Ruin probabilities for the generalised Ornstein-Uhlenbeck process and the structure of the upper and lower bounds

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Declaration

The work in this thesis is my own except where otherwise stated. Chapter 2 is based on a paper written jointly with Allan Sly.

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Abstract

For a bivariate Lévy process (ξ, η), the generalised Ornstein-Uhlenbeck (GOU) process $V = (V_t)_{t \geq 0}$, is defined by

$$V_t := e^{\xi_t} \left(V_0 + \int_0^t e^{-\xi_{s-}} \mathrm{d}\eta_s \right),$$

where V_0 is a random variable independent of (ξ, η). It is closely related to the stochastic integral process $Z = (Z_t)_{t \geq 0}$ defined by

$$Z_t := \int_0^t e^{-\xi_{s-}} \mathrm{d}\eta_s.$$

We examine the infinite horizon ruin probability for V, and the associated behaviour of Z. In particular, we define conditions under which V has zero probability of ruin, and conditions under which ruin is certain. These conditions are stated in terms of the canonical characteristics of the bivariate Lévy process and reveal the effect of the dependence relationship between ξ and η. We also present an in-depth examination of the structure of the upper and lower bound of V.

Contents

Notation and terminology

Abbreviations and acronyms

a.s.	almost surely in probability
iid	independent and identically distributed
iff	if and only if
càdlàg	right continuous and left limit exists
OU	Ornstein-Uhlenbeck process
GOU	generalised Ornstein-Uhlenbeck process
Cov	covariance
SDE	stochastic differential equation
sup	supremum
inf	infimum
max	maximum
min	minimum
lim	limit

Notation

$P(\Lambda)$	probability of event Λ
$E(X)$	expected value of random variable X
P_X	distribution of X

$\epsilon(X)$	stochastic exponential process of a semi-martingale X
$[X,Y]$	quadratic variation process of semi-martingales X and Y
$[X,Y]^c$	path-by-path continuous part of $[X,Y]$
Π_X	Lévy measure of a Lévy process X
$N_{X,t}$	random jump measure of a Lévy process X
$a \wedge b$	minimum of a and b
$a \vee b$	maximum of a and b
Y^T	process Y stopped at T, so $Y_t^T := Y_{t \wedge T}$
$f^-(x)$	negative part of f, so $f^-(x) := -(f(x)) \vee 0$
$f^+(x)$	positive part of f, so $f^+(x) := f(x) \vee 0$
$f(t-)$	left limit of f at t, so $f(t-) := \lim_{h \downarrow 0} f(t-h)$
$f(t+)$	right limit of f at t, so $f(t+) := \lim_{h \downarrow 0} f(t+h)$
$\Delta f(t)$	jump of f at t, so $\Delta f(t) := f(t) - f(t-)$
$=_D$	equality in distribution
$\rightarrow_D$	convergence in distribution
$\rightarrow_P$	convergence in probability
$\mathbb{N}$	set of positive integers
$\mathbb{R}$	set of real numbers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}^d$	$d-$dimensional Euclidean space
$\langle x, y \rangle$	inner product in Euclidean space, so $\langle x, y \rangle := \sum_{i=1}^n x_i y_i$
$\|x\|$	norm in Euclidean space, so $\|x\| := \left(\sum_{i=1}^n x_i^2\right)^{1/2}$
$\bar{\Lambda}$	closure of set Λ in Euclidean space
1_Λ	indicator function of the Λ

$\Lambda \setminus \Gamma$	$\{x : x \in \Lambda, x \notin \Gamma\}$
$f(x) = O(\phi(x))$	real valued function f is big order ϕ, so there exists constants c and x_0 such that$\lvert f(x)\rvert < c\phi(x)$ for some constant c, for all $x > x_0$
$f(x) = o(\phi(x))$	f is little order ϕ, so $\lim_{x\to\infty} \frac{f(x)}{\phi(x)} = 0$
$f(x) \sim \phi(x)$	f is asymptotically equivalent to ϕ, so $\lim_{x\to\infty} \frac{f(x)}{\phi(x)} = 1$
$\mu_1 * \mu_2$	convolution of finite measures μ_1 and μ_2
(ξ, η)	a bivariate Lévy process
V	the GOU $V_t := e^{\xi_t}\left(V_0 + \int_0^t e^{-\xi_{s-}} \mathrm{d}\eta_s\right)$
Z	the stochastic integral process $Z_t := \int_0^t e^{-\xi_{s-}} \mathrm{d}\eta_s$
$\psi(z)$	infinite horizon ruin probability of V

General

For a real function f, increasing means $f(s) \leq f(t)$ for $s < t$ and decreasing means $f(s) \geq f(t)$ for $s < t$. When we wish to exclude equality we say strictly increasing and strictly decreasing.

For a real number x, positive means $x > 0$ and negative means $x < 0$.

The integral $\int_a^b$ is interpreted as $\int_{[a,b]}$ and the integral $\int_{a+}^b$ as $\int_{(a,b]}$.

The integral of a $\mathbb{R}^d$-valued function or the expectation of a random variable in $\mathbb{R}^d$ is the vector in $\mathbb{R}^d$ with componentwise integrals or expectations.

A distribution is spread out if it has a convolution power with an absolutely continuous component.

Chapter 1

The generalised Ornstein-Uhlenbeck process

1.1 Introduction

This thesis is mainly concerned with the infinite horizon ruin probability of the generalised Ornstein-Uhlenbeck process. We now define our objects of interest. Let $(\xi_t, \eta_t)_{t\geq 0}$ be a bivariate Lévy process with $\xi_0 = \eta_0 = 0$, adapted to a filtered complete probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{0\leq t\leq\infty}, P)$ satisfying the "usual hypotheses" (see Protter [60] p.3), where ξ and η are not identically zero. Assume the σ-algebra $\mathscr{F}$ and the filtration $\mathbb{F}$ are generated by (ξ, η), that is, $\mathscr{F} := \sigma\left((\xi,\eta)_t : 0 \leq t < \infty\right)$ and $\mathscr{F}_t := \sigma\left((\xi,\eta)_s : 0 \leq s \leq t\right)$. The generalised Ornstein-Uhlenbeck (GOU) process $V = (V_t)_{t\geq 0}$, where V_0 is a random variable independent of (ξ, η), is defined by

$$V_t := e^{\xi_t}\left(V_0 + \int_0^t e^{-\xi_{s-}} \mathrm{d}\eta_s\right). \tag{1.1}$$

It is closely related to the stochastic integral process $Z = (Z_t)_{t\geq 0}$ defined by

$$Z_t := \int_0^t e^{-\xi_{s-}} \mathrm{d}\eta_s. \tag{1.2}$$

For a Lebesgue set Λ, let $T_{z,\Lambda}$ denote the hitting time of Λ for V when $V_0 = z$. Thus, $T_{z,\Lambda} := \inf\{t > 0 : V_t \in \Lambda | V_0 = z\}$, where $T_{z,\Lambda} := \infty$ whenever $V_t \notin \Lambda$ for all $t > 0$ and $V_0 = z$. Define the infinite horizon ruin probability for the GOU by

$$\psi(z) := P\left(\inf_{t>0} V_t < 0 | V_0 = z\right) = P\left(\inf_{t>0} Z_t < -z\right) = P\left(T_{z,(-\infty,0)} < \infty\right). \tag{1.3}$$

Stochastic integrals and stochastic differential equations are interpreted according to Protter [60] and all required stochastic integral results are referenced. General probability, measure theory and stochastic process results, as found in Chung [13], Billingsley [5], and Kallenberg [34], are used consistently, often without reference when basic. More specialised theory required throughout the bulk of the thesis is placed in the present chapter. The main Lévy process texts we have used are Sato [62], Bertoin [3] and Cont and Tankov [14].

Chapter 1 contains an overview of Lévy process theory and a comprehensive introduction to the GOU, including an extensive literature review. All known ruin probability results for the GOU are described in detail, and analysed. Chapter 2 contains new results on conditions for zero ruin for the GOU. Chapter 3 contains new results on conditions for certain ruin for the GOU, and an analysis of the structure of the upper and lower bounds. A large amount of material is placed in the Appendix. Appendix A contains an alternative, less sophisticated method for proving conditions for zero ruin for a special case of the GOU. Appendix B and C contain proofs of certain statements made in Chapters 2 and 3. Appendix D states simplified versions of the major results, which hold when ξ and η are independent. Appendix E discusses some of the conditions and assumptions made in existing papers, and relates them to the results in Chapters 2 and 3. Appendix F states and proves some asymptotic results on the behaviour of the GOU.

1.2 Lévy processes

Note that this is not a full account, but only a list of basic properties which will be needed in this thesis.

Definition 1.1 (d-dimensional Lévy process). An $\mathbb{R}^d$-valued stochastic process $X := (X_t)_{t\geq 0})$ on a probability space $(\Omega, \mathscr{F}, P)$ is called a Lévy process on $\mathbb{R}^d$ if it possesses the following five properties:

1. It has independent increments: for every $n \geq 1$ and $0 \leq t_0 < t_1 < \ldots < t_n$, the random variables X_{t_0}, $X_{t_1} - X_{t_0}$,...,$X_{t_n} - X_{t_{n-1}}$ are independent;

2. It has stationary increments: the distribution of $X_{t+h} - X_t$ does not dependent on t;

3. It is stochastically continuous: $\forall \epsilon > 0$, $\lim_{h\to 0} P\left(|X_{t+h} - X_t| \geq \epsilon\right) = 0$.

4. It starts at the origin: $X_0 = 0$ a.s.

5. It has càdlàg (right-continuous and left limit) sample paths: $\forall \omega \in \Omega\ X_t(\omega)$ is right continuous with left limits as a function of t.

A Lévy process is a time-homogenous strong Markov process. It is a consequence of property 3 that the probability of a jump $\Delta X_t := X_t - X_{t-}$ occurring at a fixed time t is zero, that is, given $t \geq 0$, $P(\omega : \Delta X_t(\omega) \neq 0) = 0$. Thus jumps occur at random times. It is a consequence of property 5 that a.s. each sample path $X_t(\omega)$ has, at most, countably many jumps, whilst given $c > 0$, $X_t(\omega)$ has finitely many jumps of size $|\Delta X_t(\omega)| \geq c$ on a compact time interval. If X and Y are independent Lévy processes on $\mathbb{R}^d$ defined on the same probability space $(\Omega, \mathscr{F}, P)$ then $X + Y$ is a Lévy process on the same space and the probability of X and Y jumping together is 0, that is,

$$P(\omega : \exists t > 0 \text{ such that } \Delta X_t(\omega) \neq 0, \Delta Y_t(\omega) \neq 0) = 0.$$

The mean of X_t is a well-defined vector $E(X_t) \in \mathbb{R}^d$ iff $E(|X_t|) < \infty$. For all $t > 0$, $E(X_t) = tE(X_1)$. If X has bounded jumps, that is, $\sup_{t>0} |\Delta X_t| \leq c$ for some real constant c, then $E(|X_1|^n) < \infty$ for all integers n and so X has finite moments of all orders.

A Lévy process X is a kind of continuous time analogue of a random walk. Given any time interval $\Delta > 0$ we can define the discrete time process $S_n := X_{n\Delta} = \sum_{i=1}^{n-1} Y_i$ for $Y_i = X_{(i+1)\Delta} - X_{i\Delta}$. Since the Y_i are iid (independent and identically distributed) random variables, S_n is a random walk. A probability distribution F on $\mathbb{R}^d$ is called *infinitely divisible* if for any integer n there exists iid random variables $Y_1, \ldots, Y_n$ such that $Y_1 + \ldots + Y_n$ has distribution F.

Proposition 1.2 (infinite divisibility and Lévy processes). *If X is a Lévy process on $\mathbb{R}^d$ then for every $t > 0$ the distribution of X_t is infinitely divisible. If F is an infinitely divisible distribution on $\mathbb{R}^d$ then there exists, uniquely in distribution, a Lévy process X such that $X_1 =_D F$.*

The three fundamental examples of Lévy processes are the Poisson process, the compound Poisson process and Brownian motion.

Definition 1.3 (Poisson process). Let $(\tau_i)_{i\in\mathbb{N}}$ be a sequence of iid exponential random variables with parameter λ and let $T_n = \sum_{i=1}^{n} \tau_i$ with $T_0 = 0$ a.s. With $1_{t\geq T_n}$ denoting the indicator function, the process $M := (M_t)_t \geq 0$ defined by

$$M_t := \sum_{n\geq 1} 1_{t\geq T_n}$$

is called a Poisson process with intensity $\lambda := E(M_1)$.

Definition 1.4 (Compound Poisson process). Let M be a Poisson process with intensity λ and let $(Y_i)_{i\in\mathbb{N}}$ be an iid sequence of random vectors in $\mathbb{R}^d$ with common distribution F, independent of M. The process $X := (X_t)_{t\geq 0}$ defined by $X_t := \sum_{i=1}^{M_t} Y_i$ is called the compound Poisson process with intensity λ and jump size distribution F.

The Poisson process is a Lévy process on $\mathbb{R}$ with piecewise constant paths increasing by jumps of size 1. For any $t > 0$, M_t has a Poisson distribution with parameter λt. The compound Poisson process is a Lévy process on $\mathbb{R}^d$ with piecewise constant paths.

Definition 1.5 (Brownian motion). A stochastic process $B := (B_t)_{t\geq 0}$ on $\mathbb{R}^d$ is called d-dimensional Brownian motion if it has independent increments, continuous paths $B_t(\omega)$ for all $\omega \in \Omega$ and for any $0 < s < t$, $B_t - B_s$ is a Gaussian random variable with mean zero and covariance matrix $(t-s)A$ for a deterministic matrix A.

When we set $B_0 = 0$ a.s, Brownian motion is a Lévy process. We now examine the jumps of a Lévy process X. Since paths of a Lévy process are càdlàg the only type of discontinuities possible are jump discontinuities of form $\Delta X_t = X_t - X_{t-}$. Let Λ be a Borel set in $\mathbb{R}^d$ and define

$$N_t^\Lambda(\omega) := \sum_{0<s\leq t, \Delta X_s \neq 0} 1_\Lambda(\Delta X_s),$$

which counts the number of jumps of the sample path $X_t(\omega)$ occurring in time $(0, t]$ with size in Λ. When $0 \notin \bar{\Lambda}$, where $\bar{\Lambda}$ is the closure of Λ, $(N_t^\Lambda)_{t\geq 0}$ is a Poisson process. It can be written in the form $N_t^\Lambda = \sum_{n=1}^\infty 1_{t \geq T_\Lambda^n}$ where $T_\Lambda^1 = \inf\{t > 0 : \Delta X_t \in \Lambda\}$ and $T_\Lambda^n = \inf\{t > T_\Lambda^{n-1} : \Delta X_t \in \Lambda\}$ for integers n. For any Borel set Λ define $\Pi(\Lambda) := E\left(N_1^\Lambda\right)$, the intensity of the Poisson process. Note that $\Pi(\Lambda)$ is the expected number of jumps which occur up to time 1, with size in Λ. If $0 \in \bar{\Lambda}$ then $\Pi(\Lambda)$ and $N_t^\Lambda(\omega)$ may be infinite, since X can have an infinite number of small jumps in any time interval.

Proposition 1.6. *The set function $\Pi(\Lambda)$ is a measure on $\mathbb{R}^d$ and a σ-finite measure on $\mathbb{R}^d \setminus \{0\}$. For each fixed (t, ω) the set function $\Lambda \to N_t^\Lambda(\omega)$ is a measure on $\mathbb{R}^d$ and a σ-finite measure on $\mathbb{R}^d \setminus \{0\}$.*

Definition 1.7 (Random measure, Lévy measure). Let X be a Lévy process on $\mathbb{R}^d$. The set of measures $N_t(\omega)$ on $\mathbb{R}^d$ defined by $N_t^\Lambda(\omega) := \sum_{0<s\leq t} 1_\Lambda(\Delta X_s)$ is called the random measure of X. The measure Π on $\mathbb{R}^d$ defined by $\Pi(\Lambda) := E\left(N_1^\Lambda\right)$ is called the Lévy measure of X.

When necessary we will denote the random and Lévy measures of X by N_X and Π_X respectively. If $\Pi((-1,1)^d) = \infty$ then X is called an *infinite activity* Lévy process and if $\Pi((-1,1)^d) < \infty$ then X is *finite activity.* A compound Poisson process with intensity λ is a finite activity Lévy process with $\Pi(\mathbb{R}^d) = \lambda$. Conversely, for a Borel set Λ in $\mathbb{R}^d$ define J_t^Λ to be the sum of all jumps of X with size in Λ up to time t, namely

$$J_t^\Lambda := \sum_{0<s\leq t} \Delta X_s 1_\Lambda(\Delta X_s).$$

If $0 \notin \bar{\Lambda}$, or if $\Pi(\Lambda) < \infty$, then J^Λ is finite a.s. and is a compound Poisson process with intensity $\Pi(\Lambda)$. It can be written as $J_t^\Lambda = \int_\Lambda z N_t(\cdot, \mathrm{d}z) = \sum_{i=1}^{N_t^\Lambda} Y_i$ for some iid sequence $(Y_i)_{i\in\mathbb{N}}$ with common distribution Y, where $Y \in \Lambda$ and is independent of N^Λ. For disjoint Borel sets Λ_1 and Λ_2 with finite Lévy measure, the processes J^{Λ_1} and J^{Λ_2} are independent.

Proposition 1.8 (Lévy-Itô decomposition). *Let X be a Lévy process on $\mathbb{R}^d$ with random measure N and Lévy measure Π. Then there exists a vector $\gamma \in \mathbb{R}^d$ and a Brownian motion B on $\mathbb{R}^d$ with covariance matrix Σ such that*

$$X_t = \gamma t + B_t + X_t^1 + X_t^2 \tag{1.4}$$

where

$$X_t^1(\omega) = \int_{\{|z|<1\}} z\left(N_t(\omega, \mathrm{d}z) - t\Pi(\mathrm{d}z)\right), \tag{1.5}$$

$$X_t^2(\omega) = \int_{\{|z|\geq 1\}} z N_t(\omega, \mathrm{d}z) = J_t^{\{|z|\geq 1\}} \tag{1.6}$$

and

$$\gamma = E\left(X_1 - \int_{|z|\geq 1} z N_1(\cdot, \mathrm{d}z)\right). \tag{1.7}$$

The four processes in (1.4) are mutually independent Lévy processes, X_t^1 is a mean-zero martingale, and Π satisfies

$$\int_{\{|z|\leq 1\}} |z|^2 \Pi(\mathrm{d}z) < \infty \text{ and } \int_{\{|z|\geq 1\}} \Pi(\mathrm{d}z) < \infty.$$

The triplet (γ, Σ, Π) is called the *characteristic triplet* of X and uniquely determines the distribution of the process. We call Σ the *Gaussian covariance matrix* and γ the *adjustment coefficient.* Sample paths of X are continuous a.s. iff $\Pi = 0$. Note that the process X^1 has jumps with size less than 1, whilst X^2 has jumps of size greater than or equal to 1. The choice of 1 as the cut-off value is arbitrary, however the value of γ depends on this choice. Also note that $E(X_1)$ may not exist as a finite or well-defined vector in $\mathbb{R}^d$, so γ cannot be split into two separate expectations in the equation (1.7). If X is a univariate Lévy process we denote the variance of the Brownian component by σ^2 rather than the matrix Σ.

Proposition 1.9 (Lévy-Khinchin representation). *Let X be a Lévy process on $\mathbb{R}^d$ with characteristic triplet (γ, Σ, Π). Then for $z \in \mathbb{R}^d$*

$$E\left(e^{i\langle z, X_t\rangle}\right) = e^{t\psi(z)}$$

where

$$\psi(z) = -\frac{1}{2}\langle z, \Sigma z\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle 1_{\{|x|\le 1\}}\right) \Pi(\mathrm{d}x) + i\langle \gamma, z\rangle$$

Proposition 1.10. *Let X be a Lévy process on $\mathbb{R}^d$ and let $\Lambda \subset \mathbb{R}^d$ be Borel. If a real function f on $\mathbb{R}^d$ satisfies $\int_\Lambda |f(z)|\Pi(\mathrm{d}z) < \infty$ then*

$$E\left(\int_\Lambda f(z)N_t(\cdot, \mathrm{d}z)\right) = t\int_\Lambda f(z)\Pi(\mathrm{d}z).$$

The *total variation* of an $\mathbb{R}^d$-valued function over the interval $[a, b]$ is defined by

$$V_f([a,b]) = \sup \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|,$$

where the supremum is taken over all finite partitions $a = t_0 < t_1 < \cdots < t_n = b$. A Lévy process X is said to be of *finite variation* if, with probability 1, its sample paths $X_t(\omega)$ are of finite variation on $[0, t]$ for every $t > 0$.

Proposition 1.11 (Finite variation Lévy process). *A Lévy process X is of finite variation iff its characteristic triplet satisfies $\Sigma = 0$ and*

$$\int_{|z|\le 1} |z|\Pi(\mathrm{d}z) < \infty. \tag{1.8}$$

Corollary 1.12. *If X is a finite variation Lévy process on $\mathbb{R}^d$ then*

$$X_t = dt + \int_{\mathbb{R}^d} zN_t(\cdot, \mathrm{d}z) = dt + \sum_{0<s\le t} \Delta X_s$$

where

$$d = \gamma - \int_{|z|<1} z\Pi(\mathrm{d}z) = E\left(X_1 - \int_{\mathbb{R}^d} zN_1(\cdot, \mathrm{d}z)\right) \tag{1.9}$$

is called the drift vector.

By definition, a finite activity Lévy process has a Lévy measure satisfying (1.8) and hence has a finite variation jump process, however the converse does not hold in general.

In this thesis we deal mainly with bivariate and univariate Lévy processes. We make some comments specific to these situations. Throughout this thesis we let (ξ,η) be a bivariate Lévy process and denote its characteristic triplet by $((\tilde{\gamma}_\xi,\tilde{\gamma}_\eta),\Sigma_{\xi,\eta},\Pi_{\xi,\eta})$. The characteristic triplets of ξ and η as one-dimensional Lévy processes are denoted by $(\gamma_\xi,\sigma_\xi^2,\Pi_\xi)$ and $(\gamma_\eta,\sigma_\eta^2,\Pi_\eta)$ respectively, where

$$\Pi_\xi(\Gamma)=\Pi_{\xi,\eta}(\Gamma\times\mathbb{R}) \quad \text{and} \quad \Pi_\eta(\Gamma)=\Pi_{\xi,\eta}(\mathbb{R}\times\Gamma) \tag{1.10}$$

for Γ a Borel subset of $\mathbb{R}$ with $0\notin\bar{\Gamma}$,

$$\gamma_\xi=\tilde{\gamma}_\xi+\int_{\{|x|<1\}\cap\{x^2+y^2\geq 1\}} x\Pi_{\xi,\eta}(\mathrm{d}(x,y)), \tag{1.11}$$

$$\gamma_\eta=\tilde{\gamma}_\eta+\int_{\{|y|<1\}\cap\{x^2+y^2\geq 1\}} y\Pi_{\xi,\eta}(\mathrm{d}(x,y)), \tag{1.12}$$

and σ_ξ^2 and σ_η^2 are the upper left and lower right entries respectively, in the matrix $\Sigma_{\xi,\eta}$. Analogous to (1.4), we can decompose ξ into the sum of four independent Lévy processes

$$\xi_t=\gamma_\xi t+B_{\xi,t}+\xi_t^1+\xi_t^2, \tag{1.13}$$

where

$$\xi_t^1=\int_{|x|<1} x\left(N_{\xi,t}(\cdot,\mathrm{d}x)-t\Pi_\xi(\mathrm{d}x)\right)\ ,\ \xi_t^2=\int_{|x|\geq 1} xN_{\xi,t}(\cdot,\mathrm{d}x),$$

and

$$\gamma_\xi=E\left(\xi_1-\int_{|x|\geq 1} xN_{\xi,1}(\cdot,\mathrm{d}x)\right),$$

and similarly for η. Note that the processes ξ^1 and ξ^2 are not the same as the first coefficient of the processes $(\xi,\eta)^1$ and $(\xi,\eta)^2$ from the bivariate Lévy-Itô decomposition, as stated in Proposition 1.8.

The quadratic variation of two one-dimensional semimartingales X and Y on the same probability space is denoted by $[X,Y]=([X,Y]_t)_{t\geq 0}$. Definitions and properties are found in Protter [60]. The path-by-path continuous part of $[X,Y]$ is denoted by $[X,Y]^c$ and if $[X,X]^c=0$ then X is called *quadratic pure jump*. If ξ is a Lévy process on $\mathbb{R}$ with Brownian motion component B_ξ then $\xi_t-B_{\xi,t}$ is quadratic pure jump. The function $(X,Y)\to[X,Y]$ is bilinear and symmetric, if X and Y are independent Lévy processes on $\mathbb{R}$ then $[X,Y]$ is identically zero, whilst if (B, C) is 2-dimensional Brownian motion then $[B, C]_t = \mathrm{Cov}(B_t, C_t) =: \sigma_{B,C}$ where Cov denotes the covariance. By Theorem 28 of Protter [60], if (ξ, η) is a bivariate Lévy process then

$$[\xi, \eta]_t = \sigma_{\xi,\eta} + \sum_{0<s\leq t} \Delta\xi_s \Delta\eta_s.$$

Definition 1.13 (Subordinator). A 1-dimensional Lévy process X is said to be a subordinator if $X_t(\omega)$ is an increasing function of t, a.s.

Proposition 1.14. *Let X be a Lévy process on $\mathbb{R}$. The following conditions are equivalent:*

1. *X is a subordinator.*
2. *$X_t \geq 0$ a.s. for some $t > 0$.*
3. *$X_t \geq 0$ a.s. for every $t > 0$.*
4. *The characteristic triplet satisfies $\sigma^2 = 0$, $\int_{(-\infty,0]} \Pi(\mathrm{d}x) = 0$, $\int_{(0,1)} x\Pi(\mathrm{d}x) < \infty$, and $d \geq 0$. That is, there is no Brownian component, no negative jumps, the positive jumps are of finite variation and the drift is non-negative.*

In this 1-dimensional case, that the condition $\int_{(0,1)} x\Pi(\mathrm{d}x) < \infty$ is actually implied by the remaining conditions in part 4 of the above proposition. The condition $\int_{(-\infty,0]} \Pi(\mathrm{d}x) = 0$ implies, by equation (1.9), that $d \in [-\infty, \infty)$ and $d = -\infty$ iff $\int_{(0,1)} x\Pi(\mathrm{d}x) = \infty$.

A 1-dimensional Lévy process X will drift to ∞, drift to $-\infty$ or oscillate, namely;

$$\lim_{t\to\infty} X_t = \infty; \tag{1.14}$$

$$\lim_{t\to\infty} X_t = -\infty; \tag{1.15}$$

$$-\infty = \liminf_{t\to\infty} X_t < \limsup_{t\to\infty} X_t = \infty. \tag{1.16}$$

Necessary and sufficient conditions for these cases are given in [18]. Whenever the expected value of X_1 is well-defined, and hence contained in $[-\infty, \infty]$, cases (1.14) (1.15) and (1.16) equate respectively to $E(X_1) > 0$, $E(X_1) < 0$ and $E(X_1) = 0$. For the case in which the expected value of X_1 does not exist as a well-defined member of the extended reals, we need more notation. For $x > 0$, denote the tails functions of the Lévy measure by

$$\overline{\Pi}_X^+(x) := \Pi_X((x, \infty)), \quad \overline{\Pi}_X^-(x) := \Pi_X((-\infty, -x)), \quad \overline{\Pi}_X(x) := \overline{\Pi}_X^+(x) + \overline{\Pi}_X^-(x).$$

Define, for $x \geq 1$,

$$A_X^+(x) := \overline{\Pi}_X^+(1) + \int_1^x \overline{\Pi}_X^+(u)\mathrm{d}u, \quad A_X^-(x) := \overline{\Pi}_X^-(1) + \int_1^x \overline{\Pi}_X^-(u)\mathrm{d}u \tag{1.17}$$

where it suffices to assume $\overline{\Pi}_X^+(1) > 1$ and $\overline{\Pi}_X^-(1) > 1$. Define the integrals

$$J_X^+ = \int_1^\infty \frac{x}{A_X^-(x)} |\overline{\Pi}_X^+(\mathrm{d}x)| \text{ and } J_X^- = \int_1^\infty \frac{x}{A_X^+(x)} |\overline{\Pi}_X^-(\mathrm{d}x)|.$$

In [18] it is shown that if $E(X_1)$ does not exist then (1.14) occurs iff $J_X^- < \infty$, (1.15) occurs iff $J_X^+ < \infty$ and (1.16) occurs iff $J_X^- = J_X^+ = \infty$.

1.3 Generalising the Ornstein-Uhlenbeck process

As its name suggests, the GOU is a generalisation of the well known Ornstein-Uhlenbeck process, (which we will denote by OU). The OU can be defined as the solution $X := (X_t)_{t \geq 0}$ to the stochastic differential equation (SDE)

$$X_t = X_0 + \alpha \int_0^t X_{s-}\mathrm{d}s + \sigma B_t \tag{1.18}$$

where α and σ are real constants, B is Brownian motion on $\mathbb{R}$ with variance equal to 1, and X_0, the initial value of X, is a random variable independent of B. Alternatively, the OU can be defined as the stochastic integral

$$X_t = e^{\alpha t}\left(X_0 + \int_0^t \sigma e^{-\alpha s}\mathrm{d}B_s\right). \tag{1.19}$$

This stochastic integral process is well-defined and can easily be shown to be the *unique strong solution* (see Protter p.253) to equation (1.18).

The GOU is obtained from the OU by replacing the process $(\alpha t, \sigma B_t)$ with a general two dimensional Lévy process. However, the resulting process differs depending on which form of the OU is chosen as a starting point.

The counterpart of equation (1.18) for the GOU is

$$V_t = V_0 + U_t + \int_0^t V_{s-}\mathrm{d}R_s, \tag{1.20}$$

where (R, U) is a bivariate Lévy process independent of the random variable V_0. Alternatively, the counterpart of equation (1.21) for the GOU is

$$V_t := e^{\xi_t}\left(V_0 + \int_0^t e^{-\xi_{s-}}\mathrm{d}\eta_s\right) \tag{1.21}$$

where (ξ, η) is a bivariate Lévy process independent of the random variable V_0. This is the more common definition of the GOU, and the definition we shall use throughout this work. Note that the two approaches are not equivalent. Namely, if we set $(R, U) = (\xi, \eta)$, equation (1.21) does not solve equation (1.20). To describe the link between the two equations, we first recall the definition of the stochastic exponential.

For a semimartingale γ with $\gamma_0 = 0$, the *stochastic exponential* of γ, denoted $\epsilon(\gamma)$, is the unique strong solution of the SDE

$$Y_t = 1 + \int_0^t Y_{s-} \mathrm{d}\gamma_s. \tag{1.22}$$

The stochastic exponential of γ is a well-defined semimartingale and is given by

$$\epsilon(\gamma) = \exp\left\{\gamma_t - \frac{1}{2}[\gamma, \gamma]_t\right\} \prod_{0<s\leq t} (1 + \Delta\gamma_s) \exp\left\{-\Delta\gamma_s + \frac{1}{2}(\Delta\gamma_s)^2\right\} \tag{1.23}$$

where the infinite product converges.

The following proposition is due to Alex Lindner, and obtained by private communication with Ross Maller.

Proposition 1.15. *If (R, U) is a bivariate Lévy process with $\Pi_R((-\infty, -1]) = 0$ then the SDE (1.20) has the unique strong solution (1.21) where (ξ, η) is the bivariate Lévy process given by*

$$\xi_t := \ln \epsilon(R)_t, \tag{1.24}$$

$$\eta_t := U_t - \sum_{0<s\leq t} \left(1 - e^{\Delta\xi_s}\right) \Delta U_s - t\mathrm{Cov}\left(B_{\xi,1}, B_{U,1}\right). \tag{1.25}$$

On the other hand, if (ξ, η) is a general bivariate Lévy process then the stochastic integral process (1.21) is the unique strong solution of the SDE (1.20) where (R, U) is the bivariate Lévy process given by

$$R_t := \xi_t + \frac{1}{2}\sigma_\xi^2 t + \sum_{0<s\leq t} \left(e^{\Delta\xi_s} - \Delta\xi_s - 1\right), \tag{1.26}$$

$$U_t := \eta_t + \sum_{0<s\leq t} \left(e^{\Delta\xi_s} - 1\right) \Delta\eta_s + t\mathrm{Cov}(B_{\xi,1}, B_{\eta,1}). \tag{1.27}$$

Equation (1.26) is equivalent to (1.24). Further, $\Pi_R((-\infty, -1]) = 0$.

1.3.1 Literature on the GOU

There are only a few papers which consider the GOU, as defined in (1.20) or (1.21), in its full generality. The process appears implicitly in the work of de Haan and Karandikar [15] as a continuous generalisation of a stochastic recurrence equation. Basic properties are given by Carmona et al. [12]. A general survey of the GOU and its applications is given by Maller et al. [47]. The stationarity of the GOU is examined by Lindner and Maller [44], and we further explain these results in Section 1.4. Aspects of these stationarity results are generalised by Endo and Matsui [21], to the case in which η is an d-dimensional Lévy process.

The study of the GOU is closely related to the study of integrals of form Z, defined in equation (1.2). The first obvious link is that Z is contained in the definition of V, and the one-sided hitting probability for Z determines the ruin probability for V, as shown in equation (1.3). Secondly, the stationarity of V is related to the convergence of an integral of the form Z. The exact link is presented in Proposition 1.19. There have been relatively few papers dealing with Z in its full generality. Erickson and Maller [22], present necessary and sufficient conditions for the almost sure convergence of Z_t to a random variable Z_∞ as $t \to \infty$. Bertoin et al. [4], present necessary and sufficient conditions for continuity of the law of Z_∞ given it exists. Both these results are explained further in Section 1.4. In [52], Nyrhinen presents asymptotic equivalences for the one-sided hitting probability for Z. These results are explained further in Section 1.6. Kondo et al. [40] generalise the results in [22] to the case in which η is an d-dimensional Lévy process, and examine properties of the limit distribution Z_∞.

There are a large number of papers dealing with V when (ξ, η) is subject to restrictions.We first mention those papers which deal with the ruin probability of V, in which either ξ or η remains a reasonably general Lévy process. Harrison [31] presents results on the ruin probability of V when ξ is a linear deterministic function and η is a Lévy process with finite variance. Paulsen [54] generalises Harrison's results, and presents new ruin probability results for V in the case that ξ and η are independent, finite activity Lévy process. Results on the ruin probability of V for the case in which ξ and η are independent general Lévy processes are presented in Kalashnikov and Norberg [32] and Paulsen [56, 57]. Chiu and Yin [68] generalise some of Paulsen's results on the ruin probability of V, to the case in which ξ is a Lévy process and η is an independent jump-diffusion process. Cai [10] and Yuen et al [70, 71] present ruin probability results for V when ξ is a Lévy process and η is an independent compound Poisson process.

The above ruin probability results which have assumptions general enough to be relevant to our present work will be discussed further in Section 1.6. Note that when we restrict further to the case in which both ξ and η are assumed to be specific types of Lévy processes, there are numerous papers in the insurance mathematics area which deal with the ruin probability of V, however we do not discuss these. Survey papers exist which describe this ruin probability literature. The situation as of 1998 is described in Paulsen [55] and the situation as of 2008 is described in Paulsen [58].

We mention papers with restrictions on (ξ, η), which deal with V and focus on mathematical topics other than ruin probability. When ξ is deterministic linear and η is Brownian motion plus drift, Wolfe [67] presents results on the stationarity of V. There are several papers which examine first passage times and martingales for V in the case that ξ is a linear deterministic function and η is a Lévy process with no positive jumps, notably Hadjiev [30], Novikov [50], Patie [53] and Borovkov and Novikov [6]. When ξ is linear deterministic and η is an d-dimensional Lévy process, Masuda [48] examines various stability properties of V.

There is a significant amount of literature on the process Z when (ξ, η) is subject to restrictions, with attention mainly focused on the case in which Z_t converges to Z_∞ as $t \to \infty$. Notable is Yor [69] and Carmona et al. [11]. Gjessing and Paulsen [23] study the distribution of Z_∞ when ξ and η are independent finite activity Lévy processes, and obtain exact distributions for some special cases. Hove and Paulsen [59] use Markov chain Monte Carlo simulation to find the distribution of Z_∞ in some special cases. The work of Klüppelberg and Kostadinova [37] and Brokate et al. [7] provides results on the tail distribution of Z_∞ in the case that ξ and η are independent and η is a compound Poisson process plus drift. In the case that (ξ, η) is a Poisson process, Lindner and Sato [45] investigate continuity properties and the infinite divisibility of Z_∞.

The GOU has significant economic applications. The stochastic integral process Z can be interpreted as the discounted value of a continuous discounted perpetuity, and V can be interpreted as the forward value of a continuous perpetuity. The GOU has also found application in more specialised financial time series, with a particular form of V used as a constituent of the COGARCH process, introduced by Klüppelberg et al. [38] and studied further in Klüppelberg et al. [39] and Lindner [43]. In addition, Kostadinova [42] defines the properties of an insurance risk process which is closely associated with a particular form of V and develops an optimal investment strategy. We further discuss these economic interpretations in Section 1.7.

1.4 Stability of the GOU

We describe known results relating to the stability properties of the GOU. These results are stated without proof and will be referred to throughout the work. We state some of these results with different notation from the originals in order to fit with our requirements. We assume a bivariate Lévy process (ξ, η) with associated processes V and Z as defined in (1.1) and (1.2) respectively.

It was proved in [12], p.44, that V is a time-homogenous Markov process. When we replace a fixed time t with a stopping time $T < \infty$ a.s., the same proof establishes the strong Markov property of V. Specifically, the Lévy process

$$(\bar{\xi}_u, \bar{\eta}_u) := (\xi_{T+u} - \xi_T, \eta_{T+u} - \eta_T), \quad u \geq 0, \tag{1.28}$$

is independent of the stopping time σ-algebra

$$\mathscr{F}_T := \{\Lambda \in \mathscr{F} : \Lambda \cap \{T \leq t\} \in \mathscr{F}_t, \ \forall t \geq 0\},$$

and is equal in distribution to (ξ, η). Furthermore, the equation (1.1) and a simple change of variables argument proves that for all $r \geq 0$,

$$V_{T+r} = e^{\bar{\xi}_r} \left(V_T + \int_0^r e^{-\bar{\xi}_{u-}} \mathrm{d}\bar{\eta}_u \right).$$

These two observations establish the following result.

Proposition 1.16. *V is a time-homogenous strong Markov process.*

In [22], necessary and sufficient conditions are stated for a.s. convergence of Z_t to a finite random variable Z_∞ as t approaches ∞, whilst in [44], necessary and sufficient conditions are stated for stationarity of V. To describe these conditions, we need some notation.

For a bivariate Lévy process (X, Y), recall the definitions (1.17) and define the integral

$$I_{X,Y} := \int_{(e,\infty)} \frac{\ln(y)}{A_X^+(\ln(y))} |\overline{\Pi}_Y(\mathrm{d}y)|$$

and the Lévy process $K^{X,Y}$ by

$$K_t^{X,Y} := Y_t + \sum_{0<s\leq t} \left(e^{\Delta X_s} - 1\right) \Delta Y_s + t\mathrm{Cov}(B_{X_1}, B_{Y_1}),$$

where $\mathrm{Cov}(B_{X_1}, B_{Y_1})$ denotes the covariance of the Brownian components of X and Y. We now state [22], Theorem 2.

Theorem 1.17. *Z_t converges a.s. to a finite random variable Z_∞ as $t \to \infty$ if and only if $\lim_{t\to\infty} \xi_t = \infty$ a.s. and $I_{\xi,\eta} < \infty$. If $\lim_{t\to\infty} \xi_t = \infty$ a.s. but $I_{\xi,\eta} = \infty$ then $|Z_t| \to_P \infty$. If ξ_t does not tend to $+\infty$ as $t \to \infty$, then $|Z_t| \to_P \infty$ or there exists a constant $c \in \mathbb{R} \setminus \{0\}$ such that*

$$P\left(Z_t = c\left(e^{-\xi_t} - 1\right) \forall t > 0\right) = 1. \tag{1.29}$$

Consequently, Z_t converges in distribution to a finite random variable as $t \to \infty$ if and only if it converges a.s. to a finite random variable.

Note that if (1.29) holds and $\lim_{t\to\infty} \xi_t = \infty$ a.s. then Z_t converges a.s. to the constant random variable $Z_\infty = -c$. We now state Theorem 2.2 of [4], which proves that this is the only case in which Z_t converges to a non-continuous random variable.

Theorem 1.18. *Suppose $\lim_{t\to\infty} \xi_t = \infty$ a.s. as $t \to \infty$. The following are equivalent:*

1. *Z_t converges a.s. to a finite random variable Z_∞ as $t \to \infty$, where Z_∞ has an atom.*
2. *There exists $c \in \mathbb{R} \setminus \{0\}$ such that Z_t converges a.s. to $-c$ as $t \to \infty$.*
3. *There exists $c \in \mathbb{R} \setminus \{0\}$ such that (1.29) holds.*
4. *There exists $c \in \mathbb{R} \setminus \{0\}$ such that $e^{-\xi} = \epsilon(\eta/c)$.*

Note that Z is of form (1.29) iff V is of form

$$P\left(V_t = e^{\xi_t}(V_0 - c) + c \; \forall t > 0\right) = 1. \tag{1.30}$$

We now state [44], Theorem 2.1, which makes use of Theorem 1.17 above.

Theorem 1.19. *Suppose V is strictly stationary. Then one of the following two conditions is satisfied.*

1. *$\int_0^t e^{\xi_{s-}} \mathrm{d}K_s^{\xi,\eta}$ converges a.s. as $t \to \infty$ to a finite random variable, or equivalently, $\lim_{t\to\infty} \xi_t = -\infty$ and $I_{-\xi,K^{\xi,\eta}} < \infty$.*
2. *V is of form (1.30) for some constant $c \in \mathbb{R} \setminus \{0\}$.*

Conversely, if (i) or (ii) holds then there is a finite random variable V_∞, unique in distribution, such that the process V, starting with $V_0 =_D V_\infty$, is strictly stationary. Furthermore, if (i) holds, then V_∞ satisfies $V_\infty =_D \int_0^\infty e^{\xi_{s-}} \mathrm{d}K_s^{\xi,\eta}$.

Note that, regardless of the asymptotic behaviour of ξ, if (1.30) holds then V is strictly stationary iff $V_0 = c$.

In [22] the authors use exactly the same definition of Z as we have used above. However, in [44], the sign of the process ξ is reversed in their definition of V, that is, they define $V_t := e^{-\xi_t}\left(z + \int_0^t e^{\xi_{s-}}\mathrm{d}\eta_s\right)$. This version of V is stationary depending on the behaviour of $\int_0^t e^{-\xi_{s-}}\mathrm{d}K_s^{-\xi,\eta}$, and this process fits the form of Z used in [22]. It seems likely that the authors in [44] chose their definition of the GOU in order to use Theorem 2 of [22] directly, without any sign change. However, it suits our purposes to have the GOU in the form $V_t = e^{\xi_t}(z + Z_t)$ and study the behaviour of V in terms of Z. This is because we are examining ruin probability of the GOU rather than stationarity. With our definition of the GOU we can, without any sign change, use Theorem 2 of [22] to study Z. In addition, when our version of the GOU is considered as an economic model, as will be explained later in the chapter, the rate of return is ξ_t and the forwarding term e^{ξ_t}. This fits with convention, for example [32], [56] and [42]. Note that the version in [44] implies that the rate of return and forwarding term are $-\xi_t$ and $e^{-\xi_t}$ respectively.

