Soliton, rational, and periodic solutions for the infinite hierarchy of defocusing nonlinear Schrödinger equations

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Analysis of short-pulse propagation in positive dispersion media, e.g., in optical fibers and in shallow water, requires assorted high-order derivative terms. We present an infinite-order "dark" hierarchy of equations, starting from the basic defocusing nonlinear Schrödinger equation. We present generalized soliton solutions, plane-wave solutions, and periodic solutions of all orders. We find that "even"-order equations in the set affect phase and "stretching factors" in the solutions, while "odd"-order equations affect the velocities. Hence odd-order equation solutions can be real functions, while even-order equation solutions are complex. There are various applications in optics and water waves.

I. INTRODUCTION

In silica optical fibers, the group velocity dispersion is anomalous (i.e., negative) for long wavelengths (above about 1.3 microns, thus including the attenuation minimum at 1.55 microns). Propagation of nonlinear pulses in this range is covered by the well-known "fundamental" focusing nonlinear Schrödinger equation when only the Kerr nonlinearity is included, and by higher-order equations in this family when higher-order dispersion and forms of nonlinearity are included. The forms of solutions in these cases, relevant to optical fibers, deep water waves, and other phenomena, have been presented in [1].

On the other hand, group velocity dispersion in fibers is normal (i.e., positive) for short wavelengths (below about 1.3 microns, thus including the visible range). The transmission of stationary nonlinear pulses along fibers in the normal regime was first considered by Hasagawa and Tappert [2] in 1973. This regime is described by the defocusing nonlinear Schrödinger equation (DNLSE). In this case, the information is sent using dark "holes" on a constant background. These dark solitons have been used to carry information for a great distance [3]. Various studies have shown that dark solitons suffer less dispersion and loss than bright solitons in fibers [4]. Picosecond dark soliton propagation is not confined to the visible spectrum and has been demonstrated over a 1-km fiber in the infrared at a wavelength of 850 nm [5]. The analysis of such short pulses requires modeling with equations containing numerous high-order derivative terms. This present paper can provide a basis for such an endeavor.

Dark solitons have also recently been observed in water [6,7]. They occur when $kh < 1.363$ for water of depth h and waves of wave number k, and the waves correspond approximately to solutions of the DNLSE. The coupling of dark solitons in optical fiber devices has also been considered [8].

The second derivative term represents dispersion or diffraction, while the cubic nonlinearity comes from the Kerr effect in glass fibers. So the basic "dark" equation represents the situation in which the two terms have opposite signs. For example, we could have anomalous (i.e., negative) dispersion with a "negative" nonlinearity, i.e., where the refractive index decreases locally with high intensity.

As pointed out in [9], more higher-order terms are needed as pulses are made shorter. For example, a dark soliton solution for the complex cubic Ginzburg-Landau equation has been given in [10]. When delayed (Raman) contributions of the nonlinear material response are included, the equation describing fiber pulse propagation includes an integral term. A first-order expansion of it separates out the term $\tau_R \frac{\partial}{\partial t}(|\psi|^2)$ for Raman time delay, τ_R. For accuracy with short pulses, more higher-order time derivative terms should be included [11].

The analysis of pulses consisting of just a few optical cycles, and of some supercontinuum phenomena, can be carried out only when many high derivative factors are included. These involve dispersive and other effects. The results herein should assist in this work.

Though studied much less than the familiar focusing equation, the DNLSE has many applications. It is well known that the fundamental DNLSE is integrable. A classification of its exact solutions has been given in [12]. Its various soliton and periodic solutions and other families of solutions have been presented in [13]. In this paper, we greatly extend the range of phenomena of normal dispersion that can be described by related equations by introducing the whole infinite hierarchy associated with the fundamental DNLSE.

II. INFINITE SET OF FUNCTIONALS

A. Derivation

We define the invariant integrands of the fundamental defocusing NLSE as

$$p_{j+1} = \psi \frac{\partial}{\partial t}\left(\frac{p_j}{\psi}\right) + \sum_{j_1+j_2=j}(p_{j_1} p_{j_2}), \tag{1}$$

taking $p_1 = -|\psi|^2$ to start. The minus sign allows for the defocusing, as we deal with an "energy deficit" from a constant background, and it introduces quite important differences into the equations. Hence, the first few integrands are

$$p_2 = -\psi \psi_t^*,$$
$$p_3 = |\psi|^4 - \psi \psi_{tt}^*,$$
$$p_4 = \psi[\psi_t(\psi^*)^2 + 4\psi_t^*|\psi|^2 - \psi_{ttt}^*]. \tag{2}$$

With this formulation, some signs are negative, in contrast to the focusing NLSE.

The defocusing NLSE has infinitely many invariants. After accounting for finite backgrounds [14], these can be defined as

$$I_j = \int_{-\infty}^{\infty} p_j \, dt, \quad j = 1, 2, 3, \ldots, \infty$$

with I_1 being energy, I_2 is the momentum, and I_3 is the Hamiltonian, all relative to background. However, we do not use these integrals in this paper.

Now, we define the jth "dark" operator in the defocusing NLS hierarchy as

$$D_j(\psi, \psi^*) = (-1)^j \frac{\delta}{\delta \psi^*} \left[\int p_{j+1} dt \right],$$

where we have taken the functional derivative of the invariant integrand to get the higher-order operator. Not all signs are positive in each D_j, so the equations arising support solutions that are very different from those of the focusing NLS hierarchy, which were presented in [1].

