

Expansion of Co-Compact Convex Spacelike Hypersurfaces in Minkowski Space by their Curvature

BEN ANDREWS, XUZHONG CHEN, HANLONG FANG &
JAMES MCCOY

ABSTRACT. We consider the expansion of co-compact convex hypersurfaces in Minkowski space by functions of their curvature, and prove under quite general conditions that solutions are asymptotic to the self-similar expanding hyperboloid. In particular, this implies a convergence result for a class of special solutions of the cross-curvature flow of negatively curved Riemannian metrics on three-manifolds.

1. INTRODUCTION

There is a great deal of literature concerning the evolution of hypersurfaces by geometric heat equations in Euclidean space, particularly concerning evolution by mean curvature [Hu1, Hu2, W2, W3, HS1, HS2, HS3, SW, A7, CM, W1], but also including many other evolution equations in which the speed is a nonlinear function of the principal curvatures [T, C, Ge1, U, A1, A3, A4, HI, A5, AMZ, ALM]. There is an analogous situation for spacelike hypersurfaces in Minkowski space, and in particular the evolution of spacelike hypersurfaces by mean curvature flow has received considerable attention (see, e.g., [E1, E2, E3]); see also [Ge2, Ge3, Ge4], where spacelike hypersurfaces are deformed by inverse mean curvature flows in more general Lorentzian background spaces.

In this paper, we are interested in the evolution of convex spacelike hypersurfaces by nonlinear functions of the principal curvatures. We restrict our attention to the situation of co-compact hypersurfaces, where the hypersurface is invariant under a discrete group of ambient isometries, and the quotient with respect to this group is compact. This is a rather special situation, which excludes many

difficult cases studied in previous work on the mean curvature flow (see [E3], for example). In particular, such hypersurfaces are always comparable to a hyperboloid at infinity. Our motivation to understand this case arises in part from an interesting relation between these evolution equations for hypersurfaces and the cross-curvature flow, a fully nonlinear parabolic evolution equation introduced by Chow and Hamilton [CH] for negatively curved metrics on compact three-manifolds. Our results on evolving spacelike hypersurfaces (specifically, where the hypersurface is of dimension three, and the speed in the direction of the unit normal vector ν is equal to the Gauss curvature K), amount to analysis of an interesting class of special solutions for the cross-curvature flow. Rather little is known about solutions of the cross-curvature flow in general (see [B, MC, KY, DKY, GI, CNSC, CSC]). We discuss the relation between Gauss curvature flow and cross curvature flow in Section 12.

The flows we consider are as follows: denote the first fundamental form of an embedding X by g , and the second fundamental form by h . We consider functions $F(h, g)$ of the form $F = f(\kappa_1, \dots, \kappa_n)$, where κ_i is the i th principal curvature (i.e., the i th eigenvalue of h with respect to g), and f is a smooth symmetric function defined on the positive cone $\Gamma_+ = \{(x_1, \dots, x_n) \mid \min_i x_i > 0\}$, which is increasing in each argument and homogeneous of degree one. We define the “dual” function F_* by $F_*(A, g) = F(g, A)^{-1}$ for A positive definite, which means that $F_*(\tau, \bar{g}) = f_*(\tau_1, \dots, \tau_n)$, where τ_1, \dots, τ_n are the eigenvalues of the matrix τ with respect to the inner product \bar{g} , and $f_*(x_1, \dots, x_n) = f(x_1^{-1}, \dots, x_n^{-1})^{-1}$.

Our primary result is the following theorem.

Theorem 1.1. *Let $S = F(h, g)^\alpha$, where*

$$F = f(\kappa_1, \dots, \kappa_n) = \frac{1}{f_*(\kappa_1^{-1}, \dots, \kappa_n^{-1})},$$

and suppose that one of the following is true:

- (1) *Either $0 < \alpha \leq 1$, f is concave, and $f = 0$ on the boundary of Γ_+ ;*
- (2) *or $0 < \alpha \leq 1$ and both f and f_* are concave;*
- (3) *or $0 < \alpha \leq 1$, f_* is concave and $f_* = 0$ on the boundary of Γ_+ ;*
- (4) *or $0 < \alpha \leq 1$ and $n = 2$;*
- (5) *or $\alpha > 1$, f_* is concave and zero on the boundary of the positive cone, and $\kappa_i > \kappa_j$ implies $(\partial f / \partial \kappa_i) \kappa_i \geq (\partial f / \partial \kappa_j) \kappa_j$.*

It follows that, for any co-compact spacelike uniformly locally convex initial embedding $X_0 : \tilde{M} \rightarrow \mathbb{R}^{n,1}$, there exists a unique co-compact solution $X : \tilde{M} \times [0, \infty) \rightarrow \mathbb{R}^{n,1}$ of the flow

$$(1.1) \quad \frac{\partial X}{\partial t} = S\nu.$$

The rescaled embeddings given by $\tilde{X}(p, t) = X(p, t) / ((1 + \alpha)F(I, I)^\alpha t)^{1/(1+\alpha)}$ converge in C^∞ to a limiting embedding with image equal to the future timelike hyperboloid \mathbb{H}^n .

The condition in case (5) is unusual, but includes several cases of interest, including $f = E_k^{1/k}$ for $k = 1, \dots, n$ (see Section 11 for details). In particular, the application to the cross-curvature flow involves the Gauss curvature flow, which is contained in case (5) with $\alpha = n$ and $f = E_n^{1/n}$, in which case $\dot{f}_*^i \tau_i = \dot{f}_*^j \tau_j$ for all i and j .

Theorem 1.1 is in some regards less difficult to prove than the corresponding results for Euclidean hypersurfaces (see, e.g., [AMZ]), although each case requires slightly different treatment, and the estimation of principal curvatures in cases (2), (3), and especially (5) is very delicate. The results are somewhat stronger than those known in the Euclidean setting: for example, speeds given by powers of ratios of elementary symmetric functions of principal curvatures

$$S = (E_k/E_\ell)^{\alpha/(k-\ell)}, \quad 0 \leq \ell < k \leq n,$$

are contained in case (2) for $\alpha \leq 1$, and in case (5) for $\alpha > 1$ if $\ell = 0$. In the Euclidean setting, it was shown in [AMZ] that such flows are definitely not well behaved in the Euclidean setting (in the sense that smooth solutions can develop curvature singularities before they shrink to a point) unless either $\alpha = 1$ or $\ell = 0$, and the results for $\ell = 0$ are established only for $\alpha = 1$ except in a few special cases. The examples given in [AMZ] can be modified to show that the assumptions of Theorem 1.1 are close to optimal: for example, the condition that F_* must be zero on the boundary of the positive cone for $\alpha > 1$ is essentially necessary, and for $\alpha \leq 1$, at least the restriction of F to a boundary face of the positive cone must be inverse concave. It is not clear whether the extra condition in case (5) can be removed, but an inspection of the proof shows that it can certainly be weakened.

One reason for the comparative simplicity of the evolution of spacelike hypersurfaces is the fact that the self-similar expanding hyperboloids become comparable to each other for large times, in contrast to the situation for contracting Euclidean spheres.

We have not included in Theorem 1.1 some cases that are somewhat more general but require considerable work to prove, such as cases where f_* may not be concave, but the restriction of f to a boundary face of Γ_+ is inverse concave. The proofs and statements of these cases are analogous to those in the Euclidean case proved in [AMZ, Theorems 4–7].

2. SPACELIKE HYPERSURFACES

We consider the Minkowski space $\mathbb{R}^{n,1}$, which is the vector space \mathbb{R}^{n+1} equipped with the Minkowski inner product $x \cdot y = -x_0 y_0 + \sum_{i=1}^n x_i y_i$. A vector v in $\mathbb{R}^{n,1}$ is called spacelike, timelike, or null if the “length” $v \cdot v$ is positive, negative, or zero. A timelike vector is called *future timelike* if it has positive first component, and *past timelike* if the first component is negative. The hyperbolic space \mathbb{H}^n embeds naturally into $\mathbb{R}^{n,1}$ as the future timelike unit sphere, that is,

$$\mathbb{H}^n = \{x \mid x \cdot x = -1, x_0 > 0\}.$$

2.1. Computations on the hypersurface. We consider complete hypersurfaces described by embeddings $X : \tilde{M} \rightarrow \mathbb{R}^{n,1}$. The embedding is assumed to be spacelike, so the induced inner product $g(u, v) = X_*(u) \cdot X_*(v)$ is positive definite at each point. In this case, we can associate with each point $x \in \tilde{M}$ a unique future timelike normal vector $\nu \in \mathbb{H}$.

We briefly describe the local differential geometry of spacelike hypersurfaces, indicating in particular where differences occur with the Euclidean setting. We define the second fundamental form $h \in \text{Sym}_2(TM)$ by the following formula: for any tangent vector fields U and V on \tilde{M} ,

$$(2.1) \quad U(X_*V) = h(U, V)\nu + X_*(\nabla_U V).$$

Here, as in the Euclidean case, ∇ is the Levi-Civita connection of the metric g . Similarly, we define the Weingarten map as the derivative of the normal: since $\nu \cdot \nu = -1$, we have $\nu \cdot \nu_*(U) = 0$ for any vector U , so that $\nu_*(U)$ is entirely tangential to $X(\tilde{M})$, and we can write

$$(2.2) \quad \nu_*U = X_*(A(U)),$$

where $A \in \text{End}(TM)$ is the Weingarten map. Differentiating the identity $X_*(U) \cdot \nu = 0$ yields a relation between A and h (the Weingarten relation):

$$(2.3) \quad \begin{aligned} 0 &= V(X_*U) \cdot \nu + X_*U \cdot \nu_*V = -h(U, V) + X_*U \cdot X_*(A(V)) \\ &= -h(U, V) + g(U, A(V)). \end{aligned}$$

The second fundamental form satisfies the Codazzi identity $\nabla_i h_{jk} = \nabla_j h_{ik}$ as in the Euclidean case. However, a crucial difference arises in the Gauss equations relating the intrinsic curvature of the metric g to the second fundamental form: we have

$$(2.4) \quad R_{ijkl} = h_{jk}h_{il} - h_{ik}h_{jl},$$

which differs from the Euclidean Gauss equation in the sign of the right-hand side.