1.5 Discretizing the GOU

We describe two stochastic difference equations, and show how V can be expressed as a solution of either one. We also give the associated discrete stochastic series for Z. Throughout this section, and the rest of the work, we take equation (1.1) as our definition of the GOU.

For $n > 0$ define the stochastic difference equation

$$Y_n = A_n Y_{n-1} + B_n \tag{1.31}$$

where (A_n, B_n) is an iid sequence independent of a random variable Y_0. Using induction, the solution is the stochastic series

$$Y_n = Y_0 \prod_{j=1}^{n} A_j + \sum_{i=1}^{n} \prod_{j=i+1}^{n} A_j B_i, \tag{1.32}$$

where we use the convention that $\prod_{j=1}^{0} a_j = 1$. Using equation (1.1) we can write, for an integer $n > 0$,

$$V_n = e^{\xi_n - \xi_{n-1}}\left(e^{\xi_{n-1}}\left(V_0 + \int_0^{n-1} e^{-\xi_{s-}}\mathrm{d}\eta_s\right)\right) + e^{\xi_n}\int_{(n-1)+}^{n} e^{-\xi_{s-}}\mathrm{d}\eta_s. \tag{1.33}$$

Thus, if we define the random variables

$$(A_n, B_n) := \left(e^{\xi_n - \xi_{n-1}}, e^{\xi_n} \int_{(n-1)+}^{n} e^{-\xi_{s-}} \mathrm{d}\eta_s \right) \tag{1.34}$$

and let $V_0 = Y_0$ then V_n is a solution of equation (1.31). By equation (1.1), $Z_n = e^{-\xi_n} V_n - V_0$, and so equation (1.32) implies that

$$Z_n = \sum_{i=1}^{n} \prod_{j=1}^{i} A_j^{-1} B_i. \tag{1.35}$$

It is proved in Appendix C that $(A_n, B_n)_{n \geq 1}$ is an iid sequence under these definitions. Note that even when ξ and η are independent, the random variables A_n and B_n may be dependent.

Alternatively, for $n > 0$ define the stochastic difference equation

$$Y_n = C_n Y_{n-1} + C_n D_n \tag{1.36}$$

where (C_i, D_i) is an iid sequence independent of Y_0. The solution is the stochastic series

$$Y_n = Y_0 \prod_{j=1}^{n} C_i + \sum_{i=1}^{n} \prod_{j=i}^{n} C_j D_i. \tag{1.37}$$

Using equation (1.33) it is clear that V_n is a solution of equation (1.36) if we let $V_0 = Y_0$ and define

$$(C_n, D_n) := \left(e^{\xi_n - \xi_{n-1}}, e^{\xi_{n-1}} \int_{(n-1)+}^{n} e^{-\xi_{s-}} \mathrm{d}\eta_s \right). \tag{1.38}$$

With these definitions, it is clear that

$$Z_n = \sum_{i=1}^{n} \prod_{j=1}^{i-1} C_j^{-1} D_i. \tag{1.39}$$

It is proved in Appendix C that $(C_n, D_n)_{n \geq 1}$ is an iid sequence under these definitions. Again, when ξ and η are independent, the random variables C_n and D_n may be dependent.

When discretizing V in this work, we will use the first approach, namely via the difference equation (1.31) and the series (1.32). When discretizing Z we will use the second approach, namely via the series (1.39). We do this in order to directly access particular results from papers on these objects. There has been significant attention paid to the two series (1.32) and (1.39) and they are linked via the fixed point of the same random equation, as we briefly explain.

Let (M, Q) be a two dimensional random variable on $(\Omega, \mathscr{F}, P)$ and let ϕ be a random affine map from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$\phi(t, \omega) := M(\omega)t + Q(\omega). \tag{1.40}$$

A (distributional) fixed point of ϕ is an a.s. finite distribution R on $\mathbb{R}$, independent of (M, Q), such that $R =_D MR + Q$. We can find R using an iteration method. We can suppose the existence of an iid sequence of random vectors, $(M_n, Q_n)_{n\geq 1}$, with common distribution (M, Q). Let $(\phi_n)_{n\geq 1}$ be the associated iid sequence of random affine maps, so $\phi_n(t) := M_n t + Q_n$. For integers $n \geq 1$, define the *outer iteration sequence* by

$$O_n(t) := \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1(t) = \phi_n(\phi_{n-1}(\cdots \phi_1(t) \cdots)),$$

and the *inner iteration sequence* by

$$I_n(t) := \phi_1 \circ \phi_2 \circ \cdots \circ \phi_n(t).$$

By induction, the solutions are

$$O_n(t) = t \prod_{j=1}^{n} M_j + \sum_{i=1}^{n} \prod_{j=i+1}^{n} M_j Q_i \tag{1.41}$$

and

$$I_n(t) = t \prod_{j=1}^{n} M_j + \sum_{i=1}^{n} \prod_{j=1}^{i-1} M_j Q_i. \tag{1.42}$$

In these solutions, we can replace the initial value t with a random variable L independent of $(M_n, Q_n)_{n\geq 1}$, in which case $O_n(L)$ and $I_n(L)$ are identically distributed. Note that $O_n(L)$ is the sequence defined in (1.32), for $(M_n, Q_n) = (A_n, B_n)$ and $L = Y_0$. Also, $I_n(0)$ is the sequence defined in (1.39), for $(M_n, Q_n) = (C_n^{-1}, D_n)$. The relationship between these sequences and the fixed point of the random equation (1.40) is the following: If $O_n(L)$ converges in distribution to an a.s. finite distribution R, as $n \to \infty$, then R is a fixed point for (1.40). If $P(M = 0) = 0$, then if $I_n(L)$ converges in distribution to an a.s. finite distribution R, then R is a fixed point for (1.40), and $O_n(L)$ converges in distribution to R as well. This result is due to Vervaat [66], and Goldie and Maller [25]. For more general random equations, the convergence of inner and outer iterations sequences to a fixed point is given by Letac's principle, as expressed in Theorem 2.1 of Goldie [24].

Finally, it is worth noting that the above discretization schemes are valid when the integer time increments are replaced with iid random times $(T_i)_{i\in\mathbb{N}}$ where $T_i - T_{i-1}$ are positive iid random variables for all $i \in \mathbb{N}$.

1.5.1 Literature on related stochastic difference equations

There is a large amount of literature on the above difference equations and series. We first mention some papers which focus on mathematical properties other than ruin probability. Note this is a list of papers which have been useful and of interest during this work, rather than a definitive list.

Various convergence, stability and recurrence properties of equation (1.31) are presented by Kesten [35], Vervaat [66], de Haan et al. [16], Babillot et al. [2] and Buraczewski [8]. Limit behaviour and rates of convergence for both equation (1.31) and (1.36) are given by Rachev and Samorodnitsky [61]. Convergence properties and continuity of the limit for the series (1.39) are examined by Grincevičius in [28], [27] and [29] and Dufresne [19]. Goldie [24] and Grey [26] present results on the tail of fixed points of various random equations, including equation (1.40). A general review of perpetuities and random equations is given by Embrechts and Goldie [20]. Goldie and Maller [25] explain the link between the series (1.32) and (1.39) and the random equation (1.40), and describe necessary and sufficient conditions for convergence of (1.39).

The literature on ruin probability for equations (1.31) and (1.36) is described in the survey paper by Paulsen [58]. The notable papers are Nyrhinen [52] and Konstantinides and Mikosch [41]. Nyrhinen [51], Cai [9], Tang and Tsitsiashvili [64, 65] and Tang [63] examine various aspects of the ruin probability of equation (1.31) with the extra assumption that (A_n) and (B_n) are independent sequences. These papers are not particularly useful to us however, since the GOU cannot be embedded into such a discrete model.

1.6 Relevant ruin probability results

We describe the existing results on the ruin probability of the GOU. We only mention results which allow both ξ and η to be general Lévy processes, either dependent or independent.

Paulsen 1998 [56] This paper provides results on the infinite horizon ruin probability of the GOU. Paulsen uses the alternative form of the GOU, described in equation (1.20), and assumes independence between the Lévy processes. He also has an inflation process within the model, however this makes no mathematical difference and we can ignore it. His model can be described as follows: let R

and U be independent Lévy processes and define

$$V_t = z + U_t + \int_0^t V_{s-} \mathrm{d}R_s, \tag{1.43}$$

where $V_0 = z \geq 0$, along with the condition that $\Pi_R((-\infty, -1]) = 0$. As noted in Proposition 1.15, this condition allows the SDE to be solved. Using the independence of R and U, the solution simplifies to

$$V_t = \epsilon(R)_t \left(V_0 + \int_0^t \epsilon(R)_{s-}^{-1} \mathrm{d}U_s \right).$$

Paulsen's main result, Theorem 3.1, provides conditions under which the infinite horizon ruin probability is one. We write out the complete theorem, however to fit with our upcoming work we use the more common definition of the GOU, equation (1.21). Thus, we assume independent Lévy processes ξ and η and let V, Z and $\psi(z)$ be defined respectively using equations (1.1), (1.2) and (1.3). We let $T_z := \inf\{t > 0 : V_t < 0\}$ denote the time of ruin. All conditions on R and U are transferred into equivalent conditions on ξ and η, using the fact that $\xi = \ln \epsilon(R)$ and $\eta = U$.

Theorem 1.20. *Let ξ and η be independent Lévy processes. Assume that η is not a subordinator and $\int_{|x|>1} |x| \Pi_\xi(\mathrm{d}x) < \infty$, so ξ has finite mean.*

(a) If $E(\xi_1) < 0$ and for some $\delta > 0$,

$$\int_1^\infty x^\delta \Pi_\eta(\mathrm{d}x) < \infty \quad \text{and} \quad \int_1^\infty (e^x - 1)^\delta \Pi_\xi(\mathrm{d}x) < \infty,$$

then for all $z \geq 0$ there exists $\alpha > 0$ such that $E(e^{\alpha T_z}) < \infty$. In particular $\psi(z) = 1$ and all moments of T_z are finite.

(b) If ξ is not identically zero and $E(\xi_1) = 0$ and for some $\delta > 0$

$$\int_1^\infty (\ln x)^{2+\delta} \Pi_\eta(\mathrm{d}x) < \infty, \quad \int_{-\infty}^{-1} |x|^{2+\delta} \Pi_\xi(\mathrm{d}x) < \infty$$

and

$$\int_1^\infty (e^x - 1)^\delta \Pi_\xi(\mathrm{d}x) < \infty,$$

then $\psi(z) = 1$ for all $z \geq 0$.

(c) If $E(\xi_1) > 0$ and either

(i)

$$\int_{-\infty}^{\infty} |\ln(1+x)|\Pi_\eta(\mathrm{d}x) < \infty \quad \textit{and} \quad \int_{-\infty}^{-1} x^4 \Pi_\xi(\mathrm{d}x) < \infty,$$

or

(ii)

$$\int_{|x|>1} |x|\Pi_\eta(\mathrm{d}x) < \infty \quad \textit{and} \quad \int_{-\infty}^{-1} e^{-2x}\Pi_\xi(\mathrm{d}x) < \infty$$

hold, then the following are true.

As $t \to \infty$, Z_t converges a.s. to a finite continuous random variable $Z_\infty = \int_0^\infty e^{-\xi_{s-}}\mathrm{d}\eta_s$ with distribution function H. For all $z \geq 0$ the ruin function satisfies

$$\psi(z) = \frac{H(-z)}{E\left(H(-V_{T_z}) \mid T_z < \infty\right)}. \tag{1.44}$$

Finally $\psi(z) < 1$ unless $\eta_t = \gamma_\eta t$, $\xi_t = \gamma_\xi t$ and $\gamma_\eta < \gamma_\xi z$.

Paulsen also gives a theorem stating conditions under which Z_∞ is a.s. finite and the characteristic function of Z_∞ satisfies a particular integro-differential equation. As Paulsen notes, this result is of limited practical value in finding the distribution of Z_∞, and we do not discuss it. Our interest lies in Theorem 1.20.

The first question that arises is whether the moment conditions in (a) and (c) can be replaced with the precise iff conditions for stationarity of V, and convergence of Z, respectively. Also desirable would be the removal of the finite mean condition for ξ. The main question however, is how dependence changes the result. It would be desirable to have precise iff conditions on the Lévy measure of (ξ, η) under which ruin is zero, or ruin is one. This is done in Chapters 2 and 3 respectively.

In proving the above result, Paulsen, either implicitly or explictly, accepts certain results as true without giving a proof. Some of these assumptions are false, and we discuss them in Appendix E. These problems are minor however, and make little difference to the statement of the theorem. Alternative proofs are available. The only adjustment that needs to be made to the theorem statement is in the final sentence of part (c), in which it must be assumed that $-\eta$ is not a subordinator. The main assumption of interest is the following: If ξ and η are independent and η is not a subordinator then $P(Z_t < 0) > 0$ for all $t > 0$, or equivalently, $\psi(0) > 0$. This statement is true in both the independent and dependent case. In the independent case the result seems intuitively clear, however the proof is not obvious. The proof of the statement in the dependent case is given in Chapter 2 and requires a change of measure argument and some analytic lemmas. It will be of use throughout this work.

Nyrhinen 2001 [52] This is the only paper which contains ruin probability results for the GOU in which (ξ, η) is allowed to be a 2-dimensional, infinite activity Lévy process. Firstly, this paper proves asymptotic results for the one-sided hitting probabilities, in finite and infinite time, of a discrete stochastic process X_n. Secondly, the paper discretizes the stochastic integral process Z and relates the discrete asymptotic results to the continuous setting, thus establishing equivalent hitting probabilities for Z. As equation (1.3) shows, the one-sided hitting probabilities for Z establish the ruin probability for V. We describe Nyrhinen's results in detail, and then make some comments.

Let (M, Q, L) and $(M_n, Q_n, L_n)_{n\geq 1}$ be iid random vectors on $(\Omega, \mathscr{F}, P)$ where $P(M > 0) = 1$. Define the process $\{X_n | n = 1, 2, \cdots\}$ by

$$X_n = \sum_{i=1}^{n} \prod_{j=1}^{i-1} M_j Q_i + \prod_{j=1}^{n} M_j L_n. \tag{1.45}$$

For real $m > 0$ define the hitting time of (m, ∞) by $T_m := \inf\{n : X_n > m\}$ where $T_m := \infty$ if $X_n \leq m$ for all $n \in \mathbb{N}$. Define the function $c(t) := \ln E\left(M^t\right)$, and let $\mathscr{D} := \{t \in \mathbb{R} : c(t) < \infty\}$. Define

$$w := \sup\{t \in \mathbb{R} : c(t) \leq 0\} \in [0, \infty]$$

and

$$t_0 := \sup\left\{t \in \mathbb{R} : c(t) < \infty,\ E\left(|Q|^t\right) < \infty,\ E\left((ML^+)^t\right) < \infty\right\} \in [0, \infty].$$

and

$$\bar{y} := \sup\left\{y \in \mathbb{R} : P\left(\sup_{n\in\mathbb{N}} X_n > y\right) > 0\right\} \in (-\infty, \infty].$$

Nyrhinen provides asymptotic results for X_n under the following hypothesis, which we call *hypothesis* H: Suppose that $0 < w < t_0 \leq \infty$ and $\bar{y} = \infty$.

In Lemma 1, Nyrhinen shows that whenever hypothesis H holds, then $P(M > 1) > 0$, the function c is strictly convex and continuously differentiable on the interior of $\mathscr{D}$ and the derivative at the point w is positive, so $c'(w) > 0$. To describe the asymptotic result, we need some more definitions, which are well-defined under hypothesis H by Lemma 1. Define $\mu := 1/c'(w) \in (0, \infty)$ and

$$x_0 := \lim_{t \to t_0 -} (1/c'(t)) \in [0, \infty)$$

Let c^* be the Fenchel-Legendre transform of c, namely, define $c^*(v) = \sup\{tv - c(t) : t \in \mathbb{R}\}$ for $v \in \mathbb{R}$. We can now define the function $R : (x_0, \infty) \to \mathbb{R} \cup \{\pm\infty\}$ by

$$R(x) := \begin{cases} xc^*(1/x) & \text{for } x \in (x_0, \mu), \\ w & \text{for } x \geq \mu. \end{cases}$$

This function has been analysed in traditional ruin theory. In particular, in this situation it is known that R is finite and continuous on (x_0, ∞) and strictly decreasing on $(x_0, \mu.)$ We now state Nyrhinen's main asymptotic result, which is his Theorem 2.

Theorem 1.21. *If hypothesis H holds then*

$$\lim_{m\to\infty} (\ln m)^{-1} \ln P(T_m \leq x \ln m) = -R(x) \tag{1.46}$$

for every $x > x_0$ *and*

$$\lim_{m\to\infty} (\ln m)^{-1} \ln P(T_m < \infty) = -w. \tag{1.47}$$

In the comments below Theorem 2, Nyrhinen states a second asymptotic result using a result by Goldie. Theorem 6.3 of Goldie [24] is a second-order asymptotic result for the tail of a random variable R when R is a fixed point for the random equation

$$R =_D Q + M \max(L, R) \tag{1.48}$$

and when a further set of conditions hold. Nyrhinen observes that under hypothesis H, $\sup_{n\in\mathbb{N}} X_n$ is a fixed point for (1.48), and all but one of the conditions of Goldie Theorem 6.3 are satisfied for the random variable $R = \sup_{n\in\mathbb{N}} X_n$. By assuming this extra condition, Nyrhinen is able to give the following result, which we state as a proposition. Recall that a distribution is *spread out* if it has a convolution power with an absolutely continuous component.

Proposition 1.22. *If hypothesis H holds and the distribution of* $\ln M$ *is spread out, then*

$$m^w P(T_m < \infty) = C_+ + o(m^{-\gamma}) \tag{1.49}$$

when m *tends to infinity, where* C_+ *and* γ *are positive real constants.*

Although not mentioned explicitly by Nyrhinen, C_+ is given by the formula defined in Theorem 6.2 and (2.18) of Goldie, namely

$$C_+ = \frac{1}{w\alpha} E\left(\left(\left(Q + M \max\left(L, \sup_{n\in\mathbb{N}} X_n\right)\right)^+\right)^w - \left(\left(M \sup_{n\in\mathbb{N}} X_n\right)^+\right)^w\right) \tag{1.50}$$

where $\alpha := E\left(|M|^w \ln|M|\right)$. The exact equation implied by Goldie Theorem 6.3 is

$$m^w P(T_m < \infty) = C_+ - f(t) + O(m^{-\beta/2})$$

where $0 < \beta < \min\{1, t_0 - w\}$ and $f(t)$ is a contour integral in the complex plane with domain depending on β, However, $0 < \gamma \leq \beta/2$ can be chosen small enough such that the contour integral is zero and (1.49) holds. The fact that C_+ is strictly positive under the conditions of Proposition 1.22 is not a result of the Goldie theorem but instead follows from Theorem 1.21. Specifically, it follows from the fact that $C_+ > 0$ iff equation (1.47) holds, which is easy to show using basic logarithm calculations.

The next result by Nyrhinen is Theorem 3, in which iff conditions are given for the condition $\bar{y} = \infty$. Namely, if hypothesis H holds, then $\bar{y} = \infty$ iff there exists $n \geq 1$ such that

$$P\left(Q + ML + \left(\prod_{j=1}^{n} M_j - 1\right)^{-1}\left(\sum_{i=1}^{n}\prod_{j=1}^{i-1} M_j Q_i\right) > 0,\ \prod_{j=1}^{n} M_j > 1\right) > 0. \tag{1.51}$$

Nyrhinen comments that the verification of this condition is generally difficult. In Example 1, he notes that if $0 < w < t_0 \leq \infty$, and further

$$P(Q + ML \geq 0) > 0 \text{ and } P(M > 1, Q > 0) > 0, \tag{1.52}$$

then (1.51) holds, and so $\bar{y} = \infty$. However, he comments that this sufficient condition is not sharp. In Example 2 he gives a simple example of (M, Q, L) which satisfies $0 < w < t_0 \leq \infty$, fails (1.52) but still satisfies $\bar{y} = \infty$. This concludes the discrete section of the paper. We now describe the continuous result. We use our own notation rather than Nyrhinen's, so (ξ, η) is a bivariate Lévy process and $Z_t := \int_0^t e^{-\xi_{s-}} \mathrm{d}\eta_s$, as usual. Define

$$M_n : = e^{-(\xi_n - \xi_{n-1})} \tag{1.53}$$

$$Q_n : = e^{\xi_{n-1}} \int_{(n-1)+}^{n} e^{-\xi_{s-}} \mathrm{d}\eta_s \tag{1.54}$$

$$L_n : = e^{\xi_n}\left(\sup_{n-1<t\leq n} \int_{(n-1)+}^{t} e^{-\xi_{s-}} \mathrm{d}\eta_s - \int_{(n-1)+}^{n} e^{-\xi_{s-}} \mathrm{d}\eta_s\right). \tag{1.55}$$

Nyrhinen implicitly assumes that (M_n, Q_n, L_n) is an iid sequence. This claim follows by an easy extension of our proof in Appendix C. Note that with these allocations Z can be written as a discrete stochastic series of the form (1.39), namely $Z_n = \sum_{i=1}^{n} \prod_{j=1}^{i-1} M_j Q_i$

Nyrhinen proves the following result.

Proposition 1.23. *With (ξ, η), Z and (M_n, Q_n, L_n) defined as above,*

$$\sup_{n-1<t\leq n} Z_t = X_n,$$

and

$$\sup_{0\leq t\leq n} Z_t = \max_{m=1,\cdots,n} X_m$$

for

$$X_n := \sum_{i=1}^{n}\prod_{j=1}^{i-1} M_j Q_i + \prod_{j=1}^{n} M_j L_n.$$

Actually, Nyrhinen proves equality in distribution, using an induction argument. However, with direct calculation it is easily seen that the exact equality holds.

For real $m > 0$ define the hitting time of (m, ∞) for Z by $T_m^c := \inf\{t \geq 0 : Z_t > m\}$ where $T_m^c := \infty$ if $Z_t \leq m$ for all $t \geq 0$. Then the above proposition implies that for all $n \in \mathbb{R}$,

$$P(T_m^c \leq n) = P(T_m \leq n)$$

and

$$P(T_m^c < \infty) = P(T_m < \infty).$$

Thus, for a Lévy processes (ξ, η), the asymptotic result in Theorem 1.21 holds for T_m^c when hypothesis H is satisfied for the associated values of (M_n, Q_n, L_n). If further, the distribution of $\ln M$ is spread out then the asymptotic result in Theorem 1.21 holds for T_m^c. This is the content of Nyrhinen's Theorem 4 and Corollary 5. Immediately following Corollary 5, an example is given. The example is very simple, using independent Lévy processes $\xi_t := \alpha t + B_t$ and $\eta_t := \beta t + Y_t$ where α and β are positive reals, B is Brownian motion and Y is a compound Poisson process. Nyrhinen states conditions under which $0 < w < t_0 \leq \infty$, ξ_1 is absolutely continuous and the conditions (1.52) hold, thus implying that $\bar{y} = \infty$. Note that when M_n is defined as above, the condition $0 < w < \infty$ implies that ξ is non-deterministic.

We make some comments on this paper, beginning with the discrete results. It is not immediately clear whether the sequence X defined in (1.45) converges under hypothesis H. Note that if we choose $L_n = L$ then X_n is the inner iteration sequence $I_n(L)$ defined in (1.42) for the random equation $\phi(t) = Mt + Q$. Goldie and Maller [25] prove that $I_n(L)$ converges a.s. to a finite random variable iff $\prod_{j=1}^{n} M_j \to 0$ a.s. as $n \to \infty$ and $I_{M,Q} < \infty$, where $I_{M,Q}$ is an integral involving the marginal distributions of M and Q. Since the distribution of L has no effect on the convergence of $I_n(L)$, it is clear that the above conditions are precisely those under which X_n converges a.s. for iid (M_n, Q_n, L_n). We show these conditions are satisfied under under hypothesis H, and thus the sequences X and $\sum_{i=1}^{n}\prod_{j=1}^{i-1} M_j Q_i$ converge a.s. to the same finite random variable.

Suppose $0 < w < t_0 \leq \infty$. In the proof of Lemma 1, Nyrhinen shows that the condition $w > 0$ implies that the function $c(t)$ is strictly convex and continuously differentiable on the interior of $\mathscr{D}$. Since $c(0) = 0$ and $c(w) = 0$ there must exist $t \in (0, w)$ such that $\ln E(M^t) < 0$. By Jensen's inequality, $E(\ln M) < 0$, which is well-defined by the assumption $P(M = 0) = 0$. Hence, the random walk $S_n := \sum_{j=1}^{n}(-\ln M_j) \to \infty$ a.s. as $n \to \infty$. Since $S_n = -\ln\prod_{j=1}^{n} M_j$ it follows that $\prod_{j=1}^{n} M_j \to 0$ a.s. as $n \to \infty$, as required. Further, since $E(\ln M) \in [-\infty, 0)$, Corollary 4.1 of Goldie and Maller [25] implies that if $E(\ln^+|Q|) < \infty$ then the integral condition $I_{M,Q} < \infty$ is satisfied. Since $0 < t_0$ there exists $0 < s$ such that $E(|Q|^s) < \infty$. Hence $E(\ln|Q|) < \infty$ which implies that $E(\ln^+|Q|) < \infty$ as required.

The discrete results of Nyrhinen are both interesting and useful. However, in their stated form, the corresponding results for the continuous case are not especially useful. This is because the conditions in hypothesis H are quite inaccessible when (ξ, η) is a reasonably complicated Lévy process and (M_n, Q_n, L_n) are defined according to equations (1.53), (1.54) and (1.55). Correspondingly, the only continuous example Nyrhinen gives is extremely simple, involving independent finite activity Lévy processes. The most serious offender is the condition $\bar{y} = \infty$ where

$$\bar{y} = \sup\left\{y \in \mathbb{R} : P\left(\sup_{t>0} Z_t > y\right) > 0\right\}.$$

We discuss this condition further in Appendix E. Verifying the condition $0 < w < t_0 \leq \infty$ is also problematic, because it requires knowledge of $E|Z_1|$ and $E(\sup_{0<t\leq 1}|Z_t|)$. To make this result accessible, these conditions must be stated in terms of the characteristic triplet of (ξ, η) or the marginal distributions of ξ and η. This is done in Appendix F.

Paulsen 2002 [57] The first main result in this paper, Theorem 3.2 (a), is a modification of part of Nyrhinen's work in [52], specifically, the continuous version of Nyrhinen's adaptation of Goldie [24] Theorem 6.3, which we state above as Proposition 1.22. Paulsen assumes ξ and η are independent Lévy processes and and states an asymptotic result for $P(\inf_{t>0} Z_t < -m)$ as $m \to \infty$. Note that this is the reciprocal approach to Nyrhinen, who gives asymptotics for $P(\sup_{t>0} Z_t >$

m) as $m \to \infty$ (without assuming independence). Paulsen's approach makes more sense from a GOU ruin probability perspective since $\psi(m) = P(\inf_{t>0} Z_t < -m)$, as shown in equation (1.3). Paulsen defines M_n and Q_n as in equations (1.53) and (1.54) respectively, and defines L_n as the reciprocal version of equation (1.55), namely

$$L_n := e^{\xi_n}\left(\inf_{n-1<t\leq n}\int_{(n-1)+}^{t} e^{-\xi_{s-}}\mathrm{d}\eta_s - \int_{(n-1)+}^{n} e^{-\xi_{s-}}\mathrm{d}\eta_s\right). \tag{1.56}$$

Paulsen states a set of conditions on the distributions of ξ and η, which include the conditions of Theorem 1.20 part (c) above. He proves that these conditions imply that the reciprocal conditions of Proposition 1.22 hold for his chosen values of (M_n, Q_n, L_n). In particular, $\inf_{t>0} Z_t$ satisfies the reciprocal versions of hypothesis H and the random equation (1.48). Thus the reciprocal version of the asymptotic result (1.49) holds, namely

$$m^w P(\inf_{t>0} Z_t < -m) = C_- + o(m^{-\gamma}) \tag{1.57}$$

when m tends to infinity, where C_- and γ are positive real constants. The value C_- is given by the formula defined in (2.19) of Goldie, namely

$$C_- = \frac{1}{w\alpha} E\left(\left(\left(Q + M\min\left(L, \inf_{t>0} Z_t\right)\right)^-\right)^w - \left(\left(M \inf_{t>0} Z_t\right)^-\right)^w\right) \tag{1.58}$$

where $\alpha := E\left(|M|^w \ln |M|\right)$ and L is defined by (1.56).

Note that Paulsen actually gives a different form for the associated constant rather than simply quoting the above. Since he assumes the conditions of Theorem 1.20 part (c) hold, the ruin probability formula (1.44) must hold, namely

$$\psi(m) = \frac{H(-m)}{h(m)} \tag{1.59}$$

where we define $h(m) := E\left(H(-V_{T_m}) \mid T_m < \infty\right) \in [0,1]$ and $h := \lim_{m\to\infty} h(m)$. Note that h may not exist. Using (1.57),

$$\lim_{m\to\infty} m^w P(\inf_{t>0} Z_t < -m) = C_-,$$

and combining with (1.59) we have

$$\lim_{m\to\infty} m^w P(Z_\infty < -m) = hC_-.$$

where h must exist. However, it is now a consequence of Goldie [24] Theorem 4.1 that

$$\lim_{m\to\infty} m^w P(Z_\infty < -m) = \frac{1}{w\alpha} E\left(\left((MZ_\infty + Q)^-\right)^w - \left((MZ_\infty)^-\right)^w\right),$$

and thus an alternative value for C_- is obtained. We state the precise result.

Theorem 1.24. *Suppose the conditions of Theorem 1.20 part (c) hold. Further, assume there exists $w > 0$ and $\epsilon > 0$ such that $\ln E(e^{-w\xi_1}) = 0$, $\ln E(e^{-(w+\epsilon)\xi_1}) < \infty$ and $E(|\eta|^{w+\epsilon}) < \infty$. Also, assume that the distribution of ξ_T is spread out when T is uniformly distributed on $[0,1]$ and independent of ξ. Then*

$$m^w P(\inf_{t>0} Z_t < -m) = C_- + o(m^{-\epsilon})$$

as $m \to \infty$. The positive real constant C is given by

$$C_- = \frac{1}{w\alpha h} E\left(\left((MZ_\infty + Q)^-\right)^w - \left((MZ_\infty)^-\right)^w\right) \tag{1.60}$$

where $(M, Q) := (e^{-\xi_1}, \int_0^1 e^{-\xi_{s-}} \mathrm{d}\eta_s)$, and α and h are defined above.

This result is not quite correct. If we assume the above conditions on w and ϵ, and quote the result directly from Nyrhinen, the conclusion should be that there exists some $\gamma > 0$ such that $m^w P(\inf_{t>0} Z_t < -m) = C_- + o(m^{-\gamma})$ as $m \to \infty$. Using Goldie Theorem 6.3 implies the same. In Appendix F we present a more precise version of the above theorem which also holds for general (ξ, η) and has simpler conditions.

The above theorem shows that under the relevant conditions, $\psi(m)$ behaves essentially like m^{-w}, where the value w depends principally on ξ. Note that the condition $\ln E(e^{-w\xi_1}) = 0$ for some $w > 0$, requires that ξ be non-deterministic, which is in line with Nyrhinen's conditions. Also note that Paulsen requires that ξ_T be spread out, whereas Nyrhinen requires that ξ_1 be spread out. Using the random time gives more generality. For example, if ξ_t is a compound Poisson process with arithmetic jump size distribution, then ξ_T has an absolutely continuous distribution whilst ξ_1 has no absolutely continuous component.

The next result by Paulsen is Theorem 3.2 (b) which provides asymptotics for $\psi(m)$ when the negative jumps of η are heavy-tailed and dominate ξ. It shows that $\psi(m)$ behaves essentially like $m^{-\kappa_1}$ where the value of κ_1 depends principally on η.

Theorem 1.25. *Suppose the conditions of Theorem 1.20 part (c) hold. Further, assume there exists $\kappa_1 > 0$ and $\epsilon > 0$ such that $\Pi_\eta((-\infty, -x]) \sim x^{-\kappa_1} G(x)$ where G is slowly varying, and $\ln E(e^{-(\kappa_1+\epsilon)\xi_1}) < 0$. Then*

$$\psi(m) \sim \frac{1}{h(m)} \frac{1}{-\ln E(e^{-\kappa_1\xi_1})} m^{-\kappa_1} G(m) \quad \text{as } m \to \infty.$$

The proof is quite simple. Using equation (1.59), it suffices to prove that

$$H(-m) \sim \frac{1}{-\ln E(e^{-\kappa_1\xi_1})} m^{-\kappa_1} G(m)$$

as $m \to \infty$. This is achieved by expressing Z_∞ as the fixed point of the random equation (1.40), and then using discrete rate results from Grey [26]. In Remark 3.2 part (b), Paulsen comments that under the conditions of Theorem 1.25, $h = \lim_{m\to\infty} h(m)$ has not been proven to exist. He conjectures that h exists and equals 1, in which case $\psi(m) \sim \frac{1}{-\ln E(e^{-\kappa_1 \xi_1})} m^{-\kappa_1} G(m)$ as $m \to \infty$.

Immediately following this theorem is an examination of $\psi(m)$ when the negative jumps of ξ are large enough in absolute value that the conditions of Theorem 1.24 or 1.25 do not hold. Several specific cases are examined and upper and lower bounds for $\psi(m)$ are established. Note that in all these cases Z_t still converges to a finite random variable Z_∞ as $t \to \infty$. In each case, Paulsen's method is to define new Lévy processes $\bar{\xi}$ and $\bar{\eta}$ which satisfy the conditions of Theorem 1.24 or 1.25, and for which $\psi(m) \geq \bar{\psi}(m)$, or $\psi(m) \leq \bar{\psi}(m)$, where $\bar{\psi}$ is the ruin probability function for the GOU associated with $\bar{\xi}$ and $\bar{\eta}$. Application of the relevant theorem on $\bar{\psi}(m)$ thus produces upper and lower bounds for ψ. The paper finishes with an examination of a special case in which ξ is compound Poisson plus drift, and η is Brownian motion plus drift.

We make some comments. In this paper Paulsen has certainly made Nyrhinen's continuous result more accessible, with conditions stated on the characteristic triplet of ξ and η. However his conditions include those of Theorem 1.20, so the comments we made earlier on that situation apply to the current paper also. Namely, it would be desirable to remove the finite mean assumption for ξ and replace his moment conditions, which are sufficient for convergence of Z_t, with the precise iff conditions. Of course, the main question is how the result changes when dependence between ξ and η is permitted. Note that with ξ and η independent and η not a subordinator, the condition of Nyrhinen's which is most difficult to verify, $\bar{y} = -\infty$, is assumed by Paulsen to be true. This assumption is false even in the independent case, as discussed in Appendix E. The assumption holds only if extra conditions are imposed, in line with Proposition D.4. The theorem statement must be adjusted accordingly.

Kalashnikov and Norberg 2002 [32] This paper provides various asymptotic results and bounds for the infinite horizon ruin probability of the GOU when the underlying Lévy processes are independent. The results are achieved by discretizing V and Z into the discrete sequences described in Section 1.5. The bulk of the paper consists of asymptotic results for the infinite horizon ruin probability of these discrete processes. We explain the discrete results first, and then discuss the continuous implications. Let

$$Y_n := Y_0 \prod_{j=1}^{n} A_j + \sum_{i=1}^{n} \prod_{j=i+1}^{n} A_j B_i, \tag{1.61}$$

where (A_n, B_n) is an iid sequence of random vectors, independent of the random variable Y_0, with common distribution (A,B). Define

$$\psi^*(m) = P\left(\inf_{n\in\mathbb{N}} Y_n < 0 \middle| Y_0 = m \right).$$

Throughout the paper, a set of conditions is assumed, which we call *hypothesis G*: Suppose that $A > 0$ a.s., $P(A < 1) > 0$ and $P(B \leq mA) > 0$ for all $-\infty < m < \infty$. The first major result of the paper, Theorem 1, shows that under these conditions $\psi^*(m)$ is greater than a certain power function. We need to define some terms. Let $\alpha < 1$ and $\beta > 0$ be constants such that

$$\bar{q} := P(A \leq \alpha, B \leq \beta) > 0.$$

The fact that $P(A < 1) > 0$ ensures that such α and β exist. Let

$$q^* := P\left(B \leq -\frac{2\beta}{1-\alpha} A \right),$$

which is strictly positive under hypothesis G.

Theorem 1.26. *If hypothesis G holds then $\psi^*(m) \geq \frac{a}{m^b}$ for all $m > c$ where a,b,c are strictly positive constants defined by*

$$b = \frac{\ln \bar{q}}{\ln \alpha}, \; c = \frac{\beta}{1-\alpha}, \; a = \bar{q} q^* c^b.$$

The next major result, Theorem 2, deals with

$$\psi^*(m, s^*) := P\left(\inf_{n\in\mathbb{N}} Y_n < s^* \middle| Y_0 = m \right).$$

Theorem 1.27. *Suppose hypothesis G holds, A has a non-lattice distribution and there exists $w > 0$ such that $E(A^{-w}) = 1$. If*

1. *If $E\left(\left(\frac{B^-}{A}\right)^w\right) < \infty$ then, for any $\delta > 0$, there exist constants $s^* > 0$ and $0 < \bar{k} < 1$ such that*

$$\psi(m, s^*) \leq \bar{k} \left(\frac{s^*}{m} \right)^{w-\delta}$$

2. *If $E(A^{-w-\delta_0}) < \infty$ for some $\delta_0 > 0$ then for any $\delta > 0$ there exist constants $s^* > 0$ and $0 < \bar{k} \leq 1$ such that*

$$\bar{k}\left(\frac{s^*}{m}\right)^{w+\delta} \lesssim \psi^*(m, s^*).$$

The paper then examines the sequence

$$D_n := \sum_{i=1}^{n} \prod_{j=1}^{i-1} M_j Q_i. \tag{1.62}$$

where $M_n := A_n^{-1}$ and $Q_n := B_n/A_n$. Asymptotics for the tail of the supremum, and infimum, are found using Theorem 6.2 of Goldie [24], which is a first-order asymptotic result for the tail of a random variable R when R is a fixed point for the random equation

$$R =_D Q + M \max(L, R) \tag{1.63}$$

and when a further set of conditions hold. It is clear from Goldie's work that when $L = 0$, and certain moment conditions hold, the random variable $\sup_{n\in\mathbb{N}} D_n$ is a fixed point of (1.63). We now present Norberg's Theorem 3, which is a slight restatement of Goldie's Theorem 6.2 for the case in which $L = 0$. Conditions are stated in terms of A and B to fit with the previous theorems.

Theorem 1.28. *Suppose hypothesis G holds, the distribution of A is spread out and there exists $w > 0$ such that $E(A^{-w}) = 1$, $E\left(A^{-w}\ln^+(A)\right) < \infty$ and $E\left(\left|\frac{B}{A}\right|^w\right) < \infty$. Then there exist constants $C_- \geq 0$ and $C_+ \geq 0$ such that*

$$P\left(\sup_{n\in\mathbb{N}} D_n > m\right) \sim \frac{C_+}{m^w}, \quad z \to \infty,$$

$$P\left(\inf_{n\in\mathbb{N}} D_n < -m\right) \sim \frac{C_-}{m^w}, \quad z \to \infty.$$

Note that the formula for C_+ is stated above in (1.50). The final theorem in the paper, Theorem 4, provides a power function lower bound for $\psi^*(m, s^*)$.

Theorem 1.29. *Under the conditions of Theorem 1.28, for any sufficiently large $s^* \geq 0$, there exists a constant $C(s^*) > 0$ such that $\psi^*(m, s^*) \geq C(s^*)x^{-w}$.*

This ends the explanation of the discrete results. We now let ξ and η be independent Lévy processes and recall our usual definitions for V, Z and ψ, namely equations (1.1), (1.2) and (1.3), respectively. The processes V and Z are discretized, via stopping times, into the sequences (1.61) and (1.62) respectively.

Accordingly, let $0 = T_0 < T_1 < T_2 < \cdots$ be an increasing sequence of stopping times such that the increments $T_i - T_{i-1}$ are iid with distribution T. As explained in Section 1.5, if we let

$$(A_n, B_n) := \left(e^{\xi_{T_n} - \xi_{T_{n-1}}}, e^{\xi_{T_n}} \int_{T_{(n-1)}+}^{T_n} e^{-\xi_{s-}} \mathrm{d}\eta_s \right)$$

then for all $n \in \mathbb{N}$, $Y_n = V_n$, and $D_n = Z_n$ for the corresponding (M_n, Q_n). With this choice of vectors, we have

$$\psi^*(m) = P\left(\inf_{n\in\mathbb{N}} V_n < 0 \middle| V_0 = m \right) = P\left(\inf_{n\in\mathbb{N}} Z_n < -m \right).$$

The continuous analogue of hypothesis G is as follows: Suppose that ξ is not a subordinator and, for the stopping time $T > 0$ defined above, $P(Z_T \leq u) > 0$ for all $u \in \mathbb{R}$. It is also noted that this hypothesis can be replaced by the slightly stronger assumption $P(\xi_T < 0, Z_T < 0) > 0$. The continuous analogues of the various moment conditions are obvious.

The continuous version of Theorem 1.26 is obtained using the obvious fact that $\psi(m) \geq \psi^*(m)$. The authors note that this inequality may be strict, for example, if ξ has a Brownian motion component, or if η has an infinite activity Lévy measure. They comment that in such cases, the partition $T_{v,n} := nv$ can be used, where $v > 0$. If $\psi_v^*(m)$ is defined to be the ruin probability associated with $T_{v,n}$ then $\psi(m) = \lim_{v\to 0} \psi_v^*(m)$.

The continuous version of Theorem 1.27 (stated as Corollary 2) is obtained using the following statement: Under hypothesis G, whenever $0 < s^* \leq m$

$$\frac{a}{(s^*)^b}\psi^*(m, s^*) \leq \psi(m) \leq \psi^*(m, s^*). \tag{1.64}$$

It is claimed that these inequalities follow from the Markov property, the fact that $\psi(m) \geq \psi^*(m)$, and Theorem 1.26. This is easily seen for the inequality on the left, however the proof is not obvious for the inequality $\psi(m) \leq \psi^*(m, s^*)$.

As noted by the authors, the equality $\psi(m) = \psi^*(m)$ does not hold in general. However, it is claimed that under the conditions of Theorem 1.28, the equality holds, and accordingly, Theorem 1.28 can be converted to a statement on $\psi(m)$, in particular, there exists $C_- \geq 0$ such that

$$\psi(m) \sim \frac{C_-}{m^w}, \quad z \to \infty.$$

Note that it would be an interesting task to examine the precise conditions under which the statements $\psi(m) \leq \psi^*(m, s^*)$ and $\psi(m) = \psi^*(m)$ are true, especially for general (ξ, η). The final result in the paper uses Theorem 1.29 and the inequalities (1.64) to give conditions under which C_- is strictly positive, in the continuous situation.

