with the $b_{j,m}$ rational at infinity and $b_{m,m} = d_0(d_0 - 1)\dots(d_0 - m + 1)$.

To prove the second part, suppose that $d_0 \not\equiv 0, 1$ and write $P = p'/p$ and $Q = q'/q$ so that $Q = AP + B$, with A, B rational at infinity and $A \not\equiv 0, 1$. This yields

$$0 = A'P + AP' + B' + A^2P^2 + 2ABP + B^2 + \nu_1(AP + B) + \nu_0$$
$$= (A^2 - A)P^2 + A(P' + P^2) + (A' + 2AB + \nu_1 A)P + B' + B^2 + \nu_1 B + \nu_0,$$

and so

$$(A - A^2)(P^2 + e_1P + e_0) = A(P' + P^2) + (A' + 2AB + \nu_1 A + (A - A^2)e_1)P$$
$$+ B' + B^2 + \nu_1 B + \nu_0 + (A - A^2)e_0. \qquad \square$$

Lemma 3.4. *Suppose that u and v are linearly independent (both formal or both locally analytic) solutions of an equation*

$$y''' + B_2y'' + B_1y' + B_0y = 0,$$

with the B_j rational at infinity, and assume that $W = W(u, v) = uv' - u'v$ is such that $W' + E_1W = 0$, where E_1 is rational at infinity. Then u and v solve an equation

$$(3.4) \qquad y'' + E_1y' + E_0y = 0,$$

where E_0 is also rational at infinity.

Proof. Since u and v are solutions of the equation $W(u, v, y) = 0$, it is enough to prove that $W(u', v') = E_2W(u, v)$ with E_2 rational at infinity. But $W' = uv'' - u''v = -E_1W$ leads to

$$(-E_1' + E_1^2)W = W'' = u'v'' - u''v' + uv''' - u'''v$$
$$= W(u', v') + v(B_2u'' + B_1u' + B_0u) - u(B_2v'' + B_1v' + B_0v)$$
$$= W(u', v') - B_2W' - B_1W = W(u', v') + (E_1B_2 - B_1)W. \qquad \square$$

4. Asymptotics for linear differential equations

As in [2] a fundamental role will be played by formal and asymptotic expansions for solutions of linear differential equations. For an equation $L[y] = 0$, with L as in (1.1) and the a_j rational at infinity, classical results (see [24, Theorem 19.1] or

[2, 16])) show that there exist $p \in \mathbf{N}$ and a fundamental set of k linearly independent formal solutions

$$(4.1) \qquad \widetilde{h}_j(z) = \exp(P_j(z^{1/p}))z^{\gamma_j}\sum_{\mu=0}^{n_j}U_{j,\mu}(z^{1/p})(\log z)^\mu$$

which satisfy the following: γ_j is a complex number; n_j is a non-negative integer; the *exponential part* $P_j(z^{1/p})$ is a polynomial in $z^{1/p}$; the $U_{j,\mu}(z^{1/p})$ are formal series in descending integer powers of $z^{1/p}$, that is, in which at most finitely many positive powers occur; the lead series U_{j,n_j} is not the zero series. Formal solutions (not necessarily linearly independent) with these properties will be referred to henceforth as *canonical formal solutions*.

A standard approach [24] to obtaining these $\widetilde{h}_j(z)$ is to transform a solution h of $L[y] = 0$ into a vector $\mathbf{h} = (h, h', \ldots, h^{(k-1)})$, so that a fundamental solution set for $L[y] = 0$ corresponds to the first row of a matrix solution $V(z) = U(z)z^G e^{Q(z)}$ of an equation $Y' = A(z)Y$, where $Q(z)$ is a diagonal matrix, its entries polynomials in $z^{1/p}$, while G is a constant matrix, which may be assumed to be in Jordan form, and $U(z)$ is a matrix with entries which are formal series in descending integer powers of $z^{1/p}$. Furthermore, for each $\theta \in \mathbf{R}$ there exists $\delta = \delta(\theta) > 0$ such that $L[y] = 0$ has a fundamental set of analytic solutions

$$(4.2) \qquad h_j(z) = \exp(P_j(z^{1/p}))z^{\gamma_j}\sum_{\mu=0}^{n_j}V_{j,\mu}(z^{1/p})(\log z)^\mu$$

on a sector S given by $|z| > R_0 > 0$, $|\arg z - \theta| < \delta$, in which each $V_{j,\mu}(z^{1/p})$ is analytic on S and satisfies $V_{j,\mu}(z^{1/p}) \sim U_{j,\mu}(z^{1/p})$ as $z \to \infty$ on S, in the sense of asymptotic series (see [24, Theorem 19.1] or [22]). Here $W(z) \sim U(z) = \sum_{m=M}^\infty U_m z^{-m/p}$ as $z \to \infty$ on S means that, for each $n \geq M$,

$$W(z) - \sum_{m=M}^{n}U_m z^{-m/p} = o(|z|^{-n/p}) \quad \text{as } z \to \infty \text{ in } S.$$

It may be assumed that the exponential parts $P_j(z^{1/p})$ have zero constant term, and this convention will be used throughout. Given any exponential part $P_j(z^{1/p})$ arising for $L[y] = 0$, there is always a canonical formal solution with exponential part $P_j(z^{1/p})$ which is free of logarithms, that is, has $n_j = 0$; this holds because the matrix G may be chosen to be in Jordan form. The following lemma is well known [2, 16, 24].

Lemma 4.1. *Given k linearly independent canonical formal solutions of $L[y] = 0$ with exponential parts $q_1, \ldots, q_k$, their formal Wronskian has exponential part $\sum_{j=1}^{k} q_j$, and the exponential parts of any fundamental set of canonical formal solutions of $L[y] = 0$ form a permutation of the q_j.* $\square$

For the special case of a second order equation, suppose that A^* is rational at infinity, with $A^*(z) = (1 + o(1))c_n z^n$ as $z \to \infty$, where $c_n \in \mathbf{C}^*$ and $n \geq -1$. Then infinity is an irregular singular point for

$$(4.3) \qquad w'' + A^* w = 0,$$

and asymptotics are developed via Hille's method [14] as follows. The *critical rays* are given by $\arg z = \theta^*$, where $c_n e^{i(n+2)\theta^*}$ is real and positive. If $0 < \beta < 2\pi/(n+2)$

then, in a sector given by $|z| > r_1$, $|\arg z - \theta^*| < \beta$, there exist linearly independent analytic solutions, for $j = 1, 2$,

$$\phi_j(z) = A^*(z)^{-1/4}(1 + o(1))\exp((-1)^j iZ),$$

(4.4)

$$Z = \int^z A^*(t)^{1/2}\, dt = \frac{2c_n^{1/2} z^{(n+2)/2}}{n + 2} + \dots.$$

If $n = -1$ then this sector should be understood as lying on the Riemann surface of $\log z$. To one side of the critical ray, one of these solutions is large and the other small, and these roles are reversed as the critical ray is crossed. Any linear combination $D_1 \phi_1 + D_2 \phi_2$ with $D_1, D_2 \in \mathbf{C}^*$ has a sequence of zeros tending to infinity near the critical ray. Moreover, the corresponding formal solutions, to which the ϕ_j are asymptotic, may be calculated readily from $A^*(z)$ and Z, with $n_j = 0$ and $p \in \{1, 2\}$ in (4.1) (see [19] for details).

Lemma 4.2. *Suppose that $F_1, \dots, F_k$ are formal expressions, each of which is given by*

$$F_j(z) = U_j(z) z^{\gamma_j} r_j(z) \exp(q_j(z)),$$

with $\gamma_j \in \mathbf{C}$, r_j a rational function, q_j a polynomial and $U_j(z) = 1 + O(1/z)$, in which $O(1/z)$ denotes a formal series in negative integer powers of z. Assume that none of the r_j vanishes identically, and that $q_j - q_{j'}$ is non-constant for $j \neq j'$. Then the formal Wronskian $W = W(F_1, \dots, F_k)$ has an expansion

$$W(z) = \left(1 + O\left(\frac{1}{z}\right)\right)\left(\prod_{j=1}^{k}[z^{\gamma_j} r_j(z)\exp(q_j(z))]\right)\left(\prod_{k \geq m > n \geq 1}[q_m'(z) - q_n'(z)]\right).$$

Proof. This is standard, and is proved by induction on k, using

$$W(F_1, \dots, F_k) = F_1^k W\left(1, \frac{F_2}{F_1}, \dots, \frac{F_k}{F_1}\right) = F_1^k W\left(\left(\frac{F_2}{F_1}\right)', \dots, \left(\frac{F_k}{F_1}\right)'\right),$$

$$\left(\frac{F_j}{F_1}\right)'(z) = \left(1 + O\left(\frac{1}{z}\right)\right) z^{\gamma_j - \gamma_1}\left(\frac{r_j(z)}{r_1(z)}\right)(q_j'(z) - q_1'(z))\exp(q_j(z) - q_1(z)). \quad \square$$

5. Beginning the proof of Theorem 1.3: Frank's method

Assume that f is as in the hypotheses of Theorem 1.3. Let $f_1, \dots, f_k$ be linearly independent locally analytic solutions of $L[y] = 0$. Frank's method [3, 4] defines g, h, w_j and Y locally by

(5.1) $\quad g^k = \dfrac{f}{F}, \quad h = -\left(\dfrac{f'}{f}\right)g, \quad w_j = f_j'g + f_j h = fg\left(\dfrac{f_j}{f}\right)', \quad \dfrac{Y'}{Y} = -a_{k-1}.$

Note that g might not be meromorphic in $\Omega(r_1)$, but g'/g is, and has a simple pole with residue 1 at every pole of f; moreover, at a pole of f of multiplicity m_0, calculating the leading Laurent coefficient of F/f gives

(5.2) $\quad (g')^{-k} = (-1)^k m_0(m_0 + 1)\cdots(m_0 + k - 1).$

Now write locally, using Abel's identity, $W(f_1, \dots, f_k) = cY$ and

$$\frac{Y}{(fg)^k} = \frac{YF}{f^{k+1}} = \frac{cW(f_1, \dots, f_k, f)}{f^{k+1}} = cW\left(\frac{f_1}{f}, \dots, \frac{f_k}{f}, 1\right) = cW\left(\left(\frac{f_1}{f}\right)', \dots, \left(\frac{f_k}{f}\right)'\right),$$

so that $Y = cW(w_1, \ldots, w_k)$ by (5.1). Hence the w_j are linearly independent (local) solutions of an equation $M[y] = 0$, where

$$(5.3) \quad M = D^k + A_{k-1}D^{k-1} + A_{k-2}D^{k-2} + \ldots + A_0, \quad A_{k-1} = a_{k-1}, \quad D = \frac{d}{dz}.$$

Here a pivotal role is played by whether or not the differential operators L and M are the same, and Brüggemann's method in [2] depends on reducing the problem to the case $L = M$. It will be proved in Proposition 6.1 below that if $L = M$ then all poles z of f with $|z|$ sufficiently large have the same multiplicity. Thus Example II in Section 2 demonstrates that L and M can indeed be different operators.

By Frank's method, the A_j are analytic in some annulus $\Omega(r_1)$ and satisfy $T(r, A_j) = S(r, f'/f)$, where $S(r, f'/f)$ denotes any term which is $O(\log T(r, f'/f) + \log r)$ as $r \to \infty$, possibly outside a set of finite measure [11]: see [6, Section 2 and Lemmas A, B and 5] for details, including the Nevanlinna characteristic in $\Omega(r_1)$. Denote by Λ the field generated by the a_j, A_j and their derivatives: then $T(r, \lambda) = S(r, f'/f)$ for all $\lambda \in \Lambda$.

To simplify the subsequent calculations it is convenient to write

$$(5.4) \quad \begin{aligned} -k\frac{X'}{X} &= a_{k-1} = A_{k-1}, \quad p = \frac{f}{X}, \quad p_j = \frac{f_j}{X}, \\ q &= -\left(\frac{p'}{p}\right)g, \quad t_j = p'_j g + p_j q = \frac{w_j}{X}. \end{aligned}$$

It is then well known that there exist equations

$$(5.5) \qquad\qquad y^{(k)} + c_{k-2}y^{(k-2)} + \ldots + c_0 y = 0,$$

and

$$(5.6) \qquad\qquad y^{(k)} + C_{k-2}y^{(k-2)} + \ldots + C_0 y = 0,$$

in which the c_j and C_j all belong to Λ, and $c_{k-2} = C_{k-2}$ if and only if $a_{k-2} = A_{k-2}$, such that $L[Xy] = 0$ if and only if y solves (5.5), and $M[Xy] = 0$ if and only if y solves (5.6). In particular, the p_j and t_j are linearly independent local solutions of (5.5) and (5.6) respectively. The following lemma [4] is key to Frank's method: see also [6, Lemma C].

Lemma 5.1. [4] *Let G, Φ, $p_1, \ldots, p_k$, $c_0, \ldots, c_{k-2}$ and $C_0, \ldots, C_{k-2}$ be analytic functions on a plane domain U, such that $p_1, \ldots, p_k$ are linearly independent solutions of (5.5). Then the functions $p'_1 G + p_1 \Phi, \ldots, p'_k G + p_k \Phi$ are solutions in U of the equation (5.6) if and only if, with the notation $C_k = 1$ and $c_{k-1} = C_{k-1} = c_{-1} = C_{-1} = 0$ and*

$$M_{k,\mu}[w] = \sum_{m=\mu}^{k} \frac{m!}{\mu!(m-\mu)!} C_m w^{(m-\mu)} \quad (0 \le \mu \le k), \quad M_{k,-1}[w] = 0,$$

the functions G and Φ satisfy, for $0 \le \mu \le k-1$,

$$(5.7) \qquad M_{k,\mu}[\Phi] - c_\mu \Phi = -M_{k,\mu-1}[G] + c_\mu M_{k,k-1}[G] + (c'_\mu + c_{\mu-1})G. \qquad \square$$

Because the proof of Lemma 5.1 is based on purely formal calculations, an analogous statement holds linking formal solutions of (5.5), (5.6) and (5.7). Since the coefficient of Φ in $M_{k,\mu}[\Phi] - c_\mu \Phi$ is $c_{0,\mu} = D_\mu = C_\mu - c_\mu$, the equations (5.7) may be

written in the form

$$(5.8) \qquad T_\mu[G] = S_\mu[\Phi] = \sum_{j=0}^{k-\mu} c_{j,\mu} \Phi^{(j)}, \quad c_{0,\mu} = D_\mu = C_\mu - c_\mu, \quad 0 \le \mu \le k-1,$$

in which T_μ and S_μ are linear differential operators with coefficients in Λ. In particular these equations are satisfied by $G = g$, $\Phi = q$. Taking $\mu = k - 1$ in (5.7) produces

$$(5.9) \qquad -\Phi' = -U[G] = \frac{(k-1)G''}{2} + \frac{D_{k-2}G}{k}.$$

Now $\mu = k - 2$ and (5.9) give (as in [7, p. 162] or [17, Lemma 8, pp. 307-8])
$$(5.10)$$
$$D_{k-2}\Phi = \frac{k(k^2-1)}{12}G''' + G'\left(-\frac{(k+1)D_{k-2}}{2} + 2C_{k-2}\right) + G\left(\frac{k-1}{2}D'_{k-2} + c'_{k-2} - D_{k-3}\right).$$

Next, combining (5.9) with (5.7) for $\mu = k - 3$ yields, with d_j denoting elements of Λ,

$$(5.11) \qquad \frac{2D_{k-3}}{k-2}\Phi = \frac{k(k^2-1)}{12}G^{(4)} + G''\left(\frac{(k-1)D_{k-2}}{3} + 2C_{k-2}\right) + d_1 G' + d_2 G.$$

Note that (5.11) holds even if $k = 3$, in which case $M_{k,k-4}[G] = 0$. Differentiating (5.10) and using (5.9) and (5.11) leads to

$$(5.12) \qquad D^*\Phi = \left(\frac{2D_{k-3}}{k-2} - D'_{k-2}\right)\Phi = \frac{(k+2)D_{k-2}}{3}G'' + d_3 G' + d_4 G.$$

Lemma 5.2. *There exists a non-trivial homogeneous linear differential equation $N_1[y] = 0$, of order at most 3 and with coefficients in Λ, with the property that if the pair $\{G, \Phi\}$ solves the system (5.8) then G solves $N_1[y] = 0$.*

Proof. If $D_{k-2} \equiv 0$ this is clear from (5.10), so assume that $D_{k-2} \not\equiv 0$. If D^* vanishes identically in (5.12) then a second order equation arises for G, while otherwise combining (5.12) with (5.10) yields a third order equation. $\qquad \square$

Consider now two cases.

Case 1. Assume that $c_{0,\mu} = C_\mu - c_\mu \equiv 0$ for $0 \le \mu \le k-1$ in (5.8). This is equivalent to the equations (5.5) and (5.6) being the same, and hence equivalent to the operators L and M being identical. In this case, $t_j = p'_j g + p_j q$ is a solution of (5.5) for $1 \le j \le k$. Since p_1 and p_2 are linearly independent, $p_1 p'_2 - p'_1 p_2$ does not vanish identically and so (5.4) yields

$$H_1 = \frac{f'}{f} + \frac{a_{k-1}}{k} = \frac{p'}{p} = -\frac{q}{g} = \frac{p'_1 t_2 - p'_2 t_1}{p_1 t_2 - p_2 t_1}.$$

For $\theta \in \mathbf{R}$ and $\kappa \in \mathbf{C}$ the number of distinct zeros of $\kappa - H_1$ in $r_1 + 1 \le |z| \le r$, $|\arg z - \theta| \le \pi/4$, is at most the number of zeros of $p'_1 t_2 - p'_2 t_1 - \kappa(p_1 t_2 - p_2 t_1)$ there, which is bounded by a power of r as $r \to \infty$, by [6, Lemma 2] or standard sectorial methods. Hence f'/f has finite order of growth, by the second fundamental theorem, and every $\lambda \in \Lambda$ is rational at infinity.

Case 2. Assume that the coefficient of Φ in at least one of the S_μ in (5.8) is not identically zero, this being equivalent to $L \neq M$.

Let ν be the largest integer with $0 \leq \nu \leq k - 1$ such that $c_{0,\nu} \not\equiv 0$. Then every pair $\{G, \Phi\}$ satisfying the system (5.8) (including $\{g, q\}$) has

$$(5.13) \qquad \Phi = (c_{0,\nu})^{-1} \left(T_\nu[G] - \sum_{j=1}^{k-\nu} c_{j,\nu} \frac{d^{j-1}}{dz^{j-1}}(U[G]) \right) = T^*[G],$$

by (5.9). Observe that the operator T^* has order at least 1, and in particular is not the zero operator, since otherwise (5.4) leads to $(-p'/p)g = q = \eta_1 g$, with $\eta_1 \in \Lambda$, so that $f'/f \in \Lambda$, and hence f'/f is rational at infinity, contrary to assumption. It follows from (5.8), (5.9) and (5.13) that if $\{G, \Phi\}$ solves (5.8) then G solves the system

$$(5.14) \qquad U[G] = \frac{d}{dz}(T^*[G]), \quad S_\mu(T^*[G]) = T_\mu[G], \quad 0 \leq \mu \leq k - 2,$$

as does, in particular, g. Conversely, if G solves the system (5.14) (in the analytic or formal sense), then (5.8) is satisfied by setting $\Phi = T^*[G]$. This system (5.14) cannot be trivial, because otherwise (5.8) holds with $\Phi = T^*[G]$ and an arbitrary choice of G, which would then have to solve the equation $N_1[y] = 0$ of Lemma 5.2. A standard reduction procedure [15, p.126] now generates a non-trivial homogeneous linear differential equation $N[y] = 0$, with coefficients in the field Λ, whose (analytic or formal) solution space coincides with that of the system (5.14). Here every solution G of $N[y] = 0$ is such that the pair $\{G, T^*[G]\}$ solves (5.8), and so G solves $N_1[y] = 0$, from which it follows that N has order at most 3.