For example,

$$D_2 = -\psi_{tt} + 2\psi |\psi|^2, \tag{3}$$

$$D_3 = -\psi_{ttt} + 6|\psi|^2 \psi_t, \tag{4}$$

etc.

B. The form of the equations

We can use the above to write down one equation that includes the whole higher-order hierarchy of the DNLSE:

$$F[\psi(x,t)] = i\psi_x + \sum_{j=1}^{\infty} (\alpha_{2j} D_{2j} - i\,\alpha_{2j+1} D_{2j+1}) = 0,$$

where each α_j, $j = 2, 3, 4, 5, \ldots, \infty$, is an arbitrary real number. In Ref. [13], we have taken $\alpha_2 = \frac{1}{2}$. However, Ref. [12] uses $\alpha_2 = 1$, and, indeed, any value of α_2 can be used here, including zero. Hence, our solutions cover equations such as $\psi_x = \alpha_3(-\psi_{ttt} + 6|\psi|^2 \psi_t)$, which do not involve the basic DNLSE operator at all.

Then the equation takes the form

$$\begin{aligned} F[\psi(x,t)] = {}& i\psi_x + \alpha_2 D_2[\psi(x,t)] - i\alpha_3 D_3[\psi(x,t)] \\ & + \alpha_4 D_4[\psi(x,t)] - i\alpha_5 D_5[\psi(x,t)] \\ & + \alpha_6 D_6[\psi(x,t)] - i\alpha_7 D_7[\psi(x,t)] \\ & + \alpha_8 D_8[\psi(x,t)] - i\alpha_9 D_9[\psi(x,t)] \\ & + \cdots = 0, \end{aligned} \tag{5}$$

where $F[\psi(x,t)]$ represents the whole hierarchy of DNLSE integrable equations. For convenience, in the following, we represent $\frac{\partial^j \psi}{\partial t^j}$ by ψ_{jt} and $\frac{\partial^j \psi^*}{\partial t^j}$ by ψ_{jt}^* when $j > 3$.

In the lowest one, viz. second order, we can write S as the fundamental defocusing nonlinear Schrödinger operator:

$$S[\psi(x,t)] = i\psi_x + \alpha_2 D_2 = i\psi_x + \alpha_2(-\psi_{tt} + 2\psi|\psi|^2).$$

Keeping the highest third-order term D_3, we have the Hirota operator:

$$D_3[\psi(x,t)] = -\psi_{ttt} + 6|\psi|^2 \psi_t,$$

which generates the Hirota equation. In the next generalization, we keep D_4 as the fourth-order ($j = 4$) operator (starting with fourth-order derivative):

$$\begin{aligned} D_4[\psi(x,t)] = {}& -\psi_{tttt} + 8|\psi|^2 \psi_{tt} - 6\psi|\psi|^4 + 4\psi|\psi_t|^2 \\ & + 6\psi_t^2 \psi^* + 2\psi^2 \psi_{tt}^*. \end{aligned} \tag{6}$$

Continuing the process, we can keep K_5 as the fifth order ($j = 5$), i.e., quintic "dark" operator (starting with fifth-order derivative):

$$\begin{aligned} D_5[\psi(x,t)] = {}& -\psi_{ttttt} + 10|\psi|^2 \psi_{ttt} - 30|\psi|^4 \psi_t + 10\psi\psi_t\psi_{tt}^* \\ & + 10\psi\psi_t^*\psi_{tt} + 20\psi^*\psi_t\psi_{tt} + 10|\psi_t|^2\psi_t. \end{aligned}$$

This expression can be written in a shorter form:

$$\begin{aligned} D_5[\psi(x,t)] = {}& -\psi_{ttttt} + 10[|\psi|^2 \psi_{ttt} + (\psi|\psi_t|^2)_t \\ & + 2\psi^*\psi_t\psi_{tt} - 3|\psi|^4\psi_t]. \end{aligned} \tag{7}$$

Further, K_6 is the sixth-order ($j = 6$), i.e., sextic, operator (starting with sixth-order derivative):

$$\begin{aligned} & D_6[\psi(x,t)] \\ & = 2\{\psi[9\psi_t^*\psi_{ttt} + 4\psi_t\psi_{ttt}^* - 35(\psi^*)^2\psi_t^2 \\ & \quad + 11|\psi_{tt}|^2 + 6\psi^*\psi_{tttt}] \\ & \quad + 5[\psi_t(5\psi_t^*\psi_{tt} + 3\psi^*\psi_{ttt}) + 2\psi_t^2\psi_{tt}^* + 2\psi^*\psi_{tt}^2] \\ & \quad + \psi^2[\psi_{tttt}^* - 5\psi^*(6|\psi_t|^2 + 5\psi^*\psi_{tt})] \\ & \quad - 5\psi^3[(\psi_t^*)^2 + 2\psi^*\psi_{tt}^*] + 10\psi|\psi|^6\} - \psi_{6t}. \end{aligned}$$

Operators D_7 and D_8 are longer and they are presented in the Appendixes.

In all the above expressions, x is the propagation variable and t is a transverse variable (time in a moving frame), with the function $|\psi(x,t)|$ being the envelope of the waves.