In section 3.4 of the paper, two examples are given of ξ and η which satisfy hypothesis G. In the first example, both ξ and η are Brownian motion plus drift, where the Brownian motions are independent. The partition is taken as $T_n = nv$ where $v > 0$ is fixed. In the second, ξ is Brownian motion plus drift and η is compound Poisson plus drift, where η is not a subordinator. The partition is taken as the jump times of the compound Poisson process. No proof is given that these examples satisfy hypothesis G, however, in these simple cases it is obvious. Thus, the lower bound specified in Theorem 1.26 holds for each case. The authors comment that examining the accuracy of the lower bound is difficult, and one must resort to numerical studies. Later in the paper, it is also claimed that both these examples satisfy the extra conditions of the later theorems iff ξ has positive drift $d_\xi > 0$, in which case $\epsilon = 2d_\xi/\sigma_\xi^2$. A final example is given in which ξ and η are independent compound Poisson processes plus drift, where ξ and η are not subordinators. It is commented that all conditions mentioned in the theorems are automatically satisfied, except the existence of $w > 0$ such that $E(\xi_1^{-w}) = 1$, and iff conditions are given on ξ such that this holds.

We make some comments on this paper, beginning with the convergence implications, in the continuous case, of the conditions in Theorem 1.28. In particular, there exists $w > 0$ such that $E(\xi_T^{-w}) = 1$, which implies that $E(\xi_T) > 0$, and hence $\lim_{t\to\infty} \xi_t = \infty$. Hence, by Theorem 1.19, V_t cannot converge a.s. to a finite random variable V_∞ as $t \to \infty$, unless the degenerate case (1.29) holds. Since ξ and η are independent, this cannot occur. Note that Z_t converges a.s. to finite Z_∞ as $t \to \infty$, iff the associated sequence D_n converges as $n \to \infty$. For the values (M_n, Q_n) defined as above, Theorem 4.1 of Goldie [24], implies that D_n converges under the conditions of Theorem 1.28.

As in Nyrhinen [52], the conditions in this paper are quite inaccessible when (ξ, η) is a reasonably complicated Lévy process. Correspondingly, the examples are limited to the independent, finite variation cases. In hypothesis G it must be assumed that $P(Z_T \le u) > 0$ for all $u \in \mathbb{R}$, which we discuss in Appendix E. This is a stronger version of Nyrhinen's condition that $\bar{y} = -\infty$ where

$$\bar{y} = \inf\left\{y \in \mathbb{R} : P\left(\inf_{t>0} Z_t < y\right) > 0\right\}.$$

The replacement hypothesis mentioned earlier involves the joint distribution of ξ_T and Z_T, which is equally problematic. Further, Theorems 1.28 and 1.29 require the condition $E(|Z_T|^w) < \infty$. To make the results in this paper more accessible, all the conditions must be stated in terms of the characteristic triplet of (ξ, η) or the marginal distributions of ξ and η.

Lindner and Maller 2005 [44] This paper is not about ruin probability, and has already been discussed in Section 1.4. However we mention a result which relates closely to the topics discussed above. In Proposition 4.1, the authors give conditions on the marginal measures of ξ and η under which moments exist for the stationary distribution of the GOU. We have modified this result into a statement on the existence of moments for $\sup Z_t$, and presented it as Lemma 3.24. This lemma can be used to make some of the conditions in [52] and [32] more accessible, which we have noted above is desirable.

1.7 Economic applications

The first, and most basic, economic application of the GOU is as a continuous perpetuity. However, the interpretation of the underlying bivariate Lévy process is slightly different depending on whether equation (1.20) or (1.21) is taken as the definition of V. In a sense, equation (1.20) arises more naturally. We can consider R to be an accumulated investment returns process, and U to be an accumulated income process in a non-economic (no interest) environment. For example, U could be the the income stream, consisting of premiums minus pay outs, of an insurance company. If we suppose the insurance company continually invests all of its income stream η into a risky asset with accumulated returns R, and continually reinvests any profits, then the total surplus V_t of the company at time t will be the sum of the initial surplus V_0 at time zero, plus the income stream η_t, plus the accumulated gains/losses at time t from the investing process. Since all of the current surplus is invested in the risky asset, the gains/loss from investment is described by the process $\int_0^t V_{s-}\mathrm{d}R_s$, and hence the total surplus at time t is exactly the SDE (1.20). Note that the value at time t of one dollar, invested at time zero in a risky asset with accumulated returns R, is given by the SDE $S_t = \int_0^t S_{s-}\mathrm{d}R_s$ and the solution is the stochastic exponential $S_t = \epsilon(R_t)$. For example, the traditional Black-Scholes model uses $R_t := \mu t + \sigma B_t$ where μ and σ are real numbers and B is standard Brownian motion. In this case the stochastic exponential simplifies to $\epsilon(R)_t = e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$.

When the stochastic integral process (1.20) is interpreted economically, the exponential Lévy process e^{ξ_t} is considered to be the value at time t of one dollar invested into a risky asset at time zero. Further, η is considered to be an accumulated income process in a non-economic environment. Thus, with (R, U) as above, $e^{\xi_t} = \epsilon(R)_t$ and $\eta_t = U_t$. With ξ and η so defined, the process $Z_t := \int_0^t e^{-\xi_{s-}} \mathrm{d}\eta_s$ is the discounted value of the perpetuity η at time t under the price process e^{ξ_t}. The equation $V_t = e^{\xi_t}(z + Z_t)$ is the forward value of an initial fortune z and the continuous perpetuity η, under price process e^{ξ_t}. It is assumed that η is continually invested in a risky asset following the price process e^{ξ_t}, and all profits are continually reinvested. When considered as a model of an insurance company, V is often called the integrated risk process, and Z is called the discounted net loss process.

It is interesting that under this economic interpretation, the equation (1.20) has e^{ξ_t} as the forwarding term, and $e^{-\xi_{t-}}$ as the discounting term. The use of the $t-$ in the discounting term is necessary in order for the integrand in the stochastic integral process Z to be predictable. However, it now seems consistent for $e^{\xi_{t-}}$ to be used as a forwarding term. As it is, using simple deterministic functions it is easy to see the inconsistencies which arise. Suppose $\eta_t = 0$ on [0,1) and $\eta_1 = 1$, and suppose $e^{\xi_t} = 1$ on [0,1) and $e^{\xi_1} = 2$. Then $Z_1 = \int_0^1 e^{-\xi_{s-}} \mathrm{d}\eta_s = 1$ and $V_1 = e^{\xi_1} Z_1 = 2$. It seems logical that the a surplus at time 1 which is discounted to zero, and then forwarded back to time 1, should be the original value. This would be achieved using either e^{ξ_t} or $e^{\xi_{t-}}$ as both the forwarding and discounting term. However, this inconsistency in the deterministic situation does not occur when (ξ, η) are Lévy processes. The probability of ξ jumping at a fixed time t is zero, so the process (1.20) is a.s. equal (a modification) of the version which uses $e^{\xi_{t-}}$ as the forwarding term. As an economic model, it is favourable to use (1.20) since it uniquely solves (up to indistinguishability) the SDE described in Proposition 1.15.

In [42] and [37], the GOU (1.20) is used to model the total surplus of an insurance company in a more specialised way. As above, this model uses an exponential Lévy process e^{ξ_t} to model the price of a risky asset, and uses η as the income stream of the insurance company in a non-economic world. However, ξ and η are assumed to be independent, and η is assumed to be a compound Poisson process plus drift. Rather than investing all the current surplus in the risky asset, the company is allowed to invest part of its money in a riskless bond with the price process $e^{\delta t}$, where $\delta > 0$ is the riskless interest rate. The proportion of the surplus invested in the risky asset is denoted by $\theta \in [0, 1]$, and is assumed to remain constant over a predetermined time. Thus, as the Lévy process ξ fluctuates, the portfolio must be rebalanced to maintain a fixed θ. The combined investment process can be written as e^{ξ_θ} for a Lévy process ξ_θ and, for each θ, the total surplus is given by the GOU

$$V_{\theta,t} = e^{\xi_{\theta,t}} \left(V_0 + \int_0^t e^{-\xi_{\theta,s-}} \mathrm{d}\eta_s \right).$$

Another economic application of the GOU involves the COGARCH model, introduced in [38], which can be used to model the price of a risky asset. If L is a one-dimensional Lévy process then the COGARCH is defined to be the process $S := (S_t)_{t\geq 0}$ given by

$$S_t = \int_0^t \sigma_{s-} \mathrm{d}L_s$$

where σ^2 is a special case of the GOU (1.21), in which η is linear deterministic drift and ξ is defined in terms of L. The process σ can be interpreted as the volatility process and is defined by

$$\sigma_t^2 := e^{\xi_t} \left(\sigma_0^2 + \int_0^t e^{-\xi_{s-}} \mathrm{d}(\beta s) \right)$$

where $\beta > 0$ and ξ is a Lévy jump process defined by

$$\xi_t := at + \sum_{0<s\leq t} \ln\left(1 + b(\Delta L_s)^2\right)$$

for parameters $a < 0$ and $b > 0$. There is only one source of randomness underlying both the price process S and the volatility process σ, and that is the Lévy process L.

Chapter 2

No ruin for the Generalised Ornstein-Uhlenbeck process

2.1 Introduction

In this chapter we examine when the GOU has zero probability of ruin. We also present some basic foundational results on the behaviour of Z and V. As mentioned in Chapter 1, there are only a few papers dealing with ruin probability, or with passage-time problems, for the GOU. Patie [53], and Novikov [50], give first passage-time distributions in the special case that $\xi_t = \lambda t$ for $\lambda \in \mathbb{R}$, and η has no positive jumps. With regard to ruin probability, Nyrhinen [52] and Kalashnikov and Norberg [32] discretize the GOU into a stochastic recurrence equation. Under a variety of conditions, they produce some asymptotic equivalences for the infinite horizon ruin probability. The main results on GOU ruin probability come from Paulsen [56]. In the special case that ξ and η are independent, Paulsen gives conditions for certain ruin for the GOU and a formula for the ruin probability under conditions which ensure that the integral process Z_t converges almost surely as $t \to \infty$. Since these papers were written, the theory relating to the GOU, and to the process Z, has advanced. We have described these results in Section 1.4.

Our main results of the chapter are presented in Section 2.2. The first result, Theorem 2.1, presents exact necessary and sufficient conditions under which the infinite horizon ruin probability for the GOU is zero. These conditions do not relate to the convergence of Z or stationarity of V or to any moment conditions. Instead they are are expressed at a more basic level, directly on the Lévy measure of (ξ, η). This theorem shows that when ξ and η are dependent, the ruin probability function for the GOU behaves very differently to the case, described by Paulsen, in which ξ and η are independent. The second result, Theorem 2.3, shows that $P(Z_t < 0) > 0$ for all $t > 0$ as long as η is not a subordinator. This result is an important building block in the proof of Theorem 2.1, as well as being of interest in its own right. Finally in Section 2.2, Theorem 2.4 extends a ruin probability formula in Paulsen [56], presenting a slightly different version which deals with the general dependent case, and applies whenever Z_t converges almost surely to a random variable Z_∞ as $t \to \infty$.

Section 2.3 contains technical results of interest, which characterise what we call the lower bound function of the GOU, and are used to prove the main ruin probability theorem. Section 2.4 contains proofs of the results stated in Sections 2.2 and 2.3.

When we specialise to the situation that (ξ, η) is a compound Poisson process with deterministic drift, Theorem 2.1 can be proved using a different method. This method is less sophisticated and cannot be extended to the general case. Rather than utilising Theorem 2.3 and the theorems in section 2.3, it relies on a "brute force" approach. We present this theorem, and proof, in Appendix A

2.2 Ruin Probability Results

Our results are given in terms of regions of support of the Lévy measure $\Pi_{\xi,\eta}$. We define some notation, beginning with the following quadrants of the plane. Let $A_1 := \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$, and similarly, let A_2, A_3 and A_4 be the quadrants in which $\{x \geq 0, y \leq 0\}$, $\{x \leq 0, y \leq 0\}$ and $\{x \leq 0, y \geq 0\}$ respectively. For each $i = 1, 2, 3, 4$ and $u \in \mathbb{R}$ define

$$A_i^u := \big\{(x, y) \in A_i : y - u(e^{-x} - 1) < 0\big\}. \tag{2.1}$$

These sets are defined such that if $(\Delta\xi_t, \Delta\eta_t) \in A_i^u$ and $V_{t-} = u$, then $\Delta V_t < 0$, as we see from the equation

$$\begin{aligned}
\Delta V_t &= V_t - V_{t-} \\
&= e^{\xi_t}\Big(z + \int_0^{t-} e^{-\xi_{s-}} \mathrm{d}\eta_s + e^{-\xi_{t-}} \Delta\eta_t\Big) - e^{\xi_{t-}}\Big(z + \int_0^{t-} e^{-\xi_{s-}} \mathrm{d}\eta_s\Big) \\
&= (e^{\xi_t} - e^{\xi_{t-}})\Big(z + \int_0^{t-} e^{-\xi_{s-}} \mathrm{d}\eta_s\Big) + e^{\xi_t} e^{-\xi_{t-}} \Delta\eta_t \\
&= (e^{\Delta\xi_t} - 1)V_{t-} + e^{\Delta\xi_t}\Delta\eta_t & (2.2) \\
&= e^{\Delta\xi_t}\left(\Delta\eta_t - V_{t-}\left(e^{-\Delta\xi_t} - 1\right)\right). & (2.3)
\end{aligned}$$

If $u \leq 0$ then $A_2^u = A_2$ and $A_4^u = \emptyset$. As u decreases to $-\infty$, the sets A_1^u expand, whilst A_3^u shrink. Define

$$\theta_1 := \begin{cases} \sup\{u \leq 0 : \Pi_{\xi,\eta}(A_1^u) > 0\} \\ -\infty \quad \text{if } \Pi_{\xi,\eta}(A_1) = 0 \end{cases}, \theta_3 := \begin{cases} \inf\{u \leq 0 : \Pi_{\xi,\eta}(A_3^u) > 0\} \\ 0 \quad \text{if } \Pi_{\xi,\eta}(A_3) = 0 \end{cases}.$$

If $u \geq 0$ then $A_3^u = A_3$ and $A_1^u = \emptyset$. As u increases to ∞, the sets A_2^u shrink, whilst A_4^u expand. Define

$$\theta_2 := \begin{cases} \sup\{u \geq 0 : \Pi_{\xi,\eta}(A_2^u) > 0\} \\ 0 \quad \text{if } \Pi_{\xi,\eta}(A_2) = 0, \end{cases}, \theta_4 := \begin{cases} \inf\{u \geq 0 : \Pi_{\xi,\eta}(A_4^u) > 0\} \\ \infty \quad \text{if } \Pi_{\xi,\eta}(A_4) = 0 \end{cases}.$$

For each $i = 1, 2, 3, 4$, note that $\Pi_{\xi,\eta}(A_i^{\theta_i}) = 0$, since in the definitions of A_i^u we are requiring that $y - u(e^{-x} - 1)$ be strictly less than zero.

Theorem 2.1 (Exact conditions for no ruin for the GOU). *The ruin probability function satisfies $\psi(0) = 0$ if and only if η is a subordinator. If η is not a subordinator then there exists $c > 0$ such that the ruin probability function satisfies $\psi(c) = 0$ if and only if the Lévy measure satisfies $\Pi_{\xi,\eta}(A_3) = 0$, $\theta_2 \leq \theta_4$, and:*

- *when $\sigma_\xi^2 \neq 0$ the Gaussian covariance matrix is of form*

$$\Sigma_{\xi,\eta} = \begin{bmatrix} 1 & -u \\ -u & u^2 \end{bmatrix} \sigma_\xi^2$$

  for some $u \in [\theta_2, \theta_4]$ satisfying

$$g(u) := \tilde{\gamma}_\eta + u\tilde{\gamma}_\xi - \frac{1}{2}u\sigma_\xi^2 - \int_{\{x^2+y^2<1\}} (ux + y)\Pi_{\xi,\eta}(\mathrm{d}(x,y)) \geq 0; \quad (2.4)$$

- *when $\sigma_\xi^2 = 0$ the Gaussian covariance matrix is of form $\Sigma_{\xi,\eta} = 0$ and there exists $u \in [\theta_2, \theta_4]$ satisfying $g(u) \geq 0$.*

If $\sigma_\xi^2 \neq 0$ and the conditions of the theorem hold, then $\psi(z) = 0$ for all $z \geq c := -\frac{\sigma_{\xi,\eta}}{\sigma_\xi^2}$, whilst $\psi(z) > 0$ for all $z < c$.

If $\sigma_\xi^2 = 0$ and the conditions of the theorem hold, then $\psi(z) = 0$ for all $z \geq c := \inf\{u \in [\theta_2, \theta_4] : g(u) \geq 0\}$, whilst $\psi(z) > 0$ for all $z < c$.

We now discuss some examples and special cases which illustrate and amplify the results in Theorem 2.1.

Remark 2.2. 1. Suppose that (ξ,η) is continuous. By the Lévy-Itô decomposition in Proposition 1.8 we can write $(\xi_t,\eta_t) = (\gamma_\xi t, \gamma_\eta t) + (B_{\xi,t}, B_{\eta,t})$. Theorem 2.1 states that $\psi(z) = 0$ for all $z \geq u$ and $\psi(z) > 0$ for all $z < u$, if and only if there exists $u \geq 0$ such that $B_\eta = -uB_\xi$, and $(\gamma_\xi - \frac{1}{2}\sigma_\xi^2)u + \gamma_\eta \geq 0$. For example we could have

$$(\xi_t,\eta_t) := (B_t + ct, -B_t + (1/2 - c)t), \tag{2.5}$$

where $c \in \mathbb{R}$ and $\sigma_\xi^2 = 1$. Then Theorem 2.1 implies that $\psi(z) = 0$ for all $z \geq -\frac{\sigma_{\xi,\eta}}{\sigma_\xi^2} = 1$ whilst $\psi(z) > 0$ for all $z < 1$. In this simple case we can check the result directly and we present these calculations in Appendix B.

2. Suppose that (ξ,η) is a finite variation Lévy process. Then $\Sigma_{\xi,\eta} = 0$ and $\int_{|z|<1} |z|\Pi_{\xi,\eta}(\mathrm{d}z) < \infty$. We can define the drift vector as

$$(d_\xi, d_\eta) := (\tilde{\gamma}_\xi, \tilde{\gamma}_\eta) - \int_{|z|<1} z\Pi_{\xi,\eta}(\mathrm{d}z) \tag{2.6}$$

and write

$$(\xi_t,\eta_t) = (d_\xi, d_\eta)t + \int_{\mathbb{R}^2} zN_{\xi,\eta,t}(\cdot,\mathrm{d}z).$$

In this situation, Theorem 2.1 simplifies to the following statement: $\psi(0) = 0$ iff η is a subordinator. If η is not a subordinator then $\psi(c) = 0$ for some $c > 0$ iff $\Pi_{\xi,\eta}(A_3) = 0$, $\theta_2 \leq \theta_4$, and at least one of the following is true:

- $d_\xi = 0$, and $d_\eta \geq 0$; or
- $d_\xi > 0$ and $-\frac{d_\eta}{d_\xi} \leq \theta_4$; or
- $d_\eta > 0$, and $d_\xi < 0$, such that $-\frac{d_\eta}{d_\xi} \geq \theta_2$.

If the second property holds, then $\psi(z) = 0$ for all $z \geq c := \max\{\theta_2, -\frac{d_\eta}{d_\xi}\}$ and $\psi(z) > 0$ for all $z < c$. If the other properties hold, then $\psi(z) = 0$ for all $z \geq c := \theta_2$ and $\psi(z) > 0$ for all $z < c$.

These results are easily obtained by using (2.6) to transform condition (2.4) into the equation $g(u) = d_\eta + ud_\xi \geq 0$. For a simple example, let N_t be a Poisson process with parameter λ, let $c > 0$ and let

$$(\xi_t,\eta_t) := (-ct + N_t, 2ct - N_t). \tag{2.7}$$

Then we are in the third case above, and $\psi(z) = 0$ for all $z \geq \theta_2 = \frac{e}{e-1}$, and $\psi(z) > 0$ for all $z < \frac{e}{e-1}$. In this simple case, we can verify the results by direct but tedious calculations which we omit here.

3. Suppose that ξ and η are independent. This implies that ξ and η jump separately, which means that all jumps occur at the axes of the sets A_i. Further, there is zero covariance between the Brownian components of ξ and η, namely $\sigma_{\xi,\eta} = 0$. With a little work, Theorem 2.1 simplifies to the following statement: $\psi(0) = 0$ iff η is a subordinator. If η is not a subordinator then $\psi(z) = 0$ for $z > 0$ iff ξ and η are each of finite variation and have no negative jumps, and $g(z) = d_\eta + zd_\xi \geq 0$. Note that for this situation to occur, it must be that $d_\eta < 0$ (since η is not a subordinator), which implies that $d_\xi > 0$. Hence $E(\xi_1) > 0$.

4. Paulsen [56] states conditions for certain ruin when ξ and η are independent. In the cases $E(\xi_1) < 0$ and $E(\xi_1) = 0$, and under certain moment conditions, he shows that $\psi(z) = 1$ for all $z \geq 0$. Theorem 2.1 shows that the situation changes when dependence is allowed. The continuous process defined in (2.5), and the jump process defined in (2.7), illustrate this difference. Each process trivially satisfies Paulsen's moment conditions and can satisfy $E(\xi_1) < 0$, or $E(\xi_1) = 0$, depending on the choices of c and λ, however it is not the case that $\psi(z) = 1$ for all $z \geq 0$. Note that Paulsen does not comment on the possibility of zero ruin in the independent case. The above statement (3) completely explains this situation.

5. We make some comments on subordinators and explain why Theorem 2.1 has to have a separate statement for the simple case in which η is a subordinator. As noted in Proposition 1.14, η is a subordinator if and only if the following three conditions hold:

   - $\sigma_\eta^2 = 0$, so η has no Brownian component;
   - $\Pi_\eta((-\infty, 0)) = 0$, so η has no negative jumps;
   - $d_\eta \geq 0$, where

$$d_\eta := \gamma_\eta - \int_{(0,1)} y\Pi_\eta(\mathrm{d}y) = E\left(\eta_1 - \int_{(0,\infty)} yN_{\eta,1}(\cdot, \mathrm{d}y)\right).$$

   Note, by definition, $d_\eta \in [-\infty, \infty)$, and $d_\eta = -\infty$ iff $\int_{(0,1)} y\Pi_\eta(\mathrm{d}y) = \infty$.

Suppose that η is a subordinator. Since $\sigma_\eta^2 = 0$ the covariance matrix is of form $\Sigma_{\xi,\eta} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \sigma_\xi^2$. Using (1.12), and the fact that η has no negative jumps, we obtain

$$\begin{aligned}\tilde{\gamma}_\eta - \int_{\{x^2+y^2<1\}} y\Pi_{\xi,\eta}(\mathrm{d}(x,y)) &= \gamma_\eta - \int_{\mathbb{R}\times\{|y|<1\}} y\Pi_{\xi,\eta}(\mathrm{d}(x,y)) \\ &= d_\eta.\end{aligned}$$

Thus, the fact that $d_\eta \geq 0$, implies that (2.4) is satisfied for $u = 0$. Also, since η has no negative jumps, $\theta_2 = 0$, and hence the condition $\theta_2 \leq \theta_4$ is satisfied. However there is one condition that is not satisfied. Even though η has no negative jumps, we cannot say $\Pi_{\xi,\eta}(A_3) = 0$, since it may be the case that $\Pi_{\xi,\eta}((-\infty, 0) \times \{0\}) > 0$. Namely, ξ may make a negative jump at the same time that η has no jump.

6. If $\Pi_{\xi,\eta}(A_3) = 0$, and $\theta_2 \leq \theta_4$, then the function $g(u)$ from (2.4) exists for any $u \in [\theta_2, \theta_4]$, and $g(u) \in [-\infty, \infty)$. Under such conditions, the domain of integration for the integral component of g can be decreased using the fact that

$$\Pi_{\xi,\eta}\left(\{y - u(e^{-x} - 1) \leq 0\}\right) = 0. \tag{2.8}$$

Further, if $g(u)$ is finite for some $u \in [\theta_2, \theta_4]$, then

$$\int_{\{y-u(e^{-x}-1)\in(0,1)\}} \left(y - u(e^{-x} - 1)\right) \Pi_{\xi,\eta}(\mathrm{d}(x,y)) < \infty. \tag{2.9}$$

On first viewing, (2.9) may seem counterintuitive, as it places a constraint on the size of the positive jumps of V. However, if (2.9) does not hold, and all the other conditions, excluding (2.4), are satisfied, then the Lévy properties of (ξ, η) imply that V_t can drift negatively when $V_{t-} = u$. These statements, and the equations (2.8) and (2.9), are discussed further in Remark 2.10 following Theorem 2.9.

Theorem 2.3. *The Lévy process η is not a subordinator if and only if $P(Z_T < 0) > 0$ for any fixed time $T > 0$.*

One direction of this result is trivial and has been noted above, namely, if η is a subordinator then $P(Z_T < 0) = 0$ for any $T > 0$. The other direction seems quite intuitive and in fact is implicitly assumed by Paulsen [56] in the case when ξ and η are independent. However even in the independent case the proof is non-trivial. We prove it in the general case using a change of measure argument and some analytic lemmas. As well as being of independent interest, this result is essential in proving Theorem 2.1.

The final theorem in this section provides a formula for the ruin probability in the case that Z converges. Recall that T_z denotes the first time V drops below zero when $V_0 = z$, or equivalently, the first time Z drops below $-z$.

Theorem 2.4. *Suppose Z_t converges a.s. to a finite random variable Z_∞ as $t \to \infty$, and let $G(z) := P(Z_\infty \le z)$. Then*

$$\psi(z) = \frac{G(-z)}{E\left(G(-V_{T_z}) \middle| T_z < \infty\right)}.$$

Remark 2.5.

1. To clarify the meaning of this formula, note that
$$G(-V_{T_z})(\omega) := P\left(\nu \in \Omega \ : \ Z_\infty(\nu) < -V_{T_z}(\omega)\right),$$
which is defined whenever $T_z(\omega) < \infty$.

2. In the case that ξ and η are independent, Paulsen [56] shows, under a number of side conditions which ensure that Z_t converges a.s. to a finite random variable Z_∞ with distribution function $H(z) := P(Z_\infty < z)$ as $t \to \infty$, that
$$\psi(z) = \frac{H(-z)}{E\left(H(-V_{T_z}) \middle| T_z < \infty\right)}.$$
This formula is a modification of a result given by Harrison [31] for the special case in which ξ is deterministic drift and η is a Lévy process with finite variance. Theorem 2.4 extends the formula to the general dependent case. Our proof is similar to those of Paulsen and Harrison, however we write it out in full because some details are different.

3. As noted in Theorem 1.17, Z_t converges a.s. to a finite random variable Z_∞ as $t \to \infty$ if and only if $\lim_{t\to\infty} \xi_t = +\infty$ a.s. and $I_{\xi,\eta} < \infty$. As noted in Theorem 1.19, Lindner and Maller [44] prove that if V is not a constant process, then V is strictly stationary if and only if $\int_0^t e^{\xi_{s-}} \mathrm{d}K_s^{\xi,\eta}$ converges a.s. to a finite random variable as $t \to \infty$. In neither of these cases do the conditions of Theorem 2.1 simplify. Each of the processes defined in (2.5) and (2.7) can belong to either of these cases, or neither, depending on the choice of constant c and parameter λ.

4. As noted in Theorem 1.18, Z_t converges a.s. to a finite random variable Z_∞ as $t \to \infty$, then Z_∞ has an atom iff Z_∞ is a constant value $-c$ iff $Z_t = c(e^{-\xi_t} - 1)$ a.s. iff $V_t = e^{\xi_t}(z - c) + c$ a.s. In this case it is trivial that $\psi(z) = 0$ for all $z \ge c$. Theorem 2.1 produces the same result, however this will not become immediately clear until Remark 2.8 (2) following Theorem 2.7.

2.3 Technical Results of Interest

This section contains technical results needed in the proofs of Theorems 2.1 and 2.3, which also have some independent interest. The proofs of these results are given in Section 4. Recall that the stochastic, or Doléans-Dade, exponential of a semimartingale X_t is denoted by $\epsilon(X)_t$. The first proposition introduces a process W which will play an important role throughout the rest of the paper. This proposition is adapted from Proposition 8.22 of [14] and is presented without proof.

Proposition 2.6. *Given a bivariate Lévy process (ξ, η) there exists a Lévy process W such that $e^{-\xi_t} = \epsilon(W)_t$ and (ξ, η, W) is a trivariate Lévy process. If ξ has characteristic triplet $(\gamma_\xi, \sigma_\xi, \Pi_\xi)$ then*

$$W_t = -\xi_t + \frac{\sigma_\xi^2 t}{2} + \sum_{0<s\le t} \left(e^{-\Delta\xi_s} + \Delta\xi_s - 1\right) \tag{2.10}$$

and the characteristic triplet of W is given by $\sigma_W^2 = \sigma_\xi^2$ and

$$\Pi_W(\Lambda) = \Pi_\xi\left(\{x : e^{-x} - 1 \in \Lambda\}\right) \tag{2.11}$$

and

$$\gamma_W = -\gamma_\xi + \frac{1}{2}\sigma_\xi^2 + \int_{\mathbb{R}} \left(x 1_{(-1,1)}(x) + (e^{-x} - 1) 1_{(-\ln 2, \infty)}(x)\right) \Pi_\xi(\mathrm{d}x), \tag{2.12}$$

where the integral converges.

We define the lower bound function δ for V in (1.1) as

$$\delta(z) = \inf\left\{u \in \mathbb{R} : P\left(\inf_{t\ge 0} V_t \le u \middle| V_0 = z\right) > 0\right\}.$$

The following theorem exactly characterizes the lower bound function.

Theorem 2.7. *The lower bound function satisfies the following properties:*

1. *For all $z \in \mathbb{R}$, $\delta(z) \le z$.*
2. *If $z_1 < z_2$ then $\delta(z_1) \le \delta(z_2)$.*
3. *For all $z \in \mathbb{R}$, $\delta(z) = z$ if and only if $\eta - zW$ is a subordinator.*
4. *For all $z \in \mathbb{R}$, $\delta(z) = \delta(\delta(z))$, and*

$$\delta(z) = \sup\left\{u : u \le z, \eta - uW \text{ is a subordinator}\right\}.$$

Remark 2.8. 1. If η is a subordinator then $\delta(0) = 0$, so V cannot drop below zero when $V_0 = z \geq 0$.

2. As noted in Theorem 1.18, if Z_t converges a.s. to a finite random variable Z_∞ as $t \to \infty$, then $Z_\infty = -a$ iff $e^{-\xi} = \epsilon(\eta/a)$. If this holds then $\delta(a) = a$ by point 3 above, since $\eta/a = W$. Thus $\psi(z) = 0$ for all $z \geq a$, as mentioned in Remark 2.5 (3).

Theorem 2.9. *Let $u \in \mathbb{R} \setminus \{0\}$ and let (ξ, η, W) be the trivariate Lévy process from Proposition 2.6. The Lévy process $\eta - uW$ is a subordinator if and only if the following three conditions are satisfied: the Gaussian covariance matrix is of the form*

$$\Sigma_{\xi,\eta} = \begin{bmatrix} 1 & -u \\ -u & u^2 \end{bmatrix} \sigma_\xi^2, \tag{2.13}$$

at least one of the following is true:

- $\Pi_{\xi,\eta}(A_3) = 0$ *and* $\theta_2 \leq \theta_4$ *and* $u \in [\theta_2, \theta_4]$;
- $\Pi_{\xi,\eta}(A_2) = 0$ *and* $\theta_1 \leq \theta_3$ *and* $u \in [\theta_1, \theta_3]$;
- $\Pi_{\xi,\eta}(A_3) = \Pi_{\xi,\eta}(A_2) = 0$ *and* $u \in [\theta_1, \theta_4]$;

and in addition, u satisfies (2.4).

Remark 2.10. In Remark 2.2 (5) we stated three necessary and sufficient conditions for a Lévy process to be a subordinator. These three conditions correspond respectively with the three conditions in Theorem 2.9, as we shall see in the proof. In particular, if one of the dot point conditions holds, and $u \in [\theta_i, \theta_j]$ for its respective i, j, then $\Pi_{\eta-uW}((-\infty, 0]) = 0$, which we will show to be equivalent to (2.8), and the function g from (2.4) satisfies $g(u) = d_{\eta-uW} \in [-\infty, \infty)$. Further, if $g(u)$ is finite for some $u \in [\theta_i, \theta_j]$ then $\int_{(0,1)} z\Pi_{\eta-uW}(\mathrm{d}z) < \infty$, which we will show to be equivalent to (2.9). Note that if $\eta - uW$ has no Brownian component, no negative jumps, but $\int_{(0,1)} z\Pi_{\eta-uW}(\mathrm{d}z) = \infty$, then, somewhat suprisingly, $\eta - uW$ is fluctuating and hence not a subordinator, regardless of the value of the shift constant $\gamma_{\eta-uW}$. This behaviour occurs since $d_{\eta-uW} = -\infty$, and is explained in Sato [62], p138.

2.4 Proofs

We begin by proving Theorem 2.3. For this proof, some lemmas are required. In these we assume that $X = (\xi, \eta)$ has bounded jumps so that X has finite absolute moments of all orders. Then, to prove Theorem 2.3 we reduce to this case.

Lemma 2.11. *Suppose $X = (\xi, \eta)$ has bounded jumps and $E(\eta_1) = 0$. If we let $T > 0$ be a fixed time then Z^T is a mean-zero martingale with respect to $\mathbb{F}$.*

Proof. Since η is a Lévy process the assumption $E(\eta_1) = 0$ implies that η is a càdlàg martingale. Since ξ is càdlàg , $e^{-\xi}$ is a locally bounded process and hence Z is a local martingale for $\mathbb{F}$ by Protter [60], p.171. If we show that $E\left(\sup_{s\le t}|Z_s^T|\right) < \infty$ for every $t \ge 0$ then Protter [60], p.38 implies that Z^T is a martingale. This is equivalent to showing $E\left(\sup_{t\le T}|Z_t|\right) < \infty$. Since Z is a local martingale and $Z_0 = 0$, the Burkholder-Davis-Gundy inequalities in Lipster and Shiryaev [46], p.70 and p.75, ensure the existence of $b > 0$ such that

$$\begin{aligned}
E\left(\sup_{0\le t\le T}\left|\int_0^t e^{-\xi_{s-}}\mathrm{d}\eta_s\right|\right) &\le bE\left(\left[\int_0^\bullet e^{-\xi_{s-}}\mathrm{d}\eta_s, \int_0^\bullet e^{-\xi_{s-}}\mathrm{d}\eta_s\right]_T^{1/2}\right)\\
&= bE\left(\left(\int_0^T e^{-2\xi_{s-}}\mathrm{d}[\eta,\eta]_s\right)^{1/2}\right)\\
&\le bE\left(\left(\int_0^T \sup_{0\le t\le T} e^{-2\xi_t}\mathrm{d}[\eta,\eta]_s\right)^{1/2}\right)\\
&= bE\left(\sup_{0\le t\le T} e^{-\xi_t}[\eta,\eta]_T^{1/2}\right)\\
&\le b\left(E\left(\sup_{0\le t\le T} e^{-2\xi_t}\right)\right)^{1/2}\left(E\left([\eta,\eta]_T\right)\right)^{1/2},
\end{aligned}$$

where the second inequality follows from the fact that $[\eta,\eta]_s$ is increasing and the final inequality follows by the Cauchy-Schwarz inequality. (The notation $[\cdot,\cdot]$ denotes the quadratic variation process.) Now, by Protter [60], p.70,

$$E\left([\eta,\eta]_T\right) = \sigma_\eta^2 T + E\left(\sum_{0\le s\le T}(\Delta\eta)^2\right) = \sigma_\eta^2 T + T\int x^2\Pi_\eta(\mathrm{d}x),$$

which is finite since η has bounded jumps. Thus it suffices to prove that

$$E\left(\sup_{0\le t\le T} e^{-2\xi_t}\right) < \infty.$$

Setting $Y_t := e^{-\xi_t}/E(e^{-\xi_t})$, a non-negative martingale, it follows by Doob's maximal inequality, as expressed in Shiryaev [1], p.765, that

$$E\left(\sup_{0\le t\le T}\frac{e^{-2\xi_t}}{(E(e^{-\xi_t}))^2}\right)\le 4\frac{E\left(e^{-2\xi_T}\right)}{(E(e^{-\xi_T}))^2},$$

which is finite since ξ has bounded jumps and hence has finite exponential moments of all orders (Sato [62], p.161). It is shown in Sato [62], p.165, that $\left(E(e^{-\xi_t})\right)^2=\left(E(e^{-\xi_1})\right)^{2t}$. Letting $c:=\left(E(e^{-\xi_1})\right)^2\in(0,\infty)$, the above inequality implies that

$$E\left(\sup_{0\le t\le T}e^{-2\xi_t}\right)\le\max\{1,c^T\}E\left(\sup_{0\le t\le T}\frac{e^{-2\xi_t}}{c^t}\right)<\infty.$$

□

We now present two lemmas dealing with absolute continuity of measures. These lemmas will be used to construct a new process W such that W^T is a mean-zero martingale which is mutually absolutely continuous with Z^T. Then $P(Z_T<0)>0$ if and only if $P(W_T<0)>0$, and the latter statement will follow immediately from the fact that W^T is a mean-zero martingale.

Lemma 2.12. *Let $X:=(\xi,\eta)$ and $Y:=(\tau,\nu)$ be bivariate Lévy processes adapted to $(\Omega,\mathscr{F},\mathbb{F},P)$, and let $Z_t:=\int_0^t e^{-\xi_{s-}}\mathrm{d}\eta_s$ and $W_t:=\int_0^t e^{-\tau_{s-}}\mathrm{d}\nu_s$. If the induced probability measures of X^T and Y^T are mutually absolutely continuous, then the induced probability measures of Z^T and W^T are mutually absolutely continuous.*

Proof. Let $D([0,T]\to\mathbb{R}^2)$ denote the set of càdlàg functions from $[0,T]$ to $\mathbb{R}^2$ and $\mathscr{B}^{2[0,T]}$ denote the σ-algebra generated in this set by the Borel cylinder sets (see Kallenberg [33]). Then the induced probability measures of X^T and Y^T can be written as P_{X^T} and P_{Y^T} on the measure space $\left(D([0,T]\to\mathbb{R}^2),\mathscr{B}^{2[0,T]}\right)$. Let $C:=(C',C'')$ be the co-ordinate mapping of $\left(D([0,T]\to\mathbb{R}^2),\mathscr{B}^{2[0,T]}\right)$ to itself. Define Z' on the probability space $\left(D([0,T]\to\mathbb{R}^2),\mathscr{B}^{2[0,T]},P_{X^T}\right)$ by $Z'_t:=\int_0^t e^{-C'_{s-}}\mathrm{d}C''_s$. Define W' on $\left(D([0,T]\to\mathbb{R}^2),\mathscr{B}^{2[0,T]},P_{Y^T}\right)$ by $W'_t:=\int_0^t e^{-C'_{s-}}\mathrm{d}C''_s$. Note that Z' and W' are different processes since they are being evaluated under different measures. Now $Z=X\circ Z'$ and $W=Y\circ W'$. Hence $P(Z^T\in\Lambda)=P_{X^T}(Z'\in\Lambda)$ and $P(W^T\in\Lambda)=P_{Y^T}(W'\in\Lambda)$. Since P_{X^T} and P_{Y^T} are mutually absolutely continuous, Protter [60], p.60 implies that Z' and W' are P_{X^T}-indistinguishable, and P_{Y^T}-indistinguishable. So $P_{X^T}(Z'\in\Lambda)=P_{X^T}(W'\in\Lambda)$. Since P_{X^T} and P_{Y^T} are mutually absolutely continuous $P_{X^T}(W'\in\Lambda)=0$ iff $P_{Y^T}(W'\in\Lambda)=0$ which proves $P(Z^T\in\Lambda)=0$ iff $P(W^T\in\Lambda)=0$, as required. □

Lemma 2.13. *If $X := (\xi, \eta)$ has bounded jumps, $E(\eta_1) \geq 0$, η is not a subordinator, and η is not pure deterministic drift, then there exists a bivariate Lévy process $Y := (\tau, \nu)$ with bounded jumps, adapted to $(\Omega, \mathscr{F}, \mathbb{F}, P)$, such that X^T and Y^T are mutually absolutely continuous for all $T > 0$, and $E(\nu_1) = 0$.*

Proof. As mentioned in Remark 2.2 (5), the Lévy process η is a subordinator if and only if the following three conditions hold: $\sigma_\eta^2 = 0$, $\Pi_\eta((-\infty, 0)) = 0$, and $d_\eta \geq 0$ where $d_\eta := \gamma_\eta - \int_{(0,1)} y \Pi_\eta(\mathrm{d}y)$. Thus it suffices to prove the lemma in the following three cases.

Case 1: Suppose $\sigma_\eta \neq 0$. Given dependent Brownian motions B_ξ and B_η there exists a Brownian motion B' independent of B_η, and constants a_1 and a_2 such that $(B_\xi, B_\eta) = (a_1 B' + a_2 B_\eta \ , \ B_\eta)$. Using the Lévy-Itô decomposition, X can be written as the sum of two independent processes as follows;

$$X_t = (\xi_t, \eta_t) = (\xi_t' + B_{\xi,t}, \eta_t' + B_{\eta,t}) =_D (\xi_t' + a_1 B_t', \eta_t') + (a_2 B_{\eta,t}, B_{\eta,t}),$$

where (ξ', η') is a pure jump Lévy process with drift, independent of (B_ξ, B_η). Let $c := E(\eta_1)$ and define the Lévy process Y by

$$Y_t := (\xi_t' + a_1 B_t', \eta_t') + (a_2(B_{\eta,t} - ct), B_{\eta,t} - ct)\,.$$

It is a simple consequence of Girsanov's theorem for Brownian motion, e.g. Klebaner [36], p.241, that the induced measures of the processes $B_{\eta,t}$ and $B_{\eta,t} - ct$ on $\big(D([0,T] \to \mathbb{R}), \mathscr{B}^{[0,T]}\big)$ are mutually absolutely continuous. It is trivial to show that this implies that the induced probability measures of $(a_2 B_{\eta,t}, B_{\eta,t})^T$ and $(a_2(B_{\eta,t} - ct), B_{\eta,t} - ct)^T$ are mutually absolutely continuous. Using independence, this implies that the induced probability measures of X^T and Y^T are mutually absolutely continuous. Note that if we write Y as $Y = (\tau, \nu)$ then $\nu_t = \eta_t - ct$ so $E(\nu_1) = 0$ as required.