2.2. Computation using support functions. The support function machinery for convex bodies is very convenient for dealing with evolution of convex hypersurfaces in Euclidean space. Similarly, there is a support function formalism for spacelike convex hypersurfaces in Minkowski space, which we now describe. Given a convex hypersurface $X(\tilde{M})$, the support function $u : \mathbb{H}^n \rightarrow \mathbb{R}$ is defined by $u(z) = \inf\{-z \cdot p \mid p \in X(\tilde{M})\}$. In particular, for a co-compact uniformly convex hypersurface (see Section 3 below), z is well defined on all of \mathbb{H}^n , and the hypersurface can be reconstructed from u by the following embedding:

$$(2.5) \quad \tilde{X} = uz - z_*(\tilde{\nabla}u),$$

where $\bar{\nabla}$ is the Levi-Civita connection of the standard (induced hyperbolic) metric on \mathbb{H}^n . The principal curvatures $\kappa_1, \dots, \kappa_n$ of $X(\bar{M})$ (defined as the eigenvalues of the second fundamental form h with respect to the induced metric g) can then be computed as the reciprocals of the eigenvalues r_1, \dots, r_n of the positive definite bilinear form $\mathfrak{r}_{ij} = u\bar{g}_{ij} - \bar{\nabla}_i\bar{\nabla}_j u$ (since the map X is the inverse of the Weingarten map, and we have $\partial_i X = \mathfrak{r}_{ik}\bar{g}^{k\ell}\partial_i z$). We accordingly refer to r_1, \dots, r_n as the principal radii of curvature. As in the Euclidean case, the matrix \mathfrak{r} satisfies a Codazzi-type identity: $\bar{\nabla}_i\mathfrak{r}_{jk} = \bar{\nabla}_j\mathfrak{r}_{ik}$.

3. CO-COMPACTNESS

We further assume that there is a subgroup G of the group $\text{Isom}_+(\mathbb{R}^{n,1})$ of future-preserving isometries of $\mathbb{R}^{n,1}$ such that, for each $T \in G$, $T(X(\bar{M})) = X(\bar{M})$, and G acts properly discontinuously on \bar{M} . G is then the fundamental group of the quotient $M = \bar{M}/G$, and acts by isometries on \bar{M} with the induced metric g . We assume that the quotient M is compact.

Example 3.1. Simple examples of this situation can be constructed as follows. Take any group G of isometries of the hyperbolic space \mathbb{H}^n which acts co-compactly (e.g., the fundamental group of a compact hyperbolic manifold acting by deck transformations on the universal cover \mathbb{H}^n). Each isometry of hyperbolic space is given by the restriction of some $\text{Isom}_+(\mathbb{R}^{n,1})$ transformation, so G is naturally a subgroup of $\text{Isom}_+(\mathbb{R}^{n,1})$ with compact quotient. Co-compact hypersurfaces with this symmetry can be constructed by taking the normal graph over \mathbb{H}^n of any smooth function defined on the quotient.

The tensors g , h and A are well defined on M : if $T \in G$, then T_* is an isometry from $T_x\bar{M}$ to $T_{T(x)}\bar{M}$, and $\nu_{Tx} = T_*(\nu_x)$ for any $x \in \bar{M}$. It follows that $h_{Tx}(T_*U, T_*V) = h(U, V)$ and $T_*(A_x(U)) = A_{Tx}(T_*U)$.

3.1. The linear representation. Note that the future-preserving isometries $\text{Isom}_+(\mathbb{R}^{n,1})$ of $\mathbb{R}^{n,1}$ have the form $z \mapsto Lz + b$, where $b \in \mathbb{R}^{n,1}$, and L is a linear transformation in $O_+(n, 1)$, the space of future-preserving linear transformations preserving the Lorentzian inner product. Thus, the inclusion of G in $\text{Isom}_+(\mathbb{R}^{n,1})$ can be written in the form

$$(3.1) \quad g(z) = L(g)z + b(g),$$

and we have

$$\begin{aligned} L(g_2g_1)z + b(g_2g_1) &= g_2g_1(z) = L(g_2)(g_1z) + b(g_2) \\ &= L(g_2)L(g_1)z + L(g_2)b(g_1) + b(g_2) \end{aligned}$$

for all $z \in \mathbb{R}^{n,1}$ and $g_1, g_2 \in G$. In particular, the linear part of this inequality implies that the map $L : G \rightarrow O_+(n, 1)$ is itself a representation. We call this the linear representation of G .

3.2. The support function and standard co-compact hypersurfaces. If \tilde{M} is such that the inclusion of G in $\text{Isom}_+(\mathbb{R}^{n,1})$ is contained in the linear part $O_+(n, 1)$ for some choice of origin—that is, $b(g) = 0$ for all $g \in G$ in equation (3.1)—we say that the hypersurface is *standard*. The standard hypersurfaces are somewhat easier to work with: in particular, the support function u is then well defined on the quotient $\mathbb{H}/L(G)$, since in this case we have

$$\begin{aligned} u(L(g)z) &= \inf\{-L(g)z \cdot p \mid p \in X(\tilde{M})\} \\ &= \inf\{-L(g)z \cdot L(g)p \mid p \in X(\tilde{M})\} \\ &= \inf\{-z \cdot p \mid p \in X(\tilde{M})\} = u(z), \end{aligned}$$

for each $g \in G$, since $L(g)p$ ranges over $X(\tilde{M})$ when p does. In the non-standard case, the support function is defined on \tilde{M} but not on M : the support function is $O_+(n, 1)$ invariant but not $\text{Isom}_+(\mathbb{R}^{n,1})$ invariant. In particular, maximum principle arguments involving the support function in the general case must take into account the behaviour at infinity.

4. EVOLUTION EQUATIONS

4.1. Evolution equations on the hypersurface. We follow the notational conventions of [AMZ], in particular using the following expressions for the speed S . We write $S(h, g) = S(A) = F(A)^\alpha$, where F is homogeneous of degree 1, smooth, invariant under change of basis, and strictly increasing on the positive cone: $(d/ds)F(A + sB) > 0$ whenever B is nonzero and positive semi-definite. We denote by \dot{F}^{ij} the derivative of F with respect to the components of A , so that

$$\frac{d}{ds}F(A + sB) = \dot{F}^{ij}B_{ij}.$$

Also, F can be expressed in the form $F = f(\kappa_1, \dots, \kappa_n)$, where f is a smooth symmetric function on \mathbb{R}^n that is increasing in each argument and homogeneous of degree one. We denote $\dot{f}^i = \partial f / \partial \kappa_i$. In an orthonormal frame of eigenvectors for A , the matrix \dot{F} is also diagonal, with diagonal entries equal to \dot{f}^i (see [A5, Theorem 5.1]). Similarly, we denote by \ddot{F} the second derivatives of F with respect to components of A , and by \ddot{f} the second derivatives of f with respect to the principal curvatures.

Evolution equations for geometric quantities on the hypersurface can be deduced by following the corresponding arguments from the Euclidean case. The evolution of the induced metric g is computed as follows:

$$\begin{aligned} (4.1) \quad \frac{\partial}{\partial t}g_{ij} &= \frac{\partial}{\partial t} \left(\frac{\partial X}{\partial x^i} \cdot \frac{\partial X}{\partial x^j} \right) = \frac{\partial}{\partial x^i}(S\nu) \cdot \frac{\partial X}{\partial x^j} + \frac{\partial X}{\partial x^i} \cdot \frac{\partial}{\partial x^j}(S\nu) \\ &= SA_i^k g_{kj} + Sg_{ik} A_j^k = 2Sh_{ij}. \end{aligned}$$

The evolution of the unit normal is as follows. We have $0 = (\partial\nu/\partial t) \cdot \nu$, and

$$\begin{aligned} \left(\frac{\partial}{\partial t}\nu\right) \cdot \frac{\partial X}{\partial x^i} &= \frac{\partial}{\partial t}\left(\nu \cdot \frac{\partial X}{\partial x^i}\right) - \nu \cdot \frac{\partial}{\partial x^i}(f\nu) \\ &= -\nu \cdot \left(\frac{\partial S}{\partial x^i}\nu + SA_i^k \frac{\partial X}{\partial x^k}\right) = \frac{\partial S}{\partial x^i}, \end{aligned}$$

from which it follows that

$$(4.2) \quad \frac{\partial \nu}{\partial t} = \frac{\partial S}{\partial x^i} g^{ij} \frac{\partial X}{\partial x^j} = \nabla S.$$

The evolution of the second fundamental form is as follows:

$$\begin{aligned} (4.3) \quad \frac{\partial}{\partial t} h_{ij} &= \frac{\partial}{\partial t} \left(\frac{\partial \nu}{\partial x^i} \cdot \frac{\partial X}{\partial x^j} \right) \\ &= \frac{\partial}{\partial x^i} \left(\frac{\partial S}{\partial x^k} g^{k\ell} \frac{\partial X}{\partial x^\ell} \right) \cdot \frac{\partial X}{\partial x^j} + \frac{\partial \nu}{\partial x^i} \cdot \frac{\partial}{\partial x^j} (S\nu) \\ &= \nabla_i \nabla_j S + SA_i^p A_j^q g_{pq}. \end{aligned}$$

If we introduce the canonical spacetime connection (as in [AH, Section 6.3] or [AB, Section 2.3]) by setting $\nabla_t \partial_i = Sh_i^p \partial_p$, this then becomes

$$(4.4) \quad \nabla_t h_{ij} = \frac{\partial}{\partial t} h_{ij} - h(\nabla_t \partial_i, \partial_j) - h(\partial_i, \nabla_t \partial_j) = \nabla_i \nabla_j S - Sh_i^p h_{pj}.$$

An evolution equation for the speed follows. Since $\nabla_t g_{ij} = 0$,

$$(4.5) \quad \frac{\partial}{\partial t} S(h, g) = \dot{S}^{ij} \nabla_t h_{ij} = \dot{S}^{ij} \nabla_i \nabla_j S - S\dot{S}(h^2).$$

Equation (4.4) may be converted to the form of a parabolic equation by using a version of Simons’s identity obtained by applying the Codazzi and Gauss identities:

$$\begin{aligned} \nabla_i \nabla_j h_{k\ell} &= \nabla_i \nabla_k h_{j\ell} = \nabla_k \nabla_i h_{j\ell} + R_{ikj}{}^p h_{p\ell} + R_{ik\ell}{}^p h_{jp} \\ &= \nabla_k \nabla_\ell h_{ij} + h_{jk} h_i^p h_{p\ell} - h_{ij} h_k^p h_{p\ell} + h_{k\ell} h_i^p h_{jp} - h_{i\ell} h_k^p h_{jp}. \end{aligned}$$

Symmetrising in (i, j) and in (k, ℓ) gives the required identity:

$$(4.6) \quad \nabla_{(i} \nabla_{j)} h_{k\ell} = \nabla_{(k} \nabla_{\ell)} h_{ij} - h_{ij} h_k^p h_{p\ell} + h_{k\ell} h_i^p h_{jp}.$$