Importantly, the coefficients α_j are arbitrary real constants and so they do not have to be small. Hence we can go far beyond simple extensions of the NLSE, which involve corrective or perturbative terms. In the specific case when only α_2 is nonzero, the equation is known to be integrable and is called the "defocusing" NLSE (see Chap. 5 of [13]). If only α_2 is nonzero, then the transform $t \to it$ converts the focusing NLSE to the DNLSE, although the same transform generally only converts finite solutions of the former to singular (nonphysical) solutions of the latter. In any case, this procedure does not convert the higher-order equations, since all odd-order derivatives would then include an additional i factor, thus disrupting the nature of the equation. To obtain the DNLSE set with the transform, we additionally need to artificially convert $\alpha_{2n+1} \to i(-1)^{n+1}\alpha_{2n+1} = -i^{2n+1}\alpha_{2n+1}$ and $\alpha_{2n} \to (-1)^{n+1}\alpha_{2n} = -i^{2n}\alpha_{2n}$ for $n = 1, 2, 3, \ldots$. Our derivation above does not have this ad hoc feature.

We note that the jth operator ($j \geqslant 3$) is

$$\begin{aligned} D_j = {}& -\frac{\partial^j \psi}{\partial t^j} + 2j|\psi|^2 \frac{\partial^{j-2}\psi}{\partial t^{j-2}} \\ & + [1 + (-1)^j]\psi^2 \frac{\partial^{j-2}\psi^*}{\partial t^{j-2}} + \cdots. \end{aligned} \tag{8}$$

Finally, the term with no derivative in D_j is

$$(-1)^{1+\frac{j}{2}} \frac{j!}{[(j/2)!]^2} \, \psi |\psi|^j \qquad (9)$$

if j is even, $j = 2,4,6,\ldots$, and zero if j is odd.

III. ANALYSIS

A. Odd-numbered equations

First, we consider some generalities. If all even-labeled coefficients are zero, i.e., $\alpha_{2n} = 0$, $n = 1,2,3,\ldots,\infty$, then we have

$$\psi_x = \sum_{j=1}^{\infty} \alpha_{2j+1} D_{2j+1}.$$

This can have real-valued solutions. For example, the lowest-order one is $\psi_x = \alpha_3(-\psi_{ttt} + 6|\psi|^2 \psi_t)$. If we assume that $\psi = f(y)$ is a real odd function where $y = t + xv_3$, then for a dark pulse solution ($f'(y) \to 0$ for $y \to \infty$) we have $v_3 f = \alpha_3(-f'' + 2f^3)$. Now, we have $f''(0) = 0$. This shows that $v_3 = 2\alpha_3$ and $[f'(y)]^2 = (1 - f^2)^2$. Hence $f = \tanh(y) = \tanh(t + 2x\,\alpha_3)$.

B. Even-numbered equations

If all odd-labeled coefficients are zero, i.e., $\alpha_{2n+1} = 0$, $n = 1,2,3,\ldots,\infty$, then we have

$$-i\psi_x = \sum_{j=1}^{\infty} \alpha_{2j} D_{2j}.$$

Now the solutions take the form $\psi = e^{i\phi x} h(t)$. For example, for $j = 1$ (DNLSE), for a pulse solution that is odd in t, we have $\phi(1 - h^2)/\alpha_2 - (h')^2 + h^4 = 1$. For convenience, we have taken $h(\infty) = 1$. This shows that $\phi = 2\alpha_2$ and $2(1 - h^2) - (h')^2 + h^4 = 1$. Hence $h = \tanh(t)$ and $\psi = e^{2i\alpha_2 x}\tanh(t)$.

IV. PLANE-WAVE SOLUTIONS

We start with plane-wave solutions of the extended DNLS equation. If a solution ψ is independent of t, then we see from Eq. (9) that

$$i\psi_x + \psi \sum_{n=1}^{\infty} \binom{2n}{n}(-1)^{n+1}\alpha_{2n}|\psi|^{2n} = 0,$$

where $\binom{2n}{n}$ is a binomial coefficient. So

$$i\psi_x + 2\alpha_2\psi|\psi|^2 - 6\alpha_4\psi|\psi|^4 + 20\alpha_6\psi|\psi|^6 - \cdots = 0.$$

Thus, for the arbitrary background forward-propagating plane-wave solution to Eq. (5), $\psi_p = k \exp(i\phi_p x)$, we have (with $j = 2n$)

$$\phi_p = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2n)!}{(n!)^2}k^{2n}\alpha_{2n} \qquad (10)$$

so that

$$\psi_p = k \exp\left(ix \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2n)!k^{2n}}{(n!)^2}\alpha_{2n}\right)$$

$$= k \exp\big[ix(2k^2\alpha_2 - 6k^4\alpha_4 + 20k^6\alpha_6 - 70k^8\alpha_8$$

$$+ 252k^{10}\alpha_{10} + \cdots)\big], \qquad (11)$$

noting that α_2 does not have to be $1/2$.

If we have $\alpha_{2n} = 1$ for each n, then

$$i\psi_x + \psi\left(1 - \frac{1}{\sqrt{1 + 4|\psi|^2}}\right) = 0$$

and

$$\phi_p = 1 - \frac{1}{\sqrt{1 + 4k^2}}.$$

If $\alpha_{2n} = \frac{1}{2n}$ for each n, then

$$i\psi_x + \psi \log\big[\tfrac{1}{2}(1 + \sqrt{1 + 4|\psi|^2})\big] = 0$$

and

$$\phi_p = \log\big[\tfrac{1}{2}(1 + \sqrt{1 + 4k^2})\big].$$

Here k is the arbitrary amplitude of the plane wave, and the series in Eq. (11) contain even coefficients of Eq. (5). The simple nature of the scaling is apparent, with arbitrary background level k causing each coefficient α_{2n} to be multiplied by k^{2n}. The expression Eq. (11) represents the solution of Eq. (5) of any order up to infinity. There are only even terms as waves forward-propagating.