Case 2: Suppose $\sigma_\eta = 0$ and $\Pi_\eta((-\infty, 0)) > 0$. We can assume that X has jumps contained in Λ, a square in $\mathbb{R}^2$, i.e for all $t > 0$

$$(\Delta\xi_t, \Delta\eta_t) \in \Lambda := \{(x, y) \in \mathbb{R}^2 : -a \leq x \leq a, -a \leq y \leq a\}.$$

For any $0 < b < a$ define the set $\Gamma \subset \Lambda$ by

$$\Gamma := \{(x, y) \in \mathbb{R}^2 : -a \leq x \leq a, -a \leq y \leq -b\}.$$

A Lévy measure is σ-finite and $\Pi_\eta((-\infty, 0)) > 0$ so there must exist a $b > 0$ small enough such that $\Pi_X(\Gamma) > 0$. By Protter [60], p.27, we can write $X =$

$\tilde{X} + \hat{X}$ where $\tilde{X}_t := (\tilde{\xi}_t, \tilde{\eta}_t)$ is a Lévy process with jumps contained in $\Lambda \setminus \Gamma$ and $\hat{X}_t := (\hat{\xi}_t, \hat{\eta}_t)$ is a compound Poisson process independent of $\tilde{X}$, with jumps in Γ and parameter $\lambda := \Pi_X(\Gamma) < \infty$. So we can write $\hat{X}_t = \sum_{i=1}^{N_t} C_i$ where N is a Poisson process with parameter λ and $(C_i)_{i\geq 1} := (C_i', C_i'')_{i\geq 1}$ is an independent identically distributed sequence of two dimensional random vectors, independent of N, with $C_i \in \Gamma$. Let M be a Poisson process independent of N, C_i and $\tilde{X}$, with parameter $r\lambda$ for some $r \geq 1$. Define the Lévy process Y by $Y_t := \tilde{X}_t + \sum_{i=1}^{M_t} C_i$. We show the induced probability measures of X^T and Y^T on $\big(D([0,T] \to \mathbb{R}), \mathscr{B}^{[0,T]}\big)$ are mutually absolutely continuous. Since $\tilde{X}$ is independent of both compound Poisson processes, this is equivalent to showing the induced probability measures of $\sum_{i=1}^{N_t} C_i$ and $\sum_{i=1}^{M_t} C_i$ are mutually absolutely continuous. Let $A \in \mathscr{B}^{[0,T]}$ and note that

$$P\left(\left(\sum_{i=1}^{N_t} C_i\right)_{0\leq t\leq T} \in A\right) = \sum_{n=0}^{\infty} P\left(\left(\sum_{i=1}^{N_t} C_i\right)_{0\leq t\leq T} \in A \middle| N_T = n\right) P\left(N_T = n\right). \tag{2.14}$$

Since N is a Poisson process, $P(N_t = n) > 0$ for all $n \in \mathbb{N}$. Thus the left hand side of (2.14) is zero if and only if $P\left(\left(\sum_{i=1}^{N_t} C_i\right)_{0\leq t\leq T} \in A \middle| N_T = n\right) = 0$ for all $n \in \mathbb{N}$.

For any Poisson processes, regardless of the parameter, Kallenberg [33], p.179, shows that once we condition on the event that n jumps have occurred in time $(0,T]$, then the jump times are uniformly distributed over $(0,T]$. This implies that

$$P\left(\left(\sum_{i=1}^{N_t} C_i\right)_{0\leq t\leq T} \in A \middle| N_T = n\right) = P\left(\left(\sum_{i=1}^{M_t} C_i\right)_{0\leq t\leq T} \in A \middle| M_T = n\right).$$

Thus $P\left(\left(\sum_{i=1}^{N_t} C_i\right)_{0\leq t\leq T} \in A\right) = 0$ if and only if $P\left(\left(\sum_{i=1}^{M_t} C_i\right)_{0\leq t\leq T} \in A\right) = 0$, which proves that the two measures are mutually absolutely continuous, as required.

Recall that $Y_t =: (\tau_t, \nu_t) = \tilde{X}_t + \sum_{i=1}^{M_t} C_i$ where $\tilde{X} := (\tilde{\xi}, \tilde{\eta})$ and $C_i := (C_i', C_i'') \in \Gamma$. Thus $\nu_t = \tilde{\eta}_t + \sum_{i=1}^{M_t} C_i''$ which implies that $tE(\nu_1) = tE(\tilde{\eta}_1) + r\lambda t E(C_i'')$ where $E(\tilde{\eta}_1) > E(\eta_1) \geq 0$. Choosing $r = E(\tilde{\eta}_1)/|\lambda E(C_i'')|$ gives $E(\nu_1) = 0$ as required.

Case 3: Suppose $\sigma_\eta = 0$, $\Pi_\eta((-\infty, 0)) = 0$, and $d_\eta < 0$, where we allow the possibility that $d_\eta = -\infty$. If $\Pi_\eta((0,\infty)) = 0$ then $\eta_t = d_\eta t$ is deterministic, and this possibility has been excluded. So $\Pi_\eta((0,\infty)) > 0$, and we can assume X has jumps contained in Λ where we define the set $\Lambda := \{(x,y) \in \mathbb{R}^2 : -a \leq x \leq a, 0 < y \leq a\}$. For any $0 < b < a$ define the set $\Gamma^{(b)} \subset \Lambda$ by $\Gamma^{(b)} := \{(x,y) \in \mathbb{R}^2 : -a \leq x \leq a, b \leq y \leq a\}$. We can write $X = \tilde{X}^{(b)} + \hat{X}^{(b)}$ where $\tilde{X}^{(b)} := (\tilde{\xi_t}^{(b)}, \tilde{\eta_t}^{(b)})$ is a Lévy process with jumps contained in $\Lambda \setminus \Gamma^{(b)}$ and $\hat{X}^{(b)} := (\hat{\xi_t}^{(b)}, \hat{\eta_t}^{(b)})$ is a compound Poisson process independent of $\tilde{X}^{(b)}$, with jumps in $\Gamma^{(b)}$ and parameter $\lambda^{(b)} := \Pi_X(\Gamma^{(b)}) < \infty$.

If $d_\eta \in (-\infty, 0)$ then we can write $E\left(\tilde{\eta_t}^{(b)}\right) = d_\eta t + t\int_{(0,b)} x\Pi_\eta(\mathrm{d}x)$. Since $\lim_{b\downarrow 0} \int_{(0,b)} x\Pi_\eta(\mathrm{d}x) = 0$, there exists $b > 0$ such that $E\left(\tilde{\eta_t}^{(b)}\right) < 0$. If $d_\eta = -\infty$ then $\int_{(0,1)} x\Pi_\eta(\mathrm{d}x) = \infty$. Note that $E(\eta_1) = E\left(\tilde{\eta_1}^{(b)}\right) + E\left(\hat{\eta_1}^{(b)}\right) \in (0, \infty)$ since jumps are bounded, whilst

$$\lim_{b\downarrow 0} E\left(\hat{\eta_t}^{(b)}\right) = \lim_{b\downarrow 0} \int_{(b,a)} x\Pi_\eta(\mathrm{d}x) = \infty.$$

Hence there again exists $b > 0$ such that $E\left(\tilde{\eta_t}^{(b)}\right) < 0$. From now on we assume $b > 0$ is small enough such that $E\left(\tilde{\eta_t}^{(b)}\right) < 0$. Since a Lévy measure is σ-finite and $\Pi_\eta((0,\infty)) > 0$ we can also assume $\Pi_X(\Gamma^{(b)}) > 0$. Thus we drop the $^{(b)}$ from our labeling. We can write $\hat{X}_t = \sum_{i=1}^{N_t} C_i$ where N is a Poisson process with parameter λ and $(C_i)_{i\geq 1} := (C'_i, C''_i)_{i\geq 1}$ is an independent identically distributed sequence of two dimensional random vectors, independent of N, with $C_i \in \Gamma$. Let M be a Poisson process independent of N, C_i and $\tilde{X}$, with parameter $r\lambda$ for some $r > 0$. Define the Lévy process Y by $Y_t := \tilde{X}_t + \sum_{i=1}^{M_t} C_i$. Then the induced probability measures of X^T and Y^T are mutually absolutely continuous by the same proof as used in Case 2. If $Y =: (\tau, \nu)$ then $\nu_t = \tilde{\eta}_t + \sum_{i=1}^{M_t} C''_i$ with $C''_i \in [b,a]$. Since $E(\tilde{\eta}_1) < 0$ for our choice of $0 < b < a$, choosing $r = |E(\tilde{\eta}_1)|/\lambda E(C''_i)$ gives the result. □

Proof of Theorem 2.3. Take a general (ξ, η), let $a > 0$ and define

$$\Lambda := \{(x,y) \in \mathbb{R}^2 : -a \leq x \leq a, -a \leq y \leq a\}.$$

We can write $X = \tilde{X} + \hat{X}$ where $\tilde{X}_t := (\tilde{\xi}_t, \tilde{\eta}_t)$ is a Lévy process with jumps contained in Λ and $\hat{X}_t := (\hat{\xi}_t, \hat{\eta}_t)$ is a compound Poisson process, independent of $\tilde{X}$, with jumps in $\mathbb{R}^2 \setminus \Lambda$, and parameter $\lambda := \Pi_X(\mathbb{R}^2 \setminus \Lambda) < \infty$. Note that

$$\hat{X}_t := \sum_{0\leq s\leq t} \Delta X_s 1_{\mathbb{R}^2\setminus\Lambda}(\Delta X_s)$$

and by Poisson properties, $P(\hat{X}_t = 0) > 0$ for any $t \geq 0$.

Suppose that $P\left(\int_0^T e^{-\tilde{\xi}_{s-}}\mathrm{d}\tilde{\eta}_s < 0\right) > 0$. Then $P(Z_T < 0) > 0$, because

$$\begin{aligned} P\left(\int_0^T e^{-\xi_{s-}}\mathrm{d}\eta_s < 0\right) &\geq P\left(\int_0^T e^{-\xi_{s-}}\mathrm{d}\eta_s < 0 \;\middle|\; \hat{X}_T = 0\right) P\left(\hat{X}_T = 0\right) \\ &= P\left(\int_0^T e^{-\tilde{\xi}_{s-}}\mathrm{d}\tilde{\eta}_s < 0 \;\middle|\; \hat{X}_T = 0\right) P\left(\hat{X}_T = 0\right) \\ &= P\left(\int_0^T e^{-\tilde{\xi}_{s-}}\mathrm{d}\tilde{\eta}_s < 0\right) P\left(\hat{X}_T = 0\right) \\ &> 0. \end{aligned}$$

Further, note that η is not a subordinator iff we can choose $a > 0$ such that $\tilde{\eta}$ is not a subordinator. If $\sigma_\eta^2 > 0$ or $d_\eta < 0$ then any $a > 0$ suffices. If $\Pi_\eta((-\infty, 0)) > 0$ then we can choose $a > 0$ large enough such that $\Pi_\eta((-a, 0)) > 0$. The converse is obvious. Thus the theorem is proved if we can prove it for the case in which the jumps are bounded. From now on assume that the jumps of $X = (\xi, \eta)$ are contained in the set Λ defined above. Note that this implies that $E(\eta_1)$ is finite.

If η is pure deterministic drift, then $\eta_t = d_\eta t$ where $d_\eta < 0$, since η is not a subordinator. In this case the theorem is trivial, since Z is strictly decreasing. Thus, assume that η is not deterministic drift. We first prove the theorem in the case that $-c := E(\eta_1) < 0$. Note that

$$\begin{aligned} P\left(Z_T < 0\right) &= P\left(\int_0^T e^{-\xi_{s-}}\mathrm{d}(\eta_s + cs) - \int_0^T e^{-\xi_{s-}}\mathrm{d}(cs) < 0\right) \\ &\geq P\left(\int_0^T e^{-\xi_{s-}}\mathrm{d}(\eta_s + cs) < 0\right) \\ &> 0. \end{aligned}$$

The final inequality follows by Lemma 2.11, which implies that $\int_0^T e^{-\xi_{s-}}\mathrm{d}(\eta_s + cs)$ is a martingale, so $E\left(\int_0^T e^{-\xi_{s-}}\mathrm{d}(\eta_s + cs)\right) = 0$. Note that $\int_0^T e^{-\xi_{s-}}\mathrm{d}(\eta_s + cs)$ is not identically zero due to our assumption that η is not deterministic drift.

Now we assume that $c := E(\eta_1) \geq 0$. Lemma 2.13 ensures there exists $Y := (\tau, \nu)$ with bounded jumps, adapted to $(\Omega, \mathscr{F}, \mathbb{F}, P)$, such that X^T and Y^T are mutually absolutely continuous for all $T > 0$, and $E(\nu_1) = 0$. If we let $W_t := \int_0^t e^{-\tau_{s-}}\mathrm{d}\nu_s$ then Lemma 2.11 ensures that W^T is a mean-zero martingale. We prove that W_T is not identically zero

Firstly, note that if ν is deterministic drift then the condition $E(\nu_1) = 0$ implies that ν is identically zero. This cannot occur, since ν is mutually absolutely continuous with η, and we have assumed that η is not identically zero. Now, since ν is not deterministic drift, the quadratic variation $[\nu, \nu]$ is an increasing process.

Hence

$$\Big[\int_0^\bullet e^{-\tau_{s-}}\mathrm{d}\nu_s, \int_0^\bullet e^{-\tau_{s-}}\mathrm{d}\nu_s\Big]_T = \int_0^T e^{-2\tau_{s-}}\mathrm{d}[\nu,\nu]_s > 0.$$

If W_T is identically zero then W_t must be identically zero for all $t \le T$, since W^T is a martingale. Thus $[W,W]_T = 0$, which gives a contradiction.

Since W is not identically zero, and $E(W_T) = 0$, we conclude $P(W_T < 0) > 0$. However, Lemma 2.12 ensures that the induced probability measures of Z^T and W^T are mutually absolutely continuous. Hence $P(Z_T < 0) > 0$. □

Theorem 2.1 follows from Theorems 2.7 and 2.9. So we now prove these theorems.

Proof of Theorem 2.7. Property 1 is immediate from the definition while Property 2 follows from the fact that V_t is increasing in z for all $t \ge 0$. Let W be the process such $e^{-\xi_t} = \epsilon(W)_t$. Then for any $u \in \mathbb{R}$,

$$\begin{aligned} V_t &= e^{\xi_t}\left(z + \int_0^t e^{-\xi_{s-}} d\eta_s\right) \\ &= e^{\xi_t}\left(z + \int_0^t e^{-\xi_{s-}} d(\eta_s - uW_s) + u\int_0^t e^{-\xi_{s-}} dW_s\right) \\ &= e^{\xi_t}\left(z + \int_0^t e^{-\xi_{s-}} d(\eta_s - uW_s) + u(e^{-\xi_t} - 1)\right) \\ &= u + e^{\xi_t}\left(z - u + \int_0^t e^{-\xi_{s-}} d(\eta_s - uW_s)\right). \end{aligned}$$

Now if $\eta - zW$ is a subordinator then $\int_0^t e^{-\xi_{s-}} d(\eta_s - zW_s) \ge 0$ so $\delta(z) = z$. By Theorem 2.3 if $\eta - zW$ is not a subordinator then for some t and some $\epsilon > 0$,

$$P\left(\int_0^t e^{-\xi_{s-}} d(\eta_s - zW_s) < -\epsilon\right) > 0$$

and so, with $V_0 = z + \epsilon$ and $u = z$,

$$\begin{aligned} & P\left(\inf_{t\ge 0} V_t < z \middle| V_0 = z + \epsilon\right) \\ = \quad & P\left(\inf_{t\ge 0}\left\{z + e^{\xi_t}\left(\epsilon + \int_0^t e^{-\xi_{s-}} d(\eta_s - zW_s)\right)\right\} < z\right) \\ > \quad & 0, \end{aligned}$$

which implies that $\delta(z) \le \delta(z+\epsilon) < z$ and establishes Property 3.

Property 3 implies Property 4 if $\eta - \delta(z)W$ is a subordinator. So suppose that $\eta - \delta(z)W$ is not a subordinator. Then from the argument above we know that for some $\epsilon > 0$, $\delta(\delta(z)+\epsilon) < \delta(z)$. Let $T_u = \inf\{t > 0 : V_t \le u\}$. By definition of δ we have that $P(T_{\delta(u)+\epsilon} < \infty) > 0$. By the strong Markov property of V_t, if $u < z$,

$$\begin{aligned}
& P\left(\inf_{t\ge 0} V_t < \delta(u) \middle| V_0 = z\right) \\
\ge\quad & P\left(\inf_{t\ge 0} V_{t+T_{\delta(u)+\epsilon}} < \delta(u) \middle| V_0 = z\right) \\
=\quad & P\left(\inf_{t\ge 0} V_{t+T_{\delta(u)+\epsilon}} < \delta(u) \middle| T_{\delta(u)+\epsilon} < \infty, V_0 = z\right) P\left(T_{\delta(u)+\epsilon} < \infty\right) \\
\ge\quad & P\left(\inf_{t\ge 0} V_t < \delta(u) \middle| V_0 = \delta(u)+\epsilon\right) P\left(T_{\delta(u)+\epsilon} < \infty\right) \\
>\quad & 0.
\end{aligned}$$

This contradiction proves Property 4. □

Proof of Theorem 2.9. The Lévy process $S^{(u)} := \eta - uW$ is a subordinator if and only if the following three conditions hold: $\sigma^2_{S^{(u)}} = 0$, $\Pi_{S^{(u)}}((-\infty, 0)) = 0$, and $d_{S^{(u)}} \ge 0$ where $d_{S^{(u)}} := E\left(S_1^{(u)} - \int_{(0,\infty)} zN_{S^{(u)},1}(\cdot, \mathrm{d}z)\right)$.

Note that $\sigma^2_{S^{(u)}} = 0$ is equivalent to $B_\eta - uB_W = 0$, which is equivalent to $B_\eta = -uB_\xi$ by (2.10),which establishes (2.13).

We show that $S^{(u)}$ has no negative jumps for $u \ne 0$ if and only at least one of the dot point conditions of the theorem hold. Using (2.10) we see that $\Delta S_t^{(u)} = \Delta\eta_t - u\left(e^{-\Delta\xi_t} - 1.\right)$. If $u > 0$ then $\Delta S_t^{(u)} < 0$ requires $(\Delta\xi_t, \Delta\eta_t)$ be contained within A_2^u, A_3, or A_4^u. Every $(\Delta\xi_t, \Delta\eta_t) \in A_3$ produces a $\Delta S_t^{(u)} < 0$. Recall that the value θ_2 is the supremum of all the values of $u \ge 0$ at which there can be a negative jump $\Delta S_t^{(u)}$ with $(\Delta\xi_t, \Delta\eta_t) \in A_2$. Note that at $u = \theta_2$ such a jump is not possible. The obvious symmetric statement holds for θ_4. Hence, if $u > 0$ then $S^{(u)}$ has no negative jumps if and only if $\Pi_{\xi,\eta}(A_3) = 0$, $\theta_2 \le \theta_4$ and $u \in [\theta_2, \theta_4]$.

If $u < 0$ then $\Delta S_t^{(u)} < 0$ requires $(\Delta\xi_t, \Delta\eta_t)$ be contained within A_1^u, A_2, or A_3^u. Every $(\Delta\xi_t, \Delta\eta_t) \in A_2$ produces a $\Delta S_t^{(u)} < 0$. Recall that the value θ_1 is the supremum of all the values of $u \le 0$ at which there can be a negative jump $\Delta S_t^{(u)}$ with $(\Delta\xi, \Delta\eta) \in A_1$, and at $u = \theta_1$ such a jump is not possible. The obvious symmetric statement holds for θ_3. Hence, if $u < 0$ then $S^{(u)}$ can have no negative jumps if and only if $\Pi_{\xi,\eta}(A_2) = 0$, $\theta_1 \le \theta_3$ and $u \in [\theta_1, \theta_3]$. Finally, if $\Pi_{\xi,\eta}(A_3) = \Pi_{\xi,\eta}(A_2) = 0$ then $\theta_3 = \theta_2 = 0$ and so both of the above are satisfied when $u \in [\theta_1, \theta_4]$.

Now suppose that at least one of the dot point conditions holds. We let $u \in [\theta_i, \theta_j]$ for suitable i, j, and prove that $g(u) = d_{S^{(u)}}$. First, note that for any Borel set Λ

$$\begin{aligned}
\int_\Lambda z N_{\eta-uW,1}(\cdot, \mathrm{d}z) &= \int_{\{x+y\in\Lambda\}} (x+y) N_{-uW,\eta,1}(\cdot, \mathrm{d}(x,y)) \\
&= \int_{\{y-ux\in\Lambda\}} (y-ux) N_{W,\eta,1}(\cdot, \mathrm{d}(x,y)) \\
&= \int_{\{y-u(e^{-x}-1)\in\Lambda\}} \left(y-u(e^{-x}-1)\right) N_{\xi,\eta,1}(\cdot, \mathrm{d}(x,y)).
\end{aligned}$$

Now

$$\begin{aligned}
& d_{S^{(u)}} \\
= \quad & \gamma_\eta - u\gamma_W + E\bigg(\int_{|y|\geq 1} y N_{\eta,1}(\cdot, \mathrm{d}y) - u\int_{|x|\geq 1} x N_{W,1}(\cdot, \mathrm{d}x) \\
& \quad - \int_{(0,\infty)} z N_{\eta-uW,1}(\cdot, \mathrm{d}z)\bigg) \\
= \quad & \gamma_\eta - u\gamma_W + E\bigg(\int_{|y|\geq 1} y N_{\eta,1}(\cdot, \mathrm{d}y) - u\int_{(-\infty,-\ln 2)} \left(e^{-x}-1\right) N_{\xi,1}(\cdot, \mathrm{d}x) \\
& \quad - \int_{\{y-u(e^{-x}-1)>0\}} \left(y-u(e^{-x}-1)\right) N_{\xi,\eta,1}(\cdot, \mathrm{d}(x,y))\bigg) \\
= \quad & \gamma_\eta + u\gamma_\xi - \frac{1}{2}u\sigma_\xi^2 + E\bigg(\int_{\mathbb{R}^2} \Big(y 1_{|y|\geq 1} - ux 1_{|x|<1} - u(e^{-x}-1) \\
& \quad - \left(y - u(e^{-x}-1)\right) 1_{\{y-u(e^{-x}-1)>0\}}\Big) N_{\xi,\eta,1}(\cdot, \mathrm{d}(x,y))\bigg) \\
= \quad & \gamma_\eta + u\gamma_\xi - \frac{1}{2}u\sigma_\xi^2 \\
& \quad - E\bigg(\int_{\{y-u(e^{-x}-1)>0\}\cap\{(-1,1)\times(-1,1)\}} (ux+y) N_{\xi,\eta,1}(\cdot, \mathrm{d}(x,y))\bigg) \\
= \quad & \tilde{\gamma}_\eta + u\tilde{\gamma}_\xi - \frac{1}{2}u\sigma_\xi^2 \\
& \quad - E\bigg(\int_{\{y-u(e^{-x}-1)>0\}\cap\{x^2+y^2<1\}} (ux+y) N_{\xi,\eta,1}(\cdot, \mathrm{d}(x,y))\bigg),
\end{aligned}$$

The first equality follows because the expected value of each of the Brownian motion components of η and W is zero, as is the expected value of the compensated small jump processes of η and W. The second equality follows using (2.11) and the method above for converting integrals. The third equality follows using (2.12). The fourth equality follows since u is contained in suitable $[\theta_i, \theta_j]$ which implies that $S^{(u)}$ has no negative jumps, and correspondingly

$N_{\xi,\eta,1}(\{y-u(e^{-x}-1)\le 0\})=0$. The final equality follows by (1.11) and (1.12). Thus we are done if we can exchange integration and expectation in the above expression. Now if $f(x,y)$ is a non-negative measurable function and Λ is a Borel set in $\mathbb{R}^2$ then the monotone convergence theorem implies that

$$E\left(\int_\Lambda f(x,y)N_{\xi,\eta,1}(\cdot,\mathrm{d}(x,y))\right)=\int_\Lambda f(x,y)\Pi_{\xi,\eta}(\mathrm{d}(x,y)).$$

For general $f(x,y)$, if $\int_\Lambda f^+(x,y)\Pi_{\xi,\eta}(\mathrm{d}(x,y))$ or $\int_\Lambda f^-(x,y)\Pi_{\xi,\eta}(\mathrm{d}(x,y))$ is finite, then the following is a well-defined member of the extended real numbers;

$$\begin{aligned}
&E\left(\int_\Lambda f(x,y)N_{\xi,\eta,1}(\cdot,\mathrm{d}(x,y))\right)\\
&= \int_\Lambda f^+(x,y)\Pi_{\xi,\eta}(\mathrm{d}(x,y))-\int_\Lambda f^-(x,y)\Pi_{\xi,\eta}(\mathrm{d}(x,y))\\
&= \int_\Lambda f(x,y)\Pi_{\xi,\eta}(\mathrm{d}(x,y)).
\end{aligned}$$

However, using the fact that $0<e^{-x}-1+x<x^2$ whenever $|x|<1$, we have

$$\begin{aligned}
&\int_{\{x^2+y^2<1\}}(ux+y)^-\Pi_{\xi,\eta}(\mathrm{d}(x,y))\\
&= \int_{\{x^2+y^2<1\}}-(ux+y)1_{\{ux+y\le 0\}}\Pi_{\xi,\eta}(\mathrm{d}(x,y))\\
&\le \int_{\{x^2+y^2<1\}}\left(y-u(e^{-x}-1)-(ux+y)\right)1_{\{ux+y\le 0\}}\Pi_{\xi,\eta}(\mathrm{d}(x,y))\\
&= \int_{\{x^2+y^2<1\}}-u(e^{-x}-1+x)1_{\{ux+y\le 0\}}\Pi_{\xi,\eta}(\mathrm{d}(x,y))\\
&\le \int_{\{x^2+y^2<1\}}|u|x^2 1_{\{ux+y\le 0\}}\Pi_{\xi,\eta}(\mathrm{d}(x,y))\\
&\le |u|\int_{\mathbb{R}}\min\left\{1,x^2\right\}\Pi_\xi(\mathrm{d}x),
\end{aligned}$$

which is finite since Π_ξ is a Lévy measure. Note that the first inequality follows from the fact that the choice of u satisfies $\Pi_{\xi,\eta}(\{y-u(e^{-x}-1)\le 0\})=0$ whilst $\Pi_{\xi,\eta}(\{y-u(e^{-x}-1)>0\})\ge 0$. □

Proof of Theorem 2.1. By Theorem 2.7, $\psi(0)=0$ iff $\delta(0)=0$ iff η is a subordinator. Suppose η is not a subordinator and let $c>0$. Clearly $\psi(c)=0$ if and only if $\delta(c)\ge 0$. By Theorem 2.7, this is equivalent to the condition that there exists $0<u\le c$ such that $\delta(u)=u$. Combining this fact with Theorem 2.9 proves Theorem 2.1. □

Proof of Theorem 2.4. Define

$$U_t := e^{\xi_t}(Z_\infty - Z_t) = e^{\xi_t}\int_{t+}^{\infty} e^{-\xi_{s-}}\mathrm{d}\eta_s.$$

Note that since we are integrating over (t,∞) there are no predictability problems moving e^{ξ_t} under the integral sign, as there would have been if we were integrating over $[t,\infty)$. Thus $U_t = \int_{t+}^{\infty} e^{-(\xi_{s-}-\xi_t)}\mathrm{d}\eta_s$, from which it follows, from Lévy properties, that U_t is independent of $\mathscr{F}_t$ and that U_{T_z} conditioned on $T_z < \infty$ is independent of $\mathscr{F}_{T_z}$.

Since (ξ,η) is a Lévy process we know that for any $u>0$ and $t>0$

$$(\hat{\xi}_{u-},\hat{\eta}_u) := (\xi_{(t+u)-}-\xi_t\ ,\ \eta_{t+u}-\eta_t) =_D (\xi_{u-},\eta_u). \tag{2.15}$$

Thus

$$\begin{aligned}
U_t &= \int_{s\in(t,\infty)} e^{-(\xi_{s-}-\xi_t)}\mathrm{d}\eta_s = \int_{u\in(0,\infty)} e^{-(\xi_{(t+u)-}-\xi_t)}\mathrm{d}\eta_{t+u} \\
&= \int_{u\in(0,\infty)} e^{-(\xi_{(t+u)-}-\xi_t)}\mathrm{d}(\eta_{t+u}-\eta_t) = \int_{u\in(0,\infty)} e^{-\hat{\xi}_{u-}}\mathrm{d}\hat{\eta}_u \\
&=_D \int_{u\in(0,\infty)} e^{-\xi_{u-}}\mathrm{d}\eta_u \quad \text{(by (2.15))} = Z_\infty \quad (\text{since } \Delta\eta_0 = 0).
\end{aligned}$$

In particular, for any Borel set A,

$$P\left(U_{T_z}\in A \middle| T_z<\infty\right) = P(Z_\infty \in A). \tag{2.16}$$

Next note that if $\omega \in \{T_z < \infty\}$ then by definition of U,

$$\begin{aligned}
z + Z_\infty &= z + Z_{T_z} + e^{-\xi_{T_z}}U_{T_z} \\
&= e^{-\xi_{T_z}}\left(e^{\xi_{T_z}}(z+Z_{T_z}) + U_{T_z}\right) \\
&= e^{-\xi_{T_z}}(V_{T_z} + U_{T_z}).
\end{aligned}$$

This implies that

$$P(T_z<\infty, z+Z_\infty<0) = P(T_z<\infty, V_{T_z}+U_{T_z}<0). \tag{2.17}$$

Finally note that $(Z_\infty < -z) \subset (T < \infty)$ since the convergence from Z_t to Z_∞ is a.s convergence. Thus

$$\begin{aligned}
P(z+Z_\infty<0) &= P(T_z<\infty, z+Z_\infty<0)\\
&= P(T_z<\infty, V_{T_z}+U_{T_z}<0) \quad \text{(by (2.17))}\\
&= E\left(P(T_z<\infty, V_{T_z}+U_{T_z}<0\big|\mathscr{F}_{T_z})\right)\\
&= \int_{T_z<\infty} P\left(V_{T_z}+U_{T_z}<0\big|\mathscr{F}_{T_z}\right)(\omega)P(\mathrm{d}\omega).
\end{aligned}$$

But if $T_z(\omega)<\infty$ then

$$\begin{aligned}
P\left(V_{T_z}+U_{T_z}<0\big|\mathscr{F}_{T_z}\right)(\omega) &= P\left(V_{T_z}(\omega)+U_{T_z}<0\big|\mathscr{F}_{T_z}\right)(\omega)\\
&= P\left(U_{T_z}<-V_{T_z}(\omega)\big|T_z<\infty\right)\\
&= P(Z_\infty<-V_{T_z}(\omega)) \quad \text{(by (2.16))}.
\end{aligned}$$

The second last equality follows since U_{T_z} conditioned on $T_z<\infty$ is independent of $\mathscr{F}_{T_z}$. Thus we obtain the required formula from

$$\begin{aligned}
G(-z) &= \int_{T_z<\infty} G(-V_{T_z})(\omega)P(\mathrm{d}\omega)\\
&= E\left(G(-V_{T_z})1_{T_z<\infty}\right)\\
&= E\left(G(-V_{T_z})1_{T_z<\infty}\big|T_z<\infty\right)P(T_z<\infty)\\
&\quad + E\left(G(-V_{T_z})1_{T_z<\infty}\big|T_z=\infty\right)P(T_z=\infty)\\
&= E\left(G(-V_{T_z})\big|T_z<\infty\right)P(T_z<\infty).
\end{aligned}$$

□

Chapter 3

Certain ruin for the Generalised Ornstein-Uhlenbeck process

3.1 Introduction

In Section 3.2, we state results on certain ruin for the GOU. Theorem 3.1 of Paulsen [56] gives conditions for certain ruin for the GOU in the special case in which ξ and η are independent. In Theorem 2.1 we showed that this theorem does not hold for the general case. Theorems 3.1 and 3.3 of Section 3.2 give the required generalization, stated in terms of the characteristic triplet of (ξ, η). Section 3.3 begins with results, in particular Proposition 3.6 and Theorem 3.9, which describe the structure of the upper and lower bounds and the sets of values on which the GOU is almost surely increasing, or decreasing. Section 3.3 then outlines the ruin probability implications of these structural results, in particular with Theorems 3.13 and 3.14, which state conditions for certain ruin in terms of upper and lower bound structure. Section 3.3 concludes with technical propositions used to prove the major theorems. Section 3.4 contains proofs of the results in Section 3.2 and 3.3, and concludes with a number of examples which illustrate and extend certain results. To avoid trivialities, assume throughout this chapter that neither ξ nor η are identically zero.

3.2 Conditions for Certain Ruin

In Chapter 2, Theorem 2.1, exact conditions were given on the characteristic triplet of (ξ, η) for the existence of $u \geq 0$ such that $\psi(u) = 0$, and a precise value was given for the value $\inf\{u \geq 0 : \psi(u) = 0\}$, where we use the convention that

$\inf\{\emptyset \cap [0,\infty)\} = \infty$. It is a consequence of Theorem 3.1 below, that when the relevant assumptions are satisfied, there exists $z \geq 0$ such that $\psi(z) < 1$ iff there exists $u \geq 0$ such that $\psi(u) = 0$. Thus, even though they are not stated explictly, Theorem 3.1 implies exact conditions on the characteristic triplet of (ξ, η) for certain ruin.

Statements (1) and (2) of Theorem 3.1 are generalizations to the dependent case of Paulsen's Theorem 3.1, parts (a) and (b), respectively. Statement (1) of Theorem 3.1 also removes Paulsen's assumption of finite mean for ξ, and replaces his moment conditions with the precise necessary and sufficient conditions for stationarity of V. For statement (2) of Theorem 3.1, a finite mean assumption and moment conditions remain necessary.

Theorem 3.1. *Let* $m := \inf\{u \geq 0 : \psi(u) = 0\}$.

1. *Suppose* $\lim_{t\to\infty} \xi_t = -\infty$ *a.s. and* $I_{-\xi,K^{\xi,\eta}} < \infty$. *Then* $0 < \psi(z) < 1$ *iff* $0 \leq z < m < \infty$.

2. *Suppose* $E(\xi_1) = 0$, $E\left(|\xi_1|^{2+\delta}\right) < \infty$ *for some* $\delta > 0$ *and there exist* $p, q > 1$ *with* $1/p + 1/q = 1$ *such that* $E\left(e^{-p\xi_1}\right) < \infty$ *and* $E\left(|\eta_1|^q\right) < \infty$. *If, for all* $c \in \mathbb{R}$, *the degenerate case (1.30) does not hold, then* $0 < \psi(z) < 1$ *iff* $0 \leq z < m < \infty$. *If there exists* $c \in \mathbb{R}$ *such that equation (1.30) holds, then* $\psi(z) < 1$ *iff* $\psi(z) = 0$, *which occurs iff* $0 \leq c \leq z$.

Remark 3.2. 1. In proving [56] Theorem 3.1 (b), Paulsen discretizes the GOU at integer time points and then uses a recurrence result from [2]. His argument uses the inequality $P(V_1 < 0|V_0 = z) > 0$ for all $z \geq 0$. When $E(\xi_1) = 0$, this inequality is valid in the independent case if either ξ or η has a Brownian component, or can have negative jumps. However, even in the independent case, this inequality can fail to hold when V_t decreases due to a deterministic drift. For example, let N and M be independent Poisson processes with parameter 1 and define $\xi_t := -t + N_t$ and $\eta_t := -t + M_t$. Note that $E(\xi_1) = 0$ and Paulsen's conditions are satisfied trivially. Let $T_z := \inf\{t > 0 : V_t < 0|V_0 = z\}$. Then $V_t \geq (z+1)e^{-t} - 1 := V_t'$ on $t \leq T_z$ and $P(V_1' < 0|V_0' = z) = 0$ whenever $z > e^1 - 1$. In proving statement (2) of Theorem 3.1 we get around this difficulty by discretizing the GOU at random times T_i and then showing that the stated conditions result in $P(V_{T_1} < 0|V_0 = z) > 0$ for all $z \geq 0$ in the general case.

2. Assume that ξ and η are independent and η is not a subordinator. In this case, whenever ξ drifts to $-\infty$ a.s. or ξ oscillates between ∞ and $-\infty$

a.s., it is a consequence of Theorem 2.1 that $\psi(u) > 0$ for all $u \geq 0$, and hence $m = \infty$. Thus, by statement (1) of Theorem 3.1, if $\lim_{t\to\infty} \xi_t = -\infty$ a.s. and $I_{-\xi,K^{\xi,\eta}} < \infty$, then $\psi(z) = 1$ for all $z \geq 0$. This result is a slight strengthening of Paulsen's Theorem 3.1 (a). Further, statement (2) simplifies exactly to Paulsen's Theorem 3.1 (b). Since ξ and η are independent the conditions in statement (2) simplify to $E(\xi_1) = 0$, $E\left(|\xi_1|^{2+\delta}\right) < \infty$ $E\left(e^{-\xi_1}\right) < \infty$ and $E(\eta_1) < \infty$. Since $m = \infty$, $\psi(z) = 1$ for all $z \geq 0$ whenever these conditions hold. The simplification of conditions occurs because Hölder's inequality is not needed in the proof, and a simpler argument using independence suffices. When transferred onto the Lévy measure, these conditions are equivalent to those in Paulsen's Theorem 3.1 (b).

We now present Theorem 3.3, which is the generalization to the dependent case of Paulsen's Theorem 3.1, part (c). In addition, Paulsen's assumption of finite mean for ξ is removed, and his moment conditions are replaced with the precise necessary and sufficient conditions for a.s. convergence of Z_t to a finite random variable Z_∞, as $t \to \infty$. A formula for the ruin probability in this situation was given in Chapter 2, Theorem 2.4, however no conditions for certain ruin were found. Theorem 3.3 gives exact conditions on the characteristic triplet of (ξ, η) for certain ruin. To state these conditions, we need to define the following terms.

Let $A_1 := \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$, and similarly, let A_2, A_3 and A_4 be the quadrants in which $\{x \geq 0, y \leq 0\}$, $\{x \leq 0, y \leq 0\}$ and $\{x \leq 0, y \geq 0\}$ respectively. For each $i = 1,2,3,4$ and $u \in \mathbb{R}$ let

$$B_i^u := \left\{(x,y) \in A_i : y - u(e^{-x} - 1) > 0\right\}$$

and define

$$\theta_1' := \begin{cases} \inf\{u \leq 0 : \Pi_{\xi,\eta}(B_1^u) > 0\} \\ 0 \quad \text{if } \Pi_{\xi,\eta}(A_1 \setminus A_2) = 0, \end{cases} \qquad \theta_3' := \begin{cases} \sup\{u \leq 0 : \Pi_{\xi,\eta}(B_3^u) > 0\} \\ -\infty \quad \text{if } \Pi_{\xi,\eta}(A_3 \setminus A_2) = 0, \end{cases}$$

$$\theta_2' := \begin{cases} \inf\{u \geq 0 : \Pi_{\xi,\eta}(B_2^u) > 0\} \\ \infty \quad \text{if } \Pi_{\xi,\eta}(A_2 \setminus A_3) = 0, \end{cases} \qquad \theta_4' := \begin{cases} \sup\{u \geq 0 : \Pi_{\xi,\eta}(B_4^u) > 0\} \\ 0 \quad \text{if } \Pi_{\xi,\eta}(A_4 \setminus A_3) = 0. \end{cases}$$

Theorem 3.3. *Suppose* $\lim_{t\to\infty} \xi_t = \infty$ *a.s. and* $I_{\xi,\eta} < \infty$. *Then* $\psi(0) = 1$ *if and only iff* $-\eta$ *is a subordinator, or there exists* $z > 0$ *such that* $\psi(z) = 1$. *The latter occurs if and only if* $\Pi_{\xi,\eta}(A_1) = 0$, $\theta_4' \leq \theta_2'$, *and there exists* $u \in [\theta_4', \theta_2']$ *such that*

$$\Sigma_{\xi,\eta} = \begin{bmatrix} 1 & -u \\ -u & u^2 \end{bmatrix} \sigma_\xi^2, \tag{3.1}$$

and

$$g(u) := \tilde{\gamma}_\eta + u\tilde{\gamma}_\xi - \frac{1}{2}u\sigma_\xi^2 - \int_{\{x^2+y^2<1\}} (ux+y)\Pi_{\xi,\eta}(\mathrm{d}(x,y)) \leq 0. \tag{3.2}$$

If there exists $z \geq 0$ such that $\psi(z) = 1$ and, for all $c \in \mathbb{R}$, the equation (1.30) does not hold, then the following hold:

1. *If $\sigma_\xi^2 = 0$ then $\psi(z) = 1$ for all $z \leq m := \sup\{u \in [\theta_4', \theta_2'] : g(u) \leq 0\}$, and $0 \leq \psi(z) < 1$ for all $z > m$;*
2. *If $\sigma_\xi^2 \neq 0$ then $\psi(z) = 1$ for all $z \leq m := -\frac{\sigma_{\xi,\eta}}{\sigma_\xi^2}$, and $0 < \psi(z) < 1$ for all $z > m$.*

If there exists $z \geq 0$ such that $\psi(z) = 1$ and there exists $c \in \mathbb{R}$ such that (1.30) holds, then $0 < c = \theta_4' = \theta_2'$, $\psi(z) = 1$ for all $z < c$, and $\psi(z) = 0$ for all $z \geq c$.

Remark 3.4.

1. When $\Pi_{\xi,\eta}(A_1) = 0$, $\theta_4' \leq \theta_2'$ and $u \in [\theta_4', \theta_2']$ the function $g(u)$ is a well-defined member of the extended reals. The existence and finiteness of g is fully analysed in point (1) of Remark 3.19.

2. Assume ξ and η are independent. Then all jumps occur at the axes of the sets A_i, and $\sigma_{\xi,\eta} = 0$. With a little work, Theorem 3.3 simplifies to the following statement: Suppose $\lim_{t\to\infty} \xi_t = \infty$ a.s. and $I_{\xi,\eta} < \infty$. Then $\psi(0) = 1$ iff $-\eta$ is a subordinator, or $\psi(z) = 1$ for some $z > 0$. The latter occurs iff ξ and η are each of finite variation and have no positive jumps, and $g(z) \leq 0$. Note that when (ξ, η) is finite variation, g simplifies to $g(u) = d_\eta + ud_\xi$, as explained in equation (3.4). Since ξ drifts to ∞ a.s., it must be that $d_\xi > 0$. Thus, $g(z) \leq 0$ for some $z > 0$ iff $d_\eta < 0$. In particular, $-\eta$ is a subordinator.

3. In Paulsen [56], Theorem 3.1 (c), it is stated that when ξ and η are independent, $E(\xi_1) > 0$, and a set of moment conditions hold, then $\psi(z) = 1$ iff $\xi_t = \alpha t$, $\eta_t = \beta t$ and $\beta < -\alpha z$ for real constants α and β. This statement contradicts the independence version of Theorem 3.3 stated above, and is false. A simple counterexample is $(\xi, \eta)_t := (t, -t - N_t)$ where N is a Poisson process. Paulsen's moment conditions are satisfied trivially. However, Theorem 3.3 implies that $\psi(z) = 1$ for all $z \leq 1$, and this is confirmed by elementary calculations. If we denote the jump times of N_t by $0 = T_0 < T_1 < T_2 < \cdots$ then

$$V_t = 1 + e^t \left(z - 1 - \sum_{i=1}^{N_t} e^{-T_i} \right).$$

Thus, if $z = 1$, then $V_{T_2} = -e^{T_2 - T_1} < 0$ a.s. and so $\psi(1) = 1$.

The following proposition fully explains the ruin probability function for the degenerate situation (1.30). It will be used to prove that Theorems 3.1 and 3.3 correctly allow for this case.