This is very similar to the corresponding Euclidean identity, but with the signs of the curvature terms reversed. Substituting this in the evolution equation (4.3) gives the following:

$$\begin{aligned}
 (4.7) \quad \nabla_t h_{ij} &= \dot{S}^{k\ell} \nabla_i \nabla_j h_{k\ell} + \ddot{S}(\nabla_i h, \nabla_j h) - S h_i^p h_{pj} \\
 &= \dot{S}^{k\ell} (\nabla_k \nabla_\ell h_{ij} - h_{ij} h_k^p h_{p\ell} + h_{k\ell} h_i^p h_{jp}) + \ddot{S}(\nabla_i h, \nabla_j h) - S h_i^p h_{pj} \\
 &= \dot{S}^{k\ell} \nabla_k \nabla_\ell h_{ij} + \ddot{S}(\nabla_i h, \nabla_j h) + (\dot{S}(h) - S) h_i^p h_{pj} - h_{ij} \dot{S}(A^2).
 \end{aligned}$$

This may also be written in the following form, since $S = F^\alpha$:

$$\begin{aligned}
 (4.8) \quad \nabla_t h_{ij} &= \alpha F^{\alpha-1} \dot{F}^{k\ell} \nabla_k \nabla_\ell h_{ij} + \alpha F^{\alpha-1} \ddot{F}(\nabla_i h, \nabla_j h) \\
 &\quad + \alpha(\alpha-1) F^{\alpha-2} \nabla_i F \nabla_j F \\
 &\quad + (\alpha-1) F^\alpha h_{ij}^2 - \alpha F^{\alpha-1} \dot{F}(A^2) h_{ij}.
 \end{aligned}$$

In this case, (4.5) takes the following form:

$$(4.9) \quad \frac{\partial}{\partial t} F^\alpha = \alpha F^{\alpha-1} \dot{F}^{ij} \nabla_i \nabla_j F^\alpha - \alpha F^{2\alpha-1} \dot{F}(h^2).$$

The evolution equations (4.8) and (4.9) are identical to the Euclidean counterparts (see [AMZ, Lemma 9]), except that all of the curvature terms have their sign reversed. Since h and g are well defined on M , equations (4.7)–(4.9) hold on M and not just on \tilde{M} .

4.2. Evolution equations in the Gauss map parametrisation. The evolution equation (1.1) yields an evolution equation for the support function u on \tilde{M} : since $u(z, t) = -X \circ \nu^{-1}(z) \cdot z$,

$$\begin{aligned}
 (4.10) \quad \frac{\partial u}{\partial t} &= \left(-\frac{\partial X}{\partial t} \Big|_{\nu^{-1}(z)} - X_* \left(\frac{\partial}{\partial t} \nu^{-1}(z) \right) \right) \cdot z \\
 &= S \Big|_{\nu^{-1}(z)} = s \left(\frac{1}{r_1}, \dots, \frac{1}{r_n} \right).
 \end{aligned}$$

Here, $S = F_*(\tau)^{-\alpha}$, where F_* is the dual function of F , defined by $F_*(A) = F(A^{-1})^{-1}$. Equation (4.10) implies evolution equations for the speed and for the matrix \mathbf{v} :

$$\begin{aligned}
 (4.11) \quad \frac{\partial}{\partial t} F_*^{-\alpha} &= -\alpha F_*^{-(1+\alpha)} \dot{F}_*^{k\ell} \left(\frac{\partial u}{\partial t} \bar{g}_{k\ell} - \bar{\nabla}_k \bar{\nabla}_\ell \frac{\partial u}{\partial t} \right) \\
 &= \alpha F_*^{-(1+\alpha)} \dot{F}_*^{k\ell} \bar{\nabla}_k \bar{\nabla}_\ell F_*^{-\alpha} - \alpha F_*^{-(1+2\alpha)} \dot{F}_*(\bar{g}).
 \end{aligned}$$

The evolution equation for τ can be computed as follows: from the definition of τ ,

$$\begin{aligned} \frac{\partial}{\partial t} \tau_{ij} &= \frac{\partial u}{\partial t} \bar{g}_{ij} - \bar{\nabla}_i \bar{\nabla}_j \frac{\partial u}{\partial t} = -\bar{\nabla}_i \bar{\nabla}_j F_*^{-\alpha} + F_*^{-\alpha} \bar{g}_{ij} \\ &= \alpha \bar{\nabla}_i (F_*^{-(1+\alpha)} \nabla_j F_*) + F_*^{-\alpha} \bar{g}_{ij} \\ &= \alpha F_*^{-(1+\alpha)} \bar{\nabla}_i \bar{\nabla}_j F_* - \alpha(1 + \alpha) F_*^{-(2+\alpha)} \bar{\nabla}_i F_* \bar{\nabla}_j F_* + F_*^{-\alpha} \bar{g}_{ij} \\ &= \alpha F_*^{-(1+\alpha)} \dot{F}_*^{k\ell} \bar{\nabla}_i \bar{\nabla}_j \tau_{k\ell} + \alpha F_*^{-(1+\alpha)} \ddot{F}_*^{k\ell, mn} \bar{\nabla}_i \tau_{k\ell} \bar{\nabla}_j \tau_{mn} \\ &\quad - \alpha(1 + \alpha) F_*^{-(2+\alpha)} \bar{\nabla}_i F_* \bar{\nabla}_j F_* + F_*^{-\alpha} \bar{g}_{ij}. \end{aligned}$$

A Simons-like identity holds:

$$\begin{aligned} \bar{\nabla}_i \bar{\nabla}_j \tau_{k\ell} &= \bar{\nabla}_i \bar{\nabla}_k \tau_{j\ell} = \bar{\nabla}_k \bar{\nabla}_i \tau_{j\ell} + \bar{R}_{ikj}{}^p \tau_{p\ell} + \bar{R}_{ik\ell}{}^p \tau_{jp} \\ &= \bar{\nabla}_k \bar{\nabla}_\ell \tau_{ij} + \bar{g}_{kj} \tau_{i\ell} - \bar{g}_{ij} \tau_{k\ell} + \bar{g}_{k\ell} \tau_{ij} - \bar{g}_{i\ell} \tau_{kp}. \end{aligned}$$

On symmetrisation, this becomes

$$(4.12) \quad \bar{\nabla}_{(i} \bar{\nabla}_{j)} \tau_{k\ell} = \bar{\nabla}_{(k} \bar{\nabla}_{\ell)} \tau_{ij} - \bar{g}_{ij} \tau_{k\ell} + \bar{g}_{k\ell} \tau_{ij}.$$

This gives the following (noting that $\dot{F}(\tau) = F$):

$$\begin{aligned} (4.13) \quad \frac{\partial}{\partial t} \tau_{ij} &= \alpha F_*^{-(1+\alpha)} \dot{F}_*^{k\ell} \bar{\nabla}_k \bar{\nabla}_\ell \tau_{ij} + \alpha F_*^{-(1+\alpha)} \ddot{F}_*^{k\ell, mn} \bar{\nabla}_i \tau_{k\ell} \bar{\nabla}_j \tau_{mn} \\ &\quad - \alpha(1 + \alpha) F_*^{-(2+\alpha)} \bar{\nabla}_i F_* \bar{\nabla}_j F_* \\ &\quad + (1 - \alpha) F_*^{-\alpha} \bar{g}_{ij} + \alpha F_*^{-(1+\alpha)} \dot{F}_*(\bar{g}) \tau_{ij}. \end{aligned}$$

Again, these are the same as in the Euclidean case (see [AMZ, Lemma 10]), but with the signs of the curvature terms reversed.

In general, equation (4.10) holds on \mathbb{H}^n , but not on the quotient $\mathbb{H}^n/L(G)$ except in the standard case. However, equations (4.11)–(4.13) hold on the quotient since τ is invariant under isometries.

5. SHORT-TIME EXISTENCE OF SOLUTIONS

Although the map X is defined on the spatially non-compact space $\tilde{M} \times [0, T)$, we can use the co-compactness to produce a co-compact solution for a short time relatively easily (i.e., without concern for the behaviour at infinity).

We will show that the evolving hypersurfaces can be written for a short time as normal graphs over the initial hypersurface, and that the graph function is defined on M rather than \tilde{M} . Specifically, we construct a solution $X : \tilde{M} \times [0, \delta) \rightarrow \mathbb{R}^{n,1}$ of the form

$$(5.1) \quad X(\varphi(x, t), t) = X(x, 0) + u(\pi(x), t)v(x, 0),$$

for some $\delta > 0$, where $\pi : \tilde{M} \rightarrow M$ is the projection, $\varphi(\cdot, t)$ is a diffeomorphism of \tilde{M} for each t (commuting with the deck transformations on \tilde{M} corresponding to the covering map π), and $u : M \times [0, \delta) \rightarrow \mathbb{R}$ is a smooth function. We produce u as the solution of a scalar parabolic equation on the compact manifold M .

The geometric invariants of a hypersurface given as a normal graph of the form (5.1) can be computed directly as follows. Writing $X_0 = X(\cdot, 0)$ for the initial embedding and $\partial_i^0 = (\partial/\partial x^i)X_0$ for the initial coordinate tangent vectors in some local coordinates for \tilde{M} , the coordinate tangent vectors of the embedding $X^t(x) = X(\varphi(x, t), t)$ are given by

$$(5.2) \quad \partial_i^t := \frac{\partial}{\partial x^i}(X_0 + \tilde{u} \nu_0) = (\delta_i^p + \tilde{u}(h^0)_i^p) \partial_p^0 + \nabla_i \tilde{u} \nu_0,$$

where h^0 is the second fundamental form of X_0 , where ν_0 is the unit normal of X_0 , and where $\tilde{u} = u \circ \pi$. From this, the following expression for the unit normal ν^t of X^t follows:

$$(5.3) \quad \nu^t = \sigma^{-1}(\nu^0 + [(\mathcal{g}^0 + \tilde{u}h^0)^{-1}]^{k\ell} \nabla_\ell \tilde{u} \partial_k^0),$$

where

$$(5.4) \quad \sigma = \sqrt{1 + \mathcal{g}_0^{pq} [(I + \tilde{u}h^0)^{-1}]_p^k [(I + \tilde{u}h^0)^{-1}]_q^\ell \nabla_k \tilde{u} \nabla_\ell \tilde{u}}.$$

The induced metric is computed from (5.2):

$$(5.5) \quad g_{ij}^t = \partial_i^t \cdot \partial_j^t = (\delta_i^p + \tilde{u}(h^0)_i^p)(\delta_j^q + \tilde{u}(h^0)_j^q) \mathcal{g}_{pq}^0 - \nabla_i \tilde{u} \nabla_j \tilde{u}.$$