Suppose that N has order 1: then $g'/g \in \Lambda$, and so p'/p and f'/f belong to Λ, by (5.4) and (5.13), so that f'/f is rational at infinity, contrary to assumption. Thus N has order at least 2, but at most 3, and the system (5.14) has a solution G with G/g non-constant. By an argument from [4] (see [6, Proof of Theorem 3, Case 1B] for details), p'/p has a representation as a rational function in the p_j and their derivatives. The same sectorial argument as used in Case 1 shows that f'/f has finite order of growth, as has $g^k = f/F$, and all members of the field Λ are rational at infinity.

Hence the fact that N has order at most 3 gives an operator V_2, having order at most 2 and coefficients which are rational at infinity, with the following property. Every solution G of (5.14) has $T^*[G] = V_2[G]$, so that the pair $\{G, V_2[G]\}$ solves (5.8), and $p_j'G + p_j V_2[G]$ solves (5.6) for $j = 1, \ldots, k$, by Lemma 5.1. Moreover, (5.13) gives $q = T^*[g] = V_2[g]$. With $V = V_2 + a_{k-1}/k$, this implies using (5.4) that each $f_j'G + f_j V[G]$ solves $M[y] = 0$, and $-f'/f = h/g = V[g]/g$. It now follows, using the Wiman-Valiron theory [12] and estimates for logarithmic derivatives [8] applied to g, that f has finite order. It also follows that f has an unbounded sequence of poles, since otherwise f'/f is rational at infinity. The following key lemma has thus been proved.

Lemma 5.3. *With the hypotheses of Theorem 1.3, the function f'/f has finite order, all elements of the field Λ are rational at infinity, and $h = -(f'/f)g$ satisfies*

$$(5.15) \qquad -h' = \left(\frac{k-1}{2} \right) g'' - \frac{a_{k-1}}{k} g' + \frac{D_{k-2} - a_{k-1}'}{k} g, \quad D_j = C_j - c_j.$$

Furthermore, if the operators L and M are not the same then the following additional conclusions hold. The function g solves a homogeneous linear differential equation $N[y] = 0$, of order 2 or 3, with coefficients which are rational at infinity. Moreover,

f has finite order and an unbounded sequence of poles, and there exist functions α, β, γ, all rational at infinity, such that

$$(5.16) \qquad h = -\left(\frac{f'}{f}\right)g = V[g], \quad V = \alpha D^2 + \beta D + \gamma.$$

Finally, if G is a locally analytic solution of $N[y] = 0$, and K is a locally analytic solution of $L[y] = 0$, then $K'G + KV[G]$ is a (possibly trivial) solution of $M[y] = 0$. □

Here (5.15) follows from (5.4) and (5.9). The last assertion of Lemma 5.3 also holds for formal solutions G and K of $N[y] = 0$ and $L[y] = 0$ respectively.

6. The first special case

Proposition 6.1. *With the hypotheses of Theorem 1.3, suppose in addition that $c_{k-2} = C_{k-2}$ in (5.5) and (5.6), which holds if and only if $a_{k-2} = A_{k-2}$ in L and M, and certainly holds if the operators L and M are the same. Then f satisfies conclusion (i) of Theorem 1.3.*

Proof. The approach here is essentially due to Frank and Hellerstein [4]. Since $D_{k-2} = 0$ in (5.15), integration gives a constant d such that

$$(6.1) \qquad \frac{f'}{f} = -\frac{h}{g} = \frac{(k-1)g'}{2g} + \frac{d}{g} - \frac{a_{k-1}}{k}.$$

If $d = 0$ then comparing residues shows that f has no poles in some $\Omega(r_2)$ and F/f, which has finite order by Lemma 5.3, satisfies $g^{-k} = F/f = R_1 e^{P_1}$ with R_1 rational at infinity and P_1 a polynomial, so that f'/f is rational at infinity, by (6.1), contrary to assumption.

Assume henceforth that $d \neq 0$ in (6.1), which makes g meromorphic of finite order in $\Omega(r_1)$. Suppose that f has no poles in some $\Omega(r_2)$. Then g has no zeros and poles there and $g = R_2 e^{P_2}$ in (6.1), with R_2 rational at infinity and P_2 a polynomial. This gives, since f'/f is not rational at infinity,

$$(6.2) \qquad \frac{f'}{f} = R + S e^P, \quad SP' \not\equiv 0,$$

where R and S are rational at infinity and P is a polynomial. It follows using [11, Lemma 3.5] that

$$(6.3) \qquad \begin{aligned} \frac{1}{g^k} &= \frac{F}{f} \\ &= S^k e^{kP} + e^{(k-1)P}\left(kS^{k-1}R + a_{k-1}S^{k-1} + \frac{k(k-1)}{2}S^{k-2}(S' + P'S)\right) + \dots. \end{aligned}$$

Since F/f has neither zeros nor poles in $\Omega(r_2)$, the coefficient of $e^{(k-1)P}$ must vanish identically, leading to the first equation of (1.5), with $H' = Se^P$, and to $F/f = S^k e^{kP} = (H')^k$. Here H''/H' does not vanish at infinity, because P' does not.

Suppose next that f has an unbounded sequence of poles. At a pole z of f, with $|z|$ large and with multiplicity m, equations (5.2) and (6.1) deliver

$$\frac{1}{d^k} = \chi(m) = \frac{m(m+1)\dots(m+k-1)}{(m+(k-1)/2)^k},$$

so that d^k must be real and greater than 1, by the arithmetic-geometric mean inequality. A further application of the same inequality to χ'/χ shows that all

poles z of f with $|z|$ sufficiently large have fixed multiplicity m. Set $T_1 = f'/f$. Since g^k and T_1 have finite order, standard estimates [8] give $M_1 > 0$ such that $T_1^{(j)}(z)/T_1(z) = O(|z|^{M_1})$ and $g^{(j)}(z)/g(z) = O(|z|^{M_1})$, for $|z|$ outside a set F_0 of finite measure and $1 \le j \le k$. If $|z| \notin F_0$ and $\log^+ |T_1(z)|/\log |z|$ is sufficiently large this leads, using (6.1) and [11, Lemma 3.5], to

$$\frac{1}{g(z)^k} = \frac{F(z)}{f(z)} = T_1(z)^k + \ldots = (1 + o(1))T_1(z)^k = (1 + o(1))\frac{d^k}{g(z)^k},$$

which is a contradiction since $d^k > 1$. Thus $\log^+ |f'(z)/f(z)| = O(\log |z|)$ for $|z|$ outside a set of finite measure and applying the Wiman–Valiron theory [12] to $1/f$ shows that f has finite order. Furthermore, since f and g have finite order and all poles z of f with $|z|$ sufficiently large have fixed multiplicity m, the function $G_0 = f'/f + mg'/g$ is rational at infinity. Substituting $f'/f = -mg'/g + G_0$ into (6.1) produces a first order linear differential equation for g of form

$$g' + \delta g = d_0,$$

with $d_0 \in \mathbf{C}$ and δ rational at infinity, and with $\delta(\infty) \ne 0$, because $f/F = g^k$ has an essential singularity at infinity. This equation may be solved to give $g = d_0 H/H'$, where $H''/H' = \delta$ and $H(z) \ne \infty$ and $H'(z) \ne 0$ for large z in a sector containing an unbounded sequence of poles of f. It follows, using (6.1) again, that

$$(6.4) \qquad \frac{g'}{g} = \frac{H'}{H} - \frac{H''}{H'}, \quad \frac{f'}{f} = -\frac{a_{k-1}}{k} - \left(\frac{k-1}{2}\right)\frac{H''}{H'} + d_1\frac{H'}{H}, \quad d_1 \in \mathbf{C}.$$

Now comparing residues shows that $d_1 = -m$ in (6.4), giving the second equation of (1.5).

To determine the solutions of $L[y] = 0$, write

$$\phi = (H')^{-k}e^H, \quad \psi = (H')^{-k}H^{-m}, \quad \delta = \frac{H''}{H'}, \quad M_k = (D + \delta)\cdots(D + k\delta), \quad D = \frac{d}{dz}.$$

Then it is easy to verify that

$$(6.5) \qquad \Phi = M_k[\phi] = e^H, \quad \Psi = M_k[\psi] = cH^{-m-k}, \quad M_k\left[(H')^{-k}P_{k-1}(H)\right] = 0,$$

where P_{k-1} denotes any polynomial of degree at most $k-1$. In fact, the action of the differential operator M_k on ϕ, ψ and $(H')^{-k}P_{k-1}(H)$ amounts to k times differentiating with respect to H the terms e^H, H^{-m} and $P_{k-1}(H)$. Define Z locally by

$$\frac{Z'}{Z} = -\frac{a_{k-1}}{k} + \left(\frac{k+1}{2}\right)\frac{H''}{H'}.$$

Then a standard change of variables gives $L_k = D^k + \ldots + \widetilde{A}_1 D + \widetilde{A}_0$, with coefficients which are readily computable and rational at infinity, such that $L_k[Zy] = ZM_k[y]$, and the last equation of (6.5) shows that $L_k[w] = 0$ has linearly independent solutions y_j given locally by (1.6).

The next step is to show that $L_k = L$. When f has no poles in some $\Omega(r_2)$, combining the first equation of (1.5) with (6.3) and the remarks immediately following it yields

$$Z\phi = cf, \quad \frac{L[f]}{f} = \frac{F}{f} = S^k e^{kP} = (H')^k = \frac{\Phi}{\phi} = \frac{M_k[\phi]}{\phi} = \frac{ZM_k[\phi]}{cf} = \frac{L_k[cf]}{cf} = \frac{L_k[f]}{f}.$$

Thus the operators L and L_k must agree: otherwise f satisfies a homogeneous linear differential equation with coefficients which are rational at infinity, and so has finite

order, contradicting (6.2). On the other hand, when f has an unbounded sequence of poles, (1.5) and (6.4) lead to

$$Z\psi = cf, \quad \frac{L[f]}{f} = \frac{F}{f} = \frac{1}{g^k} = c\left(\frac{H'}{H}\right)^k = \frac{c\Psi}{\psi} = \frac{cM_k[\psi]}{\psi} = \frac{cZM_k[\psi]}{f} = \frac{cL_k[f]}{f}.$$

Again the operators L and cL_k must agree, and c must be 1, because otherwise f cannot have an unbounded sequence of poles. Thus, in both cases, the y_j solve $L[y] = 0$. Next, using (1.5) and (1.6) shows, after multiplying y_2 by a constant if necessary, that

$$\frac{f'}{f} = \frac{y_1'}{y_1} + \left(\frac{y_2}{y_1}\right)' \quad \text{or} \quad \frac{f'}{f} = \frac{y_1'}{y_1} - m\left(\frac{y_2'}{y_2} - \frac{y_1'}{y_1}\right).$$

This gives $f = cy_1 \exp(y_2/y_1)$ or $f = cy_1^{m+1}y_2^{-m}$ as asserted.

Finally, set $\widetilde{M}_2 = (D + (k-1)\delta)(D + k\delta)$. There exists an operator $\widetilde{L}_2 = D^2 + \widetilde{a}_1 D + \widetilde{a}_0$, with coefficients which are rational at infinity, such that $\widetilde{L}_2[Zy] = Z\widetilde{M}_2[y]$ and

$$\widetilde{L}_2[Z\phi] = Z\widetilde{M}_2[\phi] = Z(H')^{2-k}e^H, \quad \widetilde{L}_2[Z\psi] = Z\widetilde{M}_2[\psi] = cZ(H')^{2-k}H^{-m-2}.$$

Since f equals $cZ\phi$ or $cZ\psi$, there exists $r_3 > 0$ such that $\widetilde{L}_2[f]$ has no zeros in $\Omega(r_3)$, and Proposition 6.1 is proved. $\quad\square$

7. Annihilators

The remainder of the proof of Theorem 1.3 focuses on the case where the operators L and M differ. In this case Lemma 5.3 ensures that if ϕ is a non-trivial solution of $L[y] = 0$, and ψ is a non-trivial solution of $N[y] = 0$, then $\chi = \phi'\psi + \phi V[\psi]$ solves $M[y] = 0$. Here χ may vanish identically, in which case ψ will be said to annihilate ϕ, and vice versa. This notion makes sense when ϕ and ψ are both analytic solutions, and also when they are both formal solutions. The terminology in this section is as in Section 4, and the convention that exponential parts have zero constant term still applies. The following variant of an auxiliary result from [2] is key to the proof of Theorem 1.3.

Lemma 7.1. [2] *Assume that $L \neq M$ and take a canonical formal solution G of $N[y] = 0$ which is free of logarithms and has exponential part κ. In addition, take a fundamental set of canonical formal solutions $f_1, \ldots, f_k$ of $L[y] = 0$, such that f_j has exponential part q_j, and a fundamental set of canonical formal solutions $w_1, \ldots, w_k$ of $M[y] = 0$, where w_j has exponential part s_j. Then the following conclusions hold.*

   (i) *Each $W_j = f_j'G + f_j V[G]$ is either identically zero or a canonical formal solution of $M[y] = 0$ with exponential part $q_j + \kappa$.*

   (ii) *There exists $\lambda = \lambda(G) \in \{1, \ldots, k\}$ such that the collection $s_1, \ldots, s_k$ consists of*

$$(7.1) \qquad\qquad q_j + \kappa \quad (j \neq \lambda), \quad q_\lambda - (k-1)\kappa.$$

   (iii) *If the W_j are linearly dependent, then G annihilates a canonical formal solution g_1 of $L[y] = 0$ with exponential part q_λ, and every formal solution of $L[y] = 0$ which is annihilated by G is a constant multiple of g_1.*

   (iv) *If κ is not identically zero, then the W_j are linearly dependent.*

Proof. Conclusion (i) follows immediately from Lemma 5.3. Next, Lemma 4.1 and Abel's identity give, since $a_{k-1}(\infty) = A_{k-1}(\infty) = 0$,

$$(7.2) \qquad \sum_{j=1}^{k} q_j = \sum_{j=1}^{k} s_j = 0.$$

Suppose first that the W_j are linearly independent. Then (i), (7.2) and Lemma 4.1 yield

$$0 = \sum_{j=1}^{k} s_j = \sum_{j=1}^{k} (q_j + \kappa) = k\kappa,$$

which implies that $\kappa = 0$ and that $\{s_1, \ldots, s_k\} = \{q_1, \ldots, q_k\}$, again by Lemma 4.1. This proves conclusion (iv), and that (7.1) applies when the W_j are linearly independent.

Now suppose that the W_j are linearly dependent: then G annihilates a non-trivial solution g_1 of $L[y] = 0$. It may be assumed that the exponential parts and formal series appearing in G and the f_j and w_j all involve integer powers of $z^{1/p}$, for some fixed $p \in \mathbf{N}$. Because G is free of logarithms, (5.16) implies that $V[G]/G$ is a formal series in descending powers of $z^{1/p}$, and therefore so is g_1'/g_1. Thus g_1 is a canonical formal solution of $L[y] = 0$, and by Lemma 4.1 it may be assumed that $g_1 = f_1$; moreover, every formal solution g_2 of $L[y] = 0$ which is annihilated by G has $W(g_1, g_2) = 0$, so that g_2 is a constant multiple of g_1. This proves (iii).

Now set $U_j = f_j'G + f_j V[G]$. Then $U_1 \equiv 0$, but $U_2, \ldots, U_k$ are linearly independent, and $M[y] = 0$ has a fundamental set $\{U^*, U_2, \ldots, U_k\}$ of canonical formal solutions, with exponential parts $s^*, q_2 + \kappa, \ldots, q_k + \kappa$ respectively. Using (7.2) twice, as well as Lemma 4.1, shows that these exponential parts have sum 0 and $s^* = q_1 - (k-1)\kappa$, which leads to (7.1). $\qquad \square$

The following lemma, in which transcendentally fast means faster than any power of z, gives a sufficient condition for an analytic solution of $N[y] = 0$ to annihilate a solution of $L[y] = 0$.

Lemma 7.2. *Suppose that $L \neq M$. Then $g(z)$ cannot tend to 0 transcendentally fast as $z \to \infty$ in a sector, and the equation $N[y] = 0$ cannot have a fundamental set of canonical formal solutions with the same exponential part. Moreover, if G is a non-trivial analytic solution of $N[y] = 0$ and $G(z)$ tends to 0 transcendentally fast as $z \to \infty$ in a sector S, then G annihilates a non-trivial analytic solution of $L[y] = 0$.*

Proof. If g tends to zero transcendentally fast on a sector, then $F/f = g^{-k}$ tends to infinity transcendentally fast there; since f has finite order by Lemma 5.3, this contradicts standard estimates [8] for logarithmic derivatives $f^{(j)}/f$.

Next, if $N[y] = 0$ has a fundamental set of canonical formal solutions with the same exponential part κ, then κ is a polynomial in z, by Lemma 4.1 and Abel's identity. Here κ cannot be the zero polynomial, because g^k is transcendental, and so there exists a sector on which every solution of $N[y] = 0$, including g, tends to zero transcendentally fast, which is a contradiction.

Assume now that G is a non-trivial analytic solution of $N[y] = 0$ which tends to 0 transcendentally fast in a sector S, but annihilates no non-trivial solution of $L[y] = 0$. Then there exist k solutions f_j of $L[y] = 0$ such that the $f_j'G + f_j V[G]$ are linearly independent solutions of $M[y] = 0$ on S. Because $N[y] = 0$ has order at most 3 and at least two distinct exponential parts, the asymptotics in Section 4

give rise to a subsector S^* of S on which $G(z) \neq 0$ and $G^{(j)}(z)/G(z) = O(|z|^q)$ as $z \to \infty$, for some $q \in \mathbf{N}$ and all $j \in \{1, \dots, k\}$. This is clear if one solution h_j as in (4.2) dominates the others on a subsector, and so evidently holds unless there are two solutions h_j as in (4.2), with the same exponential part, for which the powers γ_j differ by $\delta \in i\mathbf{R} \setminus \{0\}$; but in this case, for any given $A \in \mathbf{C}^*$, a subsector may be chosen on which $\log|z^\delta - A|$ is bounded. Define functions Y, ϕ and Φ on S^* by

$$(7.3) \qquad \frac{Y'}{Y} = -a_{k-1} = -A_{k-1}, \qquad -\frac{\phi'}{\phi} = \frac{V[G]}{G} = \alpha\frac{G''}{G} + \beta\frac{G'}{G} + \gamma, \quad \Phi = L[\phi].$$

It follows that

$$\begin{aligned}
cY &= W(f_1'G + f_1 V[G], \dots, f_k'G + f_k V[G]) \\
&= W(f_1'G - f_1(\phi'/\phi)G, \dots, f_k'G - f_k(\phi'/\phi)G) = (\phi G)^k W((f_1/\phi)', \dots, (f_k/\phi)') \\
&= (\phi G)^k W(1, f_1/\phi, \dots, f_k/\phi) = \phi^{-1} G^k W(\phi, f_1, \dots, f_k).
\end{aligned}$$

This delivers in turn

$$\frac{\Phi}{\phi} = \frac{L[\phi]}{\phi} = \frac{cW(f_1, \dots, f_k, \phi)}{Y\phi} = \frac{c}{G^k},$$

so that $\Phi(z)/\phi(z)$ tends to infinity transcendentally fast in the sector S^*. But (7.3) implies that there exist $q', q'' \in \mathbf{N}$ with $\phi'(z)/\phi(z) = O(|z|^{q'})$ as $z \to \infty$ in S^*, and hence $\Phi(z)/\phi(z) = O(|z|^{q''})$ as $z \to \infty$ on a subsector of S^*, a contradiction. $\square$

8. The second special case

Proposition 8.1. *With the hypotheses of Theorem 1.3, suppose in addition that $L \neq M$ and that there exist $E \in \mathbf{N}$ and a function R which is rational at infinity such that all poles z of $\mathfrak{f}(z) = f\left(z^E\right)$ with $|z|$ sufficiently large have multiplicity $R(z)$. Then f satisfies at least one of conclusions (i) and (ii) of Theorem 1.3.*

The proof of Proposition 8.1 will occupy the remainder of this section. Observe first that f has finite order and an unbounded sequence of poles, by Lemma 5.3. Next, it may be assumed that $E = 1$. To see this, let $\omega = \exp(2\pi i/E)$ and let z_0 be large and a pole of f of multiplicity m_0. Let $w_0^E = z_0$. Then w_0 is a pole of $\mathfrak{f}$ of multiplicity $m_0 = R(w_0)$. This is true for all E choices of w_0 and so $R(z) = R(\omega z)$ for all large z, which gives $R(z) = S\left(z^E\right)$ for some function S which is rational at infinity. Thus the multiplicity m_0 of the pole of f at z_0 satisfies $m_0 = R(w_0) = S(w_0^E) = S(z_0)$. Assume for the remainder of this section that $E = 1$.