Proposition 3.5. *Suppose that there exists $c \in \mathbb{R}$ such that $V_t = e^{\xi_t}(z - c) + c$. If $c \geq 0$ then $\psi(z) = 0$ for all $z \geq c$, and the following statements hold for all $0 \leq z < c$:*

1. *If ξ drifts to $-\infty$ a.s. then $0 < \psi(z) < 1$;*
2. *If ξ oscillates between ∞ and $-\infty$ a.s. then $\psi(z) = 1$;*
3. *If ξ drifts to ∞ a.s. then $\psi(z) = 1$.*

If $c < 0$ then the following statements hold for all $z \geq 0$:

(4) If ξ drifts to $-\infty$ a.s. then $\psi(z) = 1$;

(5) If ξ oscillates between ∞ and $-\infty$ a.s. then $\psi(z) = 1$;

(6) If ξ drifts to ∞ a.s. then $0 < \psi(z) < 1$.

3.3 Upper and lower bounds and the ruin function

Define the *lower bound function* δ for V by

$$\delta(z) := \inf \left\{ u \in \mathbb{R} : P\left(\inf_{t \geq 0} V_t \leq u \middle| V_0 = z \right) > 0 \right\}$$

and the *upper bound function* Υ by

$$\Upsilon(z) := \sup \left\{ u \in \mathbb{R} : P\left(\sup_{t \geq 0} V_t \geq u \middle| V_0 = z \right) > 0 \right\},$$

where we use the convention that $\inf\{\emptyset \cap \mathbb{R}\} = \infty$ and $\sup\{\emptyset \cap \mathbb{R}\} = -\infty$. When $V_0 = z$, the probability that the sample paths V_t will ever rise above $\Upsilon(z)$, or below $\delta(z)$, is zero. In particular, the ruin probability function ψ satisfies $\psi(z) = 0$ iff $\delta(z) \geq 0$. Define the sets L and U by

$$L := \{u \in \mathbb{R} : \delta(u) = u\} \text{ and } U := \{u \in \mathbb{R} : \Upsilon(u) = u\}.$$

It will be a consequence of Proposition 3.17 that L and U must each be of the form

$$\emptyset, \{a\}, [a,b], [a,\infty), \text{ or } (-\infty, b] \tag{3.3}$$

for some $a, b \in \mathbb{R}$. The fact that L and U are both connected sets is of great importance.

This section contains a detailed analysis of δ, Υ, U and L and their relationship with the ruin function. In particular, we are interested in which combinations of L and U can exist. For each combination we are also interested in the possible asymptotic behaviour of ξ, namely, whether ξ drifts to ∞ a.s., ξ drifts to $-\infty$ a.s. or ξ oscillates between ∞ and $-\infty$ a.s. We are interested in the asymptotic behaviour of ξ because of its link with the conditions for convergence of Z_t and stationarity of V, as discussed in Section 1.4. As well as being of independent interest, the results contained in this section are essential for the proofs of Theorems 3.1 and 3.3.

We begin with some comments on δ, and L. The analogues for Υ and U are obvious through symmetry. Firstly, note that $\delta(z) \leq z$ for all $z \in \mathbb{R}$, whilst the fact that V_t is increasing in z for all $t \geq 0$ implies that $\delta(z_1) \leq \delta(z_2)$ whenever $z_1 < z_2$. The following proposition explains the behaviour of the lower bound function outside the set L, and states that L is precisely the set of starting parts $V_0 = z$ for which almost all sample paths V_t are increasing for some time period. Recall that $T_{z,\Lambda} := \inf\{t > 0 : V_t \in \Lambda\}$, and define $L^c := \mathbb{R} \setminus L$.

Proposition 3.6. *The following statements hold for L and δ, and the symmetric statements hold for U and Υ:*

1. *If* $z \geq \sup L$ *then* $\delta(z) = \sup L$;
2. *If* $z < \inf L$ *then* $\delta(z) = -\infty$;
3. *For* $z \in L$, $P\left(V_t \text{ is increasing on } 0 < t \leq T_{z,L^c} \middle| V_0 = z\right) = 1$;
4. *For* $z \in L^c$, $P\left(V_t \text{ is increasing on } 0 < t \leq T_{z,L} \middle| V_0 = z\right) < 1$.

Recall that in Section 3.1 we assumed that neither ξ nor η are identically zero in order to avoid trivialities. The following proposition explains the nature of these trivialities.

Proposition 3.7. *1. $L = \mathbb{R}$ iff $\xi_t = 0$ a.s. for all $t > 0$ and η is a subordinator.*

2. $U = \mathbb{R}$ iff $\xi_t = 0$ a.s. for all $t > 0$ and $-\eta$ is a subordinator.

3. $L = U = \mathbb{R}$ iff $\xi_t = \eta_t = 0$ a.s. for all $t > 0$.

For the rest of this chapter we again assume that neither ξ nor η are identically zero. The following proposition explains the degenerate situation described in equation (1.30). Note that the deterministic case $(\xi, \eta)_t := (\alpha, \beta)t$ for non-zero constants α and β satisfies the conditions of this proposition for $c = -\beta/\alpha$. Recall that a Borel set $\Lambda \subsetneq \mathbb{R}$ is an *absorbing* set for V, if for all $0 \leq s \leq t$, $P(V_t \in \Lambda | V_s = x) = 1$ for all $x \in \Lambda$. That is, whenever a sample path V_t hits Λ, it never leaves. The stochastic exponential will be denoted by ϵ.

Proposition 3.8. *The following are equivalent for $c \neq 0$:*

1. $L \cap U \neq \emptyset$;

2. $L \cap U = \{c\}$;

3. $V_t = e^{\xi_t}(z - c) + c$ and $Z_t = c\left(e^{-\xi_t} - 1\right)$;

4. $\{c\}$ is an absorbing set;

5. $\Sigma_{\xi,\eta}$ satisfies (3.1) for $u = c$, $\Pi_{\xi,\eta} = 0$ or is supported on the curve $\{(x, y) : y - c(e^{-x} - 1) = 0\}$, and $g(c) = 0$;

6. $e^{-\xi_t} = \epsilon(\eta/c)_t$.

If the above conditions hold and $\Sigma_{\xi,\eta} \neq 0$ then $L = U = \{c\}$ and there exist Lévy processes (ξ, η) for this situation such that ξ drifts to ∞ a.s., ξ drifts to $-\infty$ a.s. or ξ oscillates a.s. If the above conditions hold and $\Sigma_{\xi,\eta} = 0$ then:

(a) $U = (-\infty, c]$ and $L = [c, \infty)$ iff ξ is a subordinator;

(b) $L = (-\infty, c]$ and $U = [c, \infty)$ iff $-\xi$ is a subordinator;

(c) $L = U = \{c\}$ iff neither ξ or $-\xi$ is a subordinator. There exist Lévy processes (ξ, η) for this situation such that ξ drifts to ∞ a.s., ξ drifts to $-\infty$ a.s. or ξ oscillates a.s.

Now we present a theorem which describes all possible combinations of L and U and the associated asymptotic behaviour of ξ, for the case in which $L \cap U = \emptyset$.

Theorem 3.9. *Suppose that $L \cap U = \emptyset$. If $\Sigma_{\xi,\eta} \neq 0$ then only the following cases can exist:*

1. $L = U = \emptyset$*;*

2. $L = \{a\}$ *for some* $a \in \mathbb{R}$ *and* $U = \emptyset$*;*

3. $U = \{a\}$ *for some* $a \in \mathbb{R}$ *and* $L = \emptyset$*.*

If $\Sigma_{\xi,\eta} = 0$ *then only the following cases can exist:*

(a) If $L = \emptyset$ *then* U *is of the form* $\emptyset$, $\{a\}$, $[a,b]$, $[a,\infty)$, *or* $(-\infty, b]$ *for some* $a, b \in \mathbb{R}$*;*

(b) If $U = \emptyset$ *then* L *is of the form* $\emptyset$, $\{a\}$, $[a,b]$, $[a,\infty)$, *or* $(-\infty, b]$ *for some* $a, b \in \mathbb{R}$*;*

(c) If $L \neq \emptyset$ *and* $U \neq \emptyset$ *then there exist* $a < b$ *such that* $L = (-\infty, a]$ *and* $U = [b,\infty)$, *or* $U = (-\infty, a]$ *and* $L = [b, \infty)$*.*

If $U = (-\infty, a]$ *or* $L = [b,\infty)$ *(or both with* $a < b$*) then* ξ *is a subordinator. If* $L = (-\infty, a]$ *or* $U = [b,\infty)$ *(or both) then* $-\xi$ *is a subordinator. For all of the other combinations of* L *and* U *above, there exist Lévy processes* (ξ, η) *such that* ξ *drifts to* ∞ *a.s.,* ξ *drifts to* $-\infty$ *a.s. or* ξ *oscillates a.s.*

An absorbent set $\Lambda \subsetneq \mathbb{R}$ is a *maximal absorbing* set if it is not properly contained in any other absorbing set. Note that if Λ is a maximal absorbing set, then $\mathbb{R} \setminus \Lambda$ contains no absorbing sets otherwise we could take the union of Λ with the absorbing set, and this would be an absorbing set properly containing Λ. The following corollary is immediate. For each statement (1)-(4), the claim that the sets Λ are maximal absorbing follows from Proposition 3.6. The remaining statements follow immediately from Theorem 3.9.

Corollary 3.10. *There exist Lévy processes* (ξ, η) *with* $L \cap U = \emptyset$ *such that the associated GOU has the following maximal absorbing sets* Λ *:*

1. $\Lambda = U \cup L$, *where* $U = (-\infty, a]$ *and* $L = [b, \infty)$*;*

2. $\Lambda = U$, *where* $U = (-\infty, a]$ *and* $L = \emptyset$*;*

3. $\Lambda = L$, *where* $L = [b,\infty)$ *and* $U = \emptyset$*;*

4. $\Lambda = (a, b)$ *where* $L = (-\infty, a]$ *and* $U = [b, \infty)$*.*

If (ξ, η) *has* $L \cap U = \emptyset$ *and does not have* U *and* L *satisfying one of (1)-(4), then no absorbing sets exist.*

We examine two striking cases of L and U structure, and state exact conditions on the characteristic triplet of (ξ, η) for such behaviour. Note that similar conditions can be found for each of the other L and U structures stated in Theorem 3.9, however, the statements are longer and unwieldy.

Proposition 3.11. *Suppose $L \cap U = \emptyset$. Then $U = (-\infty, a]$ and $L = [b, \infty)$ for $-\infty < a < b < \infty$ iff (ξ, η) is of finite variation and the following hold:*

- *There is no Brownian component ($\Sigma_{\xi,\eta} = 0$);*
- *The drift of ξ is non-negative ($d_\xi \geq 0$);*
- *The Lévy measure satisfies $\Pi_{\xi,\eta}(A_3) = \Pi_{\xi,\eta}(A_4) = 0$, $\theta_1' > -\infty$, and $\theta_2 < \infty$.*

If these conditions hold then ξ is a subordinator and, for any $V_0 = z \in \mathbb{R}$, $\lim_{t\to\infty} |V_t| = \infty$ a.s.

Similarly $L = (-\infty, a]$ and $U = [b, \infty)$ for $-\infty < a < b < \infty$ iff (ξ, η) is of finite variation and the following hold:

- *There is no Brownian component ($\Sigma_{\xi,\eta} = 0$);*
- *The drift of ξ is non-positive ($d_\xi \leq 0$);*
- *The Lévy measure satisfies $\Pi_{\xi,\eta}(A_1) = \Pi_{\xi,\eta}(A_2) = 0$, $\theta_4' < \infty$ and $\theta_3 > -\infty$.*

If these conditions hold then $-\xi$ is a subordinator, and V is strictly stationary and converges in distribution as $t \to \infty$ to a random variable V_∞ supported on (a, b).

We now present a theorem describing the relationship between the sets L and U, and the upper and lower bounds of the limit random variable Z_∞ of Z_t as $t \to \infty$.

Theorem 3.12. *Let $a, b \in \mathbb{R}$ and suppose $Z_t \to Z_\infty$ a.s. as $t \to \infty$, where Z_∞ is a finite random variable. If, for all $c \in \mathbb{R}$, the degenerate case (1.30) does not hold, then $a \leq \sup U$ iff $Z_\infty < -a$ a.s., whilst $b \geq \inf L$ iff $Z_\infty > -b$ a.s. Further, $-\sup U = \inf\{u \in \mathbb{R} : Z_\infty < u \text{ a.s.}\}$ and $-\inf L = \sup\{u \in \mathbb{R} : Z_\infty > u \text{ a.s.}\}$. Alternatively, if there exists $c \in \mathbb{R}$ such that equation (1.30) holds, then $Z_\infty = -c$ a.s. and $\inf L = \sup U = c$.*

The next theorem presents results on certain ruin which occur when L and U are of a particular structure.

Theorem 3.13. *Suppose that $L \cap U = \emptyset$. Then the following statements hold:*

1. *If $\sup U \geq 0$ and $L \cap [0, \sup U] = \emptyset$, then $\psi(z) = 1$ for all $z \leq \sup U$;*

2. *If $\sup L \geq 0$ and $U \cap [0, \sup L] = \emptyset$, then $0 < \psi(z) < 1$ for all $0 \leq z < \inf L$. If $\sup L \geq 0$ and $U \cap [0, \sup L] \neq \emptyset$, then $\psi(z) < 1$ for all $z > \sup U$.*

Note that in statement (2) above, when $\sup L \geq 0$ and $L \cap U \neq \emptyset$, Theorem 3.9 ensures that $\sup U < \inf L$, and statement (1) above ensures that $\psi(z) = 1$ for all $z \leq \sup U$. Also, by definition of L, $\psi(z) = 0$ whenever $z \geq \inf L$.

We now present a major theorem which utilises Theorems 3.9, 3.12 and 3.13, and is the major tool in proving Theorems 3.1 and 3.3. For the non-degenerate case, and for (ξ, η) which satisfies various asymptotic and stability criteria, this theorem presents iff conditions for certain ruin, stated in terms of L and U structure. In particular, it completely describes the L and U structures for which certain ruin occurs.

Theorem 3.14. *Suppose $L \cap U = \emptyset$.*

1. *Suppose $\lim_{t\to\infty} \xi_t = -\infty$ a.s. and $I_{-\xi, K^{\xi,\eta}} < \infty$. There exists $z \geq 0$ such that $\psi(z) < 1$ iff $L \cap [0, \infty) \neq \emptyset$. If this occurs then $0 < \psi(z) < 1$ for all $0 \leq z < \inf L$, $\psi(z) = 0$ for all $z \geq \inf L$, and one of the following must hold:*

   (a) $L = [a, b]$ and $U = \emptyset$, where $-\infty \leq a \leq b < \infty$, and $b \geq 0$;

   (b) $L = (-\infty, a]$ and $U = [b, \infty)$ where $0 \leq a < b < \infty$.

2. *Suppose $E(\xi_1) = 0$, $E\left(|\xi_1|^{2+\delta}\right) < \infty$ for some $\delta > 0$ and there exist $p, q > 1$ with $1/p + 1/q = 1$ such that $E\left(e^{-p\xi_1}\right) < \infty$ and $E\left(|\eta_1|^q\right) < \infty$. There exists $z \geq 0$ such that $\psi(z) < 1$ iff $L \cap [0, \infty) \neq \emptyset$. If this occurs then $L = [a, b]$ and $U = \emptyset$, where $-\infty < a \leq b < \infty$ and $b \geq 0$, in which case $0 < \psi(z) < 1$ for all $0 \leq z < a$ and $\psi(z) = 0$ for all $z \geq a$;*

3. *Suppose $\lim_{t\to\infty} \xi_t = \infty$ a.s. and $I_{\xi,\eta} < \infty$. There exists $z \geq 0$ such that $\psi(z) = 1$ iff $U \cap [0, \infty) \neq \emptyset$. If this occurs then one of the following must hold:*

   (c) $U = [a, b]$ and $L = \emptyset$, where $-\infty \leq a \leq b < \infty$ and $b \geq 0$, in which case $\psi(z) = 1$ for all $z \leq b$ and $0 < \psi(z) < 1$ for all $z > b$;

(d) $U = (-\infty, a]$ and $L = [b, \infty)$ where $0 \le a < b < \infty$, *in which case* $\psi(z) = 1$ *for all* $z \le a$, $0 < \psi(z) < 1$ *for all* $a < z < b$ *and* $\psi(z) = 0$ *for all* $z \ge b$.

Remark 3.15. The characteristic triplet conditions which equate to the iff result in statement (3) above, are given in Theorem 3.3, and are obtained using the forthcoming Proposition 3.20. Further, exact characteristic triplet conditions for the structure $U = (-\infty, a]$ and $L = [b, \infty)$ in case (d) above, are given in Proposition 3.11.

3.3.1 Technical results on the upper and lower bounds

We present a series of important technical propositions on δ, L, Υ and U. As well as being of independent interest, they are essential in proving the previously stated theorems. The first proposition is obtained by combining and restating parts of Proposition 2.6, Theorem 2.7 and Theorem 2.9, and no proof is given. When put into this form the proposition completely describes the relationship between the Lévy measure of (ξ, η) and the lower bound function δ. We recall some notation from Section 2.2. Let $A_1 := \{(x, y) \in \mathbb{R}^2 : x \ge 0, y \ge 0\}$, and similarly, let A_2, A_3 and A_4 be the quadrants in which $\{x \ge 0, y \le 0\}$, $\{x \le 0, y \le 0\}$ and $\{x \le 0, y \ge 0\}$ respectively. For each $i = 1, 2, 3, 4$ and $u \in \mathbb{R}$ define $A_i^u := \{(x, y) \in A_i : y - u(e^{-x} - 1) < 0\}$. For $u \le 0$ define

$$\theta_1 := \begin{cases} \sup\{u \le 0 : \Pi_{\xi,\eta}(A_1^u) > 0\} \\ -\infty \quad \text{if } \Pi_{\xi,\eta}(A_1 \setminus A_4) = 0, \end{cases} \qquad \theta_3 := \begin{cases} \inf\{u \le 0 : \Pi_{\xi,\eta}(A_3^u) > 0\} \\ 0 \quad \text{if } \Pi_{\xi,\eta}(A_3 \setminus A_4) = 0, \end{cases}$$

and for $u \ge 0$ define

$$\theta_2 := \begin{cases} \sup\{u \ge 0 : \Pi_{\xi,\eta}(A_2^u) > 0\} \\ 0 \quad \text{if } \Pi_{\xi,\eta}(A_2 \setminus A_1) = 0, \end{cases} \qquad \theta_4 := \begin{cases} \inf\{u \ge 0 : \Pi_{\xi,\eta}(A_4^u) > 0\} \\ \infty \quad \text{if } \Pi_{\xi,\eta}(A_4 \setminus A_1) = 0. \end{cases}$$

Throughout, let W be the Lévy process such that $e^{-\xi_t} = \epsilon(W)_t$.

Proposition 3.16 (lower bound). *The following statements are equivalent:*

1. *The lower bound* $\delta(z) > -\infty$ *for some* $z \in \mathbb{R}$;

2. *There exists* $u \in \mathbb{R}$ *such that* $\delta(u) = u$;

3. *There exists* $u \in \mathbb{R}$ *such that the Lévy process* $\eta - uW$ *is a subordinator.*

Statements (2) and (3) hold for a particular value $u \neq 0$ iff the following three conditions are satisfied: (i) the Gaussian covariance matrix satisfies equation (3.1); (ii) one of the following is true:

(a) $\Pi_{\xi,\eta}(A_3) = 0$, $\Pi_{\xi,\eta}(A_2) \neq 0$, $\theta_2 \leq \theta_4$ *and* $u \in [\theta_2, \theta_4]$;

(b) $\Pi_{\xi,\eta}(A_2) = 0$, $\Pi_{\xi,\eta}(A_3) \neq 0$, $\theta_1 \leq \theta_3$ *and* $u \in [\theta_1, \theta_3]$;

(c) $\Pi_{\xi,\eta}(A_3) = \Pi_{\xi,\eta}(A_2) = 0$ *and* $u \in [\theta_1, \theta_4]$;

and, (iii), in addition, u satisfies $g(u) \geq 0$ for the function g from equation (3.2).

From the definition of L it is an immediate corollary, firstly, that $L = \emptyset$ iff none of conditions (1)-(3) of Proposition 3.16 hold, and secondly, that η is a subordinator iff $0 \in L$. The next proposition adds further information concerning L. Most importantly, it shows that the set L is always connected, and gives concrete values for the endpoints.

Proposition 3.17. *If $\sigma_\xi^2 \neq 0$ and any of conditions (1)-(3) of Proposition 3.16 hold, then $L = \{-\frac{\sigma_{\xi,\eta}}{\sigma_\xi^2}\}$. If $\sigma_\xi^2 = 0$ and any of (1)-(3) hold, then $\sigma_\eta^2 = 0$ and one of the following holds:*

- *η is a subordinator and condition (ii) of Proposition 3.16 does not hold for any $u \neq 0$, in which case $L = \{0\}$;*
- *Condition (ii) is satisfied for some $u \neq 0$, in which case there exists $-\infty \leq a \leq b \leq \infty$ such that $L = [a, b]$.*

In the latter case, if condition (a) of Proposition 3.16 holds then $0 \leq a = \max\{\theta_2, m_1\}$ and $b = \min\{\theta_4, m_2\}$ for $m_1 := \inf\{u \in \mathbb{R} : g(u) \geq 0\}$ and $m_2 := \sup\{u \in \mathbb{R} : g(u) \geq 0\}$. If (b) holds then $a = \max\{\theta_1, m_1\}$ and $b = \min\{\theta_3, m_2\} \leq 0$. If (c) holds then $a = \max\{\theta_1, m_1\}$ and $b = \min\{\theta_4, m_2\}$.

Define L^* to be the set of starting values on which the GOU has no negative jumps, namely

$$L^* := \{u \in \mathbb{R} : \forall t > 0 \; P\left(\Delta V_t < 0 | V_{t-} = u\right) = 0\}.$$

It is an immediate consequence of Proposition 3.6 that $L \subseteq L^*$. The next proposition describes L^*. In particular, it shows that the set L^* is always connected, and gives concrete values for the endpoints. It also shows that whenever $V_{t-} > \sup L^*$ and a negative jump ΔV_t occurs, then the jump cannot be so negative as to cause $V_t \leq \sup L^*$. Thus, L^* acts as a barrier for negative jumps of V.

Proposition 3.18. *1. If $L^* \neq \emptyset$ then, for any $t \geq 0$, $V_{t-} > \sup L^*$ implies $V_t > \sup L^*$ a.s.;*

2. $L^* = \{u \in \mathbb{R} : \eta - uW$ has no negative jumps$\}$;

3. $L^* \neq \emptyset$ iff condition (ii) of Proposition 3.16 is satisfied for some $u \neq 0$, or η has no negative jumps;

4. $L^* = \{0\}$ iff η has no negative jumps and condition (ii) does not hold for any $u \neq 0$;

5. If condition (ii) of Proposition 3.16 holds for some $u \neq 0$ then $L^* = [\theta_2, \theta_4]$, $[\theta_1, \theta_3]$ or $[\theta_1, \theta_4]$, corresponding to conditions (a), (b) or (c) of Proposition 3.16.

Remark 3.19. 1. If (ξ, η) is an infinite variation Lévy process then, as noted in Proposition 1.11, $\int_{\{x^2+y^2<1\}} |(x,y)|\Pi_{\xi,\eta}(\mathrm{d}(x,y)) = \infty$. Thus, it may be the case that for a particular $u \in \mathbb{R}$ the integral $\int_{\{x^2+y^2<1\}}(ux+y)\Pi_{\xi,\eta}(\mathrm{d}(x,y))$, and hence the function $g(u)$ in (3.2), may not exist as a well-defined member of the extended real numbers. However, it is a consequence of the proof of Theorem 2.9, that if $u \in L^*$ then $g(u)$ is a well defined member of the extended reals, and $g(u) \in [-\infty, \infty)$. Under such conditions, it is also shown that

$$\Pi_{\xi,\eta}\left(\{y - u(e^{-x} - 1) < 0\}\right) = 0$$

and so the domain of integration for the integral component of g can be decreased to $\{x^2 + y^2 < 1\} \cap \{y - u(e^{-x} - 1) \geq 0\}$.

2. Note that g is a linear function on $\mathbb{R}$ iff the Lévy measure of (ξ, η) is of finite variation, namely

$$\int_{\{x^2+y^2<1\}} |(x,y)|\Pi_{\xi,\eta}(\mathrm{d}(x,y)) < \infty.$$

In this case the drift vector (d_ξ, d_η) is finite, and we can write

$$\begin{aligned} g(u) &= \gamma_\eta - \int_{(-1,1)} y\Pi_\eta(\mathrm{d}y) + u\left(\gamma_\xi - \frac{1}{2}\sigma_\xi^2 - \int_{(-1,1)} x\Pi_\xi(\mathrm{d}x)\right) \\ &= d_\eta + u\left(d_\xi - \frac{1}{2}\sigma_\xi^2\right), \end{aligned} \tag{3.4}$$

where the first equality follows by converting $(\tilde{\gamma}_\xi, \tilde{\gamma}_\eta)$ to $(\gamma_\xi, \gamma_\eta)$ using equation (1.11) and the symmetric version for η, and the second equality follows by converting $(\gamma_\xi, \gamma_\eta)$ to (d_ξ, d_η) using equation (1.9). It will be a consequence of the proof of Proposition 3.17, that if $a, b \in L$ and $a \neq b$ then g is a linear function on $\mathbb{R}$.

3. In Proposition 1.14 we stated exact conditions for a Lévy process to be a subordinator. When $u \neq 0$ the Lévy measure conditions in Proposition 3.16 are exactly the requirements for $\eta - uW$ to be a subordinator. Equation (3.1) is equivalent to the condition $\sigma_{\eta-uW} = 0$. The requirement that one of the conditions (a), (b) and (c) holds is equivalent to the requirement that there exists $u \neq 0$ such that $\Pi_{\eta-uW}((-\infty, 0)) = 0$. Note that this implies that $L^* \setminus \{0\}$ is precisely the set of all $u \neq 0$ such $\eta - uW$ has no negative jumps. Finally, if $u \in L^*$ then $g(u) = d_{\eta-uW}$, and hence condition (3.2) is equivalent to the requirement that $\eta - uW$ has positive drift. The fact that $\eta - uW$ is of finite variation actually follows from the two conditions $\Pi_{\eta-uW}((-\infty, 0)) = 0$ and $d_{\eta-uW} \geq 0$. To see this, note that when $\Pi_{\eta-uW}((-\infty, 0)) = 0$, the equation (1.9) simplifies to

$$d_{\eta-uW} = \gamma_{\eta-uW} - \int_{(0,1)} x\Pi_{\eta-uW}(\mathrm{d}x)$$

and hence $d_{\eta-uW}$ is a well-defined member of the extended reals regardless of whether $\eta - uW$ is finite variation. In particular, $d_{\eta-uW} \in [-\infty, \infty)$, and $d_{\eta-uW} = -\infty$ iff $\int_{(0,1)} x\Pi_{\eta-uW}(\mathrm{d}x) = \infty$ which occurs iff $\eta - uW$ is infinite variation.

Although the situation is symmetric, we explicitly state the parallel version for U and Υ, to Proposition 3.16. No proof is given. We state the parallel result explicitly because some of the statements are not obvious, and we need to use them for Theorem 3.3. Also, we will need to combine them with the statements for L and δ in order to prove Theorem 3.9, 3.13 and 3.14. If we define

$$U^* := \{u \in \mathbb{R} : \forall t > 0 \; P\left(\Delta V_t > 0 | V_{t-} = u\right) = 0\},$$

then the symmetric versions of Proposition 3.17, Proposition 3.18 and Remark 3.19 also hold. We will need to use these results, however the parallels are obvious in this case, so we do not state them explicitly.

Proposition 3.20 (upper bound). *The following are equivalent:*

1. *The upper bound $\Upsilon(z) < \infty$ for some $z \in \mathbb{R}$;*

2. *There exists $u \in \mathbb{R}$ such that $\Upsilon(u) = u$;*

3. *There exists $u \in \mathbb{R}$ such that the Lévy process $-(\eta - uW)$ is a subordinator.*

Statements (2) and (3) hold for a particular value $u \neq 0$ iff the following three conditions are satisfied: (i) the Gaussian covariance matrix satisfies equation (3.1); (ii) one of the following is true:

(a) $\Pi_{\xi,\eta}(A_1) = 0$, $\Pi_{\xi,\eta}(A_4) \neq 0$, $\theta_4' \leq \theta_2'$ *and* $u \in [\theta_4', \theta_2']$;

(b) $\Pi_{\xi,\eta}(A_4) = 0$, $\Pi_{\xi,\eta}(A_1) \neq 0$, $\theta_3' \leq \theta_1'$ *and* $u \in [\theta_3', \theta_1']$;

(c) $\Pi_{\xi,\eta}(A_1) = \Pi_{\xi,\eta}(A_4) = 0$ *and* $u \in [\theta_3', \theta_2']$;

and, (iii), in addition, u satisfies $g(u) \leq 0$ for the function g from equation (3.2).

Remark 3.21. Symmetric statements to those for L and L^* in Remark 3.19, hold for U and U^*. The following remarks relate to the combination of L and U, and L^* and U^*.

1. Parallel to 1 and 2 of Remark 3.19, whenever $u \in U^*$, $g(u)$ from (3.2) is a well-defined member of the extended reals, $g(u) \in (-\infty, \infty]$, and $-g(u) = d_{-(\eta - uW)}$. Since $d_{-(\eta-uW)} = -d_{\eta-uW}$, we know that if $u \in U^* \cup L^*$ then $g(u)$ is a well-defined member of the extended reals and $g(u) = d_{\eta-uW}$.

2. If $a \in L$, $b \in U$ and $a \neq b$ then g is linear and (ξ, η) is finite variation. This statement is proved easily using similar arguments to those in the proof of Proposition 3.17.

We state a proposition, describing the possible combinations of L^* and U^*, which will be essential for proving Theorem 3.9.

Proposition 3.22. *The following statements hold for L^*, and the symmetric statements hold for U^* :*

1. *If $L^* = \mathbb{R}$ then $U^* = \emptyset$ or $U^* = \mathbb{R}$;*

2. *If $L^* = [a, b]$ for some $-\infty < a \leq b < \infty$, then $U^* = \emptyset$ or $U^* = L^* = \{a\} = \{b\}$;*

3. *If $L^* = [b, \infty)$ for some $b \in \mathbb{R}$, then $U^* = \emptyset$ or $U^* = (-\infty, a]$ for some $-\infty < a \leq b < \infty$;*

4. If $L^* = (-\infty, a]$ for some $a \in \mathbb{R}$, then $U^* = \emptyset$ or $U^* = [b, \infty)$ for some $-\infty < a \leq b < \infty$.

We end this section with two useful lemmas. The first follows by considering the definitions of θ_i and θ_i' for $i = 1, 2, 3, 4$, and no proof is given. It will be used several times as a calculation tool. The second gives conditions on the Lévy measure of ξ and η which ensure that the random variable $\sup_{0\leq t\leq 1} |Z_t|$ has finite mean. It will be needed to prove statement (2) of Theorem 3.1.

Lemma 3.23. *1. If $\Pi_{\xi,\eta}(A_1) \neq 0$ then $\theta_1' \leq \theta_1 \leq 0$;*

2. If $\Pi_{\xi,\eta}(A_2) \neq 0$ then $0 \leq \theta_2' \leq \theta_2$;

3. If $\Pi_{\xi,\eta}(A_3) \neq 0$ then $\theta_3 \leq \theta_3' \leq 0$;

4. If $\Pi_{\xi,\eta}(A_4) \neq 0$ then $0 \leq \theta_4 \leq \theta_4'$.

Further:

(a) $\Pi_{\xi,\eta}(A_1) = 0$ iff $\theta_1 = -\infty$ and $\theta_1' = 0$;

(b) $\Pi_{\xi,\eta}(A_2) = 0$ iff $\theta_2 = 0$ and $\theta_2' = \infty$;

(c) $\Pi_{\xi,\eta}(A_3) = 0$ iff $\theta_3 = 0$ and $\theta_3' = -\infty$;

(d) $\Pi_{\xi,\eta}(A_4) = 0$ iff $\theta_4 = \infty$ and $\theta_4' = 0$.

Lemma 3.24. *Suppose there exist $r > 0$ and $p, q > 1$ with $1/p + 1/q = 1$ such that $E\left(e^{-\max\{1,r\}p\xi_1}\right) < \infty$ and $E\left(|\eta_1|^{\max\{1,r\}q}\right) < \infty$. Then*

$$E\left(\sup_{0\leq t\leq 1}\left|\int_0^t e^{-\xi_{s-}}\mathrm{d}\eta_s\right|^{\max\{1,r\}}\right) < \infty. \tag{3.5}$$

Remark 3.25. Note that if ξ and η are independent then the conditions of the above lemma simplify to the requirement of $r > 0$ such that $E\left(e^{-\max\{1,r\}\xi_1}\right) < \infty$ and $E\left(|\eta_1|^{\max\{1,r\}}\right) < \infty$.

3.4 Proofs and Examples

The proofs of the results are presented in mathematically chronological order rather than the order in which the statements of the results are presented. For all the proofs, except the proof of Proposition 3.7, we assume that neither ξ nor η are identically zero.

Proposition 3.18. We begin by proving statements (2) and (3). The proof of statements (4) and (5) follows trivially from the proof of statements (2) and (3). We finish by proving statement (1).

(2) It is a consequence of the statement of Proposition 2.6 that

$$\Delta(\eta_t - uW_t) = \Delta\eta_t - u\left(e^{-\Delta\xi_t} - 1\right).$$

Thus, equation (2.3) implies that whenever $V_{t-} = u$, a jump $(\Delta\xi_t, \Delta\eta_t)$ causes a negative jump ΔV_t iff $\Delta(\eta_t - uW_t)$ is negative. Hence L^* is precisely the set of all u such that $\eta_t - uW_t$ has no negative jumps.

(3) By (2) above, $L^* \neq \emptyset$ iff $\eta - uW$ has no negative jumps for some $u \in \mathbb{R}$. If $u = 0$, this occurs iff η has no negative jumps. If $u \neq 0$, it is noted in point (3) of Remark 3.19, that this occurs iff $u \neq 0$ satisfies condition (ii) of Proposition 3.16.

(1) Suppose $L^* \neq \emptyset$. If $0 \in L^*$ then the statement is trivial. If $0 \notin L^*$ then condition (ii) of Proposition 3.16 must hold for some $u \neq 0$. We can assume that condition (a) of Proposition 3.16 holds. If condition (b) or (c) of Proposition 3.16 holds then the proof is similar. Since (a) holds, property (5) implies that $L^* = [\theta_2, \theta_4]$. Recall that equation (2.2) states

$$\Delta V_t = (e^{\Delta\xi_t} - 1)V_{t-} + e^{\Delta\xi_t}\Delta\eta_t,$$

and suppose $V_{t-} > \theta_4$. It follows immediately from the definitions of θ_4 and A_4^u, and from equation (2.2), that there exists $(x, y) \in A_4^{V_{t-}}$ such that $(e^x - 1)\theta_4 + e^x y \geq 0$ and $(e^x - 1)V_{t-} + e^x y < 0$. Thus,

$$\begin{aligned} V_t &= V_{t-} + (e^x - 1)V_{t-} + e^x y \\ &= V_{t-} + (e^x - 1)(V_{t-} - \theta_4) + (e^x - 1)\theta_4 + e^x y \\ &\geq V_{t-} + (e^x - 1)(V_{t-} - \theta_4) \\ &> \theta_4, \end{aligned}$$

as required.

□

Proposition 3.17. Assume that $\sigma_\xi^2 \neq 0$ and statements (1)-(3) of Proposition 3.16 hold for some $u \neq 0$. Then equation (3.1) must hold for u, which implies that $u = -\frac{\sigma_{\xi,\eta}}{\sigma_\xi^2}$, and hence is the unique non-zero number satisfying statements (1)-(3) of Proposition 3.16. Since $-\frac{\sigma_{\xi,\eta}}{\sigma_\xi^2}$ satisfies condition (2), $L = \{-\frac{\sigma_{\xi,\eta}}{\sigma_\xi^2}\}$ by definition.

Now assume that $\sigma_\xi^2 \neq 0$ and statements (1)-(3) of Proposition 3.16 hold for $u = 0$. By statement (2), $0 \in L$. By statement (3), η is a subordinator, and hence $\sigma_\eta^2 = \sigma_{\xi,\eta} = 0$. Thus, by the above, no non-zero number can satisfy statements (1)-(3), and so $L = \{0\} = \{-\frac{\sigma_{\xi,\eta}}{\sigma_\xi^2}\}$.

Now assume that $\sigma_\xi^2 = 0$. If statements (1)-(3) of Proposition 3.16 hold for $u = 0$ then η is a subordinator by statement (3) and hence $\sigma_\eta^2 = 0$. Alternatively, If statements (1)-(3) of Proposition 3.16 hold for some $u \neq 0$ then equation (3.1) must hold for u, which implies that $\sigma_\eta^2 = u^2\sigma_\xi^2$, and so $\sigma_\eta^2 = 0$.

Now assume that $\sigma_\xi^2 = 0$ and condition (ii) of Proposition 3.16 does not hold for any $u \neq 0$. This immediately implies that $L \cap (\mathbb{R} \setminus \{0\}) = \emptyset$. If, further, η is a subordinator, then $0 \in L$, and hence $L = \{0\}$.

Now assume that $\sigma_\xi^2 = 0$ and condition (ii) of Proposition 3.16 holds for some $u \neq 0$. This occurs precisely when one of conditions (a), (b) or (c) of Proposition 3.16 holds, and equation (3.2) holds. It follows immediately that $\inf L = a$ and $\sup L = b$ for the values of a and b given in the proposition statement. It remains to prove that the set L is connected. Since L^* is connected, this occurs iff $\{u \in \mathbb{R} : g(u) \geq 0\}$ is connected, which follows from the analysis below.

As noted in point (1) of Remark 3.19, whenever $u \in L^*$ we know $g(u) \in [-\infty, \infty)$. There are three possibilities for behaviour of g on L^*. Firstly, it may be that $g(u) = -\infty$ for all $u \in L^*$. Secondly there may exist $v \in L^*$ such that $g(v)$ is finite and $g(u) = -\infty$ for all $u \in L^*$ with $u \neq v$. We show that the only other possibility is that g is linear on $\mathbb{R}$. Suppose there exists $u_1, u_2 \in L^*$ with $u_1 \neq u_2$, such that $g(u_1)$ and $g(u_2)$ are both finite. Then

$$g(u_1) - g(u_2) = \left(\tilde{\gamma}_\xi - \frac{1}{2}\sigma_\xi^2 - \int_{\{x^2+y^2<1\}} x\Pi_{\xi,\eta}(\mathrm{d}(x,y))\right)(u_1 - u_2)$$

is finite, which implies that $\int_{\{x^2+y^2<1\}} x\Pi_{\xi,\eta}(\mathrm{d}(x,y))$ exists, and is finite. Since $g(u_1)$ is finite, this implies that $\int_{\{x^2+y^2<1\}} y\Pi_{\xi,\eta}(\mathrm{d}(x,y))$ exists and is finite. Thus, g is a linear function on $\mathbb{R}$. □

Proposition 3.6. It is a consequence of Proposition 3.16 that $\delta(\delta(z)) = \delta(z)$ and

$$\delta(z) = \sup\{u \leq z : \delta(u) = u\}. \tag{3.6}$$

Now the first statement of Proposition 3.6 follows immediately from (3.6). To prove the second statement, assume $z < \inf L$. Suppose $-\infty < m := \delta(z)$. Since $\delta(z) \leq z$, we have $-\infty < m \leq z < \inf L$. However, equation (3.6) implies that $m \in L$, which gives a contradiction. Hence $\delta(z) = -\infty$. The third and fourth statements follow immediately from the definitions of δ and L. □

Proposition 3.7. Assume $L = \mathbb{R}$. This implies, using Proposition 3.16 and point (2) of Remark 3.19, that $\Sigma_{\xi,\eta} = 0$ and g is linear. Further, it must be the case that $\Pi_{\xi,\eta}(A_3) = \Pi_{\xi,\eta}(A_2) = 0$ and $L^* = [\theta_1, \theta_4] = (-\infty, \infty)$. Now $\theta_1 = -\infty$ iff $\Pi_{\xi,\eta}((0,\infty) \times [0,\infty))$ whilst $\theta_4 = -\infty$ iff $\Pi_{\xi,\eta}((-\infty, 0) \times [0,\infty)) = 0$. Hence ξ can have no jumps and η can only have positive jumps. By Proposition 3.16, $g(u) \geq 0$ on $\mathbb{R}$. Since $g(u) = d_\eta + ud_\xi$, this implies that $d_\xi = 0$ and $d_\eta \geq 0$, thus proving one direction of the first claim. The converse is trivial since the GOU simplifies to $V_t = z + \eta_t$. The proof of the second claim is similar. The third claim follows immediately from the first two. □

Proposition 3.22. We prove statements (1), (2) and (3). The proof of statement (4) is symmetrical to the proof of statement (3).

(1) Assume that $L^* = \mathbb{R}$. Then condition (c) of Proposition 3.16 must hold, and so $\Pi_{\xi,\eta}(A_2) = \Pi_{\xi,\eta}(A_3) = 0$, and $L^* = [\theta_1, \theta_4]$. Since $\theta_1 = -\infty$ and $\theta_4 = \infty$, it must be that $\Pi_{\xi,\eta}(A_1 \setminus A_4) = 0$ and $\Pi_{\xi,\eta}(A_4 \setminus A_1) = 0$, respectively. Thus, if $\Pi_{\xi,\eta}(A_1 \cap A_4) = 0$ then $\Pi_{\xi,\eta}(\mathbb{R}^2) = 0$, in which case condition (c) of Proposition 3.20 holds, and $U^* = \mathbb{R}$. Alternatively, if $\Pi_{\xi,\eta}(A_1 \cap A_4) \neq 0$ then η has positive jumps and so $0 \notin U^*$, and condition (ii) of Proposition 3.20 cannot hold. Hence $U^* = \emptyset$.

(2) Assume that $L^* = [a, b]$ for some $-\infty < a \leq b < \infty$. There are four ways in which this is possible, namely, when conditions (a), (b) or (c) of Proposition 3.16 hold, or when $L^* = \{0\}$. For each of these four cases we show that $U^* = \emptyset$ or $U^* = L^* = \{a\} = \{b\}$.

Suppose first that condition (a) of Proposition 3.16 holds, and $U^* \neq \emptyset$. The case in which condition (b) holds and $U^* \neq \emptyset$, is symmetric. Propositions 3.16 and 3.18 imply that $\Pi_{\xi,\eta}(A_3) = 0$, $\Pi_{\xi,\eta}(A_2) \neq 0$, $\theta_2 \leq \theta_4$ and $L^* = [\theta_2, \theta_4]$. Since $\theta_4 < \infty$, it must be that $\Pi_{\xi,\eta}(A_4 \setminus A_1) \neq 0$. Since $\Pi_{\xi,\eta}(A_3) = 0$, this implies that $-\eta$ is not a subordinator, and so $0 \notin U^*$. Thus, since we have assumed that $U^* \neq \emptyset$, it must be that condition (a) of Proposition 3.20 holds, and so $\Pi_{\xi,\eta}(A_1) = 0$, $\theta_4' \leq \theta_2'$, and $U^* = [\theta_4', \theta_2']$. However, statements (2) and (4) of Lemma 3.23 state that $\theta_2' \leq \theta_2$ and $\theta_4 \leq \theta_4'$. Hence $\theta_2' = \theta_2 = \theta_4 = \theta_4'$.