The second fundamental form is given as follows:

$$(5.6) \quad h_{ij}^t = \nu^t \cdot \nabla_i^0 \partial_j^t = \sigma^{-1} \left\{ \nabla_i^0 \nabla_j \tilde{u} + \tilde{u}(h^0)_i^p (h^0)_{pj} + (h^0)_{ij} - [(\mathcal{g}^0 + \tilde{u}(h^0))^{-1}]^{rp} \nabla_r \tilde{u} (\nabla_j \tilde{u} h_{ip}^0 + \nabla_i \tilde{u} h_{jp}^0 + u \nabla_p^0 h_{ij}^0) \right\}.$$

Differentiating equation (5.1) with respect to t , and taking the inner product with ν^0 , we obtain the following expression for the time derivative of \tilde{u} if X evolves according to (1.1):

$$(5.7) \quad \frac{\partial \tilde{u}}{\partial t} = \frac{\partial X}{\partial t} \cdot \nu^0 = \sigma S(h^t, g^t),$$

where g^t and h^t are given in terms of \tilde{u} by equations (5.5) and (5.6). In particular, this has the form $\partial \tilde{u} / \partial t = S[\nabla^0 \nabla \tilde{u}, \nabla u, u, x]$, where S is smooth and strictly monotone in the first argument, so this is a scalar fully nonlinear parabolic equation for \tilde{u} . Furthermore, the equation is invariant under the deck

transformations on \tilde{M} , and so is well defined as an equation on M . We define $\tilde{u}(x, t) = u(\pi(x), t)$, where u is the solution of the corresponding equation on the compact manifold M . We leave it to the reader to check that one can solve for the diffeomorphisms $\varphi(\cdot, t)$ to produce from this a co-compact solution of the original equation (1.1).

6. ESTIMATES ON SUPPORT FUNCTION AND SPEED
IN THE STANDARD CASE

6.1. Upper and lower bounds on the support function. From the evolution equation (4.10), we have the following. Let

$$\tilde{u}(t) = (u_0^{1+\alpha} + (1 + \alpha)t)^{1/(1+\alpha)}.$$

Then, $(d/dt)\tilde{u}(t) = \tilde{u}(t)^{-\alpha}$, so we can write

$$\begin{aligned} \frac{\partial}{\partial t}(u - \tilde{u}) &= F_*(\mathbf{r})^{-\alpha} - \tilde{u}^{-\alpha} = F_*(\mathbf{r})^{-\alpha} - F_*(\tilde{u}g)^{-\alpha} \\ &= a^{k\ell}(\bar{\nabla}_k \bar{\nabla}_\ell(u - \tilde{u}) - \bar{g}_{k\ell}(u - \tilde{u})), \end{aligned}$$

where $a^{k\ell} = \alpha \int_0^1 F_*^{-(1+\alpha)} \dot{F}_*^{k\ell} |_{(1-s)\mathbf{r}+s\tilde{u}g} ds$ is positive definite since F_* is increasing. It follows by the maximum principle that positivity or negativity of $u - \tilde{u}$ is preserved. If $u_- \leq u(x, 0) \leq u_+$ for all $x \in M$, then

$$(6.1) \quad (u_-^{1+\alpha} + (1 + \alpha)t)^{1/(1+\alpha)} \leq u(x, t) \leq (u_+^{1+\alpha} + (1 + \alpha)t)^{1/(1+\alpha)}.$$

In particular, this implies that $u((1 + \alpha)t)^{-1/(1+\alpha)}$ converges uniformly to 1 as $t \rightarrow \infty$ (if the solution exists for all time).

6.2. Upper and lower bounds on the speed. In this situation, there are upper and lower bounds on the speed that hold in great generality. It is useful to first rewrite the evolution for u as follows:

$$\frac{\partial}{\partial t}u = \alpha F_*^{-(1+\alpha)} \dot{F}_*^{k\ell} \bar{\nabla}_k \bar{\nabla}_\ell u - \alpha F_*^{-(1+\alpha)} \dot{F}_*(\bar{g})u + (1 + \alpha)F_*^{-\alpha}.$$

The latter equation together with equation (4.11) yields an evolution equation for uF_*^α :

$$\begin{aligned} \frac{\partial}{\partial t}(uF_*^\alpha) &= \alpha F_*^{-(1+\alpha)} \dot{F}_*^{k\ell} \bar{\nabla}_k \bar{\nabla}_\ell (uF_*^\alpha) \\ &\quad - 2\alpha^2 F_*^{-(2+\alpha)} \dot{F}_*^{k\ell} \bar{\nabla}_k F_* \bar{\nabla}_\ell (uF_*^\alpha) + (1 + \alpha). \end{aligned}$$

Thus, upper and lower bounds on $uF_*^\alpha - (1 + \alpha)t$ are preserved. We then let $c_- = \inf_{M \times \{0\}}(uF_*^\alpha)$ and $c_+ = \sup_{M \times \{0\}}(uF_*^\alpha)$. Then, for any $t > 0$ in the interval of existence,

$$(6.2) \quad c_- + (1 + \alpha)t \leq uF_*^\alpha(x, t) \leq c_+ + (1 + \alpha)t,$$

for all $x \in M$. When combined with (6.1), this gives strong upper and lower bounds on F_* , which imply in particular that $F_*((1 + \alpha)t)^{-1/(1+\alpha)}$ converges uniformly to 1 as $t \in \infty$ (again, if we assume long-time existence).

7. BOUNDS ON SUPPORT FUNCTION AND SPEED IN THE GENERAL CASE

In the general case, upper and lower bounds on the speed can still be obtained very generally. In order to derive such bounds, it is necessary first to obtain some control on the support function.

Lemma 7.1. *There exist $r > 0$ and $R > 0$ such that, for any $z_0 \in \mathbb{H}^n$, there exist $p_-(z_0)$ and $p_+(z_0)$ for which*

$$-p_-(z_0) \cdot z + R \leq u(z, 0) \leq -p_+(z_0) \cdot z + r$$

for every $z \in \mathbb{H}^n$, with equality holding throughout when $z = z_0$.

Proof. By compactness of M , there exist $0 < r \leq R$ such that we have that $|v|^2/R \leq h(v, v) \leq |v|^2/r$ for all $v \neq 0$ in TM (equivalently, we have $r\bar{g}(v, v) \leq \mathfrak{r}(v, v) \leq R\bar{g}(v, v)$ for all $v \in TM$). Given $z_0 \in \mathbb{H}^n$, we choose $p_+(z_0) = \bar{X}(z_0, 0) - rz_0$ and $p_-(z_0) = \bar{X}(z_0, 0) - Rz_0$. We shall also let $u_+(z) = u(z, 0) + p_+(z_0) \cdot z - r$ and $u_-(z) = u(z, 0) + p_-(z_0) \cdot z - R$. Then,

$$\begin{aligned} u_+(z) &= -\bar{X}(z, 0) \cdot z + (\bar{X}(z_0, 0) - rz_0) \cdot z - r \\ &= (\bar{X}(z_0, 0) - \bar{X}(z, 0) + r(z - z_0)) \cdot z. \end{aligned}$$

In particular, $u_+(z_0) = 0$ and $\partial_i u_+|_{z_0} = 0$, and $\mathfrak{r}_{ij}[u_+] = \mathfrak{r}_{ij}[u] - r\bar{g}_{ij} \geq 0$. Thus, along any geodesic γ from z_0 in \mathbb{H}^n ,

$$\frac{d}{ds} \left(\cosh^2 s \frac{d}{ds} \left(\frac{u_+}{\cosh(s)} \right) \right) = -\mathfrak{r}[\gamma', \gamma'] \cosh(s) \leq 0,$$

from which it follows that $u_+ \leq 0$ everywhere. Similarly, $u_- \geq 0$, as required. \square

Corollary 7.2. *With $r, R, p_-(z_0)$, and $p_+(z_0)$ as in Lemma 7.1, for every $z \in \mathbb{H}^n$ and every $t \in [0, T)$,*

$$\begin{aligned} -p_-(z_0) \cdot z + (R^{1+\alpha} + (1 + \alpha)t)^{1/(1+\alpha)} \\ \leq u(z, t) \leq -p_+(z_0) \cdot z + (r^{1+\alpha} + (1 + \alpha)t)^{1/(1+\alpha)}. \end{aligned}$$

Proof. The proof uses the maximum principle, but some care is required to control the behaviour near infinity. For any $\varepsilon > 0$ and fixed $z_0 \in \mathbb{H}^n$, let $v_\varepsilon(z, t) = u(z, t) + p_- \cdot z - \varepsilon z \cdot z_0$. Then, $v_\varepsilon(z, 0) \geq R$ for all z , and $v_\varepsilon(z, 0) \geq -\varepsilon z \cdot z_0 \rightarrow \infty$ as $z \rightarrow \infty$ in \mathbb{H}^n . Also, for each $\varepsilon > 0$,

$$\frac{\partial}{\partial t} v_\varepsilon = F_*(\tau[v_\varepsilon])^{-\alpha}.$$

In particular, while the solution remains smooth, $(\partial/\partial t)v_\varepsilon$ is bounded, and so $v(z, t)_\varepsilon \rightarrow \infty$ as $z \rightarrow \infty$ in \mathbb{H}^n for each t in the interval of existence. It follows that $v(\cdot, t)$ attains an interior minimum, and the maximum principle implies that $v_\varepsilon \geq (R^{1+\alpha} + (1 + \alpha)t)^{1/(1+\alpha)}$. Taking $\varepsilon \rightarrow 0$ gives the desired lower bound on $u(z, t)$. The proof of the upper bound is similar. \square

The speed bounds can now be proved by using an identity related to one employed for mean curvature flow by Smoczyk [S] (see also [AMZ, Theorem 14]).