Lemma 8.1. *There exist functions d_0, d_1, both rational at infinity, such that f and g satisfy*

$$(8.1) \qquad \qquad \frac{f'}{f} = d_0\frac{g'}{g} + d_1.$$

Moreover, d_0 either has $d_0(\infty) = \infty$ or is constant and equal to a negative integer.

Proof. Let $d_0 = -R$. By the remark following (5.1), there exists $r_0 > 0$ such that $f'/f - d_0 g'/g$ has no poles in $\Omega(r_0)$, and so is rational at infinity since g^k and f have finite order. The last assertion follows from the fact that f has an unbounded sequence of poles. $\square$

Lemma 8.2. *There exist functions ν_1, ν_0, both rational at infinity, such that g satisfies (1.8).*

Proof. The equation (8.1) yields, using (5.1),

$$-h = \left(\frac{f'}{f}\right) g = d_0 g' + d_1 g, \quad -h' = d_0 g'' + (d_0' + d_1) g' + d_1' g.$$

Combining this with (5.15) gives

$$(8.2) \quad 0 = \left(d_0 - \frac{k-1}{2}\right) g'' + \left(d_0' + d_1 + \frac{a_{k-1}}{k}\right) g' + \left(d_1' + \frac{a_{k-1}' + c_{k-2} - C_{k-2}}{k}\right) g$$

and an equation (1.8), as asserted, since $d_0 - (k-1)/2 \not\equiv 0$ by Lemma 8.1. $\qquad \square$

From (1.8), (5.16) and (8.1) it follows that

$$-d_0 g' - d_1 g = -\left(\frac{f'}{f}\right) g = h = V[g] = (\beta - \alpha\nu_1) g' + (\gamma - \alpha\nu_0) g$$

and so, since g^k has an unbounded sequence of zeros,

$$(8.3) \qquad\qquad -d_0 = \beta - \alpha\nu_1, \quad -d_1 = \gamma - \alpha\nu_0.$$

In the next lemma the convention that exponential parts have zero constant term is retained.

Lemma 8.3. *There exists an equation (4.3), with A^* rational at infinity, such that $yU^{-1/2}$ solves (4.3) for every solution y of (1.8), where $U'/U = -\nu_1$. The equation (1.8) has a pair of linearly independent canonical formal solutions with distinct exponential parts, and (4.3) has an irregular singular point at infinity. If κ is a non-zero exponential part for equation (1.8), then there exists a locally analytic solution u_1 of (1.8), with exponential part κ, which continues without zeros in some $\Omega(r_2)$ and annihilates a non-trivial locally analytic solution y_1 of $L[y] = 0$, where y_1 is given by*

$$(8.4) \qquad\qquad \frac{y_1'}{y_1} = d_0 \frac{u_1'}{u_1} + d_1.$$

Moreover, both $zu_1'(z^2)/u_1(z^2)$ and $zy_1'(z^2)/y_1(z^2)$ are rational at infinity.

Proof. The existence of the equation (4.3) solved by $yU^{-1/2}$ for every solution y of (1.8) is a standard consequence of Abel's identity. Now the exponential parts κ_1, κ_2 for (1.8) are polynomials in $z^{1/2}$, by (4.4), and their sum is a polynomial in z; thus $\kappa_j(z) = Q_j(z) + z^{1/2}(-1)^j Q^*(z)$ with Q^* and the Q_j polynomials in z.

Suppose that $\kappa_1 = \kappa_2 = \kappa_0$. Then κ_0 is a polynomial, and must be non-constant since g satisfies (1.8) and $f/F = g^k$ has an essential singularity at infinity. But this implies the existence of a sector on which every solution of (1.8), including g, tends to zero transcendentally fast as $z \to \infty$, which contradicts Lemma 7.2. Thus $\kappa_1 \neq \kappa_2$, so that (4.3) has an irregular singular point at infinity, and at least one canonical formal solution of (1.8) has non-zero exponential part.

Take a canonical formal solution u_1 of (1.8) with exponential part $\kappa \neq 0$. Then u_1 is given by a formal expression as in (4.1), but free of logarithms, and u_1'/u_1 is a formal series in descending powers of $z^{1/2}$. Since f has finite order and an unbounded sequence of poles, the function g'/g is not rational at infinity. Thus g cannot solve a first order homogeneous linear differential equation with coefficients which are rational at infinity, and so the division algorithm for linear differential operators [15, p.126] shows that the operator N of Lemma 5.3 must satisfy $N = N_0 \circ (D^2 + \nu_1 D + \nu_0)$, for some operator N_0 of order 1 or 0. Hence every solution of (1.8), including u_1, solves $N[y] = 0$. It follows from Lemma 7.1 that u_1 annihilates

some canonical formal solution y_1 of $L[y] = 0$. This gives, using (1.8), (5.16) and (8.3),

$$-y_1' u_1 = y_1 V[u_1] = y_1((\beta - \alpha \nu_1) u_1' + (\gamma - \alpha \nu_0) u_1) = y_1(-d_0 u_1' - d_1 u_1),$$

and hence (8.4). Thus y_1'/y_1 is also a formal series in $z^{1/2}$, and the hypotheses of Lemma 3.3 are satisfied with $p = y_1$ and $q = u_1$. Hence (3.2) holds with the b_j rational at infinity and $b_k \not\equiv 0$ by Lemma 8.1. But $L[y_1] = 0$, and so u_1'/u_1 is algebraic at infinity, that is, u_1'/u_1 solves a polynomial equation with coefficients which are rational at infinity. In particular, the series for u_1'/u_1 converges for large z in some sector, as does that for y_1'/y_1, by (8.4), and u_1 and y_1 are analytic local solutions of (1.8) and $L[y] = 0$ respectively. Since the algebraic equation for u_1'/u_1 has only finitely many branches for its solutions, and each branch has no poles in some sector $|z| > r_2$, $|\arg z| < 4\pi$, it follows that u_1 continues without zeros in $\Omega(r_2)$. This means that, as z crosses a critical ray of (4.3), the solution $u_1 U^{-1/2}$ of (4.3) must change from small to large or vice versa. Therefore continuing twice around a circle $|z| = r_3 > r_2$ brings $u_1 U^{-1/2}$ back to a constant multiple of itself, and the same is true for u_1. Thus $z u_1'(z^2)/u_1(z^2)$ is rational at infinity, and so is $z y_1'(z^2)/y_1(z^2)$ by (8.4). $\qquad\square$

Choose a critical ray $\arg z = \theta^*$ for the equation (4.3) and a sector S^*, symmetric about the critical ray, and with internal angle slightly less than $4\pi/(2 + \deg_\infty A^*)$, in which f has an unbounded sequence of poles, these being zeros of g. In the sector S^*, equation (4.3) has two linearly independent zero-free analytic solutions, by (4.4). Denote these by $u^* = u U^{-1/2}$ and $v^* = v U^{-1/2}$ say, where u and v solve (1.8). Here u and v have distinct exponential parts κ_u and κ_v, each a polynomial in $z^{1/2}$, and it may be assumed that κ_u is non-constant and

$$(8.5) \qquad \liminf_{z \to \infty, z \in S^*} \left| \frac{\kappa_v(z)}{\kappa_u(z)} \right| \leq 1, \quad g = v - u,$$

since u and v may be interchanged and multiplied by constants. Now Lemma 8.3 shows that there exist locally analytic solutions u_1 of (1.8) and y_1 of $L[y] = 0$ respectively, such that u_1 has exponential part κ_u, while (8.4) holds and both $z u_1'(z^2)/u_1(z^2)$ and $z y_1'(z^2)/y_1(z^2)$ are rational at infinity. Thus u_1 must be a constant multiple of u and so, by (8.1),

$$(8.6) \qquad T_1 = \frac{y_1'}{y_1} = d_0 \frac{u'}{u} + d_1, \quad \frac{f'}{f} = d_0 \frac{g'}{g} + d_1 = d_0 \left(\frac{g'}{g} - \frac{u'}{u} \right) + \frac{y_1'}{y_1}.$$

Poles z of f occur where $v(z)/u(z) = 1$, and have multiplicity equal to $-d_0(z)$, by (8.6). Furthermore, by (4.4), $\zeta = (1/2\pi i) \log(v^*/u^*) = (1/2\pi i) \log(v/u)$ maps S^* conformally onto a domain containing a right or left half-plane $\pm \operatorname{Re} \zeta > M_0 > 0$. Since d_0 takes integer values at all points in S^* where ζ is integer-valued, applying Lemma 3.1 shows that there exists a polynomial Q such that

$$(8.7) \qquad d_0 = Q(T), \quad T = 2\pi i \zeta = \log \left(\frac{v}{u} \right).$$

The second equation of (8.6) can now be written in the form

$$(8.8) \qquad \begin{aligned} \frac{f'}{f} &= Q(T) \left(\frac{v' - u'}{v - u} - \frac{u'}{u} \right) + \frac{y_1'}{y_1} = Q(T) \frac{v'u - u'v}{(v - u)u} + \frac{y_1'}{y_1} \\ &= Q(T) \frac{v'/v - u'/u}{1 - u/v} + \frac{y_1'}{y_1} = \frac{Q(T)T'}{1 - e^{-T}} + \frac{y_1'}{y_1} = \frac{Q(T)T'}{1 - e^{-T}} + T_1, \end{aligned}$$

which gives (1.7), and it suffices to consider two cases.

8.1. Case I. Suppose first that Q is constant and one exponential part for (1.8) is 0. Then $d_0 = Q(T)$ is constant and v has exponential part 0 in S^*, because u does not. A pole of f of multiplicity m_0 in S^* gives $v/u = 1$ and $g' = v' - u' = v' - (u'/u)v = T'v$, as well as (5.2). Since all poles of f in S^* with $|z|$ sufficiently large have fixed multiplicity $-d_0$, it follows from Lemma 3.1 and (5.2) that $(T'v)^{-k}$, which also has exponential part 0 in S^*, must be constant, and so must $v' - (u'/u)v$. But then $W(u,v)/u$ is constant, and so $\nu_1 = -u'/u$ in (1.8). Because u solves (1.8), it must be the case that $\nu_0 = -(u'/u)' = \nu_1'$. Now comparing (1.8) and (8.2) shows that, since d_0 is constant, $c_{k-2} - C_{k-2}$ must vanish, so that Proposition 6.1 may be applied, and f satisfies conclusion (i) of Theorem 1.3.

8.2. Case II. Assume now that either both exponential parts for (1.8) are non-zero, or Q is non-constant.

Lemma 8.4. *The solution v continues without zeros in some $\Omega(r_2)$, and $zv'(z^2)/v(z^2)$ is rational at infinity.*

Proof. Suppose first that both exponential parts for (1.8) are non-zero. Then Lemma 8.3 gives a solution V_1 of (1.8), such that V_1 and u are linearly independent and V_1 continues without zeros in some $\Omega(r_2)$, with $zV_1'(z^2)/V_1(z^2)$ rational at infinity. Since u and v are linearly independent and zero-free in S^*, the solution V_1 must be a constant multiple of v.

Now suppose that v has exponential part 0 in S^*: then Q is non-constant, and (8.7) implies that $T = \log(v/u)$ is algebraic at infinity. Thus v continues without zeros in some $\Omega(r_2)$, because u does, and the same argument as applied to u in the proof of Lemma 8.3 shows that $zv'(z^2)/v(z^2)$ is rational at infinity as asserted. □

The functions u'/u, v'/v and $T' = v'/v - u'/u$ are all defined for large $z \in S^*$ and given by convergent series in descending powers of $z^{1/2}$. Denote by $\widehat{\psi}$ the result of continuing a function element ψ once counter-clockwise around a circle $|z| = r_3 > r_2$, starting in S^*. Since u and v both continue without zeros, there exists $\zeta_0 \in \mathbf{C}$ such that

(8.9)
$$
\begin{aligned}
&\text{(a)} \quad \widehat{u} = cu, \quad \widehat{v} = cv, \quad \widehat{T} = T + \zeta_0 \quad \text{or} \\
&\text{(b)} \quad \widehat{u} = cv, \quad \widehat{v} = cu, \quad \widehat{T} = -T - \zeta_0.
\end{aligned}
$$

Lemma 8.5. *There exist $d_2 \in [0, 1/2]$ and functions E_0, E_1 and $E_2 \not\equiv 0$, each rational at infinity, such that*

(8.10)
$$
\frac{u'(z)}{u(z)} = (d_2 - 1)T'(z) + o(|T'(z)|), \quad \frac{v'(z)}{v(z)} = d_2 T'(z) + o(|T'(z)|)
$$

as $z \to \infty$ in S^*, while $E_2 f'' + E_1 f' + E_0 f$ has no zeros in some $\Omega(r_3)$. If subcase (a) applies in (8.9), then T' is rational at infinity, with $T'(\infty) \neq 0$, while if subcase (b) applies then $d_2 = 1/2$ and $H_0(z) = z^{1/2}T'(z)$ is rational at infinity, with $H_0(\infty) \neq 0$.

Proof. Suppose first that subcase (a) applies in (8.9). Then u'/u, v'/v and T' are all rational at infinity, and so is T_1 in (8.6). Thus applying Lemma 3.3 to f and g gives, in view of (8.1), (8.8) and Lemma 8.2, functions E_0, E_1 and E_2, each rational at infinity, such that $E_2 \not\equiv 0$ and

$$
E_2\frac{f''}{f} + E_1\frac{f'}{f} + E_0 = \left(\frac{f'}{f} - T_1\right)^2 = \left(\frac{Q(T)T'}{1 - e^{-T}}\right)^2.
$$

Hence $E_2 f'' + E_1 f' + E_0 f$ has no zeros in some $\Omega(r_3)$.

To prove the existence of d_2 in subcase (a), suppose first that $\deg_\infty(u'/u) > \deg_\infty T'$. Then, as $z \to \infty$, with $\arg z$ arbitrary,

$$\frac{v'(z)}{v(z)} = \frac{u'(z)}{u(z)} + T'(z) = (1 + o(1))\frac{u'(z)}{u(z)}.$$

Since u has non-zero exponential part, this gives a sector on which u and v both tend to zero transcendentally fast, and hence so does every solution of (1.8), including g, contradicting Lemma 7.2. Thus there exists $d_2 \in \mathbf{C}$ such that (8.10) holds as $z \to \infty$, with $\arg z$ arbitrary, and $T'(\infty) \neq 0$, since u has non-zero exponential part. If $d_2 \notin \mathbf{R}$, or if $d_2 \in \mathbf{R} \setminus [0, 1]$, then again there exists a sector on which u, v and g all tend to zero transcendentally fast, contradicting Lemma 7.2. Finally, (8.5) gives $d_2 \leq 1/2$.

Assume now that subcase (b) holds in (8.9). Because f has an unbounded sequence of poles in S^* and y_1 continues without zeros, (8.8) leads to

$$(8.11) \quad \widehat{T}' = -T', \quad \frac{f'}{f} = \frac{-Q(T)T'}{1 - e^{T+\zeta_0}} + \widehat{T}_1, \quad e^{\zeta_0} = 1, \quad Q(T)T' = \widehat{T}_1 - T_1 = T_2 - T_1.$$

Furthermore, $u'/u + v'/v = 2H_1$ and $u'v'/uv$ are rational at infinity, and so are $T_1 + T_2$ and $T_1 T_2$ by continuation of the first equation of (8.6). On the other hand (8.11) implies that $T'(z) = 2z^{1/2}H_2(z)$, with H_2 rational at infinity. This yields

$$(8.12) \quad \frac{u'(z)}{u(z)} = H_1(z) - z^{1/2}H_2(z), \quad \frac{v'(z)}{v(z)} = H_1(z) + z^{1/2}H_2(z).$$

Since u has non-zero exponential part, either $\deg_\infty H_1 \geq 0$ or $\deg_\infty H_2 \geq -1$. Moreover, $\deg_\infty H_2 \geq \deg_\infty H_1$ (and so $\deg_\infty H_2 \geq -1$) in (8.12); otherwise there again exists a sector on which every solution of (1.8), including g, tends to zero transcendentally fast, contradicting Lemma 7.2. Thus (8.10) holds with $d_2 = 1/2$.