Now suppose that condition (c) of Proposition 3.16 holds. Then $\Pi_{\xi,\eta}(A_2) = \Pi_{\xi,\eta}(A_3) = 0$, and $L^* = [\theta_1, \theta_4]$. Since $\theta_4 < \infty$ and $\theta_1 > -\infty$ it must be that $\Pi_{\xi,\eta}(A_4 \setminus A_1) \neq 0$ and $\Pi_{\xi,\eta}(A_1 \setminus A_4) \neq 0$, respectively. Hence condition (ii) of Proposition 3.20 cannot hold, and so $U^* \setminus \{0\} = \emptyset$. Further, $-\eta$ is not a subordinator, and so $U^* = \emptyset$.

Now suppose that $L^* = \{0\}$, and $U^* \neq \emptyset$. By statement (4) of Proposition 3.18,

$L^* = \{0\}$ iff η has no negative jumps and at the same time $\Pi_{\xi,\eta}(A_3 \cap A_4) \neq 0$ and $\Pi_{\xi,\eta}(A_2 \cap A_1) \neq 0$. Hence, condition (ii) of Proposition 3.20 fails to hold, which implies $U^* \setminus \{0\} = \emptyset$. Thus, since we have assumed $U^* \neq \emptyset$, it must be that $U^* = L^* = \{0\}$.

(3) Assume that $L^* = [b, \infty)$ for some $b \in \mathbb{R}$ and $U^* = \emptyset$. We show that $U^* = (-\infty, a]$ for some $-\infty < a \leq b < \infty$. By the symmetric version of statement (2) of Proposition 3.22, it is immediate that $U^* \neq \{0\}$.

Since $L^* = [b, \infty)$, condition (a) or (c) of Proposition 3.16 must hold, with $\theta_4 = \infty$. Thus, $\Pi_{\xi,\eta}(A_3) = 0$, which implies that $\theta_3' = -\infty$. Also, since $\theta_4 = \infty$, it must be that $\Pi_{\xi,\eta}(A_4 \setminus A_1) = 0$. Since $U^* \neq \emptyset$, it must be that $\Pi_{\xi,\eta}(A_1 \cap A_4) = 0$, and so $\Pi_{\xi,\eta}(A_4) = 0$. This implies that one of conditions (b) or (c) of Proposition 3.20 must hold, and so $U^* = (-\infty, \theta_1']$ or $U^* = (-\infty, \theta_2']$ respectively. Now, if condition (a) of Proposition 3.16 holds, then $L^* = [\theta_2, \infty)$. Note that Lemma 3.23 states that $\theta_1' \leq 0 \leq \theta_2' \leq \theta_2$, and hence the result is proved for either form of U^*.

Alternatively, if condition (c) of Proposition 3.16 holds, then $L^* = [\theta_1, \infty)$ where $\theta_1 > -\infty$, which implies that $\Pi_{\xi,\eta}(A_1 \setminus A_4) \neq 0$. Hence, condition (b) of Proposition 3.20 must hold and $U^* = (-\infty, \theta_1']$. Lemma 3.23 states that $\theta_1' \leq \theta_1$, and so we are done. □

Proposition 3.8. We prove the equivalence of statements (1)-(6).

(1)⇔(2) Assume $L \cap U \neq \emptyset$ and let $z_1, z_2 \in L \cap U$. We show $z_1 = z_2 \neq 0$. By Proposition 3.16, $z \in L$ iff $\eta - zW$ is increasing and by Proposition 3.20, $z \in U$ iff $\eta - zW$ is decreasing. Thus, $\eta - z_1 W = \eta - z_2 W = 0$, which implies $z_1 W = z_2 W$. Since ξ is not zero, W is not zero, and thus $z_1 = z_2$. Further, if $z_1 = z_2 = 0$, then η must be both increasing and decreasing, which requires that η be identically zero. Since we have rejected this case, it must be that $z_1 = z_2 \neq 0$.

(2)⇔(3) Suppose $L \cap U = \{c\}$. Then $V_t = c$ for all $t \geq 0$ whenever $V_0 = c$, which implies $e^{\xi_t}(c + Z_t) = c$, which implies $V_t = e^{\xi_t}(z - c) + c$, as required. Conversely, suppose $V_t = e^{\xi_t}(z-c)+c$. Clearly, $c \in L \cap U$ and so $L \cap U \neq \emptyset$, which implies $L \cap U = \{c\}$ by the above.

(2)⇔(4) By the definitions of δ and Υ, it is clear that c is an absorbing point iff $\delta(c) = \Upsilon(c) = c$, and the definitions of L and U imply that this occurs iff $c \in L \cap U$.

(2)⇒(5) Assume $L \cap U = \{c\}$ where $c \neq 0$. Propositions 3.16 and Proposition 3.20 immediately imply that equation (3.1) is satisfied for $u = c$, and imply respectively that $g(c) \geq 0$ and $g(c) \leq 0$, thus giving $g(c) = 0$. Finally, since (2) $\Rightarrow$ (3), the equation $Z_t := \int_0^t e^{-\xi_{s-}} d\eta_s = c\left(e^{-\xi_t} - 1\right)$ holds, which implies that $e^{-\xi_{t-}}\Delta\eta_t = c\left(e^{-\xi_t} - 1\right) - c\left(e^{-\xi_{t-}} - 1\right)$ and so $\Delta\eta_t = c\left(e^{-\Delta\xi_t} - 1\right)$.

(5)$\Rightarrow$(2) Assume that the conditions of statement (5) hold for $c \neq 0$. We prove $c \in L$. Since (3.1) is satisfied for $u = c$, and $g(c) = 0$ holds, we know that conditions (i) and (iii) of Proposition 3.16 are respectively satisfied for $u = c$. Thus it suffices to prove condition (ii) of Proposition 3.16 is satisfied for $u = c$, or equivalently, show $c \in L^*$. If $\Pi_{\xi,\eta} = 0$ then this is trivial since $L^* = \mathbb{R}$. Now suppose that $\Pi_{\xi,\eta}$ is supported on the curve $\{(x, y) : y - c(e^{-x} - 1) = 0\}$ for $c \in \mathbb{R}$. If $c > 0$, $\Pi_{\xi,\eta}(A_2) \neq 0$ and $\Pi_{\xi,\eta}(A_4) \neq 0$, then $\theta_2 = \theta_4 = c$ and so $L^* = \{c\}$. If $c \geq 0$, $\Pi_{\xi,\eta}(A_2) = 0$ and $\Pi_{\xi,\eta}(A_4) \neq 0$, then $\theta_2 = 0$ and $\theta_4 = c$, and so $L^* = [0, c]$. If $c \geq 0$, $\Pi_{\xi,\eta}(A_2) \neq 0$ and $\Pi_{\xi,\eta}(A_4) = 0$, then $\theta_2 = c$ and $\theta_4 = \infty$, and so $L^* = [c, \infty)$. In each of these three cases, $c \in L^*$. The proof for $c < 0$ is similar and we omit.

A symmetric argument proves that $c \in U$. Hence, $c \in L \cap U$ which, by the equivalence of statements (1) and (2), implies that $L \cap U = \{c\}$, as required.

(2)$\Leftrightarrow$(6) $L \cap U = \{c\}$ iff $\eta - cW = 0$ where $e^{-\xi_t} = \epsilon(W)_t$ which occurs iff $e^{-\xi_t} = \epsilon(\eta/c)_t$.

Now assume that the above statements (1)-(6) hold. If $\Sigma_{\xi,\eta} \neq 0$ and both L and U are non-empty, then Propositions 3.16 and 3.20 immediately imply that $L = U = \{c\}$ where $c = -\frac{\sigma_{\xi,\eta}}{\sigma_\xi^2}$. For examples of Lévy processes (ξ, η) satisfying statements (1)-(6) and such that ξ drifts to ∞ a.s., ξ drifts to $-\infty$ a.s. or ξ oscillates a.s., see Example 3.27.

If $\Sigma_{\xi,\eta} = 0$ then the statements (a), (b) and (c) follow immediately by examining the equation for V in statement (3) above. For examples of Lévy processes (ξ, η) satisfying statement (c) and such that ξ drifts to ∞ a.s., ξ drifts to $-\infty$ a.s. or ξ oscillates a.s., see Example 3.28. □

Theorem 3.9. Assume that $L \cap U = \emptyset$. Suppose, firstly, that $\Sigma_{\xi,\eta} \neq 0$. We must show that (ξ, η) exists such that (1), (2) or (3) occurs, and for each of these cases, we must show that ξ can satisfy each of the three asymptotic behaviours. For case (1), this is obvious. Choosing (ξ, η) such that $\Sigma_{\xi,\eta}$ does not satisfy equation (3.1) implies that (ξ, η) fails both propositions, and so $L = U = \emptyset$, regardless of the choice of $(\tilde{\gamma}_\xi, \tilde{\gamma}_\eta)$ and $\Pi_{\xi,\eta}$. Clearly, we can make suitable choices for these objects to obtain the desired asymptotic behaviour of ξ. For case (2), our existence claims are proven by Example 3.26, and case (3) is symmetric. It follows from Proposition 3.17, and the symmetric version for U, that whenever L and U are non-zero, they are each equal to $\{-\sigma_{\xi,\eta}/\sigma_\xi^2\}$. Hence, no cases, other than (1), (2) and (3) of Theorem 3.9, can exist.

Now suppose that $\Sigma_{\xi,\eta} = 0$. We must show that (ξ, η) exists such that (a), (b) or (c) occurs, and for each of these cases, we must show that ξ can satisfy the specified asymptotic behaviours. Examples 3.29 and 3.30 present (ξ, η) such that $L = \emptyset$, whilst U may be of form $\emptyset$, $\{a\}$ or $[a, b]$ for $-\infty < a < b < \infty$, and for each of these combinations, it is shown that ξ can satisfy the three asymptotic behaviours. In Example 3.31, $L = \emptyset$, U is of form $[b, \infty)$ for $b \in \mathbb{R}$, and ξ drifts to $-\infty$ a.s. In Example 3.33, $L = \emptyset$, U is of form $(-\infty, a]$ for $a \in \mathbb{R}$, and ξ drifts to ∞ a.s. These four examples prove the existence claims for (a), and the case (b) is symmetric. In Example 3.32, $L = (-\infty, a]$, $U = [b, \infty)$ for $-\infty < a < b < \infty$ and ξ drifts to $-\infty$ a.s. In Example 3.34, $U = (-\infty, a]$, $L = [b, \infty)$ for $-\infty < a < b < \infty$, and ξ drifts to ∞ a.s. These two examples prove the existence claims for (c).

We now assume that $\Sigma_{\xi,\eta} = 0$, $L \neq \emptyset$, $U \neq \emptyset$ and $L \cap U = \emptyset$. We prove that no cases, other than those listed in (c), can exist. As noted in point (2) of Remark 3.21, it follows from our assumptions that (ξ, η) is finite variation and g is linear.

Suppose that $L = [a, b]$ for some $-\infty < a \leq b < \infty$. We show that this causes a contradiction with our assumptions. If $L^* = [c, d]$ for some $-\infty < c \leq a \leq b \leq d < \infty$, then point (2) of Proposition 3.22 states that $U^* = \emptyset$ or $U^* = L^* = \{c\} = \{d\}$. Thus, $U = \emptyset$ or $U = L = \{a\} = \{b\}$, both of which contradict our assumptions. Hence, it must be the case that $L^* = [c, \infty)$ for some $-\infty < c \leq a$, or $L^* = (-\infty, d]$ for some $b \leq d < \infty$.

Thus, we suppose that $L = [a, b]$ and $L^* = [c, \infty)$ for some $-\infty < c \leq a \leq b < \infty$. The case in which $L^* = (-\infty, d]$ for some $b \leq d < \infty$ is symmetric. We know $g(u) = d_\eta + ud_\xi$. If $d_\xi \geq 0$ then it must be that $b = \infty$, which we have rejected. Hence $d_\xi < 0$, and we must have $b = -\frac{d_\eta}{d_\xi} \geq a$. Thus, since U is non-empty, $L \cap U = \emptyset$, and $g(u) \leq 0$ on U, it must be that $U \subset [b, \infty)$. However, point (3) of Proposition 3.22 implies that $U^* \cap [b, \infty) = \emptyset$. Hence U is empty, and we have a contradiction. This completes the proof that $L \neq [a, b]$ for some $-\infty < a \leq b < \infty$.

We now assume that $L = [b, \infty)$ for $b \in \mathbb{R}$. We first prove that ξ is a subordinator, which is another of the statements of Proposition 3.17 and point (2) of Remark 3.19, imply respectively, that (ξ, η) has no Brownian component, and

(ξ,η) is of finite variation. Thus, we can write $g(u) = d_\eta + ud_\xi$. Proposition 3.16 implies that $g(u) \geq 0$ on $[b,\infty)$ and hence $d_\xi \geq 0$. Finally, it must be that $L^* = [c,\infty)$ for some $-\infty \leq c \leq b$. It is a consequence of the proofs of statements (1) and (3) of Proposition 3.22, that ξ has no negative jumps. Thus ξ is a subordinator.

Now, we assume that $L = [b,\infty)$ for $b \in \mathbb{R}$ and $U = \emptyset$. We prove that $U = (-\infty, a]$ for some $-\infty < a < b < \infty$. Note that $L^* = [c,\infty)$ for some $-\infty \leq c \leq b$, so statement (3) of Proposition 3.22 implies that $U^* = (-\infty, d]$ for some $-\infty < d \leq c$. Since $g(u) = d_\eta + ud_\xi$ and $d_\xi \geq 0$, $U = (-\infty, a]$ for some $-\infty < a \leq d$. Since we have assumed $L \cap U = \emptyset$, $a < b$ as required.

If we assume that $U = (-\infty, a]$ for $a \in \mathbb{R}$, it can be shown, using a method of proof similar to the one above, that ξ is a subordinator, and $L = \emptyset$ or $L = [b,\infty)$ for some $-\infty < a < b < \infty$. We omit the details.

Now, if we assume $L = (-\infty, a]$ for $a \in \mathbb{R}$, then symmetric proofs to the ones above, show that $-\xi$ is a subordinator, and $U = \emptyset$ or $U = [b,\infty)$ for $-\infty < a < b < \infty$. Similarly, if we assume $U = [b,\infty)$ for $b \in \mathbb{R}$, then symmetric proofs show that $-\xi$ is a subordinator, and $L = \emptyset$ or $L = (-\infty, a]$ for $-\infty < a < b < \infty$. □

Proposition 3.11. Assume $L \cap U = \emptyset$. In the above proof of Theorem 3.9, it was shown that if $L = [b,\infty)$ for $b \in \mathbb{R}$ then (ξ,η) is of finite variation, $\Sigma_{\xi,\eta} = 0$, $d_\xi \geq 0$, $\Pi_{\xi,\eta}(A_3) = 0$, $\Pi_{\xi,\eta}(A_4 \setminus A_1) = 0$, and $\theta_2 < \infty$. It is clear from Propositions 3.16 and 3.17 that the converse also holds. A similar proof shows that $U = (-\infty, a]$ for $a \in \mathbb{R}$ iff (ξ,η) is of finite variation, $\Sigma_{\xi,\eta} = 0$, $d_\xi \geq 0$, $\Pi_{\xi,\eta}(A_4) = 0$, $\Pi_{\xi,\eta}(A_3 \setminus A_2) = 0$, and $\theta'_1 > -\infty$. Combining these two sets of iff conditions immediately gives iff conditions for the case in which $U = (-\infty, a]$ and $L = [b,\infty)$ with $-\infty < a < b < \infty$. Since V is increasing on L and decreasing on U, and V is a strong Markov process, it is clear that in this situation $\lim_{t\to\infty} |V_t| = \infty$ a.s. for any $V_0 = z \in \mathbb{R}$.

It follows by symmetric methods that $L = (-\infty, a]$ and $U = [b,\infty)$ for $-\infty < a < b < \infty$ iff the stated conditions in Proposition 3.11 hold. The only extra proof needed is to show that in this situation, V is strictly stationary. In [44] it is shown that

$$V_t =_D e^{\xi_t} z + \int_0^t e^{\xi_{s-}} \mathrm{d}K_s^{\xi,\eta}.$$

By Theorem 2 in [22] it is shown that if $\lim_{t\to\infty} \xi_t = -\infty$ and the integral condition $I_{-\xi,K^{\xi,\eta}} = \infty$ holds, then $|\int_0^t e^{\xi_{s-}} \mathrm{d}K_s^{\xi,\eta}| \to_P \infty$ as $t \to \infty$.

As noted, if $L = (-\infty, a]$ and $U = [b,\infty)$ with $-\infty < a < b < \infty$ then $-\xi$ is a subordinator and so $\lim_{t\to\infty} \xi_t = -\infty$ a.s. Now if $I_{-\xi,K^{\xi,\eta}} = \infty$ then by the above, and since $\lim_{t\to\infty} e^{\xi_t} = -\infty$ a.s, it must be that $|V_t| \to_D \infty$. However this is impossible since V is increasing on L and decreasing on U. Thus, we must have $I_{-\xi,K^{\xi,\eta}} < \infty$. Hence, by Theorem 2.1 in [44], V is strictly stationary and converges in distribution to $\int_0^\infty e^{\xi_{s-}}\mathrm{d}K_s^{\xi,\eta} := V_\infty$. Since V is increasing on L and decreasing on U, and V is a strong Markov process, it is clear that V_∞ has support (a,b). □

Lemma 3.24. For ease of notation let $k := \max\{1, r\}$. Assume there exists $r > 0$ and $p, q > 1$ with $1/p + 1/q = 1$ such that $E\left(e^{-kp\xi_1}\right) < \infty$ and $E\left(|\eta_1|^{kq}\right) < \infty$.We prove the proposition firstly for the case in which $E(\eta_1) = 0$. Since η is a Lévy process this assumption implies that η is a càdlàg martingale. Since ξ is càdlàg , $e^{-\xi}$ is a locally bounded process and hence Z is a local martingale for $\mathbb{F}$ by Protter [60], p.171. Since Z is a local martingale and $Z_0 = 0$, the Burkholder-Davis-Gundy inequalities in Lipster and Shiryaev [46], p.70 and p.75, ensure that for our choices of p, q and k there exists $b > 0$ such that

$$\begin{aligned}
E\left(\sup_{0\le t\le 1}\left|\int_0^t e^{-\xi_{s-}}\mathrm{d}\eta_s\right|^k\right) &\le bE\left(\left[\int_0^\bullet e^{-\xi_{s-}}\mathrm{d}\eta_s, \int_0^\bullet e^{-\xi_{s-}}\mathrm{d}\eta_s\right]_1^{k/2}\right)\\
&= bE\left(\left(\int_0^1 e^{-2\xi_{s-}}\mathrm{d}[\eta,\eta]_s\right)^{k/2}\right)\\
&\le bE\left(\left(\int_0^1 \sup_{0\le t\le 1} e^{-2\xi_t}\mathrm{d}[\eta,\eta]_s\right)^{k/2}\right)\\
&= bE\left(\sup_{0\le t\le 1} e^{-k\xi_t}[\eta,\eta]_1^{k/2}\right)\\
&\le b\left(E\left(\sup_{0\le t\le 1} e^{-pk\xi_t}\right)\right)^{1/p}\left(E\left([\eta,\eta]_1^{qk/2}\right)\right)^{1/q}
\end{aligned}$$

where the second inequality follows from the fact that $[\eta,\eta]_s$ is increasing and the final inequality follows for our choices of p and q by Hölder's inequality. (The notation $[\cdot,\cdot]$ denotes the quadratic variation process.) Since $k \ge 1$, $q > 1$ the Burkholder-Davis-Gundy inequalities state that there exists $c \in \mathbb{R}$ such that

$$\begin{aligned}
E\left([\eta,\eta]_1^{qk/2}\right) &\le \frac{1}{c}E\left(\sup_{0\le t\le 1}|\eta_t|^{qk}\right)\\
&\le \frac{8}{c}E\left(|\eta_1|^{qk}\right)\\
&< \infty
\end{aligned}$$

where the second inequality holds by a formulation of Doob's inequality as expressed in Sato [62], p.167 and the final inequality follows from our moment assumption.

Thus it suffices to prove $E\left(\sup_{0\le t\le 1} e^{-pk\xi_t}\right)<\infty$. Setting $Y_t=e^{-\xi_t}/E(e^{-\xi_t})$, a non-negative martingale, it follows by Doob's maximal inequality, as expressed in Shiryaev [1], p.765, that

$$E\left(\sup_{0\le t\le 1}\frac{e^{-pk\xi_t}}{(E(e^{-\xi_t}))^{pk}}\right)\le\left(\frac{pk}{pk-1}\right)^{pk}\frac{E\left(e^{-pk\xi_1}\right)}{(E(e^{-\xi_1}))^{pk}}$$

which is finite by our moment assumption. By Sato [62], p.165, we know that $\left(E(e^{-\xi_t})\right)^{pk}=\left(E(e^{-\xi_1})\right)^{pkt}$. Letting $c:=\left(E(e^{-\xi_1})\right)^{pk}\in(0,\infty)$ it is clear that

$$E\left(\sup_{0\le t\le 1} e^{-pk\xi_t}\right)\le\max\{1,c\}E\left(\sup_{0\le t\le 1}\frac{e^{-pk\xi_t}}{c^t}\right)$$

which is finite by the above inequality. Hence the proposition is proved for the case in which $E(\eta_1)=0$. Now we drop this restriction, noting that $E\left(|\eta_1|\right)<\infty$ by our moment assumptions. Thus, we have

$$\begin{aligned}
E\left(\sup_{0\le t\le 1}\left|\int_0^t e^{-\xi_{s-}}\mathrm{d}\eta_s\right|^k\right) &= E\left(\sup_{0\le t\le 1}\left|\int_0^t e^{-\xi_{s-}}\mathrm{d}(\eta_s-sE\eta_1+sE\eta_1)\right|^k\right)\\
&\le E\Bigg(\Bigg(\sup_{0\le t\le 1}\left|\int_0^t e^{-\xi_{s-}}\mathrm{d}(\eta_s-sE\eta_1)\right|+\\
&\qquad |E\eta_1|\sup_{0\le t\le 1}\left|\int_0^t e^{-\xi_{s-}}\mathrm{d}s\right|\Bigg)^k\Bigg).
\end{aligned}$$

Since the integrator is a Lévy process with zero mean, we know

$$E\left(\sup_{0\le t\le 1}\left|\int_0^t e^{-\xi_{s-}}\mathrm{d}(\eta_s-sE\eta_1)\right|^k\right)<\infty.$$

Also note that

$$\begin{aligned}
E\left(\sup_{0\le t\le 1}\left|\int_0^t e^{-\xi_{s-}}\mathrm{d}s\right|^k\right) &\le E\left(\sup_{0\le t\le 1}\left|\int_0^t \sup_{0\le v\le 1} e^{-\xi_v}\mathrm{d}s\right|^k\right)\\
&= E\left(\sup_{0\le v\le 1} e^{-\xi_v}\sup_{0\le t\le 1}\left|\int_0^t\mathrm{d}s\right|^k\right)\\
&= E\left(\sup_{0\le v\le 1} e^{-k\xi_v}\right)
\end{aligned}$$

which is finite since we showed above that $E\left(\sup_{0\le t\le 1} e^{-pk\xi_t}\right)<\infty$ for $p>1$. Now the final result holds by Minkowski's inequality. □

Theorem 3.12. Assume $Z_t \to Z_\infty$ a.s. as $t \to \infty$, where Z_∞ is a finite random variable. Suppose that for all $c \in \mathbb{R}$, equation (1.30) does not hold. This implies that Z_∞ is continuous. As noted in Proposition 1.17, a necessary condition for the convergence of Z_t, is $\lim_{t\to\infty} \xi_t = \infty$ a.s., which implies that $e^{\xi_t} \to \infty$ a.s. Since Z_∞ is finite a.s., and $e^{\xi_t} \to \infty$ a.s., it is clear from the definition $V_t := e^{\xi_t}(z + Z_t)$, that

$$P(\lim_{t\to\infty} V_t = \infty | V_0 = z) = P(Z_\infty > -z). \tag{3.7}$$

Now let $a \leq \sup U$. By definition of U, $P(\lim_{t\to\infty} V_t = \infty | V_0 = a) = 0$ which implies, by equation (3.7), that $Z_\infty < -a$ a.s., as required.

Conversely, let $a > \sup U$. We prove $P(Z_\infty > -a) > 0$. Since we have assumed that $|Z_\infty| < \infty$ a.s., we can choose $x > a$ such that $P(Z_\infty > -x) > 0$. Note that $\Upsilon(a) = \infty$ and so there exists a fixed time $T > 0$ such that $P(V_T \geq x | V_0 = a) > 0$.

Hence, using (3.7), the law of conditional probability and the Markov property,

$$\begin{aligned}
P(Z_\infty > -a) &= P(\lim_{t\to\infty} V_t = \infty | V_0 = a) \\
&\geq P(\lim_{t\to\infty} V_t = \infty | V_T \geq x) P(V_T \geq x | V_0 = a) \\
&= P(\lim_{t\to\infty} V_t = \infty | V_0 \geq x) P(V_T \geq x | V_0 = a) \\
&\geq P(\lim_{t\to\infty} V_t = \infty | V_0 = x) P(V_T \geq x | V_0 = a)
\end{aligned}$$

which is greater than zero by (3.7) and the choice of x and T. Thus we have proved

$$a \leq \sup U \text{ iff } Z_\infty < -a \text{ a.s.} \tag{3.8}$$

Now we prove $-\sup U = m$ where $m := \inf\{u \in \mathbb{R} : Z_\infty < u \text{ a.s.}\}$. By equation (3.8), $Z_\infty < -\sup U$ and thus $-\sup U \geq m$. By assumption, Z_∞ has no atoms and so $Z_\infty < m$ a.s. Thus, equation (3.8) implies that $-m \leq \sup U$. The proofs of the statements for L are symmetric.

Now we deal with the degenerate case. Assume that there exists $c \in \mathbb{R}$ such that equation (1.30) holds, and assume that $Z_t \to Z_\infty$ a.s. as $t \to \infty$. By equation (1.30) it is immediate that $Z_\infty = -c$ a.s. Further, since ξ drifts to ∞ a.s. as $t \to \infty$, Proposition 3.8 implies that $L = U = \{c\}$, or $U = (-\infty, c]$ and $L = [c, \infty)$. In both of these cases, $\inf L = \sup U = c$. □

Theorem 3.13. (1) Assume $L \cap U = \emptyset$, $\sup U \geq 0$ and $L \cap [0, \sup U] = \emptyset$, and let $0 \leq u \leq \sup U$. We want to prove that $\psi(u) = 1$. Note that there exists $z \geq u$ such that $z \in U$, and so $\Upsilon(z) = z$. Since $\psi(u) \geq \psi(z)$, it suffices to prove that $\psi(z) = 1$.

Since $L \cap [0, \sup U] = \emptyset$, we know $\delta(z) < 0$, which implies that $P_z(\inf_{t>0} V_t < 0) > 0$. Thus, there exists a fixed time $T \in \mathbb{R}$ such that $P_z(\inf_{0<t\leq T} V_t < 0) := m > 0$. Let $n \in \mathbb{N}$ and let A be the distribution of V_{nT} conditional on both $V_0 = z$ and $\inf_{0<t\leq nT} V_t \geq 0$. Since $\Upsilon(z) = z$ we know $A \leq z$ a.s. Now

$$\begin{aligned} P_z\left(\inf_{nT<t\leq(n+1)T} V_t < 0 \middle| \inf_{0<t\leq nT} V_t \geq 0\right) &= P_A\left(\inf_{0<t\leq T} V_t < 0\right) \\ &\geq P_z\left(\inf_{0<t\leq T} V_t < 0\right) \\ &= m, \end{aligned}$$

where the first equality follows from the Markov property and the inequality follows from the fact that $A \leq z$ and V_t is increasing in z. Define

$$P^n := P_z\left(\inf_{0<t\leq nT} V_t < 0\right)$$

for all $n \in \mathbb{N}$. By the law of total probability

$$P^{n+1} = P^n + P_z\left(\inf_{nT<t\leq(n+1)T} V_t < 0 \middle| \inf_{0<t\leq nT} V_t \geq 0\right)(1 - P^n)$$

and so $P^{n+1} \geq P^n + (1 - P^n)m$ where $P^1 = m \in (0, 1)$. This implies that $P^n \geq 1 - (1 - m)^n$ which implies that $\lim_{n\to\infty} P^n = 1$, and hence $P_z(\inf_{0<t} V_t < 0) = 1$ by the continuity property of measures.

(2) Assume $L \cap U = \emptyset$, $\sup L \geq 0$, and $U \cap [0, \sup L] = \emptyset$. We let $z \geq 0$ and prove that $\psi(z) < 1$. If $z \geq \inf L$ then $\psi(z) = 0$ by definition. Thus, it suffices to assume $0 \leq z < \inf L$.

Suppose that $\psi(z) = 1$. By assumption, $\Upsilon(z) > \inf L$ and so, by definition, $P(C) > 0$ where $C := \{\sup_{t\geq 0} V_t \geq \inf L\}$. By definition of L, $\lim_{t\to\infty} V_t \geq \inf L$ a.s. for all $\omega \in C$. Let $T_1 := \inf\{t > 0 : V_t < 0\}$ and $T_n := \inf\{t > T_{n-1} : V_t < V_{T_{n-1}}\}$ for integers $n > 1$. By assumption, $\psi(z) = 1$ and so T_1 is finite a.s. Further, the strong Markov property of V implies that $\{T_n\}$ is a sequence of stopping times increasing towards infinity as $n \to \infty$, and each T_i is a.s. finite. In particular, each T_i is a.s. finite on C. However $V_{T_n} < 0$ a.s. which contradicts the fact that $\lim_{t\to\infty} V_t > \inf L$ a.s. on C. Hence $\psi(z) < 1$. The proof of the case in which $U \cap [0, \sup L] \neq \emptyset$ is almost identical, and we omit.

□

Theorem 3.14. Assume $L \cap U = \emptyset$, $\lim_{t\to\infty} \xi_t = -\infty$ a.s. and $I_{-\xi,K^{\xi,\eta}} < \infty$. Suppose that $L \cap [0, \infty) \neq \emptyset$. Since ξ drifts to $-\infty$ a.s., Propositions 3.8 and

3.9 imply that one of conditions (a) or (b) must hold. Further, it follows from statement (2) of Proposition 3.13 and the definition of L, that $0 < \psi(z) < 1$ for all $0 \leq z < \inf L$, and $\psi(z) = 0$ for all $z \geq \inf L$.

Now suppose that $L \cap [0, \infty) = \emptyset$. We let $z \geq 0$ and prove that $\psi(z) = 1$. Let N be a Poisson process with parameter λ, let D_i be an iid sequence of 1-dimensional exponential random variables and let $C_i = 1$ for all i. Suppose that N, D_i and (ξ, η) are mutually independent and define the compound Poisson process $(X_t, Y_t) := \sum_{i=1}^{N_t}(C_i, D_i)$. Now define a new Lévy process $(\xi_t^\diamond, \eta_t^\diamond) := (\xi_t, \eta_t) + (X_t, Y_t)$, and denote the associated GOU by $V^\diamond$. For $V^\diamond$, denote the upper and lower bound functions, the sets of upper and lower bounds, and the ruin probability function by $\Upsilon^\diamond$, $\delta^\diamond$, $U^\diamond$, $L^\diamond$ and $\psi^\diamond$ respectively.

Define $T_z := \inf\{t > 0 : V_t < 0 | V_0 = z\}$. Since $\sup L < 0$, we know $\delta(z) < 0$ and hence T_z is finite a.s. Note that $V_0 = V_0^\diamond = z$. Also, whenever $V_{t-} \geq 0$, every jump $\Delta(X, Y)_t$ causes a non-negative jump ΔV_t. Hence $V_t \leq V_t^\diamond$ a.s. on $t \leq T_z$. This implies that $\psi(z) \geq \psi^\diamond(z)$. Thus it suffices to show that $\psi^\diamond(z) = 1$. To do this, we first need to prove that $V^\diamond$ is strictly stationary.

We show that $\lambda > 0$ can be chosen small enough such that $\lim_{t\to\infty} \xi_t^\diamond = -\infty$. Since $\lim_{t\to\infty} \xi_t = -\infty$, either $E(\xi_1) \in [-\infty, 0)$ or $E(\xi_1)$ does not exist. If $E(\xi_1) \in [-\infty, 0)$ then $E(\xi_1^\diamond) = E(\xi_1) + \lambda$ and so we can choose λ small enough such that $E(\xi_1^\diamond) < 0$, which implies that $\lim_{t\to\infty} \xi_t^\diamond = -\infty$. If $E(\xi_1)$ does not exist then we know $E(\xi_1^\diamond)$ does not exist. We show that $\lim_{t\to\infty} \xi_t^\diamond = -\infty$ holds for any $\lambda > 0$. Note that $\xi^\diamond = \xi + N$ and , as noted in Section 1.2, $J_\xi^+ < \infty$ since $E(\xi_1)$ does not exist and $\lim_{t\to\infty} \xi_t = -\infty$. Also note that $\overline{\Pi}_{\xi^\diamond}^- = \overline{\Pi}_\xi^-$ and so $A_{\xi^\diamond}^- = A_\xi^-$. Since ξ and N are independent we have $\overline{\Pi}_{\xi^\diamond}^+ = \overline{\Pi}_\xi^+ + \overline{\Pi}_N^+$. Further $\overline{\Pi}_N^+(x) = 0$ for all $x \geq 1$. Hence $J_{\xi^\diamond}^+ = J_\xi^+$ and so is finite. As noted in Section 1.2, this implies that $\lim_{t\to\infty} \xi_t^\diamond = -\infty$.

We now show that $(\xi^\diamond, \eta^\diamond)$ satisfies $I_{-\xi^\diamond, K^{\xi^\diamond,\eta^\diamond}} < \infty$. Since (ξ, η) and (X, Y) are independent, it is clear from the definitions in Section 1.4 that $K_t^{\xi^\diamond,\eta^\diamond} = K_t^{\xi,\eta} + K_t^{X,Y}$ and $\overline{\Pi}_{K^{\xi^\diamond,\eta^\diamond}}(y) = \overline{\Pi}_{K^{\xi,\eta}}(y) + \overline{\Pi}_{K^{X,Y}}(y)$. And, as above, $A_{-\xi^\diamond}^+ = A_{-\xi}^+$. Hence

$$I_{-\xi', K^{\xi^\diamond,\eta^\diamond}} = I_{-\xi, K^{\xi,\eta}} + \int_{(e,\infty)} \left(\frac{\ln(y)}{A_{-\xi}^+(\ln(y))} \right) |\overline{\Pi}_{K^{X,Y}}(dy)|.$$

By the choice of (X, Y) it is clear that $K_1^{X,Y}$ has a finite expected value which implies that $\int_{(e,\infty)} y|\overline{\Pi}_{K^{X,Y}}(dy)| < \infty$. Hence $I_{-\xi', K^{\xi^\diamond,\eta^\diamond}} < \infty$. Thus $V^\diamond$ is strictly stationary.

For a Lebesgue set Λ define $T_\Lambda^\diamond := \inf\{t > 0 : V_t^\diamond \in \Lambda\}$. Note that $\theta_1'^\diamond = -\infty$ and hence Proposition 3.20 implies that $\Upsilon^\diamond(u) = \infty$ for all $u \in \mathbb{R}$, or equivalently,

$U^\diamond = \emptyset$. Also, $\theta_1^\diamond = 0$, and so Proposition 3.16 implies that $L^\diamond \cap (-\infty, 0) = \emptyset$, whilst the fact that $L \cap (0, \infty) = \emptyset$ clearly implies that $L' \cap (0, \infty) = \emptyset$.

These facts imply that, for all a and u in $\mathbb{R}$, $P\left(T^\diamond_{(-\infty,a]} < \infty | V_0^\diamond = u\right) > 0$ and $P\left(T^\diamond_{[a,\infty]} < \infty | V_0^\diamond = u\right) > 0$. Since D is an exponential random variable, it is clear that $V_t^\diamond$ has a continuous density with respect to Lebesgue measure. Hence $P\left(T^\diamond_\Lambda < \infty\right) > 0$ for any set Λ with positive Lebesgue measure. This result, and the fact that $V^\diamond$ is strictly stationary, allows us to mimic the argument of Theorem 3.1 (a) in Paulsen [56]. Let S be an independent standard exponential variable and define the resolvent kernel

$$K(z, \Lambda) := \int_0^\infty P_z(V_t^\diamond \in \Lambda) e^{-t} \mathrm{d}t = P_z(V_S^\diamond \in \Lambda).$$

Proposition 2.1 of [49] implies that $V^\diamond$ is ϕ-irreducible for the measure $\phi = \lambda K$. Using the language of [49] p.495 and 496, it is clear that K has a continuous nontrivial component for all z and hence is a T-process. Since $V^\diamond$ is strictly stationary it is clear that $V^\diamond$ is non-evanescent, as defined in [49] p.494. Thus Theorem 3.2 of [49] p.494 implies that $V^\diamond$ is Harris recurrent, as defined in [49] p490, which clearly implies that $\psi^\diamond(z) = 1$ as required.

(2) Assume that $L \cap U = \emptyset$, $E(\xi_1) = 0$, there exists $\delta > 0$ such that $E\left(|\xi_1|^{2+\delta}\right) < \infty$ and there exist $p, q > 1$ with $1/p + 1/q = 1$ such that $E\left(e^{-p\xi_1}\right) < \infty$ and $E\left(|\eta_1|^q\right) < \infty$.

Suppose that $L \cap [0, \infty) \neq \emptyset$. Since ξ oscillates a.s., Proposition 3.9 implies that $L = [a, b]$ and $U = \emptyset$ where $-\infty < a \leq b < \infty$ and $b \geq 0$. Hence, it follows from statement (2) of Proposition 3.13 and the definition of L, that $0 < \psi(z) < 1$ for all $0 < z < a$ and $\psi(z) = 0$ for all $z \geq a$.

Now suppose that $L \cap [0, \infty) = \emptyset$. We let $z \geq 0$ and prove that $\psi(z) = 1$. We know that $P\left(\inf_{t>0} V_t < 0 | V_0 = z\right) > 0$. However, it is possible that for some $z > 0$, $P(V_1 < 0 | V_0 = z) = 0$. For example, this would happen if (ξ, η) has no Brownian component and $\sup L^* > 0$. Let $0 = T_0 < T_1 < T_2 < \ldots$ be random times such that $T_i - T_{i-1}$ are iid with exponential distribution and parameter λ. Since T_1 has infinite support it is clear that $\sup L < 0$ implies $P\left(V_{T_1} < 0 | V_0 = z\right) > 0$ for all $z \geq 0$. Equation (1.1) implies that a.s.

$$V_{T_n} = e^{\xi_{T_n} - \xi_{T_{n-1}}} \left(e^{\xi_{T_{n-1}}} \left(z + \int_0^{T_{n-1}} e^{-\xi_{s-}} \mathrm{d}\eta_s \right) \right) + e^{\xi_{T_n}} \int_{T_{n-1}+}^{T_n} e^{-\xi_{s-}} \mathrm{d}\eta_s.$$

Thus, if we define $A_n := e^{\xi_{T_n} - \xi_{T_{n-1}}}$, $B_n := e^{\xi_{T_n}} \int_{T_{n-1}+}^{T_n} e^{-\xi_{s-}} \mathrm{d}\eta_s$ and the stochastic difference equation $Y_n := A_n Y_{n-1} + B_n$ with $Y_0 := V_0 = z$ then $Y_n = V_{T_n}$ a.s. for all $n \in \mathbb{N}$. Note that the term $e^{\xi_{T_n}}$ in B_n cannot be brought under the integral sign because it is not predictable. Since a Lévy process has independent increments it is clear that (A_n, B_n) is an independent sequence. Now,

$$\begin{aligned}
(A_2, B_2) &= \left(e^{\xi_{T_2}-\xi_{T_1}}, e^{\xi_{T_2}-\xi_{T_1}} e^{\xi_{T_1}} \int_{T_1+}^{T_2} e^{-\xi_{s-}} \mathrm{d}\eta_s\right) \\
&= \left(e^{\xi_{T_2}-\xi_{T_1}}, e^{\xi_{T_2}-\xi_{T_1}} \int_{T_1+}^{T_2} e^{-\left(\xi_{s-}-\xi_{T_1}\right)} \mathrm{d}\eta_s\right) \\
&= \left(e^{\xi_{T_2}-\xi_{T_1}}, e^{\xi_{T_2}-\xi_{T_1}} \int_{T_1+}^{T_2} e^{-\left(\xi_{s-}-\xi_{T_1}\right)} \mathrm{d}(\eta_s - \eta_{T_1})\right) \\
&=_D \left(e^{\xi_{T_1}}, e^{\xi_{T_1}} \int_{T_1+}^{T_2} e^{-\xi_{(s-T_1)-}} \mathrm{d}\eta_{s-T_1}\right) \\
&= \left(e^{\xi_{T_1}}, e^{\xi_{T_1}} \int_{0}^{T_1} e^{-\xi_{s-}} \mathrm{d}\eta_s\right) = (A_1, B_1),
\end{aligned}$$

where the second equality holds because $e^{\xi_{T_1}}$ is predictable with respect to the integral, the fourth equality holds because a Lévy process has identically distributed increments and the fifth equality is obtained using a change of variables. The argument for general n is identical, and thus (A_n, B_n) is an iid sequence.

Now Proposition 1.1 and Corollary 4.2 of [2] state that if $P(A_1 z + B_1 = z) < 1$ for all $z \in \mathbb{R}$, $E(\ln A_1) = 0$, $A_1 \not\equiv 1$ and there exists $\delta > 0$ such that

$$E\left(\left(|\ln A_1| + \ln^+ |B_1|\right)^{2+\delta}\right) < \infty \tag{3.9}$$

then the discrete stochastic process W has an invariant unbounded Radon measure μ unique up to a constant factor such that the sample paths W_n, with $W_0 = z$, visit every open set of positive μ-measure infinitely often with probability 1, for every $z \in \mathbb{R}$. The first of these conditions follows from our assumption that $L \cap U = \emptyset$, using Proposition 3.8. The second and third conditions follow respectively from our assumptions that $E(\xi_1) = 0$, and ξ_1 is not identically zero. We will show later that our moment conditions on ξ and η ensure equation (3.9) holds. Note that the Babillot result implies that $\psi(z) = 1$ if we can show $\mu((-\infty, 0)) > 0$. However by the definition of an invariant measure,

$$\begin{aligned}
\mu((-\infty, 0)) &= \int_{z\in\mathbb{R}} P(A_1 z + B_1 < 0)\mu(\mathrm{d}z) \\
&\geq \int_{z\in\mathbb{R}} P(V_{T_1} < 0 | V_0 = z)\mu(\mathrm{d}z).
\end{aligned}$$

Thus if $\mu([0,\infty)) > 0$ then $\mu((-\infty, 0)) > 0$ since $P(V_{T_1} < 0 | V_0 = z) > 0$ for all $z \geq 0$. And if $\mu([0,\infty)) = 0$ then $\mu((-\infty, 0)) > 0$ since $\mu(\mathbb{R}) > 0$. Thus we are done if we can prove equation (3.9).

