Singular and supersingular operators on function spaces, approximation and extrapolation.

Sergey S. Ajiev

A thesis submitted for the Degree of Doctor of Philosophy of The Australian National University.

March, 2004.

To my close and direct relatives

The thesis is applicant's original work.

Sergey S. Ajiev

....

Canberra, March, 2004.

Acknowledgements

The thesis work was conducted at the Mathematical Sciences Institute of the Australian National University. The author is most grateful to his supervisor, Professor Alan M^{c}Intosh, for his flexible synthesis of both taking care of his students and treating them as matured fellows. The former includes even development of his students' research networks. The author has also benefited essentially from the supportive nature of Professor M^{c}Intosh's welcome personality and his phenomenal ability to introduce others into a part of his research interests engendering genuine interest.

The author owes also a lot of thanks to the members of his supervisory panel, Professor Neil Trudinger and Doctors Christian Le Merdy, Adam Sikora and Xu-Jia Wang, for not only the active, but also the passive forms of their support.

Completely and partly the draft of the thesis has been read by Professors Alan M^{c}Intosh and Terence Tao correspondingly. Their comments and suggestions have made the text much more convenient for a reader, and the author would like to appreciate their efforts.

One cannot also overestimate the role of all the teachers of mathematics in author's life, both direct and indirect, in laying the basis of his mathematical knowledge.

Special thanks are due to Doctor Denis Labutin, under whose influence the application for the PhD study itself was conceived and, later, the applicant had accommodated himself to the new environment.

It is appreciated by the author that this environment was containing authors colleagues, Doctors Christian Rao, Zengjian Lou, Julie Clutterbuck and Chris Wetherell, PhD student's convenor, Doctor Ben Andrews, academic staff members, Professor Joe Gani and Doctors Vladimir Mangazeev and Alexander Isaev, administrative staff members, Mrs Annette Hughes, Mrs Verdune Biles, IT support staff members, Mr Nick Guoth and Mr Charlie Wilson.

The author wish to thank all the people not named here which provided a piece of help during the last years.

An essential author's gratitude is expressed to the Mathematical Sciences Institute, the Australian National University and the Department of Education, Science and Training of the Australian Government for the financial support.

Abstract

Σ. In the first chapter, a uniform approach to the boundedness of singular integral operators based on several parameterized classes of new conditions covering, particularly, Hörmander condition and its variations is exposed. As applications, we show the existence of functional calculus and corresponding Littlewood-Paley-type decompositions for wide classes of ω-accretive operators.

The second chapter is devoted to anisotropic supersingular integral operators and approximation properties of their compositions with pointwise multipliers. We introduce a variant of the classification of anisotropic singular and supersingular (integral) operators. Characterizations of the class of linear differential operators with variable locally bounded coefficients are established. An approximation formula generalizing the extensions of Leibniz rule due to A. Calderon, T. Kato, M. Murray and G. Ponce but suitable for non-convolution supersingular operators is obtained. Necessary conditions for anisotropic convolution supersingular operators with homogeneous kernels and sufficient conditions for general ones are found for the residual of the approximation formula to be bounded from a Lebesgue space into another Lebesgue space, or a smooth space of Besov, or Lizorkin-Triebel type. Applications include approximation properties of general convolution supersingular operators with homogeneous kernels, an operator related to the heat operator, Liouville and fractional differentiations and their compositions with Riesz transforms, covering criteria in some cases.

For these purposes, we establish an extrapolation theorem, use properties of $\mathcal{AD}$-classes, a counterpart of Carleson inequality and introduce and study some properties of new and old anisotropic spaces of smooth functions of Besov and Lizorkin-Triebel types defined in terms of the $\mathcal{D}$-functional measuring the opportunity of a local approximation by polynomials in a Lebesgue metric.

In Chapter III, the boundedness of anisotropic singular integral operators with the domains of definition and ranges in various anisotropic spaces of Banach-valued functions is analyzed from a unified point of view in the style of Calderón-Zygmund theory. We also consider the setting of "double" vector-valued functions and deal with the most general $\mathcal{AD}$-classes covering sub-additive and non-integral operators.

Chapter IV contains results related to the extrapolation of H^∞-calculus of a particular case of the generalized Dirac operators considered by A. Axelsson, S. Keith and A. M^{c}Intosh in connection with the Square root problem of Kato.

Contents

Preface

Singular and supersingular operators, such as Hilbert and Riesz transforms, integer and fractional differentiation, Laplacian and their non-homogeneous counterparts, enter our lives as key components of a multitude of physical models. Establishing their boundedness means observing determination and stability of the corresponding model, while approximation reduces unknown objects to known ones.

The thesis intends to widen our knowledge on these properties of singular and supersingular operators. Results of such type can be applied, for example, to studying Euler, Korteweg-de Vries and Navier-Stokes equations governing, in particular, such a dangerous natural phenomena as tsunami and Amazon river reverse flow and playing one of the most important roles in meteorology, oceanography, aerodynamics, cardiovascular science, thermo-hydraulics and petroleum industry.

There is a vast literature on the subject lacking, nevertheless, the existence of a uniform approach. The intent to remove this drawback underpins this work.

The scope of the thesis include:

a) establishing new general sufficient conditions (i.e. $\mathcal{AD}$-classes covering the known ones) for the extrapolation of scalar and vector-valued singular and supersingular operators acting between a variety of anisotropic function spaces in the style of Calderón-Zygmund theory;

b) developing a novel viewpoint on the notion of singularity both in qualitative and quantitative senses;

c) finding necessary and sufficient conditions for the opportunity to approximate some important operators with simpler ones (in particular, studying commutators);

d) investigating some narrow questions in the issues mentioned above, such as subadditive operators, "double" vector-valued extension and restricted range extrapolation;

e) studying applications of our results to the extrapolation of the functional calculus of ω-accretive operators, including a generalized Dirac operator, to Lebesgue spaces and obtaining Littlewood-Paley types theorems in terms of this functional calculus.

In the first chapter, the ideas of our approach are displayed on simple examples by extending well known results in their settings. In addition, we discuss immediate applications of our results and relations of our conditions with known ones.

Beginning with the second chapter, we consider the setting of spaces of anisotropic functions. The chapter contains, in particular, general forms of the definitions of $\mathcal{AD}$ and $\mathcal{RAD}$-classes in the case of scalar-valued functions, an extrapolation theorem, the viewpoint on the notion of singularity both in qualitative and quantitative senses and necessary and sufficient conditions for the existence of a nice approximation of the composition $T \circ M_g$ of a supersingular operator T and the pointwise multiplication M_g by a function g. There are also a very general extension of the Leibniz rule and applications to convolution supersingular integral operators with homogeneous kernels and commutators including the answer to a question of S. Hofmann.

The third chapter is almost completely devoted to the extrapolation results in the setting of vector-valued functions. Particularly, we introduce the widest $\mathcal{AD}$-classes of singular operators and cover the restricted range extrapolation of sub-additive operators. There is also a scheme of the "double" vector-valued extension of singular operators extending and generalizing the corresponding results of A. Benedek, A. P. Calderón, R. Panzone, J. Bourgain, J. L. Rubio De Francia, F. J. Ruiz and J. L. Torrea.

Chapter IV contains results related to the extrapolation of H^∞-calculus of a particular case of the generalized Dirac operators considered by A. Axelsson, S. Keith and A. M^{c}Intosh in connection with the Square root problem of Kato and embedding theorems for this operator. We also show why the resolvent approach was preferred before the traditional semigroup one and derive high order counterparts of the Hilbert identity for resolvents.

Because each chapter has its independent numbering, we mention the chapter number only referring to theorems, lemmas, corollaries, examples, or remarks from other chapters. The formulas are numbered independently in every proof. At every particular occasion, constant C means a fixed positive value not depending on the other entries of the formula, unless stated in the form $C = C(\alpha, \beta, ...)$.

All the references are numbered in the order of their appearance.

CHAPTER 1

On a Uniform Approach to Singular Integral Operators

1. Introduction

The main goal of this chapter[1] is to display the idea of a unified point of view on sufficient conditions formulated in the style of the Hörmander one for the boundedness of singular integral operators and to motivate its usefulness by means of comparison with similar known results (including the theory of Calderón-Zygmund operators). In particular, we introduce $\mathcal{AD}$-classes of singular integral operators which extend and generalize Calderón-Zygmund operators and closely related operators possessing H^∞-calculus. In a line with this main purpose, formulations of assertions are partly included also in more general forms. We consider also the definitions of $\mathcal{AD}$-classes in reduced forms while the complete ones are represented in Chapter III. It also contains results on boundedness of singular integral operators (SIO) from one smooth function space into another (corresponding to the "upper case" in the sense of Section 3 below).

The theory of singular integral operators has a half-century background of intensive development. The main ingredient — decomposition lemma — appeared in 1952 thanks to A.P. Calderón and A. Zygmund (see [**2**]). In 1960, L. Hörmander (see [**3**]) introduced his "cancellation condition", and, since that time, the notion of singular integral operator (SIO) is understood as follows.

Traditionally, a SIO T is an integral operator defined, in a sense, by means of the kernel K, s.t. $T \in \mathcal{L}(L_{p_0}, L_{p_0})$:

$$Tf(x) = p.v. \int K(x,y) f(y) dy.$$

The Hörmander condition states that for any $y, z \in \mathbb{R}^n$, and some $C > 0$

$$\int_{|x-z|\geq 2|y-z|} |K(x,y) - K(x,z)| dx \leq C_H < +\infty. \qquad \text{(class } \mathcal{H}\text{)}$$

[1]The content of this chapter is published in [**1**]

Every SIO satisfying ($\mathcal{H}$) (from the class $\mathcal{H}$) admits a bounded extension from L_p to $L_p, 1 < p < \infty$, from H_1 to L_1 and from L_1 to $L_{1,\infty}$ (weak-L_1). In addition, the adjoint operator is bounded from L_∞ to BMO.

One can point out that another condition weaker then $\mathcal{H}$ was presented by X. Duong and A. M[c]Intosh (1999) in [**4**], and our approach permits to weaken it in the same settings (see $\mathcal{AAD}$-conditions in [**5**] and Chapter III).

Nowadays the following definition of Calderón-Zygmund operator (CZO) is the most commonly accepted.

A CZO is a SIO T satisfying for some $0 < \delta < 1$: *a)* $|K(x,y)| < C/|x-y|^n$;

b) $|K(x,y) - K(x,z)| \le C|y-z|^\delta |x-z|^{-(n+\delta)}$ for $|x-z| \ge 2|y-z|$

We are not imposing absolute value conditions like *a)* at all, but one of the $\mathcal{AD}$-classes introduced here contains conditions which are equivalent, or weaker than the above mentioned ones. Namely, $\mathcal{AD}_x(L_1, \infty, 0, 0, 0)$ is equivalent to the Hörmander integral condition, and $\mathcal{AD}_x(L_\infty, \infty, \delta, \delta, \delta)$ in Definition 2.5 is weaker then property *b)* of Calderón-Zygmund operators.

Anisotropic counterpart of Calderón-Zygmund theory for convolution operators was obtained by O.V. Besov, V.P. Il'in and P.I. Lizorkin in [**6**] (1966).

In 1972, C.L. Fefferman and E.M. Stein (see [**7**]) proved (particularly) $H_1 - L_1$ and $L_\infty - BMO$-boundedness of Calderón-Zygmund operators.

Let us pay more attention to the H_p-theory of SIOs.

R. Coifman (see [**8**]) obtained (1974) $H_p - H_p$ boundedness of CZO for the case of one dimension. In 1986, J. Alvarez and M. Milman (see [**9**]) established $H_p - H_p$-boundedness excluding the limiting cases (i.e. $p > n/(n+\delta), 0 < \delta < 1$). More precisely their assertion reads as follows: a CZO T satisfying the orthogonality condition $T^\star \mathcal{P}_0 = 0$ is bounded on H_p. Here the orthogonality condition $T^\star \mathcal{P}_N = 0$ means $\int x^\alpha T a = 0$ for any $|\alpha| \le N$, and $a \in C_0^\infty$ orthogonal to $\mathcal{P}_N$, where $\mathcal{P}_N$ is the space of all polynomials of the degree no greater than N.

An extension of this result to $0 < p < 1$ was pointed out by several authors: a δ-CZO satisfying $T^\star \mathcal{P}_{[\delta]}$ is bounded on H_p for $0 < p \le 1$. Next we recall the definition of δ-CZO.

Let $s = 1$ for $\delta \in \mathbb{N}$, and $s = \{\delta\} := \delta - [\delta]$ otherwise. Let T be a SIO, then it is δ-CZO if it satisfies *a*) $|K(x,y)| < C/|x-y|^n$;

b) $|D^\alpha_y K(x,y) - D^\alpha_y K(x,z)| \le C|y-z|^s|x-z|^{-(n+|\alpha|+s)}$ for $|x-z| \ge 2|y-z|, |\alpha| = [\delta]$.

Similarly to the case of CZOs, some of the presented $\mathcal{AD}$-conditions (for example, $\mathcal{AD}_x(\infty, L_\infty, l_\infty, \delta, \delta, \delta)$) in this note are weaker then condition *b*) of the δ-CZOs. We provide a direct analogs of the Hörmander condition in this case too (e.g. $\mathcal{AD}_x(u, L_q, l_\infty, \delta, \delta, \delta)$, $u, q \in [1, \infty)$).

One should add that J. Alvarez (1992) (see [**10**]) showed the lack of $H_p - L_p$ (and $H_p - H_p$) boundedness for $p = n/(n+\delta)$. In 1994, D. Fan (see [**11**]), exploiting Littlewood-Paley-theory approach, considered the limiting case for a convolution δ-CZO T, that is, he demonstrated that, under the above conditions, T is bounded from H_p to $H_{p,\infty}$.

R. Fefferman and F. Soria (1987) (see [**12**]) proved $H_{1,\infty} - L_{1,\infty}$-boundedness for a convolution SIO T satisfying the following Dini condition:

$$\int_0^{1/2} \Gamma(t)dt/t < \infty, \text{ where } \Gamma(t) = \sup_{h\neq 0} \int_{|x|>2|h|/t} |K(x-h) - K(x)|dx.$$

In 1988 (publ. 1991), using a similar approach, H. Liu (see [**13**]) investigated boundedness properties of a convolution SIO (in particular, a CZO) in the setting of homogeneous groups and obtained the following results:

a) $H_p = H_{p,\infty}$-boundedness for CZO (without condition *a*)), if $n/(n+1) < p < 1$;

b) $H_{p,\infty} - L_{p,\infty}$-boundedness, if $n/(n+1) < p \le 1$, for SIO T satisfying

$$\int_0^{1/2} \Gamma(t)^p |\log t| t^{np-1-n} dt < \infty, \text{ where}$$

$$\Gamma(t) = \sup_{h\neq 0} \int_{|h|/t<|x|<4|h|/t} |K(x-h) - K(x)|dx;$$

c) $H_{p,\infty} - H_{p,\infty}$-boundedness, if $n/(n+1) < p < 1$, for an ω-CZO (without cond. *a*)), that is for a SIO T satisfying

$$|K(x-y) - K(x)| < C|x|^{-n}\omega(|y|/|x|), |x| > 2|y|,$$

where ω is a nondecreasing function with

$$\int_0^{1/2} t^{n-n/p-1} |\log t|^{2/p+\varepsilon} \omega(t)dt < +\infty \text{ for some } \varepsilon > 0.$$

But, for $\omega(t) = t^s$, one needs $s > n/p - n$, i.e. a nonlimiting case.

The theorems in this, the second and third chapters contain extensions and additions to these results.

It is interesting to point out the "off-diagonal" case of the Calderón-Zygmund-Hörmander result on boundedness of a SIO proved in 1961 by J.T. Schwartz (see [**14**] and an extension due to H. Triebel [**15**]): for $1 < p_0 \leq r_0 < \infty, 1 \leq q,\ 1 + 1/r_0 = 1/p_0 + 1/q$, suppose that a convolution operator T with kernel K is bounded from L_{p_0} to L_{r_0}

$$\text{and satisfies} \int_{|x|>2|y|} |K(x-y) - K(x)|^q dx \leq C.$$

Then T is bounded from L_p to L_r for $1 < p \leq p_0,\ 1 + 1/r = 1/p + 1/q$ and from L_1 to $L_{q,\infty}$. But this ("off-diagonal") setting of the SIO theory has not attracted much attention since that time even despite the work [**16**] (1963) of P.I. Lizorkin on (L_p, L_q)-multipliers. All the general forms of the assertion of this note include the "off-diagonal" case.

Section 4 is devoted to some applications of the results considered here and in Chapter III to questions connected with functional calculus, namely: its existence and related Littlewood-Paley type theorems. There we consider the class of operators for which the kernels of their holomorphic semigroups satisfy Poisson-like $\mathcal{AD}$-estimates. In this terms, we provide sufficient conditions for the existence of a functional calculus of some operators in Hardy and other function spaces, extending results of D. Albrecht, X. Duong and A. M^c^Intosh (1995) (see [**17, 18**]). For the limiting values of parameters, the norm estimate (for the bounded extension) of the form

$$\|\phi(T)\|_{\mathcal{L}(X,Y)} \leq C\|\phi\|_{H_\infty}$$

is proved for X being a Hardy space and Y — a Hardy-Marcinkiewicz one, so that $X \neq Y$.

Another application is a continuous, in the sense of [**19**] (1972), form of the Littlewood-Paley theorem in terms of the above mentioned functional calculus. It should be noted that the classical approach to this theorem relied on the properties of Hilbert transforms even in weighted multiple (product) case (see [**20**]) (1967). Instead, we are following the approach of direct application of vector-valued SIO boundedness results used by O.V. Besov (1984) in [**21**] to extend the Littlewood-Paley inequality to the L_p-spaces with mixed norm of functions periodic in some directions, and by X. Duong (1990) [**18**] to extend the existence of a H^∞-functional calculus on L_2 to one in $L_p, p \neq 2$.

2. Definitions and Designations

This section is a simplification of its counterparts in other chapters that reflects the introductory nature of the first chapter.

Assume $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a set E, $n \in \mathbb{N}$, let E^n be the Cartesian product. Let A be a Banach space and denote by $\|\cdot|A\| = \|\cdot\|_A$ the norm in space A. For $t \in (0, \infty]$ let l_t be a (quasi)normed space of sequences with finite (quasi)norm $\|\{\alpha\}|l_t\| = (\sum_i |\alpha_i|^t)^{1/t}$ for $t \neq \infty$, or $\|\{\alpha\}|l_\infty\| = \sup_i |\alpha_i|$; We also use the designation $l_{t,log}$ for the (quasi)normed space of sequences with the finite norm $\|\{\alpha\}|l_{t,log}\| = \|\{\beta\}|l_t\|$, where $\beta_j = \sum_{i \geq j} |\alpha_i|$. For $p \in (0, \infty)$, let us assume that its adjoint p' is defined by the relation $l_{p'} := (l_p)^*$.

For a measurable subset G of $\mathbb{R}^n$, let $X(G, A)$ be a function space of all (strongly) measurable functions $f : G \to A$ with some finite quasiseminorm $\|\cdot|X(G, A)\|$. In particular, for $p, q \in (0, \infty]$ let $L_{p,q}(G, A)$ be the Bochner-Lebesgue-Lorentz space of all (strongly) measurable functions $f : G \to A$ with the finite norm $\|f|L_{p,q}(G, A)\| = \|\|f\|_A|L_{p,q}(G)\|$, where

$$\|g|L_{p,q}(G)\| := \left(\int_0^\infty |\{x \in G : |f(x)| > t\}|^{q/p} t^{q-1} dt\right)^q \quad \text{for } q < \infty \text{ and}$$

$$\|g|L_{p,\infty}(G)\| := \sup_{t>0} t|\{x \in G : |f(x)| > t\}|^{1/p}.$$

Let $Q_0 := [-1, 1]^n$, $Q_t(z) := z + tQ_0$ for $t > 0, z \in \mathbb{R}^n$. Let $\mathcal{P}_\lambda(A)$ be the space of polynomials $\{\sum_{|\alpha| \leq \lambda} c_\alpha x^\alpha : c_\alpha \in A\}$.

As in [**22**], we define the operators of translation $\tau_a : f(\cdot) \mapsto f(\cdot + a)$ and contraction $\sigma_\kappa : f(\cdot) \mapsto f(\kappa \cdot)$ for arbitrary $a \in \mathbb{R}^n$ and $\kappa \in \mathbb{R}_+$.

DEFINITION 2.1. For $u \in [1, \infty], t > 0, \lambda \geq 0, x \in \mathbb{R}^n$, $f \in L_{1,loc}(\mathbb{R}^n, A)$, we shall refer to the following local approximation functional by means of polynomials as to the $\mathcal{D}$–functional:

$$\mathcal{D}_u(t, x, f, \lambda, A) = t^{-n/u} \|f - P_{t,x,\lambda} f | L_u(Q_t(x), A)\|,$$

where $P_{t,x,\lambda} := \tau_x^{-1} \circ \sigma_t^{-1} \circ P_{0,0,\lambda} \circ \sigma_t \circ \tau_x$, and $P_{0,0,\lambda} : L_u(Q_1(0)) \to \mathcal{P}_\lambda(A)$ is a surjective projection. For simplicity, we shall understand $\mathcal{D}_u(t, x, f, \lambda)$ to be $\mathcal{D}_u(t, x, f, \lambda, A)$, if $A = \mathbb{R}, \mathbb{C}$. If a function f depends also on the two (vector) variables x, y, $f = f(x, y)$, and $f_{|x=w}(y) := f(w, y)$ then

$$\mathcal{D}_u^y(t, z, f(w, \cdot), \lambda, A) = \mathcal{D}_u(t, z, f_{|x=w}, \lambda, A).$$

Let $C_0^\infty(G)$ be the space of the infinitely differentiable functions compactly supported on the open set G.

Define the local maximal functional of a function f by

$$M(t, x, f) = M(\phi, t, x, f) := \sup\{|t^{-n} \varphi(\cdot/t) * f|(y) : |y - x| \leq t, \varphi \in C_0^\infty(Q_0)\}.$$

DEFINITION 2.2. For $p, q \in (0, \infty]$, let the vector-valued Hardy-Lorentz space $H_{p,q}(\mathbb{R}^n, A)$ be the completion of the quasinormed space of locally summable A-valued functions f with a finite quasinorm

$$\|f|H_{p,q}(\mathbb{R}^n, A)\| := \left\|\sup_{t>0} M(t, \cdot, f)\,|L_{p,q}(\mathbb{R}^n)\right\|.$$

REMARK 1. One should note that $H_{p,q}(\mathbb{R}^n, A) = L_{p,q}(\mathbb{R}^n, A)$ for $p > 1$.

We say that an operator T bounded from $L_{\theta_0}(\mathbb{R}^n)$ into $L_{\theta_1}(\mathbb{R}^n)$ for some $\theta_0, \theta_1 \in (0, \infty]$ is a singular integral operator (SIO) if there exists corresponding to it measurable kernel $K(x, y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ and satisfying $(Tf)(x) := \int K(x, y)f(y)dy$ for a. e. $x \in \mathbb{R}^n \setminus \operatorname{supp} f$ and every $f \in L_\infty(\mathbb{R}^n, A)$ with compact support and if it is in one of the classes defined below.

REMARK 2. More general case of operator-valued kernels corresponding operators T defined on vector-valued functions is considered in Chapter III and used for some applications in Section 4.

DEFINITION 2.3. Assume $\lambda_0, \lambda_1, \gamma \in [-n, \infty)$, $u, q, q_1 \in (0, \infty]$, $\gamma \geq 0$, $\delta > 0, b > 1$. Let $X := X(\mathbb{N}_0)$ be a (quasi)(semi)normed space of sequences, $E^w_{q,q_1,\lambda_1}(\mathbb{R}^n)$ be the weighted Lorentz space with the norm

$$\|f|E^w_{q,q_1\lambda_1}(\mathbb{R}^n)\| = \left\|f(\cdot)|\cdot - w|^{\lambda_1 + n/q'}\,|\,L_{q,q_1}(\mathbb{R}^n)\right\|,$$

and $\Delta_i(r, w) = Q_{\delta r b^{i+1}}(w) \setminus Q_{\delta r b^i}(w), i \in \mathbb{N}$. Then it will be understood that:

a) $T \in \mathcal{AD}_x(u, L_q, X, \lambda_0, \lambda_1, \gamma)$, if

$$\sup_{\substack{w\in\mathbb{R}^n \\ r>0}} \|\{\mu_i(r, w)\}_{i\in\mathbb{N}_0}|X\| =: C_{\mathcal{AD}} < +\infty, \text{ where}$$

$$\mu_i(r, w) := \|r^{-\lambda_0}\mathcal{D}^y_u(r, w, K(\cdot, y)\chi_{\Delta_i}(\cdot), \gamma)|E^w_{q,q,\lambda_1}(\mathbb{R}^n)\|, i \in \mathbb{N}_0;$$

b) $T \in \mathcal{AD}_x(L_q, u, X, \lambda_0, \lambda_1, \gamma)$, if

$$\sup_{\substack{w\in\mathbb{R}^n \\ r>0}} \|\{\mu_i(r, w)\}_{i\in\mathbb{N}_0}|X\| =: C_{\mathcal{AD}} < +\infty, \text{ where}$$

$$\mu_i(r, w) := r^{-\lambda_0}\mathcal{D}^y_u(r, w, K(\cdot, y)\chi_{\Delta_i}(\cdot), \gamma, E^w_{q,q,\lambda_1}(\mathbb{R}^n)), i \in \mathbb{N}_0;$$

c) $T \in \mathcal{AD}_x(u, L_{q,q_1}, \lambda_0, \lambda_1, \gamma)$, if

$$\sup_{\substack{w\in\mathbb{R}^n \\ r>0}} \mu(r, w) =: C_{\mathcal{AD}} < +\infty, \text{ where}$$

$$\mu(r, w) := \|r^{-\lambda_0}\mathcal{D}^y_u(r, w, K(\cdot, y)|\cdot - w|^{\lambda_1 + n/q'}, \gamma)|L_{q,q_1}(\mathbb{R}^n \setminus Q_{r\delta}(w))\|;$$

d) $T \in \underline{\mathcal{AD}_x(L_{q,q_1}, u, \lambda_0, \lambda_1, \gamma)}$, if

$$\sup_{r>0}^{w\in\mathbb{R}^n} \mu(r,w) =: C_{\mathcal{AD}} < +\infty, \text{ where}$$

$$\mu(r,w) := r^{-\lambda_0}\mathcal{D}_u^y(r,w,K(\cdot,y)\chi_{\mathbb{R}^n\setminus Q_{r\delta}(w)}(\cdot),\gamma,E^w_{q,q_1\lambda_1}(\mathbb{R}^n)).$$

The infimum of constants C in each case will be designated by means of $C_{\mathcal{AD}}$ for the corresponding $\mathcal{AD}$-condition.

REMARK 3. It can be noted that the definitions of $\mathcal{AD}$-classes have equivalent continuous forms (see Chapter 3), which means also their independence from the parameter $b > 1$.

DEFINITION 2.4. Let $\gamma_0, \gamma_1 \geq 0$. We assume that an operator T belongs to the class $ORT_x(\gamma_0, \gamma_1)$ if $\int \pi T\phi = 0$ for all $\pi \in \mathcal{P}_{\gamma_1}$ and $\phi \in C_0^\infty$, such that $\int \phi\pi = 0$ for each $\pi \in \mathcal{P}_{\gamma_0}$.

DEFINITION 2.5. For $\Omega \subset \mathbb{C}$ let $\{T(z)\}_{z\in\Omega}$ be a family of integral operators with the corresponding $\mathbb{C}$-valued kernels $\{K_z\}_{z\in\Omega}$, $K_z = K_z(x,y)$, $x, y \in \mathbb{R}^n$. We assume that the family $\{T(z)\}_{z\in\Omega}$ satisfies Poisson-type $\mathcal{AD}_x$-estimates with parameters $u \in [1,\infty]$, $\lambda \geq 0$ on the domain Ω if for some $\epsilon, m \in (0,\infty)$ and any $w, x \in \mathbb{R}^n$, $z \in \Omega$, $r \in (0,\infty)$

$$\mathcal{D}_u(r,w,K_z(x,\cdot),\lambda) \leq C\left(\frac{r}{|z|^m}\right)^\lambda |z|^{-mn}\left(1+\frac{|x-w|}{|z|^m}\right)^{-(n+\lambda+\epsilon)},$$

$T(z) \in ORT_x(\lambda,\lambda)$.

And we also understand $K_z(x,y)$ to satisfy Poisson-type $\mathcal{AD}_y$-estimate if $K_z^I(x,y) = K_z(y,x)$ satisfies Poisson-type $\mathcal{AD}_x$-estimate.

DEFINITION 2.6. We assume that operator T defined by the kernel $K(x,y)$ to be in $\mathcal{AD}_y$-class, or $ORT_y(\gamma_0,\gamma_1)$-class if, and only if, the corresponding operator T^I defined by the kernel $K^I(x,y) = K(y,x)$ is in the corresponding $\mathcal{AD}_x$-class, or, correspondingly, $ORT_x(\gamma_0,\gamma_1)$-class.

3. Counterparts of Known (Clasical) Results

Here we study the relation between $\mathcal{AD}$-classes and the Hörmander condition.

For the sake of simplicity, we shall only consider in this section $\mathcal{AD}$-classes with $q = 1$, $X = l_1$ and $\lambda_0 = \lambda_1 = \gamma = 0$.

REMARK 4. In spite of the relation

$$\mathcal{AD}_x(u, L_1, 0, 0, 0) = \mathcal{AD}_x(u, L_1, l_1, 0, 0, 0) \subset$$

$$\subset \mathcal{AD}_x(L_1, u, l_1, 0, 0, 0) \subset \mathcal{AD}_x(L_1, u, 0, 0, 0),$$

the non-coincidence of different types of $\mathcal{AD}$-classes will be discussed in the proofs separately to demonstrate the approach in more general cases.

Let us point out that the class of operators satisfying the Hörmander condition $\mathcal{H}$ is equal to the class $\mathcal{AD}_x(L_1, \infty, l_1, 0, 0, 0)$. We shall show the inclusion $\mathcal{H} \subset \mathcal{AD}_x(L_1, \infty, l_1, 0, 0, 0)$. The opposite one was pointed out to the candidate by A.M^{c}Intosh. Indeed, the corresponding kernels should have a uniformly bounded for any $z \in \mathbb{R}^n$, $r > 0$ quantity

$$A(r, z) = \inf_{c(x)} \sup_{\{y:|y-z|\leq r\}} \int_{|x-z|\geq 2r} |K(x, y) - c(x)|dx.$$

And, on choosing, for fixed z, r, $c(x)$ to be equal to $K(x, z)$, we can find that

$$A(r, z) \leq \sup_{\{y:|y-z|\leq r\}} \int_{|x-z|\geq 2r} |K(x, y) - K(x, z)|dx \leq$$

$$\leq \sup_y \int_{|x-z|\geq 2|y-z|} |K(x, y) - K(x, z)|dx \leq C_H < +\infty,$$

where C_H is the constant in the Hörmander condition (class $\mathcal{H}$).

3.1. Lower "Summability" Case. This subsection contains the case of "Lower" extrapolation, i.e. a SIO of (strong) (p_0, p_0)-type is shown to be, in particular, of (p, p) type for some $p < p_0$.

THEOREM 3.1. *For $p_0 \in (1, \infty]$, $u \in [1, \infty]$, let T be a SIO from $\mathcal{AD}_x(L_1, u, l_1, 0, 0, 0) \cup \mathcal{AD}_x(L_1, u, 0, 0, 0) \cup \mathcal{AD}_x(u, L_1, 0, 0, 0)$, bounded from L_{p_0} into itself. Then,*

a) $T \in \mathcal{L}(L_{p,q}(\mathbb{R}^n))$ for $p \in (1, p_0]$, $q \in (0, \infty]$;

b) $T \in \mathcal{L}(L_1(\mathbb{R}^n), L_{1,\infty}(\mathbb{R}^n))$ if $u = \infty$;

c) $T \in \mathcal{L}(H_1(\mathbb{R}^n), L_1(\mathbb{R}^n))$.

Proof of the Theorem 3.1. We suppose that the kernel $K(x, y)$ corresponds to the operator T. One should note that part $a)$ of the theorem is a consequence of either $b)$ or $c)$ in view of the interpolation properties of the scale of Hardy-Lebesgue spaces (see [**23**]).

To the first, let us recall that the statements of the parts b) and c) are implied as in the classical approach) by the estimate

$$\int_{\mathbb{R}^n\setminus Q_{r\delta}(z)} |Ta|dx \le C_{\mathcal{AD}}, \tag{1}$$

where $r>0, z\in\mathbb{R}^n$ and a is a $(1,L_\infty,0)$-, or a $(1,1,0)$-atom in the case of the part c), or b) correspondingly. Indeed, in the case c), the atomic decomposition result for H_1 (see [**8, 24**]) permits us to prove the boundedness of T on $(1,\infty,0)$-atoms only, that follows from (1) and

$$\int_{Q_{r\delta}(z)} |Ta|dx \le (r\delta)^{n/p_0'}\|Ta\|_{p_0} \le \delta^{n/p_0'}\|T|\mathcal{L}(L_{p_0})\|,$$

where a is an $(1,\infty,0)$-atom. In the case b), for a function $f\in L_1$ and $\lambda>0$, Calderón-Zygmund decomposition of a set $\Omega_\lambda := \{x : Mf>\lambda\} = \bigcup_{i\in\mathbb{N}} Q_i$, where the set $\{\delta Q_i\}$ possess finite intersection property and $C|\Omega_\lambda| \ge \sum_i |Q_i|$, provides representation

$$f = f_0 + C\lambda\sum_i |Q_i|a_i, \text{ where } a_i \text{ is a } (1,1,0)-\text{atom} \tag{2}$$

and $\|f_0|L_\infty\| \le C\lambda$. Therefore, Chebyshev inequality, (1) and just mentioned properties imply

$$\begin{aligned}\lambda\{|Tf_0|>c\lambda\} &\le C\lambda^{1-p_0}\|Tf_0\|_{p_0}^{p_0} \le C\|T|\mathcal{L}(L_{p_0})\|^{p_0}\|f|L_1\|,\\ \lambda\{|Tf_1|>c\lambda\} &\le C\lambda\left(|\cup_i \delta Q_i| + \sum |Q_i|\int_{\mathbb{R}^n\setminus\delta Q_i}|Ta_i|\right) \le\\ &\le C\lambda|\Omega_\lambda| \le C\|f|L_1\|.\end{aligned} \tag{3}$$

To obtain the formula (1) suppose $g(x)$ to be a function minimizing functionals

$$\inf_c \int_{Q_r(z)} |K(x,y)-c|^u dy \tag{4}$$

at a.e. x if $T\in\mathcal{AD}_x(u,L_1,l_1,0,0,0)$, or minimizing the functional

$$\int_{Q_r(z)}\left(\int_{\mathbb{R}^n\setminus Q_{r\delta}(z)}|K(x,y)-c(x)|dx\right)^u dy, \tag{5}$$

if $T\in\mathcal{AD}_x(L_{1,1},u,0,0,0)$, or $g(x)=\sum_i g_i(x)$, where functions $\{g_i(x)\}_{i\in\mathbb{N}}$ to minimize functionals

$$\inf_{c(x)}\left(\int_{Q_r(z)}\left(\int_{Q_{r2^i\delta}(z)\setminus Q_{r2^{i-1}\delta}(z)}|K(x,y)-c(x)|dx\right)^u dy\right)^{1/u} \tag{6}$$

correspondingly if $T\in\mathcal{AD}_y(L_1,u,l_1,0,0,0)$. In view of Minkowski inequality, it follows from (4), (5), or (6) that, correspondingly, for an arbitrarily $(1,1,0)$-(part b)), or $(1,\infty,0)$-atom with support $Q_r(z)$, one has due to the Hölder and Minkowski inequalities, Fubini theorem and the orthogonality of atom a to constants:

$$\int_{\mathbb{R}^n\setminus Q_{r\delta}(z)} |Ta|dx \leq$$

$$\leq Q = \int_{\mathbb{R}^n\setminus Q_{r\delta}(z)} \left| \int (K(x,y) - g(x))a(y)dy \right| dx \leq$$

$$\leq \int_{\mathbb{R}^n\setminus Q_{r\delta}(z)} r^{-n/u} \left(\inf_c \int_{Q_r(z)} |K(x,y) - c|^u dy \right)^{1/u} dx \leq C_{\mathcal{AD}}, \tag{7}$$

$$Q \leq \int_{Q_r(z)} \int_{\mathbb{R}^n\setminus Q_{r\delta}(z)} |K(x,y) - g(x)|dx|a(y)|dy \leq$$

$$\leq \inf_c r^{-n/u} \left(\int_{Q_r(z)} \left(\int_{\mathbb{R}^n\setminus Q_{r\delta}(z)} |K(x,y) - c(x)|dx \right)^u dy \right)^{1/u} \leq C_{\mathcal{AD}}, \tag{8}$$

$$Q \leq \sum_{i\in\mathbb{N}} \int_{Q_r(z)} \int_{Q_{2^i r\delta}(z)\setminus Q_{2^{(i-1)}r\delta}(z)} |K(x,y) - g_i(x)|dx|a(y)|dy \leq$$

$$\leq \sum_{i\in\mathbb{N}} \inf_c r^{-n/u} \left(\int_{Q_r(z)} \left(\int_{Q_{2^i r\delta}(z)\setminus Q_{2^{(i-1)}r\delta}(z)} |K(x,y) - c(x)|dx \right)^u dy \right)^{1/u} \leq C_{\mathcal{AD}}. \tag{9}$$

In this manner, estimates $(7-9)$ motivate (1). $\mathcal{Q.E.D.}$

3.2. Upper "Summability" Case. This subsection contains the case of "Upper" extrapolation, i.e. a SIO of (strong) (p_0, p_0)-type is shown to be, in particular, of (p,p) type for some $p > p_0$.

The next theorem can be derived from the previous one by means of duality considerations but such an approach will not work, in particular, in the case of vector-valued functions, or will require additional duality results to consider scales other than $H_1 - L_p - BMO$. Thus, we provide a proof which does not rely on duality.

THEOREM 3.2. *For* $p_0 \in (1,\infty]$, $u \in [1,\infty)$, *let* T *be a SIO from* $\mathcal{AD}_y(L_1, u, l_1, 0, 0, 0) \cup \mathcal{AD}_y(L_1, u, 0, 0, 0) \cup \mathcal{AD}_y(u, L_1, 0, 0, 0)$, *bounded from* L_{p_0} *into itself. Then,*
a) $T \in \mathcal{L}(L_{p,q}(\mathbb{R}^n))$ *for* $p \in [p_0,\infty)$, $q \in (0,\infty]$;
b) $T \in \mathcal{L}(L_\infty(\mathbb{R}^n), BMO(\mathbb{R}^n))$.

Proof of the Theorem 3.2. We suppose that the kernel $K(x,y)$ corresponds to the operator T. One should note that we need to prove part *b)* only because part *a)* follows from it with the aid of the real interpolation method.

Let us fix $Q_r(z) \subset \mathbb{R}^n$, $f \in L_\infty$ and use the representation $f = f_0 + f_1$, $f_0 = \chi_{Q_{r\delta}(z)}$, where δ is a constant in the definition of the corresponding $\mathcal{AD}_y$-classes. Then the definition of $\mathcal{D}$-functional and L_{p_0}-boundedness of T and the restriction operator $f \longrightarrow f_0$ imply

$$\mathcal{D}_{p_0}(r, z, Tf_0, 0) \leq r^{-n/p_0} \|Tf_0 | L_{p_0}(\mathbb{R}^n)\| \leq r^{-n/p_0} \|T|\mathcal{L}(L_{p_0})\| \times$$

$$\times \|f_0 | L_{p_0}(\mathbb{R}^n)\| \leq \|T|\mathcal{L}(L_{p_0})\| \|f_0|L_\infty(\mathbb{R}^n)\| \leq \|T|\mathcal{L}(L_{p_0})\| \|f|L_\infty(\mathbb{R}^n)\|. \tag{1}$$

Now suppose $g(y)$ to be a function minimizing the functionals

$$\inf_c \int_{Q_r(z)} |K(x,y) - c(y)|^u dx, \tag{2}$$

at a.e. y if $T \in \mathcal{AD}_y(u, L_1, l_1, 0, 0, 0)$, or minimizing the functional

$$\int_{Q_r(z)} \left(\int_{\mathbb{R}^n \setminus Q_{r\delta}(z)} |K(x,y) - c(y)| dy \right)^u dx, \tag{3}$$

if $T \in \mathcal{AD}_y(L_{1,1}, u, 0, 0, 0)$, or $g(y) = \sum_i g_i(y)$, where the functions $\{g_i(y)\}_{i\in\mathbb{N}}$ minimize the functionals

$$\inf_{c(y)} \left(\int_{Q_r(z)} \left(\int_{Q_{r2^i\delta}(z) \setminus Q_{r2^{i-1}\delta}(z)} |K(x,y) - c(y)| dy \right)^u dx \right)^{1/u} \tag{4}$$

correspondingly if $T \in \mathcal{AD}_y(L_1, u, l_1, 0, 0, 0)$. In view of Minkowski inequality, it follows from (2), or (3), or (4) that, correspondingly,

$$\mathcal{D}_u(r, z, Tf_1, 0) \leq r^{-n/u} \left(\int \left(\int_{\mathbb{R}^n \setminus Q_{r\delta}(z)} |(K(x,y) - g(y)) f_1(y)| dy \right)^u dx \right)^{1/u} =$$

$$= Q \leq \int_{\mathbb{R}^n \setminus Q_{r\delta}(z)} \left(\int_{Q_r(z)} |K(x,y) - g(y)|^u dx \right)^{1/u} \|f|L_\infty dy, \text{ or} \tag{5}$$

$$Q \leq \left(\int \left(\int_{\mathbb{R}^n \setminus Q_{r\delta}(z)} |K(x,y) - g(y)| dy \right)^u dx \right)^{1/u}, \text{ or} \tag{6}$$

$$Q \leq \left(\sum_i \left(\int_{Q_r(z)} \left(\int_{Q_{r2^i\delta}(z) \setminus Q_{r2^{i-1}\delta}(z)} |K(x,y) - c(y)| dy \right)^u dx \right)^{1/u} \right) \|f|L_\infty\|. \tag{7}$$

Eventually formulas (1) and one of (5, 6, 7) imply

$$\mathcal{D}_1(r, z, Tf, 0) \leq (C_{\mathcal{AD}} + \|T|\mathcal{L}(L_{p_0})\|) \|f|L_\infty\|. \ \mathcal{Q.E.D.}$$

4. Functional Calculus and Littlewood-Paley-type Theorems

Here we display some applications of the results of the thesis to the existence in a variety of function spaces of the functional calculus of ω-accretive operators with their holomorphic semigroup satisfying the Poisson-type $\mathcal{AD}$-conditions (see §2). In addition, we show the validity of some Littlewood-Paley type theorems formulated in terms of this calculus for the spaces under consideration.

All the definitions and notations regarding functional calculus are understood as in the article [**17**] due to David Albrecht, Xuan Duong and Alan McIntosh, including the following definitions.