Proposition 7.3. *If $u(z, 0) + p \cdot z \geq 0$ for all $z \in \mathbb{H}^n$, then*

$$u(z, t) - (1 + \alpha)F_*(z, t)^{-\alpha} + p \cdot z \geq 0 \quad \text{for all } z \in \mathbb{H}^n \text{ and } t \geq 0.$$

If $u(z, 0) + p \cdot z \leq 0$ for all $z \in \mathbb{H}^n$, then

$$u(z, t) - (1 + \alpha)F_*(z, t)^{-\alpha} + p \cdot z \leq 0 \quad \text{for all } z \in \mathbb{H}^n \text{ and } t \geq 0.$$

Proof. If $w = u + p \cdot z - (1 + \alpha)tF_*^{-\alpha}$ (for any p), then

$$\frac{\partial}{\partial t} w = \alpha F_*^{-(1+\alpha)} \dot{F}_*^{k\ell} (\bar{\nabla}_k \bar{\nabla}_\ell w - \bar{g}_{k\ell} w).$$

In the first case of the proposition, $w(z, 0) \geq 0$ for all $z \in \mathbb{H}^n$. Choose any fixed $z_0 \in \mathbb{H}^n$, and let $w_\varepsilon = w + \varepsilon z_0 \cdot z$ with $\varepsilon > 0$, so that $w_\varepsilon > 0$ and $w_\varepsilon(z, 0) \rightarrow \infty$ as $z \rightarrow \infty$ in \mathbb{H}^n . Since $F_*^{-\alpha}$ is bounded in the interval of existence (by compactness of M), $w_\varepsilon(z, t) \rightarrow \infty$ as $z \rightarrow \infty$ in \mathbb{H}^n for each $t > 0$ in the interval of existence. In particular, w_ε attains an interior minimum, and the maximum principle applies to prove that this minimum remains non-negative. Thus, $w_\varepsilon(z, t) \geq 0$ for each $\varepsilon > 0$, and sending ε to zero yields $w(z, t) \geq 0$ for all $z \in \mathbb{H}^n$ and all $t \geq 0$ in the interval of existence. The proof of the second case is similar. \square

Corollary 7.4. *For every $z \in \mathbb{H}^n$ and every $t > 0$ in the interval of existence of the solution,*

$$\begin{aligned} & \frac{(R^{1+\alpha} + (1 + \alpha)t)^{1/(1+\alpha)} - R}{(1 + \alpha)t} \\ & \leq F_*^{-\alpha}(z, t) \leq \frac{(r^{1+\alpha} + (1 + \alpha)t)^{1/(1+\alpha)} + R - r}{(1 + \alpha)t}. \end{aligned}$$

Proof. To prove the lower bound, fix $z_0 \in \mathbb{H}^n$. By Lemma 7.1, there is a point $p_+(z_0)$ such that $u(z, 0) + p_+(z_0) \cdot z \leq r$ for all $z \in \mathbb{H}^n$. Let $p = p_+(z) - rz_0$. Then,

$$u(z, 0) + p \cdot z = u(z, 0) + p_+(z_0) \cdot z - rz_0 \cdot z \leq r - rz \cdot z_0 \leq 0,$$

with equality holding for $z = z_0$. Hence, Proposition 7.3 applies to give that $u(z, t) + p \cdot z - (1 + \alpha)tF_*^{-\alpha} \leq 0$ for each $z \in \mathbb{H}^n$ and $t > 0$ in the interval of existence. Applying this for $z = z_0$ gives

$$F_*^{-\alpha}(z_0, t) \geq \frac{u(z_0, t) - u(z_0, 0)}{(1 + \alpha)t}.$$

Finally, Corollary 7.2 implies

$$u(z_0, t) - u(z_0, 0) \geq (R^{1+\alpha} + (1 + \alpha)t)^{1/(1+\alpha)} - R.$$

It follows that

$$F_*^{-\alpha}(z_0, t) \geq \frac{(R^{1+\alpha} + (1 + \alpha)t)^{1/(1+\alpha)} - R}{(1 + \alpha)t},$$

as claimed.

Proof of the upper bound follows a similar pattern. Fixing $z_0 \in \mathbb{H}^n$ as before, Lemma 7.1 implies the existence of $p_-(z_0)$ such that $u(z, 0) + p_-(z_0) \cdot z \geq R$, with equality for $z = z_0$. Proposition 7.3 implies that

$$u(z, t) + p_-(z_0) \cdot z - (1 + \alpha)tF_*^{-\alpha}(z, t) \geq 0 \quad \text{for all } z \text{ and } t \geq 0,$$

so that $F_*^{-\alpha}(z, t) \leq u(z, t) + p_-(z_0) \cdot z / ((1 + \alpha)t)$. Corollary 7.2 then gives $u(z, t) + p_+(z_0) \cdot z \leq (r^{1+\alpha} + (1 + \alpha)t)^{1/(1+\alpha)}$. Also note that, from Lemma 7.1, $u(z_0, 0) + p_+(z_0) \cdot z_0 = r$ and $u(z_0, 0) + p_-(z_0) \cdot z_0 = R$, and therefore $(p_+(z_0) - p_-(z_0)) \cdot z_0 = r - R$. It follows that

$$\begin{aligned} u(z_0, t) + p_-(z_0) \cdot z_0 &= u(z_0, t) + p_+(z_0) \cdot z_0 + (p_-(z_0) - p_+(z_0)) \cdot z_0 \\ &\leq (r^{1+\alpha} + (1 + \alpha)t)^{1/(1+\alpha)} + R - r, \end{aligned}$$

and hence

$$F_*^{-\alpha}(z_0, t) \leq \frac{(r^{1+\alpha} + (1 + \alpha)t)^{1/(1+\alpha)} + R - r}{(1 + \alpha)t}. \quad \square$$

The upper speed bound provided by Corollary 7.4 degenerates as $t \rightarrow 0$. However, since the short-time existence result provides an upper bound on the speed for small times, we have bounds on the speed for all times:

$$F_*^{-\alpha} \leq (C + (1 + \alpha)t)^{-\alpha/(1+\alpha)}.$$

8. BOUNDS ON PRINCIPAL CURVATURES

Different arguments are required to establish bounds on principal curvatures for the various classes of flows considered. The following subsections treat each case of Theorem 1.1 in turn.

8.1. Bounds on the principal curvatures: f concave and $f|_{\partial\Gamma_+} = 0$. If f is concave and $0 < \alpha \leq 1$, then equation (4.8) implies

$$(8.1) \quad \nabla_t h_{ij} \leq \alpha F^{\alpha-1} \dot{F}^{k\ell} \nabla_k \nabla_\ell h_{ij} - (1 - \alpha) F^\alpha h_{ij}^2 - \alpha F^{\alpha-1} \dot{F} (A^2) h_{ij}.$$

Combining this with the evolution equation (4.9) for F^α gives

$$(8.2) \quad \nabla_t \left(\frac{h_{ij}}{F^\alpha} \right) \leq \alpha F^{\alpha-1} \dot{F}^{k\ell} \nabla_k \nabla_\ell \left(\frac{h_{ij}}{F^\alpha} \right) + 2\alpha^2 F^{\alpha-2} \dot{F}^{k\ell} \nabla_k F \nabla_\ell \left(\frac{h_{ij}}{F^\alpha} \right) - (1 - \alpha) F^{2\alpha} \left(\frac{h_{ij}}{F^\alpha} \right)^2.$$

The speed bounds of Corollary 7.4 therefore imply that the maximum of

$$(1 + t)^{(1-\alpha)/(\alpha+1)} F^{-\alpha} \kappa_{\max}$$

does not increase if it is large, and it follows that there exists a constant C such that

$$\kappa_{\max} \leq C F^\alpha (1 + t)^{(\alpha-1)/(1+\alpha)} \leq C (1 + t)^{-1/(1+\alpha)}.$$

Therefore, $\kappa_i/F \leq \tilde{C}$ where \tilde{C} is a constant independent of t . Since κ_i/F is homogeneous of degree zero and approaches infinity on the boundary of Γ_+ , there exists $\varepsilon > 0$ such that $\kappa_{\min} \geq \varepsilon \kappa_{\max}$. Therefore,

$$\kappa_{\min} \geq \varepsilon (1 + t)^{-1/(1+\alpha)}$$

by the lower speed bound. It follows that there exist constants $0 < C_- < C_+$ such that

$$(8.3) \quad C_- (1 + t)^{-1/(1+\alpha)} \leq \kappa_i(x, t) \leq C_+ (1 + t)^{-1/(1+\alpha)}$$

for all $x \in M$ and t in the interval of existence.

8.2. Bounds on the principal curvatures: F and F_* concave. The key step in cases (2) and (3) of Theorem 1.1 is an estimate on the smallest principal curvature (or largest principal radius), which is closely related to the estimate proved in [AMZ, Lemma 11] in the case $\alpha = 1$ in the Euclidean setting.

We consider the evolution equation for the tensor $G_{ij} = F_*^{-\alpha} v_{ij}$, and apply the tensor maximum principle (see [Ha1, Theorem 9.1], and particularly the

refinement proved in [A5, Theorem 3.2]) to bound the maximum eigenvalue. Equations (4.13) and (4.11) yield the following evolution equation for G_{ij} :

$$\begin{aligned}
 (8.4) \quad \frac{\partial}{\partial t} G_{ij} &= \alpha F_*^{-(1+\alpha)} \dot{F}_*^{k\ell} \bar{\nabla}_k \bar{\nabla}_\ell G_{ij} - 2\alpha F_*^{-(1+\alpha)} \dot{F}_*^{k\ell} \bar{\nabla}_k F_*^{-\alpha} \bar{\nabla}_\ell \tau_{ij} \\
 &\quad + \alpha F_*^{-(1+2\alpha)} \ddot{F}_*^{k\ell mn} \bar{\nabla}_i \tau_{k\ell} \bar{\nabla}_j \tau_{mn} \\
 &\quad - \alpha(1 + \alpha) F_*^{-(2+2\alpha)} \bar{\nabla}_i F_* \bar{\nabla}_j F_* + (1 - \alpha) F_*^{-2\alpha} \bar{g}_{ij} \\
 &= \alpha F_*^{-\alpha} \dot{F}_*^{k\ell} \bar{\nabla}_k \bar{\nabla}_\ell G_{ij} - 2\alpha F_*^{-1} \dot{F}_*^{k\ell} \bar{\nabla}_k F_*^{-\alpha} \bar{\nabla}_\ell G_{ij} \\
 &\quad + 2\alpha \tau_{ij} F_*^{-1} \dot{F}_*^{k\ell} \bar{\nabla}_k F_*^{-\alpha} \bar{\nabla}_\ell F_*^{-\alpha} \\
 &\quad + \alpha F_*^{-(1+2\alpha)} \ddot{F}_*^{k\ell mn} \bar{\nabla}_i \tau_{k\ell} \bar{\nabla}_j \tau_{mn} \\
 &\quad - \alpha(1 + \alpha) F_*^{-(2+2\alpha)} \bar{\nabla}_i F_* \bar{\nabla}_j F_* + (1 - \alpha) F_*^{-2\alpha} \bar{g}_{ij}.
 \end{aligned}$$