Applying Lemma 3.3 to f and g now gives, in view of Lemma 8.2 and (8.1), (8.8) and (8.11), functions E_0, E_1 and E_2, each rational at infinity, such that $E_2 \not\equiv 0$ and

$$E_2\frac{f''}{f} + E_1\frac{f'}{f} + E_0 = \left(\frac{f'}{f} - T_1\right)\left(\frac{f'}{f} - T_2\right) = -\frac{(Q(T)T')^2}{(1 - e^T)(1 - e^{-T})},$$

and so $E_2 f'' + E_1 f' + E_0 f$ again has no zeros in some $\Omega(r_3)$. $\qquad\square$

Recall that $\zeta(z) = T(z)/2\pi i$ maps a subdomain of S^* conformally onto a right or left half-plane. If $z_1 \in S^*$ and $\zeta(z_1) \in \mathbf{Z}$ then $e^{T(z_1)} = v(z_1)/u(z_1) = 1$, while f has a pole at z_1 of multiplicity $-d_0(z_1) = -Q(T(z_1))$, by (8.8), and (5.2) gives

$$(8.13) \quad (T'(z_1)v(z_1))^{-k} = (v'(z_1) - u'(z_1))^{-k} = g'(z_1)^{-k} = Q_0(T(z_1)),$$
$$Q_0 = Q(Q - 1)\ldots(Q - k + 1).$$

Lemma 8.5 makes it possible to write, on S^*,

$$(8.14) \quad T'(z)^k v(z)^k Q_0(T(z)) = e^{kd_2 T(z)}u_0(z) = e^{2\pi i k d_2\zeta(z)}u_0(z),$$

in which $\log^+|u_0(z)| = o(|T(z)|) = o(|\zeta(z)|)$ as $z \to \infty$ in S^*. Thus (8.13), (8.14) and Lemma 3.1 together imply that $kd_2 \in \mathbf{Z}$ and $u_0 \equiv 1$, so that v and u have representations, for some branch of $Q_0(T)^{1/k}$,

$$(8.15) \quad v = \frac{e^{d_2 T}}{Q_0(T)^{1/k}T'}, \quad u = ve^{-T} = \frac{e^{(d_2-1)T}}{Q_0(T)^{1/k}T'}, \quad d_2 \in [0, 1/2], \quad kd_2 \in \mathbf{Z},$$

and if T' is not rational at infinity then $d_2 = 1/2$ and k is even. Now Abel's identity, (1.8), (8.2), (8.6), (8.7) and (8.15) lead to

$$W_0 = W(u, v) = ce^{(2d_2-1)T}Q_0(T)^{-2/k}(T')^{-1},$$

$$\nu_1 = \frac{d_0' + d_1 + a_{k-1}/k}{d_0 - (k-1)/2} = -\frac{W_0'}{W_0} = (1 - 2d_2)T' + \frac{2Q_0'(T)T'}{kQ_0(T)} + \frac{T''}{T'},$$

$$\frac{y_1'}{y_1} = Q(T)\left((d_2-1)T' - \frac{Q_0'(T)T'}{kQ_0(T)} - \frac{T''}{T'}\right)$$

$$+ \left(Q(T) - \frac{k-1}{2}\right)\left((1-2d_2)T' + \frac{2Q_0'(T)T'}{kQ_0(T)} + \frac{T''}{T'}\right) - Q'(T)T' - \frac{a_{k-1}}{k}.$$

Hence $T_3 = y_1'/y_1 + a_{k-1}/k$ is given by

(8.16)
$$T_3 = \frac{1}{k}\sum_{j=0}^{k-2}\left(\frac{j-k+1}{Q(T)-j}\right)Q'(T)T' - \left(d_2(Q(T)-k+1) + \frac{k-1}{2}\right)T'$$
$$- \left(\frac{k-1}{2}\right)\frac{T''}{T'}.$$

Thus $T_1 = y_1'/y_1$ belongs to the field $\widetilde{\Lambda}$ generated by $d_0 = Q(T)$, T', a_{k-1} and their derivatives. Since $L[y_1] = 0$, a standard change of variables gives a linear differential operator $\widetilde{L}$ with coefficients $\widetilde{c}_j \in \widetilde{\Lambda}$ such that

$$L[y_1 w] = y_1\widetilde{L}[w], \quad \widetilde{L} = \sum_{j=1}^{k}\widetilde{c}_j D^j, \quad \widetilde{c}_k = 1, \quad D = \frac{d}{dz}.$$

As T_1 is known, the $\widetilde{c}_j$ can be computed from the a_j, and vice versa. Using (8.5) and (8.8), write

(8.17)
$$f = y_1\phi, \quad \frac{1}{(v-u)^k} = \frac{1}{g^k} = \frac{L[f]}{f} = \frac{\widetilde{L}[\phi]}{\phi} = \sum_{j=1}^{k}\widetilde{c}_j\frac{\phi^{(j)}}{\phi},$$

$$\frac{\phi'}{\phi} = \frac{S_1}{Y_1}, \quad S_1 = R_{1,0} = Q(T)T', \quad Y_1 = 1 - e^{-T}, \quad Y_1' = T'(1 - Y_1).$$

There exist computable coefficients $R_{j,\mu} \in \widetilde{\Lambda}$ such that, for $j \in \mathbf{N}$,

(8.18)
$$\frac{\phi^{(j)}}{\phi} = \frac{S_j}{Y_1^j}, \quad S_j = \sum_{\mu=0}^{j-1}R_{j,\mu}Y_1^\mu,$$

$$R_{j,0} = Q(T)(Q(T)-1)\ldots(Q(T)-j+1)(T')^j.$$

The relations (8.18) hold by a straightforward induction argument, since the S_j satisfy

$$S_{j+1} = Y_1 S_j' - jY_1' S_j + S_1 S_j = Y_1 S_j' + jT'(Y_1-1)S_j + S_1 S_j, \quad R_{j+1,0} = (Q(T)-j)T'R_{j,0}.$$

Using (8.15), (8.17) and (8.18) now delivers

$$Q_0(T)(T')^k(1-Y_1)^{kd_2} = Q_0(T)(T')^k e^{-kd_2 T} = v^{-k} = \frac{Y_1^k}{(v-u)^k} = \sum_{j=1}^{k}\widetilde{c}_j S_j Y_1^{k-j}$$

$$= \sum_{j=1}^{k}\sum_{\mu=0}^{j-1}\widetilde{c}_j R_{j,\mu}Y_1^{k-j+\mu} = \sum_{\mu=0}^{k-1}\sum_{j=\mu+1}^{k}\widetilde{c}_j R_{j,\mu}Y_1^{k-j+\mu}$$

$$= \sum_{\mu=0}^{k-1} \sum_{\nu=\mu}^{k-1} \widetilde{c}_{k+\mu-\nu} R_{k+\mu-\nu,\mu} Y_1^{\nu} = \sum_{\nu=0}^{k-1} Y_1^{\nu} \sum_{\mu=0}^{\nu} \widetilde{c}_{k+\mu-\nu} R_{k+\mu-\nu,\mu},$$

in which $\nu = k - j + \mu$. Now $e^{-T} = u/v$ grows transcendentally fast on a subsector of S^*, whereas each element of the field $\widetilde{\Lambda}$ has form $\widetilde{v}(z) = v_1(z) + z^{1/2} v_2(z)$, with v_1 and v_2 rational at infinity. Thus e^{-T} is transcendental over $\widetilde{\Lambda}$ and so is $Y_1 = 1 - e^{-T}$. Since (8.7), (8.13), (8.18) and Lemma 8.1 together imply that $R_{j,0} \not\equiv 0$ for $j = 1, \ldots, k$ and that $\widetilde{c}_k R_{k,0} = Q_0(T)(T')^k$, comparing the coefficients of Y_1^{ν}, starting from $\nu = 1$, determines successively $\widetilde{c}_{k-1}, \ldots, \widetilde{c}_1$ and hence $\{a_0, \ldots, a_{k-2}\}$. $\qquad\square$

The proof of Proposition 8.1 is complete, but it is worth remarking that (2.1), (2.2) and (2.3) show that $d_2 = 0$ and $d_2 = 1/3$ are both possible when $k = 3$. Furthermore, Propositions 6.1 and 8.1 each give a solution y_0 of $L[y] = 0$ on S^*, of the form (4.2), whose exponential part is a non-constant polynomial in $z^{1/2}$, and in z if k is odd. If $d_2 = 0$ or $d_2 = 1/2$ in (8.16), or if f is given by (1.5), then $y_0 = y_1$, by (8.7), Lemma 8.5 and the fact that H''/H' does not vanish at infinity in (1.5). On the other hand, if $0 < d_2 < 1/2$ then u and v both have non-constant exponential part, by Lemma 8.5 and (8.15), and Lemma 7.2 gives a non-trivial solution y_2 of $L[y] = 0$ annihilated by v; thus $y_2'/y_2 - y_1'/y_1 = d_0(v'/v - u'/u) = Q(T)T'$, by (8.3), (8.6) and (8.7), and $y_0 \in \{y_1, y_2\}$. It follows that $y_0'(z)/y_0(z) = (1 + o(1))cz^{\lambda_0}$ and $y_0^{(j)}(z)/y_0^{(k)}(z) = (1 + o(1))cz^{(j-k)\lambda_0}$ as $z \to \infty$ in a subsector of S^*, for $j = 0, \ldots, k-1$, where $\lambda_0 \geq -1/2$, and $\lambda_0 \geq 0$ if k is odd. This implies that at least one a_j has $a_j(z) \neq o(|z|^{(k-j)\lambda_0})$ as $z \to \infty$, which is sharp by [17, (1.8)].

9. A change of variables

In order to prove Theorem 1.3 it now suffices, in view of Proposition 6.1, to show that the hypotheses of Proposition 8.1 are satisfied when $L \neq M$. Since the value of E is immaterial in Proposition 8.1, a change of variables $z \to z^n$ may now be employed to ensure that, in the terminology of Section 4, the integer p is 1, so that the exponential parts and associated asymptotic or formal series involve only integer powers. Indeed, let $k \geq 3$ and $n \geq 2$ be integers and let f, F and $\mathfrak{f}$ satisfy

$$(9.1) \quad F(z) = L[f](z) = f^{(k)}(z) + a_{k-1}(z) f^{(k-1)}(z) + \ldots + a_0(z) f(z), \quad \mathfrak{f}(z) = f(z^n),$$

where the a_j are rational at infinity with $a_{k-1}(\infty) = 0$. Take linearly independent locally analytic solutions $f_1, \ldots, f_k$ of $L[y] = 0$.

Lemma 9.1. *For each integer $m \geq 1$ there exist rational functions $c_{p,m}(z)$, depending only on m and n, such that*

$$(9.2) \qquad f^{(m)}(z^n) = \sum_{p=1}^{m} c_{p,m}(z) \mathfrak{f}^{(p)}(z), \quad c_{m,m}(z) = (nz^{n-1})^{-m}.$$

Moreover, if $m \geq 2$ then $c_{m-1,m}(z)/c_{m,m}(z) \to 0$ as $z \to \infty$.

Proof. Clearly $\mathfrak{f}(z) = f(z^n)$ gives, as $z \to \infty$,

$$f'(z^n) = (nz^{n-1})^{-1} \mathfrak{f}'(z), \quad f''(z^n) = (nz^{n-1})^{-2} \mathfrak{f}''(z) + c_{1,2}(z) \mathfrak{f}'(z), \quad c_{1,2}(z) = O(|z|^{1-2n}).$$

Next, if the assertions of the lemma hold for some $m \geq 2$ then, as $z \to \infty$,

$$nz^{n-1}f^{(m+1)}(z^n) = c_{m,m}(z)\mathfrak{f}^{(m+1)}(z) + \mathfrak{f}^{(m)}(z)(c'_{m,m}(z) + c_{m-1,m}(z)) + \dots$$

$$= c_{m,m}(z)\left[\mathfrak{f}^{(m+1)}(z) + \mathfrak{f}^{(m)}(z)O(1/|z|) + \dots\right]. \qquad \square$$

Now (9.1) and (9.2) yield, as $z \to \infty$,

$$F(z^n) = (nz^{n-1})^{-k}\left[\mathfrak{f}^{(k)}(z) + \mathfrak{f}^{(k-1)}(z)O(1/|z|) + \dots\right]$$

$$+ O(|z|^{-n})(nz^{n-1})^{1-k}\left[\mathfrak{f}^{(k-1)}(z) + \dots\right] + \dots$$

$$= (nz^{n-1})^{-k}\left[\mathfrak{f}^{(k)}(z) + \mathfrak{f}^{(k-1)}(z)O(1/|z|) + \dots\right].$$

Hence there exist functions $\mathfrak{a}_j(z)$, all rational at infinity and with $\mathfrak{a}_{k-1}(\infty) = 0$, such that

$$\mathfrak{f}^{(k)}(z) + \mathfrak{a}_{k-1}(z)\mathfrak{f}^{(k-1)}(z) + \dots + \mathfrak{a}_0(z)\mathfrak{f}(z) = \mathfrak{F}(z) = (nz^{n-1})^k F(z^n).$$

The new operator is $\mathfrak{L} = D^k + \mathfrak{a}_{k-1}D^{k-1} + \dots$, and $y(z^n)$ solves $\mathfrak{L}[y] = 0$ for every locally analytic or formal solution y of $L[y] = 0$, as does each $\mathfrak{f}_j(z) = f_j(z^n)$.

If f is as in the hypotheses of Theorem 1.3 then running Frank's method as in Section 5 for $\mathfrak{f}$ and $\mathfrak{F}$ gives rise to auxiliary functions $\mathfrak{g}$, $\mathfrak{h} = -(\mathfrak{f}'/\mathfrak{f})\mathfrak{g}$ and $\mathfrak{w}_j$, which satisfy, using (5.1),

$$\mathfrak{g}(z)^k = \frac{\mathfrak{f}(z)}{\mathfrak{F}(z)} = \frac{f(z^n)}{(nz^{n-1})^k F(z^n)} = \frac{g(z^n)^k}{(nz^{n-1})^k}, \quad \mathfrak{g}(z) = \frac{g(z^n)}{nz^{n-1}},$$

$$\mathfrak{h}(z) = -\frac{\mathfrak{f}'(z)}{\mathfrak{f}(z)}\mathfrak{g}(z) = -nz^{n-1}\frac{f'(z^n)}{f(z^n)}\frac{g(z^n)}{nz^{n-1}} = h(z^n),$$

$$\mathfrak{w}_j(z) = \mathfrak{f}'_j(z)\mathfrak{g}(z) + \mathfrak{f}_j(z)\mathfrak{h}(z) = f'_j(z^n)g(z^n) + f_j(z^n)h(z^n) = w_j(z^n).$$

Thus the $\mathfrak{w}_j$ solve the equation $\mathfrak{M}[y] = y^{(k)} + \dots = 0$ which is obtained from $M[y] = 0$ in the same way as $\mathfrak{L}[y] = 0$ arose from $L[y] = 0$. In $\mathfrak{M}[y]$ the coefficient of $y^{(k-1)}$ is $\mathfrak{a}_{k-1}$, since $a_{k-1} = A_{k-1}$. It is important to note that $\mathfrak{L} = \mathfrak{M}$ if and only if $L = M$.

Therefore n may be chosen so that in the canonical formal solutions (4.1) for the equations $\mathfrak{L}[y] = 0$ and $\mathfrak{M}[y] = 0$ the integer p is 1. Moreover, $\mathfrak{L}[y] = 0$ has linearly independent canonical formal solutions $\mathfrak{h}_1, \mathfrak{h}_2$ whose formal Wronskian $W(\mathfrak{h}_1, \mathfrak{h}_2)$ is free of logarithms. This is clear if there are solutions of $\mathfrak{L}[y] = 0$ with distinct exponential parts. On the other hand if all exponential parts for $\mathfrak{L}[y] = 0$ are the same then they are all 0, since $\mathfrak{a}_{k-1}(\infty) = 0$, and there exists a solution $\mathfrak{h}_1(z) = z^{e_1}R_1(z) \not\equiv 0$, with $e_1 \in \mathbf{C}$ and R_1 rational at infinity. The standard reduction of order method then gives an equation which is solved by $(y/\mathfrak{h}_1)' = W(\mathfrak{h}_1, y)\mathfrak{h}_1^{-2}$, for every solution y of $\mathfrak{L}[y] = 0$, and which has a canonical formal solution free of logarithms.

When $L \neq M$, and hence $\mathfrak{L} \neq \mathfrak{M}$, Lemma 5.3 applied to $\mathfrak{L}$ and $\mathfrak{M}$ gives an equation $\mathfrak{N}[y] = 0$, of order 2 or 3, which is solved by $\mathfrak{g}$, as well as a counterpart $\mathfrak{V}$ for the operator V. Choosing $\mathfrak{h}_1, \mathfrak{h}_2$ as in the previous paragraph and any canonical formal solution $\mathfrak{G}$ of $\mathfrak{N}[y] = 0$ then makes $\mathfrak{h}'_j\mathfrak{G} + \mathfrak{h}_j\mathfrak{V}[\mathfrak{G}]$ a solution of $\mathfrak{M}[y] = 0$. Solving for $\mathfrak{G}$ by Cramer's rule shows that in the canonical formal solutions (4.1) for $\mathfrak{N}[y] = 0$ it may also be assumed that $p = 1$.

10. The main step

Proposition 10.1. *Assume that f and F are as in the hypotheses of Theorem 1.3, and that f has finite order and an unbounded sequence of poles. Then there*

exist $E \in \mathbf{N}$ and a function R which is rational at infinity such that all poles z of $\mathfrak{f}(z) = f\left(z^E\right)$ with $|z|$ sufficiently large have multiplicity $R(z)$.

Once Proposition 10.1 is proved, Theorem 1.3 is established as follows. Under the hypotheses of Theorem 1.3, the first possibility is that $L = M$ and f is determined by Proposition 6.1. If this is not the case then Lemma 5.3 shows that f has finite order and an unbounded sequence of poles. In view of Proposition 10.1, Proposition 8.1 may be applied, and f is thereby determined. $\qquad \square$

Assume for the remainder of the paper that the assumptions of Proposition 10.1 are satisfied.

Lemma 10.1. *The following additional assumptions may all be made.*

(A) *In Section 5, the operators L and M are not the same.*
(B) *In the canonical formal solutions (4.1) for all of the equations $L[y] = 0$, $M[y] = 0$ and $N[y] = 0$, the integer p is 1.*
(C) *The function g solves no second order homogeneous linear differential equation with coefficients which are rational at infinity. Moreover, the operator N has order 3 and may be written in the form*

$$(10.1) \qquad\qquad N = D^3 + B_2 D^2 + B_1 D + B_0,$$

with the B_j rational at infinity, while $\alpha \not\equiv 0$ in (5.16) and $D_{k-2} = C_{k-2} - c_{k-2} \not\equiv 0$ in Section 5.

Proof. Assumption (A) is legitimate because of Proposition 6.1, while (B) is justified by taking $\mathfrak{f}(z) = f(z^{m_1})$ in place of f, for some $m_1 \in \mathbf{N}$, as in Section 9. Next, the first three assumptions of (C) are valid since otherwise (5.16) shows that f and g satisfy an equation (8.1) with d_0 and d_1 rational at infinity, in which case the conclusion of Proposition 10.1 follows from a comparison of residues. The last assumption of (C) is justified by Proposition 6.1. $\qquad \square$

Lemma 10.2. *Assume that there exists a function a^* which is rational at infinity, with the property that $-f'/f + dg'/g - a^*$ has no zeros in some $\Omega(r_2)$, where $d \in \{0, \ldots, k-1\}$ is a constant. Then d satisfies $d \neq (k-1)/2$. Assume further that $a^*(\infty) \neq 0$. Then g is given by*

$$(10.2) \quad P'g = \beta_1 e^{\omega_1 P} + \beta_2 e^{\omega_2 P} + \beta_3 e^{\omega_3 P}, \quad \beta_j, \omega_j \in \mathbf{C}^*, \quad 1 = \omega_1 \neq \omega_2 \neq \omega_3 \neq 1,$$

in which P' is rational at infinity, with $P'(\infty) \neq 0$. If, in addition, $d = 0$ or $d = k-1$ then f satisfies the conclusion of Proposition 10.1.

Proof. As in (5.4), write $p'/p = f'/f + a_{k-1}/k$ and $q = -(p'/p)g$. Then g and q solve the equations (5.7) to (5.12). Moreover, $a = a^* - a_{k-1}/k$ is rational at infinity and $-p'/p + dg'/g - a$ has no zeros in $\Omega(r_2)$. Since poles of p'/p have negative residues and are simple zeros of g, while f'/f and g^k have finite order, it is possible to write

$$(10.3) \qquad\qquad q + dg' - ag = g\left(-\frac{p'}{p} + \frac{dg'}{g} - a\right) = e^P,$$

with P' rational at infinity. Then (5.9) and (10.3) yield

$$(10.4) \qquad P'e^P = xg'' - ag' - \left(\frac{D_{k-2}}{k} + a'\right)g, \quad x = d - \frac{k-1}{2},$$

and by Lemma 10.1(C) it may be assumed that $P' \not\equiv 0$. Differentiation of this equation leads to

(10.5)
$$0 = xg''' + g'' \left(-x \left(\frac{P''}{P'} + P' \right) - a \right) + g' \left(a \left(\frac{P''}{P'} + P' \right) - \frac{D_{k-2}}{k} - 2a' \right)$$
$$+ g \left(\left(\frac{P''}{P'} + P' \right) \left(\frac{D_{k-2}}{k} + a' \right) - \frac{D'_{k-2}}{k} - a'' \right),$$

and so $x \neq 0$ and $d \neq (k-1)/2$, as asserted, again by Lemma 10.1(C).