For $0 \le \omega < \mu < \pi$ let $S_{\omega+} := \{z \in \mathbb{C} | \, |\arg z| \le \omega\} \cup \{0\}$, $S^0_{\mu+} := \{z \in \mathbb{C} | \, |\arg z| < \mu\}$, space $H(S^0_{\mu+})$ be the space of all holomorphic functions on $S^0_{\mu+}$ endowed with the $L_\infty(S^0_{\mu+})$-norm, containing subspace $\Psi(S^0_{\mu+}) := \{\psi | \psi \in H(S^0_{\mu+}), \exists s > 0, |\psi(z)| \le C|z|^s(1+|z|^{2s})^{-1}\}$. An operator T closed in $L_2(\mathbb{R}^n)$ is said to be of type $S_{\omega+}$ if $\sigma(T) \subset S_{\omega+}$ and for any $\mu > \omega$ there exist C_μ such that

$$|z| \| (T - zI)^{-1} | \mathcal{L}(L_2(\mathbb{R}^n), L_2(\mathbb{R}^n)) \| \le C_\mu, z \notin S_{\omega+}.$$

THEOREM 4.1. *Assume $q \in (0, \infty]$, $r \in (0, \infty]^n$. Let T be a one-one operator of type $S_{\omega+}$ in $L_2(\mathbb{R}^n)$, $\omega \in [0, \pi/2)$, having a bounded functional calculus in $L_2(\mathbb{R}^n)$ for all $f \in H_\infty(S^0_{\mu+})$ for some $\mu > \omega$. Assume that for some $\lambda, m, \epsilon > 0$, $[v_0, \mu] \in (\omega, \pi/2)$ and all $z \in S^0_{(\pi/2-\mu)}$, the kernel $K_z(x,y)$ of holomorphic semigroup e^{-zT} associated with T satisfies one of the conditions:*

a) Poisson-type $\mathcal{AD}_x$-estimate with $u = 1$;

b) Poisson-type $\mathcal{AD}_y$-estimate with $u = 1$;

c) both Poisson-type $\mathcal{AD}_x$- and $\mathcal{AD}_y$-estimates with $u = 2$.

Then, correspondingly, for $f \in H_\infty(S^0_\nu)$ for all $\nu > \mu$, $f(T)$ can be extended to be in:

a) $$\bigcup_{p \in ((1+\lambda/n)^{-1}, 2]} \mathcal{L}(H_{p,q}(\mathbb{R}^n), H_{p,q}(\mathbb{R}^n)) \text{ with } \|f(T)|\mathcal{L}(H_{p,q}, H_{p,q})\| \le C\|f|L_\infty\|$$

and $\mathcal{L}(H_{p_0}(\mathbb{R}^n), H_{p_0,\infty}(\mathbb{R}^n))$ *with* $\|f(T)|\mathcal{L}(H_{p_0}(\mathbb{R}^n), H_{p_0,\infty}(\mathbb{R}^n))\| \le C\|f|L_\infty\|$ *for* $p_0 = (1 + \lambda/n)^{-1}$, $\lambda \notin \mathbb{Z}$;

b) $$\bigcup_{p \in [2,\infty)} \mathcal{L}(L_{p,q}(\mathbb{R}^n), L_{p,q}(\mathbb{R}^n)) \text{ with } \|f(T)|\mathcal{L}(L_{p,q}, L_{p,q})\| \le C\|f|L_\infty\| \text{ and}$$

$$\bigcup_{\gamma\in[0,\lambda),T\in\mathcal{ORT}_y(\gamma,\gamma)} \mathcal{L}(X^\gamma, X^\gamma) \text{ with } \|f(T)|\mathcal{L}(X^\gamma, X^\gamma)\| \le C\|f|L_\infty\|;$$

for $X^\gamma \in \left\{b^\gamma_{r,q}(\mathbb{R}^n), l^\gamma_{r,q}(\mathbb{R}^n)\right\}$;

c) if, in addition, functional $\Psi_{\nu_0}(f) = \int_{\Gamma_{\nu_0}} \frac{|f(\zeta)|}{|\zeta|}|d\zeta|$ *is finite for some*

$$\Gamma_{\nu_0} = \Theta(t)te^{(\pi-\nu_0)/2} + (\Theta(t)-1)te^{(\nu_0-\pi)/2}, t\in\mathbb{R},$$

then the following Littlewood-Paley-type estimates is true: for $\gamma\in[0,\lambda)$, $T\in\mathcal{ORT}_y(\gamma,\gamma)$, $p\in((1+\lambda/n)^{-1},\infty)$ *and*

$$X(\mathbb{R}^n)\in\left\{H_p(\mathbb{R}^n), b^\gamma_{\infty,\infty}(\mathbb{R}^n), b^\gamma_{r,q}(\mathbb{R}^n), l^\gamma_{r,q}(\mathbb{R}^n)\right\}$$

$$\left\| f(tT)g|\, X(\mathbb{R}^n, L_{2,\frac{dt}{t}}(\mathbb{R}_+))\right\| \asymp (\|f|L_\infty\| + \Psi_{\nu_0}(f))\, \|g\,|\,X(\mathbb{R}^n)\|\,.$$

Partial proof of the Theorem 4.1. We shall discuss only the proofs of the assertions concerning Hardy-Lorentz spaces ("Lower case"). The cases of the other spaces ("Upper case") are treated in the same way with the aid of the results of Chapter III (Theorem 3.3, *d*), *e*) from Chapter III).

Theorem D from [**17**] supplies an opportunity to consider only functions f from the class $\Psi(S^0_{\mu+})$ in all the parts of the theorem thanks to some limiting procedure.

By Theorem 3.1 from Chapter III and the existence of H^∞-calculus of the operator T in L_2, it is sufficient to estimate only the constants $C_{\mathcal{AD}}$ from the definitions of the appropriate $\mathcal{AD}$-classes. For this purpose we shall use the following representation of the operator $f(T)$ and its kernel, obtained by Xuan Duong (see [**18**]):

$$f(T) = \int_{\Gamma_{\nu_0}} e^{-zT}n(z)dz, \quad n(z) = \int_{\Gamma_0} e^{z\zeta}f(\zeta)d\zeta, \tag{1}$$

$$\Gamma_{\nu_0} = \Theta(t)te^{(\pi-\nu_0)/2} + (\Theta(t)-1)te^{(\nu_0-\pi)/2}, \Gamma_0 = \Theta(t)te^{\nu_0} + (\Theta(t)-1)te^{-\nu_0},$$

$$t\in\mathbb{R}, K_f(x,y) = \int_{\Gamma_0} k_z(x,y)n(z)dz, \quad |n(z)|\le c|z|^{-1}\|f\|_\infty.$$

Now using subadditivity of $\mathcal{D}$-functional we have in the case *a*) for a function f with $\|f|L_\infty\|\le 1$

$$\mathcal{D}_u(r,w,K_f(x,\cdot),\lambda) \le C\int_{\Gamma_{\nu_0}} \left(\frac{r}{|z|^m}\right)^\lambda |z|^{-mn}\left(1+\frac{|x-w|}{|z|^m}\right)^{-(n+\lambda+\epsilon)}|dz|/|z| \le$$

$$\le Cr^\lambda|x-w|^{-(n+\lambda)}. \tag{2}$$

Hence $f(T)\in\mathcal{AD}_x(1, L_\infty, l_\infty, \lambda,\lambda,\lambda)$ and

$$\mathcal{AD}_x(u, L_\infty, l_{t,log}, \gamma,\gamma,\lambda),\ \gamma\in[0,\lambda), t\in(0,\infty].$$

uniformly by f. It means the desirable estimate $C_{\mathcal{AD}}(f) \leq C\|f|L_\infty\|$. It is left to apply the part $c)$ of Theorem 3.1 from Chapter III.

To prove part $c)$ let us fix a (nonzero) function f $\|f|H_\infty\| + \Psi_{\nu_0}(f) \leq 1$, satisfying the conditions of $c)$, and such that for $z \in \Gamma_0$

$$\int_0^\infty f^2(tz)\frac{dt}{t} = c_I > 0. \tag{3}$$

Now we can define operators $\Lambda : g(x) \to \{(f(tT)g)(x)\}_{t\in\mathbb{R}_+}$,

$$\Lambda^{-1} : \{h(t,x)\}_{t\in\mathbb{R}_+} \to C_I^{-1}\int_{t\in\mathbb{R}_+} f(tT)h(t,x)\frac{dt}{t},$$

which define an isomorphism between $L_2(\mathbb{R}^n)$ and $L_2(\mathbb{R}^n, L_{2,\frac{dt}{t}}(\mathbb{R}_+))$ because of the Theorem F from [**17**]. Hence, analogously to the derivation of the formula (2), subadditivity of the $\mathcal{D}$-functional, Minkowski inequality and finiteness of $\Psi_{\nu_0}(f)$ imply both for $k_f = K_f$ and for $k_f = K_f^I$

$$\mathcal{D}_u(r, w, k_f(x,\cdot), \lambda, L_{2,\frac{dt}{t}}(\mathbb{R}_+)) \leq C\int_{\Gamma_{\nu_0}} \Big\|\left(r|z\cdot|^{-m}\right)^\lambda |z\cdot|^{-mn}\times$$
$$\times\left(1+|x-w||z\cdot|^{-m}\right)^{-(n+\lambda+\epsilon)}\Big| L_{2,\frac{dt}{t}}\Big\| \, |n(z)|\cdot|dz| \leq$$
$$\leq Cr^\lambda|x-w|^{-(n+\lambda)}. \tag{4}$$

It means that

$\Lambda, \Lambda^{-1} \in \mathcal{AD}_x(1, L_\infty, l_\infty, \lambda, \lambda, \lambda) \cap \mathcal{AD}_y(1, L_\infty, l_\infty, \lambda, \lambda, \lambda)$. Thus the proof of the part $c)$ is finished exactly as ones of the parts $a), b)$. $\mathcal{Q.E.D.}$

CHAPTER 2

Anisotropic supersingular operators and approximation formula

1. Introduction and background

Surprisingly supersingular operators were studied before (simply) singular integral operators (SIO). Indeed, S. L. Sobolev **[25, 26]** introduced and studied thoroughly the notions of generalized function, generalized derivative and Sobolev space itself as a domain of these simplest supersingular operators in 1935-36, while the celebrated article **[2]** of A. Calderón and A. Zygmund appeared in 1952. Even the first edition of Sobolev's book **[27]** was published in 1950. The abstract branch known as the theory of generalized functions was further developed by L. Schwartz **[28]** (1950) and I. M. Gel'fand and G. E. Shilov **[29]** (1958). A lot of works were devoted to the investigation of the properties of spaces of fractional smoothness, which were defined in the same way as was invented by S. L. Sobolev: a function space is the domain of a supersingular integral operator with the range in a better known space, while the operator itself is initially well defined on some space of generalized functions. Among the closely related works, one should mention the ones of S. M. Nikol'skii **[30, 31, 32, 34, 33, 35]**, L. N. Slobodeckii **[36]**, O. V. Besov **[37]**, I. A. Kipriyanov **[38]**, S. V. Uspenskii **[39]**, N. Aronszain, K. T. Smith **[40]**, E. M. Stein **[41]**, P. I. Lizorkin **[42, 43, 44, 45]**, N. Aronszain, F. Mulla, P. Szeptycki **[46]**.

A theorem of E. M. Stein from **[41]** can be interpreted as: $[M(g), |\mathrm{D}|^{\beta}]1 \in L_p \iff g \in L^{\beta}_{p,2}$ for $\beta, p^{-1} \in (0,1)$ (cf. the proof of Theorem 5.3, *a*)).

We should note the article **[43]** of P. I. Lizorkin, where he defined and utilized the notion of Liouville derivative as $\mathrm{D}^{\alpha} f = \widehat{(ix)^{\alpha}\tilde{f}}$. Observe that, for fractional vectors $\alpha \geq 0$, function $(ix)^{\alpha}$ is no longer a multiplier in $\mathcal{S}'$. Thus, P. I. Lizorkin needed to introduce other spaces $\Phi \subset S$ and Φ' of fundamental and generalized functions, where, in particular, the elements of Φ are orthogonal to the polynomials on all the hypersurfaces containing the origin. This fact has some relation with Theorem 10.2.

The balance between the desires to consider spaces of not only positive smoothness and to preserve the convenience of using an extended notion of Marcinkiewicz multiplier was achieved by S. M. Nikol'skii [**34, 33**], who had introduced into the usage the notion of generalized function regular in the sense of L_p. But, as is shown in [**35**], the presence of a convenient calculus of symbols is not sufficient, and estimates of Bessel-MacDonald kernels describing the operator $(I-\Delta)^{-l/2}$, $l>0$ are also very useful.

Boundedness properties of commutators of supersingular integral operators (SSIO) were investigated by M. Murray [**47, 48**], T. Kato, G. Ponce [**49**] and S. Hofmann [**50, 51**] together with X. Li and D. Yang [**52**].

Let $|\mathrm{D}|^\beta$ be the fractional differentiation operator of order $\beta>0$. M. Murray [**47, 48**] proved that the condition $|\mathrm{D}|^\beta g \in BMO(\mathbb{R})$ is necessary and sufficient for the relation

$$\||\mathrm{D}|^\beta(gf) - g|\mathrm{D}|^\beta f|L_p(\mathbb{R})\| \leq C\||\mathrm{D}|^\beta g|BMO(\mathbb{R})\| \cdot \|f|L_p(\mathbb{R})\|. \tag{1}$$

She obtained also multilinear variants of this result and considered the same issues for the composition with the Hardy transform $H|\mathrm{D}|^\beta$ instead of $|\mathrm{D}|^\beta$. Here we use Riesz transforms similarly.

T. Kato, G. Ponce [**49**] proved the estimate: for $J=(I-\Delta)^{-1/2}$, $s>0$, $p\in(1,\infty)$,

$$\|J^s(gf)-g(J^s f)|L_p(\mathbb{R}^n)\| \leq C(\|g|w^1_\infty(\mathbb{R}^n)\|\cdot\|J^{s-1}f|L_p(\mathbb{R}^n)\|+\|J^s g|L_p(\mathbb{R}^n)\|\cdot\|f|L_\infty(\mathbb{R}^n)\|),$$

but their approach permits to show that

$$\|J^s(gf)-g(J^s f)|L_p(\mathbb{R}^n)\| \leq C(\|g|w^1_\infty(\mathbb{R}^n)\|\cdot\|J^{s-1}f|L_p(\mathbb{R}^n)\|+\|J^s g|L_\infty(\mathbb{R}^n)\|\cdot\|f|L_p(\mathbb{R}^n)\|)$$

that implies $\|J^s(gf)-g(J^s f)|L_p(\mathbb{R}^n)\| \leq C\|g|w^1_\infty(\mathbb{R}^n)\|\cdot\|f|L_p(\mathbb{R}^n)\|$ for $s\in(0,1)$.

Results of this form found applications in the theories of Euler, generalized Korteweg-de Vries and Navier-Stokes equations in works of T. Kato, G. Ponce [**49**], M. Christ, M. I. Weinstein [**53**] and C. E. Kenig, G. Ponce, L. Vega [**54**].

S. Hofmann [**51**] improved Murray's Leibniz rule for fractional derivatives $|\mathrm{D}|^\beta, \beta\in(0,1)$ up to

$$\||\mathrm{D}|^\beta(gf) - g|\mathrm{D}|^\beta f|L_r(\mathbb{R})\| \leq C\||\mathrm{D}|^\beta g|L_p(\mathbb{R}^n)\| \cdot \|f|L_q(\mathbb{R}^n)\|, 1/r = 1/p+1/q. \tag{2}$$

He, X. Li and D. Yang [**52**] established necessary and sufficient conditions on function g for the boundedness of commutators with different convolution homogeneous supersingular integral operators including anisotropic ones.

Later S. Hofmann also raised *the question* on the validity of the estimate

$$\| |\mathrm{D}|^\gamma (|\mathrm{D}|^\beta (gf) - g|\mathrm{D}|^\beta f) | L_p(\mathbb{R}^n)\| \leq C\|f|L_p(\mathbb{R}^n)\|. \tag{3}$$

We provide an extended answer to this question in Theorem 9.3.

There exist counterparts for the Hilbert transform H in place of fractional differentiation, established by C. Segovia and R. L. Wheeden in [**55**]:

$$\|\mathrm{D}^\alpha(H(gf) - gHf)|L_r(\mathbb{R}^n)\| \leq C\||\mathrm{D}|^\alpha g|L_p(\mathbb{R}^n)\|\|f|L_q(\mathbb{R}^n)\|, \alpha \in (0,1), 0 \leq 1/p = 1/r - 1/q.$$

These have been further developed by J. Cohen and J. Gosselin [**56**].

There are several results related to the boundedness of Calderón-Zygmund SIOs in Lebesgue spaces. Note that the commutator of a SIO T possessing the kernel $K(x,y)$ with the pointwise multiplier on g has kernel $(g(y) - g(x))K(x,y)$. Thus, the operator defined by $(g(y) - T_m(y,x))\Omega(x-y)|x-y|^{-(n+m-\varepsilon)}$, where $T_m(y,x)$ is the mth degree Taylor polynomial of g, is a variant of the formal extension of the Calderón commutator of the operator with a convolution $(n+m-\varepsilon)$-homogeneous kernel. Boundedness of operators of this form in Lebesgue spaces were established by S. Hofmann [**57**] for $m = 1, \varepsilon = 0$, by Sh. Lu and Qi. Wu [**58**] for $\varepsilon > 0$, and by G. Hu and D. Yang [**59**] for $\varepsilon = 0$. The latter authors provided also multilinear variants of their results.

While traditionally the singularity of an operator with kernel $K(x,y)$ was defined by the estimate $|K(x,y)| \leq C|x-y|^{-(n+\alpha)}$, our classifications in Section 3 are based heavily on the idea that, in difference with purely integral operators, a supersingular integral operator is not defined by its kernel only, whose presence only means that it is integral, but also by its local part. Structural characterizations of this part are obtained too. There we introduce variants of weak and strong type classifications of singularity of an operator T based on homogeneity of the corresponding bilinear form $(T\psi, \psi)$, or similarity of T to a (fractional) differentiation operator in the sense of possessing the same boundedness properties as the differentiation operator.

In Section 4, we describe the idea and construct an approximation formula for the composition of a pointwise multiplier and a SSIO from a wide range of supersingular integral operators that is suitable in non-convolution case and is counterpart of the Leibnitz rule, including commutators as a particular case. It possesses an ability to annihilate the inconvenient local part of our SSIOs providing also a criterion for the locality of these operators. Therefore, to study the boundedness of the residual of the approximation formula, we can deal with the kernels of the corresponding operators only.

Section 5 contains necessary conditions on the pointwise multiplier for a convolution supersingular integral operator with $n+\beta > n$-homogeneous kernel $\Omega(x-y)|x-y|^{-(n+\beta)}$ to have its corresponding residual to be bounded from a Lebesgue space into some Lebesgue, or Lizorkin-Triebel space. For this purpose, we introduce and study some properties of the notions of Lebesgue- and Chebyshev-regularity of functions and operators and admissibility of functions. Particular examples considered include Lizorkin's Liouville differentiation and an operator from [**60, 50, 52**]. The steps of the proof are very similar to those used in one-dimensional case by M. A. M. Murray in [**47**] and different from those suggested in [**48**] and realized in [**50, 52**]. As an auxiliary lemma, we establish estimates of derivatives of some kernels with asymptotics of the constants appropriate for our purposes.

In Section 7, we obtain sufficient conditions on the pointwise multiplier for the boundedness of the residual of a general (non-convolution) supersingular operator contained in the appropriate $\mathcal{AD}$-classes (from [**1, 5**]) to be bounded from a Lebesgue space into some Besov-Nikol'skii, or Lizorkin-Triebel space in terms of the inclusion of the pointwise multiplier function into some special local function spaces. To accomplish it, one needs to prove an extrapolation theorem in Section 6 and direct estimates of $\mathcal{D}$-functionals and use a counterpart of the Carleson embedding theorem in Section 11. In this and other sections, we use the relations between these local function spaces and more traditional ones established in Section 10 to derive related results written in terms of different groups of function spaces.

In Sections 8 and 9, we apply results established earlier to the case of general convolution SSIO and, respectively, its particular important examples, answering, in particular, to Hofmann's question and discussing the optimality of the isotropic setting and a difference between the cases of integer and fractional differential operators.

Section 10 is devoted to properties of function spaces under consideration, while Section 11 contains the rest of the auxiliary results.

The content of this chapter will be included in [**61**].

2. Definitions and designations

In this section, we introduce, in a general form, the notations of the thesis work related to the setting of scalar-valued functions belonging to anisotropic function spaces.

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ be the sets of natural, rational and real numbers correspondingly, $\mathbb{N}_0 = \mathbb{N} \cup 0$, $\mathbb{R}_+ = (0, +\infty)$. For $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n_+ \cup \mathbb{N}^n_0$, $s \in \mathbb{R}$, $\beta \in \mathbb{R}^n_+$, we designate $|\alpha| = \sum_{i=1}^n \alpha_i$, $\alpha^s = (\alpha_1^s, \cdots, \alpha_n^s)$, $\alpha\beta = (\alpha_1\beta_1, \cdots, \alpha_n\beta_n)$, $\alpha/\beta = (\alpha_1/\beta_1, \cdots, \alpha_n/\beta_n)$, $(\alpha, \beta) = |\alpha\beta|$. **Many objects dealt with depend on the anisotropy vector γ_a with $|\gamma_a| = n$, which is mostly supposed to be fixed.** For $x, y \in \mathbb{R}^n$, $\alpha \in \mathbb{N}^n_0$, $D \subset \mathbb{N}^n_0$ and $s \geq 0$ we shall understand that $x \leq y \Leftrightarrow x_i \leq y_i$ for $1 \leq i \leq n$; $\overset{\square}{D}_\alpha := \{\beta \in \mathbb{N}^n_0 : \beta \leq \alpha\}$, $\hat{D} = \cup_{\beta \in D} \overset{\square}{D}_\beta$, $D^*_s := \{\beta \in \mathbb{N}^n_0 : (\beta, \gamma_a) \leq s\}$, $\lambda_{\min}(D) := \min_{\beta \in D}(\beta, \gamma_a)$, $\lambda_{\max}(D) := \max_{\beta \in D}(\beta, \gamma_a)$. and that $|D|$ is the number of the elements of D. For $D \subset \mathbb{Z}^n$, let $D_+ := D \cap \mathbb{N}^n_0$. In the definitions of function spaces below, we shall always require $D = \hat{D}$ for the sake of their shift invariance. Usually we shall deal with space $\mathbb{R}^n$ for some $n \in \mathbb{N}$ endowed with a γ_a-anisotropic q-, or $q*$-quasi-metric $\rho_{\gamma_a}(x, y) := |x - y|_q$, or, correspondingly, $\rho_{\gamma_a}(x, y) := |x - y|_q$, where $|x|_q := (\sum_{i=1}^n |x_i|^{q/(\gamma_a)_i})^{q^{-1}}$, and $|x|_{q*} \geq 0$ is defined by $\sum_{i=1}^n |x_i|^q |x|_{q*}^{-q(\gamma_a)_i} = 1$ for some $q \in (0, \infty)$. Lebesgue measure is $|E| = \int_{\mathbb{R}^n} \chi_E(x)dx$. Symbol Q will mean a cube $Q = Q_r(w) = \{y \in \mathbb{R}^n : \rho_{\gamma_a}(y, w) < r\}$, $r(Q) := r$, $O(Q) := w$, $\kappa Q := Q_{\kappa r}(w)$ for $\kappa > 0$.

For (quasi)(semi)normed spaces A, B, $g \in A$, $f \in A^*$, let $\|\cdot|A\|$ be its (quasi)(semi)norm, A^* be its adjoint space, $(f, g) := f(g)$, and $\mathcal{L}(A, B)$ be the space of all bounded linear operators from A into B. Let also $\{A\}$ be the set of the elements of A. We understand $A \subset B \Leftrightarrow I \in \mathcal{L}(A, B)$, $A = B \Leftrightarrow A \subset B,\ B \subset A$.

For $\alpha \in \mathbb{N}^n_0$ and measurable g, let $M(g) : f \longrightarrow gf$ be an operator of the pointwise multiplication, and D^α be a Sobolev generalized differentiation.

For $A \subset \mathbb{N}^n_0$, $|A| < \infty$, let $\mathcal{P}_A$ be the space of all polynomials $\{\pi : \pi = \sum_{\alpha \in A} c_\alpha x^\alpha\}$.

For $p \in (0, \infty)$, $q \in (0, \infty]$ and Lebesgue measurable $E \subset \mathbb{R}^n$, let $L_{p,q}(E)$ be the Lorentz space of functions on space E with the Lebesgue measure, $L_p(E) := L_{p,p}(E)$. For a measurable function f on $E \subset \mathbb{R}^n$, denote $\sigma(f, t) := |\{x \in E : |f(x)| > t\}$, $f^*(t) := \inf\{\lambda > 0 : \sigma(f, \lambda) \leq t\}$.

We need to introduce the notions of Lebesgue and Chebyshev regularity of functions and operators.

DEFINITION 2.1. For a finite $A \subset \mathbb{N}_0^n$, we say that a measurable function f is Lebesgue--regular on $\mathbb{R}^n$ with respect to $\mathcal{P}_A$ if $f-\pi \in \bigcup_{p\in(0,\infty)}\{L_p(\mathbb{R}^n)\}$, and that it is Chebyshev-regular on $\mathbb{R}^n$ with respect to $\mathcal{P}_A$ if $\lim_{t\to\infty}(f-\pi)^*(t) = 0$ for some $\pi \in \mathcal{P}_A$.

We say that an operator $T \in \mathcal{L}(\mathcal{D}(\mathbb{R}^n), \mathcal{D}'(\mathbb{R}^n))$ is Lebesgue-, or Chebyshev-regular on $\mathbb{R}^n$ with respect to $\mathcal{P}_A$ if $T\psi$ is, correspondingly, Lebesgue-, or Chebyshev-regular with respect to $\mathcal{P}_A$ for any $\psi \in \mathcal{D}(\mathbb{R}^n)$. Space $\mathcal{P}_A$ will not be mentioned if $A = \emptyset$.

REMARK 1. Note that Lebesgue regularity of a function, or an operator, implies its Chebyshev regularity.

DEFINITION 2.2. For $p, q \in (0,\infty]$ and $\Omega \subset \mathbb{R}_+ \times \mathbb{R}^n$, by means of $L_{(p,q*)}(\Omega)$ and, correspondingly, $L_{(q*,p)}(\Omega)$, we designate the following (quasi-) normed spaces of measurable functions on Ω with the finite (quasi-) norms

$$\|f|L_{(p,q*)}(\Omega)\| := \left(\int_0^\infty \|f(t,\cdot)\chi_\Omega(t,\cdot)|L_p(\mathbb{R}^n)\|^q dt/t\right)^{1/q} \quad \text{and}$$

$$\|f|L_{(q*,p)}(\Omega)\| := \left\| \left(\int_0^\infty |f(t,\cdot)\chi_\Omega(t,\cdot)|^q dt/t\right)^{1/q} \middle| L_p(\mathbb{R}^n)\right\|;$$

$$L_{(p,q*)} := L_{(p,q*)}(\mathbb{R}_+ \times \mathbb{R}^n),\ L_{(q*,p)} := L_{(q*,p)}(\mathbb{R}_+ \times \mathbb{R}^n).$$

DEFINITION 2.3. For $r > 0$, $a \in (0,\infty]^n$, $w \in \mathbb{R}^n$, an open set $G \subset \mathbb{R}^n$, $s \geq 0$, $D = \hat{D} \subset \mathbb{N}_0^n$, $|D| < \infty$ and $f \in L_{a,loc}(G)$, let us define the following $\mathcal{D}$–functional:

$$\mathcal{D}_{a,G}(r,w,f,D) := \begin{cases} \inf_{\pi\in\mathcal{P}_D} |Q_r(w)|^{-1/a}\|f-\pi|L_a(Q_r(w))\|, & \text{if } Q_r(w) \subset G, \\ 0, \text{ otherwise;} \end{cases}$$

$$\mathcal{D}_a(r,w,f,D) := \mathcal{D}_{a,\mathbb{R}^n}(r,w,f,D).$$

Let us also define the following maximal functions:

$$M_a^s f(x) := \sup_{x\in Q_{t\kappa}(z)} t^s \mathcal{D}_a(t,z,f,\emptyset),\ M_a f := M_a^0 f.$$

For $Q = Q_r(w)$, $a \geq 1$, we shall designate by $p_{r,w} = p_{r,w,D} = p_{Q,D}$ a bounded projector from $L_a(Q)$ onto $\mathcal{P}_D(A)$ defined as in Section 2 of Chapter I.

REMARK 2. It is shown in [**22**] that, in our settings,

$$|Q_r(w)|^{-1/a}\|f - p_{Q,D}f|L_a(Q_r(w))\| \asymp \mathcal{D}_a(r,w,f,D) \text{ for } a \in [1,\infty].$$

DEFINITION 2.4. For $r > 0$, $a \in (0,\infty]^n$, $w \in \mathbb{R}^n$, $s \geq 0$, $D = \hat{D} \subset \mathbb{N}_0^n$, $|D| < \infty$ and $f \in L_{a,loc}(\mathbb{R}^n)$, let us define the following $\mathcal{D}^c$–functional:

$$\mathcal{D}_a^c(r,w,f,D) := |Q_r(w)|^{-1/a} \|f(\cdot) - T_D(\cdot,w,f)|L_a(Q_r(w))\|, \text{ where}$$

$T_D(y,w,f) := \sum_{\alpha \in D} (\alpha!)^{-1} \mathrm{D}^\alpha f(w)(y-w)^\alpha$ is the Taylor polynomial of order D at w.

DEFINITION 2.5. For $p,q \in (0,\infty]$, $\gamma \geq 0$ and measurable function f on $\mathbb{R}_+ \times \mathbb{R}^n$, let $MB_{p,q}^\gamma f$ and $ML_{p,q}^\gamma f$ be, respectively, the functionals:

$$MB_{p,q}^\gamma f(x) := \sup_{x \in Q_r(w)} |Q_r(w)|^{-(1/p+\gamma)} \|f|L_{(p,q*)}((0,r) \times Q_r(w))\| \text{ and}$$

$$ML_{p,q}^\gamma f(x) := \sup_{x \in Q_r(w)} |Q_r(w)|^{-(1/p+\gamma)} \|f|L_{(q*,p)}((0,r) \times Q_r(w))\|.$$

Let us introduce local spaces of Nikol'skii-Besov and Lizorkin-Triebel types.

DEFINITION 2.6. For $u,a,p,q \in (0,\infty]$, $\gamma, s \geq 0$, a finite $D \subset \mathbb{N}_0^n$, by means of $bl_{p,u,q,a}^{s,\gamma,D}(\mathbb{R}^n)$, $ll_{p,u,q,a}^{s,\gamma,D}(\mathbb{R}^n)$, correspondingly, we shall designate the spaces of all functions $f \in L_{a,loc}(\mathbb{R}^n)$ with the following finite quasi-semi-norms

$$\|MB_{u,q}^\gamma g_{s,a}|L_p(\mathbb{R}^n)\|, \ \|ML_{u,q}^\gamma g_{s,a}|L_p(\mathbb{R}^n)\|, \text{ where}$$

$$g_{s,a}(t,x) := t^{-s}\mathcal{D}_a(t,x,f,D).$$

Sometimes we also shall use the functionals

$$\|f|bl_{p,u,q,a}^{s,\gamma,D}(\mathbb{R}^n)\|_T = \|MB_{u,q}^\gamma g_{s,a}^c|L_p(\mathbb{R}^n)\|, \ \|f|ll_{p,u,q,a}^{s,\gamma,D}(\mathbb{R}^n)\|_T = \|ML_{u,q}^\gamma g_{s,a}^c|L_p(\mathbb{R}^n)\|, \text{ where}$$

$$g_{s,a}^c(t,x) := t^{-s}\mathcal{D}_a^c(t,x,f,D).$$

DEFINITION 2.7. For an open set $G \subset \mathbb{R}^n$, $s \geq 0$, $D \subset \mathbb{N}_0^n$, $|D| < \infty$, $p,q,a \in (0,\infty]$, let us designate by means of $\widetilde{b}_{p,q,a}^{s,D}(G)$, $\widetilde{l}_{p,q,a}^{s,D}(G)$, correspondingly, the following quasi-semi-normed spaces defined by their quasi-semi-norms

$$\|f|\widetilde{b}_{p,q,a}^{s,D}(G)\| := \|h_{s,a}|L_{(p,q*)}(\mathbb{R}_+ \times G)\|,$$

$$\|f|\widetilde{l}_{p,q,a}^{s,D}(G)\| := \|h_{s,a}|L_{(q*,p)}(\mathbb{R}_+ \times G)\| \text{ for } p < \infty \text{ and}$$

$$\|f|\widetilde{l}_{\infty,q,a}^{s,D}(\mathbb{R}^n)\| := \|f|ll_{\infty,u,q,a}^{s,0,D}(\mathbb{R}^n)\| \text{ for some } u < p, \text{ where}$$

$$h_{s,a}(t,x) := t^{-s}\mathcal{D}_{a,G}(t,x,f,D).$$

By means of $\|\cdot|\widetilde{b}_{p,q,a}^{s,D}(G)\|_T$ and $\|\cdot|\widetilde{l}_{p,q,a}^{s,D}(G)\|_T$ respectively, we shall designate the functionals that coincide with the above expressions for the norms $\|\cdot|\widetilde{b}_{p,q,a}^{s,D}(G)\|$ and

$\| \cdot |\widetilde{l}^{s,D}_{p,q,a}(G)\|$ except for the usage of $\mathcal{D}^c$-functional in place of $\mathcal{D}$-functional (as in the previous definition).

By means of $\underline{\widetilde{b}^{s,D}_{p,q}(G)}$, we shall designate any of the spaces $\widetilde{b}^{s,D}_{p,q,a}(G)$, whose parameters a, p, q, s satisfy the condition: $s - \frac{n}{p} + \frac{n}{a} > 0$.

By means of $\underline{\widetilde{l}^{s,D}_{p,q}(G)}$, we shall designate any of the spaces $\underline{\widetilde{l}^{s,D}_{p,q,a}(G)}$, whose parameters a, p, q, s satisfy the condition: $s - \frac{n}{\min(p,q)} + \frac{n}{a} > 0$.

Let also $\underline{\widetilde{b}^{s}_{p,q}(G)} := \widetilde{b}^{s,D^*_s}_{p,q}(G)$ and $\underline{\widetilde{l}^{s}_{p,q}(G)} := \widetilde{l}^{s,D^*_s}_{p,q}(G)$.

Assume that $\mathcal{D}(\mathbb{R}^n)$, $\{\mathcal{D}(\mathbb{R}^n)\} = C_0^\infty(\mathbb{R}^n)$ is the space of basic (fundamental) functions and $\mathcal{D}'(\mathbb{R}^n)$ is its dual one of generalized functions. For $f \in \mathbb{D}'(\mathbb{R}^n)$, $\psi \in C_0^\infty$, let

$$\psi_t(f)(x) := t^{-n}(f, \psi_t(x - \cdot)), \text{ while } \psi_t(x) := \psi(\frac{x}{t^{\gamma_a}}).$$

Definition 2.8. For $s \geq 0$, $D^*_s \subset D \subset \mathbb{N}^n_0$, $p, q \in (0, \infty]$, some $\varepsilon > 0$ and a function $\psi \in C^\infty$, $\psi(x) \leq C(1+|x|_{\gamma_a})^{-(n+s+\varepsilon)}$, $\psi \perp \mathcal{P}_D$ and $\psi \not\perp \mathcal{P}_B$ for any $B \supset A$, let us designate by means of $\underline{b^{s,D}_{p,q}(\mathbb{R}^n), l^{s,D}_{p,q}(\mathbb{R}^n)}$, correspondingly, the following quasi-semi-normed spaces of generalized functions defined by their quasi-semi-norms

$$\|f|b^{s,D}_{p,q}(\mathbb{R}^n)\| := \|\sigma_{s,\psi}|L_{(p,q*)}(\mathbb{R}_+ \times G)\|,$$

$$\|f|l^{s,D}_{p,q}(\mathbb{R}^n)\| := \|\sigma_{s,\psi}|L_{(q*,p)}(\mathbb{R}_+ \times G)\| \text{ for } p < \infty,$$

$$\|f|l^{s,D}_{\infty,q}(\mathbb{R}^n)\| := \|ML^0_{u,q}\sigma_{s,\psi}|L_\infty(\mathbb{R}^n)\| \text{ for some } u < p, \text{ where}$$

$$\sigma_{s,\psi}(t, x) := t^{-s}\psi_t(f)(x).$$

Let also $\underline{b^{s}_{p,q}(\mathbb{R}^n)} := b^{s,\mathbb{N}^n_0}_{p,q}(\mathbb{R}^n)$ and $\underline{l^{s}_{p,q}(\mathbb{R}^n)} := l^{s,\mathbb{N}^n_0}_{p,q}(\mathbb{R}^n)$.

Remark 3. In Definition 2.8, the corresponding spaces defined with the aid of different functions ψ coincide with the equivalence of their norms (see [**62**]).

For any $x(G) = x^{\cdots}_{p,q}(G)$ of the (quasi)(semi)-normed spaces defined above, we designate by means of the capital symbol $X^{\cdots}_{p,q}(G)$ its submanifold $x(G) \cap L_p(G)$ endowed with the (quasi)-norm

$$\|f|X^{\cdots}_{p,q}(G)\| := \|f|L_p(G)\| + \|f|x(G)\|.$$

Definition 2.9. For $p \in (0, \infty]$, $s \in \mathbb{N}$, we designate the corresponding seminormed (homogeneous) and classical Sobolev spaces with the aid of the (semi)norms

$$\|f|\underline{w^s_p(\mathbb{R}^n)}\| = \sum_{|\alpha|=s} \|D^\alpha f|L_p(\mathbb{R}^n)\|,$$

$$\|f|\underline{W_p^s(\mathbb{R}^n)}\| = \|f|L_p(\mathbb{R}^n)\| + \|f|w_p^s(\mathbb{R}^n)\|.$$

Later we shall sometimes assume that all functions are defined on $\mathbb{R}^n$ and omit this space.

3. Supersingular operators

In this section we define singular and supersingular operators of strong and weak types, as well as local and integral operators, and describe the class of all local singular and supersingular integral operators of weak order $s \geq 0$.

DEFINITION 3.1. For $s \in \mathbb{R}$, we say that an operator $T \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$ is singular of the weak type s if $|(T\varphi_t(\cdot - x), \varphi_t(\cdot - x))| \leq C(\varphi)t^{n-s}$ for any $t \in (0,1)$, $x \in \mathbb{R}^n$ and $\varphi \in C_0^\infty(Q_0)$. If $s > 0$, we say that the operator T is a supersingular operator of weak type.

In the next definition, we assume that $z_{p_i}^{s_i}$, $i = 0, 1$ are two Sobolev, Besov-Nikol'skii, or Lizorkin-Triebel spaces of the same type.

DEFINITION 3.2. For $s_0, s_1 \geq 0$, $p_0, p_1 \in (0, \infty]$, we say that an operator $T \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$ is singular of the strong type $(z_{p_0}^{s_0}, (z_{p_1}^{s_1})^*)$ if, for any $\psi \in \mathcal{D}(\mathbb{R}^n)$, $|(T\psi_t, \psi_t)| \leq C(T, \psi)\|\psi_t|z_{p_0}^{s_0}\| \cdot \|\psi_t|z_{p_1}^{s_1}\|$. If $s_0 - n/p_0 + s_1 - n/p_1 + n > 0$, we say that the operator T is a supersingular operator of strong type.

COROLLARY 3.3. *An operator of the strong type* $(z_{p_0}^{s_0}, (z_{p_1}^{s_1})^*)$ *is of weak type* $s = s_0 - n/p_0 + s_1 - n/p_1 + n > 0$.

DEFINITION 3.4. An operator $T \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$ is understood to be local if $(T\varphi, \psi) = 0$ for any $\varphi, \psi \in \mathcal{D}$ with $\operatorname{supp} \varphi \cap \operatorname{supp} \psi = \emptyset$.

DEFINITION 3.5. We say that an operator $T \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$ is integral if there is a kernel $K(x, y)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x) : x \in \mathbb{R}^n\}$ satisfying

$$(T\varphi, \psi) = \int_{\mathbb{R}^{2n}} K(x, y)\varphi(x)\psi(y)\,dx\,dy$$

for any $\varphi, \psi \in \mathcal{D}$ with $\operatorname{supp} \varphi \cap \operatorname{supp} \psi = \emptyset$. An integral operator T is local if its kernel $K(x, y) = 0$ for a.e. $x \neq y$.

We shall consider singular integral operators from a particular subclass of $\mathcal{AD}$-classes from [**1, 5**] and introduce reverse $\mathcal{AD}$-class (or $\mathcal{RAD}$-class). For $i \in \mathbb{N}_0$, $Q_r(w) \subset \mathbb{R}^n$ and some $b > 1$, $\delta > 0$, let $\Delta_i(r, w) := Q_{rb^{i+1}\delta}(w) \Delta Q_{rb^i\delta}(w)$.

DEFINITION 3.6. For $u, q \in [1, \infty]$, $\lambda_0, \lambda_1 \in \mathbb{R}$, finite $D \subset \mathbb{N}_0^n$ and a singular integral operator T associated with the kernel $K(x, y)$, we say that T is in $\underline{\mathcal{AD}_y(u, L_q, l_\infty, \lambda_0, \lambda_1, D)}$, if the following relations take place: there are $\delta > 0$ and $b > 1$ satisfying

$$\sup_{Q_r(z)\subset\mathbb{R}^n} \sup_{i\in\mathbb{N}_0} w_i(r, z) = C_{\mathcal{AD}} < \infty, \text{ where}$$

$$w_i(r, z) = \left(\int_{\Delta_i(r,z)} (r^{-\lambda_0}\mathcal{D}_u(r, z, K(\cdot, y), D)|y - z|^{\lambda_1 + n/q'})^q dy\right)^{1/q}, \text{ and}$$

that T is in $\underline{\mathcal{RAD}(L_q, u, l_\infty, \lambda_0, \lambda_1, \emptyset)}$, if the following relations take place: there are $\delta > 0$ and $b > 1$ satisfying

$$\sup_{Q_r(z)\subset\mathbb{R}^n} \sup_{-i\in\mathbb{N}_0} w_i(r, z) = C_{\mathcal{RAD}} < \infty, \text{ where}$$

$$w_i(r, z) = r^{-(\lambda_0 + n/u)} \left(\int_{Q_r(z)} \|K(x, \cdot)|x - \cdot|^{\lambda_1 + n/q'}|L_q(\Delta_i(r, z))\|^u dx\right)^{1/u},$$

with natural modifications for p, or u being equal to ∞.

REMARK 4. For $u_0 \le u_1$, $q_0 \le q_1$, $\mu_0 \le \lambda_0$, $\mu_0 - \mu_1 = \lambda_0 - \lambda_1$, $D_0 \supset D_1$, let us note the inclusions

$$\mathcal{AD}_y(u_0, L_{q_0}, l_\infty, \mu_0, \mu_1, D_0) \supset \mathcal{AD}_y(u_1, L_{q_1}, l_\infty, \lambda_0, \lambda_1, D_1);$$

$$\mathcal{RAD}_y(u_0, L_{q_0}, l_\infty, \lambda_0, \lambda_1, D_0) \supset \mathcal{RAD}_y(u_1, L_{q_1}, l_\infty, \mu_0, \mu_1, D_1).$$

Classes of supersingular integral operators considered in this section are defined, roughly speaking, to be similar in the sense of having some same properties to the differentiation ones.