At a point where a maximum of the largest eigenvalue of G_{ij} occurs, choose an orthonormal basis for $T\mathbb{H}^n$ that diagonalises τ (hence also G) with eigenvalues in decreasing order, so that the maximum eigenvalue of G_{ij} is $\tau_1 F_*^{-\alpha}$. Note that \dot{F}_* is then also diagonal at the maximum point, and the second derivatives \ddot{F}_* can be written in terms of the derivatives of f_* by using the expression in [A1, Equation 2.23]. The speed bound of Corollary 7.4 and the result of [A5, Theorem 3.2] imply that there exists C such that the maximum eigenvalue of $G_{ij} - C(1 + t)^{(1-\alpha)/(1+\alpha)} \bar{g}_{ij}$ is decreasing, provided the following holds at the maximum point:

$$\begin{aligned}
 Q := &\alpha F_*^{-(1+2\alpha)} \ddot{F}_*^{k\ell mn} \bar{\nabla}_1 \tau_{k\ell} \bar{\nabla}_1 \tau_{mn} - \alpha(1 + \alpha) F_*^{-(2+2\alpha)} (\bar{\nabla}_1 F_*)^2 \\
 &+ 2\alpha \tau_1 F_*^{-1} \sum_k \dot{F}_*^k (\bar{\nabla}_k F_*^{-\alpha})^2 \\
 &- 2\alpha F_*^{-(1+\alpha)} \sup_\Gamma \dot{F}_*^k (2\Gamma_k^p \bar{\nabla}_k G_{1p} - (\Gamma_k^p)^2 (G_{pp} - G_{11})) \leq 0.
 \end{aligned}$$

The optimal choice of Γ is obtained by completing the square, giving

$$\begin{aligned}
 Q = &\alpha F_*^{-(1+2\alpha)} \ddot{F}_*^{k\ell mn} \bar{\nabla}_1 \tau_{k\ell} \bar{\nabla}_1 \tau_{mn} - \alpha(1 + \alpha) F_*^{-(2+2\alpha)} (\bar{\nabla}_1 F_*)^2 \\
 &+ 2\alpha \tau_1 F_*^{-1} \sum_k \dot{F}_*^k (\bar{\nabla}_k F_*^{-\alpha})^2 + 2\alpha F_*^{-(1+\alpha)} \sum_{k,p} \dot{F}_*^k \frac{(\bar{\nabla}_k G_{1p})^2}{G_{pp} - G_{11}}.
 \end{aligned}$$

Multiplying through by $\alpha^{-1} F_*^{1+2\alpha}$ gives the following:

$$\begin{aligned}
 \alpha F_*^{1+2\alpha} Q = &\sum_{k,\ell} \dot{f}_*^{k\ell} \bar{\nabla}_1 \tau_{kk} \bar{\nabla}_1 \tau_{\ell\ell} + 2 \sum_{k>\ell} \frac{\dot{f}_*^k - \dot{f}_*^\ell}{\tau_k - \tau_\ell} \bar{\nabla}_1 \tau_{k\ell}^2 - (1 + \alpha) \frac{(\bar{\nabla}_1 F_*)^2}{F_*} \\
 &+ 2\tau_1 \sum_k \dot{f}_*^k \left(\frac{\bar{\nabla}_k F_*^{-\alpha}}{F_*^{-\alpha}} \right)^2 + 2F_*^\alpha \sum_{k,p} \dot{f}_*^k \frac{(\bar{\nabla}_k G_{1p})^2}{F_*^{-\alpha} (\tau_p - \tau_1)}.
 \end{aligned}$$

We note that $\bar{\nabla}_k G_{1p} = \bar{\nabla}_k(F_*^{-\alpha} \mathbf{r}_{1p}) = F_*^{-\alpha} \bar{\nabla}_k \mathbf{r}_{1p}$ for $p \neq 1$ at the maximum point, while $\bar{\nabla}_k G_{11} = 0$. This gives the following:

$$\begin{aligned} & \sum_{k,\ell} \ddot{f}_*^{k\ell} \bar{\nabla}_1 \mathbf{r}_{kk} \bar{\nabla}_1 \mathbf{r}_{\ell\ell} + 2 \sum_{k>1} \frac{\dot{f}_*^k - \dot{f}_*^1}{\mathbf{r}_k - \mathbf{r}_1} (\bar{\nabla}_k \mathbf{r}_{11})^2 + 2 \sum_{k>\ell>1} \frac{\dot{f}_*^k - \dot{f}_*^\ell}{\mathbf{r}_k - \mathbf{r}_\ell} (\bar{\nabla}_1 \mathbf{r}_{k\ell})^2 \\ & \quad - \frac{(1 + \alpha)F_*}{\alpha \mathbf{r}_1^2} (\bar{\nabla}_1 \mathbf{r}_{11})^2 + 2 \sum_k \frac{\dot{f}_*^k}{\mathbf{r}_1} (\bar{\nabla}_k \mathbf{r}_{11})^2 + 2 \sum_k \sum_{p>1} \frac{\dot{f}_*^k}{\mathbf{r}_p - \mathbf{r}_1} (\bar{\nabla}_1 \mathbf{r}_{kp})^2 \\ & = \sum_{k,\ell} \ddot{f}_*^{k\ell} \bar{\nabla}_1 \mathbf{r}_{kk} \bar{\nabla}_1 \mathbf{r}_{\ell\ell} + 2 \sum_{k>1} \frac{\dot{f}_*^k - \dot{f}_*^1}{\mathbf{r}_k - \mathbf{r}_1} (\bar{\nabla}_k \mathbf{r}_{11})^2 + 2 \sum_{k>\ell>1} \frac{\dot{f}_*^k - \dot{f}_*^\ell}{\mathbf{r}_k - \mathbf{r}_\ell} (\bar{\nabla}_1 \mathbf{r}_{k\ell})^2 \\ & \quad - \frac{(1 + \alpha)F_*}{\alpha \mathbf{r}_1^2} (\bar{\nabla}_1 \mathbf{r}_{11})^2 + 2 \sum_k \frac{\dot{f}_*^k}{\mathbf{r}_1} (\bar{\nabla}_k \mathbf{r}_{11})^2 \\ & \quad + 2 \sum_{k>1} \frac{\dot{f}_*^1}{\mathbf{r}_k - \mathbf{r}_1} (\bar{\nabla}_k \mathbf{r}_{11})^2 + 2 \sum_{k,p>1} \frac{\dot{f}_*^k}{\mathbf{r}_p - \mathbf{r}_1} (\bar{\nabla}_1 \mathbf{r}_{kp})^2. \end{aligned}$$

The concavity of f_* implies that the terms on the first line are all non-positive. The remaining terms are all manifestly non-positive except for the fifth term. Consider the terms which involve $(\bar{\nabla}_k \mathbf{r}_{11})^2$ with $k > 1$: these are

$$\begin{aligned} & 2 \sum_{k>1} \frac{\dot{f}_*^k - \dot{f}_*^1}{\mathbf{r}_k - \mathbf{r}_1} (\bar{\nabla}_k \mathbf{r}_{11})^2 + 2 \sum_{k>1} \frac{\dot{f}_*^1}{\mathbf{r}_k - \mathbf{r}_1} (\bar{\nabla}_k \mathbf{r}_{11})^2 + 2 \sum_{k>1} \frac{\dot{f}_*^k}{\mathbf{r}_1} (\bar{\nabla}_k \mathbf{r}_{11})^2 \\ & = 2 \sum_{k>1} \dot{f}_*^k \left(\frac{1}{\mathbf{r}_k - \mathbf{r}_1} + \frac{1}{\mathbf{r}_1} \right) (\bar{\nabla}_k \mathbf{r}_{11})^2 = 2 \sum_{k>1} \dot{f}_*^k \frac{\mathbf{r}_k}{\mathbf{r}_1(\mathbf{r}_k - \mathbf{r}_1)} (\bar{\nabla}_k \mathbf{r}_{11})^2. \end{aligned}$$

This leaves only the terms involving $\bar{\nabla}_1 \mathbf{r}_{11}$ to consider: these are (omitting those in the \dot{f}_* term, which are non-positive)

$$\begin{aligned} & \left(-\frac{1 + \alpha}{\alpha} \frac{f_*}{\mathbf{r}_1^2} + 2 \frac{\dot{f}_*^1}{\mathbf{r}_1} \right) (\bar{\nabla}_1 \mathbf{r}_{11})^2 \\ & = \left(-\frac{1 + \alpha}{\alpha} \sum_{k>1} \frac{\dot{f}_*^k \mathbf{r}_k}{\mathbf{r}_1^2} + \frac{\alpha - 1}{\alpha} \frac{f_*}{\mathbf{r}_1} \right) (\bar{\nabla}_1 \mathbf{r}_{11})^2. \end{aligned}$$

In cases (2) and (3), this term is again non-positive since $\alpha \leq 1$. Therefore, the maximum principle applies to show that $\mathbf{r}_{11} F_*^{-\alpha} \leq C(1+t)^{(1-\alpha)/(1+\alpha)}$, and hence $\mathbf{r}_{11} \leq C(1+t)^{1/(1+\alpha)}$.

In case (2), the argument of Section 8.1 gives $\kappa_{\max} \leq C(1+t)^{-1/(1+\alpha)}$. The argument above gives $\mathbf{r}_{\max} \leq C(1+t)^{1/(1+\alpha)}$, so that $\kappa_{\min} \geq C(1+t)^{-1/(1+\alpha)}$. Thus, all principal curvatures of the hypersurface are comparable to $(1+t)^{-1/(1+\alpha)}$, and an estimate of the form (8.3) holds.

8.3. Bounds on the principal curvatures: F_* concave and $F_*|_{\partial\Gamma_+} = 0$. In case (3), the estimate $\tau_{\max} \leq C(1+t)^{1/(1+\alpha)}$ holds by the argument of the previous section. Since the speed bounds also give $F_* \geq C(1+t)^{1/(1+\alpha)}$, it follows that $\tau_{\max}/F_* \leq C$. This quantity is scaling invariant, and approaches infinity on the boundary of the positive cone; thus, it defines a properly contained sub-cone of the positive cone. It follows that $\tau_{\max}/\tau_{\min} \leq C$ for some C , so all principal curvatures are comparable to each other and to F_* . In particular, an estimate of the form (8.3) holds.