Now assume that $a^*(\infty) \neq 0$, which implies that $a(\infty) \neq 0$. The following is an extension of a method from [7]. Since $C_{k-2} = D_{k-2} + c_{k-2}$, formula (5.10) becomes, in view of (10.3),

(10.6)
$$D_{k-2}e^P = \frac{k(k^2-1)}{12}g''' + g' \left((x+1)D_{k-2} + 2c_{k-2} \right)$$
$$+ g \left(\frac{k-1}{2}D'_{k-2} + c'_{k-2} - D_{k-3} - aD_{k-2} \right),$$

and (5.12) may be written as

$$\left(\frac{2D_{k-3}}{k-2} - D'_{k-2} \right) e^P = \frac{(k+2)D_{k-2}}{3}g'' + d_5g' + d_6g,$$

with d_5, d_6 rational at infinity. Comparing the last equation with (10.4) delivers

(10.7)
$$x \left(\frac{2D_{k-3}}{k-2} - D'_{k-2} \right) = \frac{(k+2)D_{k-2}P'}{3},$$

again using Lemma 10.1(C). Combining (10.4) with (10.6) and (10.7) leads to

(10.8)
$$0 = \frac{k(k^2-1)}{12}g''' - g'' \left(\frac{xD_{k-2}}{P'} \right) + g' \left((x+1)D_{k-2} + 2c_{k-2} + \frac{aD_{k-2}}{P'} \right)$$
$$+ g \left(\frac{D'_{k-2}}{2} + c'_{k-2} - \frac{(k^2-4)D_{k-2}P'}{6x} - aD_{k-2} + \frac{D_{k-2}}{P'} \left(\frac{D_{k-2}}{k} + a' \right) \right).$$

Lemma 10.1(C) implies that (10.8) must be (10.5) multiplied by $k(k^2-1)/12x$. Comparing the coefficients of g'' yields

(10.9)
$$\frac{P''}{P'} + P' = -\frac{a}{x} + \frac{12xD_{k-2}}{k(k^2-1)P'}.$$

Next, matching the coefficients of g' and using (10.9) results in

(10.10)
$$c_{k-2} = -\frac{(12x^2 + 12x + k^2 - 1)D_{k-2}}{24x} - \frac{k(k^2-1)a'}{12x} - \frac{k(k^2-1)a^2}{24x^2}.$$

Examining the coefficients of g in (10.5) and (10.8) in the light of (10.9) and (10.10) leads to

$$aD_{k-2} \left(\frac{k^2 - 1 - 12x^2}{12x^2} \right) = D'_{k-2} \left(\frac{12x^2 + 1 - k^2}{24x} \right) + \frac{(k^2-4)D_{k-2}P'}{6x}.$$

Because $k \geq 3$ and $D_{k-2}P' \not\equiv 0$, this forces $k^2 - 1 - 12x^2 \neq 0$ and

(10.11)
$$a = -\frac{xD'_{k-2}}{2D_{k-2}} + \frac{2x(k^2-4)P'}{k^2 - 1 - 12x^2}.$$

Therefore $P'(\infty) \neq 0$, since $a(\infty) \neq 0$, and using (10.11) to eliminate a from (10.9) delivers

$$\frac{P''}{P'} + \frac{(3k^2 - 12x^2 - 9)P'}{k^2 - 1 - 12x^2} - \frac{D'_{k-2}}{2D_{k-2}} = \frac{12xD_{k-2}}{k(k^2 - 1)P'}.$$

Setting $Z = 1/D_{k-2}$ yields in turn a linear differential equation of form

$$(10.12) \qquad \left(\frac{2P''}{P'} + \eta_1 P'\right) Z + Z' = \frac{24x}{k(k^2 - 1)P'}, \quad \eta_1 = \frac{6(k^2 - 4x^2 - 3)}{k^2 - 1 - 12x^2}.$$

If $k^2 - 4x^2 - 3 = 0$, then $(k + 2x)(k - 2x) = 3$; because $k, 2x \in \mathbf{Z}$, this forces $2k = \pm 4$, a contradiction. Assume henceforth that $k^2 - 4x^2 - 3 \neq 0$: then the integrating factor for (10.12) is $(P')^2 e^{\eta_1 P}$, with $\eta_1 \neq 0$, and the general solution to (10.12) is

$$Z = (P')^{-2}\left(\eta_2 + d_7 e^{-\eta_1 P}\right), \quad d_7 \in \mathbf{C}, \quad \eta_2 = \frac{24x}{k(k^2 - 1)\eta_1}.$$

Since P' and Z are rational at infinity with $P'(\infty) \neq 0$, this yields $Z = \eta_2 (P')^{-2}$ and

$$(10.13) \qquad \begin{aligned} \frac{-D_{k-2}}{k} &= \frac{-(P')^2}{k\eta_2} = \eta_3 (P')^2, \\ \eta_3 &= \frac{-(k^2 - 1)\eta_1}{24x} = \frac{(4x^2 + 3 - k^2)(k^2 - 1)}{4x(k^2 - 1 - 12x^2)} \neq 0, \end{aligned}$$

as well as

$$(10.14) \qquad a = -x\left(\frac{P''}{P'} + \eta_4 P'\right), \quad \eta_4 = -\frac{2(k^2 - 4)}{k^2 - 1 - 12x^2} \neq 0,$$

using (10.11). Combining (10.4), (10.13) and (10.14) shows that g solves the equation

$$(10.15) \qquad P' e^P = x\left(y' + \left(\frac{P''}{P'} + \eta_4 P'\right) y\right)' + \eta_3 (P')^2 y.$$

Now write $\zeta = P(z)$ and $Y_0(\zeta) = y(z)P'(z)$ so that

$$y' + \left(\frac{P''}{P'} + \eta_4 P'\right) y = \frac{dY_0}{d\zeta} + \eta_4 Y_0, \quad \left(y' + \left(\frac{P''}{P'} + \eta_4 P'\right) y\right)' = P'\left(\frac{d^2 Y_0}{d\zeta^2} + \eta_4 \frac{dY_0}{d\zeta}\right).$$

Thus (10.15) becomes

$$(10.16) \qquad e^\zeta = xY_0''(\zeta) + x\eta_4 Y_0'(\zeta) + \eta_3 Y_0(\zeta).$$

The auxiliary equation for the complementary function of (10.16) is then

$$(10.17) \qquad x\lambda^2 + x\eta_4 \lambda + \eta_3 = 0, \quad x\eta_3\eta_4 \in \mathbf{C}^*.$$

Suppose that 1 is a double root of (10.17). Then g has a representation $g = (P')^{-1}e^P Q_2(P)$, for some polynomial $Q_2 \not\equiv 0$ of degree at most two. Since $P'(\infty) \neq 0$, there cannot exist a sector on which g has an unbounded sequence of zeros, contradicting the assumption that f has an unbounded sequence of poles.

Now suppose that 1 is a simple root of (10.17), or that (10.17) has a repeated root. Then the fact that $\eta_3 \neq 0$ gives

$$P'g = (\beta_1 + \beta_2 P)e^{\omega_1 P} + \beta_3 e^{\omega_3 P}, \quad \beta_j, \omega_j \in \mathbf{C}, \quad 0 \neq \omega_1 \neq \omega_3 \neq 0.$$

Here $\beta_2 \neq 0$ by Lemma 10.1(C), since otherwise g satisfies a second order linear differential equation, and $\beta_3 \neq 0$ by the assumption that f has an unbounded sequence of poles. Denote by $\widehat{\psi}$ the result of analytically continuing a function element ψ once

around a given circle $|z| = r_3 > r_2$. Then there exists $\zeta_0 \in \mathbf{C}$ such that $\widehat{P} = P + \zeta_0$ and

$$(\beta_1 + \beta_2\zeta_0 + \beta_2 P)e^{\omega_1 P + \omega_1 \zeta_0} + \beta_3 e^{\omega_3 P + \omega_3 \zeta_0} = P'\widehat{g}$$
$$= P'\omega g = \omega\left((\beta_1 + \beta_2 P)e^{\omega_1 P} + \beta_3 e^{\omega_3 P}\right),$$

where $\omega^k = 1$. Because $\beta_3 P'(\infty) \neq 0$, examining the coefficients of $e^{\omega_3 P}$ and $e^{\omega_1 P}$ leads to

$$e^{\omega_3 \zeta_0} = \omega, \quad (\beta_1 + \beta_2\zeta_0 + \beta_2 P)e^{\omega_1 \zeta_0} = \omega(\beta_1 + \beta_2 P).$$

Differentiating the last relation then shows that $e^{\omega_1 \zeta_0} = \omega$, since $\beta_2 \neq 0$, and

$$\omega\beta_1 = (\beta_1 + \beta_2\zeta_0)e^{\omega_1 \zeta_0} = \omega(\beta_1 + \beta_2\zeta_0),$$

so that $\zeta_0 = 0$ and P is rational at infinity, which forces g to solve a second order equation, contradicting Lemma 10.1(C). Thus (10.2) holds, with the $\omega_j \in \mathbf{C}^*$ pairwise distinct, since $\eta_3 \neq 0$, and $\omega_1 = 1$, and none of the β_j can vanish, again by Lemma 10.1(C). The proof of (10.2) is now complete.

Next, suppose that $d = 0$ or $d = k - 1$, so that $x = \pm(k-1)/2$. Now (10.13) and (10.14) imply that (10.16) takes the form

$$e^\zeta = xY_0''(\zeta) + x\left(\frac{k+2}{k-1}\right)Y_0'(\zeta) + x\left(\frac{k+1}{(k-1)^2}\right)Y_0(\zeta).$$

The auxiliary equation for the complementary function has roots $\lambda_j = 1 - jk/(k-1)$, for $j = 1, 2$, and (10.2) becomes, in view of Lemma 10.1(C),

$$(10.18) \qquad g = \frac{e^P}{P'}\left(e_1 + e_2 e^{\eta P} + e_3 e^{2\eta P}\right), \quad \eta = -\frac{k}{k-1}, \quad e_j \in \mathbf{C}^*.$$

Thus (5.4), (10.3), (10.18) and partial fractions deliver

$$g = \frac{e_4 e^P}{P'}\left(e^{\eta P} - e_5\right)\left(e^{\eta P} - e_6\right),$$

$$(10.19) \qquad \frac{f'}{f} + \frac{a_{k-1}}{k} + a = \frac{p'}{p} + a = e_7 P'\left(\frac{1}{e^{\eta P} - e_5} - \frac{1}{e^{\eta P} - e_6}\right) + \frac{dg'}{g}, \quad e_j \in \mathbf{C}.$$

Again the e_j are all non-zero, and $e_5 \neq e_6$ since g cannot have multiple zeros. If r_4 is large and some continuation of $e^{\eta P}$ takes the value e_5 at some $z_0 \in \Omega(r_4)$ then z_0 is a zero of g, and so is a pole of f of multiplicity m_1 satisfying $-m_1 = d + e_7/\eta e_5$, so that (5.2), (10.18) and (10.19) imply that at the point z_0 the following equations are satisfied:

$$e_5^{k-1} = e^{(k-1)\eta P} = e^{-kP};$$

$$\frac{(-1)^k}{m_1(m_1+1)\dots(m_1+k-1)} = (g')^k = \left(e_4 e^P \eta e_5(e_5 - e_6)\right)^k$$
$$= e_4^k e_5^{1-k}\eta^k e_5^k(e_5 - e_6)^k = e_4^k e_5 \eta^k(e_5 - e_6)^k.$$

Similarly, all zeros of continuations of $e^{\eta P} - e_6$ to $\Omega(r_4)$ are poles of f of multiplicity m_2, where $-m_2 = d - e_7/\eta e_6$, and

$$\frac{m_2(m_2+1)\dots(m_2+k-1)}{m_1(m_1+1)\dots(m_1+k-1)} = (-1)^k\frac{e_5}{e_6} = (-1)^{k-1}\frac{m_2+d}{m_1+d}.$$

But $d = 0$ or $d = k - 1$, so that $m_1 = m_2$ (and k is odd). $\qquad \square$

11. The exponential parts for the equation $N[y] = 0$

Let p and q be polynomials in z, and let $\theta \in \mathbf{R}$. Write $p \prec q$ (respectively, $p \preceq q$, $p \simeq q$) to indicate that $\operatorname{Re} p(re^{i\theta}) < \operatorname{Re} q(re^{i\theta})$ (respectively $\operatorname{Re} p(re^{i\theta}) \leq \operatorname{Re} q(re^{i\theta})$, $\operatorname{Re} p(re^{i\theta}) = \operatorname{Re} q(re^{i\theta})$) as $r \to +\infty$. Since each $P_\theta(r) = \operatorname{Re} p(re^{i\theta})$ is a polynomial in r, every $\theta \in \mathbf{R}$ has $p \prec 0$ or $p \simeq 0$ or $0 \prec p$, and if p is not constant then all but finitely many $\theta \in [0, 2\pi]$ have either $p \prec 0$ or $0 \prec p$.

Suppose that $N[y] = 0$ has linearly independent canonical formal solutions with exponential parts $\kappa_1, \kappa_2, \kappa_3$. The κ_j are polynomials in z by Lemma 10.1, and it will be assumed as before that $\kappa_j(0) = 0$ for all j, from which it follows that if $\kappa_j - \kappa_{j'}$ is constant then $\kappa_j - \kappa_{j'} \equiv 0$.

Lemma 11.1. *The κ_j are not all the same polynomial, and there does not exist $\theta \in \mathbf{R}$ with $\kappa_j \prec 0$ on $\arg z = \theta$ for $j = 1, 2, 3$.*

Proof. The first assertion is proved in Lemma 7.2, and the second holds because otherwise $g^k = f/F$ tends to zero transcendentally fast on a sector, contradicting Lemma 7.2. $\qquad\square$

Lemma 11.1 does not exclude two of the κ_j being the same polynomial, possibly identically zero, and this case will be dealt with in Sections 15 and 18. When there is no repetition among the κ_j, the next lemma shows that there are two subcases to handle.

Lemma 11.2. *Suppose that the κ_j are pairwise distinct. Then it is possible to label the κ_j and choose a ray $\arg z = \theta \in \mathbf{R}$ such that*

(11.1) $\qquad$ (A) $\quad \kappa_1 \prec \kappa_2 \prec 0 \prec \kappa_3 \quad$ or $\quad$ (B) $\quad \kappa_1 \prec \kappa_2 \prec 0, \quad \kappa_3 \equiv 0$

or

(11.2) $\qquad\qquad\qquad$ (C) $\quad \kappa_1 \prec 0 \prec \kappa_3, \quad \kappa_2 \equiv 0.$

Proof. If one of the κ_j is identically zero label the other two as κ_a and κ_b, and choose $\theta \in [0, 2\pi]$ such that $\kappa_a \prec 0$ on $\arg z = \theta$. A small change to θ delivers either $\kappa_b \prec \kappa_a$, which leads to (B), or $\kappa_a \prec \kappa_b \prec 0$ or $\kappa_a \prec 0 \prec \kappa_b$, leading to (B) or (C).

Assume now that none of the κ_j is the zero polynomial. Let m^* be the largest of the degrees of the κ_j and, with no loss of generality, write

$$\kappa_j(z) = \alpha_j z^{m^*} + \dots, \quad \alpha_1 \neq 0.$$

If $\alpha_2 = 0$ then it is easy to choose a ray $\arg z = \theta$ on which $\kappa_1 \prec \kappa_2 \prec 0$ and, by varying θ slightly if necessary, either $\kappa_3 \prec 0$ or $0 \prec \kappa_3$. Lemma 11.1 then implies that (A) must hold.

Next, suppose that $\alpha_j \neq 0$ for all j. If α_2/α_1 is not a negative real number choose a ray on which $\alpha_1 z^{m^*}$ and $\alpha_2 z^{m^*}$ both have negative real part and $\kappa_3 \not\simeq 0$. Shifting θ slightly gives either $\kappa_1 \prec \kappa_2 \prec 0$ or $\kappa_2 \prec \kappa_1 \prec 0$, and Lemma 11.1 forces (A) to hold, subject to re-labelling if necessary.

Thus the proof is complete, after re-labelling if necessary, unless both α_2/α_1 and α_3/α_1 are negative real numbers, in which case the argument of the previous paragraph applies with κ_2 and κ_3 in place of κ_1 and κ_2. $\qquad\square$

12. A decomposition of the operators N and V

By Lemma 10.1(C), the equation $N[y] = 0$ in Lemma 5.3, which is satisfied by g, has order 3, and so the asymptotics for its solutions may be complicated. However,

the following lemma gives a condition under which two linearly independent solutions of $N[y] = 0$ must together solve a second order equation, for which the asymptotics are then considerably simpler.

Lemma 12.1. *With N and V as in Lemma 5.3, suppose that g_1 and g_2 are linearly independent (both formal or both locally analytic) solutions of $N[y] = 0$ such that*

$$(12.1) \qquad g_1 V[g_2] - g_2 V[g_1] = d(g_2 g_1' - g_1 g_2'),$$

where d is rational at infinity. Then g_1 and g_2 solve an equation (3.4) and

$$(12.2) \qquad W' + E_1 W = 0, \quad E_1 = \frac{\beta + d}{\alpha}, \quad W = W(g_1, g_2),$$

$$(12.3) \qquad N = (D + \delta) \circ (D^2 + E_1 D + E_0),$$

where E_1, E_0 and δ are rational at infinity. If, in addition, d is constant then

$$(12.4) \quad V = \alpha(D^2 + E_1 D + E_0) - dD - xE_1 - \frac{X'}{X}, \quad x = d - \frac{k-1}{2}, \quad \frac{X'}{X} = -\frac{a_{k-1}}{k}.$$

Proof. Differentiating $W = W(g_1, g_2) = g_1 g_2' - g_1' g_2$ gives $W' = g_1 g_2'' - g_1'' g_2$. Thus equation (12.1) can be rewritten, using (5.16), in the form (12.2), with E_1 rational at infinity. Applying Lemma 3.4 shows that g_1 and g_2 solve an equation (3.4), with E_0 also rational at infinity. Because g_1 and g_2 are linearly independent solutions of $N[y] = 0$ and (3.4), the operator N factorises using (10.1) and the division algorithm for linear differential operators [15, p. 126] as

$$(12.5) \qquad \begin{aligned} N &= D^3 + B_2 D^2 + B_1 D + B_0 = (D + \delta) \circ (D^2 + E_1 D + E_0) \\ &= D^3 + (E_1 + \delta)D^2 + (E_0 + E_1' + \delta E_1)D + E_0' + \delta E_0, \end{aligned}$$

where δ is again rational at infinity.