EXAMPLE 3.7. Arbitrary Calderón-Zygmund operator is a singular integral operator of weak type 0 and strong types (L_p, L_p), (L_∞, BMO), (H_1, L_1), $(L_1, L_{1,\infty})$.

EXAMPLE 3.8. For $\gamma_a = (1, \dots, 1)$, $s = \alpha + \beta > 0$, $p, q \in (1, \infty)$, a linear differential operator (LDO) with constant coefficients of order s,

$$Tf(x) = \sum_{|\alpha|\le s}^{\alpha\in\mathbb{N}_0^n} a_\alpha D^\alpha f(x),$$

is supersingular local integral operator of strong type $(l_{p,q}^\alpha, (l_{p',q'}^\beta)^*)$ and, therefore, of weak type s.

EXAMPLE 3.9. For $\gamma_a = (1, \dots, 1)$, $s = \alpha+\beta > 0$, the fractional differentiation operator $|D|^\beta$ possesses the kernel $K(x,y) = c_{n,\beta}|x-y|^{-(n+\beta)}$ and is supersingular non-local integral operator of strong type $(l^\alpha_{p,q}, (l^\beta_{p',q'})^*)$ and, therefore, of weak type s.

Coifman and Meyer's book [**63**] contains the following unnumbered proposition.

ASSERTION 3.10. [**63**] *Arbitrary Calderón-Zygmund operator with zero kernel is an operator $M(g)$ with $g \in L_\infty(\mathbb{R}^n)$.*

The first result is its counterpart in our settings.

THEOREM 3.11. *Assume $s \geq 0$ and T is local supersingular integral operator of weak type s. Then*

$$T = \sum_{(\alpha+\beta,\gamma_a)\leq s}^{\alpha,\beta\in\mathbb{N}_0^n} \mathrm{D}^\alpha M(a_{\alpha,\beta})\mathrm{D}^\beta,$$

where $a_{\alpha,\beta} \in L_{\infty,loc}$ for each pair (α, β).

The proof of Theorem 3.11. Let us note that, thanks to the Schwartz kernel theorem, operator T admits bounded continuous extension $f \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$, $(f, \phi \otimes \psi) = (T\phi, \psi)$ satisfying $(f, \psi) = 0$ for any $\psi \in \mathcal{D}(\mathbb{R}^{2n})$ with supp $\psi \cap \{(x,x) : x \in \mathbb{R}^n\}$. Choosing system $\{\psi_i\}_{i\in\mathbb{N}} \subset \mathcal{D}'(\mathbb{R}^{2n})$ with the properties $\sum_{i\in\mathbb{N}} \psi_i(x,x) = 1$ for any $x \in \mathbb{R}^n$ and $\sum_{i\in\mathbb{N}} \chi_{\text{supp } \psi_i} \in L_\infty(\mathbb{R}^{2n})$, we decompose $f \overset{\mathcal{D}'}{=} \sum_{i\in\mathbb{N}} f_i$, $f_i = f \circ M(\psi_i)$ into the sum of finitely supported generalized functions on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$.

This construction shows that, without loss of generality, we can assume that the support of $f \subset \mathcal{D}'(\mathbb{R}^{2n})$ is finite. Changing variables $z := 2^{-1/2}(x-y)$, $w := 2^{-1/2}(x+y)$ in the function f, one obtains $g \in \mathcal{D}'(\mathbb{R}^{2n})$. Then, applying Theorem 11.1 to the function g, we gain a representation

$$(g, \phi) = \sum_{|\alpha|\leq m} \int_{\mathbb{R}^{2n}} h_\alpha(z,w)\mathrm{D}^\alpha\phi(z,w)dzdw, \text{ where } m \in \mathbb{N}_0,\ \{h_\alpha\}_{|\alpha|\leq m} \subset L_\infty. \tag{1}$$

Let us observe that in the new variables z, w, we have $(g, \phi) = 0$ for any $\phi \in \mathcal{D}(\mathbb{R}^{2n})$ with $\{(z,w) \in \text{supp } \phi : z = 0\} = \emptyset$. Analogously to considerations from [**29**], formula (1) shows that g can be continuously extended to the closure of $\mathcal{D}(\mathbb{R}^{2n})$ in the space of functions having continuous derivatives D^β_z for any $|\beta| \leq m$ by z. Thus, taking a function $\phi(z) \in \mathcal{D}(\mathbb{R}^n)$ equal to 1 in the vicinity of the origin, we obtain the identity

$$(g, \psi) = \sum_{|\beta|\leq m} (g_\beta, \mathrm{D}^\beta_z\psi(0,\cdot)), \text{ where } g_\beta \in \mathcal{D}'(\mathbb{R}^n),\ g_\beta = (\beta!)^{-1} g \circ M(z^\beta\phi). \tag{2}$$

Another application of Theorem 11.1 to each g_β provides, for some $m' \in \mathbb{N}_0$, the following identities

$$g_\beta = \sum_{|\alpha|\le m'} h_{\alpha,\beta}(w)\mathrm{D}_w^\alpha \text{ for } |\beta| \le m. \tag{3}$$

Combining relations (2) and (3), one obtains, for any $\varphi \in \mathcal{D}(\mathbb{R}^{2n})$,

$$(g, \varphi) = \int_{\mathbb{R}^n} \sum_{|\gamma|\le m+m'} h_\gamma(w)\mathrm{D}^\gamma\varphi(0, w)dw, \text{ where } \gamma = (\beta, \alpha). \tag{4}$$

Thus, returning to (x, y) and using coordinate x as the parameter on the diagonal $\{(x, x) : x \in \mathbb{R}^n\}$, we have, for $\phi(x), \psi(y) \in \mathcal{D}(\mathbb{R}^n)$,

$$(\phi, T\psi) = (f, \phi\psi) = \int_{\mathbb{R}^n} \sum_{|\alpha+\beta|\le m_f} f_{\alpha,\beta}(x)\mathrm{D}^\alpha\phi(x)\mathrm{D}^\beta\psi(x)dx, \text{ where } m_f = m + m'. \tag{5}$$

This means that $T = (-1)^{|\alpha|}\mathrm{D}^\alpha M(f_{\alpha,\beta})\mathrm{D}^\beta$. Now the condition of the weak singularity of the type s finishes the proof providing the restriction

$$f_{\alpha,\beta} = 0 \text{ for } (\alpha + \beta, \gamma_a) \le s. \quad \mathcal{Q.E.D.}$$

REMARK 5. Requiring the operator T to be of some strong type under the conditions of the previous theorem, we would have obtained different restrictions on α, β as can be seen from the examples below.

EXAMPLE 3.12. Arbitrary local supersingular integral operator of strong type $(w_p^1, (w_{p'}^1)^*)$ possesses the form

$$T = \mathrm{div}(\mathbf{A}\mathrm{grad}f),$$

where $\mathbf{A}$ is a matrix with $L_{\infty,loc}$-coefficients.

Theorem 3.11 means, in particular, that a supersingular integral operator which is bounded, for instance, from the Sobolev space W_p^α into the Lebesgue space L_p is defined by its kernel $K(x, y)$ (for $x \ne y$) modulo a linear differential operator (LDO) based on Sobolev generalized derivatives with variable locally essentially bounded coefficients.

Let us obtain another characteristic property of these operators. To approximate properly a supersingular integral operator, we need to find a way to neutralize its local part.

4. Approximation formula as a counterpart of the Leibniz rule

Here we derive the author's approximation formula, discuss its correctness and obtain immediate applications. In particular, we establish a characterization of the class of the linear differential operators with locally bounded coefficients.

We begin with the following observations. Let g be a differentiable function, $\mathrm{D} : f \longrightarrow f'$ denote differentiation. Then both the composition $\mathrm{D}M(g) : f \longrightarrow (gf)'$ and its easier-to-calculate counterpart

$M(g)\mathrm{D} : f \longrightarrow gf'$ are well defined if f is differentiable. But the classical Leibniz rule

$$\mathrm{D}M(g)f - M(g)\mathrm{D}f = (gf)' - gf' = g'f$$

shows that their difference can be extended to non-differentiable functions f too. The original Leibniz rule demonstrates that the remainder mapping

$$\mathrm{D}^n M(g) - \sum_{i=0}^{n-1} \binom{n}{i} M(g^{(i)})\mathrm{D}^{n-i} : f \longrightarrow (gf)^{(n)} - \sum_{i=1}^{n-1} \binom{n}{i} g^{(i)} f^{(n-i)} = g^{(n)} f$$

possesses the same property.

What should one do if we replace the nth derivative by a more general operator?

In 1967, A.P. Calderón established the following version of the Leibniz rule for convolution pseudo-differential operators providing us with, even, some sufficient conditions answering Hofmann's question. A convolution operator T is defined by symbol $\sigma(\xi)$ if

$$\hat{Tf}(\xi) := \sigma(\xi)\hat{f}(\xi).$$

THEOREM 4.1. (A.P. Calderón, 1967) *For $m \in \mathbb{N}$, $s \in [0, m]$, $\gamma \in \mathbb{N}_0^n$, $|\gamma| = m - s$, let the convolution operator T be defined by a symbol σ satisfying $|\mathrm{D}^\alpha \sigma(\xi)| \leq C_\alpha |\xi|^{s-|\alpha|}$ for $|\alpha| \leq m$. Define also*

$$R_m := TM(g) - \sum_{|\alpha| \leq m} M(\mathrm{D}^\alpha g) T_\alpha,$$

where $g \in W_\infty^m$ and T_α is defined by the symbol $(-i)^{|\alpha|}(\alpha!)^{-1}\mathrm{D}^\alpha \sigma$. Then we have

$$\|R_m \mathrm{D}^\gamma f | L_2\| \leq C \sum_{|\alpha| \leq m} \|\mathrm{D}^\alpha b | L_\infty\| \|f | L_2\|.$$

The original Leibniz formula shares with Calderón's extension the following inconveniences. On the one hand, to approximate the composition $TM(g)$, we need to know some properties of not only T, but also of the other $n - 1$ operators, and this could be essential in numerical applications. On the other hand, both formulas are compatible with translation-invariant (convolution) operators T only.

Let us derive a different counterpart of the Leibniz rule compatible with non-convolution operators as follows. Assume that the domain of an operator T is invariant under multiplication by the coordinate functions x_i and some function g, $\alpha = (\alpha_1, \dots, \alpha_n)$ with each α_i a natural or zero number, $M(g) : f \to gf$, $C_i^1(T) := [M(x_i), T] := TM(x_i) - M(x_i)T$, $C^\alpha(T) := C_1^{\alpha_1}(\dots(C_n^{\alpha_n}(T))\dots)$. For a finite $A \subset \mathbb{N}_0^n$, let us define the residual and the adjoint residual of T:

$$R_A(g,T) := TM(g) - \sum_{\alpha \in A} (\alpha!)^{-1} M(\mathrm{D}^\alpha g) C^\alpha(T), \;\; R_m(g,T) := R_{D_m^*}(g,T),$$

$$\alpha! := \prod_{i=1}^n \alpha_i!, \;\; |\alpha| := \sum_{i=1}^n \alpha_i, \tag{4}$$

$$R_A^*(g,T) := M(g)T - \sum_{\alpha \in A} (-1)^{|\alpha|} (\alpha!)^{-1} C^\alpha(T) M(\mathrm{D}^\alpha g), \;\; R_m^*(g,T) := R_{D_m^*}^*(g,T),$$

$$\text{i.e. } (R_A(g,T))^* = R_A(\bar{g}, T^*). \tag{5}$$

From here, we shall assume $\mathrm{D}^\alpha g \in L_{1,loc}$ for any $\alpha \in A$.

For operators A, B, let us introduce the commutator mapping $C_B(A) : A \to C_B(A) = [B, A] = AB - BA$.

To see the correctness of the definition of $C^\alpha(T)$, we use the following lemma. Indeed, thanks to the well known relations $[M(x_i), \mathrm{D}^{e_j}] = \delta_{i,j} I$,
$[M(x_i), M(x_j)] = 0$, $[\mathrm{D}^{e_i}, \mathrm{D}^{e_j}] = 0$ and $C_{\lambda I}(A) = 0$, one can see that the order of $\{C_i^{\alpha_i}\}_{i=1}^n$ is not important in the definition of $C_\alpha(T)$.

The next lemma is an alternative form of writing of the Jacobi identity:

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

LEMMA 4.2. *Let A, B be some operators. Then one has the identity*

$$[C_C, C_B](A) := C_B(C_C(A)) - C_C(C_B(A)) = C_{[B,C]}(A).$$

The proof of Lemma 4.2. follows immediately from definitions. $\mathcal{Q.E.D.}$

To deal with linear differential operators (LDO), we prove the following useful relation. Let us recall that $\alpha \le \beta \Leftrightarrow \alpha_i \le \beta_i$, $1 \le i \le n$.

LEMMA 4.3.

$$(\alpha!)^{-1}C^{\alpha}(\mathrm{D}^{\beta}) = \begin{cases} \binom{\beta}{\alpha}\mathrm{D}^{\beta-\alpha}, & \text{if } \alpha \le \beta, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$\binom{\beta}{\alpha} := \prod_{i=1}^{n} \binom{\beta_i}{\alpha_i}.$$

<u>The proof of Lemma 4.3.</u> For $f \in \mathcal{C}^1(\mathbb{R})$, one has $(xf(x))^{(n)} - xf^{(n)}(x) = nf^{(n-1)}(x)$ and $C^k(\mathrm{D}^n) = \frac{n!}{k!}\mathrm{D}^{(n-k)}$ if $k \le n$, or $C^k(\mathrm{D}^n) = 0$ for $k > n$. Above mentioned correctness considerations and these identities applied by each of the coordinates prove the lemma. $\mathcal{Q.E.D}$.

Let us recall that $\overset{\square}{D}_{\beta} := \{\alpha \in \mathbb{N}_0^n : \alpha_i \le \beta_i,\ 1 \le i \le n\}$.

COROLLARY 4.4. *For $m \in \mathbb{N}$, let D_B be a linear differential operator with variable coefficients, $T = \sum_{\alpha\in B} M(a_\alpha)\mathrm{D}^\alpha$. Then*

a) $R_A(g, \mathrm{D}_B) = \sum_{\beta\in B} M(a_\beta) \sum_{\alpha\in \overset{\square}{D}_\beta \setminus A} M(\binom{\beta}{\alpha}\mathrm{D}^\alpha g)\mathrm{D}^{\beta-\alpha}$;

b) $R_A(g, \mathrm{D}_B) = 0$ *for* $\hat{B} \subset A \subset \mathbb{N}_0^n$;

c) $R_{m-1}(g, \mathrm{D}_{D_m^*}) := M(\sum_{|\alpha|=m} a_\alpha \mathrm{D}^\alpha g)$.

<u>The proof of Corollary 4.4.</u> Part *a*) follows from Lemma 4.3 and the linearity of $R_A(g,\cdot)$. The other parts follow from *a*). $\mathcal{Q.E.D}$.

The next corollary is a sort of $T1$-theorem for some LDO.

COROLLARY 4.5. *For $m \in \mathbb{N}$, $p, q \in (0, \infty]$, $r^{-1} = p^{-1} + q^{-1}$, let $S_m = \sum_{|\alpha|=m} M(a_\alpha)\mathrm{D}^\alpha$. Then*

$$a)\ R_{m-1}(g, S_m) \in \mathcal{L}(L_q, L_r) \Leftrightarrow S_m(g) \in L_p; \tag{6}$$

$$b)\ R_{m-2}(g, S_m) \in \mathcal{L}(W_q^1, L_r) \Leftrightarrow S_m(g) \in L_p,\ \sum_{|\alpha+\gamma|=m} \prod_{i=1}^{n} (\alpha_i + \gamma_i)\mathrm{D}^\alpha g a_\alpha \in L_p. \tag{7}$$

<u>The proof of Corollary 4.5.</u> It follows from homogeneity considerations and the following identities

$$R_m(g, S_m) = 0,\ R_{m-1}(g, S_m) = M(S_m(g)),$$

$$R_{m-2}(g, S_m) = M(S_m(g)) + \sum_{|\gamma|=1} M(\sum_{|\alpha+\gamma|=m} \prod_{i=1}^{n} (\alpha_i + \gamma_i) D^\alpha g a_\alpha) D^\gamma.\ \mathcal{Q.E.D}.$$

The next lemma is a criterion for a SSIO to be linear differential operator with locally bounded coefficients.

LEMMA 4.6. *Assume $s > 0$, $p \in [1,\infty]$ and T is supersingular integral operator of the strong type (w_p^s, L_p). Then $T = \sum_{(\alpha,\gamma_a)<s} M(a_\alpha)\mathrm{D}^\alpha$ with $\{a_\alpha\}_{(\alpha,\gamma_a)<s} \subset L_{\infty,loc}$ if, and only if,*

$$R_m(\phi, T) = 0 \text{ for some } m \in \mathbb{N}_0 \text{ and any } \phi \in \mathcal{D}.$$

The proof of Lemma 4.6. The necessity is proved in Corollary 4.4. The sufficiency is followed from Theorem 3.11 and Remark 5 due to the locality of T provided by the identity

$$0 = (R_m(\phi, T)\psi_1, \psi_2) = (T\phi, \psi_2)$$

for any $\phi, \psi_1, \psi_2 \in \mathcal{D}$ satisfying $\phi\psi_1 = \phi$, $\operatorname{supp}\phi \cap \operatorname{supp}\psi_2 = \emptyset$. $\mathcal{Q.E.D.}$

This explains why sometimes, in comparison with T itself, one of its residuals $R_m(g, T)$ is defined by the kernel $K(x, y), x \neq y$ of T, and, therefore, can appear not to be a supersingular, and can even be a smoothing integral operator.

5. Necessary conditions for a convolution SSIO with a homogeneous kernel

In this section, we provide necessary conditions for a convolution SSIO with a homogeneous kernel to have a good approximation using the notions of admissability and Lebesgue (Marcinkiewicz) regularity defined in Section 2.

For $\beta > 0$ and a 0-homogeneous measurable function Ω defined on $\mathbb{R}^n$, by means of T_Ω^β, or, correspondingly, $T_\Omega^{\beta*}$, we designate an anisotropic supersingular integral operator of weak type β with the kernel $\Omega_\beta(x-y)|x-y|_q^{-(n+\beta)}$, or $\Omega_\beta(x-y)|x-y|_{q*}^{-(n+\beta)}$.

DEFINITION 5.1. We say that the function Ω is admissible (with parameter b_1) if there are a point $z \in \mathbb{R}^n$ with $\prod_{i=1}^n z_i \neq 0$ and an (open) vicinity G of z such that

$$\sup_{\alpha \in \mathbb{N}_0^n} \|\mathrm{D}^\alpha \Omega^{-1}|L_\infty(G)\| b_1^{-|\alpha|}/|\alpha|! < \infty$$

for some parameter $b_1 > 0$.

REMARK 6. *a*) Sobolev embedding theorems permit to substitute the L_∞-norm in Definition 5.1 for any L_σ-norm with $\sigma \in [1,\infty]$.

b) If functions $\{\Omega_j\}_{1\leq j\leq m}$ are admissible with parameters $\{b_j\}_{1\leq j\leq m}$, then their product

$\Omega := \prod_{1\le j\le m}\Omega_j$ is admissible too with parameter $b := \sum_{1\le j\le m} b_j$. This follows from the Leibniz rule.

Recall that $\bar{1} = (1,\dots,1)$ and $|\cdot|$ is either $|\cdot|_q$, or $|\cdot|_{q*}$.

EXAMPLE 5.2. *a)* Function $\Omega(x) := |x|^{n+\frac{n}{n+1}} x_n^{-1-n/2}\exp\left(\frac{\sum_{i=1}^{n-1} x_i^2}{4x_n}\right)\chi_{x_n>0}$ corresponding to the kernel $K(x) = x_n^{-(1+n/2)}\exp\left(-\frac{x_1^2+\dots+x_{n-1}^2}{4x_n}\right)\chi_{x_n>0}$ considered in [**60, 50, 52**] is admissible.
b) For $\alpha\in(0,\infty)^n$, function $\Omega(x) := C(\alpha)\frac{|x|^{(\alpha,\gamma_a)+n}}{x_+^{\alpha+\bar{1}}}$ corresponding to Lizorkin's Liouville differentiation D^α defined by the symbol $(i\xi)^\alpha := \prod_{i=1}^n |\xi_i|^{i(\pi\alpha_i \mathrm{sign}\ \xi_i)/2}$ is admissible.

THEOREM 5.3. *For $p_0, p_1, \theta\in[1,\infty)$, $q\in[1,\infty)$, $\beta>0$, $A\subset \overset{\circ}{D}{}^*_\beta$, $\lambda/n = p_0^{-1}-p_1^{-1}$ and a measurable function g, let operator $T\in\{T_\Omega^\beta, T_\Omega^{\beta*}\}$ be with $\mathcal{P}_A\subset KerT$ and an admissible 0-homogeneous function Ω. Then*
a) inclusion $R_A(g,T)\in\mathcal{L}(L_{p_0}(\mathbb{R}^n), \tilde{l}^{\lambda,A}_{p_0,\theta}(\mathbb{R}^n))$ implies
$\|g|\tilde{l}^{\lambda+\beta,A}_{\infty,\theta}(\mathbb{R}^n)\| \le C\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), \tilde{l}^{\lambda,A}_{p_0,\theta}(\mathbb{R}^n))\|$;
b) inclusion $R_A(g,T)\in\mathcal{L}(L_{p_0}(\mathbb{R}^n), L_{p_1}(\mathbb{R}^n))$ and Lebesgue-regularity of $R_A(g,T)$ imply
$\|g|\tilde{b}^{\lambda+\beta,A}_{\infty,\infty}(\mathbb{R}^n)\| \le C\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), L_{p_1}(\mathbb{R}^n))\|$.

REMARK 7. *a)* The condition of Lebesgue-regularity of $R_A(g,T)$ in part *b)* of Theorem 5.3 can be substituted with the one of Chebyshev-regularity.
b) It is well known in the isotropic case that $\|\cdot|\tilde{l}^{\lambda+\beta}_{p,2}(\mathbb{R}^n)\|$ is an equivalent semi-norm for the BMO-Sobolev space of all functions f with finite $\||D|^{\lambda+\beta}f|BMO(\mathbb{R}^n)\|$.
c) Under the conditions of Theorem 5.3, we do not need to require admissability of Ω in the case $A=\{0\}$ (traditional commutator) because one can adopt the approach from [**48, 50, 52**] to our settings. Thus, it will be sufficient, for example, to require only $\Omega^{-1}\in L_\infty(\mathbb{R}^n\setminus\{0\})$. While, according to [**52**], if an operator T with $T1=0$ is defined by a symbol $m(\xi)\in C^\infty(\mathbb{R}^n\setminus\{0\})$ even in each variable, $Tf = \widehat{m\tilde{f}}$, with $m(t^{\gamma_a}\xi) = t^\kappa m(\xi)$, $\kappa\in(0,1)$ and $m(\xi)>0$ for $\xi\neq 0$, then the 0-homogeneous factor of its (convolution) kernel $K(x) = \Omega(x)|x|_{2*}^{-(n+\kappa)}$ is in $C^\infty(\mathbb{R}^n\setminus\{0\})$ and $\Omega(x)\neq 0$ for $x\neq 0$.

The proof of Theorem 5.3. Assuming $T := T_\Omega^\beta$, we begin with part *b)*. Under the conditions of the theorem, let us choose some constants ξ, ξ_1, κ_1 and, then, fix g and a cube $Q_r(w_x)\subset\mathbb{R}^n$ and choose $\rho\in(0,1)$ and another cube $Q_{r\rho}(w_y)$ satisfying $|(w_y)_i - x_i| \ge$

$(r\rho\xi)^{(\gamma_a)_i}$ for any $x \in Q_r(w_x)$, $1 \le i \le n$,

$$\|\mathrm{D}^\alpha(\Omega^{-1})(w_y - \cdot)|L_\infty(Q_r(w_x))\| \le Cb^{|\alpha|} \text{ for some } b > 0 \text{ and any } \alpha \in \mathbb{N}_0^n,$$

$$\xi \ge ((1+\varepsilon)(2n\kappa_1 + b)^{((\gamma_a)_{\min})^{-1}}, \ \varepsilon > 0,$$

$$\kappa_1 = \max_{1 \le i \le n}(q, |q/(\gamma_a)_i - 1| + 1), \ Q_r(w_y) \subset Q_{r\xi_1}(w_x). \tag{1}$$

For a function f, let also $T_A(y,a,f) := \sum_{\alpha \in A} \frac{\mathrm{D}^\alpha f(a)}{\alpha!}(y-a)^\alpha$ be its Taylor polynomial of the order A. Note that we can choose a family of integral projectors $\{p_{r,w}\}_{Q_r(w) \subset \mathbb{R}^n}$, $p_{r,w} = p_{r,w,A} \in \mathcal{L}(L_1(Q_r(w)), \mathcal{P}_A)$ satisfying

$$p_{r,w}f(y) = \sum_{\alpha \in A} \left(\frac{y-w}{r^{\gamma_a}}\right)^\alpha r^{-n} \int \varphi_\alpha \left(\frac{\cdot - w}{r^{\gamma_a}}\right) f \text{ with } \{\phi_\alpha\}_{\alpha \in A} \subset C_0^\infty(Q_1(0)). \tag{2}$$

Using the identities $p_{r,w_y}(f - T_A(\cdot,a,f))(x) = p_{r,w_y}f(x) - T_A(x,a,f)$, $T_A(x,x,f) = f(x)$ and the formal expansion into Taylor series, we obtain

$$p_{r\rho,w_y}f(x) - f(x) =$$

$$= \sum_{\alpha \in \mathbb{N}_0^n} (r\rho)^{(\alpha,\gamma_a)-n} \frac{(-1)^{m_\alpha}}{\alpha!} \mathrm{D}^\alpha \left(\frac{|\cdot - x|_q}{\Omega(\cdot - x)}\right)(w_y) \sum_{\gamma \in A} R_A(g, T_\Omega^\beta)((\cdot)^\alpha \varphi_\gamma)\left(\frac{\cdot - w_y}{(r\rho)^{\gamma_a}}\right)(x). \tag{3}$$

With the aid of the induction procedure by $|\alpha|$, one can observe that

$$\mathrm{D}^\alpha|\cdot - x|_q(w_y) = \sum_{i=1}^{2^{|\alpha|}} c_{i,\alpha}|w_y - x|_q^{n+\beta-q|\gamma_i|}(w_y - x)^{\frac{q\beta}{\gamma_a} - \gamma_i} \text{ with } \gamma_i \le \alpha \text{ and} \tag{4}$$

$$|c_{i,\alpha}|/|\alpha|! \le Cq^{|\gamma_i|}(\max_{1 \le j \le n}(|q/(\gamma_a)_j - 1|) + 1)^{|\alpha - \gamma_i|} \le C\kappa_1^{|\alpha|} \text{ for any } 1 \le i \le 2^{|\alpha|}. \tag{5}$$

These estimates, the simple inequality $m_1!m_2! \le (m_1+m_2)!$, 0-homogeneity of Ω^{-1} and the high order Leibniz rule deliver the formulas

$$|\mathrm{D}^\alpha|\cdot - x|_q(w_y)| \le C|\alpha|!(2\kappa_1)^{|\alpha|}(r\xi_1)^{n+\beta}(r\xi\rho)^{-|\alpha|\gamma_{a\min})} \text{ for } x \in Q_r(w_x),$$

$$\sup_{x \in Q_r(w_x)} |\mathrm{D}^\alpha\left(\frac{|\cdot - x|_q}{\Omega(\cdot - x)}\right)(w_y)| \le C(r\xi_1)^{n+\beta}(r\xi\rho)^{-|\alpha|\gamma_{a\min})} \times$$

$$\times \sum_{\beta \le \alpha} \binom{\alpha}{\beta} |\beta|!b^{|\beta|}|\alpha - \beta|!(2\kappa_1)^{|\alpha-\beta|} \le C|\alpha|!(r\xi_1)^{n+\beta}(r\xi\rho)^{-|\alpha|\gamma_{a\min})}(b + 2\kappa_1)^{|\alpha|}. \tag{6}$$

Thanks to the $L_{p_0} - L_{p_1}$-boundedness of $R_A(g,T)$, Lemma 5.5, formulae $(1,3,4)$ and estimate (6), inclusion $g \in \tilde{b}_{\infty,\infty}^{\lambda+\beta,A}(\mathbb{R}^n)$ follows from the estimates

$$r^{-(\lambda+\beta)}\mathcal{D}_1(r, w_x, g, A) \le r^{-(\beta+n/p_0)}\|f - p_{r,w_y}f|L_{p_1}(Q_r(w_x))\| \le$$

$$\le \sum_{\alpha \in \mathbb{N}_0^n} \frac{r^{(\alpha,\gamma_a)-n/p_0}}{\alpha!}(2\kappa_1 + b)^{|\alpha|}|\alpha|!(r\xi\rho)^{-(\alpha,\gamma_a)}\xi_1^{n+\beta} \sum_{\gamma \in A} \left\| ((\cdot)^\alpha \varphi_\gamma)\left(\frac{\cdot - w_y}{(r\rho)^{\gamma_a}}\right) \right| L_{p_0}(Q_r(w_y))\Big\| \times$$

$$\times C\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), L_{p_1}(\mathbb{R}^n))\| \le C\sum_{k=0}^{\infty}(n(2\kappa_1+b)\xi^{-(\gamma_a)_{\min}})^k. \tag{7}$$

The case $T = T_\Omega^{\beta *}$ is treated in the same way with the aid of Lemma 5.4, b). To consider part a), let us observe that, under the assumptions of this part, boundedness condition of part b) holds for functions from $\mathcal{D}$ because the condition of Lebesgue regularity of $R_A(g,T)$ and Theorem 10.1, together with the appropriate Sobolev embedding, imply the estimate $\|R_A(g,T)\phi|L_{p_1}(\mathbb{R}^n)\| \le C\|R_A(g,T)\phi|\tilde{l}^{\lambda,A}_{p_0,\theta}(\mathbb{R}^n)\|$ for any $\phi \in \mathcal{D}$. One sees also that the kernel corresponding to $R_A(g,T)$ is $K_R(x,y) = (g(y) - T_A(y,x,g))K(x,y)$, where $K(x,y)$ corresponds to T. Using the identity $ab - cd = (a-c)b + c(b-d)$, both parts of Theorem 11.3 and Lagrange theorem, we obtain the inclusion $K_R \in \mathcal{AD}_y(\infty, L_\infty, l_\infty, \varepsilon, \varepsilon+\lambda, A)$ for any $\varepsilon \in (0, \lambda+\beta-\lambda_{\max}(A))$ from the membership $g \in \tilde{b}^{\lambda+\beta,A}_{\infty,\infty}(\mathbb{R}^n)$ provided by b) in the following way:

$$\mathcal{D}_\infty(r,w,K_R(\cdot,y),A) \le C\sum_{\alpha\in A}\mathcal{D}_\infty(r,w,\mathrm{D}^\alpha_x K(\cdot,y),A)\frac{|(y-w)^\alpha|}{|y-w|^{n+\beta}}+$$

$$+\mathcal{D}^c_\infty(r,w,K(\cdot,y),A)\sum_{i=1}^{n} r^{(\gamma_a)_i}\|\mathrm{D}^{e_i}|\cdot - y|_q|L_\infty(Q_r(w))\| \le$$

$$\le C\|g|\tilde{b}^{\lambda+\beta,A}_{\infty,\infty}(\mathbb{R}^n)\|\frac{r^{\lambda+\varepsilon}}{|y-w|_q^{n+\varepsilon}} \text{ for } y \notin Q_{r\delta}(w) \subset \mathbb{R}^n, \tag{8}$$

where δ is chosen to satisfy $|x-y|_q \asymp |y-w|_q$ for $x \in Q_r(w)$. Thus, Theorem 6.1 finishes the proof of part a) thanks to the identity $R_A(g,T)1 = g$. $\mathcal{Q.E.D.}$

LEMMA 5.4. *Let $\beta \in \mathbb{R}$, $q \in \mathbb{R}_+$. Then, for any $\alpha \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$ with $\prod_{i=1}^n x_i \ne 0$, one has*

$$a)\ |\mathrm{D}^\alpha|x|_q^\beta| \le C|\alpha|!|\alpha|^{(-[\beta])_+}(2\max_{1\le i\le n}(q, |q/(\gamma_a)_i - 1| + 1))^{|\alpha|}|x|_q^\beta|x^{-\alpha}|;$$

$$b)\ |\mathrm{D}^\alpha|x|_{q*}^\beta| \le C|\alpha|!|\alpha|^{(-[\beta])_+}((n+3)\frac{\gamma_{a\max}}{\gamma^2_{a\min}}\max(q, |q-1|))^{|\alpha|}|x|_{q*}^\beta|x^{-\alpha}|.$$

The proof of Lemma 5.4. The proof of part a) is a simplification of the one of part b). So we shall consider the latter. For simplicity we assume that $x_i > 0$ for $1 \le i \le n$. The implicit function theorem provides

$$\mathrm{D}^{e_i}|x|_{q*} = x_i^{q-1}|x|_{q*}^{-q(\gamma_a)_i}(\gamma_a, \frac{x^q}{|x|_{q*}^{q\gamma_a}})^{-1}. \tag{1}$$

Thanks to the Leibniz and composition rules, conducting induction by $|\alpha|$, we see that

$$\mathrm{D}^\alpha |x|_{q*}^\beta = \sum_{j=1}^{(n+3)^{|\alpha|}} c_j x^{(q-1)(\gamma_j+\lambda_{j,0}+\sum_{i=1}^n e_i|\lambda_{j,i}|)-\delta_j} |x|_{q*}^{-(\beta+q|\alpha|)} \times$$

$$\times (\gamma_a, \frac{x^q}{|x|_{q*}^{q\gamma_a}})^{-(|\gamma_j|+|\lambda_{j,0}|+2\sum_{i=1}^n |\lambda_{j,i}|)}, \tag{2}$$

where integer vectors δ_j, γ_j, $\lambda_{j,0}$ and $\lambda_{j,i}$ are the exponents of the derivatives fallen during our calculation process on the factors of the form $x^{(\cdot)}$, $|x|_{q*}^{(\cdot)}$, the numerator of the corresponding summand during the differentiation of $(\gamma_a, \frac{x^q}{|x|_{q*}^{q\gamma_a}})^{(\cdot)}$ and the denominator of the *ith* summand of the same sum respectively. The estimates $(\gamma_a, \frac{x^q}{|x|_{q*}^{q\gamma_a}}) \geq \gamma_{a\,\min}$ and

$$|c_j| \leq C(\gamma_{a\,\max} \max(q, |q-1|))^{|\alpha|} \prod_{k=0}^{|\alpha|-1} |\beta - k| \leq C(\gamma_{a\,\max} \max(q, |q-1|))^{|\alpha|} |\alpha|^{(-[\beta])_+} |\alpha|! \tag{3}$$

are used to finish the proof of part *b*). $\mathcal{Q.E.D.}$

LEMMA 5.5. *Let* $|a_\alpha| \leq C_0 b^{|\alpha|} \frac{|\alpha|!}{\alpha!}$ *for any* $\alpha \in \mathbb{N}_0^n$. *Then the series* $\sum_{\alpha \in \mathbb{N}_0^n} a_\alpha x^\alpha$ *converges absolutely on* $Q_\kappa := \{x \in \mathbb{R}^n : |x_i| < \kappa(nb)^{-1}\}$ *for* $\kappa \leq 1$ *and, in particular, uniformly for* $\kappa < 1$.

The proof of Lemma 5.5. The decisive ingredient is the estimate

$$|\sum_{|\alpha|=k} a_\alpha x^\alpha| \leq C_0 (\frac{\kappa}{n})^{|\alpha|} \sum_{|\alpha|=k} \frac{k!}{\alpha!} = C_0 \kappa^k. \ \mathcal{Q.E.D.}$$

6. An extrapolation theorem

In this section, we prove an extrapolation theorem used both to obtain necessary conditions for the good approximation in the previous section and to establish the most general sufficient conditions in the next section. Not possessing its most general form, this theorem is still sufficient for our purposes achieved in the next section.

THEOREM 6.1. *Assume* $A \subset \mathbb{N}_0^n$, $|A| < \infty$, $\lambda, \lambda_0, \lambda_1 \in \mathbb{R}$, $\lambda_1 > \lambda \geq 0$, $q \geq 1$, $v_0 \in [p_0, p_\infty]$, $p_0, p_1, v_0, v_1, u, \eta, \theta \in (0, \infty]$, $n/p_\infty = n(v_0^{-1} - v_1^{-1}) = \lambda_0 - \lambda_1 - \lambda \geq 0$, $T \in \mathcal{AD}_y(u, L_q, l_\infty, \lambda_0, \lambda_1, A) \cap \mathcal{L}(L_{p_0}(\mathbb{R}^n), \tilde{l}_{p_1,\theta}^{\lambda,A}(\mathbb{R}^n))$ *and either* $q^{-1} + p_\infty^{-1} \leq 1$, *or* $p_0 = p_\infty$. *Let also either* $v_0 = p_\infty$, *or* $A \supset D_\lambda^*$. *Then*

a) $T \in \mathcal{L}(L_{v_0}(\mathbb{R}^n), \tilde{l}_{v_1,\theta}^{\lambda,A}(\mathbb{R}^n))$;

b) $T \in \mathcal{L}(L_{v_0,\eta}(\mathbb{R}^n), \tilde{b}_{v_1,\eta}^{\lambda,A}(\mathbb{R}^n))$ *for* $v_0 \in (p_0, p_\infty)$.

The proof of Theorem 6.1. Part $b)$ follows from $a)$ due to the real interpolation method. One can assume that $p_0 < v_0 \le p_\infty$. Let us fix arbitrary function $f \in L_{p_\infty}(\mathbb{R}^n)$, cube $Q_r(w) \subset \mathbb{R}^n$ and choose $\kappa > 1$ to satisfy $\bigcup_{z \in Q_r(w)} Q_{r\delta}(z) \subset Q_{r\kappa}(w)$. Assume also that kernel $K(x,y)$ is associated with T. One can use the following decomposition of f: $f = f_1 + f_2$, $f_1 = \chi_{Q_{r\kappa}(w)} f$. Boundedness assumption and Hölder inequality imply

$$(r\kappa)^{-n/p_1} \|Tf_1|\tilde{l}^{\lambda,A}_{p_1,\theta}(\mathbb{R}^n)\| \le (r\kappa)^{n(p_\infty^{-1}-p_0^{-1})} \|T|\mathcal{L}(L_{p_0}(\mathbb{R}^n), \tilde{l}^{\lambda,A}_{p_1,\theta}(\mathbb{R}^n))\| \|f_1|L_{p_0}(\mathbb{R}^n)\| \le$$
$$\le \|T|\mathcal{L}(L_{p_0}(\mathbb{R}^n), \tilde{l}^{\lambda,A}_{p_1,\theta}(\mathbb{R}^n))\| \|f_1|L_{p_\infty}(\mathbb{R}^n)\|. \tag{1}$$

Let us observe that, similar to [**1**], Hölder inequality and the definition of the $\mathcal{AD}$-class with $k \in \mathbb{N}_0$ and $b > 1$ provide the estimates

$$b^{k\lambda} r^{-\lambda} \mathcal{D}_u(b^{-k}r, z, Tf_2, A) \le$$
$$\le C \sum_{i=k}^{\infty} b^{i(\lambda-\lambda_0)} \left(\int_{Q_{\delta b^{i-k+1}r}(z) \setminus Q_{\delta b^{i-k}r}(z)} (b^{k\lambda_0} r^{-\lambda_0} \mathcal{D}_u(b^{-k}r, z, K(\cdot,y), A)|z-y|^{\lambda_1+n/q'})^q dy \right)^{1/q} \|f_2|L_{p_\infty}(\mathbb{R}^n)\|,$$

$$\|\{b^{k\lambda} r^{-\lambda} \mathcal{D}_u(b^{-k}r, z, Tf_2, A)\}_{k\in\mathbb{N}_0}|l_\theta\| \le C C_{\mathcal{AD}} \|\{b^{-k\lambda_1}\}_{k\in\mathbb{N}_0}|l_\theta\| \|f_2|L_{p_\infty}(\mathbb{R}^n)\|. \tag{2}$$

Combination of estimates $(1,2)$ and Hölder inequality prove the case $v_0 = p_\infty$:

$$r^{-n/\min(p,u)} \| \|\{b^{k\lambda} r^{-\lambda} \mathcal{D}_u(b^{-k}r, \cdot, Tf, A)\}_{k\in\mathbb{N}_0}|l_\theta\| f|L_{\min(p,u)}(Q_r(w))\| \le$$
$$\le C(\|T|\mathcal{L}(L_{p_0}(\mathbb{R}^n), \tilde{l}^{\lambda,A}_{p_1,\theta}(\mathbb{R}^n))\| + C_{\mathcal{AD}}) \|f|L_{p_\infty}(\mathbb{R}^n)\|.$$

Complex interpolation method finishes the proof of part $a)$ for $v_0 \in (p_0, p_\infty)$. $\mathcal{Q.E.D.}$

7. Sufficient conditions for the approximation of a general SSIO

This section is devoted to studying sufficient conditions for a SSIO to be well approximated. It begins with Proposition 1, where some ideas of our approach are presented in particular case of the extended form of Hofmann's question. Theorem 7.3 contains the most general form of sufficient conditions expressed in terms of the local variants of the local approximation spaces of Besov and Lizorkin-Triebel types. In Corollary 7.2, we reduce the generality at the expense of the usage of the ordinary local approximation spaces of Besov and Lizorkin-Triebel types. Theorem 7.3 deals with yet other sufficient conditions not covered by the previous theorems. In the end of the section, we provide the way of further generalizations of the results of this section with the aid of Theorem 6.1.

In the following proposition, we consider a model case of mappings into a seminormed local approximation space of Sobolev-Slobodeckiy type, $\tilde{b}^{\lambda}_{p,p,1}(\mathbb{R}^n) = \tilde{l}^{\lambda}_{p,p,1}(\mathbb{R}^n)$.