V. FIRST-ORDER SOLITON OF THE FULL EQUATION

The "black" soliton of the basic DNLSE is well known (e.g., see Eq. 5.55 of [13]). This fundamental dark soliton has been observed in fibers [15] as a dip on a broad background. In that experiment, its quoted width of 185 fs is very narrow, and this would imply that higher-order terms, beyond the usual defocusing NLSE, would be required to analyze it. The first-order soliton of Eq. (5), taking α_2 and all other coefficients α_j to be arbitrary, is

$$\psi_b = k \exp(i\phi_b x)\tanh(kt + xv_b), \qquad (12)$$

where the phase term is

$$\phi_b = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2n)!k^{2n}}{(n!)^2}\alpha_{2n}$$

$$= 2k^2\alpha_2 - 6k^4\alpha_4 + \cdots, \qquad (13)$$

and where the velocity is

$$v_b = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2n)!k^{2n+1}}{(n!)^2}\alpha_{2n+1}$$

$$= 2k^3\alpha_3 - 6k^5\alpha_5 + \cdots. \qquad (14)$$

The background level k is arbitrary. It is clear from these expressions that velocity depends on third-, fifth-, seventh-, and other odd-order coefficients, α_{2n+1}, while the phase depends on
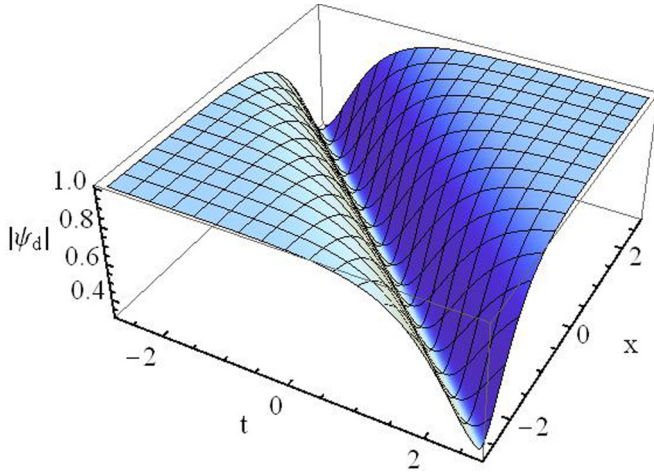
FIG. 1. Plot of a gray soliton, Eq. (15), with $k = 1$, $A = 0.95$, $\alpha_2 = 1/2$, $\alpha_3 = 0.4$, $\alpha_4 = -0.3$, $\alpha_5 = 0.1$, and all other α_j's zero; the minimum is 0.31225.

the second-, fourth-, sixth-, and other even-order coefficients α_{2n}.

So this solution applies for infinitely many orders in the original equation. It confirms and generalizes the brief derivations in Secs. III A and III B.

VI. GENERALIZED GRAY SOLITONS

Gray solitons (Fig. 1) have been observed in optical fibers and on water surfaces [7]. The water wave observation was approximately described as being governed by the DNLSE alone (i.e., only α_2 is nonzero) in [7]. The DNLSE gray soliton (e.g., see Eq. 5.58 of [13]) is well known. Here we extend this class for the whole hierarchy. The generalized gray soliton for the system with all α_j's arbitrary is

$$\psi_d = k \exp(i\phi_g x)\{\sqrt{1-A^2} - iA\tanh[A(kt + v_g x)]\},$$
(15)

where $0 < A \leqslant 1$ and where the phase term is the same as in Eq. (13), i.e.,

$$\phi_g = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2n)!k^{2n}}{(n!)^2}\alpha_{2n} = 2k^2\alpha_2 - 6k^4\alpha_4 + \cdots.$$

However, the velocity is more complicated:

$$v_g = \frac{\sqrt{1-A^2}}{A^2}\sum_{n=1}^{\infty}2^{2n-1}\alpha_{2n}k^{2n}\left\{A(A^2-1)^{n-\frac{1}{2}}\right.$$

$$\left. - \frac{1}{\sqrt{\pi}}(-1)^n\Gamma\left(n+\frac{1}{2}\right){}_2\tilde{F}_1\left(\frac{1}{2},1;n+1;\frac{1}{A^2}\right)\right\}$$

$$- \frac{1}{\sqrt{\pi}A^2}\sum_{n=1}^{\infty}4^n\alpha_{2n+1}k^{2n+1}\left[(-1)^n\Gamma\left(n+\frac{3}{2}\right)\right.$$

$$\times {}_2\tilde{F}_1\left(\frac{1}{2},1;n+2;\frac{1}{A^2}\right) + \sqrt{\pi}A(A^2-1)^{n+\frac{1}{2}}\right], \quad (16)$$

where ${}_2\tilde{F}_1$ is the regularized hypergeometric function [16]. Thus the minimum amplitude is $k\sqrt{1-A^2}$. Apart from the

$\sqrt{1-A^2}$ factor, these turn out to be polynomials. Now, the velocity involves each value of α_j, i.e., all the even and odd coefficients. There is no corresponding result for the focusing NLSE.