8.4. Bounds on the principal curvatures: $n = 2$. The argument to control principal curvatures in this case is closely related to the main result of [A6]: the key step is to compute the evolution equation for $G = (\kappa_2 - \kappa_1)^2/(\kappa_2 + \kappa_1)^2$. The equations are identical, except that the sign is reversed in the “reaction” terms involving curvature alone. Thus, the calculation in Section 5 of [A6] gives

$$\begin{aligned} \frac{\partial G}{\partial t} &= \dot{S}^{ij}\nabla_i\nabla_j G + (\dot{G}^{ij}\ddot{S}^{klmn} - \dot{S}^{ij}\ddot{G}^{klmn})\nabla_i h_{kl}\nabla_j h_{mn} \\ &\quad - \dot{G}(h)\dot{S}(h^2) + (\alpha - 1)S\dot{G}(h^2). \end{aligned}$$

Since G is homogeneous of degree zero, the term involving $\dot{G}(h)$ is zero. The reaction terms are therefore

$$-(1 - \alpha)S\dot{G}(h^2) = -4(1 - \alpha)S\frac{\kappa_1\kappa_2(\kappa_2 - \kappa_1)^2}{(\kappa_2 + \kappa_1)^3} \leq 0.$$

The gradient terms $Q = (\dot{G}\ddot{S} - \dot{S}\ddot{G})(\nabla h, \nabla h)$ are the same as in the situation of [A6], and so are given at the maximum point by

$$\begin{aligned} Q &= S\dot{G}^1 \left(\frac{\alpha(\alpha - 1)}{\kappa_2^2} + \frac{2\alpha}{\kappa_2(\kappa_2 - \kappa_1)} \right) (\nabla_1 h_{22})^2 \\ &\quad + S\dot{G}^2 \left(\frac{\alpha(\alpha - 1)}{\kappa_1^2} - \frac{2\alpha}{\kappa_1(\kappa_2 - \kappa_1)} \right) (\nabla_2 h_{11})^2. \end{aligned}$$

If $\kappa_2 > \kappa_1$, then \dot{G}^2 is non-negative, while the terms in the second bracket are both non-positive; and \dot{G}^1 is non-positive, while the terms in the first bracket are

$$\begin{aligned} &\frac{\alpha}{\kappa_2^2(\kappa_2 - \kappa_1)} ((\alpha - 1)(\kappa_2 - \kappa_1) + 2\kappa_2) \\ &= \frac{\alpha}{\kappa_2^2(\kappa_2 - \kappa_1)} ((\alpha + 1)\kappa_2 - (1 - \alpha)\kappa_1) \leq 0, \end{aligned}$$

since $\kappa_2 \geq \kappa_1$. Hence, $Q \leq 0$ and the maximum of G is non-increasing. This implies a bound on the ratio $\kappa_{\max}/\kappa_{\min}$, and combining this with the speed bound gives estimates above and below on principal curvatures of the form (8.3).

8.5. Bounds on the principal curvatures: the case $\alpha > 1$. In case (5), the computation from Section 8.2 can again be applied: the argument there shows that the maximum eigenvalue of $F_*^{-\alpha} \tau_{ij} - C(1+t)^{-(\alpha-1)/(\alpha+1)} \bar{g}_{ij}$ is non-increasing, provided we can show that the terms involving $\bar{\nabla}_1 \tau_{11}$ are non-positive (the condition $\alpha \leq 1$ was not used in controlling any of the other terms). These terms are given by

$$\left(-\frac{1+\alpha}{\alpha} \sum_{k>1} \frac{f_*^k \tau_k}{\tau_1^2} + \frac{\alpha-1}{\alpha} \frac{f_*^1}{\tau_1} \right) (\bar{\nabla}_1 \tau_{11})^2.$$

The assumption of case (5) is that $f^i \kappa_i > f^j \kappa_j$ for $\kappa_i > \kappa_j$. This can be translated to an inequality between the derivatives of f_* , as follows: since $f_*(\tau_1, \dots, \tau_n) = f(\tau_1^{-1}, \dots, \tau_n^{-1})^{-1}$, the derivative is given by $f_*^i = f^{-2} \dot{f}^i \tau_i^{-2}$. Therefore, we have $\dot{f}_*^i \tau_i = f^{-2} \dot{f}^i \tau_i^{-1} = f^{-2} \dot{f}^i \kappa_i$. If $\tau_i > \tau_j$, then $\kappa_i < \kappa_j$ and so

$$\dot{f}_*^i \tau_i - \dot{f}_*^j \tau_j = f^2 (\dot{f}^i \kappa_i - \dot{f}^j \kappa_j) \leq 0.$$

In particular, this implies that $\dot{f}_*^k \tau_k \geq \dot{f}_*^1 \tau_1$ for all $k > 1$. Substituting this in the terms above gives the following upper bound:

$$\left(-(n-1) \frac{1+\alpha}{\alpha} + \frac{\alpha-1}{\alpha} \right) \frac{\dot{f}_*^1}{\tau_1} (\bar{\nabla}_1 \tau_{11})^2.$$

This is non-positive, so we obtain $\tau_{\max} \leq C(1+t)^{1/(1+\alpha)}$, applying the maximum principle. The upper bound on κ_{\max} follows as in case (3) since f_* is zero on the boundary of the positive cone, and the estimate (8.3) holds as before.

9. HIGHER REGULARITY AND LONG-TIME EXISTENCE

In this section, we establish bounds on higher derivatives of curvature and all derivatives of the embedding for a solution of (1.1), and deduce that the solutions exist for all time. The estimates will also be important in the next section in proving convergence to hyperboloids.

The basic regularity result is as follows.

Proposition 9.1. *Let $X : \tilde{M} \times [0, a] \rightarrow \mathbb{R}^{n,1}$ be a co-compact solution of (1.1) satisfying the hypotheses of Theorem 1.1, and suppose there exists $C > 0$ such that*

$$C^{-1} \leq \kappa_i(x, t) \leq C$$

for all $x \in \tilde{M}$ and $t \in [0, a]$, and $i \in \{1, \dots, n\}$, Then, there exists a constant \tilde{C}_k for each $k \geq 1$, depending only on a, C , and F , such that on $\tilde{M} \times [a/2, a]$, the following estimates hold:

$$\tilde{C}_1^{-1} \bar{g} \leq g_{(x,t)} \leq C_1 \bar{g} \quad \text{and} \quad |\bar{\nabla}^{(k)} X|_{\bar{g}} \leq \tilde{C}_k$$

for each $k \geq 2$, where \bar{g} and $\bar{\nabla}$ are the metric and connection induced on \tilde{M} the embedding at time $a/2$.

Proof. The argument is similar to that given in [AMZ, Theorem 5], and employs the same strategy as [Ha1] and [Hu1], beginning with estimates on higher derivatives of curvature, then deducing estimates on derivatives of the embedding itself. The details differ slightly in the various cases of Theorem 1.1, and we indicate the steps in each case briefly.

In cases (2), (3), and (5), where f_* is concave, the assumed bounds on curvature make the evolution equation (4.10) uniformly parabolic. The estimates of Evans [E4] or Krylov [K] apply to give Hölder continuity of the second derivatives. Schauder estimates then apply to the evolution equations for higher derivatives to produce $C^{k,\alpha}$ estimates for the solution away from $t = 0$. In particular, there are estimates on all derivatives of curvature.

The bounds on the derivatives of the embedding follow. First, the evolution equation (4.1) for the metric g and the bounds on principal curvatures show that the metrics at different times are all comparable. The evolution equation for the k th derivative of the embedding involves only derivatives of curvature up to order k , and so gives bounds on the k th derivative for each $k \geq 2$ (see, e.g., the argument given in [AB, Theorem 3]).

In case (1) where f is concave, Hölder continuity of second derivatives can be deduced by writing the evolving hypersurfaces as graphs: the graph function then satisfies a uniformly parabolic fully nonlinear evolution equation with concave dependence on the second derivatives from the evolution equation, and the Evans-Krylov estimates apply. The higher regularity again follows using Schauder estimates, and the argument to bound the derivatives of the embedding is unchanged.

Finally, in case (4) the special regularity results of [A2] apply (to either equation (4.10) or the graphical evolution equation) to give Hölder continuity of second derivatives. The rest of the argument is the same as for the other cases. \square

Corollary 9.2. *Under the assumptions of Theorem 1.1, the solution X exists on $\tilde{M} \times [0, \infty)$.*

Proof. If the maximal time of existence T is finite, the estimates above give uniform bounds on all derivatives of the embedding, and also on the time derivatives of these. In particular, the embeddings X_t are Cauchy in C^k for each k as t approaches T , and hence converge in C^∞ to an embedding X_T . Applying the short-time existence result to X_T extends the solution to a longer time interval, contradicting the assumption that T is the maximal time of existence. \square

10. UNIFORM HIGHER REGULARITY AND CONVERGENCE

Scaling-invariant bounds on all higher derivatives of the curvature for the evolving hypersurfaces can now be established, by using the scaling properties of the equation. Let $X : \tilde{M} \times [0, \infty) \rightarrow \mathbb{R}^{n,1}$ be a solution of (1.1). For fixed $\tau \in (0, T)$, define $X_\tau : \tilde{M} \times [0, \infty) \rightarrow \mathbb{R}^{n,1}$ by

$$X_\tau(z, t) = \tau^{-1/(1+\alpha)} X(z, \tau t).$$

Then, X_τ is again a solution of equation (1.1), which is co-compact with the inclusion of G in $\text{Isom}_+(\mathbb{R}^{n,1})$ given as in equation (3.1) by

$$g_\tau(z) = L(g)z + \tau^{-1/(1+\alpha)}b(g).$$

The bounds established in the previous sections imply the following bounds for X_τ : the bounds on the support function from Corollary 7.2 give

$$\begin{aligned} & -\frac{p_-(z_0) \cdot z}{\tau^{1/(1+\alpha)}} + \left(\frac{R^{1+\alpha}}{\tau} + (1 + \alpha)t\right)^{1/(1+\alpha)} \\ & \leq u_\tau(z, t) \leq -\frac{p_+(z_0) \cdot z}{\tau^{1/(1+\alpha)}} + \left(\frac{r^{1+\alpha}}{\tau} + (1 + \alpha)t\right)^{1/(1+\alpha)}. \end{aligned}$$

The speed bounds of Corollary 7.4 give the following for $t \geq 0$:

$$\begin{aligned} (10.1) \quad & \frac{\left(\frac{R^{1+\alpha}}{\tau} + (1 + \alpha)t\right)^{1/(1+\alpha)}}{(1 + \alpha)t} - \frac{R}{\tau^{1/(1+\alpha)}} \\ & \leq F_\tau(x, t)^\alpha \leq \frac{\left(\frac{r^{1+\alpha}}{\tau} + (1 + \alpha)t\right)^{1/(1+\alpha)}}{(1 + \alpha)t} + \frac{R - r}{\tau^{1/(1+\alpha)}}. \end{aligned}$$

In particular, the speed F_τ^α has positive bounds above and below independent of τ on $\tilde{M} \times (a, b)$ (for τ not close to zero) for any $0 < a < b < \infty$, and converges to $((1 + \alpha)t)^{-\alpha/(1+\alpha)}$ as τ approaches infinity. Similarly, the principal curvature bound (8.3) implies that the principal curvatures of X_τ have positive upper and lower bounds independent of τ on $\tilde{M} \times (a, b)$ (away from $\tau = 0$).