PROPOSITION 1. For $\gamma_a = \bar{1}$, $p \in (0, \infty)$, integer $0 < \beta < \beta + \lambda < 1$, the boundedness of the commutator $[|D|^\beta, M(g)] \in \mathcal{L}(L_p(\mathbb{R}^n), \tilde{b}^{\lambda,\{0\}}_{p,p,1})$ follows from the inclusion $g \in \tilde{l}^{\lambda+\beta,\{0\}}_{\infty,p,1}(\mathbb{R}^n)$. In particular, we have the following estimate:

$$\||\mathrm{D}|^\lambda[|D|^\beta, M(g)]|\mathcal{L}(L_2(\mathbb{R}^n))\| \asymp \||\mathrm{D}|^{\gamma+\lambda} g|BMO(\mathbb{R}^n)\|.$$

The proof of Proposition 1. One sees that the kernel of $[|D|^\beta, M(g)]$ is $(g(y) - g(x))|y - x|^{-(n+\beta)}$. Let us fix a cube $Q_t(z) \subset \mathbb{R}^n$, a function $f \in L_{p_0}(\mathbb{R}^n)$ and estimate the corresponding $\mathcal{D}$-functional. Assume that $\Delta_i(t, z) := Q_{\delta b^{i+1} t}(z) \Delta Q_{\delta b^i t}(z)$ for $i \in \mathbb{N}_0$, some $b > 1$ and sufficiently large $\delta > 1$. Designating $C_g^\beta := [|D|^\beta, M(g)]$, $f_1 := f\chi_{Q_{t\delta}(z)}$, $f_2 = f - f_1$, we obtain with the aid of the identity $\varphi\psi - p_{t,z,\{0\}}\varphi p_{t,z,\{0\}}\psi = \psi(I - p_{t,z,\{0\}})\varphi + p_{t,z,\{0\}}\varphi(I - p_{t,z,\{0\}})\psi$ that

$$\begin{aligned} &t^{-\lambda}\mathcal{D}_1(t, z, [|D|^\beta, M(g)]f_2, \{0\}) \le \\ &\le t^{-\lambda} \int_{\mathbb{R}^n \setminus Q_{t\delta}(z)} \mathcal{D}_1(t, z, g, \{0\})\mathcal{D}_\infty(t, z, |\cdot - y|^{-(n+\beta)}, \emptyset)dy + \\ &+ t^{-\lambda} \int_{\mathbb{R}^n \setminus Q_{t\delta}(z)} \mathcal{D}_\infty(t, z, p_{t,z,\{0\}}(g(y) - g(\cdot)), \emptyset)\mathcal{D}_1(t, z, |\cdot - y|^{-(n+\beta)}, \{0\})|f(y)|dy = \\ &=: I_1(t, z) + I_2(t, z). \end{aligned} \tag{1}$$

Thus, we obtain

$$\begin{aligned} &I_1(t, z) \le \\ &\le Ct^{-\lambda}\mathcal{D}_1(t, z, g, \{0\}) \sum_{i=0}^{\infty} \int_{\mathbb{R}^n} |z - y|^{-(n+\beta)} \chi_{\Delta_i(t,z)}|f(y)|dy \le \\ &\le Ct^{-\lambda-\beta}\mathcal{D}_1(t, z, g, \{0\}) \sum_{i=0}^{\infty} b^{i\beta}\mathcal{D}_1(\delta b^i t, z, f, \emptyset) \le \\ &\le Ct^{-\lambda-\beta}\mathcal{D}_1(t, z, g, \{0\}) \sup_{i\in\mathbb{N}_0} \mathcal{D}_1(\delta b^i t, z, f, \emptyset). \end{aligned} \tag{2}$$

With the aid of the Minkowski inequality and the Lagrange theorem, we obtain in a similar manner that

$$\begin{aligned} &I_2(t, z) \le Ct^{1-\lambda} \sum_{i=0}^{\infty} (b^i\delta t)^{\lambda+\beta} \times \\ &\times (b^i\delta t)^{-(\lambda+\beta)} \int_{\mathbb{R}^n} \mathcal{D}_\infty(t, z, p_{t,z,\{0\}}(g(y) - g(\cdot)), \emptyset)|z - y|^{-(n+\beta+1)} \chi_{\Delta_i(t,z)}|f(y)|dy \le \\ &\le C \sum_{i=0}^{\infty} b^{i(\lambda-1)}\mathcal{D}_1(t, z, (b^i\delta t)^{-(\lambda+\beta)}\mathcal{D}_1^c(b^i\delta t, x, g, \{0\}), \emptyset) \times \\ &\times \mathcal{D}_1(b^i\delta t, z, f, \emptyset). \end{aligned} \tag{3}$$

Together with this estimate and the definition of $\mathcal{AD}$-classes, Hardy and Minkowski inequalities imply

$$\int_{\mathbb{R}_+\times\mathbb{R}^n}(I_2(t,z))^p\frac{dtdy}{t}\leq C\left(\mathcal{D}_1(t,z,f,\emptyset)M_1^0(t)^{-(\alpha+\beta)}\mathcal{D}_1^c(t,\cdot,g,\{0\})(z)\right)^p\frac{dtdy}{t}. \quad (4)$$

To deal with I_3, we take some $a\in(1,p)$. Let us also choose δ_2 satisfying $Q_{t\delta}(z)\subset\cap_{w\in Q_t(z)}Q_{t\delta_2}(w)$ and use Hölder and Minkowski inequalities, along with the decomposition $Q_r(w):=\cup_{i\in-\mathbb{N}_0}\Delta_i(r,w)$, to see that

$$I_3(t,z):=t^{-\lambda}\mathcal{D}_1(t,z,R_A(g,T)f_1,\{0\})\leq$$

$$\leq t^{-\lambda}\mathcal{D}_1\left(t,z,\int_{Q_{t\delta_2}(\cdot)}|g(y)-g(\cdot)||\cdot-y|^{-(n+\beta)}f(y)|dy,\emptyset\right)\leq$$

$$\leq C\mathcal{D}_1\left(t,z,\sum_{i=0}^{\infty}b^{-i\lambda}(b^{-i}t\delta_2)^{-(\lambda+\beta)}\mathcal{D}_{a'}^c(b^{-i}t\delta_2,\cdot,g,\{0\})\mathcal{D}_a(b^{-i}t\delta_2,\cdot,f,\emptyset),\emptyset\right), \text{ because} \quad (5)$$

$$\sum_{i=0}^{\infty}\int_{Q_{b^{-i}t\delta_2}(x)\setminus Q_{b^{-i}t\delta_2}(x)}|g(y)-g(x)||x-y|^{-(n+\beta)}|f(y)|dy\leq$$

$$\leq C\sum_{i=0}^{\infty}b^{-i\lambda}(b^{-i}t\delta_2)^{-(\lambda+\beta)}\mathcal{D}_{a'}^c(b^{-i}t\delta_2,\cdot,g,\{0\})\mathcal{D}_a(b^{-i}t\delta_2,\cdot,f,\emptyset) \quad (6)$$

Thus, Hardy, Minkowski and Fefferman-Stein inequalities provide the estimate

$$\int_{\mathbb{R}_+\times\mathbb{R}^n}(I_3(t,z))^p\frac{dtdy}{t}\leq C\left(t^{-(\lambda+\beta)}\mathcal{D}_{a'}^c(t,y,g,\{0\})\mathcal{D}_a(t,y,f,\emptyset)\right)^p\frac{dtdy}{t}. \quad (7)$$

Eventually, thanks to estimates $(1,2,4,7)$ and the Carleson embedding theorem, we establish the key estimate

$$\|[|D|^\beta,M(g)]f|\tilde{l}_{p,p}^{\lambda,\{0\}}(\mathbb{R}^n)\|\leq C(\|g|\tilde{l}_{\infty,p,1}^{\lambda+\beta,\{0\}}(\mathbb{R}^n)\|+\|g|\tilde{l}_{\infty,p,a'}^{\lambda+\beta,\{0\}}(\mathbb{R}^n)\|_T)\|f|L_{p_0}(\mathbb{R}^n)\|. \quad (8)$$

Therefore, the proof of the proposition is finished by Theorem 10.1, c), d) and k) showing that

$$\|g|\tilde{l}_{\infty,p,1}^{\lambda+\beta,\{0\}}(\mathbb{R}^n)\|+\|g|\tilde{l}_{\infty,p,a'}^{\lambda+\beta,\{0\}}(\mathbb{R}^n)\|_T\leq C\|g|\tilde{l}_{\infty,p,1}^{\lambda+\beta,\{0\}}(\mathbb{R}^n)\|. \ \mathcal{Q.E.D.}$$

It the rest of this section, we use the following assumptions:

A_1 : for $0\leq i\leq 2$, $0\leq j\leq 5$, $w_i,\sigma_i,\sigma\in[1,\infty]$, $\lambda_j\in\mathbb{R}$, $\lambda,\gamma,s\geq 0$, $\beta>0$, $p_0,p_1,q,u,a\in(0,\infty]$, $r\in[1,\infty]$, $a_1\in(0,1]$, $A,B,D\subset\mathbb{N}_0^n$, $|B|<\infty$, assume $sp_0\leq 1$, $p_2^{-1}=p_1^{-1}-\lambda/n$, $D_\beta^*\subset A\subset\overset{\circ}{D}{}_{\min(\lambda,\lambda_2)+\beta}^*$, $A+D\subset B$, $a^{-1}+w_1^{-1}\leq 1$, $\zeta^{-1}+w_2^{-1}\leq 1$, $p_0^{-1}+\sigma_1^{-1}<1$, $(\max(a,\zeta))^{-1}+p_0^{-1}+(\min(\sigma_0,\sigma_2))^{-1}<1$, $\lambda\in(\lambda_4,\lambda_0)$, $s+\gamma=p_0^{-1}+r^{-1}-p_1^{-1}$, $\lambda_1-\lambda_0=\lambda_3-\lambda_2=\lambda_5-\lambda_4=\beta-ns$;

$A_2 : (q^{-1} - p_1^{-1})_+ \leq \gamma$.

THEOREM 7.1. *Let A_1 and A_2 be valid, as well as one of the following groups of conditions:*

i) $\zeta = u$, $Y^{\lambda,B}_{p_1,q,a_1} = \tilde{b}^{\lambda,B}_{p_1,q,a_1}(\mathbb{R}^n)$ *and* $X^{\lambda+\beta,\gamma,A}_{r,u,q,a} = bl^{\lambda+\beta,\gamma,A}_{r,u,q,a}(\mathbb{R}^n)$;

ii) $\zeta = \min(u,q)$, $\zeta^{-1} + w_2^{-1} < 1$, $Y^{\lambda,B}_{p_1,q,a_1} = \tilde{l}^{\lambda,B}_{p_1,q,a_1}(\mathbb{R}^n)$ *and* $X^{\lambda+\beta,\gamma,A}_{r,u,q,a}(\mathbb{R}^n) = ll^{\lambda+\beta,\gamma,A}_{r,u,q,a}(\mathbb{R}^n)$.

Assume also that either $w_2 = \infty$, $a_1 < 1$, or $a_1 = 1$. Let a supersingular integral operator T of the strong type $(\tilde{l}^{\beta}_{\sigma,1}(\mathbb{R}^n), L_\sigma(\mathbb{R}^n))$ be in the classes $\mathcal{AD}_y(w_0, L_{\sigma_0}, l_\infty, \lambda_0, \lambda_1, D)$, $\mathcal{AD}_y(w_1, L_{\sigma_1}, l_\infty, \lambda_2, \lambda_3, \emptyset)$ and $\mathcal{RAD}_y(L_{\sigma_2}, w_2, l_\infty, \lambda_4, \lambda_5, \emptyset)$, which definitions use, correspondingly, the constants $\delta_0, \delta_1, \delta_2$ satisfying

$\bigcup_{w\in Q_t(z)} Q_{t\max(\delta_0,\delta_1)}(w) \subset Q_{t\delta}(z) \subset \bigcap_{w\in Q_t(z)} Q_{t\delta_2}(w)$ *for some $\delta > 0$ and any $Q_t(z) \subset \mathbb{R}^n$.*

Then one has:

a) $\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), Y^{\lambda,B}_{p_1,q,a_1})\| \leq C\|g|X^{\lambda+\beta,\gamma,A}_{r,u,q,a}\|$;

b) $\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), Y^{\lambda,D^*_\lambda}_{p_1,q,\tilde{a}_1})\| \leq C\|g|X^{\lambda+\beta,\gamma,A}_{r,u,q,a}\|$ *and*

$\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), L_{p_2}(\mathbb{R}^n))\| \leq C\|g|X^{\lambda+\beta,\gamma,A}_{r,u,q,a}\|$ *for $p_1\lambda \leq n$ and one of the following groups of conditions to be valid:*

b_i) $p_0^{-1} + \lambda/n \geq p_1^{-1}$, $n/r \in (\lambda_4, \lambda_2)$, $\zeta^{-1} + p_0^{-1} + (\min(\sigma_1,\sigma_2)^{-1} < 1 + (\lambda+\beta)/n)$, $(\lambda+\beta)/n + w_1^{-1} > \zeta^{-1}$;

b_{ii}) $r = w_1 = w_2 = \sigma_1 = \sigma_2 = \infty$, $\lambda_2 = \lambda_4 = 0$ *and* $\lambda/n + p_0^{-1} - p_1^{-1} \in (0,1)$.

The proof of Theorem 7.1. Let us begin with part a). Theorem 3.4, Lemma 4.6 and Corollary 4.4 show that the residual operation annihilates all local components of T. As in the proof of Theorem 5.3, a), we see that the kernel of $R_A(g,T)$ possesses the form $K_R(x,y) = (g(y) - T_A(y,x,g))K(x,y)$, where $T_A(y,x,g) := \sum_{\alpha\in A} \frac{D^\alpha f(a)}{\alpha!}(y-a)^\alpha$. Let us fix a cube $Q_t(z) \subset \mathbb{R}^n$, a function $f \in L_{p_0}(\mathbb{R}^n)$ and estimate the corresponding $\mathcal{D}$-functional. Assume that w_j^0, w_j^1, w_j^2 are the functions from the definitions of the classes $\mathcal{AD}_y(w_0, L_{\sigma_0}, l_\infty, \lambda_0, \lambda_1, D)$, $\mathcal{AD}_y(w_1, L_{\sigma_1}, l_\infty, \lambda_2, \lambda_3, \emptyset)$, $\mathcal{RAD}_y(L_{\sigma_2}, w_2, l_\infty, \lambda_0, \lambda_1, \emptyset)$ and $\Delta_i^j(t,z) := Q_{\delta_j b^{i+1}t}(z)\Delta Q_{\delta_j b^i t}(z)$ for $i \in \mathbb{N}_0$, $0 \leq j \leq 2$. Designating $f_1 := f\chi_{Q_{t\delta}(z)}$, $f_2 = f - f_1$, we obtain with the aid of the identity $\varphi\psi - p_{t,z,A}\varphi p_{t,z,D}\psi = \psi(I - p_{t,z,A})\varphi + p_{t,z,A}\varphi(I - p_{t,z,D})\psi$ that

$$t^{-\lambda}\mathcal{D}_{a_1}(t,z,R_A(g,T)f_2,B) \leq t^{-\lambda}\mathcal{D}_1(t,z,R_A(g,T)f_2,B) \leq$$

$$\leq t^{-\lambda}\int_{\mathbb{R}^n\setminus Q_{t\delta}(z)} \mathcal{D}_a(t,z,T_A(y,\cdot,g),A)\mathcal{D}_{w_1}(t,z,K(\cdot,y),\emptyset)|f(y)|dy+$$

$$+t^{-\lambda}\int_{\mathbb{R}^n\setminus Q_{t\delta}(z)}\mathcal{D}_\infty(t,z,p_{t,z,A}(g(y)-T_A(y,\cdot,g)),\emptyset)\mathcal{D}_{w_0}(t,z,K(\cdot,y),D)|f(y)|dy=$$

$$=: I_1(t,z)+I_2(t,z). \tag{1}$$

Thanks to conditions of the theorem, one can find σ_5 satisfying $\sigma_1^{-1}+\sigma_5^{-1}=1$, $\sigma_5<p_0$. Thus, we obtain

$$I_1(t,z)\le$$

$$\le t^{-\lambda}\sum_{\alpha\in A}\frac{1}{\alpha!}\mathcal{D}_a(t,z,\mathrm{D}^\alpha g,(A-\alpha)_+)\times$$

$$\times\sum_{i=0}^{\infty}\int_{\mathbb{R}^n}\mathcal{D}_\infty(t,z,(\cdot-y)^\alpha)\mathcal{D}_{u_1}(t,z,K(\cdot,y)\chi_{\Delta_i^1(t,z)},\emptyset)|f(y)|dy\le$$

$$\le C\sum_{\alpha\in A}t^{(\alpha,\gamma_a)-(\lambda+\beta)}\mathcal{D}_a(t,z,\mathrm{D}^\alpha g,(A-\alpha)_+)\times$$

$$\times\sum_{i=0}^{\infty}b^{i((\alpha,\gamma_a)-(\lambda_3+ns))}w_i^1(t,z)(\delta_1 b^i t)^{ns}\mathcal{D}_{\sigma_5}(\delta_1 b^i t,z,f,\emptyset)\le$$

$$\le CC^1_{\mathcal{AD}}\sum_{\alpha\in A}t^{(\alpha,\gamma_a)-(\lambda+\beta)}\mathcal{D}_a(t,z,\mathrm{D}^\alpha g,(A-\alpha)_+)\sup_{i\in\mathbb{N}_0}(\delta_1 b^i t)^{ns}\mathcal{D}_{\sigma_5}(\delta_1 b^i t,z,f,\emptyset). \tag{2}$$

There are parameters σ_3,σ_4 satisfying $\sigma_0^{-1}+\sigma_3^{-1}+\sigma_4^{-1}=1$, $\sigma_3<\max(a,\zeta)$, $\sigma_4<p_0$. With the aid of Hölder and Minkowski inequalities, we obtain using $\dim\mathcal{P}_A=|A|<\infty$, for $a_2=\min(a_1,1/2)$,

$$I_2(t,z)\le C\sum_{i=0}^{\infty}b^{i(\lambda+\beta-\lambda_1-ns)}\mathcal{D}_{a_2}(t,z,(b^i\delta_0 t)^{-(\lambda+\beta)}\mathcal{D}^c_{\sigma_3}(b^i\delta_0 t,x,g,A),\emptyset)\times$$

$$\times w_i^0(t,x)(b^i\delta_0 t)^{ns}\mathcal{D}_{\sigma_4}(b^i\delta_0 t,z,f,\emptyset). \tag{3}$$

Together with this estimate and the definition of $\mathcal{AD}$-classes, Hardy and Minkowski inequalities imply

$$\|I_2|Z\|\le CC^0_{\mathcal{AD}}\|(\cdot)^{ns}\mathcal{D}_{\sigma_4}(\cdot,\cdot,f,\emptyset)M^0_{a_2}(\cdot)^{-(\alpha+\beta)}\mathcal{D}^c_{\sigma_3}(\cdot,\cdot,g,A)|Z\|\text{ for}$$

$$Z\in\{L_{(p,q*)},L_{(q*,p)}\}. \tag{4}$$

According to conditions of the theorem, let us choose parameters σ_6,σ_7,w_3 to satisfy $\sigma_2^{-1}+\sigma_6^{-1}+\sigma_7^{-1}=1$, $\sigma_7<p_0$, $\sigma_6<\max(a,\zeta)$, $w_3^{-1}+w_2^{-1}=1$ and use Hölder and Minkowski inequalities to see that, for $w_2=\infty$, or, correspondingly, $a_1=1$, one has

$$I_3(t,z):=t^{-\lambda}\mathcal{D}_{a_1}(t,z,R_A(g,T)f_1,B)\le$$

$$\le t^{-\lambda}\mathcal{D}_{a_1}\left(t,z,\int_{Q_{t\delta_2}(\cdot)}|g(y)-T_A(y,\cdot,g)||K(\cdot,y)f(y)|dy,\emptyset\right)=:S\le$$

$$\leq C\mathcal{D}_{a_1}\Bigg(t,z,\sum_{i=0}^{\infty} b^{-i(\lambda+\beta-(\lambda_5+ns))}(b^{-i}t\delta_2)^{-(\lambda+\beta)}\mathcal{D}^c_{\sigma_6}(b^{-i}t\delta_2,\cdot,g,A)\sup_{y\in Q_t(z)} w_i^2(t,y)\times$$

$$\times(b^{-i}t\delta_2)^{ns}\mathcal{D}_{\sigma_7}(b^{-i}t\delta_2,\cdot,f,\emptyset),\emptyset\Bigg), \text{ or} \quad (5)$$

$$S\leq C\sum_{i=0}^{\infty} b^{-i(\lambda+\beta-(\lambda_5+ns))}\mathcal{D}_{a_1}\big(t,z,(b^{-i}t\delta_2)^{-(\lambda+\beta)}\mathcal{D}^c_{\sigma_6}(b^{-i}t\delta_2,\cdot,g,A)\times$$

$$\times(b^{-i}t\delta_2)^{ns}\mathcal{D}_{\sigma_7}(b^{-i}t\delta_2,\cdot,f,\emptyset),\emptyset\big)\, w_i^2(t,z). \quad (6)$$

Thus, Hardy, Minkowski and Fefferman-Stein inequalities provide the estimate

$$\|I_3|Z\|\leq CC^2_{\mathcal{AD}}\|(\cdot)^{-(\lambda+\beta)}\mathcal{D}^c_{\sigma_6}(\cdot,\cdot,g,A)(\cdot)^{ns}\mathcal{D}_{\sigma_7}(\cdot,\cdot,f,\emptyset)|Z\| \text{ for } Z\in\{L_{(p,q*)},L_{(q*,p)}\}. \quad (7)$$

Eventually, thanks to estimates $(1,2,4,7)$ and Theorems 11.3 and 11.4, we establish the key estimate

$$\|R_A(g,T)f|Y^{\lambda,B}_{p_1,q,a_1}(\mathbb{R}^n)\|\leq C\max_{0\leq j\leq 2} C^j_{\mathcal{AD}}\Big(\sum_{\alpha\in A}\|\mathrm{D}^\alpha g|X^{\lambda+\beta-(\alpha,\gamma_a),\gamma,(A-\alpha)+}_{r,u,q,a}(\mathbb{R}^n)\|+$$

$$+\|g|X^{\lambda+\beta,\gamma,A}_{r,u,q,\max(\sigma_3,\sigma_7)}(\mathbb{R}^n)\|_T\Big)\|f|L_{p_0}(\mathbb{R}^n)\|. \quad (8)$$

Therefore, the proof of part a) is finished by Theorem 10.1, a), b), e), f) showing that

$$\sum_{\alpha\in A}\|\mathrm{D}^\alpha g|X^{\lambda+\beta-(\alpha,\gamma_a),\gamma,(A-\alpha)+}_{r,u,q,a}(\mathbb{R}^n)\|+\|g|X^{\lambda+\beta,\gamma,A}_{r,u,q,\max(\sigma_3,\sigma_7)}(\mathbb{R}^n)\|_T\leq C\|g|X^{\lambda+\beta,\gamma,A}_{r,u,q,a}(\mathbb{R}^n)\|.$$

To consider part b), we choose $a_2,a_3,\sigma_8,\sigma_9,\sigma_{10},q_1,s_1\in(0,\infty]$ and $\rho=\lambda/n+p_0^{-1}-p_1^{-1}\geq 0$ to satisfy $a_2^{-1}\geq w_1^{-1}+r^{-1}$, $a_2^{-1}\geq w_2^{-1}+\sigma_{10}^{-1}+r^{-1}$, $w_1\leq a_3$, $(\lambda+\beta)/n+a_3^{-1}>\zeta^{-1}$, $a_3^{-1}+p_0^{-1}+(\min(\sigma_1,\sigma_2))^{-1}<1$, $q_1^{-1}=p_1^{-1}-\lambda$, $s_1=\lambda+\beta+n\gamma$, $\max(\sigma_8,\sigma_9)<p_0$, $\sigma_{10}<q_1$. In the first case of part b), definitions of the $\mathcal{AD}$ and $\mathcal{RAD}$ classes, along with Hölder and Minkowski inequalities, deliver the following estimates

$$\mathcal{D}_{a_2}(t,z,R_A(g,T)f_2,\emptyset)\leq\int_{\mathbb{R}^n\setminus Q_{t\delta}(z)}\mathcal{D}_{a_2}(t,z,(g(y)-T_A(y,\cdot,g))K(\cdot,y),\emptyset)|f(y)|dy\leq$$

$$\leq C\sum_{i=0}^{\infty}\int_{\Delta_i^1(t,z)}\mathcal{D}_{a_2}(t,z,(g(y)-T_A(y,\cdot,g))K(\cdot,y),\emptyset)|f(y)|dy\leq$$

$$\leq C\sum_{i=0}^{\infty} b^{i(n/r-\lambda_2)}w_i^1(t,z)\|\sup_{\xi>0}\xi^{-s_1}\mathcal{D}^c_{a_3}(\xi,\cdot,g,A)|L_r(Q_t(z))\|M^\rho_{\sigma_8}f(z); \quad (9)$$

$$\mathcal{D}_{a_2}(t,z,R_A(g,T)f_1,\emptyset)\leq\mathcal{D}_{a_2}\Big(t,z,\int_{Q_{t\delta_2}(\cdot)}(g(y)-T_A(y,\cdot,g))K(\cdot,y)f(y)dy,\emptyset\Big)\leq$$

$$\leq\mathcal{D}_{a_2}\Big(t,z,\sum_{i=0}^{\infty}\int_{\Delta_i^2(t,\cdot)}(g(y)-T_A(y,\cdot,g))K(\cdot,y)f(y)dy,\emptyset\Big)\leq$$

$$\leq C\sum_{i=0}^{\infty} b^{-i(n/r-\lambda_4)}w_i^2(t,z)\|\sup_{\xi>0}\xi^{-s_1}\mathcal{D}^c_{a_3}(\xi,\cdot,g,A)|L_r(Q_t(z))\|M_{\sigma_{10}}(M^\rho_{\sigma_9}f)(z); \quad (10)$$

$$M_{a_2}R_A(g,T)f(z) \le C(C^1_{\mathcal{AD}} + C^2_{\mathcal{AD}})\|g|\tilde{l}^{s_1,A}_{r,\infty}(\mathbb{R}^n)\|(M^{\rho}_{\sigma_8}f(z) + M_{\sigma_{10}}(M^{\rho}_{\sigma_9}f)(z)). \quad (11)$$

Thus, thanks to the boundedness of fractional and Hardy-Littlewood maximal functions, we obtain

$$\|R_A(g,T)f|L_{q_1}(\mathbb{R}^n)\| \le C(C^1_{\mathcal{AD}} + C^2_{\mathcal{AD}})\|g|\tilde{l}^{s_1,A}_{r,\infty}(\mathbb{R}^n)\|\|f|L_{p_0}(\mathbb{R}^n)\|. \quad (12)$$

In the second case of part b), Theorem 11.2 provide the pointwise estimate

$$R_A(g,T)f(x) \le C(C^1_{\mathcal{AD}} + C^2_{\mathcal{RAD}})\|g|\tilde{b}^{s_1,A}_{\infty,\infty}(\mathbb{R}^n)\| \int_{\mathbb{R}^n} |y-x|^{n(\rho-1)}|f(y)|dy,$$

implying (12) too due to the boundedness property of the potential operator. Therefore, to prove the second inequality of part b), it is sufficient to observe that Theorem 10.3 provide the embedding $X^{\lambda+\beta,\gamma,A}_{r,u,q,a}(\mathbb{R}^n) \subset \tilde{l}^{s_1,A}_{r,\infty}(\mathbb{R}^n)$. Eventually, with the aid of Theorem 10.2, we see that, thanks to its Lebesgue regularity, $R_A(g,T)f$ is such a representative of the class $R_A(g,T)f + \mathcal{P}_B$ that is also in $Y^{\lambda,D^*_\lambda}_{p_1,q,a_1}(\mathbb{R}^n))$. Due to part a), this observation finishes the proof. $\mathcal{Q.E.D.}$

Using Theorem 10.3, we obtain the following corollary.

COROLLARY 7.2. *Let A_1 and A_2 be valid with $\gamma + u^{-1} > r^{-1}$, $(\lambda+\beta)/n + \gamma - (\min(r,q))^{-1} + a^{-1} > 0$, as well as one of the following groups of conditions:*
i) $\zeta = u$, $Y^{\lambda,B}_{p_1,q,a_1} = \tilde{b}^{\lambda,B}_{p_1,q,a_1}(\mathbb{R}^n)$ and $\gamma^{-1} + u^{-1} > q^{-1}$;
ii) $\zeta = \min(u,q)$, $\zeta^{-1} + w_2^{-1} < 1$, $Y^{\lambda,B}_{p_1,q,a_1} = \tilde{l}^{\lambda,B}_{p_1,q,a_1}(\mathbb{R}^n)$.
Assume also that either $w_2 = \infty$, $a_1 < 1$, or $a_1 = 1$. Let a supersingular integral operator T of strong type $(\tilde{l}^{\beta}_{\sigma,1}(\mathbb{R}^n), L_\sigma(\mathbb{R}^n))$ be in the classes $\mathcal{AD}_y(w_0, L_{\sigma_0}, l_\infty, \lambda_0, \lambda_1, D)$, $\mathcal{AD}_y(w_1, L_{\sigma_1}, l_\infty, \lambda_2, \lambda_3, \emptyset)$ and $\mathcal{RAD}_y(L_{\sigma_2}, w_2, l_\infty, \lambda_4, \lambda_5, \emptyset)$, which definitions use constants $\delta_0, \delta_1, \delta_2$ correspondingly satisfying
$\bigcup_{w\in Q_t(z)} Q_{t\max(\delta_0,\delta_1)}(w) \subset Q_{t\delta}(z) \subset \bigcap_{w\in Q_t(z)} Q_{t\delta_2}(w)$ for some $\delta > 0$ and any $Q_t(z) \subset \mathbb{R}^n$.
Then one has:
a) $\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), Y^{\lambda,B}_{p_1,q,a_1})\| \le C\|g|\tilde{l}^{\lambda+\beta+n\gamma,A}_{r,q,a}(\mathbb{R}^n)\|$;
b) $\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), Y^{\lambda,D^*_\lambda}_{p_1,q,a_1})\| \le C\|g|\tilde{l}^{\lambda+\beta+n\gamma,A}_{r,q,a}(\mathbb{R}^n)\|$ and
$\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), L_{p_2}(\mathbb{R}^n))\| \le C\|g|\tilde{l}^{\lambda+\beta+n\gamma,A}_{r,q,a}\|$ for $p_1\lambda \le n$ and one of the following groups of conditions to be valid:
b_i) $p_0^{-1} + \lambda/n \ge p_1^{-1}$, $n/r \in (\lambda_4, \lambda_2)$, $\zeta^{-1} + p_0^{-1} + (\min(\sigma_1,\sigma_2)^{-1} < 1 + (\lambda+\beta)/n)$, $(\lambda+\beta)/n + w_1^{-1} > \zeta^{-1}$;
b_{ii} $r = w_1 = w_2 = \sigma_1 = \sigma_2 = \infty$, $\lambda_2 = \lambda_4 = 0$ and $p_0^{-1} - p_2^{-1} \in (0,1)$.

THEOREM 7.3. *Let A_1 and A_2 be valid with $\gamma = 0$ and either $r < \infty$, or $r = q = \infty$, as well as one of the following groups of conditions:*
i) $\zeta = r$, $Y^{\lambda,B}_{p_1,q,a_1} = \tilde{b}^{\lambda,B}_{p_1,q,a_1}(\mathbb{R}^n)$;
ii) $\zeta = \min(r,q)$, $\zeta^{-1} + w_2^{-1} < 1$, $Y^{\lambda,B}_{p_1,q,a_1} = \tilde{l}^{\lambda,B}_{p_1,q,a_1}(\mathbb{R}^n)$.
Assume also that either $w_2 = \infty$, $a_1 < 1$, or $a_1 = 1$. Let a supersingular integral operator T of strong type $(\tilde{l}^{\beta}_{\sigma,1}(\mathbb{R}^n), L_\sigma(\mathbb{R}^n))$ be in the classes $\mathcal{AD}_y(w_0, L_{\sigma_0}, l_\infty, \lambda_0, \lambda_1, D)$, $\mathcal{AD}_y(w_1, L_{\sigma_1}, l_\infty, \lambda_2, \lambda_3, \emptyset)$ and $\mathcal{RAD}_y(L_{\sigma_2}, w_2, l_\infty, \lambda_4, \lambda_5, \emptyset)$, which definitions use constants $\delta_0, \delta_1, \delta_2$ correspondingly satisfying
$\bigcup_{w\in Q_t(z)} Q_{t\max(\delta_0,\delta_1)}(w) \subset Q_{t\delta}(z) \subset \bigcap_{w\in Q_t(z)} Q_{t\delta_2}(w)$ *for some $\delta > 0$ and any $Q_t(z) \subset \mathbb{R}^n$.*
Then one has:
a) $\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), Y^{\lambda,B}_{p_1,q,a_1}(\mathbb{R}^n))\| \le C\|g|\tilde{l}^{\lambda+\beta,A}_{r,q,a}(\mathbb{R}^n)\|$;
b) $\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), Y^{\lambda,D^*_\lambda}_{p_1,q,a_1}(\mathbb{R}^n))\| \le C\|g|\tilde{l}^{\lambda+\beta,A}_{r,q,a}(\mathbb{R}^n)\|$ *and*
$\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), L_{p_2}(\mathbb{R}^n))\| \le C\|g|\tilde{l}^{\lambda+\beta,A}_{r,q,a}(\mathbb{R}^n)\|$ *for $p_1\lambda \le n$ and one of the following groups of conditions to be valid:*
b_i) $p_0^{-1} + \lambda/n \ge p_1^{-1}$, $n/r \in (\lambda_4, \lambda_2)$, $\zeta^{-1} + p_0^{-1} + (\min(\sigma_1,\sigma_2)^{-1} < 1 + (\lambda+\beta)/n)$, $(\lambda+\beta)/n + w_1^{-1} > \zeta^{-1}$;
b_{ii}) $r = w_1 = w_2 = \sigma_1 = \sigma_2 = \infty$, $\lambda_2 = \lambda_4 = 0$ *and* $p_0^{-1} - p_2^{-1} \in (0,1)$.

The proof of Theorem 7.3 is conducted as that of Theorem 7.1 except for the following step: we use Hölder inequality and Theorem 11.4 instead of Theorems 11.3 and 11.4 to derive estimate (8) from relations (1, 2, 4, 7). Parts *c*, *d*), *g*), *h*) of Theorem 10.1 are employed instead of *a*), *b*), *e*), *f*) respectively. $\mathcal{Q.E.D.}$

REMARK 8. Conditions on the relation between parameters q and p_1 in the conditions of the results of this section can be relaxed due to Theorem 6.1 and the next lemma proved similarly to estimating $I_1 + I_2$ in the proof of Theorem 7.1.

LEMMA 7.4. *For $0 \le i \le 1$, $0 \le j \le 3$, $w_i, \sigma_i \in [1,\infty]$, $\lambda_j \in \mathbb{R}$, $s > 0$, $A, B, D \subset \mathbb{N}_0^n$, $|B| < \infty$ and a function $g \in \tilde{b}^{s,A}_{\infty,\infty}(\mathbb{R}^n)$, assume $A \subset \overset{\circ}{D}{}^*_s$, $A + D \subset B$, $\lambda_1 - \lambda_0 = \lambda_3 - \lambda_2$, and integral operator T is in $\mathcal{AD}_y(w_0, L_{\sigma_0}, l_\infty, \lambda_0, \lambda_1, D) \cap \mathcal{AD}_y(w_1, L_{\sigma_1}, l_\infty, \lambda_2, \lambda_3, \emptyset)$. Then one has $R_A(g,T) \in \mathcal{AD}_y(\min(w_0,w_1), L_{\min(\sigma_0,\sigma_1)}, l_\infty, \mu_0, \mu_1, B)$ for $\mu_0 - \mu_1 = \lambda_0 - \lambda_1 + s$, $\mu_0 = \min(\lambda_0 + s - \lambda_{\max}(A), \lambda_2)$.*

8. Convolution SSIO with anisotropic homogeneous kernels

In this section, we apply the results of the previous section to a convolution SSIO with a homogeneous kernel defined by a 0-homogeneous function Ω contained in some Nikol'skii-type space on an annulus of the origin.

For $x \in \mathbb{R}^n$, let $|x|$ be either $|x|_q$, or $|x|_{q*}$. Here we consider approximation properties of the translation-invariant supersingular integral operators T_Ω^β of strong type $(l^\beta_{po,1}, L_{po})$ with the associated kernel $K_\Omega^\beta(x,y) = \Omega(x-y)|x-y|^{-(n+\beta)}$, where Ω is a 0-homogeneous function $\Omega(r^{\gamma_a}x) = \Omega(x)$ for any $r \in \mathbb{R}_+$ and a member of Nikol'skii space $B^{s_0}_{p,\infty}(Q_\xi(0) \setminus Q_1(0))$ with some $\xi > 1$.

LEMMA 8.1. *Assume* $\beta, \lambda, s_0 > 0$, $p \in [1,\infty]$, T_Ω^β *is the operator described above. Then one has*

a) $T \in \mathcal{AD}_y(w_0, L_{\sigma_0}, l_\infty, \lambda_0, \lambda_1, D)$ *for* $s_0/n + w_0^{-1} - p^{-1} > 0$, $\sigma_0 \le p$, $\lambda_1 - \beta = \lambda_0 \le s_0$, $D \supset D^*_{s_0}$;

b) $T \in \mathcal{AD}_y(w_1, L_{\sigma_1}, l_\infty, \lambda_2, \lambda_3, \emptyset)$ *for* $\max(w_1, \sigma_1) \le p$, $\lambda_3 - \beta = \lambda_2 \le 0$;

c) $T \in \mathcal{RAD}_y(L_{\sigma_2}, w_2, l_\infty, \lambda_4, \lambda_5, \emptyset)$ *for* $w_2 \in (0,\infty]$, $\sigma_2 \le p$, $\lambda_5 - \beta = \lambda_4 \ge 0$.

The proof of Lemma 8.1. In [**64**], the identity $B^{s_0}_{p,\infty}(Q_\xi(0) \setminus Q_1(0)) = \widetilde{B}^{s_0}_{p,\infty,w_0}(Q_\xi(0) \setminus Q_1(0))$ was established for $s_0/n + w_0^{-1} - p^{-1} > 0$. One can also note that $|x|^{-(n+\beta)}$ is a bounded pointwise multiplier in this space. There are $k_\xi \in \mathbb{N}$ and $\delta, b > 1$ satisfying

$$\bigcup_{w \in Q_{b^{(k_\xi+1)}\delta}(0) \setminus Q_{b^{k_\xi}\delta}(0)} Q_{b^{(k_\xi+1)}\delta}(w) \subset Q_\xi(0) \setminus Q_1(0).$$

Therefore, with the aid of the following homogeneity observations

$$\|\Omega| \cdot |^{-(n+\beta)}|\widetilde{b}^{s_0}_{p,\infty,w_0}(Q_{r\xi}(w) \setminus Q_r(0))\| = r^{-(n/p'+\beta+s_0)}\|\Omega| \cdot |^{-(n+\beta)}|\widetilde{b}^{s_0}_{p,\infty,w_0}(Q_\xi(0) \setminus Q_1(0))\|,$$

$$\|\Omega|\cdot|^{-(n+\beta)}|L_p(Q_{r\xi}(0)\setminus Q_r(0))\| = r^{-(n/p'+\beta)}\|\Omega|\cdot|^{-(n+\beta)}|L_p(Q_\xi(0)\setminus Q_1(0))\| \text{ for } r > 0,\ w \in \mathbb{R}^n,$$

using designations of the proof of Theorem 7.1, we obtain that

$$w_i^0(t,z) \le C\|\Omega|\widetilde{B}^{s_0}_{p,\infty,w_0}(Q_\xi(0) \setminus Q_1(0))\| \text{ for } \lambda_0 = s_0,\ \lambda_1 = s_0 + \beta,\ \sigma_0 = p;$$

$$w_i^1(t,z) \le C\|\Omega|L_p(Q_\xi(0) \setminus Q_1(0))\| \text{ for } w_1 = \sigma_1 = p,\ \lambda_2 = 0,\ \lambda_3 = \beta;$$

$$w_i^2(t,z) \le C\|\Omega|L_p(Q_\xi(0) \setminus Q_1(0))\| \text{ for } w_2 = \infty,\ \sigma_2 = p,\ \lambda_4 = 0,\ \lambda_5 = \beta.$$

The assertions of the lemma for the other values of the parameters of the $\mathcal{AD}$-classes under consideration follow from Remark 4. $\mathcal{Q.E.D.}$

It the rest of this section, we use the following assumptions:

A_3: for $\beta, \lambda, s_0 \in (0,\infty)$, $p \in (1,\infty]$, $\gamma \geq 0$, $p_0, p_1, q, u, a \in (0,\infty]$, $r \in [1,\infty]$, $a_1 \in (0,1]$, $B \subset \mathbb{N}_0^n$, assume $A + D_{s_0}^* \subset B$, $|B| < \infty$, $p_2^{-1} = p_1^{-1} - \lambda/n$, $a^{-1} + p^{-1} \leq 1$, $\zeta \geq 1$, $(\max(a,\zeta))^{-1} + p_0^{-1} + p^{-1} < 1$, $\lambda \in (0, s_0)$;

A_4: $A = D_\beta^* = \overset{\circ}{D_\beta^*} = \overset{\circ}{D_{\lambda+\beta}^*}$;

A_5: $\gamma = p_0^{-1} + r^{-1} - p_1^{-1}$;

A_6: $(q^{-1} - p_1^{-1})_+ \leq \gamma$.

This lemma, Theorems 7.1, 7.3 and Corollary 7.2 imply the following three assertions.

Corollary 8.2. *Let A_3, A_4, A_5 and A_6 be valid, as well as one of the following groups of conditions:*

i) $\zeta = u$, $Y_{p_1,q,a_1}^{\lambda,B} = \tilde{b}_{p_1,q,a_1}^{\lambda,B}(\mathbb{R}^n)$ *and* $X_{r,u,q,a}^{\lambda+\beta,\gamma,A} = bl_{r,u,q,a}^{\lambda+\beta,\gamma,A}(\mathbb{R}^n)$;

ii) $\zeta = \min(u,q) > 1$, $Y_{p_1,q,a_1}^{\lambda,B} = \tilde{l}_{p_1,q,a_1}^{\lambda,B}(\mathbb{R}^n)$ *and* $X_{r,u,q,a}^{\lambda+\beta,\gamma,A} = ll_{r,u,q,a}^{\lambda+\beta,\gamma,A}(\mathbb{R}^n)$.

Assume also that $T \in \{T_\Omega^\beta, T_\Omega^{\beta *}\}$. *Then one has:*

a) $\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), Y_{p_1,q,a_1}^{\lambda,B})\| \leq C\|g|X_{r,u,q,a}^{\lambda+\beta,\gamma,A}\|$;

b) $\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), Y_{p_1,q,a_1}^{\lambda,D_\lambda^*})\| \leq C\|g|X_{r,u,q,a}^{\lambda+\beta,\gamma,A}\|$ *and* $\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), L_{p_2}(\mathbb{R}^n))\| \leq C\|g|X_{r,u,q,a}^{\lambda+\beta,\gamma,A}\|$ *for* $p = r = \infty$, $p_1\lambda \leq n$, $p_0^{-1} - p_2^{-1} \in (0,1)$.

Corollary 8.3. *Let A_3, A_4, A_5 and A_6 with $(\lambda+\beta)/n + \gamma - (\min(r,q))^{-1} + a^{-1} > 0$, $p_0^{-1} + u^{-1} > p_1^{-1}$ be valid, as well as one of the following groups of conditions:*

i) $\zeta = u$, $Y_{p_1,q,a_1}^{\lambda,B} = \tilde{b}_{p_1,q,a_1}^{\lambda,B}(\mathbb{R}^n)$ *and* $p_0^{-1} + u^{-1} + r^{-1} > p_1^{-1} + q^{-1}$;

ii) $\zeta = \min(u,q) > 1$, $Y_{p_1,q,a_1}^{\lambda,B} = \tilde{l}_{p_1,q,a_1}^{\lambda,B}(\mathbb{R}^n)$.