So we have

$$v_g = 2\sqrt{1-A^2}[\alpha_2 k^2 + 2(2A^2 - 3)\alpha_4 k^4$$

$$+ 2(8A^4 - 20A^2 + 15)\alpha_6 k^6$$

$$+ 4(16A^6 - 56A^4 + 70A^2 - 35)\alpha_8 k^8$$

$$+ 2(128A^8 - 576A^6 + 1008A^4 - 840A^2$$

$$+ 315)\alpha_{10}k^{10}\cdots] - 2(2A^2 - 3)\alpha_3 k^3 - 2(8A^4$$

$$- 20A^2 + 15)\alpha_5 k^5 - 4(16A^6 - 56A^4 + 70A^2$$

$$- 35)\alpha_7 k^7 - 2(128A^8 - 576A^6 + 1008A^4$$

$$- 840A^2 + 315)\alpha_9 k^9 - \cdots. \qquad (17)$$

If $A = 1$, then the velocity reduces to the result of the black soliton, Eq. (14). We stress that this is quite a simple result for an equation that can contain hundreds of terms, each with various derivatives.

VII. GENERALIZED RATIONAL SOLUTIONS

Again, we allow for all operator coefficients (α_j, $j = 3, 4, 5, \ldots, \infty$) to be arbitrary. Then

$$\psi(x,t) = c\left[4\frac{1 + 2i\,B_r\,x}{D(x,t)} - 1\right]e^{i\phi_r\,x}, \qquad (18)$$

where

$$D(x,t) = 1 + 4B_r^2 x^2 - 4(ct + v_r x)^2 \qquad (19)$$

and

$$B_r = \sum_{n=1}^{\infty}\frac{n(2n)!}{(n!)^2}(-1)^{n+1}\alpha_{2n}c^{2n}$$

$$= 2c^2\left(\alpha_2 - 6c^2\alpha_4 + 30c^4\alpha_6 - 140c^6\alpha_8\right.$$

$$\left. + 630c^8\alpha_{10} + \cdots\right). \qquad (20)$$

Here c is the arbitrary background level. The coefficient ϕ in the exponential factor is then equal to

$$\phi_r = \sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}(-1)^{n+1}\alpha_{2n}c^{2n}$$

$$= 2c^2\left(\alpha_2 - 3c^2\alpha_4 + 10c^4\alpha_6 - 35c^6\alpha_8\right.$$

$$\left. + 126c^8\alpha_{10} + \cdots\right). \qquad (21)$$

Finally, the velocity is

$$v_r = \sum_{n=1}^{\infty}\frac{(2n+1)!}{(n!)^2}(-1)^{n+1}\alpha_{2n+1}c^{2n+1}$$

$$= 2c^3\left(3\alpha_3 - 15c^2\alpha_5 + 70c^4\alpha_7\right.$$

$$\left. - 315c^6\alpha_9 + 1386c^8\alpha_{11} + \cdots\right). \qquad (22)$$

In contrast to the NLSE set, these are rational but are not of the form of "rogue waves." Clearly, when $1 + 4B_r^2 x^2 = 4(ct + v_r x)^2$, we have $D(x,t) = 0$ and the solution is singular on the two curves thus indicated. This includes the points

FIG. 2. Plot of elliptic "sn" soliton, Eq. (23), with $c = 1$, $r = 0.55$, $\alpha_2 = 1/2$, $\alpha_3 = -1$, $\alpha_4 = -1/6$, $\alpha_5 = 0.17$, and all other α_j's zero.

$x = 0, t = \pm \frac{1}{2c}$. The velocity clearly depends only on the coefficients of odd-order operators, but it can be zero for various combinations (e.g., with $\alpha_3 = 5c^2\alpha_5$ when the other α_j's are zero). B_r, ϕ_r can also be zero for some combinations. This occurs, for example, when $\alpha_4 = \frac{2\alpha_2}{3c^2}, \alpha_6 = \frac{\alpha_2}{10c^4}$ for arbitrary α_2 when the other α_j's are zero. Then $B_r, \phi_r = 0$, and ψ is real.

We note that the exponential factor, ϕ_r, and the stretching factor, B_r, here depend only on the coefficients of even-order operators. If we have only odd-numbered equations (see Sec. III A), then $\phi_r = B_r = 0$, and we obtain the real-valued solution $\psi(x,t) = c\left[\frac{4}{1 - 4(ct + v_r x)^2} - 1\right]$.

VIII. PERIODIC SOLUTIONS

The single operator defocusing NLSE "sn" solution has been given in Eq. 5.24 of [13]. We now give the elliptic "sn" solutions of the full infinite-order equation (Fig. 2). We can write them in a convenient way, as follows:

$$\psi_s = ce^{i\phi_s x} \, \text{sn}\left[\frac{ct}{\sqrt{r}} + x v_s, r\right], \tag{23}$$

We note that this is a very simple result for an equation that can contain hundreds of terms, each with various derivatives. It could help to explain the appearance of "dark"-type solitons in a variety of physical, biological, financial, and social situations, going well beyond the DNLSE cases that have been previously analyzed.

IX. COLLISION OF TWO DARK SOLITONS

For the basic DNLSE, the result for the collision of two dark solitons is known (e.g., see Eq. 5.52 of [13]). Here, we consider

where sn is a Jacobi elliptic function [16], with r real. With our modulus definition, $\text{sn}(y,m) = y - \frac{1}{3!}(1 + m)y^3 + \cdots$.