To do this, it suffices to assume $T_1 = 1$ and $(A_1, B_1) := \left(e^{\xi_1}, e^{\xi_1}\int_0^1 e^{-\xi_{s-}}\mathrm{d}\eta_s\right)$ since we can choose the parameter λ of the increments to be arbitrarily small. Note that if $x, y > 0$ and $\alpha > 0$ then there exists $c_1 > 0$ such that

$$(x+y)^\alpha \le c_1\left(x^\alpha + y^\alpha\right). \tag{3.10}$$

Hence, to prove (3.9), it suffices to prove that $E\left(|\xi_1|^{2+\delta}\right) < \infty$ and

$$E\left(\left(\ln^+\left|e^{\xi_1}\int_0^1 e^{-\xi_{s-}}\mathrm{d}\eta_s\right|\right)^{2+\delta}\right) < \infty. \tag{3.11}$$

Note that the former inequality is assumed as a condition. If $x, y > 0$ then $\ln^+(xy) \le \ln^+(x) + \ln^+(y)$, and hence, using (3.10), equation (3.11) holds if

$$E\left(\left(\ln^+\left|\int_0^1 e^{-\xi_{s-}}\mathrm{d}\eta_s\right|\right)^{2+\delta}\right) < \infty. \tag{3.12}$$

Note that whenever $0 < \delta \le 1$ and $x > 0$, then there exists $c_2 > 0$ such that $(\ln^+ x)^{2+\delta} \le c_2 x^\delta$. Without loss of generality, we can assume that $0 < \delta \le 1$, and hence (3.12) holds if $E\left(\left|\int_0^1 e^{-\xi_{s-}}\mathrm{d}\eta_s\right|^\delta\right) < \infty$. However, with our assumptions on p and q, this follows from Lemma 3.24.

(3) Assume that $\lim_{t\to\infty}\xi_t = \infty$ a.s. and $I_{\xi,\eta} < \infty$. Suppose that $-\infty \le \sup U < z$. Assume, for the sake of contradiction, that $\psi(z) = 1$. Theorem 3.12 implies that $P(C) > 0$ where $C := \{Z_\infty > -z\}$. Since $\lim_{t\to\infty}\xi_t = \infty$, we know that $\lim_{t\to\infty} V_t = \infty$ a.s. on C. Now, the same strong Markov property argument used in the proof of statement (2) of Theorem 3.13, gives a contradiction. Hence $\psi(z) < 1$.

Now suppose $U \cap [0,\infty) \ne \emptyset$. Since ξ drifts to ∞ a.s., Theorem 3.9 implies that either $U = [a, b]$ and $L = \emptyset$ where $-\infty \le z \le b < \infty$ and $b \ge 0$, or $U = (-\infty, a]$ and $L = [b, \infty)$ for some $0 \le a < b < \infty$. In both of these cases, statement (1) of Theorem 3.13 implies that $\psi(z) = 1$ for all $z \le \sup U$. Using the definition of L, and the above result, it is clear that $0 < \psi(z) < 1$ for all $\sup U < z < \inf L$ and $\psi(z) = 0$ for all $z \ge \sup L$. □

Proposition 3.5. Assume that $V_t = e^{\xi_t}(z-c)+c$. By definition of L, if $c \ge 0$ then $\psi(z) = 0$ for all $z \ge c$.

Let $0 \le z < c$. If ξ drifts to $-\infty$ a.s. then $\lim_{t\to\infty} V_t = c$ a.s. Thus, the strong Markov property of V implies that $\psi(z) < 1$, using a proof similar to that used for statement (2) of Theorem 3.13. If ξ oscillates a.s. then $-\infty = \liminf_{t\to\infty} V_t <$

$\limsup_{t\to\infty} V_t = c$, and so $\psi(z) = 1$. If ξ drifts to ∞ a.s. then $\lim_{t\to\infty} V_t = -\infty$ a.s. which implies $\psi(z) = 1$.

Let $c < 0 \le z$. If ξ drifts to $-\infty$ a.s. then $\lim_{t\to\infty} V_t = c$ a.s. and so $\psi(z) = 1$. If ξ oscillates a.s. then $c = \liminf_{t\to\infty} V_t < \limsup_{t\to\infty} V_t = \infty$, and so $\psi(z) = 1$. If ξ drifts to ∞ a.s. then $\lim_{t\to\infty} V_t = \infty$ a.s. which implies $\psi(z) < 1$, using a strong Markov property argument. □

Theorem 3.1. Suppose that for all $c \in \mathbb{R}$ the degenerate case (1.30) does not hold. Then, by Proposition 3.8, $L \cap U = \emptyset$. It follows immediately from Theorem 3.14 that $0 < \psi(z) < 1$ iff $0 \le z < m < \infty$ whenever the assumptions for statement (1), or statement (2), of Theorem 3.1 are satisfied. Now suppose that there exists $c \in \mathbb{R}$ such that equation (1.30) holds. Then it follows immediately from Proposition 3.5 that $0 < \psi(z) < 1$ iff $0 \le z < m < \infty$ whenever the assumptions for statement (1), or statement (2), of Theorem 3.1 are satisfied. In both these situations, $m = c$. □

Theorem 3.3. Assume that $\lim_{t\to\infty} \xi_t = \infty$ a.s. and $I_{\xi,\eta} < \infty$. Assume that for all $c \in \mathbb{R}$ the degenerate equation (1.30) does not hold, or equivalently, $L \cap U = \emptyset$.

Theorem 3.3 claims that $\psi(0) = 1$ iff $-\eta$ is a subordinator, or there exists $z > 0$ such that $\psi(z) = 1$. This claim follows easily by combining two known results: $\psi(z) = 1$ iff $\sup U \ge 0$ and $z < \sup U$, which is implied by statement (3) of Theorem 3.13; secondly, $0 \in U$ iff $-\eta$ is a subordinator, which is stated in Proposition 3.20.

Theorem 3.3 also states conditions on the characteristic triplet of (ξ, η) and claims these are equivalent to the fact that there exists $z > 0$ such that $\psi(z) = 1$. However, using statement (3) of Theorem 3.13, we know there exists $z > 0$ such that $\psi(z) = 1$ iff $\sup U > 0$. And Proposition 3.20 gives iff conditions on the characteristic triplet of (ξ, η) for the case $\sup U > 0$ to occur. These conditions are precisely the conditions stated in Theorem 3.3.

Finally, statements (1) and (2) of Theorem 3.3 contain values for $\sup\{z \ge 0 : \psi(z) = 1\}$. However, these are an immediate consequence of the unstated parallel version of Proposition 3.17 which gives exact values for the endpoints of U.

Now, assume that there exists $c \in \mathbb{R}$ such that the degenerate equation (1.30) holds, and $L = U = \{c\}$. Since ξ drifts to ∞ a.s., Proposition 3.8 implies that $\sup U = c$. Thus, Proposition 3.5 implies that $\psi(z) = 1$ iff $\sup U \ge 0$ and $z < \sup U$. Theorem 3.3 is proved for the degenerate case by combining this statement with Proposition 3.20 and the parallel version of Proposition 3.17, in an identical manner to the above. The only difference is that the set $\{z \ge 0 : \psi(z) = 1\}$ does not contain its supremum in the degenerate case, since $\sup\{z \geq 0 : \psi(z) = 1\} = U = L$, and is an absorbing point. $\square$

3.4.1 Examples

Propositions 3.8, 3.9 and 3.11 make claims that Lévy processes (ξ, η) exist which satisfy particular combinations of L and U, and particular asymptotic behaviour for ξ. In this subsection we present examples which prove these claims. We use the simplest Lévy processes possible. Thus, the Lévy measures will always be finite activity, namely $\Pi_{\xi,\eta}(\mathbb{R}^2) < \infty$. Hence, we can always write (ξ, η) in the form $(\xi, \eta)_t = (d_\xi, d_\eta)t + (B_{\xi,t}, B_{\eta,t}) + \sum_{i=1}^{N_t} Y_i$ where $(B_{\xi,t}, B_{\eta,t})$ is a two-dimensional Brownian motion with covariance matrix $\Sigma_{\xi,\eta}$, N is a Poisson process with parameter Λ and $\{Y_i\}_{i=1}^{\infty}$ is an iid sequence of two dimensional random variables with common distribution Y.

Examples with Brownian component The first example is of a Lévy process (ξ, η) for which $L = \{a\}$, $U = \emptyset$. The second example is of a Lévy process for which $L = U = \{a\}$. For both examples we show how variables can be chosen so that ξ drifts to ∞ a.s., ξ drifts to $-\infty$ a.s. or ξ oscillates a.s.

Example 3.26. Let $(\xi, \eta)_t := (d_\xi, 2)t + (B_t, B_t) + \sum_{i=1}^{N_t} Y_i$ where B is a one-dimensional Brownian motion with variance 1, and $P(Y = (10, 10)) = 1/2$ and $P(Y = (-10, 10)) = 1/2$. The covariance matrix equation (3.1) is satisfied for $u = -1$. Condition (ii) of Proposition 3.16 is satisfied for $u = -1$, whilst condition (ii) of Proposition 3.20 is not satisfied. By equation (3.4), $g(-1) = 3/2 - d_\xi$, and so choosing $d_\xi \leq 3/2$ implies that $L = \{-1\}$ and $U = \emptyset$. However, $E(\xi_1) = d_\xi$, so if $0 < d_\xi < 3/2$ then ξ drifts to ∞ a.s., if $d_\xi < 0$ then ξ drifts to $-\infty$ a.s., and if $d_\xi = 0$ then ξ oscillates a.s.

Example 3.27. Let $(\xi, \eta)_t := (d_\xi, d_\eta)t + (B_t, -B_t)$. Equation (3.1) is satisfied for $u = 1$, whilst condition (ii) of Proposition 3.16 and condition (ii) of Proposition 3.20 are satisfied trivially. Equation (3.4) implies $g(1) = d_\eta + d_\xi - 1/2$. Thus, choosing $d_\xi = 1/2 - d_\eta$ implies that $L = U = \{1\}$. Note $E(\xi_1) = d_\xi$, so if $d_\eta < 1/2$ then ξ drifts to ∞ a.s., if $d_\eta > 1/2$ then ξ drifts to $-\infty$ a.s., and if $d_\eta = 1/2$ then ξ oscillates a.s.

Examples with no Brownian component We now present seven examples of Lévy processes (ξ, η) with no Brownian component. In Example 3.28, $L = U = \{a\}$ and we indicate how the parameters can be changed in order to obtain each of the three asymptotic behaviours for ξ. In Examples 3.29 and 3.30, $L = \emptyset$, whilst U may be of form $\emptyset$, $\{a\}$ or $[a,b]$ for $-\infty < a < b < \infty$. We indicate how parameters can be changed in order to obtain these different sets, and for each set, to obtain the three possible asymptotic behaviours for ξ. In Example 3.31, $L = \emptyset$ whilst U is of form $[b,\infty)$ for $b \in \mathbb{R}$. In Example 3.32, $L = (-\infty, a]$ and $U = [b,\infty)$ for $-\infty < a < b < \infty$. For both these examples we show that ξ drifts to $-\infty$ a.s. In Example 3.33, $L = \emptyset$ whilst U is of form $(-\infty, a]$ for $a \in \mathbb{R}$. In Example 3.34, $U = (-\infty, a]$ and $L = [b,\infty)$ for $-\infty < a < b < \infty$. For both these examples we show that ξ drifts to ∞ a.s.

Example 3.28. Let $(\xi,\eta)_t := (d_\xi, d_\eta)t + \sum_{i=1}^{N_t} Y_i$ where $P\left(Y = (3, 2e^{-3} - 2)\right) = 1/2$ and $P\left(Y = (-3, 2e^{3} - 2)\right) = 1/2$. Then $\theta_2 = \theta_2' = \theta_4 = \theta_4' = 2$ and $L^* = U^* = \{2\}$. Note that $g(u) = d_\eta + u d_\xi$, so choosing $d_\eta = -2d_\xi$ implies that $g(2) = 0$ and hence $L = U = \{2\}$. Since $E(\xi_1) = d_\xi$, choosing $d_\xi > 0$, $d_\xi < 0$, and $d_\xi = 0$, implies that ξ drifts to ∞ a.s., ξ drifts to $-\infty$ a.s. and ξ oscillates a.s., respectively.

Example 3.29. Let $(\xi,\eta)_t := (d_\xi, d_\eta)t + \sum_{i=1}^{N_t} Y_i$ where $P(Y = (4,-2)) = 1/3$ and $P(Y = (-2,-3)) = 1/3$ and $P(Y = (-2,1)) = 1/3$. Then $L = \emptyset$ since $\Pi_{\xi,\eta}(A_2)$ and $\Pi_{\xi,\eta}(A_3)$ are both non-zero, whilst $U^* = [\theta_4', \theta_2'] = [\frac{1}{e^2-1}, \frac{-2}{e^{-4}-1}] \cong [0.2, 2]$. Now $U = \{u \in U^* : g(u) \leq 0\}$ and g simplifies to $g(u) = d_\eta + u d_\xi$. Note that $E(\xi_1) = d_\xi$.

Choosing $d_\xi = 0$ and $d_\eta > 0$ implies that $U = \emptyset$ and ξ oscillates a.s. Choosing $d_\xi > 0$ and $d_\eta > -\theta_4' d_\xi$ implies that $U = \emptyset$ and ξ drifts to ∞ a.s. Choosing $d_\xi < 0$ and $d_\eta > -\theta_2' d_\xi$ implies that $U = \emptyset$ and ξ drifts to $-\infty$ a.s.

Choosing $d_\xi = 0$ and $d_\eta < 0$ implies that $U = U^* \cong [0.2, 2]$ and ξ oscillates a.s. Choosing $d_\xi > 0$ and $d_\eta < -\theta_2' d_\xi$ implies that $U = U^* \cong [0.2, 2]$ and ξ drifts to ∞ a.s. Choosing $d_\xi < 0$ and $d_\eta < -\theta_4' d_\xi$ implies that $U = U^* \cong [0.2, 2]$ and ξ drifts to $-\infty$ a.s.

Choosing $d_\xi > 0$ and $d_\eta = -\theta_4' d_\xi$ implies that $U = \{\theta_4'\} \cong \{0.2\}$ and ξ drifts to ∞ a.s. Choosing $d_\xi < 0$ and $d_\eta = -\theta_2' d_\xi$ implies that $U = \{\theta_2'\} \cong \{2\}$ and ξ drifts to $-\infty$ a.s.

Note that for Example 3.33, no adjustment of d_ξ and d_η can result in $U = \{a\}$ with ξ oscillating a.s. We now present a different example with this behaviour.

Example 3.30. Let $(\xi,\eta)_t := (0,-2)t + \sum_{i=1}^{N_t} Y_i$ where $P(Y = (2, e^{-2} - 1)) = 1/3$ and $P(Y = (-1, e - 1)) = 1/3$ and $P(Y = (-1,-2)) = 1/3$. Then $L = \emptyset$, $\theta_2 = \theta_2' = \theta_4 = \theta_4' = 1$, and $U^* = \{1\}$. Since g simplifies to $g(u) = -2$ for all $u \in \mathbb{R}$ we obtain $U = \{1\}$. Since $E(\xi_1) = 0$, ξ oscillates a.s.

Example 3.31. Let $(\xi, \eta)_t := (0, -2)t + \sum_{i=1}^{N_t} Y_i$ where $P(Y = (-1, 2)) = 1/3$ and $P(Y = (-2, -3)) = 1/3$ and $P(Y = (0, -5)) = 1/3$. Then $L^* = \emptyset$ whilst $U^* = [\theta'_4, \theta'_2] = [\frac{2}{e-1}, \infty) \cong [1.2, \infty)$. Since $g(u) = -2$ for all $u \in \mathbb{R}$ we obtain $L = \emptyset$ and $U = U^*$ Since $E(\xi_1) = -1.5$, ξ drifts to $-\infty$ a.s.

Example 3.32. Let $(\xi, \eta)_t := (d_\xi, d_\eta)t + \sum_{i=1}^{N_t} Y_i$ where $P(Y = (-1, 2)) = 1/2$ and $P(Y = (-2, -3)) = 1/2$. Then $L^* = [\theta_1, \theta_3] = (-\infty, \frac{-3}{e^2-1}] \cong (-\infty, -0.5]$ and $U^* = [\theta'_4, \theta'_2] = [\frac{2}{e-1}, \infty) \cong [1.2, \infty)$. Note that g simplifies to $g(u) = d_\eta + ud_\xi$ and hence choosing $d_\xi \leq 0$ and $d_\eta = 0$ gives $L = L^*$ and $U = U^*$. Since $E(\xi_1) = -1.5 + d_\xi$, ξ drifts to $-\infty$ a.s.

Example 3.33. Let $(\xi, \eta)_t := \sum_{i=1}^{N_t} Y_i$ where $P(Y = (1, 2)) = 1/3$ and $P(Y = (1, 8)) = 1/3$ and $P(Y = (0, -5)) = 1/3$. Then $L^* = \emptyset$ whilst $U^* = [\theta'_3, \theta'_1] = (-\infty, \frac{8}{e^{-1}-1}] \cong (-\infty, -12.6]$. Note that $g(u) = 0$ for all $u \in \mathbb{R}$ so $L = L^*$ and $U = U^*$. Since $E(\xi_1) = 1$, ξ drifts to ∞ a.s.

Example 3.34. Let $(\xi, \eta)_t := \sum_{i=1}^{N_t} Y_i$ where $P(Y = (1, 2)) = 1/2$ and $P(Y = (1, 8)) = 1/2$. Then $L^* = [\theta_1, \theta_4] = [\frac{2}{e^{-1}-1}, \infty) \cong [-3.2, \infty)$ and $U^* = [\theta'_3, \theta'_1] = (-\infty, \frac{8}{e^{-1}-1}] \cong (-\infty, -12.6]$. Note that $g(u) = 0$ for all $u \in \mathbb{R}$ so $L = L^*$ and $U = U^*$. Since $E(\xi_1) = 1$, ξ drifts to ∞ a.s.

Appendix A

Direct method for no ruin when (ξ, η) is Compound Poisson with drift

We recall equation (2.2), and note another useful formulation for the jump of the GOU.

$$\begin{aligned} \Delta V_t &= (e^{\Delta \xi_t} - 1)V_{t-} + e^{\Delta \xi_t}\Delta \eta_t & \text{(A.1)} \\ &= e^{\Delta \xi_t}(V_{t-} + \Delta \eta_t) - V_{t-}. & \text{(A.2)} \end{aligned}$$

We make an important note. By the definition of the sets A_i^u in (2.1) and the values θ_i, we know the following; if $V_{t-} > \theta_4$ then it is possible for a negative jump ΔV_t to occur, resulting from $(\Delta \xi_t, \Delta \eta_t) \in A_4^{V_{t-}}$. However, such a jump cannot result in $V_t \leq \theta_4$. To see this, let $V_{t-} > \theta_4$ and observe that we can choose $(x, y) \in A_4^{V_{t-}}$ such that $(e^x - 1)\theta_4 + e^x y \geq 0$ and $(e^x - 1)V_{t-} + e^x y < 0$. Thus,

$$\begin{aligned} V_t &= V_{t-} + (e^x - 1)V_{t-} + e^x y \\ &= V_{t-} + (e^x - 1)(V_{t-} - \theta_4) + (e^x - 1)\theta_4 + e^x y \\ &\geq V_{t-} + (e^x - 1)(V_{t-} - \theta_4) \\ &> \theta_4. \end{aligned}$$

Now we let (ξ, η) be a two dimensional compound Poisson process with deterministic drift component $(d_\xi t, d_\eta t)$ where $d_\xi \neq 0$. Thus, we can write

$$(\xi_t, \eta_t) := (d_\xi, d_\eta)t + \sum_{i=1}^{N_t}(C_i, D_i)$$

for some iid sequence of two dimensional random vectors (C_i, D_i) with common distribution (C, D). We calculate a formula for the associated GOU. The jumps $(\Delta\xi_t, \Delta\eta_t)$ occur at the jump times $0 < R_1 < R_2 < \ldots$ of N_t. Thus if $t < R_1$,

$$\begin{aligned} V_t &= e^{d_\xi t}\left(z + \int_0^t e^{-d_\xi s}\mathrm{d}(d_\eta s)\right) \\ &= \left(z + \frac{d_\eta}{d_\xi}\right)e^{d_\xi t} - \frac{d_\eta}{d_\xi}, \end{aligned}$$

and hence

$$V_{R_1} = \left(z + \frac{d_\eta}{d_\xi}\right)e^{d_\xi R_1} - \frac{d_\eta}{d_\xi} + \Delta V_{R_1}.$$

We thus obtain the following recursive formula for V_t when $R_i \le t < R_i + 1$;

$$V_t = \left(V_{R_i} + \frac{d_\eta}{d_\xi}\right)e^{d_\xi(t-R_i)} - \frac{d_\eta}{d_\xi}. \tag{A.3}$$

By expanding this formula we obtain a general formula for V_t when $R_i \le t < R_{i+1}$;

$$V_t = f(t) + \Delta V_{R_1}e^{d_\xi(t-R_1)} + \Delta V_{R_2}e^{d_\xi(t-R_2)} + \ldots + \Delta V_{R_i}e^{d_\xi(t-R_i)}, \tag{A.4}$$

where

$$f(t) := \left(z + \frac{d_\eta}{d_\xi}\right)e^{d_\xi t} - \frac{d_\eta}{d_\xi}. \tag{A.5}$$

In the case that $d_\xi = 0$ a simpler set of calculations gives us the formula

$$V_t = z + d_\eta t + \Delta V_{R_1} + \ldots + \Delta V_{R_i}. \tag{A.6}$$

The following theorem is an analogue of Theorem 2.1 for the compound Poisson case with drift. It will be proved using the comments above. It has been noted in point 2 of Remark 2.2 that this theorem follows from Theorem 2.1 in the finite variation case.

Theorem A.1. *Let (ξ, η) be a two dimensional compound Poisson process with drift where η is not a subordinator, and suppose the associated GOU satisfies $V_0 = z \ge 0$. Then $\psi(z) = 0$ for large enough z if and only if $\Pi_{\xi,\eta}(A_3) = 0$, $\theta_2 \le \theta_4$, and at least one of the following is true:*

(a) $d_\xi = 0$, and $d_\eta \ge 0$;

(b) $d_\xi > 0$ and $-\frac{d_\eta}{d_\xi} \le \theta_4$;

(c) $d_\eta > 0$, and $d_\xi < 0$, such that $-\frac{d_\eta}{d_\xi} \ge \theta_2$.

If (b) holds, then $\psi(z) = 0$ *for all* $z \geq \max\{\theta_2, -\frac{d_\eta}{d_\xi}\}$ *and* $\psi(z) > 0$ *for all* $z < \max\{\theta_2, -\frac{d_\eta}{d_\xi}\}$. *If (a) or (c) holds, then* $\psi(z) = 0$ *for all* $z \geq \theta_2$ *and* $\psi(z) > 0$ *for all* $z < \theta_2$.

Before proving this theorem, we define some terms and prove a lemma which will be needed for the proof. For a real number $c > 0$ define the processes

$$Y_t := e^{d_\xi(t-R_1)} + e^{d_\xi(t-R_2)} + \ldots + e^{d_\xi(t-R_i)}, \text{ for } R_i \leq t < R_{i+1}, \tag{A.7}$$

$$U'_t := f(t) - cY_t \tag{A.8}$$

and

$$U''_t := z + d_\eta t - cN_t. \tag{A.9}$$

where f is from (A.5) and U' is defined for $d_\xi \neq 0$. Note the similarity between the definitions U' and U'' and the equations for V in (A.4) and (A.6) respectively. For a real $a \geq 0$ define

$$R'_z := \inf\{t > 0 : U'_t < 0\}, \tag{A.10}$$

and

$$R''_z := \inf\{t > 0 : U''_t < 0\} \tag{A.11}$$

Lemma A.2. *For any* $t > 0$ *and* $c > 0$, $P(R'_z \leq t) > 0$ *and* $P(R''_z < t) > 0$.

Proof. Note first that $P(R''_z \leq t) \geq P(U''_t < 0) = P\left(N_t > (z + d_\eta t)/c\right) > 0$. We now show that for all $t > 0$ and $x > 0$ there exists $n > 0$ such that $P(Y_t > x|N_t = n) = 1$. If $d_\xi > 0$ then $e^{d_\xi(t-R_i)} > 1$ for all $t > 0$ and all i. Hence

$$P(Y_t > x|N_t = n) \geq P(n > x) = 1$$

whenever $n > x$. If $d_\xi < 0$ then $e^{d_\xi(t-R_i)} > e^{d_\xi t} > 0$ all $t > 0$ and all i. Hence

$$P(Y_t > x|N_t = n) \geq P\left(ne^{d_\xi t} > x\right) = 1$$

whenever $n > xe^{-d_\xi t}$. Using this fact, there exists $n > 0$ such that the following is true;

$$\begin{aligned} P(R_z \leq t) \geq P(U_t < 0) &= P\left(Y_t > \frac{f(t)}{c}\right) \\ &> P\left(Y_t > \frac{f(t)}{c}\Big| N_t = n\right) P(N_t = n) \\ &= P(N_t = n) > 0. \end{aligned}$$

□

Proof of Theorem A.1. Suppose $\Pi_{\xi,\eta}(A_3) = 0$ and $\theta_2 \leq \theta_4$. If $V_{t-} \in [\theta_2, \theta_4]$ then it is an immediate consequence of the definition of θ_2 and θ_4 that there can be no negative jump ΔV_t . If $V_{t-} \in (\theta_4, \infty)$ there can be a negative jump ΔV_t but it can only be caused by $(\Delta\xi_t, \Delta\eta_t) \in A_4^{V_{t-}}$ and, as we have noted above, such a jump cannot be sufficiently negative to result in $V_t \leq \theta_4$. Thus when $V_{t-} \geq \theta_2$ we have shown $V_t \geq \theta_2$. We now show that assuming (a), (b) or (c) implies zero probability of ruin when z is above the stated thresholds. The fact that there is positive probability of ruin when z is below the stated thresholds, will be proved later.

We deal with (a) first. If $d_\xi = 0$ then equation (A.6) shows that V_t has a deterministic linear drift between its jumps, given by $z + d_\eta t$. If $d_\eta = 0$ then V_t is constant between its jumps, and if $d_\eta > 0$ then V_t is strictly increasing. In both of these cases, starting the process with $z \geq \theta_2$ means that V_t can never drop below θ_2 as a result of drift or jumps, and hence $\psi(z) = 0$.

Note that if $d_\xi \neq 0$ the recursive equation (A.3) shows that V_t drifts exponentially between jumps, that is, for $t \in [R_i, R_{i+1})$. The direction of this drift depends on two factors; the sign of $V_{R_i} + \frac{d_\eta}{d_\xi}$, and the sign of d_ξ.

Now we deal with (b). Suppose $d_\xi > 0$ such that $-\frac{d_\eta}{d_\xi} \leq \theta_4$. Since $d_\xi > 0$, if $V_{R_i} + \frac{d_\eta}{d_\xi} > 0$ then V_t will drift upwards towards ∞ on $t \in [R_i, R_{i+1})$, whilst if $V_{R_i} + \frac{d_\eta}{d_\xi} = 0$ then V_t remains constant on $t \in [R_i, R_{i+1})$. If $V_{R_i-} > \theta_4$ then a negative jump $\Delta V_{R_i} < 0$ may occur, however, since $-\frac{d_\eta}{d_\xi} \leq \theta_4$ the jump cannot result in $V_{R_i} < -\frac{d_\eta}{d_\xi}$, and so the subsequent drift on $t \in [R_i, R_{i+1})$ is non-negative. Hence, if we start the process with $z \geq \max\{\theta_2, -\frac{d_\eta}{d_\xi}\}$ then V_t can never drop below $\max\{\theta_2, \frac{d_\eta}{d_\xi}\}$ and so $\psi(z) = 0$.

Now we deal with (c). Suppose $d_\eta > 0$ and $d_\xi < 0$ such that $-\frac{d_\eta}{d_\xi} \geq \theta_2$. If $V_{R_i} < -\frac{d_\eta}{d_\xi}$ then V_t will drift upwards on $t \in [R_i, R_{i+1})$ approaching the asymptote $-\frac{d_\eta}{d_\xi}$. If $V_{R_i} > -\frac{d_\eta}{d_\xi}$ then V_t will drift downwards on $t \in [R_i, R_{i+1})$ approaching $-\frac{d_\eta}{d_\xi}$. If $V_{R_i} < -\frac{d_\eta}{d_\xi}$ then V_t will remain constant on $t \in [R_i, R_{i+1})$. Thus if we start with $z \geq \theta_2$ then V_t can never move below θ_2 during a drift interval. And from our jump analysis, we know that the only possible negative jumps will occur when $V_{t-} > \theta_4$ and such jumps cannot result in $V_t \leq \theta_4$. Thus if $z \geq \theta_2$ then $\psi(z) = 0$.

To prove the sufficiency part of the theorem, it suffices to show that if η is not a subordinator then $\psi(z) > 0$ for all $z \geq 0$ if any one of the following conditions holds:

1. $\Pi_{\xi,\eta}(A_3) \neq 0$;
2. $\theta_4 < \theta_2$;
3. $d_\xi \leq 0$ and $d_\eta < 0$;
4. $d_\xi > 0$, and $-\frac{d_\eta}{d_\xi} > \theta_4$;
5. $d_\eta > 0$, $d_\xi < 0$ and $-\frac{d_\eta}{d_\xi} < \theta_2$.

We first show that it is possible to reduce the problem. Suppose that Γ is a closed square in $\mathbb{R}^2 \setminus \{0\}$ such that $\Pi_{\xi,\eta}(\Gamma) > 0$. We can write (ξ, η) as the sum of drift terms and two independent compound Poisson processes

$$(\xi_t, \eta_t) = (d_\xi t, d_\eta t) + (\xi'_t, \eta'_t) + (\xi''_t, \eta''_t)$$

where (ξ', η') has jumps in Γ and (ξ'', η'') has jumps in $\mathbb{R}^2 \setminus \Gamma$. The jumps for these processes occur at the the jump times of independent Poisson processes which we denote by M' and M'' respectively. Define

$$V'_t := e^{d_\xi t + \xi'_t}\left(z + \int_0^t e^{-(d_\xi s + \xi'_{s-})}\mathrm{d}(d_\eta s + \eta'_s)\right)$$

and define the stopping times $T_z := \inf\{t > 0 : V_t < 0\}$ and $S_z := \inf\{t > 0 : V'_t < 0\}$. Suppose we have proven that $P(S_z \leq t) > 0$ for a particular $t > 0$. Then

$$\begin{aligned} P(T_z \leq t) &\geq P(T_z \leq t, M_t = 0) \\ &= P(S_z \leq t, M_t = 0) \\ &= P(S_z \leq t)P(M_t = 0) \\ &> 0 \end{aligned}$$

which ensures that $\psi(z) := P(T_z < \infty) > 0$.

Proof of point 1 We prove that if when η is not a subordinator then $\Pi_{\xi,\eta}(A_3) > 0$ implies that $\psi(z) > 0$ for all z and for all values of d_ξ and d_η. This will be split into two cases. For case 1, we assume that

$$\Pi_{\xi,\eta}((-\infty, 0] \times (-\infty, 0)) > 0,$$

which means η can have negative jumps. For case 2, we assume that

$$\Pi_{\xi,\eta}\left((-\infty,0]\times(-\infty,0)\right)=0,$$

and $\Pi_{\xi,\eta}\left((-\infty,0]\times\{0\}\right)>0$, which means that η cannot have negative jumps and ξ can have negative jumps without η jumping at the same time. Note that this implies that $\theta_4=0$. Since η is not a subordinator we must have $d_\eta<0$. Thus, if $d_\xi\le 0$ then the result will follow from point 3 and if $d_\xi>0$ then the result will follow from point 4. So we can leave this case and return to case 1.

Suppose the conditions of case 1 hold. Since $\Pi_{\xi,\eta}$ is a measure there must exist a closed square $\Gamma\subset(-\infty,0]\times(-\infty,0)$ of arbitrary side length, such that $\Pi_{\xi,\eta}(\Gamma)>0$. By the above comments it suffices to assume that $\Pi_{\xi,\eta}(\mathbb{R}^2\setminus\Gamma)=0$ and prove $P(T_z\le t)>0$ for all $t>0$. Thus we can suppose that all the jumps $(\Delta\xi,\Delta\eta)$ are in Γ and occur at the jump times $0<R_1<R_2<\ldots<\infty$ of a Poisson process N.

For a fixed $(\Delta\xi_t,\Delta\eta_t)\in\Gamma$, the corresponding jump ΔV_t becomes less negative, approaching $e^{\Delta\xi_t}\Delta\eta_t$, as V_{t-} decreases to zero, and becomes more negative, approaching $-\infty$, as V_{t-} increases to ∞. We see this by using equation (A.1). In particular $\Delta V_t<e^{\Delta\xi_t}\Delta\eta_t$ on $t\le T_z$.

Define (x',y') to be the point in the top right corner of Γ, that is, $x':=\sup\{x\le 0:(x,y)\in\Gamma\}$ and $y':=\sup\{y<0:(x,y)\in\Gamma\}$. Then $e^{\Delta\xi_t}\Delta\eta_t\le e^{x}y=:-c<0$ for all $(\Delta\xi_t,\Delta_t)\in\Gamma$.

Note that for this choice of c an examination of equations (A.4) and (A.8) shows that whenever $d_\xi\ne 0$ we have $V_t<U'_t$ on $t\le T_z$. Thus $T_z<R'_z$ and, by Lemma A.2 we know that $P(R'_z\le t)>0$ for all $t>0$. Hence $P(T_z\le t)>0$ as required. Next, note that equations (A.6) and (A.9) show that whenever $d_\xi=0$ we have $V_t<U''_t$ on $t\le T_z$. Thus Lemma A.2 again gives the result. This completes the proof of case 1.

The following two lemmas will be used to prove the remaining points.

Lemma A.3. *If $z<\theta_2$ then $P(T_z\le t)>0$ for all $d_\xi,d_\eta\in\mathbb{R}$.*

Proof. Using the definition of θ_2, and the assumption that $z<\theta_2$, we know that there exists $\epsilon>0$ such that there exists a closed square $\Gamma\in A_2^{z+\epsilon}\setminus A_2^{\theta_2}$ with $\Pi_{\xi,\eta}(\Gamma)>0$. It suffices to assume $\Pi_{\xi,\eta}(\mathbb{R}^2\setminus\Gamma)=0$.

Thus, every $(\Delta\xi_t,\Delta\eta_t)\in\Gamma$ causes a negative jump $\Delta V_t<0$ whenever $0\le V_{t-}\le z+\epsilon$. By (A.1), for a fixed $(\Delta\xi_t,\Delta\eta_t)\in\Gamma$, jumps get more negative approaching $e^{\Delta\xi_t}\Delta\eta_t$, as V_{t-} decreases to zero, and less negative, approaching $e^{\Delta\xi_t}(z+\epsilon+\Delta\eta_t)-(z+\epsilon)$, as V_{t-} increases to $z+\epsilon$.

Define (x',y') to be the point in the top right corner of Γ, that is, $x' := \sup\{x \leq 0 : (x,y) \in \Gamma\}$ and $y' := \sup\{y < 0 : (x,y) \in \Gamma\}$. Equation (A.2) implies that for any fixed value $0 \leq V_{t-} \leq z+\epsilon$, we will be in one of two cases. Firstly, the least negative ΔV_t will occur at the point (x',y') and this point will not cause ruin. Secondly, every jump $(\Delta\xi_t,\Delta\eta_t) \in \Gamma$ will be negative enough to cause immediate ruin. This will occur if $\Delta\eta < -V_{t-}$ for every $(\Delta\xi,\Delta\eta) \in \Gamma$. In this case the most negative jump will occur at the top left corner of Γ, though this fact is not necessary for our proof.

These points show that for all $(\Delta\xi_t,\Delta\eta_t) \in \Gamma$ and any $0 \leq V_{t-} \leq z+\epsilon$ we have $\Delta V_t < e^{x'}(z+\epsilon+y')-(z+\epsilon) =: -c < 0$.

Suppose $d_\xi \neq 0$. Note that if $f(t)$ is an increasing function we can choose $t' > 0$ such that $f(t') = z+\epsilon$. Hence, with the above choice of $c > 0$, we have that $V_t < U'_t$ from (A.8) whenever $t \leq \min\{t', T_z\}$, and Lemma A.2 implies that $P(R'_z \leq t') > 0$. Hence $P(T_z \leq t') > 0$ and since ϵ, and hence t', can be chosen arbitrarily small, $P(T_z \leq t) > 0$ for all $t > 0$. If $f(t)$ is a decreasing function, then $V_t < U'_t$ for all $t \leq T_z$, and Lemma A.2 give the result.

Suppose $d_\xi = 0$. If $d_\eta > 0$ we choose a $t' > 0$ such that $z + d_\eta t' = z+\epsilon$. For our above choice of $c > 0$, $V_t < U''_t$ from (A.9) whenever $t \leq \min\{t', T_z\}$. Alternatively, if $d_\eta \leq 0$ then $V_t < U''_t$ whenever $t \leq T_z$. Thus Lemma A.2 gives the result. □

Lemma A.4. *If $z > \theta_4$ and $S_z := \inf\{t > 0 \ : \ V_t < \theta_4 + \epsilon\}$ then $P(S_z \leq t) > 0$ for all $d_\xi, d_\eta \in \mathbb{R}$ and any $\epsilon > 0$.*

Proof. Using the definition of θ_4, and the assumption that $z > \theta_4$, we know that there exists $\epsilon > 0$ such that there exists a closed square $\Gamma \in A_4^{\theta_4+\epsilon} \setminus A_4^{\theta_4}$ with $\Pi_{\xi,\eta}(\Gamma) > 0$. We assume that $\Pi_{\xi,\eta}(\mathbb{R}^2 \setminus \Gamma) = 0$.

Thus, every $(\Delta\xi_t,\Delta\eta_t) \in \Gamma$ causes a negative jump $\Delta V_t < 0$ whenever $\theta_4 + \epsilon \leq V_{t-} < \infty$. By (A.1), for a fixed $(\Delta\xi_t,\Delta\eta_t) \in \Gamma$, jumps get less negative approaching $e^{\Delta\xi_t}(\theta_4+\epsilon+\Delta\eta_t)-(\theta_4+\epsilon)$, as V_{t-} decreases to zero, and more negative, approaching $-\infty$ as V_{t-} increases to ∞.

Define (x',y') to be the point in the top right corner of Γ, that is, $x' := \sup\{x \leq 0 : (x,y) \in \Gamma\}$ and $y' := \sup\{y < 0 : (x,y) \in \Gamma\}$. Equation (A.2) implies that for any fixed value $\theta_4 + \epsilon \leq V_{t-} < \infty$, we attain the least negative ΔV_t when $(\Delta\xi_t,\Delta\eta_t)$ is the point (x',y').

These points show that for all $(\Delta\xi_t, \Delta\eta_t) \in \Gamma$ and any $\theta_4 + \epsilon \leq V_{t-} < \infty$, we have

$$\Delta V_t < e^{x'}(\theta_4 + \epsilon + y') - (\theta_4 + \epsilon) =: -c < 0.$$

Now the method of proof is almost identical to the proof of point 1, case 1. Define $S'_z := \inf\{t > 0 : U'_t < \theta_4 + \epsilon\}$ and $S''_z := \inf\{t > 0 : U''_t < \theta_4 + \epsilon\}$ and note that Lemma A.10 implies that $P(S'_z \leq t) > 0$ and $P(S''_z \leq t) > 0$. For the above choice of $c > 0$, if $d_\xi \neq 0$ then $V_t < U'_t$ on $t \leq S_z$, and hence $S_z < S'_z$. Similarly, if $d_\xi = 0$ then $V_t < U''_t$ on $t \leq S_z$, and hence $S_z < S''_z$. This proves the result. □

Proof of point 2 Suppose $\theta_4 < \theta_2$ and let $z > \theta_2$. Define $S_z := \inf\{t > 0 : V_t < \theta_2\}$ and $T_z := \inf\{t > 0 : V_t < 0\}$. We show $P(T_z \leq t) > 0$ for all $t > 0$. Since $S_z \leq T_z$, we have

$$P(T_z \leq t) = P(T_z \leq t \mid S_z \leq t)P(S_z \leq t).$$

Since $\theta_4 < \theta_2 < z$, Lemma A.4 implies $P(S_z \leq t) > 0$. Thus, it suffices to prove

$$P(T_z \leq t | S_z \leq t) =: P'(T_z \leq t) > 0.$$

However,

$$\begin{aligned} P'(T_z \leq t) &= \int_{\theta_4+}^{\theta_2-} P'(T_z \leq t | V_{S_z} = x) P'(V_{S_z} \in \mathrm{d}x) \\ &= \int_{\theta_4+}^{\theta_2-} P'(T_x \leq t - S_z) P'(V_{S_z} \in \mathrm{d}x) \end{aligned}$$

This holds because V is a time-homogenous strong Markov process with respect to the probability measure P, and hence is also a time-homogenous strong Markov process with respect to the conditional probability measure P'. Note that Lemma A.3 tells us that $P(T_x \leq r) > 0$ for all $r > 0$ and $x < \theta_2$, which implies that $P(T_x \leq t - S_z | S_z \leq t) > 0$ for all $x \in< (\theta_4, \theta_2)$. Hence $P'(T_z \leq t) > 0$ as required.

Proof of point 3 Suppose $d_\xi < 0$ and $d_\eta < 0$, and let $z > 0$. Then the function

$$f(t) := \left(z + \frac{d_\eta}{d_\xi}\right) e^{d_\xi t} - \frac{d_\eta}{d_\xi}$$

will decrease towards the asymptote $-\frac{d_\eta}{d_\xi} < 0$. Thus there exists $t' > 0$ such that $f(t') < 0$. So (A.4) tells us

$$\begin{aligned} \psi(z) \geq P(V_{t'} < 0) \geq P(V_{t'} < 0, N_{t'} = 0) &= P(f(t') < 0)P(N_{t'} = 0) \\ &= P(N_{t'} = 0) \\ &> 0. \end{aligned}$$

The same argument works in the case $d_\xi = 0$ and $d_\eta < 0$, since the function $z + d_\eta$ decreases to $-\infty$.

Proof of point 4 We have already proved the case when $\theta_4 < \theta_2$ so in the proof of points 4 and 5 we assume $\theta_2 \leq \theta_4$.

Suppose that $d_\xi > 0$, and $-\frac{d_\eta}{d_\xi} > \theta_4$. Define $S_z := \inf\{t > 0 : V_t < -\frac{d_\eta}{d_\xi}\}$ and $T_z := \inf\{t > 0 : V_t < 0\}$. We show that $\psi(z) > 0$ for all $z \geq 0$.

Firstly, we use equation (A.4) to show $P(T_z < \infty) > 0$ when $z < -\frac{d_\eta}{d_\xi}$. Note that the function

$$f(t) := \left(z + \frac{d_\eta}{d_\xi}\right) e^{d_\xi t} - \frac{d_\eta}{d_\xi}$$

decreases towards $-\infty$ so there exists $t' > 0$ such that $f(t') < 0$. Thus,

$$\begin{aligned} P(T_z \leq t') \geq P(V_{t'} < 0) \geq P(V_{t'} < 0, N_{t'} = 0) &= P(f(t') < 0)P(N_{t'} = 0) \\ &= P(N_{t'} = 0) \\ &> 0. \end{aligned}$$

If $z > \theta_4$ then Lemma A.4 implies that $P(S_z \leq t) > 0$ for all $t > 0$.