Assume also that $T \in \{T_\Omega^\beta, T_\Omega^{\beta *}\}$. *Then one has:*

a) $\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), Y_{p_1,q,a_1}^{\lambda,B})\| \leq C\|g|\tilde{l}_{r,q,a}^{\lambda+\beta+n\gamma,A}(\mathbb{R}^n)\|$;

b) $\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), Y_{p_1,q,a_1}^{\lambda,D_\lambda^*})\| \leq C\|g|\tilde{l}_{r,q,a}^{\lambda+\beta+n\gamma,A}(\mathbb{R}^n)\|$ *and* $\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), L_{p_2}(\mathbb{R}^n))\| \leq C\|g|\tilde{l}_{r,q,a}^{\lambda+\beta+n\gamma,A}(\mathbb{R}^n)\|$ *for* $p = r = \infty$, $p_1\lambda \leq n$, $p_0^{-1} - p_2^{-1} \in (0,1)$.

Corollary 8.4. *Let A_3, A_4 and A_5 with $\gamma = 0$ and either $r < \infty$, or $r = q = \infty$ be valid, as well as one of the following groups of conditions:*

i) $\zeta = r$, $Y_{p_1,q,a_1}^{\lambda,B} = \tilde{b}_{p_1,q,a_1}^{\lambda,B}(\mathbb{R}^n)$;

ii) $\zeta = \min(r,q) > 1$, $Y_{p_1,q,a_1}^{\lambda,B} = \tilde{l}_{p_1,q,a_1}^{\lambda,B}(\mathbb{R}^n)$.

Assume also that $T \in \{T_\Omega^\beta, T_\Omega^{\beta *}\}$. *Then one has:*

a) $\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), Y^{\lambda,B}_{p_1,q,a_1})\| \leq C\|g|\tilde{l}^{\lambda+\beta,A}_{r,q,a}(\mathbb{R}^n)\|$;

b) $\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), Y^{\lambda,D^*_\lambda}_{p_1,q,a_1})\| \leq C\|g|\tilde{l}^{\lambda+\beta,A}_{r,q,a}(\mathbb{R}^n)\|$ *and*

$\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), L_{p_2}(\mathbb{R}^n))\| \leq C\|g|\tilde{l}^{\lambda+\beta,A}_{r,q,a}(\mathbb{R}^n)\|$ *for* $p = r = \infty$, $p_1\lambda \leq n$, $p_0^{-1} - p_2^{-1} \in (0,1)$.

In particular, combining, correspondingly, each of corollaries 8.3, 8.4 and Theorem 10.4, we obtain the next two results.

Corollary 8.5. *Let* A_3 *be valid, as well as* $u \in (0,\infty]$, $s = \lambda+\beta+n(p_0^{-1}-p_1^{-1}+r^{-1})$, $A = D^*_\beta = \overset{\circ}{D}{}^*_s$, $p_0^{-1} - p_1^{-1} + u^{-1} > 0$, $(\min(p_1,q))^{-1} \leq p_0^{-1} + r^{-1}$, $s/n + r^{-1} + 1/p' - (\min(r,q))^{-1} > 0$, $s \geq \lambda+\beta$,*Assume also that either* $s/n - (\min(r,q))^{-1} + 1 > p_0^{-1} + p^{-1}$, *or* $\zeta^{-1} + p_0^{-1} + p^{-1} < 1$, *and that operator* $T \in \{T_\Omega^\beta, T_\Omega^{\beta *}\}$. *Then*

for $\zeta = u$, $p_0^{-1} + u^{-1} + r^{-1} > p_1^{-1} + q^{-1}$, *one has*

a) $\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), b^{\lambda,B}_{p_1,q}(\mathbb{R}^n))\| \leq C\|g|l^{s,A}_{r,q}(\mathbb{R}^n)\|$,

b) $\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), b^{\lambda,D^*_\lambda}_{p_1,q}(\mathbb{R}^n))\| \leq C\|g|l^{s,A}_{r,q}(\mathbb{R}^n)\|$ *and*

$\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), L_{p_2}(\mathbb{R}^n))\| \leq C\|g|l^{s,A}_{r,q}(\mathbb{R}^n)\|$ *if, in addition,* $p = r = \infty$, $p_1\lambda \leq n$, $p_0^{-1} - p_2^{-1} \in (0,1)$;

for $\zeta = \min(u,q) > 1$, *one has*

c) $\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), l^{\lambda,B}_{p_1,q}(\mathbb{R}^n))\| \leq C\|g|l^{s,A}_{r,q}(\mathbb{R}^n)\|$,

d) $\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), l^{\lambda,D^*_\lambda}_{p_1,q}(\mathbb{R}^n))\| \leq C\|g|l^{s,A}_{r,q}(\mathbb{R}^n)\|$ *and*

$\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), L_{p_2}(\mathbb{R}^n))\| \leq C\|g|l^{s,A}_{r,q}(\mathbb{R}^n)\|$ *if, in addition,* $p = r = \infty$, $p_1\lambda \leq n$, $p_0^{-1} - p_2^{-1} \in (0,1)$.

Corollary 8.6. *Let* A_3, A_4 *and* A_5 *be valid with* $\gamma = 0$ *and either* $r < \infty$, *or* $r = q = \infty$, *as well as* $\zeta^{-1} + p_0^{-1} + p^{-1} < 1 + (\lambda+\beta)/n$ *and* $T \in \{T_\Omega^\beta, T_\Omega^{\beta *}\}$. *Then*

for $\zeta = r$, *one has*

a) $\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), b^{\lambda,B}_{p_1,q}(\mathbb{R}^n))\| \leq C\|g|l^{s,A}_{r,q}(\mathbb{R}^n)\|$,

b) $\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), b^{\lambda,D^*_\lambda}_{p_1,q}(\mathbb{R}^n))\| \leq C\|g|l^{s,A}_{r,q}(\mathbb{R}^n)\|$ *and*

$\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), L_{p_2}(\mathbb{R}^n))\| \leq C\|g|l^{s,A}_{r,q}(\mathbb{R}^n)\|$ *if, in addition,* $p = r = \infty$, $p_1\lambda \leq n$, $p_0^{-1} - p_2^{-1} \in (0,1)$;

for $\zeta = \min(r,q) > 1$, *one has*

c) $\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), l^{\lambda,B}_{p_1,q}(\mathbb{R}^n))\| \leq C\|g|l^{s,A}_{r,q}(\mathbb{R}^n)\|$,

d) $\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), l^{\lambda,D^*_\lambda}_{p_1,q}(\mathbb{R}^n))\| \leq C\|g|l^{s,A}_{r,q}(\mathbb{R}^n)\|$ *and*
$\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), L_{p_2}(\mathbb{R}^n))\| \leq C\|g|l^{s,A}_{r,q}(\mathbb{R}^n)\|$ *if, in addition,* $p = r = \infty$, $p_1\lambda \leq n$, $p_0^{-1} - p_2^{-1} \in (0,1)$.

REMARK 9. As was pointed out in the end of Section 7, one can use Lemmas 7.4, 8.1 and Theorem 6.1 to extend results of the current section.

9. Fractional derivatives. Optimality of the isotropic case

To see the optimality of the isotropic case $\gamma_a = (1,\dots,1)$, let us note that, under the conditions of Corollaries 8.2 – 8.4, we have $0 < \lambda < \gamma_{a\min}$, while one always has $\gamma_{a\min} \leq 1$.

Noting that, for $\lambda + \beta \leq \gamma_{a\min}$, the anisotropic fractional differentiation operator $|\mathrm{D}|^\beta$ with symbol $|\xi|^\beta_{2*}$, $\beta > 0$ is associated with the kernel of the form $\Omega_\beta(x-y)|x-y|^{-(n+\beta)}$, and that, in accordance with [**52**], $\Omega_\beta(x)$ is not equal to 0 on $\mathbb{R}^n \setminus \{0\}$ and is in $C^\infty(\mathbb{R}^n \setminus \{0\})$, we use Corollary 8.4 and Remark 7, *c)* to establish the following result.

THEOREM 9.1. *Assume* $p_0, p_1, \theta \in [1,\infty)$, $q \in [1,\infty)$, $\beta, \lambda > 0$, $\beta + \lambda \in (0, \gamma_{a\min}]$, $\lambda/n = p_0^{-1} - p_1^{-1}$. *Then*
a) the inclusion $|\mathrm{D}|^\beta M(g) - M(g)|\mathrm{D}|^\beta \in \mathcal{L}(L_{p_0}(\mathbb{R}^n), \tilde{l}^{\lambda,A}_{p_0,\theta}(\mathbb{R}^n))$ *implies*
$\|g|\tilde{l}^{\lambda+\beta,\{0\}}_{\infty,\theta}(\mathbb{R}^n)\| \leq C\||\mathrm{D}|^\beta M(g) - M(g)|\mathrm{D}|^\beta|\mathcal{L}(L_{p_0}(\mathbb{R}^n), \tilde{l}^{\lambda,A}_{p_0,\theta}(\mathbb{R}^n))\|$;
b) the inclusion $|\mathrm{D}|^\beta M(g) - M(g)|\mathrm{D}|^\beta \in \mathcal{L}(L_{p_0}(\mathbb{R}^n), L_{p_1}(\mathbb{R}^n))$ *implies*
$\|g|\tilde{b}^{\lambda+\beta,\{0\}}_{\infty,\infty}(\mathbb{R}^n)\| \leq C\||\mathrm{D}|^\beta M(g) - M(g)|\mathrm{D}|^\beta|\mathcal{L}(L_{p_0}(\mathbb{R}^n), L_{p_1}(\mathbb{R}^n))\|$.

The situation with necessary conditions is essentially better with Lizorkin's Liouville fractional differentiation from Example 5.2 (with $\prod_{i=1}^n \alpha_i \neq 0$) thanks to Theorem 5.3.

THEOREM 9.2. *Let* $p_0, p_1, \theta \in [1,\infty)$, $\beta/\gamma_a = \alpha$, $q \in [1,\infty)$, $\alpha \in (0,\infty)^n$, $A \subset \mathring{D}^*_{(\alpha,\gamma_a)}$, $\lambda/n = p_0^{-1} - p_1^{-1}$, *and* g *be a measurable function. Then*
a) the inclusion $R_A(g, \mathrm{D}^\alpha) \in \mathcal{L}(L_{p_0}(\mathbb{R}^n), \tilde{l}^{\lambda,A}_{p_0,\theta}(\mathbb{R}^n))$ *implies*
$\|g|\tilde{l}^{\lambda+\beta,A}_{\infty,\theta}(\mathbb{R}^n)\| \leq C\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), \tilde{l}^{\lambda,A}_{p_0,\theta}(\mathbb{R}^n))\|$;
b) the inclusion $R_A(g, \mathrm{D}^\alpha) \in \mathcal{L}(L_{p_0}(\mathbb{R}^n), L_{p_1}(\mathbb{R}^n))$ *and Lebesgue-regularity of* $R_A(g, \mathrm{D}^\alpha)$ *imply*
$\|g|\tilde{b}^{\lambda+\beta,A}_{\infty,\infty}(\mathbb{R}^n)\| \leq C\|R_A(g,T)|\mathcal{L}(L_{p_0}(\mathbb{R}^n), L_{p_1}(\mathbb{R}^n))\|$.

A theorem of P. I. Lizorkin **[43]** shows that his variant of Riemann-Liouville differentiation (see Example 5.2, *b*)) is of strong type $(l_{p,2}^{(\alpha,\gamma_a)}(\mathbb{R}^n), L_p(\mathbb{R}^n))$. But it is an object for a multilinear approximation which will be considered in a future article.

For $1 \le i \le n$, let R_i be a Riesz transform described by the multiplier $x_i/|x|_{2*}$ and $R^\alpha := \prod_{i=1}^n R_i^{\alpha_i}$ for $\alpha \in \mathbb{N}_0^n$.

In the isotropic case, results of Sections 5, 8 imply the following extended answer to S. Hofmann's question mentioned in Introduction.

THEOREM 9.3. *For* $\beta, \lambda > 0$, $p \in (0,\infty)$, $m \in \mathbb{N}_0$, $m < \beta < \beta + \lambda < m+1$ *and* $T \in \{R^\alpha|\mathrm{D}|^\beta\}_{1\le i\le n}^{\alpha\in\mathbb{N}_0^n}$, *the boundedness* $|\mathrm{D}|^\lambda R_{D_m^*}(g,T) \in \mathcal{L}(L_p(\mathbb{R}^n))$ *is equivalent to the inclusion* $|\mathrm{D}|^{\beta+\lambda} g \in BMO(\mathbb{R}^n)$. *In particular, we have the following estimate:*

$$\| |\mathrm{D}|^\lambda (R_{D_m^*}(g,T)) | \mathcal{L}(L_p(\mathbb{R}^n))\| \asymp \| |\mathrm{D}|^{\gamma+\lambda} g | BMO(\mathbb{R}^n)\|.$$

Let us note that, in particular, considering, for example, both $|\mathrm{D}|^\beta M(g)$ and $R_{D_m^*}(g, |\mathrm{D}|^\beta)$ as operators from $L_{p,2}^\beta$ into L_p, we are approximating the inconvenient operator $|\mathrm{D}|^\beta M(g)$ by the linear combination of pseudodifferential operators with variable coefficients modulo compact residual $R_{D_m^*}(|\mathrm{D}|^\beta)$, when function g is contained in BMO-Sobolev space.

These results reveal the following two differences between the case, when T is a purely differential operator, for instance, $T = \sum_{|\alpha|\le m} \mathrm{D}^\alpha, \Delta^k$, and operators like fractional differentiation. Firstly, Corollary 4.5 show that approximation properties of the residual operator is of a coercive nature for the former case: it does not depend on the Sobolev norm of g but only on $\|S_m g | L_p\|$. But the necessary condition related to Hofmann's question shows that the latter case depends on the norm of g in the BMO-Sobolev space. Secondly, the residuals of the former operators do not add smoothness in distinction with the residuals of the latter ones.

10. Some properties of function spaces

This section contains the auxiliary results pertaining to the theory of function spaces. The next theorem contains results in a generality sufficient for our purposes.

THEOREM 10.1. *For* $s, \gamma \ge 0$, $p, q, u, a, a_1 \in (0,\infty]$, $\emptyset \ne D \subset \overset{\circ}{D}{}_s^*$ *and* $\beta \in D$, *one has:*

a) $\|f | bl_{p,u,q,a}^{s,\gamma,D}(\mathbb{R}^n)\| \asymp \|f | bl_{p,u,q,a}^{s,\gamma,D}(\mathbb{R}^n)\|_T$;

b) $\|f|ll^{s,\gamma,D}_{p,u,q,a}(\mathbb{R}^n)\| \asymp \|f|ll^{s,\gamma,D}_{p,u,q,a}(\mathbb{R}^n)\|_T$;

c) $\|f|\widetilde{b}^{s,D}_{p,q,a}(\mathbb{R}^n)\| \asymp \|f|\widetilde{b}^{s,D}_{p,q,a}(\mathbb{R}^n)\|_T$;

d) $\|f|\widetilde{l}^{s,D}_{p,q,a}(\mathbb{R}^n)\| \asymp \|f|\widetilde{l}^{s,D}_{p,q,a}(\mathbb{R}^n)\|_T$;

e) $\mathrm{D}^\beta \in \mathcal{L}(bl^{s,\gamma,D}_{p,u,q,a}(\mathbb{R}^n), bl^{s-(\alpha,\gamma_a),\gamma,(D-\beta)_+}_{p,u,q,a}(\mathbb{R}^n))$ *for a finite* $a \leq u$;

f) $\mathrm{D}^\beta \in \mathcal{L}(bl^{s,\gamma,D}_{p,u,q,a}(\mathbb{R}^n), bl^{s-(\alpha,\gamma_a),\gamma,(D-\beta)_+}_{p,u,q,a}(\mathbb{R}^n))$ *for* $a < \min(u, q, \infty)$;

g) $\mathrm{D}^\beta \in \mathcal{L}(\widetilde{b}^{s,D}_{p,q,a}(\mathbb{R}^n), \widetilde{b}^{s-(\alpha,\gamma_a),(D-\beta)_+}_{p,q,a}(\mathbb{R}^n))$ *for a finite* $a \leq p$;

h) $\mathrm{D}^\beta \in \mathcal{L}(\widetilde{l}^{s,D}_{p,q,a}(\mathbb{R}^n), \widetilde{l}^{s-(\alpha,\gamma_a),(D-\beta)_+}_{p,q,a}(\mathbb{R}^n))$ *for* $a < \min(p, q, \infty)$;

i) $\widetilde{b}^{s,D}_{p,q,a}(\mathbb{R}^n) \subset \widetilde{b}^{s,D}_{p,q,a_1}(\mathbb{R}^n)$ *for a finite* $a_1 \leq \max(a, p)$;

k) $\widetilde{l}^{s,D}_{p,q,a}(\mathbb{R}^n) \subset \widetilde{l}^{s,D}_{p,q,a_1}(\mathbb{R}^n)$ *for either* $a_1 \leq a$, *or* $a_1 < \min(p, q)$.

The proof of Theorem 10.1. Let us designate $\varepsilon = s - \lambda_{\max}(D) > 0$ and $r := \min(p, q, u, a, 1)$ and note that, thanks to Hölder inequality, we may assume $a \leq a_1$ in parts *i)*, *k)* without real loss of generality. Immediate consequences of the definitions are

$$\|f|bl^{s,\gamma,D}_{p,u,q,a}(\mathbb{R}^n)\| \leq \|f|bl^{s,\gamma,D}_{p,u,q,a}(\mathbb{R}^n)\|_T, \quad \|f|ll^{s,\gamma,D}_{p,u,q,a}(\mathbb{R}^n)\| \leq \|f|ll^{s,\gamma,D}_{p,u,q,a}(\mathbb{R}^n)\|_T, \tag{1}$$

$$\|f|\widetilde{b}^{s,D}_{p,q,a}(\mathbb{R}^n)\| \leq \|f|\widetilde{b}^{s,D}_{p,q,a}(\mathbb{R}^n)\|_T, \quad \|f|\widetilde{l}^{s,D}_{p,q,a}(\mathbb{R}^n)\| \leq \|f|\widetilde{l}^{s,D}_{p,q,a}(\mathbb{R}^n)\|_T. \tag{2}$$

In the same time, Lemma 11.5 provides the key estimates

$$r^{-s}\mathcal{D}^c_{a_1}(r, w, f, A) \leq C(\mathcal{D}_{a_1}(r, w, f, A) + \|\{b^{-i\varepsilon}(rb^{-i})^{-s}\mathcal{D}_a(rb^{-i}, w, f, A)\}_{i\in\mathbb{N}_0}|l_r\|), \tag{3}$$

$$r^{(\beta,\gamma_a)-s}\mathcal{D}_{a_1}(r, w, \mathrm{D}^\beta f, (A-\beta)_+) \leq C(\mathcal{D}_{a_1}(r, w, \|\{b^{-i\varepsilon}(rb^{-i})^{-s}\mathcal{D}_a(rb^{-i}, \cdot, f, A)\}_{i\in\mathbb{N}_0}|l_r\|, \emptyset) + $$
$$+\mathcal{D}_a(r\kappa, w, f, A)). \tag{4}$$

The rest of the proof is conducted in a routine way with the aid of (3, 4) and Hardy, Minkowski and Fefferman-Stein inequalities as is presented in [**22**]. Note that the cases $a_1 = a$ and $\beta = 0$ corresponds, respectively, to parts *a)* – *h)* and *i)*, *k)*. $\mathcal{Q.E.D.}$

THEOREM 10.2. *For* $D^*_s \subset A \subset B \subset \mathbb{N}^n_0$, $|B| < \infty$, $s, \gamma \geq 0$, $a \in [0, \infty)$ *and* $p, q \in (0, \infty]$, *let the space* $X^{s,D}_{p,q,a}$ *be either* $\widetilde{b}^{s,D}_{p,q,a}(\mathbb{R}^n)$, *or* $\widetilde{l}^{s,D}_{p,q,a}(\mathbb{R}^n)$. *Then, for any function* $f \in X^{s,B}_{p,q,a}$, *there is a unique polynomial* $pf \in \mathcal{P}_{B\setminus A}$ *satisfying*

a) $\|f - pf|X^{s,A}_{p,q,a}\| \leq C\|f|X^{s,B}_{p,q,a}\|$;

b) $pf = 0$ *if* f *is Lebesgue-regular with respect to* $\mathcal{P}_A$.

REMARK 10. Part *b)* of Theorem 10.2 remains true with the requirement of Chebyshev regularity instead of Lebesgue one.

The proof of Theorem 10.2. Designations from the proof of Lemma 11.6 will be used. Let us begin with an attempt to find pf for a fixed $f \in X^{s,B}_{p,q,a}$. Choosing $\kappa > 0$ to satisfy $Q_r(w) \subset \bigcap_{z\in Q_r(z)} Q_{r\kappa}(z)$, we observe that, for any $w \in \mathbb{R}^n$ and f from the Nikol'skii-type space $\tilde{b}^{s,B}_{p,\infty,a}(\mathbb{R}^n) \supset \tilde{b}^{s,D}_{p,q,a}(\mathbb{R}^n) \cap \tilde{l}^{s,D}_{p,q,a}(\mathbb{R}^n)$, one has

$$r^{-s}\mathcal{D}_a(r, w, f, A) \leq C\mathcal{D}_p'(r, w, (r\kappa)^{-s}\mathcal{D}_a(r\kappa, \cdot, f, A), \emptyset) \leq$$

$$\leq Cr^{-n/a}\|f|\tilde{b}^{s,B}_{p,\infty,a}(\mathbb{R}^n)\| \overset{r\to\infty}{\longrightarrow} 0. \tag{1}$$

For $D \subset \mathbb{N}^n_0$ and any polynomial $\pi \in \mathcal{P}_B$, $\pi(x) = \sum_{\alpha\in B} c_\alpha x^\alpha$, let $p_D\pi(x)$ be its part $\sum_{\alpha\in B\cap D} c_\alpha x^\alpha$. Thus, due to Lemma 11.6, a), b), we have, for $t > t/b > r$, $b > 1$ and $D = B \setminus A$,

$$\|p_D\pi_{t,w}f - p_D\pi_{t/b,w}f|L_\infty(Q_r(w))\| \leq C(r/t)^{\lambda_{\min}(B\setminus A)}\mathcal{D}_a(t, w, f, B). \tag{2}$$

Therefore, there exists $p_Df := p_D\pi_{r,w}f + \sum_{i\in\mathbb{N}_0} p_D(\pi_{rb^{i+1},w}f - \pi_{rb^i,w}f)$ uniformly on $Q_r(w)$ satisfying

$$\mathcal{D}_a(r, w, f - p_Df, A) \leq C(\mathcal{D}_a(r, w, f, B) + \|\{b^{-i\lambda_{\min}(B\setminus A)}\mathcal{D}_a(rb^i, w, f, B)\}_{i\in\mathbb{N}_0}|l_{\min(a,1)}\|). \tag{3}$$

On one hand, p_Df does not depend on r. On the other hand, for some $\kappa > 0$ and any other $w_1 \in \mathbb{R}^n$, there exists sufficiently big r_0 such that, for $t > r_0$, due to Lemma 11.6, a), c) and (2), we have $Q_{t\kappa}(w) \supset Q_t(w_1) \supset Q_r(w)$ and

$$\|p_D(\pi_{t,w}f - \pi_{t,w_1}f)|L_\infty(Q_r(w))\| \leq C(r/t)^{\lambda_{\min}(B\setminus A)-s}(t\kappa)^{-s}(\mathcal{D}_a(t\kappa, w, f, B). \tag{4}$$

Thus, it can be seen from (1) that the right-hand side of (4) tends to 0 with $t \to \infty$, and p_Df does not depend on w too. With the aid of (3), we obtain, for $r = \min(p, q, a, 1)$,

$$r^{-s}\mathcal{D}_a(r, w, f - p_Df, A) \leq C(r^{-s}\mathcal{D}_a(r, w, f, B) +$$

$$+\|\{b^{i(s-\lambda_{\min}(B\setminus A))}(rb^i)^{-s}\mathcal{D}_a(rb^i, w, f, B)\}_{i\in\mathbb{N}_0}|l_r\|). \tag{5}$$

Applying Hardy and Minkowski inequalities to (5), we finish the proof of part a).

Under the conditions of part b), the norm $\|f|L_u(\mathbb{R}^n)\|$ is finite for some $u \in (0, \infty)$. Because the dimension of $\mathcal{P}_B$ is finite, we see from Lemma 11.6, b) that, for any $w \in \mathbb{R}^n$,

$$\|p_D\pi_{r,w}|L_\infty(Q_r(w))\| \leq C\|\pi_{r,w}|L_\infty(Q_r(w))\| \leq$$

$$\leq C\mathcal{D}_u(r, w, f, \emptyset) \leq Cr^{-n/u}\|f|L_u(\mathbb{R}^n)\| \overset{r\to\infty}{\longrightarrow} 0, \tag{6}$$

while $p_Df = \lim_{r\to\infty} p_D\pi_{r,w}f$. It proves the theorem. $\mathcal{Q.E.D}$.

The next theorem is established in [**65**].

THEOREM 10.3. [**65**] *For* $D \subset \mathbb{N}_0^n$, $|D| \in (0,\infty)$, $s,\gamma \geq 0$, $a,u \in [0,\infty)$, $p,q \in (0,\infty]$, *assume* $\gamma + u^{-1} > p^{-1}$, $s/n + \gamma - (\min(p,q))^{-1} + a^{-1} > 0$. *Then the following relations take place:*

a) $\widetilde{l}^{s+n\gamma,D}_{p,q}(\mathbb{R}^n) = ll^{s,\gamma,D}_{p,u,q,a}(\mathbb{R}^n)$,

for $\gamma = 0$ *and* $u > 0$;

b) $\widetilde{l}^{s+n\gamma,D}_{p,\infty}(\mathbb{R}^n) = ll^{s,\gamma,D}_{p,u,q,a}(\mathbb{R}^n)$ *for* $\gamma, u > 0$;

c) $\widetilde{l}^{s+n\gamma,D}_{p,\infty}(\mathbb{R}^n) = bl^{s,\gamma,D}_{p,u,q,a}(\mathbb{R}^n)$ *for* $\gamma, u > 0$, $\gamma + u^{-1} > q^{-1}$;

d) $\widetilde{l}^{s+n\gamma,D}_{p,q}(v) \subset bl^{s,\gamma,D}_{p,u,q,a}(\mathbb{R}^n)$ *for* $\gamma = 0$, $u < \min(p,q)$;

e) $bl^{s,\gamma,D}_{p,u,q,a}(\mathbb{R}^n) \subset \widetilde{l}^{s+n\gamma,D}_{p,q}(\mathbb{R}^n)$ *for* $\gamma = 0$, $q \leq u$.

The following theorem is a simple consequence of [**64, 66, 62**].

THEOREM 10.4. *For* $p,q,a \in (0,\infty]$, $s > 0$, $A \subset \mathbb{N}_0^n$, $|A| < \infty$, *let* $s/n + (\max(a,1))^{-1} > \zeta^{-1}$, $A \supset D^*_s$. *Then one has*

$$b^{s,A}_{p,q}(\mathbb{R}^n) = \tilde{b}^{s,A}_{p,q,a}(\mathbb{R}^n) \text{ for } \zeta = p \text{ and } l^{s,A}_{p,q}(\mathbb{R}^n) = \tilde{l}^{s,A}_{p,q,a}(\mathbb{R}^n) \text{ for } \zeta = \min(p,q).$$

11. Auxiliary results

The first theorem here is proved by I. M. Gel'fand and G. E. Shilov.

THEOREM 11.1. [**29**] *Let* $f \in \mathcal{D}'$ *be compactly supported. Then there are* $m \in \mathbb{N}_0$, $\{f_\alpha\}_{|\alpha| \leq m} \subset L_\infty$ *satisfying* $f = \sum_{|\alpha| \leq m} M(f_\alpha)\mathrm{D}^\alpha$.

The next result is a well-known addition to Theorem 10.1.

THEOREM 11.2. *For* $s > 0$ *and* $A \overset{\circ}{\subset} D^*_s$, *space* $\widetilde{b}^{s,A}_{\infty,\infty}(\mathbb{R}^n)$

a) possesses the following equivalent norm

$$\|f|\widetilde{b}^{s,A}_{\infty,\infty}(\mathbb{R}^n)\|_T := \sup_{x,y \in \mathbb{R}^n} |y-x|^{-s}|f(y) - \sum_{\alpha \in A}(\alpha!)^{-1}(y-x)^\alpha \mathrm{D}^\alpha f(x)|,$$

b) and one has $\mathrm{D}^\alpha \in \mathcal{L}(\widetilde{b}^{s,A}_{\infty,\infty}(\mathbb{R}^n), \widetilde{b}^{s-(\alpha,\gamma_a),(A-\alpha)_+}_{\infty,\infty}(\mathbb{R}^n))$ *for any* $\alpha \in A$.

The following theorem is obtained in [**65**].

THEOREM 11.3. [**65**] *For* $s,\gamma \geq 0$, $p_0, p_1, q, r \in (0,\infty)$, $b > 1$, *let* $\sigma^{-1} = \gamma + p_1^{-1} \geq (\min(p_1,q))^{-1} > 0$, $\gamma = p_0^{-1} - p_1^{-1} + r^{-1}$. *Then there is a constant* $C > 0$ *satisfying, for* *any functions f, g measurable on $\mathbb{R}_+ \times \mathbb{R}^n$,*

$$a)\ \|fg|L_{(p_1,q*)}\| \le C\|MB^{\gamma}_{p,q}g|L_r(\mathbb{R}^n)\|\|A_b(f)|L_{p_0}(\mathbb{R}^n)\|,$$

$$b)\ \|fg|L_{(q*,p_1)}\| \le C\|ML^{\gamma}_{p,q}g|L_r(\mathbb{R}^n)\|\|A_b(f)|L_{p_0}(\mathbb{R}^n)\|,$$

where $A_b(f)(x) = \sup^{t>0}_{x\in Q_{bt}(y)} \|f(t,\cdot)|L_\infty(Q_{bt}(y))\|$.

The last assertion contains the results on the boundedness of fractional and Hardy-Littlewood maximal functions.

THEOREM 11.4. *For $s \ge 0$, $a, p, q \in (0, \infty]$, let $p^{-1} = s + q^{-1}$ and either $a < p < \infty$, or $a^{-1} = p^{-1} = s$. Then one has*

$$\|M^s_a f|L_q(\mathbb{R}^n)\| \le C\|f|L_p(\mathbb{R}^n)\|.$$

The rest of this section is constituted of results proved with the aid of ideas utilized in [**27**, **67**, **22**].

For $A \subset \mathbb{N}^n_0$, $a \in (0,\infty]$, $Q_r(w) \subset \mathbb{R}^n$, we assume that $\pi_{r,w} = \pi_{r,w,A}$ is a (non-linear) approximation operator from $L_a(Q_r(w))$ onto $\mathcal{P}_A$ satisfying $r^{-n/a}\|f - \pi_{r,w}f|L_a(Q_r(w))\| \le (1+\varepsilon)\mathcal{D}_a(r, w, f, A)$ for some $\varepsilon > 0$. Let also $\|\pi|Q_r(w)\|_* := \sum_{\alpha\in A} |a_\alpha| r^{(\alpha,\gamma_a)}$.

LEMMA 11.5. *For $a, a_1 \in (1, \infty]$, $s > 0$, $b > 1$, $\beta \in \mathbb{N}_0$, $\{\alpha \in A : \alpha \ge \beta\} \ne \emptyset$, $f \in \tilde{b}^{s,A}_{p,q,a}(\mathbb{R}^n) \cap \tilde{l}^{s,A}_{p,q,a}(\mathbb{R}^n)$ and $(\beta, \gamma_a) < s$, the following estimates hold:*
a) $\mathcal{D}^c_{a_1}(r, w, f, A) \le C(\mathcal{D}_{a_1}(r, w, f, A) + \|\{b^{\lambda_{\max}(A)i}\mathcal{D}_a(rb^{-i}, w, f, A)\}_{i\in\mathbb{N}_0}|l_{\min(a_1,1)}\|)$;
b) $\mathcal{D}_{a_1}(r, w, \mathrm{D}^\beta f, (A-\beta)_+) \le Cr^{-(\beta,\gamma_a)}(\mathcal{D}_{a_1}(r, w, \|\{b^{i(\beta,\gamma_a)}\mathcal{D}_a(rb^{-i}, \cdot, f, A)\}_{i\in\mathbb{N}_0}|l_1\|, \emptyset) +$ $+\mathcal{D}_a(r\kappa, w, f, A))$, *where* $Q_{r\kappa}(w) \supset \bigcup_{z\in Q_r(w)} Q_r(z)$.

The proof of Lemma 11.5. To prove part *a*), we note that

$$f(x) - T_A(x, w, f) = f(x) - \pi_{r,w}f(x) + \sum_{\alpha\in A}(x-w)^\alpha/\alpha! \sum_{k\in\mathbb{N}_0} \mathrm{D}^\alpha(\pi_{b^{-k}r,w}f - \pi_{b^{-k-1}r,w}f)(x). \quad (1)$$

Thus, a rough estimate $\|(\cdot - w)^\alpha|L_\infty(Q_r(w))\| \le Cr^{(\alpha,\gamma_a)}$ and Lemma 11.6, *a*), *c*) prove *a*). To consider *b*), we need a counterpart of (1) at each point x instead of w:

$$\mathrm{D}^\beta(f - \pi_{r,w}f)(x) = \sum_{k\in\mathbb{N}_0} \mathrm{D}^\beta(\pi_{b^{-k}r,x}f - \pi_{b^{-k-1}r,x}f)(x) +$$

$$+\mathrm{D}^\beta(\pi_{r,x}f - \pi_{r\kappa,w}f)(x) + \mathrm{D}^\beta(\pi_{r\kappa,w}f - \pi_{r,w}f)(x), \quad (2)$$

which is true thanks to Corollary 11.7. Another application of Lemma 11.6, *a*), *c*) to (2) is followed by the pointwise estimate implying part *b*):

$$|\mathrm{D}^{\beta}(f-\pi_{r,w}f)(x)| \le C\sum_{k\in\mathbb{N}_0}(b^i/r)^{(\beta,\gamma_a)}\mathcal{D}_a(rb^{-i},x,f,A)+\mathcal{D}_a(r\kappa,w,f,A).\ \mathcal{Q.E.D.} \quad (3)$$

LEMMA 11.6. *Assume* $a\in(0,\infty]$, $\kappa>1$, $\beta\in\mathbb{N}_0^n$. *Then, for any* $Q_r(w)\subset Q_t(z)$, $t\le\kappa r$, *one has*

a) $\|\pi_{r,w}f-\pi_{t,z}f|L_\infty(Q_r(w))\|\le C(\kappa,a)\mathcal{D}_a(t,z,f,A)$;

b) $\|\pi|Q_r(w)\|_*\asymp\|\pi|L_a(Q_r(w))\|$ *for any* $\pi\in\mathcal{P}_A$;

c) $\|\mathrm{D}^{\alpha}\pi|L_\infty(Q_r(w))\|\le C(\alpha,a)r^{-(n/a+(\alpha,\gamma_a))}\|\pi|L_a(Q_r(w))\|$ *for any* $\pi\in\mathcal{P}_A$.

COROLLARY 11.7. *Assume* $s,\varepsilon>0$, $\beta\in A\subset\mathbb{N}_0^n$, $|A|<\infty$, $a\in(0,\infty]$, $(\beta,\gamma_a)<s$. *Then, at any point* $x\in\mathbb{R}^n$ *with* $\sup_{r\in(0,\varepsilon)}r^{-s}\mathcal{D}_a(r,x,f,A)<\infty$, *there exists Peano derivative* $\lim_{r\to0}\mathrm{D}^{\beta}\pi_{r,x}f(x)$ *coinciding with the ordinary and generalized ones.*

CHAPTER 3

Banach-valued functions

1. Introduction

In this chapter[1], we study the opportunity of the bounded extension of linear and sub-linear anisotropic singular operators, which are bounded from one of the vector-valued anisotropic spaces of Besov, Lebesgue, Lizorkin-Triebel or Lorentz type into another. After [**5**], where the "lower case" of extrapolation towards Hardy-Lorentz and Lorentz spaces was considered, it is its complemention developing universal approach presented in [**1**] and based, in particular, on the introduction of $\mathcal{AD}$-classes of singular integral operators. In the classical vector-valued Calderon-Zygmund theory the case of extrapolation towards BMO has been studied relatively recently in [**69**]. An account on previous developments related to our results (but not presented here) and comparisons with known sufficient conditions can be found in Chapters I,II and [**1, 5**].

Statements contained here generalize and add to results of [**6, 2, 3, 14, 15, 70, 71, 69, 72, 7**]and have the following distinctive features, even in the isotropic setting:

a) a wide range of the couples (X, Y) of Banach-valued anisotropic spaces is considered, for which T admits the extension $T \in \mathcal{L}(X, Y)$;

b) an inhomogeneity of the dependence on some parameter is revealed;

c) results are sharp in a sense;

d) the results established include "weak" (in the sense of [**69**]) variants of sufficient conditions;

e) duality considerations are used on particular occasions only;

f) the usage of the interpolation theory during consideration of intermediate couples of spaces is very limited;

g) we obtain results which are "off-diagonal" by summability and other parameters covering, in particular, boundedness properties of the operators of potential type.

[1]The content of this chapter will be mainly included in [**68**].

In [**73**](1988), Nakai and Yabuta relaxed the sufficient conditions for the boundedness of isotropic SIOs from a Morrey-Campanato, or a Lipschitz space of generalized smoothness (including the case of the ordinary power smoothness < 1) of scalar-valued functions into another such space established by Peetre ([**74**], 1966). Their sufficient conditions were formulated in the style of [**69**] with the addition of an orthogonality requirement. Later in [**75, 76, 77**] (1987-88), they found sufficient conditions on the kernel of an operator T bounded in a Lebesgue space for the existence of its bounded extension on isotropic Besov and Lizorkin-Triebel spaces of generalized smoothness covering arbitrary power one. Their conditions were written in terms of an estimate of the residual of an appropriate Taylor expansion of its kernel $K(x,y)$ by the variable x, and, similarly to our settings, they were different in the cases of integer and fractional power smoothness. In the most general form, their conditions were used by K. Yabuta in [**77**]: for $1 \le r \le \infty$,

$$\left(\int_{A(z,d)} \left| K(x,y) - \sum_{|\alpha|\le l} \partial_x^\alpha K(z,y) \frac{(x-z)^\alpha}{\alpha!} \right|^r dy \right)^{1/r} \le C d^{-d/r'} \left(\frac{|x-z|}{d}\right)^l \omega\left(\frac{|x-z|}{d}\right) \qquad (*)$$

with $A(z,d) = \{y : d \le |y-z| \le 2d\}$, $d > 0$, and positive and nondecreasing ω satisfies $\int_0^1 \omega \frac{dt}{t} < \infty$. Condition $(*)$ implies inclusion in our class (see Def. 2.4) $\mathcal{AD}_y(L_r, \infty, l_1, l, l, D_l)$, where $D_s = \{\alpha \in \mathbb{N}_0^n : |\alpha| \le s\}$ (in the isotropic case).

In Subsection 3.1, we use a different approach relying on proper decompositions of the operators under consideration and properties of functions spaces involved to obtain counterparts of these results and their extensions in our settings.

Progress in the study of the diagonal case (for example, $T : L_p \longrightarrow L_q$ with $p = q$) of SIOs with operator-valued kernels was made in the form of an extension scheme as the tensor product $T \otimes \mathrm{Id}$ of an SIO T with a scalar-valued kernel in the works of Benedek, Calderón, and Panzone [**70**] and Bourgain [**71**]. Bourgain considered an extension of T to Bochner–Lebesgue spaces of functions with the values in a UMD space with unconditional basis. A perfect form of this case was obtained in the study of Rubio de Francia, Ruiz, and Torrea [**69**], which is rich in applications. Their study includes the range of Banach-valued spaces of nonsmooth functions including Bochner-Lebesgue spaces and $H_1(\mathbb{R}^n, A)$, $L_1(\mathbb{R}^n, A)$, $L_{1,\infty})(\mathbb{R}^n, A)$, $L_\infty(\mathbb{R}^n, A)$ and $\mathrm{BMO})(\mathbb{R}^n, A)$ and contains sufficient conditions for the boundedness of SIOs that are expressed in difference terms (as in the Hörmander condition) and are equivalent to the inclusion in the classes $\mathcal{AD}_z(L_q, \infty, l_1, 0, 0, \{0\})$ and

$\mathcal{AD}_z^{\omega*}(L_q, \infty, l_1, 0, 0, \{0\})$, defined in Section 2, where $z = x$ for the "lower" case (i.e., for the extrapolation to the $(H_1 \to L_1)$ or $(L_1 \to L_{1,\infty})$ boundedness) and $z = y$ for the "upper" case (i.e., for the extrapolation to the $(L_\infty \to \text{BMO})$ boundedness).

In Subsection 3.2, we generalize results from [**70, 71, 69**], providing also some additions to Subsection 3.1, by means of an appropriate modification of the corresponding approach from [**69**] involving, in particular, some new tools to avoid some "underwater rocks" connected with the usage of R. Fefferman-Stein-Strömberg type inequalities.

In Section 6, we study the opportunity of the bounded extension of anisotropic subadditive singular operators (SO) which are bounded from one of the vector-valued anisotropic spaces of Lebesgue type into Bochner-Marcinkiewicz spaces.

An account on the previous closest developments related to our results (but not presented here) and comparisons with known sufficient conditions can be found in [**4, 78**] and Chapter I. Let us comment shortly on the recent development.

Hebisch [**79**] applied a method for deriving estimates that were based on the homogeneity of the Hardy–Littlewood maximal function. Later, Duong, M[c]Intosh, and Robinson [**4, 80**] obtained new sufficient conditions for weak $(1, 1)$-type estimate for an operator T of strong type (p, p), $p \in (1, \infty)$. Their conditions are formulated in terms of global approximation, and the proof is based on the Hebisch technique and an analogue of the Harnack inequality. In [**5, 1**], both the usage of the Harnack inequality was excluded and the role of p-convexity and local approximation was investigated. $\mathcal{AD}$ and $\mathcal{AAD}$-classes introduced there (playing the same role as Hörmander condition in the Calderón–Zygmund theory) permitted to cover the extrapolation to Hardy-Lorentz, Lorentz, Besov and Lizorkin-Triebel anisotropic spaces of vector-valued functions. Another direction of the development was undertaken by Blunck and Kunstmann who found an opportunity to show weak (p, p)-type of a (nonintegral) operator of strong (θ, θ)-type with $p \in (1, \theta)$ which did not possess weak $(1, 1)$-type. They used a Calderón–Zygmund decomposition of an L_p-function for $p > 1$. A counterpart of Blunck and Kunstmann's approach suitable for the extrapolation of Kato's Square Root operator and involving an analog of the Calderón–Zygmund decomposition for a function from isotropic seminormed Sobolev space was considered by Auscher in [**81**]. One should note that the last two articles are dealing with the „diagonal" by summability case only, i.e. with operators from L_p or from w_p^l into L_q with $p = q$, with isotropic spaces of scalar-valued functions and with linear operators.