Again, we allow for all operator coefficients (α_j, $j = 3, 4, 5, \ldots, \infty$) to be arbitrary. The phase term can be expressed in terms of P_n, the set of orthogonal Legendre polynomials (of the first kind): Here c is the arbitrary background level. The coefficient ϕ_s in the exponential factor is then equal to

$$\phi_s = -\sum_{n=1}^{\infty} \alpha_{2n} c^{2n} \left(\frac{1}{r} - 1\right)^n P_n\left(\frac{r+1}{r-1}\right)$$

$$= r^{-1}\alpha_2 c^2(r+1) - r^{-2}\alpha_4 c^4(r^2 + 4r + 1)$$
$$+ r^{-3}\alpha_6 c^6(r^3 + 9r^2 + 9r + 1)$$
$$- r^{-4}\alpha_8 c^8(r^4 + 16r^3 + 36r^2 + 16r + 1)$$
$$+ r^{-5}\alpha_{10} c^{10}(r^5 + 25r^4 + 100r^3 + 100r^2 + 25r + 1)$$
$$+ \cdots. \tag{24}$$

The well-known polynomials are $P_1(y) = y$, $P_2(y) = \frac{1}{2}(3y^2 - 1)$, $P_3(y) = \frac{1}{2}y(5y^2 - 3)$, $P_4(y) = \frac{1}{8}(35y^4 - 30y^2 + 3)$, $P_5(y) = \frac{1}{8}y(63y^4 - 70y^2 + 15)$, etc.

Finally, the velocity is

$$v_s = -\frac{1}{\sqrt{r}} \sum_{n=1}^{\infty} \alpha_{2n+1} c^{2n+1} \left(\frac{1}{r} - 1\right)^n P_n\left(\frac{r+1}{r-1}\right)$$

$$= r^{-3/2}\alpha_3 c^3(r+1) - r^{-5/2}\alpha_5 c^5(r^2 + 4r + 1)$$
$$+ r^{-7/2}\alpha_7 c^7(r^3 + 9r^2 + 9r + 1)$$
$$- r^{-9/2}\alpha_9 c^9(r^4 + 16r^3 + 36r^2 + 16r + 1)$$
$$+ r^{-11/2}\alpha_{11} c^{11}(r^5 + 25r^4 + 100r^3 + 100r^2 + 25r + 1)$$
$$+ \cdots. \tag{25}$$

Then, the velocity clearly depends only on the coefficients of odd-order operators while the phase depends only on the coefficients of even-order operators. Interestingly, the coefficients are squared values of the elements of the (symmetric) Pascal's triangle, i.e., they are squares of binomial coefficients, $\binom{n}{k}$ for $k = 0, 1, 2, \ldots, n$. Thus

$n = 1$:					(1	1)		
$n = 2$:				(1	4	1)		
$n = 3$:			(1	9	9	1)		
$n = 4$:		(1	16	36	16	1)		
$n = 5$:	(1	25	100	100	25	1)		
$n = 6$: (1	36	225	400	225	36	1)		

all odd- and even-order equations. The odd-order equations basically modify the velocity, while the even-order equations basically modify the phase and introduce a "stretching factor" in x in the nonphase part of the solution. We take an arbitrary background, i.e., any real c, while noting that the scaling is relatively simple once the $c = 1$ case is known. Thus a general "collision" with any background (c) is given by

$$\psi_m = ce^{i\phi_m x} \frac{-\cosh[\sqrt{2}(ct + v_m x)] + i\sqrt{2}\sinh(B_m x)}{\sqrt{2}\cosh(B_m x) + \cosh[\sqrt{2}(ct + v_m x)]}. \tag{26}$$

Now, the velocity is

$$v_m = c \sum_{n=1}^{\infty} (-1)^{n+1} (2c^2)^n \alpha_{2n+1} P_n^{\left(\frac{1}{2}, -n-1\right)}(3)$$

$$= 2\left(2c^3\alpha_3 - 7c^5\alpha_5 + 24c^7\alpha_7 - 83c^9\alpha_9 \right.$$
$$\left. + 292c^{11}\alpha_{11} - \cdots\right). \tag{27}$$

The "stretching factor" is

$$B_m = \sum_{n=0}^{\infty} (-1)^n (2c^2)^{n+1} \alpha_{2n+2} P_n^{\left(\frac{1}{2}, -1-n\right)}(3)$$

$$= 2\left(c^2\alpha_2 - 4c^4\alpha_4 + 14c^6\alpha_6 - 48c^8\alpha_8 \right.$$
$$\left. + 166c^{10}\alpha_{10} - 584c^{12}\alpha_{12} + \cdots\right). \tag{28}$$

In these results, $P_n^{(a,b)}(x)$ represents a Jacobi polynomial [16]. Then

$$\phi_m = \sum_{n=1}^{\infty} \alpha_{2n}(-1)^{n+1} c^{2n} \frac{(2n)!}{(n!)^2}$$

$$= 2c^2\left(\alpha_2 - 3c^2\alpha_4 + 10c^4\alpha_6 - 35c^6\alpha_8 \right.$$
$$\left. + 126c^8\alpha_{10} + \cdots\right). \tag{29}$$

Note that the phase matches that of the plane-wave solution, Eq. (10), the "tanh" solution of Eq. (13), and the rational solution, Eq. (24). The central amplitude is always $|\psi(0,0)| = (\sqrt{2} - 1)|c| \approx 0.4142\,|c|$.

An example is given in Fig. 3. It shows a collision, with an additional velocity factor. If we have only odd-numbered equations, then $\phi_m = B_m = 0$, and the solution $\psi_m(x,t)$ of Eq. (26) becomes real-valued, and it resembles a moving gray soliton, since there is no "collision." If there is at least one even-numbered equation, then the solution is complex, and it does represent a collision. If at least one even and one odd coefficient are nonzero, then ϕ_m, B_m, v_m are all nonzero.