Now suppose that d is constant. (5.15) and (5.16) yield

$$h' = \alpha g''' + (\beta + \alpha')g'' + (\beta' + \gamma)g' + \gamma' g = -\left(\frac{k-1}{2}\right)g'' + \frac{a_{k-1}}{k}g' + \frac{a_{k-1}' - D_{k-2}}{k}g,$$

where D_{k-2} is rational at infinity, so that

$$0 = \alpha g''' + \left(\beta + \alpha' + \frac{k-1}{2}\right)g'' + \left(\beta' + \gamma - \frac{a_{k-1}}{k}\right)g' + \left(\gamma' + \frac{D_{k-2} - a_{k-1}'}{k}\right)g.$$

Comparing coefficients with (12.5) leads, using Lemma 10.1(C), to

$$(12.6) \qquad E_1 + \delta = \frac{\beta + \alpha' + (k-1)/2}{\alpha}, \quad E_0 + E_1' + \delta E_1 = \frac{\beta' + \gamma - a_{k-1}/k}{\alpha}.$$

Now (12.2) and (12.6) deliver, with $x = d - (k-1)/2$,

$$\beta = \alpha E_1 - d, \quad \delta = \frac{\beta + \alpha' + (k-1)/2}{\alpha} - E_1 = \frac{\alpha'}{\alpha} - \frac{x}{\alpha},$$

$$\gamma = \alpha(E_0 + E_1' + \delta E_1) - \beta' + \frac{a_{k-1}}{k} = \alpha(E_0 + E_1' + \delta E_1) - \alpha' E_1 - \alpha E_1' + \frac{a_{k-1}}{k}$$

$$= \alpha E_0 + E_1(\alpha' - x) - \alpha' E_1 + \frac{a_{k-1}}{k} = \alpha E_0 - xE_1 + \frac{a_{k-1}}{k},$$

and the representations given here for β and γ yield (12.4). $\qquad \square$

Lemma 12.2. *Suppose that g_1 and g_2 are linearly independent (both formal or both locally analytic) solutions of $N[y] = 0$, such that (12.1) holds, with $d \in \{0, \ldots, k-1\}$ a constant. Then g_1 and g_2 solve an equation (3.4), and formulas (12.2) to (12.4) hold, with E_1, E_0 and δ rational at infinity, and d satisfies $d \neq (k-1)/2$. Suppose further that $E_1(\infty) \neq 0$ in (3.4). Then g is given by (10.2). If, in addition, $d = 0$ or $d = k-1$, then f satisfies the conclusion of Proposition 10.1.*

Proof. Lemma 12.1 gives the equation (3.4) solved by g_1 and g_2, as well as formulas (12.2) to (12.4). Since $N[g] = 0$, but $g'' + E_1 g' + E_0 g \not\equiv 0$, (5.16), (12.3) and (12.4) deliver

$$g'' + E_1 g' + E_0 g = e^\tau, \quad \tau' = -\delta,$$

and

$$-\frac{f'}{f} = \frac{h}{g} = \frac{V[g]}{g} = \frac{\alpha e^\tau}{g} - \frac{dg'}{g} - xE_1 + \frac{a_{k-1}}{k} = \frac{\alpha e^\tau}{g} - \frac{dg'}{g} + a^*.$$

Here the function $a^* = -xE_1 + a_{k-1}/k$ is rational at infinity and $-f'/f + dg'/g - a^*$ continues without zeros in some $\Omega(r_2)$. Hence Lemma 10.2 and (12.4) imply that $d \neq (k-1)/2$ and $x \neq 0$. Finally, if $E_1(\infty) \neq 0$ in (3.4) then $a^*(\infty) \neq 0$ and the remaining assertions of Lemma 12.2 follow from Lemma 10.2. $\qquad\square$

13. Analytic solutions decaying in the same sector

This section determines conditions under which Lemma 12.2 may be applied with analytic solutions of $N[y] = 0$.

Lemma 13.1. *Assume that there exist linearly independent analytic solutions g_1, g_2 of $N[y] = 0$, such that both tend to 0 transcendentally fast as $z \to \infty$ in the same sector S. Then g_1 and g_2 satisfy the hypotheses of Lemma 12.2, for some constant $d \in \{0, \ldots, k-1\}$, and solve an equation (3.4), with E_1 and E_0 rational at infinity and $E_1(\infty) \neq 0$. Moreover, formulas (12.2) to (12.4) hold, and d and g satisfy $d \neq (k-1)/2$ and (10.2). Finally, if $d = 0$ or $d = k-1$ then the conclusion of Proposition 10.1 holds.*

In the context of Section 11, Lemma 13.1 applies if there is a repeated non-trivial exponential part among the κ_j, or if (11.1) holds for some ray $\arg z = \theta$.

Proof. Choose $z_0 \in S$ such that z_0 is not a singular point for any of the operators L, M, N, and such that

(13.1) $$g_1(z_0)g_2(z_0) \neq 0, \quad W(g_1, g_2)(z_0) \neq 0.$$

Let w lie close to z_0. Then $g_w(z) = g_2(w)g_1(z) - g_1(w)g_2(z)$ tends to 0 transcendentally fast as $z \to \infty$ in S and, by Lemma 7.2, g_w annihilates a solution $f_w \not\equiv 0$ of $L[y] = 0$, with

(13.2) $$\frac{f_w'(z)}{f_w(z)} = -\frac{V[g_w](z)}{g_w(z)} = \frac{g_1(w)V[g_2](z) - g_2(w)V[g_1](z)}{g_2(w)g_1(z) - g_1(w)g_2(z)}.$$

Let

$$G_0(z) = \frac{g_1(z)V[g_2](z) - g_2(z)V[g_1](z)}{g_2(z)g_1'(z) - g_1(z)g_2'(z)}.$$

The second condition of (13.1) implies that w is a simple zero of g_w, and by (13.2) the residue of f_w'/f_w at w is $G_0(w)$, which must belong to the set $\{0, \ldots, k-1\}$. Since this holds for all w near z_0, the function G_0 is a constant $d \in \{0, \ldots, k-1\}$, and so g_1 and g_2 satisfy the hypotheses of Lemma 12.2, and hence solve an equation (3.4). Cauchy's estimate for derivatives shows that $W(g_1, g_2)$ tends to 0 transcendentally

fast in a subsector of S, which gives $E_1(\infty) \neq 0$ by Abel's identity. The remaining assertions hold by Lemma 12.2. $\qquad\square$

14. Decaying solutions with different exponential parts

This section will deal with one case of the situation in Section 13, in which two linearly independent solutions of $N[y] = 0$ decay in the same sector and have different exponential parts, corresponding to (11.1) in Lemma 11.2. The case of a repeated non-trivial exponential part will be addressed in Section 15. The methods of this section are heavily influenced by [2], but a decisive role will be played by Lemma 13.1 and the second order equation (3.4).

Proposition 14.1. *Assume that there exists a ray $\arg z = \theta$ on which the exponential parts κ_j for the equation $N[y] = 0$ satisfy (11.1). Then f satisfies the conclusions of Proposition 10.1.*

To prove Proposition 14.1, note first that if θ is varied slightly, then (11.1) continues to hold. Take canonical formal solutions g_1, g_2 of $N[y] = 0$ with exponential parts κ_1, κ_2. By (11.1) the exponential parts for $N[y] = 0$ are pairwise distinct, and there exist linearly independent analytic solutions G_1, G_2 of $N[y] = 0$ which are asymptotic to g_1 and g_2 respectively on a sector centred on the ray $\arg z = \theta$, and so tend to 0 transcendentally fast there, by (11.1). Thus the hypotheses of Lemma 13.1 are satisfied, and therefore so are those of Lemma 12.2, for some d in $\{0, \ldots, k-1\}$, which gives rise to an equation (3.4) satisfied by the G_j. Computing series representations for $0 = G_j'' + E_1 G_j' + E_0 G_j$ shows that the g_j also solve (3.4). Thus the exponential parts κ_1 and κ_2 correspond to the equation (3.4), while κ_3 is from the third canonical formal solution of $N[y] = 0$. Furthermore, V satisfies (12.4).

Next, let the operators L, M have canonical formal solutions with exponential parts q_j, s_j respectively, labelled so that

$$(14.1) \qquad q_1 \preceq q_2 \preceq \ldots \preceq q_k, \quad s_1 \preceq s_2 \preceq \ldots \preceq s_k,$$

on $\arg z = \theta$ (the last phrase will be omitted henceforth). The q_j and s_j are polynomials in z with zero constant term. It may be assumed that θ is chosen so that if $\widetilde{p}_1, \widetilde{p}_2 \in \{q_1, \ldots, q_k, s_1, \ldots, s_k\}$ and $\widetilde{p}_1 - \widetilde{p}_2 \not\equiv 0$ then $\widetilde{p}_1 \prec \widetilde{p}_2$ or $\widetilde{p}_2 \prec \widetilde{p}_1$.

Lemma 14.1. *There exists $\lambda \in \{1, \ldots, k\}$ such that the canonical formal solution g_1 of (3.4) with exponential part κ_1 annihilates a canonical formal solution f_λ of $L[y] = 0$ with exponential part q_λ, and the exponential parts for $M[y] = 0$ are*

$$(14.2) \qquad q_j + \kappa_1 \quad (j \neq \lambda), \quad q_\lambda - (k-1)\kappa_1.$$

Moreover, this f_λ may be assumed to be $g_1^d W^{-x} X$, where W, x and X are as in Lemma 12.1. Furthermore, there exists $\mu \in \{1, \ldots, k\}$ such that the canonical formal solution g_2 of (3.4) with exponential part κ_2 annihilates a canonical formal solution $f_\mu = g_2^d W^{-x} X$ of $L[y] = 0$, with exponential part q_μ, while the exponential parts for $M[y] = 0$ are

$$(14.3) \qquad q_j + \kappa_2 \quad (j \neq \mu), \quad q_\mu - (k-1)\kappa_2.$$

Proof. Since the exponential parts for $N[y] = 0$ are pairwise distinct, g_1 and g_2 both have non-zero exponential parts and are free of logarithms. Thus Lemma 7.1 gives a canonical formal solution f_λ of $L[y] = 0$, with exponential part q_λ, such that g_1 annihilates f_λ and the exponential parts for $M[y] = 0$ are given by (14.2). Moreover, since g_1 is a solution of (3.4), solving $0 = f_\lambda' g_1 + f_\lambda V[g_1]$ in the light of (12.4) shows

that f_λ is a constant multiple of $g_1^d W^{-x} X$. The same argument works for g_2 and f_μ. $\square$

Lemma 14.2. *The integer λ is 1.*

Proof. Suppose not: then an exponential part $q_1 + \kappa_1$ occurs in the list (14.2). But this term, in view of (11.1) and (14.1), cannot be realised as $q_j + \kappa_2$ or $q_\mu - (k - 1)\kappa_2$. $\square$

Lemma 14.3. *The q_j satisfy $q_j + \kappa_1 \preceq q_1 - (k - 1)\kappa_1$ for $2 \leq j \leq k$.*

Proof. Suppose that this is not the case. Then the term $q_k + \kappa_1$, which does occur in the list (14.2), must be maximal according to the ordering $\preceq$. But (11.1) implies that

$$(14.4) \qquad q_k + \kappa_1 \prec q_k + \kappa_2 \prec q_k - (k - 1)\kappa_2.$$

This is a contradiction since the second or third term in (14.4) occurs in the list (14.3). $\square$

Thus by (14.2) the s_j in (14.1) can now be written as

$$(14.5) \qquad s_1 = q_2 + \kappa_1, \quad \ldots, \quad s_{k-1} = q_k + \kappa_1, \quad s_k = q_1 - (k - 1)\kappa_1.$$

Note that each of these relations initially holds with $\simeq$ in place of $=$, but may be assumed to be an identity, by the remark following (14.1). The same property will subsequently be used on a number of occasions without explicit reference.

Lemma 14.4. *The exponential part s_μ satisfies $s_\mu = q_\mu - (k - 1)\kappa_2$.*

Proof. Suppose first that $q_\mu - (k - 1)\kappa_2 \prec s_\mu$. Then (11.1) and (14.1) give $\mu > 1$ and

$$q_1 + \kappa_2 \preceq \ldots \preceq q_{\mu-1} + \kappa_2 \prec q_\mu - (k - 1)\kappa_2 \prec s_\mu,$$

in which all of the first μ terms occur in the list (14.3). Hence the second list in (14.1) includes μ terms $\widetilde{s}$ all satisfying $\widetilde{s} \prec s_\mu$, which is a contradiction.

Now suppose that $s_\mu \prec q_\mu - (k - 1)\kappa_2$. Then $\mu < k$ and in the list (14.3) there are at least μ terms $\widetilde{s}$ all satisfying $\widetilde{s} \preceq s_\mu \prec q_\mu - (k - 1)\kappa_2$. Of these, $\mu - 1$ are $q_1 + \kappa_2, \ldots, q_{\mu-1} + \kappa_2$ (this list being void if $\mu = 1$), and it must be the case that $q_{\mu+1} + \kappa_2 \preceq s_\mu \prec q_\mu - (k - 1)\kappa_2$. But then (11.1) and (14.5) yield a contradiction via

$$s_\mu = q_{\mu+1} + \kappa_1 \prec q_{\mu+1} + \kappa_2 \preceq s_\mu. \qquad\qquad \square$$

Lemma 14.4 implies that among the $q_j + \kappa_2$ ($j \neq \mu$) there are at least $\mu - 1$ terms $\widetilde{s}$ with $\widetilde{s} \preceq q_\mu - (k - 1)\kappa_2$, and if $\mu > 1$ these must include $q_1 + \kappa_2, \ldots, q_{\mu-1} + \kappa_2$; similarly, there are at least $k - \mu$ terms with $q_\mu - (k - 1)\kappa_2 \preceq \widetilde{s}$, and if $\mu < k$ these must include $q_{\mu+1} + \kappa_2, \ldots, q_k + \kappa_2$. It follows that

$$(14.6) \qquad s_j = q_j + \kappa_2 \quad (j \neq \mu), \quad s_\mu = q_\mu - (k - 1)\kappa_2.$$

Lemma 14.5. *The integers d and μ are related by $d = \mu - 1$.*

Proof. By Lemmas 14.1 and 14.2 the canonical formal solutions f_λ and f_μ of $L[y] = 0$ annihilated by g_1 and g_2 have exponential parts q_1 and q_μ respectively. The quotient $f_\mu / f_\lambda = (g_2/g_1)^d$ has exponential part $d(\kappa_2 - \kappa_1)$, which implies that $d(\kappa_2 - \kappa_1) = q_\mu - q_1$. If $\mu = 1$ this gives $d = 0$ since $\kappa_1 \neq \kappa_2$. For $\mu > 1$, (14.5) and (14.6) yield

$$s_1 = q_2 + \kappa_1, \quad \ldots, \quad s_{\mu-1} = q_\mu + \kappa_1, \quad s_1 = q_1 + \kappa_2, \quad \ldots, \quad s_{\mu-1} = q_{\mu-1} + \kappa_2,$$

and so

$$\kappa_2 - \kappa_1 = q_2 - q_1 = \ldots = q_\mu - q_{\mu-1}, \quad d(\kappa_2 - \kappa_1) = q_\mu - q_1 = (\mu - 1)(\kappa_2 - \kappa_1). \quad \square$$

By (11.1) there exists a canonical formal solution g_3 of $N[y] = 0$ which is free of logarithms and has exponential part κ_3.

Lemma 14.6. *The exponential part κ_3 is not the zero polynomial, and case (A) applies in (11.1).*

Proof. Suppose that $\kappa_3 \equiv 0$. Then Lemma 7.1 and (14.6) give at least one j with $q_j = s_j = q_j + \kappa_2 \prec q_j$, a contradiction. $\quad \square$

By Lemmas 7.1 and 14.6, there exists ν such that g_3 annihilates a canonical formal solution of $L[y] = 0$ with exponential part q_ν, and the exponential parts for $M[y] = 0$ are

$$(14.7) \qquad\qquad q_j + \kappa_3 \quad (j \neq \nu), \quad q_\nu - (k-1)\kappa_3.$$

Lemma 14.7. *Assume that $2 \leq \mu \leq k-1$. Then $\nu = k$ and $s_1 = q_k - (k-1)\kappa_3$.*

Proof. Suppose first that $\nu < k$. Then the list (14.7) includes $q_k + \kappa_3$, which must be maximal with respect to the ordering $\preceq$, since $0 \prec \kappa_3$. But $\mu \neq k$ by assumption, which gives

$$q_k + \kappa_3 = s_k = q_k + \kappa_2$$

using (14.6), and this contradicts (11.1). Thus $\nu = k$ in (14.7).

Now suppose that $s_1 \neq q_k - (k-1)\kappa_3$. Then $s_1 \prec q_k - (k-1)\kappa_3$ and so $s_1 = q_1 + \kappa_3$, whereas (14.6) gives $s_1 = q_1 + \kappa_2$ since $\mu \neq 1$, again contradicting (11.1). $\quad \square$

Lemma 14.8. *If $k \geq 4$ then $\mu = 1$ or $\mu = k$.*

Proof. Suppose instead that $2 \leq \mu \leq k-1$. Then, by Lemma 14.7, the list (14.7) consists of

$$(14.8) \qquad s_1 = q_k - (k-1)\kappa_3, \quad s_2 = q_1 + \kappa_3, \quad \ldots, \quad s_k = q_{k-1} + \kappa_3.$$

Using (14.5), (14.6) and (14.8) gives

$$(14.9) \qquad\qquad s_{\mu+1} = q_\mu + \kappa_3 = q_{\mu+1} + \kappa_2, \quad q_\mu - q_{\mu+1} = \kappa_2 - \kappa_3,$$

and

$$(14.10) \qquad s_\mu = q_{\mu+1} + \kappa_1 = q_\mu - (k-1)\kappa_2, \qquad q_\mu - q_{\mu+1} = \kappa_1 + (k-1)\kappa_2.$$

Define τ as follows: if $2 \leq \mu \leq k-2$ take $\tau = k-1$, and if $\mu = k-1$ choose $\tau = 1$. In either case $\mu \neq \tau, \tau+1$, since $k \geq 4$ by assumption. Thus (14.5), (14.6) and (14.8) deliver

$$(14.11) \qquad\qquad s_{\tau+1} = q_{\tau+1} + \kappa_2 = q_\tau + \kappa_3, \quad q_{\tau+1} - q_\tau = \kappa_3 - \kappa_2,$$

in addition to

$$(14.12) \qquad\qquad s_\tau = q_\tau + \kappa_2 = q_{\tau+1} + \kappa_1, \quad q_{\tau+1} - q_\tau = \kappa_2 - \kappa_1.$$

Combining (14.9), (14.10), (14.11) and (14.12) yields

$$\kappa_2 - \kappa_1 = \kappa_3 - \kappa_2 = -\kappa_1 - (k-1)\kappa_2,$$

contradicting the fact that $\kappa_2 \prec 0$. $\quad \square$

Thus d must be 0 or $k-1$: this follows from Lemmas 14.5 and 14.8 when $k \geq 4$, while if $k = 3$ then Lemma 13.1 forces $d \neq (k-1)/2 = 1$. Hence the

conclusion of Proposition 10.1 holds by Lemma 13.1 and the proof of Proposition 14.1 is complete. ☐

15. The case of a repeated non-trivial exponential part

Suppose that κ is a repeated non-trivial exponential part for the equation $N[y] = 0$. Then it is possible to choose a ray $\arg z = \theta \in \mathbf{R}$ on which $\kappa \prec 0$, and linearly independent analytic solutions g_1, g_2 of $N[y] = 0$, each with exponential part κ near $\arg z = \theta$. It then follows from Lemma 13.1 that g is given by (10.2), which yields

$$0 = N[g] = \beta_1 H_1 e^{\omega_1 P} + \beta_2 H_2 e^{\omega_2 P} + \beta_3 H_3 e^{\omega_3 P}, \quad \beta_j, \omega_j \in \mathbf{C}^*,$$

in which the ω_j are pairwise distinct, while P' and the H_j are rational at infinity and $H_j e^{\omega_j P} = N[e^{\omega_j P}/P']$. This forces each H_j to vanish identically, so that the equation $N[y] = 0$ has three pairwise distinct exponential parts for its solutions, which is a contradiction. ☐

16. Two lemmas concerning trivial exponential parts

If at least one of the three exponential parts arising from the equation $N[y] = 0$ is trivial (that is, the zero polynomial), then it is not necessarily the case that $N[y] = 0$ will have two linearly independent solutions decaying in the same sector, so that a second order equation (3.4) may not be available. The approach to this case will combine Lemma 3.1 with some ideas from [2].