2. Definitions and designations

In this section, we introduce, in a general form, the notations of the thesis work related to the setting of Banach-valued functions belonging to anisotropic function spaces.

Let $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ be the sets of natural, integer and real numbers respectively, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = (0, \infty)$. If $p \in (0, \infty]$, then let its conjugate be $p' := (1 - (\max(p,1))^{-1})^{-1}$. For $t \in (0, \infty]$, assume l_t to be the (quasi)normed space of the sequences $\{\alpha_i\}$ with the bounded (quasi)norm $\|\{\alpha_i\}|l_t\| = (\sum_i |\alpha_i|^t)^{1/t}$ for $t \neq \infty$, or $\|\{\alpha_i\}|l_\infty\| = \sup_i |\alpha_i|$; $c_0 = \{\{\alpha_i\}_i \in l_\infty : \lim_{|i|\to\infty} |\alpha_i| = 0\}$. Let l_t^0 be l_t for $t \in (0, \infty)$ and $l_\infty^0 := c_0$.

For a Banach space A, A^* denotes the adjoint to A, and, for $g \in A^*, a \in A$, the corresponding bilinear form is $g(a) = (g, a)$. Let also $\|\cdot|A\| = \|\cdot\|_A$ be the norm of A. For (quasi) Banach spaces A, B, let $\mathcal{L}(A, B)$ be the space of all linear continuous operators from A into B, $\mathcal{L}(A) := \mathcal{L}(A, A)$. For a (quasi)(semi)normed space A and its closed subspace A_1, we designate by means of $N(A)$ and A/A_1 its null-space $\{y \in A : \|y|A\| = 0\}$ and its factor-space by A_1 respectively. Let us recall that a Banach space A possesses Radon-Nikodim property, if the theorem with the same name is valid for A-valued measures. For a measure space (E, μ), the symbol $M(E)$ denotes the space of all real-valued functions measurable on E by the measure μ. Let us recall that an ideal space (structure, lattice) $X(E)$ is a Banach function space embedded into $M(E)$ and satisfying the property

$$|f| \leq |g|,\ g \in X(E) \Longrightarrow f \in X(E),\ \|f|X(E)\| \leq \|g|X(E)\|.$$

For a subset $G \subset \mathbb{R}^n$ measurable by a σ-additive measure on $\mathbb{R}^n$, the symbol $X(G, A)$ means some subspace of the space of the strongly measurable functions $f : G \to A$ with finite (quasi)norm $\|f|X(G, A)\| := \|\|f\|_A|X(G)\|$, where $X(G)$ is an ideal space of functions on G, $X(G) = X(G, \mathbb{R})$. In particular, for $p, q \in (0, \infty]$, $L_{p,q}(G, A)$ is the Bochner-Lorentz space with the (quasi)norm $\|f|L_{p,q}(G, A)\| = \|\|f\|_A|L_{p,q}(G)\|$. Sometimes we shall omit parameters G, or A in the absence of ambiguity. Let Ψ be a set. Then, for an arbitrary space $X(\mathbb{N}_0 \times \Psi)$ of functions on $\mathbb{N}_0 \times \Psi$, we assume that function $f = f(i, \psi)$, $i \in \mathbb{N}_0$, $\psi \in \Psi$ is in $X_{log}(\mathbb{N}_0 \times \Psi)$ ($X_{sup}(\mathbb{N}_0 \times \Psi)$) if the function $f'(i, \psi) := \sum_{k \geq i} f(k, \psi)$ ($f'(i, \psi) := \sup_{k \geq i} f(k, \psi)$) is contained in $X(\mathbb{N}_0 \times \Psi)$; similarly $f \in X_{dlog}(\mathbb{N}_0 \times \Psi)$ if

$f'(i,\psi) := \sum_{k\geq i}(k-i+1)f(k,\psi)$ is in $X(\mathbb{N}_0 \times \Psi)$ (see remark 4). By means of $M(\Omega,\mu,A)$, we designate the space of all Bochner-measurable A-valued functions defined on the measure space (Ω,μ), $M(\Omega,\mu) := M(\Omega,\mu,\mathbb{R})$, $M = M(\mathbb{R}^n, dx, \mathbb{R})$ (dx is Lebesgue measure).

We introduce the class $HL = HL(\Omega,\mu)$ of all ideal spaces, in which the maximal operator $f \mapsto (M|f|^\sigma)^{\sigma^{-1}}$ for some $\sigma \in (0,\infty]$. Let $|E|$ and $|D|$ be, correspondingly, the Lebesgue measure $\int \chi_E(x)dx$ of $E \subset \mathbb{R}^n$ and the number of the elements of $D \subset \mathbb{N}_0^n$. For a function $f \in M(\Omega,\mu)$, one has $\sigma(f,t) := |\{x : |f(x)| > t\}|$, and f^* means the nonincreasing right-continuous rearrangement of f.

For two sequences $\alpha = \{\alpha_i\}_{i\in\mathbb{Z}}, \beta = \{\beta_i\}_{i\in\mathbb{Z}}$, their convolution is understood to be $\delta = \{\delta_i\}_{i\in\mathbb{Z}} = \alpha * \beta$, $\delta_l = \sum_{i\in\mathbb{Z}} \alpha_{l-i}\beta_i$.

The anisotropy of $\mathbb{R}^n$ is characterized by means of anisotropy vector $\gamma_a \in \mathbb{R}^n_+$, $|\gamma_a| = n$, which we suppose to be fixed. It generates γ_a-distance $\rho_{\gamma_a}(x,y) := |x-y|_{\gamma_a}$ with $|x|_{\gamma_a} := \max_{1\leq i\leq n} |x_i|^{1/(\gamma_a)_i}$, $|x+y|_{\gamma_a} \leq c_{\gamma_a}(|x|_{\gamma_a} + |y|_{\gamma_a})$.
We assume the presence of the partial order $\alpha \leq \beta \overset{\text{def}}{\Longleftrightarrow} \alpha_i \leq \beta_i, i = 1,\dots,n$ in $\mathbb{N}_0^n$. Let $Q_0 := [-1,1]^n$ and $Q_t(z) := z + t^{\gamma_a}Q_0$ for $t > 0, z \in \mathbb{R}^n$. For $\gamma > 0$ and a finite $D \subset \mathbb{N}_0^n$, let us denote $D^*_\gamma := \{\alpha : \alpha \in \mathbb{N}_0^n, (\alpha,\gamma_a) \leq \gamma\}$, $\lambda_{\max}(D) = \max_{\alpha\in D}(\alpha,\gamma_a)$, $\lambda_{\min}(D) = \min_{\alpha\in D}(\alpha,\gamma_a)$, for $\alpha \in \mathbb{N}_0^n$ $\overset{\square}{D}_\alpha = \{\beta : \alpha \geq \beta \in \mathbb{N}_0^n\}$, $\partial D = \{\alpha : \alpha \in D, \alpha$ is maximal in D $(\beta \in D, \beta \geq \alpha \Rightarrow \beta = \alpha)\}$. For the sake of the shift invariance of any space $\mathcal{P}_D = \mathcal{P}_D(A) := \{\sum_{\alpha\in D} c_\alpha x^\alpha : c_\alpha \in A\}$ of A-valued polynomials, we always impose the restriction: $D \subset \mathbb{N}_0^n$ $D = \hat{D} := \cup_{\alpha\in D} \overset{\square}{D}_\alpha$. We understand that $f \perp \mathcal{P}_D$ if $\int f\pi = 0$ for every $\pi \in \mathcal{P}_D$.

Let $C_0^\infty(G)$ be the space of all infinitely differentiable functions with compact support included in G.

For some $\kappa > 1$, some Banach space A, some $\phi \in C_0^\infty(Q_1(0))$ with $\int_{\mathbb{R}^n} \phi = 1$, and any function $f \in L_{1,loc}(\mathbb{R}^n, a)$, we shall use the following maximal function:

$$M^*f(x) := \sup\{\|\phi_t * f(y)|A\| : t > 0,\ |y-x|_{\gamma_a} \leq \kappa t,\ \phi \in C_0^\infty(Q_1(0)),\ \int_{\mathbb{R}^n} \phi = 1\}.$$

For $u, s \in (0,\infty]$, $\lambda \in \mathbb{R}$ by means of $M^{\lambda,D}_{u,s,A}$, we designate the functional:

$$(M^{\lambda,D}_{u,s,A}f)(x) = \left(\int_0^\infty (r^\lambda \mathcal{D}_u(r,x,f,D,A))^s dr/r\right)^{1/s}, \quad f \in L_{u,loc}(\mathbb{R}^n, A).$$

DEFINITION 2.1. [22] For $u, u_1 \in [1,\infty], t > 0, D \subset \mathbb{N}_0^n, x \in \mathbb{R}^n$ and a Banach space A, we introduce $\mathcal{D}$-functional as

$$\mathcal{D}_{u,u_1}(t,x,f,D,A) := t^{-n/u} \inf_{g\in\mathcal{P}_D(A)} \|f-g|L_{u,u_1}(Q_t(x),A)\|, \mathcal{D}_u(\dots) := \mathcal{D}_{u,u}(\dots).$$

By means of $\pi_{t,x} = \pi_{t,x,D} : L_{u,u_1}(Q_t(x),A) \to \mathcal{P}_D(A)$, we designate a (nonlinear) operator satisfying, for some fixed $\varepsilon_0 > 0$, $\|f - \pi_{t,x}f|L_{u,u_1}(Q_t(x),A)\| \le (1+\varepsilon_0)\mathcal{D}_{u,u_1}(t,x,f,D,A)$ for every $f \in L_{u,u_1}(Q_t(x),A)$. Let also $\mathcal{D}_{u,u_1}(t,x,f,D) := \mathcal{D}_{u,u_1}(t,x,f,D,\mathbb{R})$.

If a function $f = f(x,y)$ depends on two (possibly multidimensional) variables x, y, and $f_w(y) := f(w,y)$, $f'_w(x) := f(x,w)$, then we assume

$$\mathcal{D}_u^y(t,z,f(x,y),D,A) = \mathcal{D}_u(t,z,f_x,D,A),\ \pi^y_{t,z,D}f(x,y) := \pi_{t,z,D}f_x,$$

$$\mathcal{D}_u^x(t,z,f(x,y),D,A) = \mathcal{D}_u(t,z,f'_y,D,A),\ \pi^x_{t,z,D}f(x,y) := \pi_{t,z,D}f'_y.$$

DEFINITION 2.2. Quasi-normed Abelian group A is understood to be p-convex for some $p \in (0,1]$ if one has $\|\sum_i a_i|A\| \le \|\{\|a_i|A\|\}|l_p\|$ for any $\{a_i\}_{i\in\mathbb{N}} \subset A$.

We define vector-valued anisotropic local approximation spaces of Besov and Lizorkin-Triebel type with the aid of $\mathcal{AD}$-functional as follows.

DEFINITION 2.3. For $p \in (0,\infty]^n$, $q \in (0,\infty]$, $u \in (0,\infty]$, $\gamma \in [0,\infty)$, $D \subset \mathbb{N}_0^n$, $|D| < \infty$ and an ideal space $Y = Y(\mathbb{R}^n)$ by means of $\widetilde{b}^{\gamma,D}_{Y,q,u}(\mathbb{R}^n,A)$, or $\widetilde{l}^{\gamma,D}_{Y,q,u}(\mathbb{R}^n,A)$, we designate, correspondingly, the anisotropic (quasi)(semi)normed space of A-valued functions $f \in L_u^{loc}(\mathbb{R}^n,A)$ with the finite (quasi)(semi)norm

$$\|f|\widetilde{b}^{\gamma,D}_{Y,q,u}(\mathbb{R}^n,A)\| = \left(\int_0^\infty \|t^{-\gamma}\mathcal{D}_u(t,\cdot,f,D,A)|Y(\mathbb{R}^n)\|^q \frac{dt}{t}\right)^{\frac{1}{q}},$$

$$\widetilde{b}^{\gamma,D}_{p,q,u}(\mathbb{R}^n,A) := \widetilde{b}^{\gamma,D}_{L_p,q,u}(\mathbb{R}^n,A), \text{ or}$$

$$\|f|\widetilde{l}^{\gamma,D}_{Y,q,u}(\mathbb{R}^n,A)\| = \left\| \left(\int_0^\infty (t^{-\gamma}\mathcal{D}_u(t,\cdot,f,D,A))^q \frac{dt}{t}\right)^{\frac{1}{q}} \middle| Y(\mathbb{R}^n)\right\|,$$

$$\widetilde{l}^{\gamma,D}_{p,q,u}(\mathbb{R}^n,A) := \widetilde{l}^{\gamma,D}_{L_p,q,u}(\mathbb{R}^n,A).$$

As in [5] and Chapter I, we deal with one of the following cases. Let A, B be Banach spaces. We say that operator T bounded from $L_{\theta_0}(\mathbb{R}^n,A)$ into $L_{\theta_1,\theta_2}(\mathbb{R}^n,B)$ for some $\theta_0,\theta_1,\theta_2 \in (0,\infty]$ is a singular integral operator (SIO) if there exists corresponding to it strongly measurable operator-valued kernel $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}(A,B)$ satisfying $(Tf)(x) :=$

$\int K(x,y)f(y)dy$ for a. e. $x \in \mathbb{R}^n \setminus \mathrm{supp} f$ and every $f \in L_\infty(\mathbb{R}^n, A)$ with compact support and if it is in one of the classes defined below.

For $q, q_1 \in (0,\infty]$, $\lambda_1 \in \mathbb{R}$ symbol E^w_{q,q_1,λ_1} designates the weighted Lorentz space defined by the (quasi)norm $\left\|(f(\cdot)|\cdot - w|^{\lambda_1+n/q'}|L_{q,q_1}(\mathbb{R}^n, A)\right\|$, $E^w_{q,\lambda_1} := E^w_{q,q,\lambda_1}$.

DEFINITION 2.4. [**5, 1**] Let $X := X(\mathbb{N}_0 \times \mathbb{R}_+ \times \mathbb{R}^n)$ be (quasi)normed space of real-valued functions, $\lambda_0, \lambda_1 \in [-n, \infty)$, $u, q, q_1 \in (0,\infty]$, $\Delta_i(r,w) = Q_{\delta r b^{i+1}}(w) \setminus Q_{\delta r b^i}(w), i \in \mathbb{N}_0$, and a finite $D \subset \mathbb{N}_0^n$. If, for some $\delta > 0, b > 1$ the function $\mu = \mu(T) : \mathbb{N}_0 \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ described below is in X, then we say that operator T is a member of class $\digamma$ for the following couples $(\mu, \digamma)$:

$$\digamma = \underline{\mathcal{AD}_x(u, L_q, X, \lambda_0, \lambda_1, D)} = \mathcal{AD}_x(u, L_q, X),\ \|\mu|X\| = C_{\mathcal{AD}} < \infty,$$

$$\mu_i(r,w) = \mu_i(r,w,T) := \|r^{-\lambda_0}\mathcal{D}^y_u(r,w,K(\cdot,y)\chi_{\Delta_i(r,w)}(\cdot), D, \mathcal{L}(A,B))|E^w_{q,\lambda_1}\|, i \in \mathbb{N}_0;$$

$$\digamma = \underline{\mathcal{AD}_x(L_q, u, X, \lambda_0, \lambda_1, D)} = \mathcal{AD}_x(L_q, u, X),\ \|\mu|X\| = C_{\mathcal{AD}} < \infty,$$

$$\mu_i(r,w) = \mu_i(r,w,T) := r^{-\lambda_0}\mathcal{D}^y_u(r,w,K(\cdot,y)\chi_{\Delta_i(r,w)}(\cdot), D, E^w_{q,\lambda_1}(\mathbb{R}^n, \mathcal{L}(A,B))), i \in \mathbb{N}_0;$$

$$\digamma = \underline{\mathcal{AD}^{\omega *}_x(L_q, u, X, \lambda_0, \lambda_1, D)} = \mathcal{AD}^{\omega *}_x(L_q, u, X),\ \|\mu(A)|X\| = C_{\mathcal{AD}} < \infty,$$

$$\mu_i(r,w) = \mu_i(r,w,T) := \sup_{\|a|A\|\le 1} r^{-\lambda_0}\mathcal{D}^y_u(r,w,K(\cdot,y)a\chi_{\Delta_i(r,w)}(\cdot), D, E^w_{q,\lambda_1}(\mathbb{R}^n, B)), i \in \mathbb{N}_0;$$

$$\digamma = \underline{\mathcal{AD}^{\omega}_x(u, L_q, X, \lambda_0, \lambda_1, D)} = \mathcal{AD}^{\omega}_x(u, L_q, X),\ \|\mu(g)|X\| \le C\|g|B^*\| \text{ for } g \in B^*,$$

$$\mu_i(r,w) = \mu_i(r,w,g,T) := \|r^{-\lambda_0}\mathcal{D}^y_u(r,w,K^*(\cdot,y)g\chi_{\Delta_i(r,w)}(\cdot), D, A^*)|E^w_{q,\lambda_1}\|, i \in \mathbb{N}_0;$$

$$\digamma = \underline{\mathcal{AD}^{\omega*\omega}_x(u, L_q, X, \lambda_0, \lambda_1, D)} = \mathcal{AD}^{\omega*\omega}_x(u, L_q, X),\ \|\mu(A,g)|X\| \le C\|g|B^*\| \text{ for } g \in B^*,$$

$$\mu_i(r,w) = \mu_i(r,w,g,T) := \left\| \sup_{\|a|A\|\le 1} r^{-\lambda_0}\mathcal{D}^y_u(r,w,(g, K(\cdot,y)a)\chi_{\Delta_i(r,w)}(\cdot), D) \mid E^w_{q,\lambda_1}(\mathbb{R}^n)\right\|, i \in \mathbb{N}_0.$$

We also say that T is in $\digamma$ defined by $\mu = \mu(T) \in L_\infty(\mathbb{R}_+ \times \mathbb{R}^n)$ for the following couples $(\mu, \digamma)$: $\digamma = \underline{\mathcal{AD}_x(u, L_{q,q_1}, \lambda_0, \lambda_1, D)} = \mathcal{AD}_x(u, L_{q,q_1})$,

$$\mu(r,w) = \mu(r,w,T) := \|r^{-\lambda_0}\mathcal{D}^y_u(r,w,K(\cdot,y)|\cdot - w|^{\lambda_1+n/q'}, D, \mathcal{L}(A,B))|L_{q,q_1}(\mathbb{R}^n \setminus Q_{r\delta b^i}(w))\|;$$

$$\digamma = \underline{\mathcal{AD}_x(L_{q,q_1}, u, \lambda_0, \lambda_1, D)} = \mathcal{AD}_x(L_{q,q_1}, u),$$

$$\mu(r,w) = \mu(r,w,T) := r^{-\lambda_0}\mathcal{D}_u^y(r,w,K(\cdot,y)|\cdot - w|^{\lambda_1+n/q'}, D, L_{q,q_1}(\mathbb{R}^n \setminus Q_{r\delta b^i}(w), \mathcal{L}(A,B))).$$

In the case $X := L_\infty(\mathbb{R}_+ \times \mathbb{R}^n, l)$, where l is a sequence space, we shall use l instead of X. The lowest bound of every constant C used in Definition 2.4 will be designated by means of $C_{\mathcal{AD}}$ for the corresponding $\mathcal{AD}$-class.

COROLLARY 2.5. [**5**] *The following inclusions take place (the parameters omitted coincide in every particular inclusion). For $D \subset D' \subset \mathbb{N}_0^n$, $\lambda_0, \lambda_1, \mu_0, \mu_1 \in \mathbb{R}$, $u_0, u_1, q_0, q_1 \in [1,\infty]$, $r, s \in (0,\infty]$, $\mu_0 < \lambda_0$, $\mu_0 - \lambda_0 = \mu_1 - \lambda_1$, $u_0 \geq q \geq u_1, q_0 \geq q_1$, one has:*

$$\mathcal{AD}_{(\cdot)}^{(\dots)}(u_0, L_{q_0}, \dots, D) \subset \mathcal{AD}_{(\cdot)}^{(\dots)}(u_1, L_{q_1}, \dots, D'),$$

$$\mathcal{AD}_{(\cdot)}^{(\dots)}(L_{q_0}, u_0, \dots, D) \subset \mathcal{AD}_{(\cdot)}^{(\dots)}(L_{q_1}, u_1, \dots, D'),$$

$$\mathcal{AD}_{(\cdot)}^{(\dots)}(\dots, l_r, \lambda_0, \lambda_1, D) \subset \mathcal{AD}_{(\cdot)}^{(\dots)}(\dots, l_{s,log}, \mu_0, \mu_1, D') \text{ for } r, s \in (0,\infty],$$

$$\mathcal{AD}_{(\cdot)}^{(\dots)}(u_0, L_q, \dots) \subset \mathcal{AD}_{(\cdot)}^{(\dots)}(L_q, u_0, \dots) \subset \mathcal{AD}_{(\cdot)}^{(\dots)}(u_1, L_q, \dots).$$

REMARK 1. Note that the definitions of $\mathcal{AD}$-classes are equivalent to their continuous forms. It shows their independence of the value of $b > 1$. For instance, the definition of class $\mathcal{AD}_x(u, L_q, X, \lambda_0, \lambda_1, D)$ for $X := X(\mathbb{N}_0 \times \mathbb{R}_+ \times \mathbb{R}^n)$ being equal to $X_1 := l_t(Y(\mathbb{R}_+ \times \mathbb{R}^n))$, or $X_1 := l_{t,log}(Y(\mathbb{R} \times \mathbb{R}^n))$, can be written with the aid of the function $\mu = \mu(T) : [1,\infty) \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$,

$$\mu(\tau, r, w, T) := \|r^{-\lambda_0}\mathcal{D}_u^y(r, w, K(\cdot,y)\chi_{\Delta_\tau}(\cdot), D, \mathcal{L}(A,B))|E_{q,\lambda_1}^w\|, i \in \mathbb{N}_0,$$

where $\Delta_\tau(r,w) = Q_{\delta r b \tau} \setminus Q_{\delta r \tau}, \tau \in [1,\infty)$, contained in $Y_1 := L_{t,\frac{d\tau}{\tau}}([1,\infty), Y(\mathbb{R} \times \mathbb{R}^n))$, or $Y_2 := L_{t,\frac{\ln\tau d\tau}{\tau}}([1,\infty), Y(\mathbb{R} \times \mathbb{R}^n))$ respectively. This explains the usage of subscript log.

DEFINITION 2.6. [**5, 1**] Let $D_0, D_1 \subset \mathbb{N}_0^n$. We say that operator T is in the class $\underline{ORT_x(D_0, D_1)}$ if $T\phi \perp \mathcal{P}_{D_1}$ for each $\phi \in C_0^\infty$, $\phi \perp \mathcal{P}_{D_0}$.

DEFINITION 2.7. [**5, 1**] We say that operator T with the kernel $K(x,y)$ is in $\underline{\mathcal{AD}_y\text{-class}}$ if an operator T^I with the kernel $K^I(x,y) = K(y,x)$ is in $\mathcal{AD}_x$-class.

We say that operator T is in $\underline{ORT_y(D_0, D_1)\text{-class}}$ if the operator T^* is in $ORT_x(D_0, D_1)$-class.

REMARK 2. Corollary 5.4 shows that $T^* \in ORT_y(D_0, D_1) \Longleftrightarrow T\mathcal{P}_{D_1}(A) \subset \mathcal{P}_{D_0}(B)$.

The next definition is a particular case of the notions of Lebesgue and Chebyshev regularity of functions and operators with respect to a class of polynomials introduced in [**61**].

DEFINITION 2.8. [**61**] Let A, B be Banach spaces, and $M(\mathbb{R}^n, B)$ be the space of all Bochner measurable functions on $\mathbb{R}^n$. We say that a function $f \in M(\mathbb{R}^n, B)$ is Lebesgue-, or Marcinkiewicz-regular if $f \in L_p(\mathbb{R}^n, B)$, or, correspondingly, $f \in L_{p,\infty}(\mathbb{R}^n, B)$ for some $p \in (0, \infty)$. We say that a function $f \in M(\mathbb{R}^n, B)$ is Chebyshev-regular if $\lim_{t\to\infty}(\|f|B\|)^*(t) = 0$. We say that an operator T is Lebesgue, Marcinkiewicz, or Chebyshev- regular if Tf is, correspondingly, Lebesgue, Marcinkiewicz, or Chebyshev regular for every f from a set dense in the domain of T.

DEFINITION 2.9. Let $p, q \in (0, \infty]$, a finite $D \subset \mathbb{N}_0^n$, A be a Banach space, and $G \subset \mathbb{R}^n$ be a measurable subset. By means of $L_{p,q}^D(G, A)$, we designate the (quasi)seminormed space of all functions f of the form $f = g + \pi$, where $g \in L_{p,q}(G, A)$ and $\pi \in \mathcal{P}_D(A)$, endowed with the finite (quasi)(semi)norm

$$\|f|L_{p,q}^D(G, A)\| = \|g|L_{p,q}(G, A)\|.$$

In particular, $L_{p,q}(\mathbb{R}^n, A)$ is the subspace of all Chebyshev regular functions in $L_{p,q}^D(\mathbb{R}^n, A)$.

3. Main results

3.1. Vector-valued functions. In this subsection, we state the main results corresponding to the setting of (single) vector-valued functions and anisotropic spaces.

Note that Hardy-Lorentz spaces of vector-valued functions

$H_{p,q}(\mathbb{R}^n, A)$ mentioned below are defined by means of quasi-norms $\|M^* f|L_{p,q}(\mathbb{R}^n)\|$.

The next two theorems, which assertions we shall extend in Subsection 3.2, are established in [**5**] and deal with the case of "lower" extrapolation (in the sense of [**1**]).

THEOREM 3.1. [**5**] *Let SIO T with $\mathcal{L}(A, B)$-valued kernel $K(x, y)$ satisfy the conditions $T \in ORT_x(D_0, D_1)$, $T \in \mathcal{L}_\theta := \mathcal{L}(L_{\theta_0}(\mathbb{R}^n, A), L_{\theta_1,\theta_2}(\mathbb{R}^n, B))$ for $\lambda_1 \in [-n, \infty)$, $\lambda_0 \geq 0$, finite $D_0, D_1 \subset \mathbb{N}_0^n$, $D_1 = D_{\lambda_1}^*$, $\theta_0, t, q_0, q_1, v_0, v_1, s, w_0 \in (0, \infty]$ and for either $\theta_1 \in (1, \infty]$, $\theta_2 \in (0, \infty]$, or $\theta_1 = \theta_2 = 1$;*
let also $\lambda_0 - \lambda_1 = n(1/\theta_0 - 1/\theta_1)$, $p_i = (1 + \lambda_i/n)^{-1}, i = 0, 1$, $u, q \in [1, \infty]$, $q > p_1$, $\theta_0 > p_0$ and $T \in \mathcal{AD}_x(u, L_q, X, \lambda_0, \lambda_1, D_0) \cup \mathcal{AD}_x(L_q, u, X, \lambda_0, \lambda_1, D_0) \cup \mathcal{AD}_x^{\omega*}(L_q, \infty, X, \lambda_0, \lambda_1, D_0)$.

Then, for $1/v_1 - 1/v_0 = 1/\theta_1 - 1/\theta_0$, *operator* T *possesses unique bounded extension to be in* $\mathcal{L}(W_1, W_2)$ *with the norm* $\|T|\mathcal{L}(W_1, W_2)\| \le C(C_{\mathcal{AD}} + \|T|\mathcal{L}_\theta\|)$ *for the following pairs of spaces* (W_1, W_2) *and groups of parameters.*

Assuming $\min\{1, t, p_1\} \ge \max(q_0, p_0)$ *and one of the following conditions to be satisfied:*

i) $X = L_\infty(\mathbb{R}_+ \times \mathbb{R}^n, l^0_{t,log})$, $\lambda_1 = \lambda_{\max}(D_1)$;

ii) $X = L_\infty(\mathbb{R}_+ \times \mathbb{R}^n, l^0_t)$, $\lambda_1 > \lambda_{\max}(D_1)$;

one has:

a) $T \in \mathcal{L}(H_{p_0,q_0}(\mathbb{R}^n, A), H_{p_1,t}(\mathbb{R}^n, B)) \cap \mathcal{L}(H_{v_0,s}(\mathbb{R}^n, A), H_{v_1,s}(\mathbb{R}^n, B))$ *for* $v_0 \in (p_0, \theta_0)$;

b) $T \in \mathcal{L}(H_{v_0,w_0}(\mathbb{R}^n, A), H_{v_1,s}(\mathbb{R}^n, B))$ *for* $v_1 < q, s \in [v_0, \infty]$, $\max(v_0, w_0) \le \min(v_1, s, 1)$ *for either* $\theta_0 > 1, v_0 \in (p_0, 1]$, *or* $\theta_0 \le 1, v_0 \in (p_0, \theta_0)$;

c) in particular, for $\lambda_0 = \lambda_1, \theta_0 = \theta_1 > 1$ $T \in \mathcal{L}(H_{p_0}(\mathbb{R}^n, A), H_{p_0,t}(\mathbb{R}^n, B)) \bigcap$

$$\bigcap_{p \in (p_0,1]} \mathcal{L}(H_{p,s}(\mathbb{R}^n, A), H_{p,s}(\mathbb{R}^n, B)) \bigcap_{p \in (1,\theta_0)} \mathcal{L}(L_{p,s}(\mathbb{R}^n, A), L_{p,s}(\mathbb{R}^n, B)).$$

Assuming $\max\{t, q_0\} \le q_1$, $p_0 \le p_1$, $(\min(1, p_1))^{-1} + 1/q_1 \le 1/t + 1/q_0$ *and one of the following groups of conditions:*

i) $p_1 < 1$, $X = l^0_{t,log}(L_\infty(\mathbb{R}_+ \times \mathbb{R}^n))$ *and* $\lambda_1 = \lambda_{\max}(D_1)$;

ii) $p_1 < 1$, $X = l^0_{t,max}(L_\infty(\mathbb{R}_+ \times \mathbb{R}^n))$ *and* $\lambda_1 > \lambda_{\max}(D_1)$;

iii) $p_1 = 1$, $X = l^0_{t,dlog}(L_\infty(\mathbb{R}_+ \times \mathbb{R}^n))$ *and* $\lambda_1 = \lambda_{\max}(D_1)$;

iv) $p_1 = 1$, $X = l^0_{t,log}(L_\infty(\mathbb{R}_+ \times \mathbb{R}^n))$ *and* $\lambda_1 > \lambda_{\max}(D_1)$;

one has:

d) $T \in \mathcal{L}(H_{p_0,q_0}(\mathbb{R}^n, A), H_{p_1,q_1}(\mathbb{R}^n, B)) \cap \mathcal{L}(H_{v_0,s}(\mathbb{R}^n, A), H_{v_1,s}(\mathbb{R}^n, B))$ *for* $v_0 \in (p_0, \theta_0)$.

THEOREM 3.2. [5] *Let SIO* T *with* $\mathcal{L}(A, B)$*-valued kernel* $K(x, y)$ *satisfy the conditions* $T \in \mathcal{L}_\theta := \mathcal{L}(L_{\theta_0}(\mathbb{R}^n, A), L_{\theta_1,\infty}(\mathbb{R}^n, B))$ *for* $\theta_0, \theta_1, t, q_0, q_1, v_0, v_1, s, w_0 \in (0, \infty]$; *let also* $\lambda_0 \ge 0$, $\lambda_1 \in [-n, \infty)$, $\lambda_0 - \lambda_1 = n(1/\theta_0 - 1/\theta_1)$, *finite* $D \subset \mathbb{N}^n_0$, $p_i = (1 + \lambda_i/n)^{-1}, i = 0, 1$, $u, q \in [1, \infty]$, $q > p_1$, $\theta_0 > p_0$, $T \in \mathcal{AD}_x(u, L_q, X, \lambda_0, \lambda_1, D) \cup \mathcal{AD}_x(L_q, u, X, \lambda_0, \lambda_1, D) \cup \mathcal{AD}^{\omega_*}_x(L_q, \infty, X, \lambda_0, \lambda_1, D)$.

Then, for $1/v_1 - 1/v_0 = 1/\theta_1 - 1/\theta_0$, *operator* T *possesses unique bounded extension to be in* $\mathcal{L}(W_1, W_2)$ *with the norm* $\|T|\mathcal{L}(W_1, W_2)\| \le C(C_{\mathcal{AD}} + \|T|\mathcal{L}_\theta\|)$ *for the following pairs of spaces* (W_1, W_2) *and groups of parameters.*

Assuming $\min\{1,t,p_1\} \ge \max(q_0,p_0)$ *and* $X = L_\infty(\mathbb{R}_+ \times \mathbb{R}^n, l_t^0)$, *one has:*

a) $T \in \mathcal{L}(H_{p_0,q_0}(\mathbb{R}^n,A), L_{p_1,t}(\mathbb{R}^n,B)) \cap \mathcal{L}(H_{v_0,s}(\mathbb{R}^n,A), L_{v_1,s}(\mathbb{R}^n,B))$ *for* $v_0 \in (p_0,\theta_0)$*;*

b) $T \in \mathcal{L}(H_{v_0,w_0}(\mathbb{R}^n,A), L_{v_1,s}(\mathbb{R}^n,B))$ *for* $v_1 \le q, s \in [v_0,\infty]$, $\max(v_0,w_0) \le \min(v_1,s,1)$ *and for either* $\theta_0 > 1, v_0 \in (p_0,1]$, *or* $\theta_0 \le 1, v_0 \in (p_0,\theta_0)$*;*

c) in particular, for $\lambda_0 = \lambda_1, \theta_0 = \theta_1$,

$$T \in \mathcal{L}(H_{p_0}(\mathbb{R}^n,A), L_{p_0,t}(\mathbb{R}^n,B)) \bigcap_{p\in(p_0,\theta_0)} \mathcal{L}(H_{p,s}(\mathbb{R}^n,A), L_{p,s}(\mathbb{R}^n,B)).$$

d) $T \in \mathcal{L}(L_1(\mathbb{R}^n,A), L_{v_1,\infty}(\mathbb{R}^n,B))$ *under the conditions of part b) with* $u=\infty$, $D=\{0\}$, $v_0 = 1$, $1-1/v_1 = 1/\theta_0 - 1/\theta_1$ *and for either* $p_1 = v_1 = 1 \ge t$, *or* $p_1 = v_1 > 1$.

Assuming $\max\{t,q_0\} \le q_1$, $p_0 \le p_1$, $(\min(1,p_1))^{-1} + 1/q_1 \le 1/t + 1/q_0$ *and one of the following groups of conditions:*

i) $X = l^0_{t,sup}(L_\infty(\mathbb{R}_+ \times \mathbb{R}^n))$ *and* $p_1 < 1$*;*

ii) $X = l^0_{t,log}(L_\infty(\mathbb{R}_+ \times \mathbb{R}^n))$ *and* $p_1 = 1$*;*

one has:

e) $T \in \mathcal{L}(H_{p_0,q_0}(\mathbb{R}^n,A), L_{p_1,q_1}(\mathbb{R}^n,B)) \cap \mathcal{L}(H_{v_0,s}(\mathbb{R}^n,A), L_{v_1,s}(\mathbb{R}^n,B))$ *for* $v_0 \in (p_0,\theta_0), s \in (0,\infty]$.

The rest of the subsection deals with "upper" cases of extrapolation.

THEOREM 3.3. *Let SIO T with* $\mathcal{L}(A,B)$*-valued kernel* $K(x,y)$ *satisfy the conditions* $T \in ORT_y(D_0,D_1)$, $\lambda_1 \in [0,\infty)$, $T \in \mathcal{L}_\theta := \mathcal{L}(L_{\theta_0}(\mathbb{R}^n,A), L_{\theta_1,\infty}(\mathbb{R}^n,B))$, $\lambda_0 \ge 0$, $D_0, D_1 \subset \mathbb{N}_0^n$, $D_1 \subset D_{\lambda_1}$, $\theta_0,\theta_1,t \in (0,\infty]$, $\lambda_0 - \lambda_1 = n(1/\theta_0 - 1/\theta_1)$, $u,q \in [1,\infty]$, $[\nu_1,\nu_0] \subset (0,\infty]$, $\nu_0 \ge t$, $u_0 \in [1,u]$, $u_0 < \theta_1$, $u_1 = \max(q',\theta_0)$. *Then operator T possesses unique bounded extension to be in* $\mathcal{L}(W_1,W_2)$ *with the norm* $\|T|\mathcal{L}(W_1,W_2)\| \le C(C_{\mathcal{AD}} + \|T|\mathcal{L}_\theta\|)$ *for the following pairs of spaces* (W_1,W_2) *and groups of parameters.*

Assuming $t \in (0,1]$ *and one of the following groups of conditions:*

i) $T \in \mathcal{AD}_y^\omega(u,L_q,X,\mu_0,\mu_1,D_0) \cup \mathcal{AD}_y^\omega(L_q,u,X,\mu_0,\mu_1,D_0) \cup \mathcal{AD}_y^{\omega_*,\omega}(u,L_\infty,X,\mu_0,\mu_1,D_0)$ *and* $\mu_0 > 0$*;*

ii) $T \in \mathcal{AD}_y(u,L_q,X,\mu_0,\mu_1,D_0) \cup \mathcal{AD}_y(L_q,u,X,\mu_0,\mu_1,D_0) \cup \mathcal{AD}_y^{\omega_*}(u,L_\infty,X,\mu_0,\mu_1,D_0)$ *and* $\mu_0 = 0$*;*

and also assuming one of the following groups of conditions:

i) $X = L_\infty(\mathbb{R}_+ \times \mathbb{R}^n, l_{t,log})$ *and* $\lambda_1 = \lambda_{\max}(D_1)$*;*

ii) $X = L_\infty(\mathbb{R}_+ \times \mathbb{R}^n, l_t)$ *and* $\lambda_1 > \lambda_{\max}(D_1)$*;*

one has:

a) $T \in \mathcal{L}(\widetilde{l}^{\mu_1,D_1}_{Y,\infty,u_0}(\mathbb{R}^n, A), \widetilde{l}^{\mu_0,D_0}_{Y,\infty,u_1}(\mathbb{R}^n, B))$ *for* $\mu_i \in [0, \lambda_i], i = 0, 1, \mu_1 - \mu_0 = \lambda_1 - \lambda_0$ *and an ideal space* $Y = Y(\mathbb{R}^n) \in HL$.

Assuming $\lambda = n(1/\theta_0 - 1/\theta_1)$, $p, s \in (0, \infty)$ *and one of the following groups of conditions:*

i) $X = L_\infty(\mathbb{R}^n, l_{t,log}(\mathbb{N}_0, L_\infty(\mathbb{R}_+)))$ *and* $\lambda_1 = \lambda_{\max}(D_1)$;

ii) $X = L_\infty(\mathbb{R}^n, l_t(\mathbb{N}_0, L_\infty(\mathbb{R}_+)))$ *and* $\lambda_1 > \lambda_{\max}(D_1)$;

one has

b) $T \in \mathcal{L}(\widetilde{l}^{\lambda,D_1}_{L_{p,s},\nu_1,u_1}(\mathbb{R}^n, A), L^{D_0}_{p,s}(\mathbb{R}^n, B))$ *for* $\lambda > 0$ *with either* $\lambda_0 = 0$ *and* $1 \le 1/t + 1/\nu_1$, *or* $\lambda_0 > 0$;

c) $T \in \mathcal{L}(L^{D_1}_{p,s}(\mathbb{R}^n, A), \widetilde{l}^{-\lambda,D_0}_{L_{p,s},\nu_0,u_0}(\mathbb{R}^n, B))$ *for* $\lambda < 0$, $u_1 < p$ *with either* $\lambda_1 = 0$ *and* $1 \le 1/t - 1/\nu_0$, *or* $\lambda_0 > 0$.

Assuming $T \in \mathcal{AD}_y(u, L_q, X, \lambda_0, \lambda_1, D_0) \cup \mathcal{AD}^{\omega_*}_y(L_\infty, \infty, X, \lambda_0, \lambda_1, D_0)$, $\mu_i \in [0, \lambda_i], i = 0, 1$, $\mu_1 - \mu_0 = \lambda_1 - \lambda_0$, $s \in (0, 1]$ *and that space* $Y = Y(\mathbb{R}^n)$ *is ideal and s-convex, one has:*

d) $T \in \mathcal{L}(\widetilde{b}^{\mu_1,D_1}_{Y,\nu_1,u_1}(\mathbb{R}^n, A), \widetilde{b}^{\mu_0,D_0}_{Y,\nu_0,u_0}(\mathbb{R}^n, B))$ *for either* $\mu_i = \lambda_i, i = 0, 1$, $1/s + 1/\nu_0 \le 1/t + 1/\nu_1$, *or* $\mu_i < \lambda_i, i = 0, 1$ *with* X *being either* $l_{t,log}(\mathbb{N}_0, L_\infty(\mathbb{R}_+ \times \mathbb{R}^n))$ *for* $\lambda_1 = \lambda_{\max}(D_1)$, *or* $l_t(\mathbb{N}_0, L_\infty(\mathbb{R}_+ \times \mathbb{R}^n))$ *for* $\lambda_1 > \lambda_{\max}(D_1)$;

e) $T \in \mathcal{L}(\widetilde{l}^{\mu_1,D_1}_{Y,\nu_1,u_1}(\mathbb{R}^n, A), \widetilde{l}^{\mu_0,D_0}_{Y,\nu_0,u_0}(\mathbb{R}^n, B))$ *for either* $\mu_i = \lambda_i, i = 0, 1$, $1 + 1/\nu_0 \le 1/t + 1/\nu_1$, *or* $\mu_i < \lambda_i, i = 0, 1$ *with* X *being either* $L_\infty(\mathbb{R}^n, l_{t,log}(\mathbb{N}_0, L_\infty(\mathbb{R}_+)))$ *for* $\lambda_1 = \lambda_{\max}(D_1)$, *or* $L_\infty(\mathbb{R}^n, l_t(\mathbb{N}_0, L_\infty(\mathbb{R}_+)))$ *for* $\lambda_1 > \lambda_{\max}(D_1)$;

d') $T \in \mathcal{L}(\widetilde{b}^{\mu_1,D_1}_{\varrho_1,\nu_1,u_1}(\mathbb{R}^n, A), \widetilde{b}^{\mu_0,D_0}_{\varrho_0,\nu_0,u_0}(\mathbb{R}^n, B))$ *for* $\varrho_0, \varrho_1, \varrho \in (0, \infty]^n$, $1/\varrho_0 = 1/\varrho + 1/\varrho_1$ *and for one of the following groups of conditions being valid:*

i) $\mu_i = \lambda_i, i = 0, 1$, $(\min(\varrho_{0\,\min}, 1))^{-1} + 1/\nu_0 \le 1/t + 1/\nu_1$;

ii) $\mu_i < \lambda_i, i = 0, 1$ *and one of the following groups of conditions:*

i) $X = l_{t,log}(\mathbb{N}_0, L_\infty(\mathbb{R}_+, L_\varrho(\mathbb{R}^n)))$ *and* $\lambda_1 = \lambda_{\max}(D_1)$;

ii) $l_t(\mathbb{N}_0, L_\infty(\mathbb{R}_+, L_\varrho(\mathbb{R}^n)))$ *and* $\lambda_1 > \lambda_{\max}(D_1)$;

e') $T \in \mathcal{L}(\widetilde{l}^{\mu_1,D_1}_{\varrho_1,\nu_1,u_1}(\mathbb{R}^n, A), \widetilde{l}^{\mu_0,D_0}_{\varrho_0,\nu_0,u_0}(\mathbb{R}^n, B))$ *for* $\varrho_0, \varrho_1, \varrho \in (0, \infty]^n$, $1/\varrho_0 = 1/\varrho + 1/\varrho_1$ *and for one of the following groups of conditions:*

i) $\mu_i = \lambda_i, i = 0, 1$ *and* $1 + 1/\nu_0 \le 1/t + 1/\nu_1$;

ii) $\mu_i < \lambda_i, i = 0, 1$;

and one of the following groups of conditions:

i) $X = L_\varrho(\mathbb{R}^n, l_{t,log}(\mathbb{N}_0, L_\infty(\mathbb{R}_+)))$ *and* $\lambda_1 = \lambda_{\max}(D_1)$;

ii) $X = L_\varrho(\mathbb{R}^n, l_t(\mathbb{N}_0, L_\infty(\mathbb{R}_+)))$ *and* $\lambda_1 > \lambda_{\max}(D_1)$.