We can also generalize to a "collision" involving two arbitrary real parameters, a_1 and a_3, with $a_3 > a_1$ (Fig. 4). Here
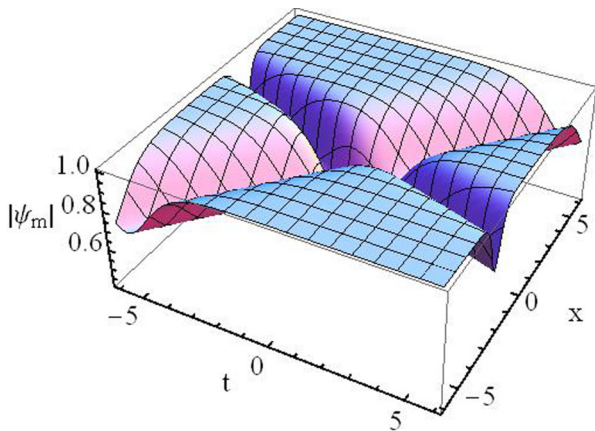
$$\psi_f = e^{i\phi_f x} \frac{T}{L}, \tag{30}$$

FIG. 3. Plot of the collision of two dark solitons, Eq. (26), solution of Eq. (5), with $c = 1$, $\alpha_4 = \frac{1}{3}$, $\alpha_5 = -\frac{1}{14}$, and all other α_j's zero.

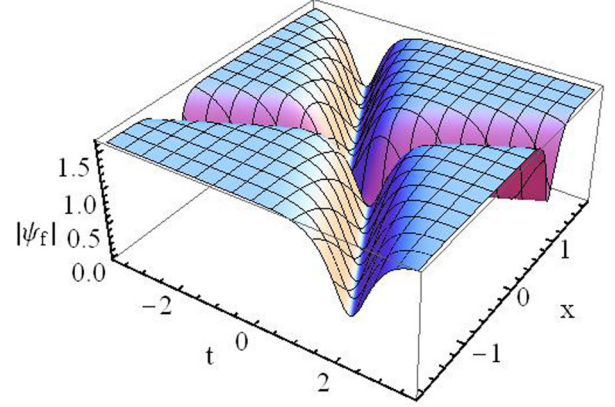FIG. 4. Plot of a collision of two dark solitons, Eq. (30), solution of Eq. (5), with parameters $a_3 = 19/7$, $a_1 = 7/17$, and coefficients $\alpha_2 = \frac{1}{2}, \alpha_3 = -0.2, \alpha_4 = -0.2$, and all other α_j's zero. The collision-induced shift is obvious in this figure. Here the value at $x = t = 0$ is 0.364, while the background is 1.6475. We have $|\psi| = 0$ when $x = 0$ and $t = \pm 0.3905$.

where

$$T = (a_3 - 2a_1)\cosh(B_f x) - \sqrt{a_1 a_3}\cosh[2p(t + v_f x)]$$
$$+ i\mu \sinh(B_f x),$$

$$L = \sqrt{a_1}\cosh[2p(t + v_f x)] + \sqrt{a_3}\cosh(B_f x),$$

and

$$v_f = \sum_{n=1}^{\infty} a_3^n \frac{(2n)!}{(n!)^2} \alpha_{2n+1} (-1)^{n+1}\, {}_2F_1\left(1, -n; \frac{1}{2} - n; \frac{a_1}{a_3}\right)$$

$$= 2(2a_1 + a_3)\alpha_3 - 2\left(8a_1^2 + 4a_1 a_3 + 3a_3^2\right)\alpha_5$$
$$+ 4\left(16a_1^3 + 8a_1^2 a_3 + 6a_1 a_3^2 + 5a_3^3\right)\alpha_7 + \cdots,$$

$$B_f = 2\mu \sum_{n=0}^{\infty} a_3^n \frac{(2n)!}{(n!)^2} \alpha_{2n+2}(-1)^n\, {}_2F_1\left(1, -n; \frac{1}{2} - n; \frac{a_1}{a_3}\right)$$

$$= 2\mu\left[\alpha_2 - 2(2a_1 + a_3)\alpha_4 + 2\left(8a_1^2 + 4a_1 a_3 + 3a_3^2\right)\alpha_6 \right.$$
$$\left. - 4\left(16a_1^3 + 8a_1^2 a_3 + 6a_1 a_3^2 + 5a_3^3\right)\alpha_8 + \cdots\right],$$

$$\phi_f = \sum_{n=1}^{\infty} \alpha_{2n}(-1)^{n+1} a_3^n \frac{(2n)!}{(n!)^2}$$

$$= 2\alpha_2 a_3 - 6\alpha_4\, a_3^2 + 20\alpha_6 a_3^3 + \cdots.$$

Here ${}_2F_1$ denotes a hypergeometric function [16]. The fixed parameters are $p = \sqrt{a_3 - a_1}$ and $\mu = 2\sqrt{a_1(a_3 - a_1)}$. The value at origin is $|\psi(0,0)| = \frac{|a_3 - \sqrt{a_1 a_3} - 2a_1|}{\sqrt{a_1} + \sqrt{a_3}}$, while the background (and hence maximum) is $\sqrt{a_3}$. Now $|\psi| = 0$ when $x = 0$ and $t = \pm\frac{1}{2}(a_3 - a_1)^{-1/2}\text{arccosh}(\frac{a_3 - 2a_1}{\sqrt{a_1 a_3}})$. The reduction to the one-parameter result is obtained by setting $a_3 \to c^2$ and $a_1 \to a_3/2$. Then B_f reduces to B_m above, ϕ_f reduces to ϕ_m, and v_f reduces to v_m/c.