Lemma 16.1. *Assume that two exponential parts κ_1, κ_2 arising from the equation $N[y] = 0$ are such that κ_2 is the zero polynomial, while*

(16.1) $$\kappa_1 \prec 0 \quad or \quad 0 \prec \kappa_1$$

on a ray $\arg z = \theta$. Let the operators L, M have canonical formal solutions with exponential parts as in (14.1). Then the exponential parts for M are as in (14.2), while

(16.2) $$s_j = q_j \quad for\ each\ j$$

and the following additional conclusions hold. If $\kappa_1 \prec 0$ in (16.1) then $\lambda = 1$ and

(16.3) $q_1 = s_1 = q_2 + \kappa_1, \quad \dots, \quad q_{k-1} = s_{k-1} = q_k + \kappa_1, \quad q_k = s_k = q_1 - (k-1)\kappa_1.$

If $0 \prec \kappa_1$ in (16.1) then $\lambda = k$ and

(16.4) $q_1 = s_1 = q_k - (k-1)\kappa_1, \quad q_2 = s_2 = q_1 + \kappa_1, \dots, \quad q_k = s_k = q_{k-1} + \kappa_1.$

Proof. First observe that $N[y] = 0$ has two canonical formal solutions which are free of logarithms and have exponential parts 0 and κ_1 respectively. Thus (14.2) and (16.2) hold by Lemma 7.1. Assume that $\kappa_1 \prec 0$ in (16.1). If $\lambda \neq 1$ then an exponential part $q_1 + \kappa_1$ occurs in the list (14.2), but this term, in view of (16.1), cannot be realised as q_j for any j, contradicting (16.2). Now suppose that $s_k \neq q_1 - (k-1)\kappa_1$; then $s_k = q_k + \kappa_1$, again contradicting (16.2).

Now assume that $0 \prec \kappa_1$ in (16.1). Then λ must be k, since otherwise an exponential part $q_k + \kappa_1$ occurs in (14.2), contradicting (16.2). Moreover, $q_1 = s_1 = q_k - (k-1)\kappa_1$, because the contrary case forces $s_1 = q_1 + \kappa_1$, which again contradicts (16.2). ☐

Lemma 16.2. *If there exists a ray $\arg z = \theta$ on which the three exponential parts arising from the equation $N[y] = 0$ satisfy (11.2), then $\kappa_3 = -\kappa_1$.*

Proof. Assuming the existence of such a ray, let the operators L, M have exponential parts as in (14.1). Now (16.3) and (16.4) yield

$$q_1 = s_1 = q_k - (k-1)\kappa_3, \quad q_k = s_k = q_1 - (k-1)\kappa_1, \quad \kappa_3 = -\kappa_1. \qquad \square$$

17. The case where (11.2) holds

This section will deal with the case where there exists a ray for which conclusion (11.2) arises in Lemma 11.2. In this situation Lemma 16.2 makes it possible to assume that the exponential parts for $N[y] = 0$ are P, 0 and $-P$, where P is a polynomial in z of positive degree ρ. Hence $N[y] = 0$ has canonical formal solutions which are free of logarithms and satisfy

$$(17.1) \quad u_1(z) = z^{\eta_1} e^{P(z)}(1 + \ldots), \quad u_2(z) = z^{\eta_2}(1 + \ldots), \quad u_3(z) = z^{\eta_3} e^{-P(z)}(1 + \ldots).$$

Since $N[g] = 0$, the order of growth of $g^k = f/F$ is $\rho(g^k) = \rho$. Choose a ray $\arg z = \theta_0$ on which $\operatorname{Re} P(z) = O(|z|^{\rho-1})$ as $|z| \to \infty$, such that f has a sequence of poles (and so g has a sequence of simple zeros) tending to ∞ in the sector $|\arg z - \theta_0| \le \pi/2\rho$. Take a sector Σ given by $|\arg z - \theta_0| \le \pi/\rho - \delta_1$, where δ_1 is small and positive, and write

$$(17.2) \qquad g = U_1 + U_2 + U_3, \quad U_j = b_j \phi_j, \quad b_j \in \mathbf{C}, \quad \phi_j \sim u_j,$$

in which the ϕ_j are analytic solutions on Σ, and the last relation holds in the sense of asymptotic series, as in Section 4. Here the fact that the asymptotics for $N[y] = 0$ may be extended to hold in Σ follows from the work of Jurkat [16]: in the present case, where the exponential parts are P, 0 and $-P$, it is relatively simple to establish, using the Phragmén–Lindelöf principle. Since g has infinitely many zeros in Σ, at least two of the b_j, and so at least one of b_1 and b_3, must be non-zero. By replacing P by $-P$, it may be assumed that $b_1 \ne 0$.

Now take a ray $\arg z = \theta$ lying in Σ, on which $P \prec 0 \prec -P$, and apply Lemma 16.1 with $\kappa_1 = P$. It follows from (16.2) and (16.3) that
(17.3)
$$q_2 = q_1 - P, \quad q_3 = q_2 - P = q_1 - 2P, \quad \ldots, \quad q_k = q_{k-1} - P = q_1 - (k-1)P, \quad s_j = q_j,$$

and so, by (7.2),
(17.4)
$$0 = q_1 + \ldots + q_k = kq_1 - \left(\frac{k(k-1)}{2}\right)P, \quad q_1 = \left(\frac{k-1}{2}\right)P, \quad q_k = -\left(\frac{k-1}{2}\right)P.$$

Hence the equations $L[y] = 0$, $M[y] = 0$ have canonical formal solutions

$$(17.5) \qquad f_j(z) = z^{\lambda_j} e^{q_j(z)}(1 + \ldots), \quad w_j(z) = z^{\mu_j} e^{q_j(z)}(1 + \ldots),$$

respectively, in which the q_j are pairwise distinct. Since $a_{k-1} = A_{k-1}$, Lemma 4.2 implies that

$$(17.6) \qquad \lambda_1 + \ldots + \lambda_k = \mu_1 + \ldots + \mu_k.$$

Write $v_j = V[u_j]$. By Lemmas 7.1 and 16.1, u_1, u_3 annihilate f_1, f_k respectively, and (17.4) gives

$$(17.7) \qquad \frac{v_1(z)}{u_1(z)} = -\frac{f_1'(z)}{f_1(z)} = \widehat{c}_1 z^{\rho-1} + \ldots, \quad \frac{v_3(z)}{u_3(z)} = -\frac{f_k'(z)}{f_k(z)} = \widehat{c}_3 z^{\rho-1} + \ldots,$$

where $\widehat{c}_1$, $\widehat{c}_3$ are non-zero constants. It follows from (5.16) and (17.1) that v_2/u_2 is given by a (possibly vanishing) formal series in descending integer powers of z of the form

$$(17.8) \qquad \frac{v_2(z)}{u_2(z)} = \alpha(z)\frac{u_2''(z)}{u_2(z)} + \beta(z)\frac{u_2'(z)}{u_2(z)} + \gamma(z) = c_N z^N + \dots.$$

Lemma 17.1. *The integer k is at least 4.*

Proof. Suppose that $k = 3$; then (17.3) and (17.4) lead to

$$q_1 = P, \quad q_2 = 0, \quad q_3 = -P.$$

Now write, using (17.1), (17.5) and (17.7),

$$f_3'(z)u_1(z) + f_3(z)v_1(z) = f_3(z)u_1(z)\left(\frac{f_3'(z)}{f_3(z)} - \frac{f_1'(z)}{f_1(z)}\right)$$
$$= z^{\lambda_3+\eta_1}(1+\dots)\left(\frac{f_3'(z)}{f_3(z)} - \frac{f_1'(z)}{f_1(z)}\right)$$

and

$$f_1'(z)u_3(z) + f_1(z)v_3(z) = f_1(z)u_3(z)\left(\frac{f_1'(z)}{f_1(z)} - \frac{f_3'(z)}{f_3(z)}\right)$$
$$= z^{\lambda_1+\eta_3}(1+\dots)\left(\frac{f_1'(z)}{f_1(z)} - \frac{f_3'(z)}{f_3(z)}\right).$$

Each of these is a formal solution of $M[y] = 0$, with zero exponential part, and so a constant multiple of w_2. But this implies that $\lambda_1 + \eta_3 = \lambda_3 + \eta_1$ and $f_3'u_1 + f_3v_1 = -(f_1'u_3 + f_1v_3)$, so that $f_1u_3 = f_3u_1$, which leads in turn to

$$\frac{V[u_3]}{u_3} - \frac{V[u_1]}{u_1} = \frac{f_1'}{f_1} - \frac{f_3'}{f_3} = \frac{u_1'}{u_1} - \frac{u_3'}{u_3}, \quad u_1V[u_3] - u_3V[u_1] = u_3u_1' - u_1u_3'.$$

Hence (12.1) holds, with $g_1 = u_1$, $g_2 = u_3$ and $d = 1$. Thus the hypotheses of Lemma 12.2 are satisfied, so that $d \neq (k-1)/2 = 1$, a contradiction. $\square$

Lemma 17.2. *One of the following two conclusions holds, in which $\rho = \deg P > 0$:*

(A) $\eta_2 = -N \leq -\rho$ and $\eta_1 + \eta_3 = -2(\rho - 1)$;
(B) $\eta_1 + \eta_3 - 2\eta_2 = 0$ and f has order of growth ρ.

Proof. (17.3) and (17.7) show that, for $j = 2, \dots, k$, the term

$$f_j'u_1 + f_jv_1 = f_ju_1\left(\frac{f_j'}{f_j} - \frac{f_1'}{f_1}\right)$$

is a canonical formal solution of $M[y] = 0$ with exponential part $q_j + P = q_{j-1}$, and so is a constant multiple of w_{j-1}. This delivers, using (17.3), (17.5) and (17.6),

$$(17.9) \qquad \begin{aligned} \mu_1 &= \lambda_2 + \eta_1 + \rho - 1, \quad \dots, \quad \mu_{k-1} = \lambda_k + \eta_1 + \rho - 1, \\ \mu_k &= \lambda_1 - (k-1)(\eta_1 + \rho - 1). \end{aligned}$$

In the same way, for $j = 1, \dots, k-1$, the term $f_j'u_3 + f_jv_3$ has exponential part $q_j - P = q_{j+1}$, and so is a constant multiple of w_{j+1}, which yields

$$(17.10) \qquad \begin{aligned} \mu_2 &= \lambda_1 + \eta_3 + \rho - 1, \quad \dots, \quad \mu_k = \lambda_{k-1} + \eta_3 + \rho - 1, \\ \mu_1 &= \lambda_k - (k-1)(\eta_3 + \rho - 1). \end{aligned}$$

Suppose first that $N \geq \rho$ and $c_N \neq 0$ in (17.8). In this case (17.3) and (17.5) show that u_2 cannot annihilate any of the f_j, and that each $f_j' u_2 + f_j v_2$ is a canonical formal solution of $M[y] = 0$ with exponential part q_j, and so a constant multiple of w_j. This implies in view of (17.6) that

$$\mu_j = \lambda_j + \eta_2 + N \quad (j = 1, \ldots, k), \quad \eta_2 = -N.$$

Moreover, (17.9) and (17.10) now lead to

$$\lambda_k = \mu_k = \lambda_1 - (k-1)(\eta_1 + \rho - 1), \quad \lambda_1 = \mu_1 = \lambda_k - (k-1)(\eta_3 + \rho - 1),$$
$$\eta_1 + \rho - 1 = -(\eta_3 + \rho - 1),$$

so that $\eta_1 + \eta_3 = -2(\rho - 1)$ and conclusion (A) holds.

Now suppose that $N \leq \rho - 1$ in (17.8): this case will lead to conclusion (B), and encompasses the possibility that v_2/u_2 vanishes identically. The first step is to show that the order of growth of f is ρ. Since g^k has order ρ it follows from (5.1) that the order of f is at least ρ. It suffices to show that in (5.16) the coefficients (which are rational at infinity) satisfy

(17.11) $\qquad \alpha(z) = O(|z|^{1-\rho}), \quad \beta(z) = O(1), \quad \gamma(z) = O(|z|^{\rho-1}) \quad \text{as } z \to \infty,$

because if this can be established then $\rho(f) \leq \rho$ follows from (5.16), the Wiman-Valiron theory [12] applied to $1/f$, and standard estimates [8] for logarithmic derivatives of g^k and g.

To prove (17.11) use (17.1), (17.7) and (17.8) to write

$$\alpha(z) P'(z)^2 (1 + O(z^{-1})) + \beta(z) P'(z)(1 + O(z^{-1})) + \gamma(z) = \frac{v_1(z)}{u_1(z)} = O(z^{\rho-1}),$$

$$\alpha(z) P'(z)^2 (1 + O(z^{-1})) - \beta(z) P'(z)(1 + O(z^{-1})) + \gamma(z) = \frac{v_3(z)}{u_3(z)} = O(z^{\rho-1}),$$

$$\alpha(z) O(z^{-2}) + \beta(z) O(z^{-1}) + \gamma(z) = \frac{v_2(z)}{u_2(z)} = O(z^N) = O(z^{\rho-1}).$$

Here $O(z^\omega)$ denotes any formal series in descending integer powers of z with leading power at most $\omega \in \mathbf{Z}$. Eliminating γ via the last equation yields

$$\alpha(z) P'(z)^2 (1 + O(z^{-1})) + \beta(z) P'(z)(1 + O(z^{-1})) = O(z^{\rho-1}),$$
$$\alpha(z) P'(z)^2 (1 + O(z^{-1})) - \beta(z) P'(z)(1 + O(z^{-1})) = O(z^{\rho-1}),$$

and now (17.11) follows from Cramer's rule.

Next, since $N \leq \rho - 1$, (17.3) and (17.8) give pairwise distinct $\widehat{d}_j \in \mathbf{C}$ with

$$\frac{f_j'(z)}{f_j(z)} + \frac{v_2(z)}{u_2(z)} = \widehat{d}_j z^{\rho-1} + \ldots.$$

If $\widehat{d}_j \neq 0$ then $f_j' u_2 + f_j v_2$ is again a canonical formal solution of $M[y] = 0$ with exponential part q_j, and so a constant multiple of w_j. Since $k \geq 4$, this implies in view of (17.5) that

(17.12) $\qquad\qquad\qquad\qquad \mu_j = \lambda_j + \eta_2 + \rho - 1$

for $j = 1$ and $j = 2$, or for $j = k - 1$ and $j = k$. If (17.12) holds for $j = 1$ and $j = 2$ then (17.9), (17.10) and (17.12) give

$$\mu_1 = \lambda_1 + \eta_2 + \rho - 1 = \lambda_2 + \eta_1 + \rho - 1, \quad \mu_2 = \lambda_2 + \eta_2 + \rho - 1 = \lambda_1 + \eta_3 + \rho - 1,$$

from which it follows that

$$\eta_1 - \eta_2 = \lambda_1 - \lambda_2 = \eta_2 - \eta_3, \quad \eta_1 + \eta_3 - 2\eta_2 = 0.$$

Similarly, if (17.12) holds for $j = k - 1$ and $j = k$, then (17.9), (17.10) and (17.12) give

$$\mu_k = \lambda_k + \eta_2 + \rho - 1 = \lambda_{k-1} + \eta_3 + \rho - 1, \quad \mu_{k-1} = \lambda_{k-1} + \eta_2 + \rho - 1 = \lambda_k + \eta_1 + \rho - 1,$$

which delivers

$$\eta_1 - \eta_2 = \lambda_{k-1} - \lambda_k = \eta_2 - \eta_3, \quad \eta_1 + \eta_3 - 2\eta_2 = 0. \qquad \square$$

Lemma 17.3. *If $b_3 = 0$ in (17.2) then f satisfies the conclusion of Proposition 10.1.*

Proof. Using (5.16) write, on Σ,

$$(17.13) \qquad g = U_1 + U_2, \quad -\frac{f'}{f} = \frac{V[g]}{g} = \frac{V[U_1] + V[U_2]}{U_1 + U_2} = \frac{V[U_1]/U_2 + V[U_2]/U_2}{e^\Phi + 1},$$

$$e^\Phi = \frac{U_1}{U_2}.$$

A zero of g arises wherever $U_1/U_2 = e^\Phi = -1$, and the multiplicity of the pole of f at such a point is

$$(17.14) \qquad m_0 = \frac{V[U_1]/U_1 - V[U_2]/U_2}{\Phi'}.$$

By (17.1) and (17.2), the function $\zeta = (1/\pi i)\Phi = (1/\pi i)\log U_1/U_2$ maps the sector Σ univalently onto a region containing a half-plane $\pm\operatorname{Re}\zeta > M_1 \in \mathbf{R}$, and (17.14) holds wherever ζ is an odd integer. Thus (17.1), (17.2), (17.7) and Lemma 3.1 give a polynomial Q^* such that

$$(17.15) \quad \frac{V[U_1]}{U_1} - \frac{V[U_2]}{U_2} = Q^*(\Phi)\Phi', \quad U_2 V[U_1] - U_1 V[U_2] = Q^*(\Phi)(U_2 U_1' - U_1 U_2').$$

Suppose first that $Q^*(\Phi)$ is rational at infinity in (17.15). Then it follows from Lemma 12.1 that U_1 and U_2 solve a second order equation (3.4) with E_1 and E_0 rational at infinity, and so does g, by (17.13), contradicting Lemma 10.1(C).

It may therefore be assumed henceforth that Q^* is non-constant. Then (17.14) and (17.15) show that the multiplicity $m_0(z)$ of a pole $z \in \Sigma$ of f tends to ∞ as $z \to \infty$, faster than $|z|^{\rho_1}$ for some $\rho_1 > 0$. Since the zeros of $g = U_1 + U_2$ in Σ have exponent of convergence ρ, this is incompatible with Case B of Lemma 17.2. Hence Case A of Lemma 17.2 must hold, and so $(\eta_1 - \eta_2) - (\eta_2 - \eta_3) = \eta_1 + \eta_3 - 2\eta_2$ is a positive integer.

Furthermore, the left-hand side of (17.15) has a meromorphic continuation along any path in $\Omega(r_1)$, as has Φ', but if a continuation of U_1/U_2 has a zero or pole at some z_0 then $\Phi(z) = \log U_1(z)/U_2(z)$ behaves like $m_1 \log(z - z_0)$ as $z \to z_0$, for some $m_1 \in \mathbf{Z} \setminus \{0\}$. Therefore (17.15) implies that $e^\Phi = U_1/U_2$ continues without poles or zeros in $\Omega(r_1)$, and so any zeros of continuations of U_1 and U_2 are shared.