Not imposing the condition $T \in ORT_y(D_0, D_1)$, the next theorem adds conclusions being true under the conditions of Theorem 3.3.

THEOREM 3.4. *Let SIO T with* $\mathcal{L}(A, B)$*-valued kernel satisfy* $T \in \mathcal{L}_\theta := \mathcal{L}(L_{\theta_0}(\mathbb{R}^n, A), L_{\theta_1,\infty}(\mathbb{R}^n, B))$, $\lambda_0, \lambda_1, \mu_0, \mu_1 \in \mathbb{R}$, $w_1, \theta_0, \theta_1, t, s, v_1, q_0, q_1 \in (0, \infty]$, $u, q \in [1, \infty]$, $\nu_i = -n/\mu_i$, $p_i = -n/\lambda_i$ *for* $i = 0, 1$, $u_0 \in [1, u]$, $u_0 < \theta_1$, $u_1 = \max(q', \theta_0)$, $\lambda_0 \geq \lambda_1$, $\nu_1, q_1, w_0 \in [u_1, \infty]$, $\lambda_0 - \lambda_1 = \mu_0 - \mu_1 = n/\theta_0 - n/\theta_1$, $\emptyset \neq D \subset \mathbb{N}_0^n$. *Then operator T possesses unique bounded extension to be in* $\mathcal{L}(W_1, W_2)$ *with the norm* $\|T|\mathcal{L}(W_1, W_2)\| \leq C(C_{\mathcal{AD}} + \|T|\mathcal{L}_\theta\|)$ *for the following pairs of spaces* (W_1, W_2) *and groups of parameters.*

Assuming $X = L_\infty(\mathbb{R}_+ \times \mathbb{R}^n, l_t)$ *and one of the following groups of conditions:*

i) $T \in \mathcal{AD}_y^w(u, L_q, X, \mu_0, \mu_1, D) \cup \mathcal{AD}_y^w(L_q, u, X, \mu_0, \mu_1, D) \cup \mathcal{AD}_y^{w*,w}(u, L_\infty, X, \mu_0, \mu_1, D)$ *and* $\mu_0 > 0$;

ii) $T \in \mathcal{AD}_y(u, L_q, X, \mu_0, \mu_1, D) \cup \mathcal{AD}_y(L_q, u, X, \mu_0, \mu_1, D) \cup \mathcal{AD}_y^{w*}(u, L_\infty, X, \mu_0, \mu_1, D)$ *and* $\mu_0 = 0$;

one has:

a) $T \in \mathcal{L}(L_{\nu_1,w_1}(\mathbb{R}^n, A), \tilde{b}^{\mu_0,D}_{\infty,\infty,u_0}(\mathbb{R}^n, B))$ *for* $\nu_1 \geq n/(\lambda_0 - \lambda_1)$ *and for either* $t \in (0, 1]$, $w_1 \leq \nu_1 = u_1$, *or* $t \in (0, 1]$, $\nu_1 > u_1$, *or* $w_1 \leq \nu_1 \leq t'$;

b) $T \in \mathcal{L}(L_{\nu_1,s}(\mathbb{R}^n, A), L_{\nu_0,s}(\mathbb{R}^n, B))$ *for* $n/(\lambda_0 - \lambda_1) > \max(\nu_1, u_1)$, $\nu_1 > \theta_0$;

c) $T \in \mathcal{L}(L_{q_1,w_1}(\mathbb{R}^n, A), \tilde{l}^{\mu_0,D}_{L_{q_0,w_0},\infty,u_0}(\mathbb{R}^n, B))$ *for* $1/q_1 - 1/q_0 = 1/\nu_1$, $\nu_1 \geq n/(\lambda_0 - \lambda_1)$, $1/w_0 + 1 \leq 1/w_1$ *and for either* $q_1 > u_1$, *or* $q_1 = u_1$, $w_0 = \infty$.

Assuming either $T \in \mathcal{AD}_y^w(u, L_{q,t}, \mu_0, \mu_1, D) \cup \mathcal{AD}_y^w(L_{q,t}, u, \mu_0, \mu_1, D)$ *for* $\mu_0 > 0$, *or* $T \in \mathcal{AD}_y(u, L_{q,t}, \mu_0, \mu_1, D) \cup \mathcal{AD}_y(L_{q,t}, u, \mu_0, \mu_1, D)$ *for* $\mu_0 = 0$, *one has:*

d) $T \in \mathcal{L}(L_{\nu_1,w_1}(\mathbb{R}^n, A), \tilde{b}^{\mu_0,D}_{\infty,\infty,u_0}(\mathbb{R}^n, B))$ *for* $\nu_1 \geq n/(\lambda_0 - \lambda_1)$, $w_1 \leq t'$;

e) $T \in \mathcal{L}(L_{\nu_1,s}(\mathbb{R}^n, A), L_{\nu_0,s}(\mathbb{R}^n, B))$ *for* $n/(\lambda_0 - \lambda_1) > \max(\nu_1, u_1)$, $\nu_1 > \theta_0$.

REMARK 3. Results from [**82**] show that interpolation theory cannot be used to prove theorems 3.3, *a*), 3.4, *a*). Duality may not be used too because spaces A^*, B^* can not possess Radon-Nikodim property.

The next observation from [**5**] helps to check some $\mathcal{AD}$-conditions.

REMARK 4. [**5**] Let $\alpha := \{\alpha_k\}_{k\in\mathbb{N}} \in l_p$ be a nonincreasing sequence, $p \in (0,\infty)$, $a \in [0,1)$, and $t > p/(p+1)$. Then,

(1) $\|\Delta\alpha|l_t\| \leq (1-a)^{-1}(\alpha_1)^{1-a}\|\alpha|l_p\|^a \leq (1-a)^{-1}\|\alpha|l_p\|$ for $1/t \leq a/p+1$, where $\Delta\alpha := \{\alpha_k - \alpha_{k+1}\}_{k\in\mathbb{N}} \in l_t$;

(2) for $\varepsilon > 0$ and a sequence $\beta(\varepsilon) := \{(k^{1/p}(\ln(k+1))^{1/p+\varepsilon})^{-1}\}_{k\in\mathbb{N}} \in l_p$, the inclusion $\Delta\beta(\varepsilon) \in l_{\frac{p}{p+1}}$ is valid if and only if $\varepsilon > 1$.

3.2. "Double" vector-valued functions. This section contains the statements of the main results in the "double" vector-valued setting extending ones due to Benedek, Calderón and Panzone, [**70**], Bourgain [**71**] and Rubio De Francia, Ruiz and Torrea [**69**].

Let Z be a Banach space with an unconditional basis $\{e_i\}_{i\in\mathbb{N}}$ and its dual basis $\{e_i'\}_{i\in\mathbb{N}}$. Let also $\mathbb{A} := \{A_k\}_{k\in\mathbb{N}}$ and $\mathbb{B} := \{B_k\}_{k\in\mathbb{N}}$ be families of Banach spaces, and $\tilde{T} := \{T_k\}_{k\in\mathbb{N}}$ be a family of SIO such that, for every $k \in \mathbb{N}$, T_k has $\mathcal{L}(A_k, B_k)$-valued kernel $K_k(x,y)$. Then, by means of $Z(\mathbb{A})$, we designate the Banach space of the sequences $a = \{a_k\}_{k\in\mathbb{N}}$ with the finite norm $\|a|Z(\mathbb{A})\| := \|\sum_{k\in\mathbb{N}} e_k\|a_k|A_k\||Z\|$. Symbol $\tilde{T}$ means SIO defined on $Z(\mathbb{A})$-valued („double" vector-valued) functions with the kernel $K(x,y)$, $K(x,y)a := \{K_i(x,y)a_i\}_{i\in\mathbb{N}}$ for $a \in \mathbb{A}$. We note the relations $(Z(\mathbb{A}))^* = Z^*(\mathbb{A}^*)$, $\mathbb{A}^* = \{A_k^*\}_{k\in\mathbb{N}}$.

THEOREM 3.5. *Let $\vartheta_0, \vartheta_1 \in (1,\infty)$ and $\theta_0^{-1} - \theta_1^{-1} = \vartheta_0^{-1} - \vartheta_1^{-1}$ and space Z be as above and an* UMD*-space. Assume also*

$$\sup_{k\in\mathbb{N}} \|T_k|\mathcal{L}(L_{\theta_0}(\mathbb{R}^n, A_k), L_{\theta_1,\infty}(\mathbb{R}^n, B_k))\| = C_L < \infty.$$

a) Let $\beth$ be either Theorem 3.3 *for $p > 1$, or parts a), b), c) of Theorem* 3.4*, and, for every $i \in \mathbb{N}$, operator T_i satisfy the conditions of $\beth$ with $A = A_i$ and $B = B_i$. Assume every B_i to have a pre-dual Banach space and to possess the Rhadon-Nikodim property, $\tilde{T} \in \mathcal{AD}_y(u, L_q, X, \lambda_0, \lambda_1, D_0) \cup \mathcal{AD}_y(L_q, u, X, \lambda_0, \lambda_1, D_0)$ as an operator with $\mathcal{L}(l_\infty(\mathbb{A}), l_\infty(\mathbb{B}))$-valued kernel, and that the conditions of one of the following groups are fulfilled:*

1) $u_1 < p_I = \sup\{p : p^{-1} + (C_{p,A}C_1C_2^2)^{-1} > 1\}$, *where $C_{p,A}, C_1, C_2$ are constants from Lemmas* 5.6, 5.8, *and either $t \in (0,1]$, $\theta_0 = \theta_1$, or $\theta_0 < \theta_1$;*

2) $Z = l_\varsigma$, $\varsigma \in [u_1,\infty) \cap [\theta_1, \theta_0]$, $1 < \varsigma \leq \nu_0, q_0$.

Then the assertions of $\beth$ remain true if we substitute spaces A, B, operator T and parameters θ_0, θ_1 in their statements for $Z(\mathbb{A}), Z(\mathbb{B})$, $\tilde{T}$ and parameters ϑ_0, ϑ_1 respectively.

b) Let $\daleth$ *be either Theorem* 3.1*, or Theorem* 3.2*, and, for every* $i \in \mathbb{N}$*, operator* T_i *satisfy the conditions of* $\daleth$ *with* $A = A_i$ *and* $B = B_i$*. Assume also that spaces* $\{A_i^*\}_{i\in\mathbb{N}}$ *possesses the Rhadon-Nikodim property,*
$\tilde{T} \in \mathcal{AD}_x(u, L_q, X, \lambda_0, \lambda_1, D_0) \cup \mathcal{AD}_x(L_q, u, X, \lambda_0, \lambda_1, D_0)$ *as an operator with* $\mathcal{L}(l_\infty(\mathbb{A}), l_\infty(\mathbb{B}))$*-valued kernel. Let the conditions of one of the following groups be satisfied:*
1) $\max(q', \theta_1') < p_I$ *and either* $t \in (0, 1]$*,* $\theta_0 = \theta_1$*, or* $\theta_0 < \theta_1$*;*
2) $Z = l_\varsigma$*,* $\varsigma \in [u_1, \infty) \cap [\theta_1, \theta_0]$*,* $1 < \varsigma \leq \nu_0, q_0$*.*
Then the assertions of $\daleth$ *remain true if we substitute spaces* A, B*, operator* T *and parameters* θ_0, θ_1 *in their statements for* $Z(\mathbb{A}), Z(\mathbb{B})$*,* $\tilde{T}$ *and* ϑ_0, ϑ_1 *respectively.*

c) Under the conditions of groups 1) *of both part a) and part b), we have*

$$\widetilde{T} \in \mathcal{L}(L_{\vartheta_0}(\mathbb{R}^n, Z(\mathbb{A})), L_{\vartheta_1}(\mathbb{R}^n, Z(\mathbb{B}))).$$

Remark 5. *a)* Let us note that the conclusions of Theorem 3.5, *a*), 1); *b*), 1) with $Z = \mathbb{R}$ are even stronger then the conclusions concerning the boundedness of the previous theorems thanks to the usage of Theorem 5.9 in its proof.

b) Assume that $Z = Z(\Omega)$, $\{T_\omega\}_{\omega\in\Omega}$, $\{A_\omega\}_{\omega\in\Omega}$ and $\{B_\omega\}_{\omega\in\Omega}$ are an ideal space on Ω and two families of Banach spaces respectively. In Theorem 3.5, the membership of $\tilde{T}$ in some $\mathcal{AD}$-class as an operator with $\mathcal{L}(Y(\mathbb{A}), Y(\mathbb{B}))$-valued kernel $Y = L_\infty(\Omega)$ is equivalent to the inclusion $\sup_{\omega\in\Omega} \mu_i(r, w, T_\omega) \in X$, where $\mu_i(r, w, T_\omega)$ is the functional corresponding to the $\mathcal{AD}$-class of the same type. Hence, in the case $A_\omega = A$, $B_\omega = B$, $T_\omega = T$, the inclusion of T in the same $\mathcal{AD}$ class is needed.

c) Teorem 3.5 remains true when $Z = Z(\Omega)$ and $Z^* = Z^*(\Omega)$ are ideal spaces and $\tilde{T} = T \otimes Id_Z$.

4. The proofs of the main results

4.1. Vector-valued functions. In a majority of the proofs of this section, we use the property, which is common for every space $X = X(\mathbb{N}_0 \times \Psi)$ under consideration: for $c \in (0, 1)$, the inclusion $f \in X$ implies that $f'(i, \psi) := \sum_{k\geq i} c^{i-k} f(k, \psi)$ and $f''(i, \psi) := \sum_{k\geq i} c^{i-k}(k - i + 1)f(k, \psi)$ are in X too, and $\max(\|f'|X\|, \|f''|X\|) \leq C(c)\|f|X\|$.

The proof of Theorem 3.3. We shall follow the same method of the decomposition $T = \sum_{i\in\mathbb{N}_0} T_i$ as used in [**5**] but by the other variable. Let us fix parallelepiped $Q_r(w) \subset \mathbb{R}^n$

and designate

$$\varphi = \chi_{Q_\delta(0)\setminus Q_{\delta/b}(0)},\ \varphi_i(x) = \varphi\left(\frac{\cdot - w}{(rb^i)^{\gamma_a}}\right),\ \varphi_0(x) := 1 - \sum_{i=1}^{\infty} \varphi_i(x), x \in \mathbb{R}^n.$$

Let $K_i(x,y) = \varphi_i(x)K(x,y)$, $i \in \mathbb{N}$. We consider an orthogonal normalized basis $\{p_l\}_{l=1}^{L}$ of polynomials in the Hilbert space $\mathcal{P}_{D^*_{\lambda_1}}$ endowed with the scalar product $(P,Q) := \int R(x)Q(x)\varphi(x)dx$. Thus we designate

$$K(x,y) = \sum_{i\in\mathbb{N}_0} K_i'(x,y), K_0'(x,y) = K_0(x,y) + \xi_0(x,y),$$

$$K_i'(x,y) = K_i(x,y) + \xi_i(x,y) - \xi_{i-1}(x,y), i \in \mathbb{N},$$

$$\xi_i(x,y) = \sum_{j=i+1}^{\infty}\sum_{l=1}^{L}\int K_j(x,z)p_l\left(\frac{z}{(rb^i)^{\gamma_a}}\right)dz(rb^i)^{-n}\varphi_i(y)p_l\left(\frac{y}{(rb^i)^{\gamma_a}}\right),$$

$$(T_if)(x) := \int K_i(x,y)f(y)dy,\ (T_i'f)(x) := \int K_i'(x,y)f(y)dy. \tag{1}$$

Note that every operator T_i' has its support in the closure of $Q_{\delta rb^i}(w)$, and tensor product

$$A \otimes \mathcal{P}_{D_{\lambda_1}} \subset (T_i')^{-1}(\mathcal{P}_{D_0}(B)) \text{ for } i \in \mathbb{N}_0$$

due to the conditions of $ORT_y(D_0,D_1)$. Let $f \in \widetilde{l}^{\lambda_1,D_1}_{Y,\infty,u_1}(\mathbb{R}^n, A)$. To prove the boundedness of $T - T_0' = \sum_{i\in\mathbb{N}} T_i'$, Theorem 5.5 permits the usage of the (quasi)(semi)norms

$$\|h|\widetilde{l}^{\lambda_1,D_1}_{Y,\infty,u_1}(\mathbb{R}^n,A)\| := \|\sup_{t>0} t^{-\lambda_1}\mathcal{D}_{q'}(t,\cdot,h,D_1,A)|Y(\mathbb{R}^n)\|; \tag{2}$$

$$\|h|\widetilde{l}^{\lambda_0,D_0}_{Y,\infty,u_0}(\mathbb{R}^n,B)\|_w := \|\sup_{\substack{t>0\\ \|g\|_{B^*}\le 1}} t^{-\lambda_0}\mathcal{D}_{u_0}(t,\cdot,(g,h),D_0)|Y(\mathbb{R}^n)\|$$

for $\lambda_0 > 0$, or $\|h|\widetilde{b}^{\lambda_0,D_0}_{\infty,\infty}(\mathbb{R}^n,B)\|_w := \sup\{t^{-\lambda_0}\mathcal{D}_u(t,z,h,D_0,B) : t>0, z\in\mathbb{R}^n\}$ for $\lambda_0 = 0$. (3)

For $g \in B^*$, we derive from the subadditivity of $\mathcal{D}$-functional that

$$\mathcal{D}_u(r,w,(g,(T-T_0')f),D_0) \le \sum_{i=1}^{\infty}\mathcal{D}_u(r,w,(g,T_i'f),D_0). \tag{4}$$

Similarly to the proof of Theorem 3.1 in [**5**] using, in addition, Minkowski and Hölder inequalities and definitions of $\mathcal{AD}^{\omega}$- and $\mathcal{AD}^{\omega*\omega}$-classes, we obtain

$$r^{-\lambda_0}\mathcal{D}_u\left(r,w,\left(g,\int\int K_j(\cdot,z)(f-\pi_{\delta rb^i,w}f)(y)p_l\left(\frac{z}{(rb^i)^{\gamma_a}}\right)dz\ \times\right.\right.$$
$$\left.\left.\times\ (rb^i)^{-n}\varphi_i(y)p_l\left(\frac{y}{(rb^i)^{\gamma_a}}\right)dy\right),D_0\right) =$$
$$= Q_1 \le Cr^{-\lambda_0}\int\mathcal{D}_u(r,w,K_j^*(\cdot,z)g,D_0,A^*)\left|p_l\left(\frac{z}{(rb^i)^{\gamma_a}}\right)\right|dz\times$$

$$\times \int \|f - \pi_{\delta rb^i,w} f\|_A(y)\varphi_i(y)(rb^i)^{-n}dy$$

for $T \in \mathcal{AD}_y^\omega(u, L_q, X, \lambda_0, \lambda_1, D_0)$, or, for $T \in \mathcal{AD}_y^{\omega*\omega}(u, L_\infty, X, \lambda_0, \lambda_1, D_0)$,

$$Q_1 \leq \int\int \left\| r^{-(\lambda_0+n/u)}(g, K_j - \pi_{r,w}K_j)(\cdot, z)\frac{a(y)}{\|a(y)\|_A} \right| L_u(Q_r(w))\left\| \left| p_l\left(\frac{z}{(rb^i)^{\gamma_a}}\right)\right| dz \times\right.$$

$\times \|a(y)\|_A\varphi_i(y)(rb^i)^{-n}dy, a(y) = (f - \pi_{\delta rb^i,w}f)(y)$, or, for $T \in \mathcal{AD}_y^\omega(L_q, u, X, \lambda_0, \lambda_1, D_0)$,

$$Q_1 \leq r^{-(\lambda_0+n/u)}\left\| \int \|(I - \pi_{r,w})K_j^*(\cdot, z)g\|_{A^*}\left|p_l\left(\frac{z}{(rb^i)^{\gamma_a}}\right)\right| dz \right| L_u(Q_r(w))\right\| \times$$

$$\times \|a(y)\|_A\varphi_i(y)(rb^i)^{-n}\left|p_l\left(\frac{y}{(rb^i)^{\gamma_a}}\right)\right| dy;$$

$$r^{-\lambda_0}\mathcal{D}_u(r, w, (g, T_i(f - \pi_{\delta rb^i,w}f))), D_0) = Q_2 \leq$$

$$\leq \int r^{-\lambda_0}\mathcal{D}_u\left(r, w, \left(g, K_i(\cdot, y)\frac{a(y)}{\|a(y)\|_A}\right), D_0\right)\|a(y)\|_A dy$$

for $T \in \mathcal{AD}_y^\omega(u, L_q, X, \lambda_0, \lambda_1, D_0) \cup \mathcal{AD}_y^{\omega*\omega}(u, L_\infty, X, \lambda_0, \lambda_1, D_0)$, or

for $T \in \mathcal{AD}_y^\omega(L_q, u, X, \lambda_0, \lambda_1, D_0)$,

$$Q_2 \leq r^{-(\lambda_0+n/u)}\left\| \int \|(I - \pi_{r,w})K_i^* g(y)|A^*\| \cdot \|(I - \pi_{\delta rb^i,w})f(y)|A\|dy \mid L_u(Q_r(w))\right\|. \quad (5)$$

For $T \in \mathcal{AD}_y^\omega(u, L_q, X, \lambda_0, \lambda_1, D_0) \cup \mathcal{AD}_y^\omega(u, L_q, X, \lambda_0, \lambda_1, D_0) \cup$ $\cup \mathcal{AD}_y^{\omega*\omega}(u, L_\infty, X, \lambda_0, \lambda_1, D_0)$, Hölder and Minkowski inequalities applied to (5) provide, for

$$\mu_i'(r, w) = \sum_{j=i}^{\infty} b^{(i-j)(\lambda_1-\lambda_{\max}(D_1))}\mu_j(r, w),$$

$$r^{-\lambda_0}\mathcal{D}_u(r, w, (g, T_i'f), D_0) \leq C\mu_i'(r, w)(\delta rb^i)^{-\lambda_1}\mathcal{D}_{q'}(\delta rb^i, w, f, D_1, A), \quad (6)$$

where $\mu_i(r, w)$ corresponds to the appropriate $\mathcal{AD}$-class. Now subadditivity of $\mathcal{D}$-functional implies

$$r^{-\lambda_0}\mathcal{D}_u(r, w, (g, (T - T_0')f), D_0) \leq C\sum_{i=0}^{\infty}\mu_i'(r, w)(\delta rb^i)^{-\lambda_1}\mathcal{D}_{q'}(\delta rb^i, w, f, D_1, A). \quad (7)$$

For $\lambda_0 = 0$, the same considerations are followed by

$$r^{-\lambda_0}\mathcal{D}_u(r, w, (T - T_0')f, D_0) \leq C\sum_{i=0}^{\infty}\mu_i'(r, w)(\delta rb^i)^{-\lambda_1}\mathcal{D}_{q'}(\delta rb^i, w, f, D_1, A). \quad (8)$$

Corollary 5.4 delivers $T_0'f - \pi_{r,w}T_0'f = T_0'(f - \pi_{r,w}f)$. (9)

Hölder inequality and $L_{\theta_0}(\mathbb{R}^n, A) \to L_{\theta_1,\infty}(\mathbb{R}^n, B)$-boundedness of T clarify the estimates

$$r^{-\lambda_0}\mathcal{D}_{u_0}(r, w, (g, T_0\chi_{Q_{\delta r,w}}(f - \pi_{\delta r,w}f)), D_0) \leq r^{-\lambda_0}\mathcal{D}_{\theta_1,\infty}(r, w, (g, T_0\chi_{Q_{\delta r,w}}(f - \pi_{\delta r,w}f)), \emptyset) =$$

$$= r^{-(\lambda_0+n/\theta_1)}\|\chi_{Q_{r,w}}(\cdot)(g, T_0\chi_{Q_{\delta r,w}}(f - \pi_{\delta r,w}f))|L_{\theta_1}(\mathbb{R}^n)\| \leq$$

$$\leq C\|T|\mathcal{L}(L_{\theta_0}(\mathbb{R}^n, A), L_{\theta_1}(\mathbb{R}^n, B))\|(r\delta)^{-\lambda_1}\mathcal{D}_{\theta_0}(r\delta, w, f, D_1, A)$$

$$(\lambda_0 + n/\theta_1 = \lambda_1 + n/\theta_0). \tag{10}$$

For $a), b), c)$, formulas $(6-10)$ and Hölder inequality if $u_1 = \max(q', \theta_0)$ are followed by

$$r^{-\lambda_0}\mathcal{D}_{u_0}(r, w, Tf, D_0, B) \le C(\sum_{i=1}^{\infty} \mu_i'(r,w)(\delta r b^i)^{-\lambda_1}\mathcal{D}_{u_1}(\delta r b^i, w, f, D_1, A)+$$

$$(\mu_0'(r,w) + \|T|\mathcal{L}(L_{\theta_0}(\mathbb{R}^n, A), L_{\theta_1,\infty}(\mathbb{R}^n, B))\|)\,(\delta r)^{-\lambda_1}\mathcal{D}_{u_1}(\delta r, w, f, D_1, A)). \tag{11}$$

Taking supremum by $r > 0$, or using discrete Hardy inequality if $\lambda_1 > \lambda_{\max}(D_1)$, one can establish the formula proving $a)$ for $\mu_i = \lambda_i$ with the aid of Theorem 5.5 and ideality of Y:

$$\sup_{r>0, \|g|B^*\|\le 1} r^{-\lambda_0}\mathcal{D}_{u_0}(r, w, (g, Tf), D_0) \le$$

$$\le C(\|T|\mathcal{L}_\theta\| + C_{\mathcal{AD}}) \sup_{r>0} r^{-\lambda_0}\mathcal{D}_{u_1}(r, w, f, D_1, A). \tag{12}$$

We repeat the same for $\mu_i \in [0, \lambda_i]$ under the conditions of $a)$ due to the shift property (Corollary 2.5):

$$\mathcal{AD}_y^{(\cdot)}(\cdot,\cdot, l_t, \lambda_0, \lambda_1, D) \subset \mathcal{AD}_y^{(\cdot)}(\cdot,\cdot, l_{t',log}, \lambda_0', \lambda_1', D) \text{ for } t, t' \in (0,\infty], \lambda_0 - \lambda_1 = \lambda_0' - \lambda_1'.$$

Parts $b)$ and $c)$ follow from $a)$ with the aid of Lemma 5.5.

Using s-convexity of Y, we derive from (11) that

$$\|r^{-\lambda_0}\mathcal{D}_{u_0}(r, \cdot, Tf, D_0, B)|Y\|^s \le C(\sum_i (\mu_i'(r,w))^s \|(\delta r b^i)^{-\lambda_1}\mathcal{D}_{u_1}(\delta r b^i, \cdot, f, D_1, A)|Y\|^s +$$

$$+\|T|\mathcal{L}(L_{\theta_0}(\mathbb{R}^n, A), L_{\theta_1,\infty}(\mathbb{R}^n, B))\|^s \|(\delta r b)^{-\lambda_1}\mathcal{D}_{u_1}(\delta r b, \cdot, f, D_1, A))|Y\|^s). \tag{13}$$

Now Lemma 5.1 finishes the proof of part $d)$.

Part $e)$ is obtained from (11) with the aid of Lemma 5.1 and, then, the ideality of Y.

For $Y = L_{\varrho 0}$, part $d')$ follows from (13) and the Hölder inequality $\|hg|L_{\varrho 0}\| \le \|h|L_{\varrho_1}\| \cdot \|g|L_\varrho\|$ applied to each summand of the right-hand side of (13). The last step is the same as for part $d)$.

The proof of $e')$ is the combination of the one of e with $Y = L_{\varrho 0}$ and, then, the usage of the Hölder inequality from the proof of part $d')$. $\mathcal{Q.E.D.}$

The proof of the Theorem 3.4. We shall add $*$ referring to the formulas from the proof of Theorem 3.3. Note that we can percept the absence of ORT-conditions in comparison with the conditions of Theorem 3.3 as the degeneration $D_1 = \emptyset$. In designations used there, the desirable decomposition

(1∗) becomes $Tf(x) = \sum_{i\in\mathbb{N}_0} (T_i f)(x) = \sum_{i\in\mathbb{N}_0} \int K_i(x,y) f(y) dy.$ (1)

Following step by step the proof of formula (11*) with natural simplifications, we have

$$r^{-\mu_0}\mathcal{D}_{u_0}(r,w,Tf,D,B) \le C(\sum_{i=1}^{\infty}\mu_i(r,w)(\delta rb^i)^{-\mu_1}\mathcal{D}_{u_1}(\delta rb^i,w,\varphi_i f,\emptyset,A)+$$

$$+\|T|\mathcal{L}(L_{\theta_0}(\mathbb{R}^n,A),L_{\theta_1,\infty}(\mathbb{R}^n,B))\|(\delta r)^{-\mu_1}\mathcal{D}_{u_1}(\delta r,w,\varphi_0 f,D_1,A)). \tag{2}$$

As before, the shift property from Corollary 2.5, means the inclusions of T in all the $\mathcal{AD}_y$-classes mentioned in the theorem. Hölder inequality implies

$$(\delta rb^i)^{-\mu_1}\mathcal{D}_{u_1}(\delta rb^i,w,\varphi_i f,\emptyset,A) \le \|\varphi_i f|L_{\nu_1,\infty}(\mathbb{R}^n,A)\|, i\in\mathbb{N}_0. \tag{3}$$

Thus embeddings $l_t \subset l_1$ and $L_{\nu_1,w_1} \subset L_{\nu_1,\infty}$ (Chebyshev) prove the first two cases of part a) (i.e. $t\in(0,1]$). The rest of a) follows from $(2,3)$, Hölder inequality for sequences and the identity $\|f|L_{\nu_1}(\mathbb{R}^n,A)\| = \|\{\|\varphi_i f|L_{\nu_1}(\mathbb{R}^n,A)\|\}_{i\in\mathbb{N}_0}|l_{\nu_1}\|$.

The condition $n/(\lambda_0-\lambda_1) > \nu_1$ in b), along with Theorem 5.5, b), means that part a) contains, in particular, the inclusion $T\in\mathcal{L}(L_{n/(\lambda_0-\lambda_1),w}(\mathbb{R}^n,A),BMO(\mathbb{R}^n,B))$ that, in turn, together with $T\in\mathcal{L}(L_{\theta_0}(\mathbb{R}^n,A),L_{\theta_1,\infty}(\mathbb{R}^n,B))$, implies part b) with the aid of the interpolation theory. Under the conditions of c), a corollary of (2) is

$$C\sup_{r>0} r^{-\mu_0}\mathcal{D}_{u_0}(r,w,Tf,D,B) \le M^{\mu_1}_{u_1}f(w) = \sup_{r>0} r^{-\mu_1}\mathcal{D}_{u_1}(r,w,f,\emptyset,A). \tag{4}$$

Now part c) follows from estimate (4), part a) of Theorem 5.5 and the boundedness of fractional maximal function $M^{\mu_1}_{u_1}$ from $L_{q_1,s}$ into $L_{q_0,s}$ for $1/q_1 - 1/q_0 = \mu_1/n$, $q_1 > u_1$ and from L_{u_1} into $L_{(u_1^{-1}+\mu_1/n)^{-1},\infty}$. Thanks to Hölder inequality, designating $T_\infty = T - T_0$ under the conditions of d), we have, for $\xi = (\mu_1/n+1-q^{-1})^{-1}$,

$$r^{-\mu_0}\mathcal{D}_{u_0}(r,w,T_\infty f,D,B) \le C_{\mathcal{AD}}(\mu_0,\mu_1)\||\cdot - w|_{\lambda_a}^{-(\mu_1+n/q')}|L_{\xi,\infty}(\mathbb{R}^n)\|\|f|L_{\nu_1,t'}(\mathbb{R}^n,A)\|.$$

this estimate and a counterpart of (10*) prove d). Part e) follows from d) in the same way as b) from a). $\mathcal{Q.E.D}$.

4.2. „Double" vector-valued functions. The proof of Theorem 3.5. Let us consider part a) and observe that, to prove the theorem, it is sufficient to show that operator $\tilde{T}$ satisfies the conditions of Theorems 3.3 and 3.4. The implication $T_i \in ORT_y(D_0,D_1), i\in\mathbb{N} \Rightarrow \tilde{T}\in ORT_y(D_0,D_1)$ is transparent. With the aid of formulas

$$\|\tilde{K}(x,y)|\mathcal{L}(Z(\mathbb{A}),Z(\mathbb{B}))\| \le C_Z\|\tilde{K}(x,y)|\mathcal{L}(l_\infty(\mathbb{A}),l_\infty(\mathbb{B}))\|,$$

$$\|(I-\pi^x_{r,w,D})\tilde{K}(x,y)|\mathcal{L}(Z(\mathbb{A}),Z(\mathbb{B}))\| \le C_Z\|(I-\pi^x_{r,w,D})\tilde{K}(x,y)|\mathcal{L}(l_\infty(\mathbb{A}),l_\infty(\mathbb{B}))\|, \tag{1}$$

we obtain $T\in\mathcal{AD}_y(u,L_q,X,\lambda_0,\lambda_1,D)\cup\mathcal{AD}_y(L_q,u,X,\lambda_0,\lambda_1,D) \Rightarrow$

$$\Rightarrow \tilde{T} \in \mathcal{AD}_y(u, L_q, X, \dots) \cup \mathcal{AD}_y(L_q, u, X, \dots). \quad (2)$$

It is left to establish the inclusion of part c), $r_0 = r_2 = \vartheta_0$, $r_1 = \vartheta_1$ and of group 2) of parts a) and b), $r_0 = \theta_1$, $r_2 = \infty$, $r_1 = \theta_0$:

$$\widetilde{T} \in \mathcal{L}(L_{r_1}(\mathbb{R}^n, Z(\mathbb{A})), L_{r_0,r_2}(\mathbb{R}^n, Z(\mathbb{B}))). \quad (3)$$

In the case of group 1), it follows from Theorem 5.9 due to the observation that the application of Hölder inequality for sequences to formulas (11) and (2) from the proofs of Theorems 3.3 and 3.4 respectively with $\mu_0 = 0$ implies

$$\sup_{r>0} \mathcal{D}_{u_0}(r, w, T_i f_i, D, B_i) \leq C M^{\lambda_0 - \lambda_1, \emptyset}_{u_1, t', A_i} f_i(w), w \in \mathbb{R}^n, i \in \mathbb{N}.$$

For group 2), it follows from the uniform $L_{\theta_0} - L_{\theta_1,\infty}$-boundedness of $\{T_i\}_{i\in\mathbb{N}}$ and Minkowski inequality:

$$\| \| \{ \|T_i f_i | B_i\| \}_{i\in\mathbb{N}} \mid l_\varsigma \| \mid L_{r_0,\infty}(\mathbb{R}^n) \| \leq \| \{ \|T_i f_i | L_{r_0,\infty}(\mathbb{R}^n, B_i)\| \} i \in \mathbb{N} \mid l_\varsigma \| \leq$$

$$\leq C_L \| \{ \|f_i | L_{r_1}(\mathbb{R}^n, A_i)\| \}_{i\in\mathbb{N}} \mid l_\varsigma \| \leq C_L \| \| \{ \|f_i | A_i\| \}_{i\in\mathbb{N}} \mid l_\varsigma \| \mid L_{r_1}(\mathbb{R}^n) \| . \quad (4)$$

Under the conditions of part b), the counterparts of formulas (1), (2) take place, and the counterpart of (3) for $Z = l_\varsigma$ follows from (4). Further let us observe that, for every natural i, the restriction T_i' of T_i^* on $L_{\theta_1'}(\mathbb{R}^n, B_i^*)$ is defined by $\mathcal{L}(B^*, A^*)$-valued kernel $(K(y,x))^*$ because of the consequence $(L_{\theta_0}(\mathbb{R}^n, A_i))^* = L_{\theta_0'}(\mathbb{R}^n, A_i^*)$ of the possession of Radon-Nikodim property by every A_i^*.

In the case of group 1) of part b), Theorem 5.9 applied to $\tilde{T}'$ and relation (2) from its proof deliver the counterpart of formula (3) because one has $| \int (g, \tilde{T} f) dx| = | \int (\tilde{T}' g, f) dx|$ for $f \in L_{r_1}(\mathbb{R}^n, A)$, $g \in L_{r_0,r_2}(\mathbb{R}^n, B^*)$ and

$$\|h | L_{r_0,r_2}(\mathbb{R}^n, B)\| := \sup_{\|g | L_{r_0', r_2'}(\mathbb{R}^n, B^*)\| \leq 1} \int_{\mathbb{R}^n} (g, h) dx. \; \mathcal{Q.E.D.}$$

5. Auxiliary results

The next lemma contains discrete counterpart of Hausdorf-Young inequality in the form of [5].

LEMMA 5.1. [5] *Let $p, q, r \in (0, \infty]$, $r \geq \max(p, q)$. Then,*

a) *both for $r \geq 1, 1 + 1/r \leq 1/p + 1/q$ and*

b) *for $r \leq 1$ one has $\|\alpha | l_p\| \cdot \|\beta | l_q\| \geq \|\alpha * \beta | l_r\|$;*

c) the condition $\max(p, q) \leq r$ *is necessary in both cases, while the condition* $1 + 1/r \leq 1/p + 1/q$ *is necessary in the case of a) only.*

In the next two propositions, let us designate $\mathcal{D}^D := \{\phi \in C_0^\infty(\mathbb{R}^n) : \phi \perp \mathcal{P}_D\}$.

THEOREM 5.2. *a) if* $|\int_{\mathbb{R}^n}(\psi, f)dx| \leq C\|\psi|L_p(\mathbb{R}^n, A^*)\| \cdot \|f|L_{p'}(\mathbb{R}^n, A)\|$ *for some* $f \in L_{p'}(\mathbb{R}^n, A)$ *and every* $\psi \in \mathcal{D}^D \otimes A^*$ *and* A^{**} *possesses Radon-Nikodim property, then there is the unique representation* $f = g + \pi$*, where* $g \in L_{p'}(\mathbb{R}^n, A^{**})$ *and* $\pi \in \mathcal{P}_D(A^{**})$*;*

b) if, under the conditions of a), A *is reflexive, then there is the unique representation* $f = g + \pi$*, where* $g \in L_{p'}(\mathbb{R}^n, A)$ *and* $\pi \in \mathcal{P}_D(A)$*;*

c) if $|\int_{\mathbb{R}^n}(f, \psi)dx| \leq C\|\psi|L_p(\mathbb{R}^n, A)\| \cdot \|f|L_{p'}(\mathbb{R}^n, A^*)\|$ *for some* $f \in L_{p'}(\mathbb{R}^n, A^*)$ *and every* $\psi \in \mathcal{D}^D \otimes A$ *and* A^* *possesses Radon-Nikodim property, then there is the unique representation* $f = g + \pi$*, where* $g \in L_{p'}(\mathbb{R}^n, A^*)$ *and* $\pi \in \mathcal{P}_D(A^*)$*.*

The proof of Theorem 5.2. In the case of part a), Radon-Nikodim property implies $(L_p(\mathbb{R}^n, A^*))^* = L_{p'}(\mathbb{R}^n, A^{**})$. Thus, the unique extension $F \in (L_p(\mathbb{R}^n, A^*))^*$ satisfying

$$(F, \psi) = \int_{\mathbb{R}^n}(g, \psi)dx = \int_{\mathbb{R}^n}(\psi, f)dx \text{ for any } \psi \in \mathcal{D}^D \otimes A^* \tag{1}$$

is represented by $g \in L_{p'}(\mathbb{R}^n, A^{**})$ Now the inclusion $g - f \in \mathcal{P}_D(A^{**})$ follows from Lemma 5.3.

Part b) is a consequence of a) and $A = A^{**}$.

In the case of part c), Radon-Nikodim property means $(L_p(\mathbb{R}^n, A))^* = L_{p'}(\mathbb{R}^n, A^*)$. Thus, the unique extension $F \in (L_p(\mathbb{R}^n, A^*))^*$ satisfying

$$(F, \psi) = \int_{\mathbb{R}^n}(g, \psi)dx = \int_{\mathbb{R}^n}(\psi, f)dx \text{ for any } \psi \in \mathcal{D}^D \otimes A^* \tag{2}$$

is represented by $g \in L_{p'}(\mathbb{R}^n, A^*)$. As before, Lemma 5.3 finishes the proof implying the inclusion $g - f \in \mathcal{P}_D(A^*)$. $\mathcal{Q.E.D.}$

LEMMA 5.3. *Let* D *be a subset of* $\mathbb{N}_0^n$*, and* A *be a Banach space. Then:*

a) if $\int_{\mathbb{R}^n} f\psi dx = 0$ *for some* $f \in L_{1,loc}(\mathbb{R}^n, A)$ *and every* $\psi \in \mathcal{D}^D$*, then* $f \in \mathcal{P}_D(A)$*;*

b) if $\int_{\mathbb{R}^n} f\psi dx = 0$ *for some* $f \in L_{1,loc}(\mathbb{R}^n, A^*)$ *and every* $\psi \in \mathcal{D}^D$*, then* $f \in \mathcal{P}_D(A^*)$*.*

The proof of Lemma 5.3. Beginning with part a), let us choose some $\phi \in C_0^\infty(Q_1(0))$ with $\int_{\mathbb{R}^n} \phi dx = 1$, $\beta \in \mathbb{N}_0^n$ with $D \subset \overset{\square\circ}{D}_\beta$ and fix some $h \in A^*$. At least as generalized functions, Sobolev's regularizations $\{\phi_{b^k} * (h, f)\}_{k \in \mathbb{N}} \subset C^\infty$ with some $b \in (0, 1)$ converge to (h, f), while every $\phi_{b^k} * (h, f)$ is in $\mathcal{P}_{\overset{\square_0}{D_\beta}}$ because of $\mathrm{D}_i^{\beta_i}\phi_{b^k} * (h, f) = 0$ for every $1 \leq i \leq n$. The latter observation is proved by the induction by n with the aid of the linear independence of any system of different monomials $\{x^\alpha\}_{\alpha \in I}$ in any dimension. Therefore, choosing a system $\{\psi_\alpha\}_{\alpha<\beta} \subset C_0^\infty(Q_1(0))$ to be biorthogonal to $\{x^\alpha\}_{\alpha<\beta}$, we see that, indeed, the system $\phi_{b^k} * (h, f) \in \mathcal{P}_D$ is convergent to the polynomial that is equal to $(h, f) \in \mathcal{P}_{\overset{\square_0}{D_\beta}}$ almost everywhere. Now we recall that the Bochner measurability of f means that there is separable subspace $A_1 \subset A$ containing almost all values of f. Thus, thanks to the Hahn-Banach theorem, we can find a separable subset $\{h_l\}_{l \in \mathbb{N}} \subset A$ separating the elements of A_1. Let E_l be the set of all points, where (h_l, f) differs from the corresponding polynomial. Using the induction by n, we can find a net

$$\{x_\alpha\}_{0 \leq \alpha \leq \beta} = \prod_{i=1}^{n} \{z_{i,j}\}_{j=1}^{\beta_i} \subset \mathbb{R}^n \setminus \bigcup_{l=1}^{\infty} E_l \text{ with every } z_{i,j} \in \mathbb{R}$$

and construct the polynomial $\pi(x) := \sum_{0 \leq \alpha \leq \beta} f(x_\alpha)\omega_\alpha(x)/\omega_\alpha(x_\alpha)$ of multilinear Lagrange type. The validness of the identities $(h_l, f(x) - \pi(x)) = 0$ for every $l \in \mathbb{N}$ and $x \in \mathbb{R}^n$ finish the proof of part a).