These new solutions can provide a considerable extension of applicability to problems of modulation instability soliton collisions in physics and chemistry.

X. CONCLUSION

We have presented the infinite-order hierarchy of the defocusing nonlinear Schrödinger equation (DNLSE). This includes an infinite number of equations, and each can have an arbitrary coefficient. We have given generalized soliton solutions, plane-wave solutions, "collision"-type phenomena, and periodic solutions of all orders. We have found that "even"-order equations in the set affect phase and "stretching factors" in the solutions, while "odd"-odd order equations affect the velocities. Hence odd-order solutions can be real functions, while even-order solutions are complex. These solutions should have greatly expanded applicability to physical situations, relative to those of the basic DNLSE.

APPENDIX A: SCALING THE EQUATIONS

For a solution $\psi(x,t;\alpha_2,\alpha_3,\alpha_4,\dots)$ of the full equation, we can produce a scaled solution by multiplying the function by a real constant, c, multiplying t by c, leaving x unchanged, and multiplying each α_j in the solution by c^j. So, the generated solution will be $c\,\psi(x,c\,t;c^2\alpha_2,c^3\alpha_3,c^4\alpha_4,\dots)$. When all α_j's are zero for $j \geqslant 3$, then we reduce to the basic DNLSE alone (e.g., see Chap. 5 of [13]), and the scaling $\alpha_2 \to c^2\alpha_2$ is the same as scaling x by a factor of c^2. When more operators are involved, we stress that the scaling involves the α_j's in the solution, rather than the variable x.

APPENDIX B: HEPTIC AND OPTIC OPERATORS

D_7 is the seventh-order ($j = 7$), i.e., heptic, operator (starting with seventh-order derivative):

$$D_7[\psi] = 14\{\psi^*(3\psi_t\psi_{tttt} + 5\psi_{tt}\psi_{ttt}) + \psi[2\psi_t^*\psi_{tttt}$$
$$- 20(\psi^*)^2\psi_t\psi_{tt} + \psi_t\psi_{tttt}^* + \psi^*(\psi_{5t} - 20\psi_t|\psi_t|^2)$$

$$+ 3\psi_{tt}^*\psi_{ttt} + 2\psi_{tt}\psi_{ttt}^*] - 5\psi^2\{\psi^*(2\psi_t^*\psi_{tt}$$
$$+ \psi^*\psi_{ttt}) + \psi_t[(\psi_t^*)^2 + 2\psi^*\psi_{tt}^*]\}$$
$$+ \psi_t(7\psi_t^*\psi_{ttt} + 2\psi_t\psi_{ttt}^* + 8|\psi_{tt}|^2)$$
$$+ 5\psi_t^*\psi_{tt}^2 - 5(\psi^*)^2\psi_t^3 + 10|\psi|^6\psi_t\} - \psi_{7t}.$$

There is an infinite number of higher-order operators. The highest one that we present here is D_8, which is the eighth-order ($j = 8$), i.e., octic, operator (starting with eighth-order derivative):

$$D_8[\psi] = 2\psi\{\psi_t[6\psi_{5t}^* - 14\psi^*(52\psi_t^*\psi_{tt} + 17\psi^*\psi_{ttt})]$$
$$- 21\psi_t^2[9(\psi_t^*)^2 + 14\psi^*\psi_{tt}^*] + 8\psi^*\psi_{6t} + 20\psi_{5t}\psi_t^*$$
$$+ 36\psi_{tt}^*\psi_{tttt} + 22\psi_{tt}\psi_{tttt}^* - 161(\psi^*)^2\psi_{tt}^2$$
$$+ 34|\psi_{ttt}|^2\} + 14\psi^*(-30|\psi_t|^2\psi_t^2 + 4\psi_{5t}\psi_t$$
$$+ 8\psi_{tt}\psi_{tttt} + 5\psi_{ttt}^2) + 14\psi^3[-4\psi_t^*\psi_{ttt}^*$$
$$+ 40(\psi^*)^2|\psi_t|^2 + 20(\psi^*)^3\psi_{tt} - 3(\psi_{tt}^*)^2$$
$$- 2\psi^*\psi_{tttt}^*] + 2\psi^2[-14\psi^*(11\psi_t^*\psi_{ttt} + 6\psi_t\psi_{ttt}^*$$
$$+ 14|\psi_{tt}|^2) - 7\psi_t^*(17\psi_t^*\psi_{tt} + 24\psi_t\psi_{tt}^*)$$
$$+ 280(\psi^*)^3\psi_t^2 + \psi_{6t}^* - 49(\psi^*)^2\psi_{tttt}]$$
$$+ 14[3\psi_t(4\psi_t^*\psi_{tttt} + \psi_t\psi_{tttt}^* + 6\psi_{tt}^*\psi_{ttt})$$
$$+ 2\psi_{tt}(11\psi_t^*\psi_{ttt} + 7\psi_t\psi_{ttt}^*) + 13|\psi_{tt}|^2\psi_{tt}]$$
$$- 490(\psi^*)^2\psi_t^2\psi_{tt} + 140|\psi|^2\psi^3[(\psi_t^*)^2 + \psi^*\psi_{tt}^*]$$
$$- \psi_{8t} - 70|\psi|^8\psi.$$

Although we do not present the ninth-order ($j = 9$) operator ($D_9[\psi]$, with coefficient α_9) here to save space, the results we give for the solutions do include it and all higher orders to infinity.

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