Take any sector Σ^* given by $|\arg z - \theta^*| \leq \pi/\rho - \delta_1$, where $\operatorname{Re} P(re^{i\theta^*}) = O(r^{\rho-1})$ as $r \to \infty$, let $\widetilde{U}_1, \widetilde{U}_2$ be continuations of U_1, U_2 to Σ^*, and write

$$(17.16) \qquad \widetilde{U}_1 = d_1\psi_1 + d_2\psi_2 + d_3\psi_3, \quad \widetilde{U}_2 = e_1\psi_1 + e_2\psi_2 + e_3\psi_3, \quad d_j, e_j \in \mathbf{C},$$

on Σ^*, in which the ψ_j are analytic solutions of $N[y] = 0$ which satisfy, as $z \to \infty$ on Σ^*,

(17.17)
$$\psi_1(z) = z^{\eta_1} e^{P(z)}(1 + o(1)), \quad \psi_2(z) = z^{\eta_2}(1 + o(1)),$$
$$\psi_3(z) = z^{\eta_3} e^{-P(z)}(1 + o(1)).$$

Suppose that $\widetilde{U}_1$ and $\widetilde{U}_2$ have a sequence $\zeta_\mu \to \infty$ of common zeros in Σ^*. The matrix with rows (d_1, d_2, d_3) and (e_1, e_2, e_3) has rank 2, since U_1 and U_2 are linearly independent, and so Cramer's rule gives $e_4, e_5 \in \mathbf{C}$ and a permutation (j, j', j'') of $(1, 2, 3)$ such that

$$\psi_{j'}(\zeta_\mu) = e_4 \psi_j(\zeta_\mu), \quad \psi_{j''}(\zeta_\mu) = e_5 \psi_j(\zeta_\mu) \quad \text{as } \mu \to \infty.$$

Here $e_4 e_5 \neq 0$, as $\psi_j(\zeta_\mu) \neq 0$ for large μ. But this gives a contradiction, since the fact that $(\eta_1 - \eta_2) - (\eta_2 - \eta_3)$ is positive implies that $\psi_2(\zeta_\mu)/\psi_3(\zeta_\mu) = o(|\psi_1(\zeta_\mu)/\psi_2(\zeta_\mu)|)$ as $\mu \to \infty$.

It follows that U_1 and U_2 continue without zeros in some annulus $\Omega(r^*)$. Lemma 3.2 shows that there exists $\rho_2 > 0$ such that any continuation of U_2 to any sector in $\Omega(r^*)$ satisfies $\log|U_2(z)| = O(|z|^{\rho_2})$ as $z \to \infty$ there. Take a sector Σ^{**} given by $\theta_1 < \arg z < \theta_2$, where these θ_j are such that no $\theta \in [\theta_1, \theta_2]$ has $\text{Re } P(re^{i\theta}) = O(r^{\rho-1})$ as $r \to \infty$. For any continuation of U_2 to Σ^{**} there exist $P^* \in \{-P, 0, P\}$ and a matching $\eta^* \in \{\eta_1, \eta_2, \eta_3\}$ such that $U_2(z) \sim cz^{\eta^*} \exp(P^*(z))$ as $z \to \infty$ in Σ^{**}. Since $U_2(z) \sim cz^{\eta_2}$ as $z \to \infty$ in Σ, repeated application of the Phragmén–Lindelöf principle to the continuations of $U_2(z)z^{-\eta_2}$ or its reciprocal shows that $P^* = 0$, and so $\eta^* = \eta_2$. Examining (17.16) in the light of (17.17), first on a subsector of Σ^* on which e^P is large and subsequently on a subsector where e^{-P} is large, forces $0 = e_1 = e_3$. Choosing $\Sigma^* = \Sigma$ gives $e_0 \in \mathbf{C}$ such that $z^{e_0} U_2(z)$ is analytic and zero-free of finite order of growth in some annulus $\Omega(r^{**})$. This, coupled with almost identical reasoning applied to U_1, shows that U_1'/U_1, U_2'/U_2 and Φ' are rational at infinity, as is $Q^*(\Phi)$ by (17.15), and this case has already been dealt with. $\qquad \square$

Assume henceforth that $b_1 b_3 \neq 0$ in (17.2), and write this formula for g as

(17.18)
$$g = Ae^{-P}((e^P - B)^2 - C^2), \quad U_1 = Ae^P,$$
$$U_2 = -2AB, \quad U_3 = A(B^2 - C^2)e^{-P}.$$

By (17.1), this initially formal expression for g results in, as $z \to \infty$ in Σ,

(17.19)
$$A(z) = b_1 z^{\eta_1} \chi_1(z), \quad B(z) = -\frac{b_2}{2b_1} z^{\eta_2 - \eta_1} \chi_2(z),$$
$$B(z)^2 - C(z)^2 = \frac{b_3}{b_1} z^{\eta_3 - \eta_1} \chi_3(z), \quad \chi_j(z) = 1 + o(1).$$

Here the χ_j have asymptotic series on Σ in descending integer powers of z and, by Lemma 17.2, $\eta_3 - \eta_1 - 2(\eta_2 - \eta_1) = \eta_1 + \eta_3 - 2\eta_2$ is a non-negative even integer. Evidently A, B and $E = C^2$ are analytic on Σ, and E does not vanish identically, since zeros of g are simple. Furthermore, it is clear from (17.18) that, at a zero of g in Σ,

(17.20) $\qquad (e^P - B)^2 = E = C^2, \quad g' = Ae^{-P}(2(e^P - B)(P'e^P - B') - E').$

Lemma 17.4. *Let $d = \pm 1$. Then there exist $r_2 > 0$ and $\sigma_d, \tau_d \in \mathbf{C}^*$, as well as $\gamma_d, \zeta_d \in \mathbf{C}$, such that $B + dC$ is analytic on $\Sigma \cap \Omega(r_2)$ and*

(17.21) $\qquad C(z)^2 = \sigma_d z^{\gamma_d} \psi_1(z), \quad \psi_1(z) = 1 + o(1),$

and

(17.22) $B(z) + dC(z) = \tau_d z^{\zeta_d} \psi_2(z), \quad \psi_2(z) = 1 + o(1),$

as $z \to \infty$ in Σ, in which the $\psi_j(z)$ have asymptotic series on Σ in descending integer powers of $z^{1/2}$. Furthermore, if conclusion (A) of Lemma 17.2 holds, then $\gamma_d = \eta_3 - \eta_1$.

Proof. Note first that $B + dC$ does not vanish identically, since $B^2 - C^2$ does not. All conclusions of the lemma clearly follow from (17.19) if $b_2 = 0$ or $\eta_3 - \eta_1 - 2(\eta_2 - \eta_1) > 0$, and in particular if conclusion (A) of Lemma 17.2 holds.

Assume therefore that $b_2 \neq 0$ and $\eta_3 - \eta_1 = 2(\eta_2 - \eta_1)$. Then (17.19) implies that $\widetilde{C}(z) = C(z)^2 z^{2(\eta_1 - \eta_2)}$ has an asymptotic series on Σ in descending non-positive integer powers of z. If this asymptotic series for $\widetilde{C}(z)$ vanishes identically then, by making Σ slightly narrower if necessary, it may be assumed that $E(z) = C(z)^2$ and $E'(z)$ both tend to zero in Σ transcendentally fast, that is, faster than any negative power of z, but f still has infinitely many poles there. This implies using (5.2) and (17.20) that if M_1 is a positive integer and z is a pole of f of multiplicity $m_0(z)$ in Σ, with $|z|$ large, then

$$g(z) = 0, \quad e^{P(z)} = B(z) + O\left(|z|^{-2M_1}\right), \quad g'(z) = O\left(|z|^{-M_1}\right), \quad |z|^{M_1} = O(m_0(z)),$$

which is a contradiction since f has finite order. Hence there must exist an integer $m_1 \leq 0$ such that (17.21) holds with $\gamma_d = 2(\eta_2 - \eta_1) + m_1$, in which $\psi_1(z)$ has an asymptotic series in descending integer powers of z. It is now clear from (17.19) and (17.21) that $\widetilde{B}(z) = (B(z) + dC(z))z^{\eta_1 - \eta_2}$ has an asymptotic series on Σ in descending integer powers of $z^{1/2}$; thus (17.22) holds unless this series for $\widetilde{B}(z)$ vanishes identically, in which case $B(z) + dC(z)$ tends to zero transcendentally fast on Σ, and so does $B(z)^2 - C(z)^2$, by the second equation of (17.19), which forces $b_3 = 0$ in (17.19), contrary to assumption. $\square$

Lemma 17.5. *For $d = \pm 1$ there exists a polynomial $Q_d \not\equiv 0$ such that*

(17.23) $\left[2dCA \left(P' - \dfrac{B' + dC'}{B + dC} \right) \right]^{-k} = Q_d(P - \log(B + dC)).$

Proof. The function g has a zero in Σ wherever $e^P = B + dC$, and at such a zero (17.20) gives

(17.24)
$$g' = Ae^{-P}(2dC(P'e^P - B') - 2CC') = 2dCAe^{-P}(P'e^P - B' - dC')$$
$$= 2dCA \left(P' - \frac{B' + dC'}{B + dC} \right).$$

Here (17.22) shows that $\zeta = (1/2\pi i)(P(z) - \log(B(z) + dC(z)))$ maps a subdomain of Σ univalently onto a half-plane $\pm \mathrm{Re}\, \zeta > M_1 \in \mathbf{R}$. Because (5.2) implies that $(g')^{-k}$ is integer-valued at each zero of g, and so at points where ζ is integer-valued, it follows from (17.19), (17.21) and Lemma 3.1 that a polynomial Q_d exists as asserted. $\square$

Lemma 17.6. *For $d = \pm 1$ the polynomial Q_d in (17.23) is constant.*

Proof. Assume that Q_d is non-constant. Then it follows from (5.2), (17.22), (17.23) and (17.24) that the multiplicity $m_0(z)$ of the pole of f at $z \in \Sigma$ tends to ∞ faster than some positive power of $|z|$ and, since the exponent of convergence of the zeros of $e^P - (B + dC)$ in Σ is ρ, this implies that $N(r, f)$ has order greater than ρ, which is incompatible with conclusion (B) of Lemma 17.2.

Hence conclusion (A) of Lemma 17.2 must hold. In view of (17.19) and Lemma 17.4, it follows that $\eta_1 + \eta_3 = -2(\rho - 1)$ and $\gamma_d = \eta_3 - \eta_1$, and that

$$C(z)A(z) \sim cz^{\gamma_d/2+\eta_1} = cz^{(\eta_1+\eta_3)/2} = cz^{1-\rho}$$

as $z \to \infty$ in Σ. But then the left-hand side of (17.23) is bounded as $z \to \infty$ in Σ, which is a contradiction. $\qquad\square$

Lemma 17.7. *There exist a large positive r_3 and an analytic function K such that*

(17.25)
$$K' = \frac{1}{U_1}, \quad U_1 = Ae^P, \quad U_2 = -2AB = e_3U_1K,$$
$$U_3 = A(B^2 - C^2)e^{-P} = e_4U_1K^2,$$

on $\Sigma \cap \Omega(r_3)$, where $e_3, e_4 \in \mathbf{C}$ and $e_4 \neq 0$.

Proof. Suppose first that $B \not\equiv 0$. Then (17.23) holds for $d = 1$ and $d = -1$, with Q_1 and Q_{-1} both constant by Lemma 17.6. Hence, by (17.22) and (17.23),

$$CA\left(P' - \frac{B'+C'}{B+C}\right), \quad CA\left(P' - \frac{B'-C'}{B-C}\right)$$

are both constant, and so identically equal. Thus $(B+C)/(B-C)$ must be constant and so must B/C. Now (17.18), (17.21), (17.23) and Lemma 17.6 yield, with $c \in \mathbf{C}^*$ as before,

(17.26) $\quad CA\left(P' - \dfrac{C'}{C}\right) = c, \quad C' - P'C = \dfrac{c}{A}, \quad Ce^{-P} = c\displaystyle\int \frac{1}{Ae^P} = c\int \frac{1}{U_1} = cK,$

from which (17.25) follows, using (17.18) again. On the other hand, if $B \equiv 0$ then the first equation of (17.26) still holds, by Lemma 17.6, and the formula for U_2 in (17.25) is trivially satisfied with $e_3 = 0$. $\qquad\square$

Lemma 17.8. *The function K of Lemma 17.7 continues meromorphically along any path in the annulus $\Omega(r_3)$, its continuations locally univalent. Moreover, all zeros of any continuation of U_1 into $\Omega(r_3)$ are simple poles of K.*

Proof. Since $e_4 \neq 0$ in (17.25), writing

(17.27)
$$\Phi = K^2 = \frac{U_3}{e_4U_1}, \quad \frac{1}{U_1^2} = (K')^2 = \frac{(\Phi')^2}{4\Phi},$$

shows that Φ continues meromorphically along any path in $\Omega(r_3)$. Any zero of any continuation of U_1 is either simple or double, since U_1 solves $N[y] = 0$, and must be a pole of Φ, by (17.27). Comparing multiplicities in (17.27) excludes simple zeros of U_1, and double zeros of U_1 have to be triple poles of Φ' and so double poles of Φ. Furthermore, any zeros of any continuation of Φ must be double, again by (17.27). Thus $K = \Phi^{1/2}$ continues meromorphically along paths in $\Omega(r_3)$, and is locally univalent since $K'(z) = 1/U_1(z) \neq 0$. $\qquad\square$

Again because $e_4 \neq 0$ in (17.25), there exists a polynomial Q_2 of degree 2 such that (17.2) and continuation of g into $\Omega(r_3)$ give $g = Q_2(K)/K'$, whether or not $U_2 \equiv 0$, where K is as in Lemma 17.8. Hence $g = 0$ forces $K = a$, where $Q_2(a) = 0$, and so $g' = Q_2'(a) = \pm b$ for some $b \in \mathbf{C}^*$, by elementary properties of quadratics. It now follows using (5.2) that all poles of f in $\Omega(r_3)$ have the same multiplicity, and f satisfies the conclusion of Proposition 10.1. $\qquad\square$

18. The case of a repeated trivial exponential part

There remains only one case to deal with, in which the equation $N[y] = 0$ has two linearly independent formal solutions g_1, g_2 with trivial exponential part. The third exponential part κ must be non-zero, by Lemma 11.1. Take a ray $\arg z = \theta_0$ on which $\kappa \prec 0$, and label the exponential parts arising from L and M to be consistent with (14.1) on $\arg z = \theta_0$. It then follows from Lemma 16.1 that the exponential parts q_j for the equation $L[y] = 0$ are pairwise distinct, and the same is true for $M[y] = 0$, and so the formal solutions of these equations are free of logarithms. This implies that any formal solution G of $N[y] = 0$ is also free of logarithms; to see this, take a fundamental set of canonical formal solutions f_j of $L[y] = 0$, write $f_j' G + f_j V[G] = w_j$, where the w_j are formal solutions of $M[y] = 0$, and solve for G.

Therefore $N[y] = 0$ has linearly independent canonical formal solutions g_1, g_2 each having the form $g_j(z) = z^{m_j}(1 + \ldots)$, with $m_j \in \mathbf{C}$. There exists a third canonical formal solution g_3, which has exponential part κ and, by Lemma 7.1, annihilates some canonical formal solution h_μ of $L[y] = 0$, with exponential part q_μ say. Consider the terms $R_j = V[g_j]/g_j$, for $j = 1, 2$; these are formal series in descending integer powers of z. Hence

$$S_j = h_\mu' g_j + h_\mu V[g_j] = h_\mu g_j \left(h_\mu'/h_\mu + R_j \right)$$

is a formal solution of $M[y] = 0$, for $j = 1, 2$, and either is identically zero or has exponential part q_μ. Since the exponential parts for M are all different, S_1 and S_2 must be linearly dependent, and some non-trivial linear combination g_4 of g_1 and g_2 must annihilate h_μ, as does g_3. Therefore $g_3 V[g_4] = g_4 V[g_3]$ and Lemma 12.2, with $d = 0$, gives an equation (3.4) solved by g_3 and g_4. Furthermore, g_4 must be a canonical formal solution of $N[y] = 0$; this is obvious unless $g_4 = d_1 g_1 - d_2 g_2$ with $d_1, d_2 \in \mathbf{C}^*$, in which case

$$d_1 S_1 = d_2 S_2, \quad d_1 g_1 \left(h_\mu'/h_\mu + R_1 \right) = d_2 g_2 \left(h_\mu'/h_\mu + R_2 \right).$$

Thus $W(g_3, g_4)$ has non-zero exponential part, so that $E_1(\infty) \neq 0$ in (3.4), and the conclusion of Proposition 10.1 follows from Lemma 12.2. $\square$

References

[1] BRANNAN, D. A., and W. K. HAYMAN: Research problems in complex analysis. - Bull. London Math. Soc. 21, 1989, 1–35.

[2] BRÜGGEMANN, F.: Proof of a conjecture of Frank and Langley concerning zeros of meromorphic functions and linear differential polynomials, - Analysis 12:1/2, 1992, 5–30.

[3] FRANK, G.: Eine Vermutung von Hayman über Nullstellen meromorpher Funktionen. - Math. Zeit. 149, 1976, 29–36.

[4] FRANK, G., and S. HELLERSTEIN: On the meromorphic solutions of nonhomogeneous linear differential equations with polynomial coefficients. - Proc. London Math. Soc. (3) 53, 1986, 407–428.

[5] FRANK, G., W. HENNEKEMPER, and G. POLLOCZEK: Über die Nullstellen meromorpher Funktionen und ihrer Ableitungen. - Math. Ann. 225, 1977, 145–154.

[6] FRANK, G., and J. K. LANGLEY: Pairs of linear differential polynomials. - Analysis 19, 1999, 173–194.

[7] GAO, S., and J. K. LANGLEY: On the zeros of certain linear differential polynomials. - J. Math. Anal. Appl. 153, 1990, 159–178.

[8] GUNDERSEN, G.: Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates. - J. London Math. Soc. 37, 1988, 88–104.

[9] HARDY, G. H., and E. M. WRIGHT: The theory of numbers. - Oxford Univ. Press, 1938.

[10] HAYMAN, W. K.: Picard values of meromorphic functions and their derivatives. - Ann. of Math. 70, 1959, 9–42.

[11] HAYMAN, W. K.: Meromorphic functions. - Oxford at the Clarendon Press, 1964.

[12] HAYMAN, W. K.: The local growth of power series: a survey of the Wiman–Valiron method. - Canad. Math. Bull. 17, 1974, 317–358.

[13] HECKNER, M.: On meromorphic functions and linear differential polynomials which have few zeros. - Mitt. Math. Sem. Giessen 211, 1992, 1–14.

[14] HILLE, E.: Ordinary differential equations in the complex domain. - Wiley, New York, 1976.

[15] INCE, E. L.: Ordinary differential equations. - Dover, New York, 1956.

[16] JURKAT, W.: Meromorphe Differentialgleichungen. - Lecture Notes in Math. 637, Springer, Berlin, 1978.

[17] LANGLEY, J. K.: An application of the Tsuji characteristic. - J. Fac. Sci. Uni. Tokyo 38, 1991, 299–318.

[18] LANGLEY, J. K.: Proof of a conjecture of Hayman concerning f and f''. - J. London Math. Soc. (2) 48, 1993, 500–514.

[19] LANGLEY, J. K.: On second order linear differential polynomials. - Resultate der Mathematik 26, 1994, 51–82.

[20] LANGLEY, J. K.: Integer-valued analytic functions in a half-plane. - Comput. Methods Funct. Theory 7, 2007, 433–442.

[21] LANGLEY, J. K.: Second order linear differential polynomials and real meromorphic functions. - Resultate der Mathematik 63, 2013, 151–169.

[22] LUTZ, D. A.: Asymptotic behavior of solutions of linear systems of ordinary differential equations near an irregular singular point. - Amer. J. Math. 91, 1969, 95–105.

[23] STEINMETZ, N.: On the zeros of $(f^{(p)} + a_{p-1}f^{(p-1)} + \ldots + a_0 f)f$. - Analysis 7, 1987, 375–389.

[24] WASOW, W.: Asymptotic expansions for ordinary differential equations. - Dover, New York, 1987.