As in the proof of point 2, we combine these two results, and use the fact that V is a time homogenous strong Markov process. Let $z > \theta_4$. Since $S_z \leq T_z$ we have

$$P(T_z < \infty) = P(T_z < \infty \mid S_z < \infty)P(S_z < \infty).$$

Thus, it suffices to prove

$$P(T_z < \infty \mid S_z < \infty) =: P'(T_z < \infty) > 0.$$

However,

$$\begin{aligned} P'(T_z < \infty) &= \int_{\theta_4+}^{-\frac{d_\eta}{d_\xi}-} P'(T_z < \infty \mid V_{S_z} = x)P'(V_{S_z} \in \mathrm{d}x) \\ &= \int_{\theta_4+}^{-\frac{d_\eta}{d_\xi}-} P'(T_x < \infty)P'(V_{S_z} \in \mathrm{d}x) \\ &> 0. \end{aligned}$$

Proof of point 5 Suppose that $\theta_2 \leq \theta_4$, $d_\eta > 0$, $d_\xi < 0$ and $-\frac{d_\eta}{d_\xi} < \theta_2$.

We let $S_z := \inf\{t > 0 : V_t < \theta_2\}$ and $T_z := \inf\{t > 0 : V_t < 0\}$ and prove that $\psi(z) > 0$. If $z > -\frac{d_\eta}{d_\xi}$ then the function

$$f(t) := \left(z + \frac{d_\eta}{d_\xi}\right) e^{d_\xi t} - \frac{d_\eta}{d_\xi}$$

will decrease towards the asymptote $-\frac{d_\eta}{d_\xi}$. Since $-\frac{d_\eta}{d_\xi} < \theta_2$ we can show that $P(S_z < \infty) > 0$ using the familiar argument of conditioning on $N_t = 0$. And for $z < \theta_2$, Lemma A.3 implies $P(T_z \leq t) > 0$ for any $t > 0$. We combine the two as above, using the strong Markov property, to conclude that $\psi(z) > 0$.

It remains to note that there is positive probability of ruin when all conditions of the theorem are satisfied, but z is below the stated thresholds. Thus, we assume that $\Pi_{\xi,\eta}(A_3) = 0$ and $\theta_2 \leq \theta_4$. Now if (a) or (c) holds and $z < \theta_2$ then $\psi(z) > 0$, as shown in Lemma A.3. If (b) holds, we need to show that $\psi(z) > 0$ if $z < \max\{\theta_2, -\frac{d_\eta}{d_\xi}\}$. However, in the proof of point 4 it was shown that if $d_\xi > 0$ and $z < -\frac{d_\eta}{d_\xi}$ then $\psi(z) > 0$, and so we are done. $\square$

Appendix B

Direct calculation of examples

We present calculations for Examples (2.5) and (2.7), which serve to verify the ruin probability results obtained by Theorem 2.1.

Example (2.5) calculations: Let $(\xi_t, \eta_t) := (B_t + ct, -B_t + (1/2 - c)t)$, where $c \in \mathbb{R}$ and $\sigma_\xi^2 = 1$. Then

$$Z_t = -\int_0^t e^{-(B_s+cs)} \mathrm{d}B_s + (1/2 - c)\int_0^t e^{-(B_s+cs)} \mathrm{d}s.$$

By Ito's formula we know that

$$\begin{aligned} e^{-(B_s+cs)} &= 1 + \int_0^t -e^{-(B_s+cs)} \mathrm{d}(B_s + cs) + 1/2\int_0^t e^{-(B_s+cs)} \mathrm{d}s \\ &= 1 - \int_0^t e^{-(B_s+cs)} \mathrm{d}B_s + (1/2 - c)\int_0^t e^{-(B_s+cs)} \mathrm{d}s. \end{aligned}$$

Combining these two formulas we obtain

$$Z_t = e^{-(B_t+ct)} - 1.$$

Thus, for all $t > 0$ we have $Z_t > -1$ almost surely, which implies that $\psi(z) = 0$ whenever $z \geq 1$. Since B is Brownian motion, for any $c \in \mathbb{R}$ we have that

$$\inf\left\{ u \in \mathbb{R} : P\left(\inf_{t\geq 0}(-B_t - ct) \leq u \right) > 0 \right\} = -\infty,$$

and thus $P(\inf_{t\geq 0} Z_t < u) > 0$ for all $u > -1$, which implies that $\psi(z) > 0$ whenever $z < 1$.

Example (2.7) calculations: Let N be a Poisson process with parameter λ, let $c > 0$ and let $(\xi_t, \eta_t) := (-ct + N_t, 2ct - N_t)$. Then

$$Z_t = 2c \int_0^t e^{(cs-N_s)-} \mathrm{d}s - \int_0^t e^{(cs-N_s)-} \mathrm{d}N_s.$$

By Ito's formula for semimartingales [60] p.79,

$$\begin{aligned} e^{ct-N_t} &= 1 + \int_0^t e^{(cs-N_s)-} \mathrm{d}(cs - N_s) \\ &\quad + \sum_{0<s\le t} \left(e^{cs-N_s} - e^{(cs-N_s)-} - e^{(cs-N_s)-} \Delta(cs - N_s)\right). \end{aligned}$$

We rearrange this formula and combine with the previous formula to obtain

$$\begin{aligned} Z_t &= e^{ct-N_t} - 1 + c\int_0^t e^{(cs-N_s)-} \mathrm{d}s \\ &\quad - \sum_{0<s\le t} \left(e^{cs-N_s} - e^{(cs-N_s)-} - e^{(cs-N_s)-} \Delta(cs - N_s)\right). \end{aligned}$$

Let the jump times of N be $0 = T_0 < T_1 < T_2 < \ldots$ and note that

$$\sum_{0<s\le t} \left(e^{cs-N_s} - e^{(cs-N_s)-} - e^{(cs-N_s)-} \Delta(cs - N_s)\right) = \sum_{i=1}^{N_t} e^{cT_i - i}.$$

Next note that

$$\begin{aligned} \int_0^t e^{(cs-N_s)-} \mathrm{d}s &= \sum_{i=1}^{N_t} \int_{T_{i-1}}^{T_i} e^{cs-(i-1)} \mathrm{d}s + \int_{T_{N_t}}^t e^{cs-N_t} \mathrm{d}s \\ &= \frac{1}{c}\left(\sum_{i=1}^{N_t} \left(e^{cT_i-(i-1)} - e^{cT_{i-1}-(i-1)}\right) + e^{ct-N_t} - e^{cT_{N_t}-N_t}\right) \\ &= \frac{1}{c}\left(-1 + \sum_{i=1}^{N_t} \left(e^{cT_i-(i-1)} - e^{cT_i-i}\right) + e^{ct-N_t}\right) \\ &= \frac{1}{c}\left(-1 + (e-1)\sum_{i=1}^{N_t} e^{cT_i-i} + e^{ct-N_t}\right). \end{aligned}$$

We substitute these two formulas into the formula for Z_t to obtain

$$\begin{aligned} Z_t &= 2e^{ct-N_t} - 2 + (e-1)\sum_{i=1}^{N_t} e^{cT_i-i} - \sum_{i=1}^{N_t} e^{cT_i-i} \\ &= 2e^{ct-N_t} - 2 + (e-2)\sum_{i=1}^{N_t} e^{cT_i-i}. \end{aligned}$$

Thus, for all $t > 0$ we have $Z_t > -2$ almost surely, which implies that $\psi(z) = 0$ whenever $z \geq 2$. Note that $\lim_{t\to\infty} \sum_{i=1}^{N_t} e^{cT_i - i} = 0$ a.s. whilst

$$\inf \left\{ u \in \mathbb{R} : P\left(\inf_{t\geq 0}(ct - N_t) \leq u \right) > 0 \right\} = -\infty.$$

Thus, $P(\inf_{t\geq 0} Z_t < u) > 0$ for all $u > -2$, which means $\psi(z) > 0$ whenever $z < 2$.

Appendix C

Proof that sequences are iid

We prove that the sequence of random vectors $(C_n, D_n)_{n\geq 1}$ defined as

$$(C_n, D_n) := \left(e^{\xi_n - \xi_{n-1}}, e^{\xi_{n-1}} \int_{(n-1)+}^{n} e^{-\xi_{s-}} \mathrm{d}\eta_s \right)$$

in equation (1.38), is iid. This sequence is used to discretize V into the solution of the stochastic difference equation (1.37).

The Lévy process (ξ, η) has independent increments, that is, $(\xi_t - \xi_s, \eta_t - \eta_s)$ is independent of $(\xi_v - \xi_r, \eta_v - \eta_r)$ whenever $0 \leq r < v \leq s < t$. Note that we can bring the term $e^{\xi_{n-1}}$ through the integral sign and write $D_n = \int_{(n-1)+}^{n} e^{-(\xi_{s-} - \xi_{n-1})} \mathrm{d}\eta_s$. Thus (C_n, D_n) is independent of (C_m, D_m) for any $n \neq m$.

We now prove that $(C_2, D_2) =_D (C_1, D_1)$, and the argument for general n is identical. The Lévy process (ξ, η) has stationary increments, that is, $(\xi_t - \xi_s, \eta_t - \eta_s) =_D (\xi_{t-s}, \eta_{t-s})$ whenever $0 \leq s < t$. Also note that for any $t > 0$, $\xi_t = \xi_{t-}$ a.s. and $\eta_t = \eta_{t-}$ a.s. It follows from these two facts and the independent increments property of (ξ, η) that, whenever $1 < s \leq 2$,

$$\begin{aligned}
(\xi_2 - \xi_1, \xi_{s-} - \xi_1, \eta_s - \eta_1) &= (\xi_s - \xi_1, \xi_{s-} - \xi_1, \eta_s - \eta_1) + (\xi_2 - \xi_s, 0, 0) \\
&=_D \mathscr{L}(\xi_s - \xi_1, \xi_{s-} - \xi_1, \eta_s - \eta_1) * \mathscr{L}(\xi_2 - \xi_s, 0, 0) \\
&=_D \mathscr{L}(\xi_{s-1}, \xi_{(s-1)-}, \eta_{s-1}) * \mathscr{L}(\xi_1 - \xi_{s-1}, 0, 0) \\
&=_D (\xi_{s-1}, \xi_{(s-1)-}, \eta_{s-1}) + (\xi_1 - \xi_{s-1}, 0, 0) \\
&= (\xi_1, \xi_{(s-1)-}, \eta_{s-1}). \qquad \text{(C.1)}
\end{aligned}$$

Now

$$
\begin{aligned}
(C_2, D_2) &= \left(e^{\xi_2-\xi_1}, \int_{1+}^{2} e^{-(\xi_{s-}-\xi_1)} \mathrm{d}\eta_s\right) \\
&= \left(e^{\xi_2-\xi_1}, \int_{1+}^{2} e^{-(\xi_{s-}-\xi_1)} \mathrm{d}(\eta_s - \eta_1)\right) \\
&=_D \left(e^{\xi_1}, \int_{1+}^{2} e^{-\xi_{(s-1)-}} \mathrm{d}\eta_{s-1}\right) \\
&= \left(e^{\xi_1}, \int_{0+}^{1} e^{-\xi_{s-}} \mathrm{d}\eta_s\right) \\
&= (C_1, D_1),
\end{aligned}
$$

where the third (distributional) equality follows using the distributional equality (C.1), the fourth equality is obtained using a change of variables and the final equality follows from the fact that $\Delta(\xi_0, \eta_0) = 0$. Thus we have proved that (C_n, D_n) is an iid sequence.

The fact that $(C_n, D_n)_{n\geq 1}$ is an iid sequence immediately implies that the sequence defined by $(C_n, C_nD_n)_{n\geq 1}$ is also an iid sequence. However,

$$
(C_n, C_nD_n) = \left(e^{\xi_n-\xi_{n-1}}, e^{\xi_n} \int_{(n-1)+}^{n} e^{-\xi_{s-}} \mathrm{d}\eta_s\right) = (A_n, B_n),
$$

when (A_n, B_n) is defined as in equation (1.34). This sequence is used to discretize V into the solution of the stochastic difference equation (1.32).

Appendix D

Examination of independent case

We write out summaries of the major results for the case in which ξ and η are independent. No proofs are given, but a lemma is stated and proved. Using this lemma, the independent results follow easily from the general versions. Propositions D.2 and D.3 are derived from the results in Section 3.3.1. Proposition D.4 is derived from Theorem 2.1 and Proposition D.5 is derived from Theorems 3.1 and 3.3. The terms δ, Υ, L and U are defined as in Chapter 3. As usual, we assume that neither ξ nor η are identically zero.

Lemma D.1. *Suppose ξ and η are independent. If there exists $u \neq 0$ such that $u \in L$, or $u \in U$, then ξ and η are of finite variation.*

Proof. Assume ξ and η are independent. Suppose that $u \in L$ with $u \neq 0$. We prove that ξ and η are finite variation. Since ξ and η are independent, it must be that $\sigma_{\xi,\eta} = 0$, which implies, by Proposition 3.16, that $\Sigma_{\xi,\eta} = 0$. By Proposition 1.11 it suffices to show

$$\int_{|(x,y)|<1} |(x,y)| \Pi_{\xi,\eta}(\mathrm{d}(x,y)) < \infty. \tag{D.1}$$

Suppose that $u > 0$. The case in which $u < 0$ is symmetric. By Proposition 3.16, the equation $g(u) > 0$ must hold. Further, case (a) or (c) of Proposition 3.16 must occur, and thus ξ and η both have no negative jumps. Thus the function g can be written

$$g(u) = \tilde{\gamma}_\eta + u\tilde{\gamma}_\xi - u \int_0^1 x \Pi_\xi(\mathrm{d}x) - \int_0^1 y \Pi_\eta(\mathrm{d}y).$$

Since $g(u) \geq 0$, the integrals must both be finite, which implies that (D.1) holds. $\square$

Proposition D.2. *Suppose ξ and η are independent and η is not a subordinator. Then there exists a finite lower bound, $\delta(z) > -\infty$, for some $z \in \mathbb{R}$, iff η is finite variation with $d_\eta < 0$ and no negative jumps, and one of the following holds:*

1. *ξ is a subordinator with $d_\xi > 0$, in which case $L = [-d_\eta/d_\xi, \infty] \subset (0, \infty)$;*

2. *$-\xi$ is a subordinator with $d_\xi < 0$, in which case $L = (-\infty, -d_\eta/d_\xi] \subset (-\infty, 0)$*

Proposition D.3. *Suppose ξ and η are independent and $-\eta$ is not a subordinator. Then there exists a finite upper bound, $\Upsilon(z) < \infty$, for some $z \in \mathbb{R}$, iff η is finite variation with positive drift and no positive jumps, and one of the following holds:*

1. *$-\xi$ is a subordinator with $d_\xi < 0$, in which case $U = [-d_\eta/d_\xi, \infty] \subset (0, \infty)$;*

2. *ξ is a subordinator with $d_\xi > 0$, in which case $U = (-\infty, -d_\eta/d_\xi] \subset (-\infty, 0)$.*

Proposition D.4. *Suppose ξ and η are independent. Then $\psi(0) = 0$ iff η is a subordinator. If η is not a subordinator then $\psi(z) = 0$ for some $z > 0$ iff ξ is a subordinator with $d_\xi > 0$ and η is of finite variation with $d_\eta < 0$. If this case occurs then $\psi(z) = 0$ for all $z \geq -d_\eta/d_\xi$.*

Proposition D.5. *Suppose ξ and η are independent and η is not a subordinator.*

1. *If $\lim_{t\to\infty} \xi_t = -\infty$ a.s. and $I_{-\xi, K^{\xi,\eta}} < \infty$ then $\psi(z) = 1$ for all $z \geq 0$;*

2. *If $E(\xi_1) = 0$, $E\left(e^{|\xi_1|}\right) < \infty$ and $E(\eta_1) < \infty$ then $\psi(z) = 1$ for all $z \geq 0$;*

3. *Suppose $\lim_{t\to\infty} \xi_t = \infty$ a.s. and $I_{\xi,\eta} < \infty$. Then $\psi(z) = 1$ for some $z \geq 0$ iff $-\eta$ is a subordinator and ξ is of finite variation with positive drift and no positive jumps. In such a case, $\psi(z) = 1$ for all $z \leq -d_\eta/d_\xi$.*

Appendix E

Comments

Paulsen's assumptions in [56] As noted in Section 1.6, certain faulty assumptions are made in Paulsen [56]. If ξ and η are independent and η is not a subordinator then Paulsen assumes:

1. If $E(\xi_1) = 0$ and ξ is not identically zero then for all $t > 0$, $P(V_t < 0|V_0 = z) > 0$ for all $z \geq 0$;

2. If $E(\xi_1) < 0$ and Λ is a Lebesgue measurable set in $(-\infty, 0)$ then $P(V_t \in \Lambda \text{ for some } 0 < t < \infty) > 0$ for all $V_0 = z \geq 0$;

3. If Z_t converges a.s. to a finite continuous random variable Z_∞ as $t \to \infty$, then for all $z \geq 0$, $P(Z_\infty > -z) > 0$.

Statements 1,2 and 3 are used by Paulsen in his proof of Theorem 1.20 part (b), (a) and (c) respectively. However, even in the independent case they are not true. For statement 2, a counterexample is presented in point 1 of Remark 3.2. Paulsen's proof can be salvaged by using the replacement inequality

$$P(V_T < 0|V_0 = z) > 0 \quad \forall z \geq 0 \tag{E.1}$$

where T is an exponential random variable independent of ξ and η. Since T has infinite support, for any $z \geq 0$, $P(V_T < 0|V_0 = z) > 0$ iff $\psi(z) := P(\inf_{t>0} V_t < 0|V_0 = z) > 0$. Thus Proposition D.4 ensures that (E.1) holds under the stated conditions.

For statement 2 to hold in the independent case, it must be that $L = \emptyset$. However, by Proposition D.2, there exist independent ξ and η such that η is not a subordinator, $E(\xi_1) < 0$ and $L = (-\infty, -d_\eta/d_\xi]$. Simple examples confirm this. If $\xi_t := -t$ and $\eta_t := -t$ then $V_t = e^{-t}(z + 1) - 1$. Whenever $z \leq -1$ the function

V_t increases towards the asymptote -1. For a simple non-deterministic example, let $\xi_t := -t$ and $\eta_t := -t + N_t$ where N is a Poisson process with jump times $0 < T_0 < T_1 < \cdots$. Then

$$V_t = -1 + (z+1)e^{-t} + e^{-t}\sum_{i=1}^{N_t} e^{T_i}.$$

Thus, whenever $z \leq -1$ all paths of V_t are increasing.

Statement 3 is also false in the independent case. If the conditions of statement 3 hold and $-\eta$ is a subordinator then, by Proposition 3.20, $0 \in U$, which implies, by Theorem 3.12, that $Z_\infty < 0$ a.s. For example, let $(\xi, \eta)_t := (t, -t - N_t)$ where N is as above. Then

$$Z_t = e^{-t} - 1 - \sum_{i=1}^{N_t} e^{-T_i}$$

and so $Z_\infty < -1$ a.s. Using Proposition D.3, it is clear that statement 3 holds when we add the extra condition that $-\eta$ is not a subordinator.

Paulsen's assumption in [57] In the proof of Theorem 3.2, Paulsen states that when ξ and η are independent and η is not a subordinator, then

$$\inf\left\{z \in \mathbb{R} : P\left(\inf_{t>0} Z_t < z\right) > 0\right\} = -\infty.$$

By (1.3), this is equivalent to assuming $\psi(z) > 0$ for all $z \geq 0$. Proposition D.4 indicates that Paulsen's statement is wrong. An example is $(\xi, \eta)_t := (t + N_t, -t)$ where N is as above. This example trivially satisfies all the conditions in Paulsen's Theorem 3.2. However, using the calculations from Appendix B it is clear that

$$Z_t = -1 + (e-1)\sum_{i=1}^{N_t} e^{-T_i - i} + e^{-t-N_t}$$

and hence $\inf_{t>0} Z_t \geq -1$ a.s.

Nyrhinen's condition in [52] As noted in Section 1.6, the continuous versions of Nyrhinen's asymptotic results require that the condition

$$\sup\left\{z : P\left(\sup_{t>0} Z_t > z\right) > 0\right\} = \infty$$

holds, where dependence between ξ and η is allowed. Clearly, this is equivalent to the condition $U \cap (-\infty, 0] = \emptyset$. Using Proposition 3.20 we can define iff conditions on the Lévy measure of (ξ, η) such that this situation occurs. The conditions will be the symmetric versions of the conditions in Theorem 2.1.

Kalashnikov and Norberg's conditions in [32]

As noted in Section 1.6, the continuous versions of the asymptotic results in this paper require hypothesis G to be satisfied for a particular stopping time $T > 0$. Namely, ξ is not a subordinator and $P(Z_T \leq u) > 0$ for all $u \in \mathbb{R}$. The authors note that when ξ and η are independent, this hypothesis can be replaced by the slightly stronger condition, $P(\xi_T < 0, Z_T < 0) > 0$.

Staying in the independent situation, if we choose an unbounded random time $T > 0$, with support $[0, \infty)$, then hypothesis G can be replaced with the simpler condition that ξ and η are both not subordinators. To see this, note that when ξ and η are independent and neither are subordinators, then $P(\inf_{t>0} Z_t < -z) > 0$ for all $z \geq 0$, by Proposition D.4. And by the choice of $T > 0$, whenever $z \geq 0$, $P(\inf_{t>0} Z_t < -z)$ iff $P(Z_T < -z) > 0$.

This result doesn't hold in general for unbounded $T > 0$. For an example of independent ξ and η, neither of which are subordinators, such that $P(Z_1 < 0) > 0$, see point 1 of Remark 3.2.

It is important to note that when ξ and η are dependent, hypothesis G cannot be replaced with the alternative statement. Specifically, when ξ and η are dependent, assuming that ξ and η are not subordinators certainly does not imply $P(Z_T \leq u) > 0$, whether or not T is unbounded. For general (ξ, η) and $T > 0$ with support $[0, \infty)$, $P(Z_T < u) > 0$ for all $u \in \mathbb{R}$ iff $P(\inf_{t>0} Z_t < -z) > 0$ for all $z \geq 0$, which is satisfied iff the conditions in Theorem 2.1 do not hold. To determine whether all the continuous asymptotic results in this paper hold in the general case we must examine under what conditions $P(\inf_{t>0} Z_t < -z) = P(\inf_{n\in\mathbb{N}} Z_n < -z)$.

Appendix F

Asymptotic Results

In this section we describe an asymptotic result by Grincevičius [28] on the absolute maximum of a discrete stochastic sequence. We state, and prove, a continuous version of this result. We also present modified versions of some of the asymptotic results described in Section 1.6. Throughout this section we let (ξ, η) be a Lévy process, and V, Z and $\psi(z)$ be defined as in equations (1.1), (1.2) and (1.3) respectively. Let

$$T_z := \inf\{t > 0 : V_t < 0 | V_0 = z\} = \inf\{t > 0 : Z_t < -z\}.$$

Let (M_n, Q_n) be an iid sequence of random vectors with common distribution (M, Q), where $M > 0$ a.s. Define $D_n := \sum_{i=1}^{n} \prod_{j=1}^{i-1} M_j Q_i$. Define the random walk $S_n := \ln\left(\prod_{j=1}^{n} M_j\right) = \sum_{j=1}^{n} \ln M_j$, where $S_0 = 0$. Grincevičius [28] provides results for the case in which $E(\ln M) = 0$, so the random walk S_n is oscillating. By Theorem 2.1 of Goldie and Maller [25], the condition $E(\ln M) = 0$, together with the non-degenerate condition $P(Q + Mc = c) < 1$ for all $c \in \mathbb{R}$, implies that $|D_n| \to_P \infty$ as $n \to \infty$. The following theorem, which is a consequence of Theorems 1 and 2 of [28], shows that $\ln|D_n|$ is asymptotically the same as $\max_{0 \le v \le n} S_v$ as $n \to \infty$. This is a useful result, since the maximum of an oscillating random walk is a well-known and well-studied process.

Theorem F.1. *Suppose $M > 0$ a.s., $E(\ln M) = 0$, $E\left(|\ln M|^2\right) < \infty$, $P(Q + Mc = c) < 1$ for all $c \in \mathbb{R}$ and one of the following conditions holds:*

1. $E\left(|\ln M|^{2+\epsilon_1}\right) < \infty$ *and* $E\left(\left(\ln^+|Q|\right)^{\frac{2}{\epsilon_1}+\epsilon_2}\right) < \infty$ *for some* $0 < \epsilon_1 \le 1$ *and* $\epsilon_2 > 0$;
2. $\ln M$ *has a continuous symmetric distribution and* $E\left(\left(\ln^+|Q|\right)^{2+\epsilon}\right) < \infty$ *for some* $\epsilon > 0$.

Then

$$\lim_{n\to\infty} \frac{\ln|D_n|}{\sqrt{n}} =_D \lim_{n\to\infty} \frac{\max_{0\le v\le n} S_v}{\sqrt{n}}$$

Further,

$$\lim_{n\to\infty} |D_n| e^{-\max_{0\le v\le n} S_v} =_D F$$

where F is a continuous proper distribution.

We do not state an explicit expression for F since it requires a lot of notation. We simply note that F is the convolution of the distributions of the stochastic series (6) and (8) in [28], which converge with probability 1.

We now prove a continuous version of this theorem. Note that we have modified the theorem in order to state conditions on the marginal distributions of ξ and η rather than on the distribution of Z.

Theorem F.2. *Suppose $E(\xi_1) = 0$, the degenerate equation (1.29) does not hold, there exist $p, q > 1$ with $1/p + 1/q = 1$ such that $E\left(e^{-p\xi_1}\right) < \infty$ and $E\left(|\eta_1|^q\right) < \infty$, and one of the following holds:*

1. *$E\left(|\xi_1|^3\right) < \infty$;*

2. *$E\left(\xi_1^2\right) < \infty$ and ξ_1 has a continuous symmetric distribution.*

Then

$$\lim_{t\to\infty} \frac{\ln|Z_t|}{\sqrt{t}} =_D \lim_{t\to\infty} \frac{-\inf_{s\le t} \xi_s}{\sqrt{t}} \tag{F.1}$$

Further,

$$\lim_{t\to\infty} |Z_t| e^{\inf_{s\le t} \xi_s} =_D F \tag{F.2}$$

where F is a continuous proper distribution.

Proof. Suppose that the assumptions of Theorem F.2 hold. For a Lévy process (ξ, η) define

$$(M_n, Q_n) := \left(e^{-(\xi_n - \xi_{n-1})}, e^{\xi_{n-1}} \int_{(n-1)+}^{n} e^{-\xi_{s-}} \mathrm{d}\eta_s\right) \tag{F.3}$$

as in (1.38). It is immediately clear that the assumptions in the first sentence of Theorem F.1 hold for $(M, Q) := \left(e^{-\xi_1}, Z_1\right)$. Using our assumptions on p and q, Lemma 3.24 implies that

$$E\left(\sup_{0\le s\le 1} |Z_s|\right) < \infty,$$

and hence $E|Z_1| < \infty$. Thus, there exists $0 < \epsilon \le 1$ such that $E\left(|Z_1|^\epsilon\right) < \infty$. Note that whenever $0 < \delta \le 1$ and $x > 0$, then there exists $c > 0$ such that

$(\ln^+ x)^{2+\delta} \le cx^\delta$. Hence $E\left(\left(\ln^+|Z_1|\right)^{2+\epsilon}\right) < \infty$. It is now clear that statement 2 of Theorem F.1 holds whenever statement 2 of Theorem F.2 holds. Further, statement 1 of Theorem F.1 holds, for $\epsilon_1 = 1$ and $\epsilon_2 = \epsilon$, whenever statement 1 of Theorem F.2 holds.

With our choices of (M_n, Q_n), it is clear that $Z_n = D_n$ for all $n \in \mathbb{N}$. Thus, Theorem F.1 implies that

$$\lim_{n\to\infty, n\in\mathbb{N}} \frac{\ln|Z_n|}{\sqrt{n}} =_D \lim_{n\to\infty, n\in\mathbb{N}} \frac{-\min_{v\in\{0,1,\cdots,n\}} \xi_v}{\sqrt{n}}$$

and

$$\lim_{n\to\infty, n\in\mathbb{N}} |Z_n| e^{\min_{v\in\{0,1,\cdots,n\}} \xi_v} =_D F.$$

We need to "fill in the gaps," and extend these equations into statements for real $t > 0$.

Define a sequence of partitions $\lambda^{(m)}$ for $m \in \mathbb{N}$ by $\lambda^{(0)} = \mathbb{N}$ and

$$\lambda^{(m)} = \left\{0, \frac{1}{2^m}, \frac{2}{2^m}, \cdots\right\}.$$

For each $m \in \mathbb{N}$ define an iid sequence of random vectors $\left(M_n^{(m)}, Q_n^{(m)}\right)_{n\in\mathbb{N}}$ by

$$\left(M_n^{(m)}, Q_n^{(m)}\right) := \left(e^{-\left(\xi_{\frac{n}{2^m}} - \xi_{\frac{n-1}{2^m}}\right)}, e^{\xi_{\frac{n-1}{2^m}}} \int_{\frac{n-1}{2^m}+}^{\frac{n}{2^m}} e^{-\xi_{s-}} \mathrm{d}\eta_s\right).$$

We prove, for each $m \in \mathbb{N}$, that the conditions of Theorem F.1 are satisfied for

$$\left(M^{(m)}, Q^{(m)}\right) := \left(e^{-\xi_{\frac{1}{2^m}}}, Z_{\frac{1}{2^m}}\right).$$

By the above comments, it suffices to show that the conditions of Theorem F.2 hold, with (ξ_1, η_1) replaced by $(\xi_{\frac{1}{2^m}}, \eta_{\frac{1}{2^m}})$. Since ξ is a Lévy process, $E(\xi_1) = 0$ iff $E\left(\xi_{\frac{1}{2^m}}\right) = 0$, and ξ_1 has a continuous symmetric distribution iff $\xi_{\frac{1}{2^m}}$ has a continuous symmetric distribution. The remaining moment conditions follow immediately from Sato [62], p.159, which states that whenever g is a submultiplicative, locally bounded, measurable function, then finiteness of the g-moment is not a time-dependent distributional property in the class of Lévy processes.

Thus, for all $m \in \mathbb{N}$, Theorem F.1 implies that

$$\lim_{n\to\infty, n\in\mathbb{N}} \frac{\ln|Z_{\frac{n}{2^m}}|}{\sqrt{\frac{n}{2^m}}} =_D \lim_{n\to\infty, n\in\mathbb{N}} \frac{-\min_{v\in\{0,\frac{1}{2^m},\cdots,\frac{n}{2^m}\}} \xi_v}{\sqrt{\frac{n}{2^m}}} \tag{F.4}$$

and

$$\lim_{n\to\infty, n\in\mathbb{N}} |Z_{\frac{n}{2^m}}| e^{\min_{v\in\{0,\frac{1}{2^m},\cdots,\frac{n}{2^m}\}} \xi_v} =_D F^{(m)}, \tag{F.5}$$

where $F^{(m)}$ is a continuous proper distribution. Since $\lambda^{(m)} \subset \lambda^{(m')}$ whenever $m < m'$, it is immediate that $F^{(m)} = F$ for all $m \in \mathbb{N}$. For each real $t > 0$, define

$$[t]^{(m)} := \max\{x \in \lambda_m | x \leq t\}.$$

Thus, for each $m \in \mathbb{N}$,

$$\lim_{t\to\infty} |Z_{[t]^{(m)}}| = \lim_{n\to\infty, n\in\mathbb{N}} |Z_{\frac{n}{2^m}}|$$

and

$$\lim_{t\to\infty} e^{\inf_{s\leq t}\xi_s} = \lim_{n\to\infty, n\in\mathbb{N}} e^{\min_{v\in\{0,\frac{1}{2^m},\cdots,\frac{n}{2^m}\}}\xi_v}.$$

However, by the càdlàg property of Lévy processes, for each $t > 0$,

$$\lim_{m\to\infty, m\in\mathbb{N}} |Z_{[t]^{(m)}}| = Z_{t-} \quad \text{a.s.}$$

and

$$\lim_{m\to\infty, m\in\mathbb{N}} e^{\min_{v\in\{0,\frac{1}{2^m},\cdots,[t]^{(m)}\}}\xi_v} = e^{\inf_{s\leq t-}\xi_s} \quad \text{a.s.}$$

Using these limits, equations F.1 and F.2 follow from equations F.4 and F.5 respectively. □

In Section 1.6 we presented some asymptotic results on $\sup_{t>0} Z_t$ which are stated in Nyrhinen [52]. We commented that the conditions are stated in terms of the distribution of Z, rather than the marginal measures of ξ and η, which makes the results quite inaccessible. We now present a modified version of these results in which moment conditions are given on ξ and η. In addition, the asymptotic results are given for $\inf_{t>0} Z_t$ rather than $\sup_{t>0} Z_t$, which fits better with our work on ruin probability for the GOU. In this context, Nyrhinen's condition $\bar{y} = \infty$ becomes the condition $\psi(z) > 0$ for all $z \geq 0$. Note that Theorem 2.1 states iff conditions on the Lévy measure of (ξ, η) such that this condition holds.

We first need to define some notation. Let $c(\alpha) := \ln E\left(e^{-\alpha\xi_1}\right)$, $\alpha \in \mathbb{R}$. By Proposition 2.3 of [17], and our assumption that ξ is not identically zero, the function $c(\alpha)$ is strictly convex and continuously differentiable on the interior of its domain of finiteness. Let c^* be the Fenchel-Legendre transform of c, namely $c^*(v) := \sup\{\alpha v - c(\alpha) : \alpha \in \mathbb{R}\}$, $v \in \mathbb{R}$. Let

$$(M, Q, L) := \left(e^{-\xi_1}, Z_1, e^{\xi_1}\left(\inf_{0\leq s\leq 1} Z_s - Z_1\right)\right). \tag{F.6}$$

and define the constant

$$t_0 := \sup\left\{\alpha \in \mathbb{R} : c(\alpha) < \infty,\ E\left(|Q|^\alpha\right) < \infty,\ E\left(\left(ML^-\right)^\alpha\right) < \infty\right\} \in [0, \infty]. \tag{F.7}$$

By the strict convexity of c, if $E\left(e^{-w\xi_1}\right)=1$ for some $w>0$ then $c'(w)>0$ and we define the constant $x_0 := \lim_{t\to t_0-}\left(1/c'(w)\right)\in[0,\infty)$. Recall that a distribution is *spread out* if it has a convolution power with an absolutely continuous component.

Theorem F.3. *Suppose $\psi(z)>0$ for all $z\geq 0$, there exists $w>0$ such that $E\left(e^{-w\xi_1}\right)=1$ and there exist $\epsilon>0$ and $p,q>1$ with $1/p+1/q=1$ such that $E\left(e^{-\max\{1,w+\epsilon\}p\xi_1}\right)<\infty$ and $E\left(|\eta_1|^{\max\{1,w+\epsilon\}q}\right)<\infty$. Then the function*

$$R(x) := \begin{cases} xc^*(1/x) & \text{for } x\in(x_0,\frac{1}{c'(w)}),\\ w & \text{for } x\geq\frac{1}{c'(w)}.\end{cases}$$

is finite and continuous on (x_0,∞) and strictly decreasing on $(x_0,\frac{1}{c'(w)})$, and we have

$$\lim_{z\to\infty}\frac{\ln P\left(T_z\leq x\ln z\right)}{\ln z}=-R(x) \tag{F.8}$$

for every $x>x_0$. In addition,

$$\lim_{z\to\infty}\frac{\psi(z)}{\ln z}=-w. \tag{F.9}$$

If, further, the distribution of ξ_1 is spread out, then there exist constants $C_->0$ and $\gamma>0$ such that

$$z^w\psi(z)=C_-+o\left(z^{-\gamma}\right)\quad\text{as } z\to\infty. \tag{F.10}$$

Proof. Suppose the conditions in the first sentence of the theorem hold. Define an iid sequence $(M_n,Q_n,L_n)_{n\geq1}$ by choosing M_n and Q_n as in equation (F.3), and letting

$$L_n := e^{\xi_n}\left(\inf_{n-1<t\leq n}\int_{(n-1)+}^{t}e^{-\xi_{s-}}\mathrm{d}\eta_s-\int_{(n-1)+}^{n}e^{-\xi_{s-}}\mathrm{d}\eta_s\right).$$

Define the sequence

$$X_n := \sum_{i=1}^{n}\prod_{j=1}^{i-1}M_jQ_i+\prod_{j=1}^{n}M_jL_n.$$

For our choices of (M_n,Q_n,L_n), we have already noted, for all $n\in\mathbb{N}$, that

$$Z_n=\sum_{i=1}^{n}\prod_{j=1}^{i-1}M_jQ_i.$$

Thus,

$$\begin{aligned} X_n &= Z_n + e^{-\xi_n} L_n \\ &= Z_{n-1} + \inf_{n-1<t\le n} \int_{(n-1)+}^{t} e^{-\xi_{s-}} \mathrm{d}\eta_s \\ &= \inf_{n-1<t\le n} Z_t, \end{aligned}$$

which implies that

$$\inf_{0\le t\le n} Z_t = \min_{m=1,\cdots,n} X_m.$$

In [52], Nyrhinen presents asymptotic results for $\max_{m=1,\cdots,n} X_m$, which we presented as Theorem 1.21. The obvious reciprocal version of this theorem ensures that equations (F.8) and (F.9) hold, if we can prove the relevant conditions are satisfied for

$$(M,Q,L) := \left(e^{-\xi_1}, Z_1, e^{\xi_1}\left(\inf_{0\le s\le 1} Z_s - Z_1\right)\right).$$

Namely, we must show that

$$\inf\left\{z\in\mathbb{R} : P\left(\inf_{t>0} Z_t < z\right) > 0\right\} = -\infty,$$

and

$$0 < \sup\{\alpha\in\mathbb{R} : c(\alpha)\le 0\} < t_0 \le \infty$$

where t_0 is defined in (F.7).

The first of these conditions is equivalent to the requirement that $\psi(z) > 0$ for all $z \ge 0$, which we have assumed as a theorem condition. By Proposition 2.3 of [17] the function $c(\alpha)$ is convex on its domain of finiteness, and the convexity is strict unless the distribution of ξ is degenerate. Thus, it is clear that our assumed moment conditions imply that $w = \sup\{\alpha\in\mathbb{R} : c(\alpha)\le 0\}$ and $w+\varepsilon \le t_0$, and hence the second condition holds. To see that $w+\epsilon \le t_0$, simply use the fact that

$$\left(\inf_{0\le s\le 1} Z_s - Z_1\right)^{-} \le 2 \sup_{0\le s\le 1} |Z_s|$$

and then apply Lemma 3.24.

If we suppose, further, that ξ_1 is spread out then the symmetric version of equation (F.10) follows immediately from Nyrhinen's comments in [52], which we have expressed as Proposition 1.22. Alternatively, it is simple to directly prove that the conditions of Theorem 6.3 in Goldie [24] hold, and then adjust the subsequent formula to obtain the reciprocal version of (F.10), as explained after Proposition 1.22. □

Remark F.4. 1. The conditions of Theorem F.3 imply that Z_t converges a.s. to a finite random variable Z_∞, as $t \to \infty$. This follows from the comments made relating to Nyrhinen [52] in Section 1.6. Specifically, with M_n and Q_n defined as above, we showed that conditions in hypothesis H imply that the sequence $\sum_{i=1}^{n} \prod_{j=1}^{i-1} M_j Q_i$, and hence the sequence Z_n, converges a.s. to a finite random variable as $n \to \infty$.

2. The final result of the theorem, equation (F.10) is also stated by Paulsen in [57], for independent ξ and η. We presented this result as Theorem 1.24. Our version of the result works in the general case, and also requires simpler and fewer conditions. In particular, we don't have to assume extra conditions that ensure the convergence of Z_t. As shown above, the existing moment conditions already ensure convergence.

3. With (M, Q, L) as above, the value C_- in equation (F.10), is given by the formula defined in (2.19) of Goldie, namely

$$C_- = \frac{1}{w\alpha} E\left(\left(\left(Q + M \min\left(L, \inf_{t>0} Z_t\right)\right)^-\right)^w - \left(\left(M \inf_{t>0} Z_t\right)^-\right)^w\right), \tag{F.11}$$

where $\alpha := E\left(|M|^w \ln |M|\right)$. When ξ and η are independent, it was pointed out by Paulsen [57], and explained in Section 1.6, that this constant can be written in a slightly different form. Using Theorem 2.4, the same approach works in the dependent case. Namely, let $G(z) := P(Z_\infty \leq z)$ and $h(z) := E\left(G(-V_{T_z}) | T_z < \infty\right) \in [0, 1]$ and $h := \lim_{z \to \infty} h(z)$. Then

$$C_- = \frac{1}{w\alpha h} E\left(\left((MZ_\infty + Q)^-\right)^w - \left((MZ_\infty)^-\right)^w\right).$$

4. As in Paulsen [57], the requirement that ξ_1 is spread out, can be replaced with the more lenient requirement that ξ_T is spread out, where T is uniformly distributed on $[0, 1]$ and independent of ξ. To see that this replacement holds, note that we can define (M_n, Q_n, L_n) in terms of increments with distribution T and repeat the proof. Since T is uniformly distributed on $[0, 1]$ and independent of ξ, the moment conditions on (ξ_T, η_T) are equivalent to the original moment conditions on (ξ_1, η_1).

Remark F.4. 1. The conditions of Theorem F.3 imply that Z_t converges a.s. to a finite random variable Z_∞ as $t \to \infty$. This follows from the comments made relating to Vervaat [52] in Section 1.6. Specifically, with M_t and Q_t defined as above, we showed that conditions in hypothesis H imply that the sequence $\sum_{i=1}^{n} \prod_{j=1}^{i-1} M_j Q_i$ and hence the sequence Z_n, converges a.s. to a finite random variable as $n \to \infty$.

2. The final result of the theorem, equation (F.10) is also stated by Paulsen in [37], for independent ξ and η. We presented this result as Theorem 1.23. Our version of the result works in the general case, and also requires simpler and fewer conditions. In particular, we don't have to assume extra conditions that ensure the convergence of Z_t. As shown above, the existing moment conditions already ensure convergence.

3. With (M, Q, L) as above, the value C_+ in equation (F.10), is given by the formula defined in (2.10) of Goldie, namely

$$C_+ = \frac{1}{\kappa\alpha} E\left(\left(Q + M \sup_{t\geq 0}\left(Z_t\right)\right)^{\kappa} - \left(M \inf_{t\geq 0} Z_t\right)^{\kappa}\right) \tag{F.11}$$

where $\alpha := E\left(|M|^\kappa \ln|M|\right)$. When ξ and η are independent, it was pointed out by Paulsen [37], and explained in Section 1.6, that this constant can be written in a slightly different form. Using Theorem 2.4, the same approach works in the dependent case. Namely, let $G(\cdot) = P(Z_\infty \leq \cdot)$ and $h(t) = E(G(-t/M); M < 0) \in [0,1]$ and $h = \lim_{t\to\infty} h(t)$. Then

$$C_+ = \frac{1}{\kappa\alpha} E\left(\left(WZ_\infty + Q\right)^{\kappa} - \left(WZ_\infty\right)^{\kappa}\right)$$

4. As in Paulsen [37], the requirement that ξ_1 is spread out can be replaced with the more lenient requirement that ξ_T is spread out, where T is uniformly distributed on $[0,1]$ and independent of ξ. To see that this replacement holds, note that we can define (M_t, Q_t, L_t) in terms of increments with distribution T and repeat the proof. Since T is uniformly distributed on $[0,1]$ and independent of ξ, the moment conditions on (ξ_T, η_T) are equivalent to the original moment conditions on (ξ_1, η_1).

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