Bounds in $C^{0,\beta}$ on the second fundamental form now follow, by slightly different arguments for each case of Theorem 1.1. In case (1), F^α is concave, and the Krylov-Safonov estimates can be applied (e.g., to the scalar equation given by describing the evolving hypersurface as a graph). In cases (2), (3), and (5), $f_*^{-\alpha}$ is a concave function of the principal radii of curvature, and the Krylov-Safonov estimate can be applied to equation (4.10). Finally, in case (4), the estimates of [A2] can be applied.

Schauder estimates now imply bounds on all higher derivatives of the curvature, and it follows that the time of existence is infinite (by the same argument as presented in [Ha1] or [Hu1]).

Finally, the estimate of Corollary 7.2 implies that $X_\tau(\cdot, \cdot)$ converges locally in Hausdorff distance to the expanding unit future hyperboloid as $\tau \rightarrow \infty$. Standard interpolation inequalities imply that the principal curvatures of these hypersurfaces converge locally uniformly to 1, and the co-compactness and higher regularity then imply that the rescaled hypersurfaces converge in C^k for every k to the

hyperboloid (an argument similar to that in [Hu1] must be used to show that the map $X_T(\cdot, 0)$ converges, as well as the image hypersurface). This completes the proof of Theorem 1.1.

11. ELEMENTARY SYMMETRIC FUNCTIONS IN CASE (5)

For application of Theorem 1.1, it is useful to know which of the commonly used examples are covered in the various cases. Many examples satisfying the conditions of cases (1)–(4) are included in [AMZ]. Here, we will establish the most important examples covered by case (5) of the theorem: namely, the flows with speed $S = E_k^{\alpha/k}$, where E_k is the k th elementary symmetric function of the principal curvatures. Note that in this case, $f = E_k^{1/k}$, and $f_* = (E_n/E_{n-k})^{1/k}$, which is concave and zero on the boundary of the positive cone. The following proposition therefore shows that these flows satisfy the requirements of case (5) of Theorem 1.1 for $\alpha > 1$:

Proposition 11.1. *Let E_k be the k th elementary symmetric function, defined on the positive cone Γ_+ . If $f = E_k^{1/k}$, then $f^i \kappa_i \geq f^j \kappa_j$ for $\kappa_i > \kappa_j$.*

Proof. By definition,

$$E_k(\kappa_1, \dots, \kappa_n) = \sum_{\substack{A \subset \{1, \dots, n\} \\ |A|=k}} \prod_{\ell \in A} \kappa_\ell.$$

The derivative with respect to κ_i can be computed as follows: E_k is a linear function of κ_i , with a factor κ_i appearing in each term in the sum corresponding to a set A with $i \in A$:

$$\dot{E}_k^i = \sum_{\substack{A \subset \{1, \dots, n\} \\ |A|=k, i \in A}} \prod_{\ell \in A \setminus \{i\}} \kappa_\ell.$$

Therefore,

$$\dot{E}_k^i \kappa_i = \sum_{\substack{A \subset \{1, \dots, n\} \\ |A|=k, i \in A}} \prod_{\ell \in A} \kappa_\ell.$$

If $i \neq j$, then this can be expanded as follows:

$$\begin{aligned} \dot{E}_k^i \kappa_i &= \sum_{\substack{A \subset \{1, \dots, n\} \\ |A|=k, i, j \in A}} \prod_{\ell \in A} \kappa_\ell + \sum_{\substack{A \subset \{1, \dots, n\} \\ |A|=k, i \in A, j \notin A}} \prod_{\ell \in A} \kappa_\ell \\ &= \kappa_i \kappa_j \left(\sum_{\substack{B \subset \{1, \dots, n\} \setminus \{i, j\} \\ |B|=k-2}} \prod_{\ell \in B} \kappa_\ell \right) + \kappa_i \left(\sum_{\substack{C \subset \{1, \dots, n\} \setminus \{i, j\} \\ |C|=k-1}} \prod_{\ell \in C} \kappa_\ell \right). \end{aligned}$$

It follows that

$$\dot{E}_k^i \kappa_i - \dot{E}_k^j \kappa_j = (\kappa_i - \kappa_j) \left(\sum_{\substack{C \subset \{1, \dots, n\} \setminus \{i, j\} \\ |C|=k-1}} \prod_{\ell \in C} \kappa_\ell \right).$$

In particular, if $\kappa_i > \kappa_j$, then $\dot{E}_k^i \kappa_i - \dot{E}_k^j \kappa_j \geq 0$. Finally, if $f = E_k^{1/k}$, then $\dot{f}^i = (1/k)E_k^{-(k-1)/k} \dot{E}_k^i$ for each i , and the result follows. \square

The reader may check that the flows defined by $f = \prod_{k=1}^n E_k^{\alpha_k/k}$ satisfy the conditions of case (5) of Theorem 1.1 for any $\alpha_1, \dots, \alpha_n \geq 0$ with $\sum_{k=1}^n \alpha_k = 1$.

12. APPLICATION TO THE CROSS-CURVATURE FLOW

In [CH], Chow and Hamilton introduce an interesting fully nonlinear heat flow for negatively (or positively) curved metrics on a three dimensional manifold, called the “cross curvature flow” (hereafter abbreviated “XCF”). We discuss the formulation of this below. The discussion in [CH] relates principally to the negatively curved case, and the authors give strong indications that the XCF should deform any negatively curved metric on a compact three-manifold to a hyperbolic metric, modulo scaling. This result is not yet known, and our purpose here is to show that this does indeed hold for metrics that are locally isometrically embeddable in Minkowski space $\mathbb{R}^{3,1}$.

Chow and Hamilton define the cross curvature tensor as follows. First, the Einstein tensor is defined by $P_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$. This is the tensor with eigenvalues equal to the principal sectional curvatures, and can alternatively be defined by

$$(12.1) \quad P^{ij} = \frac{1}{4} \mu^{iab} \mu^{jcd} R_{abcd},$$

where μ is the (dual) volume form. The cross curvature tensor (which Chow and Hamilton denote by h_{ij} , but that we denote X_{ij} to avoid confusion with the second fundamental form) is defined if P is invertible by

$$(12.2) \quad X_{ij} = (\det P)(P^{-1})_{ij}.$$

Alternatively, the following expression defines X_{ij} without the assumption that P is invertible:

$$(12.3) \quad X_{ij} = \frac{1}{2} \mu_{iab} \mu_{jcd} P^{ac} P^{bd}.$$

In particular, if we work in a frame where P is diagonal, then so is X :

$$(12.4) \quad P = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix} \implies X = \begin{bmatrix} \mu\nu & 0 & 0 \\ 0 & \lambda\nu & 0 \\ 0 & 0 & \lambda\mu \end{bmatrix}.$$

Our interest here is in the case where the metric g is embeddable (locally) with some second fundamental form h_{ij} , in either Euclidean space \mathbb{R}^4 or Minkowski space $\mathbb{R}^{3,1}$. Then, the Gauss equations give the following expression for the curvature tensor, where we take $\gamma = 1$ for embedding into \mathbb{R}^4 , and $\gamma = -1$ for embedding into $\mathbb{R}^{3,1}$:

$$(12.5) \quad R_{abcd} = \gamma(h_{ac}h_{bd} - h_{bc}h_{ad}).$$

From equation (12.1), we then have

$$P^{ij} = \frac{\gamma}{4} \mu^{iab} \mu^{jcd} (h_{ac}h_{bd} - h_{bc}h_{ad}) = \frac{\gamma}{2} \mu^{iab} \mu^{jcd} h_{ac}h_{bd}.$$

In particular, in a frame that diagonalises h , we also have P diagonal (hence, also X):

$$(12.6) \quad h = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \implies P = \gamma \begin{bmatrix} bc & 0 & 0 \\ 0 & ac & 0 \\ 0 & 0 & ab \end{bmatrix}.$$

By equation (12.4) we have (calculating in the same frame)

$$(12.7) \quad X = \begin{bmatrix} a^2bc & 0 & 0 \\ 0 & ab^2c & 0 \\ 0 & 0 & abc^2 \end{bmatrix} = Kh,$$

where $K = \det h = abc$.

The cross-curvature flow is defined by

$$(12.8) \quad \frac{\partial}{\partial t} g_{ij} = -2\gamma X_{ij}.$$

In particular, in the case where the metric is locally embeddable, this becomes

$$\frac{\partial}{\partial t} g_{ij} = -2\gamma Kh_{ij}.$$

Comparison with equation (4.1) shows that the metric induced by a solution of Gauss curvature flow in Minkowski space satisfies the cross-curvature flow with $\gamma = -1$, and Lemma 9 of [AMZ] shows that the induced metric of a solution of Gauss curvature flow in \mathbb{R}^4 satisfies the cross curvature flow with $\gamma = 1$.

Buckland [B] established that the cross-curvature flow with $\gamma = 1$ has a unique solution for any positively curved smooth initial metric on a compact three-manifold, and the cross-curvature flow with $\gamma = -1$ has a unique solution for any initial metric with negative sectional curvatures on a compact three-manifold. It follows that in the locally embeddable case, the solution of XCF with

a negatively curved initial metric is precisely the induced metric of the solution of Gauss curvature flow we construct in this paper. In particular, we proved that the evolving hypersurfaces under Gauss curvature flow approach the hyperboloid modulo scaling, and therefore the metric approaches a hyperbolic metric modulo scaling.

The corresponding result in the case $\gamma = 1$ is not known: it is not known whether convex compact hypersurfaces become spherical under the Gauss curvature flow in \mathbb{R}^{n+1} for $n \geq 3$ (although in the case of antipodally symmetric hypersurfaces, this follows from the results of Firey [F] together with the methods of [A4]). See [A3] for the case $n = 2$.

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BEN ANDREWS:
Mathematical Sciences Center
Tsinghua University
P.R. China
and Mathematical Sciences Institute
Australia National University
Australia
E-MAIL: Ben.Andrews@anu.edu.au

XUZHONG CHEN:
College of Mathematics and Econometrics AND
Hunan University
Changsha, P.R. China
E-MAIL: chenxuzhong@hnu.edu.cn

Department of Mathematics
East China Normal University
Shanghai, P.R. China

HANLONG FANG:
Department of Mathematics
Rutgers University USA
E-MAIL: longgao1106@126.com

JAMES MCCOY:
Institute for Mathematics and its Applications
School of Mathematics and Applied Statistics
University of Wollongong
Australia
E-MAIL: jamesm@uow.edu.au

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