In the case of part b), the proof of a) can be repeated with two changes only. We choose h and $\{h_l\}_{l \in \mathbb{N}}$ from A and use the definition of the norm of linear functional instead of Hahn-Banach theorem, while looking for $\{h_l\}_{l \in \mathbb{N}}$. $\mathcal{Q.E.D.}$

The next corollary was mentioned without a proof in part I ([**5**]).

COROLLARY 5.4. *If, for some finite $D_0, D_1 \subset \mathbb{N}_0^n$ and Banach spaces A, B, SIO T with the domain inside $L_{1,loc}(\mathbb{R}^n, A)$ and the range inside $L_{1,loc}(\mathbb{R}^n, A)$ and its adjoint T^* with the domain inside $L_{1,loc}(\mathbb{R}^n, B^*)$ and the range inside $L_{1,loc}(\mathbb{R}^n, A^*)$ satisfies the duality relation $(\phi, T\pi) = (T^*\phi, \pi)$ for every $\phi \in \mathcal{D}^{D_0} \otimes B^*$ and $\pi \in \mathcal{P}_{D_1}(A)$, then*

$$T^* \in ORT_y(D_0, D_1) \Longleftrightarrow T\mathcal{P}_{D_1}(A) \subset \mathcal{P}_{D_0}(B).$$

<u>The proof of Corollary 5.4.</u> Implication $\Longleftarrow$ follows immediately from the definitions and the observation $\phi \perp \mathcal{P}_D \Longleftrightarrow \phi \perp \mathcal{P}_D(A)$ provided by the notion of linear independence. Implication $\Longrightarrow$ is a consequence of Lemma 5.3. $\mathcal{Q.E.D.}$

The next theorem is established in [**65**].

THEOREM 5.5. [**65**] *Let A be a Banach space, $Y = Y(\mathbb{R}^n) \in HL$ be an ideal space, $a, b, q \in (0, \infty]$, $s \in [0, \infty)$, $p \in (a, \infty)$ and a finite $\emptyset \neq D = \hat{D} \subset \mathbb{N}_0^n$. Then one has* *a)* $\widetilde{l}^{0,D}_{L_{p,s},\infty,a}(\mathbb{R}^n, A) = L^D_{p,s}(\mathbb{R}^n, A)$ *for* $a < \infty$;

b) $\widetilde{l}^{s,D}_{Y,\infty,a}(\mathbb{R}^n, A) = \widetilde{l}^{s,D}_{Y,\infty,b}(\mathbb{R}^n, A)$ *for either* $a, b < \infty$, *or* $s > 0$.

LEMMA 5.6. *Let* $Z = Z(\Omega)$ *be a Banach ideal lattice with the modulus* $|\cdot|$, $p_0, p_1 \in [1,\infty)$, $s, q \in [1,\infty]$, $\lambda \geq 0$, $p_1^{-1} + \lambda/n = p_0^{-1}$ *and either*

a) $Z \in \mathrm{UMD}$, $s = \infty$, $\lambda = 0$, $q \geq p_0 > 1$, *or*

b) $\lambda \in (0, n)$ *and either* $p_0 > 1$, $q \geq p_1$, *or* $p_0 = 1$, $q = \infty$. *Then the following (non-linear) functional is bounded with some constant* $C = C(p_0, p_1, q)$ *from* $L_{p_0}(\mathbb{R}^n, Z)$ *into* $L_{p_1,q}(\mathbb{R}^n, Z)$:

$$f \to (\bar{M}^\lambda_s f)(\omega, x) := \left(\int_0^\infty \Big| \int_{Q_r(x)} f(\omega, y) dy \Big|^s r^{s(\lambda-n)-1} dr \right)^{1/s}.$$

Part *a)* is a celebrated result from [**71**]. Let $c_{p,Z} := C(p,p,p)$.

The proof of Lemma 5.6. Under the conditions of *b)*, Minkowski inequality and properties of the modulus of Z imply $\|(\bar{M}^\lambda_s f)(x)|Z\| \leq (|\cdot|^{\lambda-n}_{\gamma_a} * \||f||Z\|)(x)$. S. L. Sobolev's result on the $L_{p_0} \to L_{p_1,q}$-boundedness of the potential operator finishes the proof. $\mathcal{Q.E.D.}$

The next lemma extending R. Fefferman-Stein inequality is a particular case of a result from [**65**].

LEMMA 5.7. [**65**] *For* $\alpha \geq 0$, $u \in (0,\infty]$, $p \in (1,\infty)$, *a finite* $D \subset \mathbb{N}^n_0$ *and a Banach space* A, *the tensor product* $F_D(A) = \{\psi \in \mathcal{D}^D \otimes A : \operatorname{supp} \psi \text{ is finite}\}$ *is dense in* $L_p(\mathbb{R}^n, A)$ *and*

$$\int (g, f) dx \leq C \int M^* f(x) \sup_{t>0} \mathcal{D}_u(t, x, g, D, A) dx \text{ for } f \in F_D(A), g \in L_{1,loc}(\mathbb{R}^n, A^*).$$

The next lemma is implicitly obtained in [**83**].

LEMMA 5.8. *Let* $f, g \in L_{1,loc}(\mathbb{R}^n,)$, *and* $c_0 = c_0(f), c_1, c_2$ *be the minimal constants in the inequalities* $c_1^{-1}(Mg)^*(t) \leq t^{-1} \int_0^t g^*(s) ds \leq c_2 (Mg)^*(t)$ *and* $M(Mf)(x) \leq c_0 Mf(x)$ *a.e. then, for every* $Q_r(w) \subset \mathbb{R}^n$ *and* $1 \leq p < (c_0 c_1 c_2^2)'$, *we have* $\mathcal{D}_p(r, w, f, \emptyset) \leq C\mathcal{D}_1(r, w, f, \emptyset)$.

The next theorem is formulated in designations of Theorem 3.3. The case $\lambda = 0, q_0 = q_1$ was considered in [**69**], where the original form of R. Fefferman-Stein inequality was employed.

THEOREM 5.9. *Let a finite* $D \subset \mathbb{N}^n_0$, $\lambda \geq 0$, $q_0 \in (1,\infty)$, $q_1, q_2 \in [1,\infty]$, $s, u_0, u_1 \in [1,\infty]$, $1/q_1 + \lambda/n = 1/q_0$ *and either* $\lambda = 0$, $s = \infty$, *or* $\lambda > 0$. *Assume that every* B_i

possesses the Radon-Nikodim property and has a pre-dual Banach space X_i, $B_i = X_i^*$, *and* $Z \in$ UMD *has an unconditional basis* $\{e_i\}_{i\in\mathbb{N}}$ *and* $u_1 < p_I = \sup\{p \in (1,\infty] : p^{-1} + (c_{p,A}c_1c_2^2)^{-1} > 1\}$, *where* $c_{p,Z}, c_1, c_2$ *are constants from Lemmas* 5.6, 5.8. *Let also* A_i, B_i *for* $i \in \mathbb{N}$ *be Banach spaces, and every SIO* T_i *with* $\mathcal{L}(A_i, B_i)$*-valued kernel* $K_i(x,y)$ *satisfy the following estimate uniformly by* i:
for $w \in \mathbb{R}^n$ *and* $f \in L_{u_1,loc}(\mathbb{R}^n, A_i)$,

$$\sup_{r>0} \mathcal{D}_{u_0}(r, w, T_i f_i, D, B_i) \leq C M^{\lambda,\emptyset}_{u_1,s,A_i} f(w).$$

Then we have, for either $q_1 > 1$ *and* $q_0 = q_2$, *or* $q_1 = 1 < q_0$ *and* $q_2 = \infty$,
a) $\tilde{T} \in \mathcal{L}(L_{q_1}(\mathbb{R}^n, Z(\mathbb{A})), L^D_{q_0,q_2}(\mathbb{R}^n, Z(\mathbb{B})))$;
b) $\tilde{T} \in \mathcal{L}(L_{q_1}(\mathbb{R}^n, Z(\mathbb{A})), L_{q_0,q_2}(\mathbb{R}^n, Z(\mathbb{B})))$ *if* T_i *is Chebyshev regular for every* $i \in \mathbb{N}$.

The proof of Theorem 5.9. There exist $\kappa > 1$ and $u \in [p_I, u_1]$ satisfying $u^{-1} + (\kappa c_{u,Z}c_1c_2^2)^{-1} > 1$. Let us consider a vector-valued counterpart of the maximal operator from [**69**]:

$$mf = m(A)f := \|f|A\| + \sum_{i\in\mathbb{N}_0^n} ((\kappa c_{u,Z})^{-1}M)^i \|f|A\|,\ f \in L_{1,loc}(\mathbb{R}^n, A),$$

where M is Hardy-Littlewood maximal function, and A is a Banach space. Thus, we have $M(mf)(x) \leq \kappa C_{u,Z} mf(x)$ and, therefore, due to Lemma 5.8 and Hölder inequality,

$$M^{\lambda,\emptyset}_{u_1,s,A_i} f(x) \leq M^{\lambda,\emptyset}_{u,s,A_i} f(x) \leq C M^{\lambda,\emptyset}_{1,s,A_i} f(x) \text{ a.e.} \tag{1}$$

Now Lemmas 5.7, 5.6, formula (1) and the inequality from the conditions of the theorem imply, for $g_i \in \mathcal{D}^D \otimes X_i$, $i \in \mathbb{N}$ and $(\tilde{m}f)_i := m(A_i)f_i$,

$$\left|\int (g, \tilde{T}f)dx\right| \leq \left|\sum_i \int (g_i, T_i f_i)dx\right| \leq \int \sum_i (M\|g_i\|_{B_i^*})(x)(M^{0,D}_{u_0,\infty,B_i} T_i f_i)(x)dx \leq$$

$$\leq C \int \sum_i (M\|g_i\|_{B_i^*})(x)(M^{\lambda,\emptyset}_{1,s,\mathbb{R}}((\tilde{m}f)_i))(x)\, dx \leq$$

$$\leq C C_{q_0',Z^*} \|g|L_{q_0',q_2'}(\mathbb{R}^n, Z(\mathbb{B}^*))\| \cdot \|M^{\lambda,\emptyset}_{1,s,\mathbb{R}}(\tilde{m}f)|L_{q_0,q_2}(\mathbb{R}^n, Z)\|. \tag{2}$$

Because Lemma 5.6 provides both the boundedness of $\tilde{m}$ in $\mathcal{L}(L_{q_1}(\mathbb{R}^n, Z(\mathbb{A})))$ and the boundedness of $M^{\lambda,\emptyset}_{1,s,\mathbb{R}}$ from $L_{q_1}(\mathbb{R}^n, Z)$ into $L_{q_0,q_2}(\mathbb{R}^n, Z)$, estimate (2) implies the validness of the conditions of Theorem 5.2, c), which proves part a).

Let us note that, due to part a) and the assumption of Chebyshev regularity in part b), operator $\tilde{T}$ is, indeed, bounded from a dense subset of $L_{q_1}(\mathbb{R}^n, Z(\mathbb{A}))$ into not only $L^D_{q_0,q_2}(\mathbb{R}^n, Z(\mathbb{B}))$, but also $L_{q_0,q_2}(\mathbb{R}^n, Z(\mathbb{B}))$ and, therefore, possesses the unique extension on the whole space $L_{q_1}(\mathbb{R}^n, Z(\mathbb{A}))$. $\mathcal{Q.E.D}$.

6. $\mathcal{AD}$-classes of sub-additive operators

In this section, we introduce and study extrapolation properties of the most general classes of singular operators defined in the thesis. Namely, not only the operators are permitted to be sub-additive, but also the $\mathcal{AAD}$-classes considered cover the most of $\mathcal{AD}$- and $\mathcal{AAD}$-classes of SIO and SSIO dealt with in the previous chapters.

We say that a family $\{X(Q_r(w), A)\}_{Q_r(w)\subset X}$ of function spaces on the cubes of $\mathbb{R}^n$ is ζ-homogeneous if

$$\|f(\cdot/\xi)|X(Q_{r\xi}(0), A)\| = \xi^{\zeta}\|f|X(Q_r(0), A)\| \text{ and}$$

$$\|f(\cdot - w)|X(Q_r(w), A)\| = \|f|X(Q_r(0), A)\|$$

for every $Q_r(w) \subset \mathbb{R}^n$ and $\xi > 0$. In this case, we define $h(X) := \zeta$.

REMARK 6. For example, one has $h(L_p) = n/p$.

Similarly to [5] throughout the whole article, we deal with one of the following cases. Let A, B be Banach spaces. We say that operator T bounded from $L_{\theta_0}(\mathbb{R}^n, A)$ into $L_{\theta_1,\theta_2}(\mathbb{R}^n, B)$ for some $\theta_0, \theta_1, \theta_2 \in (0, \infty]$ is a singular operator (SO) if it is sub-additive,

$$\|T(f+g)(x)|B\| \le \|Tf(x)|B\| + \|Tg(x)|B\|$$

and any f, g from a domain of T and a.e. $x \in \mathbb{R}^n$, and if it is in one of the classes defined below.

DEFINITION 6.1. For $j = 1, 2$, let $X_j := X_j(\mathbb{N}_0 \cup \{-1\} \times \mathbb{R}_+ \times \mathbb{R}^n)$ be (quasi)normed space of real-valued functions, $\lambda_0, \lambda_1, \nu_0, \nu_1 \in \mathbb{R}$; $\{Z_{00}(Q_r(w), A)\}_{Q_r(w)\subset\mathbb{R}^n}$, $\{Z_{01}(Q_r(w), B)\}_{Q_r(w)\subset\mathbb{R}^n}$, $\{Z_{10}(Q_r(w), A)\}_{Q_r(w)\subset\mathbb{R}^n}$, $\{Z_{11}(Q_r(w), A)\}_{Q_r(w)\subset\mathbb{R}^n}$ be homogeneous families of vector-valued function spaces; $\Delta_i(r, w) = Q_{\delta r b^{i+1}}(w) \setminus Q_{\delta r b^i}(w)$ for $i \in \mathbb{N}_0$, $\Delta_{-1}(r, w) = Q_{\delta r}(w)$ and a finite $D \subset \mathbb{N}_0^n$. Then we say that operator T is a member of the class

$$\underline{\mathcal{AAD}_x(\vec{Z}, \vec{X}, \vec{\lambda}, \vec{\nu}, D)} = \mathcal{AAD}_x(Z_{00}, Z_{01}, Z_{10}, Z_{11}, X_1, X_2, \lambda_0, \lambda_1, \nu_0, \nu_1, D)$$

if there is a family of operators $\{G(t, w)\}_{Q_r(w)\subset\mathbb{R}^n}$ and functions $\mu^j = \mu^j(T) : \mathbb{N}_0 \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ with $j = 1, 2$ satisfying, for some $\delta > 0, b > 1$,

$$C^j_{\mathcal{AAD}} := \|\mu^j|X_j\| < +\infty \text{ for } j = 1, 2 \text{ where } \mu^1_{-1} = 0,$$

$$\mu^1_i(r, w, T) :=$$

$$= r^{h(Z_0)-\lambda_0}(\delta r b^i)^{\lambda_1-h(Z_1)}\|T(I-G(r,w))|\mathcal{B}(Z_{00}(Q_r(w),A)/\mathcal{P}_D(A),Z_{01}(\Delta_i(r,w),B))\|,\ i\in\mathbb{N}_0,$$

$$\mu_i^2(r,w,T):=$$

$$= r^{h(Z_0)-\nu_0}(\delta r b^i)^{\nu_1-h(Z_2)}\|G(r,w)|\mathcal{B}(Z_{10}(Q_r(w),A)/\mathcal{P}_D(A),Z_{11}(\Delta_i(r,w),A))\|, i\in\mathbb{N}_0\cup\{-1\}.$$

In the case $X:=L_\infty(\mathbb{R}_+\times\mathbb{R}^n,l)$, where l is a sequence space, we shall use l instead of X.

COROLLARY 6.2. [5] *The following inclusions take place (the parameters omitted coincide in every particular inclusion). For* $D\subset D'\subset\mathbb{N}_0^n$, $\lambda_0,\lambda_1,\lambda_0',\lambda_1'\in\mathbb{R}$, $Z_{00}\subset Y_{00}$, $Z_{10}\subset Y_{10}$, $Y_{01}\subset Z_{01}$, $Y_{11}\subset Z_{11}$ *and* $X_j\subset W_j$ *for* $j=1,2$, $\lambda_0'<\lambda_0$ *and* $\mu_0-\lambda_0=\mu_1-\lambda_1$, *one has:*

$$\mathcal{AAD}_x(\vec{Y},\vec{\lambda},\vec{\nu},D)\subset\mathcal{AAD}_x(\vec{Z},\vec{W},\vec{\lambda'},\vec{\nu'},D').$$

REMARK 7. Note that definitions of some $\mathcal{AAD}$-classes are equivalent to their continuous forms. It shows their independence of the value of $b>1$. For instance, the definition of class
$\mathcal{AAD}_x(\vec{Z},\vec{X},\vec{\lambda},\vec{\nu},D)$ with $X_j:=X_j(\mathbb{N}_0\times\mathbb{R}_+\times\mathbb{R}^n)$ being equal to $W_j(\mathbb{R}_+\times\mathbb{R}^n,l_{t_j})$ for $t_j\in(0,\infty])$, $j=1,2$ can be written with the aid of the functions $\mu^k=\mu^k(T)$: $[1,\infty)\times\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}$, $k=1,2$,

$$\mu_i^1(\tau,r,w):=r^{h(Z_0)-\lambda_0}(\delta r\tau)^{\lambda_1-h(Z_1)}\|T(I-G(r,w))|\mathcal{B}(Z_{00}(Q_r(w),A)/\mathcal{P}_D(A),Z_{01}(\Delta_i(r,w),B))\|,$$

$$\mu_i^2(\tau,r,w):=r^{h(Z_0)-nu_0}(\delta r\tau)^{\nu_1-h(Z_2)}\|G(r,w)|\mathcal{B}(Z_{10}(Q_r(w),A)/\mathcal{P}_D(A),Z_{11}(\Delta_\tau(r,w),A))\|,$$

where $\Delta_\tau(r,w)=Q_{\delta rb\tau}(w)\setminus Q_{\delta r\tau}(w)$ for $\tau\in(1,\infty)$ and $\Delta_1(r,w)=Q_{\delta r}(w)$ for $\tau=1$, contained in $W_j(\mathbb{R}\times\mathbb{R}^n,V_j))$ for $j=1,2$ with $V_1:=L_{t_1}(\mu,(1,\infty))$ and $V_2:=L_{t_2}(\mu,[1,\infty))$, where measure μ satisfies $d\mu=\frac{d\tau}{\tau}$ on $(1,\infty)$ and $\mu(\{1\})=1$.

THEOREM 6.3. *Let a sub-additive SO T satisfy the conditions:*
$T\in\mathcal{B}(L_{\theta_0}(\mathbb{R}^n,A),L_{\theta_1,\infty}(\mathbb{R}^n,B))$, $T\in\mathcal{B}(L_{\theta_2}(\mathbb{R}^n,A),L_{\theta_1,\infty}(\mathbb{R}^n,B))$ *for a finite* $D_0\subset\mathbb{N}_0^n$, $p_0\in(0,\theta_2)$, $\theta_0,\theta_1,\theta_2,w_0,w_1,v_0,v_1,t_1,t_2,p_1\in(0,\infty]$, $1\in[1/p_1,\lambda_1/n]$, $1\in[1/\theta_0,\nu_1/n]$, $\lambda_j,\nu_j\in\mathbb{R}$, $X_j=L_\infty(\mathbb{R}_+\times\mathbb{R}^n,l_{t_j})$ *with every* $j\in\{1,2\}$ *and both for either* $1<\lambda_1/n$, *or* $t_1\le1$ *and for either* $1<\nu_1/n$, *or* $t_2\le1$. *Assume also that*
$\lambda_0-\lambda_1=n(1/\theta_2-1/\theta_1)=n(1/p_0-1/p_1)\ge0$, $\nu_0-\nu_1=n(1/\theta_2-1/\theta_0)\ge0$, $\max(w_0,v_0)\le p_0$, $w_1\ge p_1$, $v_1\ge\theta_0$ *and* $T\in\mathcal{AAD}_x(\vec{Z},\vec{X},\vec{\lambda},\vec{\nu},D_0)$ *with* $Z_{00}(Q_1(0),A)=L_{w_0}(Q_1(0),A)$, $Z_{01}(Q_1(0),B)=L_{w_1}(Q_1(0),B)$, $Z_{10}(Q_1(0),A)=L_{v_0}(Q_1(0),A)$, $Z_{11}(Q_1(0),A)=$

$= L_{v_1}(Q_1(0), A)$.

Then operator T admits the following bounded extensions:

a) $T \in \mathcal{B}(L_{p_0}(\mathbb{R}^n, A), L_{p_1,\infty}(\mathbb{R}^n, B))$;

b) $T \in \mathcal{B}(L_{\sigma_0,\eta}(\mathbb{R}^n, A), L_{\sigma_1,\eta}(\mathbb{R}^n, B))$ for $\eta \in (0, \infty]$, $\sigma_0 \in (p_0, \theta_2)$ and $1/\sigma_0 - 1/\sigma_1 = 1/p_0 - 1/p_1$. And in all the cases, the norm of the extension is dominated by

$$\|T|\mathcal{B}(L_{\theta_2}(\mathbb{R}^n, A), L_{\theta_1,\infty}(\mathbb{R}^n, B))\| + C^1_{\mathcal{AAD}} + C^2_{\mathcal{AAD}}\|T|\mathcal{B}(L_{\theta_0}(\mathbb{R}^n, A), L_{\theta_1,\infty}(\mathbb{R}^n, B))\|.$$

REMARK 8. The case $A = B = \mathbb{C}$, $D = \emptyset$, $\gamma_a = (1, \dots, 1)$, $\theta_0 = \theta_1 = \theta_2$, $T \in \mathcal{L}(L_{\theta_0})$, $p_0 = p_1 = v_0 = w_0 \geq 1$ of Theorem 3.1 was treated in [**78**] with additional conditions $p_1 < w_1$, $\theta_0 < v_1$.

The proof of Theorem 6.3. We can choose positive constants ξ, ξ_0, ξ_1 with $\xi_0 < \xi_1$ satisfying

$$Q_r(w) \subset \bigcap_{z \in Q_{r\xi}(w)} Q_{r\xi_0}(z), \quad \bigcup_{z \in Q_{r\xi}(w)} Q_{r\xi_0\delta}(z) \subset Q_{r\xi_1}(w) \tag{1}$$

for every $Q_r(w) \subset \mathbb{R}^n$.

Let us fix a function $f \in L_{p_0}(\mathbb{R}^n, A)$ and a constant $t > 0$ and apply Lemma 6.4, *a)* to find a decomposition $f = f_\infty + \sum_{i \in \mathbb{N}} f_i$ satisfying its statement. Then we use $\mathcal{AAD}$-condition to continue the decomposition process: for $i \in \mathbb{N}$, $z_i \in Q_{r_i\xi}$, $G_i := G(r_i\xi_0, z_i)$,

$$f_i = (f_i - G_i f_i) + G_i f_i, \; T(f_i - G_i f_i) = h_i = h_{0,i} + h_{1,i}, \; h_{0,i} = \chi_{Q_{r_i\xi_1}(w_i)} h_i. \tag{2}$$

relation (1) of Lemma 6.4, formula (2) and quasi-sub-additivity of $*$-operation and T imply, for sufficiently big constant $N > 0$,

$$\|Tf|B\|^*(Nt) \leq C(\|T(\sum_{i \in \mathbb{N}} f_i - G_i f_i)|B\|^*((M\xi_1^n + 1)t) + \|T(\sum_{i \in \mathbb{N}} G_i f_i)|B\|^*(t) + \|Tf_\infty|B\|^*(t)); \tag{3}$$

$$\|T(\sum_{i \in \mathbb{N}} f_i - G_i f_i)|B\|^*((M\xi_1^n + 1)t) \leq \|\chi_{\cup_{i \in \mathbb{N}} Q_{r_i\xi_1}(w_i)} T(\sum_{i \in \mathbb{N}} f_i - G_i f_i)|B\|^*(M\xi_1^n t) +$$

$$+ \|\chi_{\mathbb{R}^n \setminus \cup_{i \in \mathbb{N}} Q_{r_i\xi_1}(w_i)} T(\sum_{i \in \mathbb{N}} f_i - G_i f_i)|B\|^*(t) = \|\chi_{\mathbb{R}^n \setminus \cup_{i \in \mathbb{N}} Q_{r_i\xi_1}(w_i)} T(\sum_{i \in \mathbb{N}} f_i - G_i f_i)|B\|^*(t). \tag{4}$$

We use the duality of Lebesgue spaces and the sub-additivity of T to derive the estimate: for some $g \in L_{p_1'}(\mathbb{R}^n)$, $g \geq 0$ with $\|g|L_{p_1'}(\mathbb{R}^n)\| = 1$,

$$\|\sum_{i \in \mathbb{N}} h_{1,i}|L_{p_1}(\mathbb{R}^n, B)\| \leq C \sum_{i \in \mathbb{N}} \int_{\mathbb{R}^n} g\|h_{1,i}|B\|. \tag{5}$$

The inclusion $T \in \mathcal{AAD}_x(\vec{Z}, \vec{X}, \vec{\lambda}, \vec{\nu}, D_0)$ and Hölder inequality deliver the estimates: for $i \in \mathbb{N}$, $k \in \mathbb{N}_0$,

$$\int_{\Delta_k(r_i\xi_0\delta, z_i)} g\|h_{1,i}|B\| \leq \|g|L_{w_1'}(\Delta_k(r_i\xi_0\delta, z_i))\|\|h_{1,i}|L_{w_1}(\Delta_k(r_i\xi_0\delta, z_i), B)\| \leq$$

$$\leq M_{w_1'}g(z_i)((r_i\xi_0\delta)^n)^{1-\lambda_1/n}\mathcal{D}_{p_0}(r_i\xi_0, z_i, f_i, D_0, A)\mu_k^1(r_i\xi_0, z_i)(r_i\xi_0)^{\lambda_0}. \tag{6}$$

Hence, conditions of the theorem, relations (1), estimate (6) and Lemma 6.4, a) imply

$$\int_{\mathbb{R}^n} g\|h_{1,i}|B\| \leq CM_{w_1'}g(z_i)\psi(t)C^1_{\mathcal{AAD}}r_i^{n/\rho}, \text{ where } \rho^{-1} = 1 + \frac{\lambda_0 - \lambda_1}{n} \tag{7}$$

and $\psi = \psi^*$, satisfying $\|\psi|L_{p_0,\infty}(\mathbb{R}_+)\| \leq C_5\|f|L_{p_0}(\mathbb{R}^n, A)\|$, is from the statement of Lemma 6.4, a). Therefore, using (7), independence of the set $\{z_i\}_{i\in\mathbb{N}} \subset \prod_{i\in\mathbb{N}} Q_{r_i\xi}(w_i)$, Hölder inequality in the form $\|\phi\psi|L_\rho\| \leq \|\phi|L_{\alpha,\infty}\| \cdot \|\psi|L_{\beta,\rho}\|$ for $\alpha^{-1} + \beta^{-1} = \rho^{-1}$ and the boundedness

$$\|M_{w_1'}g|L_{\alpha,\infty}(\mathbb{R}^n)\| \leq C\|g|L_{p_1'}(\mathbb{R}^n)\| \text{ for } \alpha = p_1', \tag{8}$$

we obtain

$$\sum_{i\in\mathbb{N}} \int_{\mathbb{R}^n} g\|h_{1,i}|B\| \leq C\sum_{i\in\mathbb{N}} \inf_{z\in Q_{r_i\xi}(w_i)} M_{w_1'}g(z)\psi(t)C^1_{\mathcal{AAD}}|Q_{r_i\xi}(w_i)|^{1/\rho} \leq$$

$$\leq C\psi(t)C^1_{\mathcal{AAD}}\|M_{w_1'}g|L_\rho(\cup_{i\in\mathbb{N}}Q_{r_i}(w_i))\| \leq C\|\psi|L_{p_0,\infty}(\mathbb{R}_+)\|C^1_{\mathcal{AAD}}. \tag{9}$$

Now formulas (5, 9) imply, uniformly by $\{z_i\}_{i\in\mathbb{N}}$,

$$\|\sum_{i\in\mathbb{N}} h_{1,i}|L_{p_1}(\mathbb{R}^n, B)\| \leq CC^1_{\mathcal{AAD}}\|f|L_{p_0}(\mathbb{R}^n, A)\|. \tag{10}$$

To estimate $\|\sum_{i\in\mathbb{N}} G_if_i|L_{\theta_0}(\mathbb{R}^n, A)\|$, we are repeating considerations related to $(5-10)$:

$$\|\sum_{i\in\mathbb{N}} G_if_i|L_{\theta_0}(\mathbb{R}^n, A)\| \leq C\sum_{i\in\mathbb{N}} \int_{\mathbb{R}^n} g\|G_if_i|A\| \text{ for some nonnegative } g \in L_{\theta_0'}(\mathbb{R}^n) \tag{11}$$

with $\|g|L_{\theta_0'}(\mathbb{R}^n)\| = 1$; for $i \in \mathbb{N}$, $k \in \mathbb{N}_0 \cup \{-1\}$, one has

$$\int_{\Delta_k(r_i\xi_0\delta, z_i)} g\|G_if_i|A\| \leq \|g|L_{v_1'}(\Delta_k(r_i\xi_0\delta, z_i))\|\|G_if_i|L_{v_1}(\Delta_k(r_i\xi_0\delta, z_i), A)\| \leq$$

$$\leq M_{v_1'}g(z_i)((r_i\xi_0\delta)^n)^{1-\nu_1/n}\mathcal{D}_{p_0}(r_i\xi_0, z_i, f_i, D_0, A)\mu_k^2(r_i\xi_0, z_i)(r_i\xi_0)^{\nu_0}; \tag{12}$$

$$\int_{\mathbb{R}^n} g\|G_if_i|A\| \leq CM_{v_1'}g(z_i)\psi(t)C^2_{\mathcal{AAD}}r_i^{n/\rho_1}, \text{ where } \rho_1^{-1} = 1 + \frac{\nu_0 - \nu_1}{n}; \tag{13}$$

$$\|M_{v_1'}g|L_{\alpha,\infty}(\mathbb{R}^n)\| \leq C\|g|L_{\theta_0'}(\mathbb{R}^n)\| \text{ for } \alpha = \theta_0'; \tag{14}$$

$$\sum_{i\in\mathbb{N}} \int_{\mathbb{R}^n} g\|G_if_i|A\| \leq C\sum_{i\in\mathbb{N}} \inf_{z\in Q_{r_i\xi}(w_i)} M_{v_1'}g(z)\psi(t)C^2_{\mathcal{AAD}}|Q_{r_i\xi}(w_i)| \leq$$

$$\leq C\psi(t)C^1_{\mathcal{AAD}}\|M_{v_1'}g|L_1(\cup_{i\in\mathbb{N}}Q_{r_i}(w_i))\| \leq Ct^{\frac{1}{\theta_0}-\frac{1}{p_0}}\|\psi|L_{p_0,\infty}(\mathbb{R}_+)\|C^2_{\mathcal{AAD}}; \tag{15}$$

$$\|\sum_{i\in\mathbb{N}} G_i f_i | L_{\theta_0}(\mathbb{R}^n, A)\| \le CC^2_{\mathcal{AAD}}\|f|L_{p_0}(\mathbb{R}^n, A)\| \text{ uniformly by}$$

$$\{z_i\}_{i\in\mathbb{N}} \subset \prod_{i\in\mathbb{N}} Q_{r_i\xi}(w_i). \tag{16}$$

Due to (16), we obtain, uniformly by $\{z_i\}_{i\in\mathbb{N}}$, that

$$\|T\sum_{i\in\mathbb{N}} G_i f_i|A\|^*(t)t^{\frac{1}{p_1}} \le t^{\frac{1}{p_1}-\frac{1}{\theta_1}}\|T|\mathcal{B}(L_{\theta_0}(\mathbb{R}^n, A), L_{\theta_1,\infty}(\mathbb{R}^n, B))\| \cdot \|\sum_{i\in\mathbb{N}} G_i f_i|L_{\theta_0}(\mathbb{R}^n, A)\| \le$$

$$\le C\|T|\mathcal{B}(L_{\theta_0}(\mathbb{R}^n, A), L_{\theta_1,\infty}(\mathbb{R}^n, B))\|C^2_{\mathcal{AAD}}\|f|L_{p_0}(\mathbb{R}^n, A)\|. \tag{17}$$

We need to use the monotonicity of the Lebesgue integral and Lemma 6.4 to obtain

$$\|Tf_\infty|B\|^*(t)t^{1/p_1} \le \|f_\infty|L_{\theta_1,\infty}(\mathbb{R}^n, A)\|t^{\frac{1}{p_1}-\frac{1}{\theta_1}} \le$$

$$\le \|T|\mathcal{B}(L_{\theta_2}(\mathbb{R}^n, A), L_{\theta_1,\infty}(\mathbb{R}^n, B))\| \cdot \|f_\infty|L_{\theta_2}(\mathbb{R}^n, A)\|t^{\frac{1}{p_1}-\frac{1}{\theta_1}} \le$$

$$\|T|\mathcal{B}(L_{\theta_2}(\mathbb{R}^n, A), L_{\theta_1,\infty}(\mathbb{R}^n, B))\|t^{\frac{1}{p_1}-\frac{1}{\theta_1}}\|f_\infty|L_{p_0}(\mathbb{R}^n, A)\|^{\frac{p_0}{\theta_2}}\|f_\infty|L_\infty(\mathbb{R}^n, A)\|^{1-\frac{p_0}{\theta_2}} \le$$

$$\le C\|T|\mathcal{B}(L_{\theta_2}(\mathbb{R}^n, A), L_{\theta_1,\infty}(\mathbb{R}^n, B))\| \cdot \|f|L_{p_0}(\mathbb{R}^n, A)\|. \tag{18}$$

Now formulas $(5, 10, 11, 16-18)$ are followed by part $a)$. Applying real interpolation, we finish the proof of the theorem. $\mathcal{Q.E.D.}$

The next lemma is a simple generalization of the Calderón-Zygmund decomposition.

LEMMA 6.4. *Assume $p \in (0, \infty)$, and A is a Banach space. Then there are constants $C_i, M > 0$, $i \in \{1, 2, 3\}$, such that, for any $t > 0$, arbitrary function $f \in L_p(\mathbb{R}^n, A)$ can be represented in the form $f = f_\infty + \sum_{i\in\mathbb{N}} f_i$, where*

$$\text{supp} f_i \subset Q_i = Q_{r_i}(w_i),\ \cup_{i\in\mathbb{N}} Q_i = G_t,\ \chi_{G_t} \le \sum_{i\in\mathbb{N}} \chi_{Q_i} \le M \text{ for } i \in \mathbb{N}, \tag{1}$$

$$\max(\|f_\infty|L_p(\mathbb{R}^n, A)\|, \|\sum_{i\in\mathbb{N}} f_i|L_p(\mathbb{R}^n, A)\|) \le C_1\|f|L_p(\mathbb{R}^n, A)\|, \tag{2}$$

and arbitrary one of the following groups of relations holds:

a) $\sup_{i\in\mathbb{N}}(\mathcal{D}_p(Q_i, f_i, \emptyset, A), \|f_\infty|L_\infty(\mathbb{R}^n, A)\| \le C_2\psi(t)$ *for some nonincreasing*

g with $\psi = \psi^*,\ \|\psi|L_{p,\infty}(\mathbb{R}_+)\| \le C_3\|f|L_p(\mathbb{R}^n, A)\|,\ |G_t| \le t;$

b) $\sup_{i\in\mathbb{N}}(\mathcal{D}_p(Q_i, f_i, \emptyset, A), \|f_\infty|L_\infty(\mathbb{R}^n, A)\| \le C_2 t,\ |G_t| \le C_3(t^{-1}\|f|L_p(\mathbb{R}^n, A)\|)^p$

and $|G_t|$ is a non-increasing function.

The proof of Lemma 6.4. To begin with a), let us fix some $f \in L_p(\mathbb{R}^n, A)$, $t > 0$ and designate

$$\psi(\tau) := ((M\|f|A\|^p)^*(\tau))^{1/p} \text{ and } G_t := \{x : M\|f(x)|A\|^p > \psi(t)\}. \quad (1)$$

Then, we choose $\{Q_i\}_{i\in\mathbb{N}}$ to be a Calderón-Zygmund decomposition of G_t satisfying

$$\cup_{i\in\mathbb{N}} Q_i = G_t,\ \chi_{G_t} \leq \sum_{i\in\mathbb{N}} \chi_{Q_i} \leq M,\ c_o Q_i \cap G_t \neq \emptyset$$

and take

$$f_i := f\chi_{Q_i \setminus \cup_{j<i} Q_j}, \quad f_\infty := f - \sum_{i\in\mathbb{N}} f_i. \quad (3)$$

Now the functions $\{f_i\}_{i\in\mathbb{N}}$, f_∞ are looked for, because

$$\mathcal{D}_p(Q_i, f_i, \emptyset, A) \leq c_o^n \mathcal{D}_p(c_o Q_i, f, \emptyset, A) \leq c_0^n \psi(t) \text{ and}$$

$$\|f(x)|A\|^p \leq M\|f|A\|^p(x) \leq \psi(t) \text{ for a.e.} x \in \mathbb{R}^n \setminus G_t.$$

The boundedness of Hardy-Littlewood maximal function finishes the proof of part a):

$$\|\psi|L_{p,\infty}(\mathbb{R}_+)\| \leq C_5 \|f|L_p(\mathbb{R}^n, A)\|.$$

The proof of part b) is conducted in the same way with $\psi(\tau) = \tau$. $\mathcal{Q.E.D.}$

CHAPTER 4

On two approaches to L_p-calculus of generalized Dirac operators

1. Introduction

The recent complete solution of the long standing Square root problem of Kato for elliptic operators and systems by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian in [**84, 85, 86**] was preceded by works of McIntosh [**87**] and Coifman, McIntosh, Meyer [**88**], as well as a book due to Auscher and Tchamitchian [**89**] is devoted to the boundedness of the square roots of elliptic operators on L_2.

Later Auscher [**81**] has extended this result for elliptic operators to a range of Lebesgue spaces developing and employing some ideas originated from the recent results of Blunck and Kunstman [**78**], improving, in turn, ones of Duong, McIntosh [**4**], Duong, Robinson [**80**] and Hebisch [**79**], and also Sobolev embedding theorems and the Calderón-Zygmund covering lemma.

All the above mentioned development are known to be partly based and intimately related to the theory of H^∞-calculus developed by McIntosh[**90**], Albrecht, Duong, McIntosh [**91**] and Cowling, Doust, McIntosh, Yagi [**92**] in the abstract setting of Hilbert and Banach spaces correspondingly.

Contemporary development is represented by the work [**93**] due to Axelsson, Keith and McIntosh, where the model of generalized Dirac operator, which is more general than one associated with Kato square root problem, was investigated. They have established the existence of H^∞-functional calculus of a perturbed Dirac operator on $L_2(\mathbb{R}^n)$ including a possible approach to the solution of the Square root problem of Kato as a particular case.

The most part of this chapter is, roughly speaking, related to [**93**] in the same way as Auscher's work [**81**] related to the recent solution of the Kato problem. More precisely, we show that assumptions on the boundedness of some projectors and results of [**93**] imply the existence of H^∞-functional calculus in L_p for a reasonably wide range of p contained in $(1, 2)$.

For this purpose, we use the resolvent approach deriving some new generalized off-diagonal estimates of two types. As a collateral results, our approach delivers an extension of the Hilbert resolvent identity to the case of the differences of a high order. To demonstrate the irrelevance of the semigroup approach in our settings, we develop a counterexample suggested by A. M^{c}Intosh.

Other main results are generalized embedding theorems for generalized Dirac operators written in the form of the boundedness of its powers from Besov and Lizorkin-Triebel spaces into Lebesgue (Lorentz) ones.

2. Definitions

In this section, define two types of off-diagonal families of functions with respect to some operators and the main assumptions on operators employed, such as properties: general **G**, ellipticity **E**, idempotentness **I** and projector boundedness $\mathbf{P}(q)$.

The following subsets of $\mathbb{C}$ is of use:

$$S_\mu := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \mu\} \text{ for } \mu \in [0, \pi/2), \; S_\mu^d := -S_\mu \cup S_\mu.$$

$H_\infty(S_\mu^d)$ means the space of all bounded and holomorphic on S_μ^d functions; $\Xi(S_\mu^d)$ means its subset of the functions $f \in H_\infty(S_\mu^d)$ such that $f(z) = \mathrm{O}|z|^{-\alpha}$ for $|z| \to \infty$ and some $\alpha > 1$. Assume $L_p = L_p(\mathbb{R}^n, \mathbb{R}^m)$, and $w_p^1 = w_p^1(\mathbb{R}^n, \mathbb{R}^m)$ is seminormed (homogeneous) Sobolev space, $\|f|w_p^1\| = \|\nabla f|L_p\|$. We say that a pair $(p, q) \in \mathbb{R}_+^2$ is in Sobolev relation $\mathcal{S}_1 \subset \mathbb{R}_+^2$ iff $W_p^1 \subset L_q$. Symbols $b_{p,q}^s$ and $l_{p,q}^s$ designate, correspondingly, the spaces of Besov and Lizorkin-Triebel type of functions defined on $\mathbb{R}^n$.

DEFINITION 2.1. We say that function a family $\{f_z\}_{z\in S_{\mu_1}} \subset H_\infty(S_\mu)$ is in $\bar{D}(A, S_{\mu_1}, p, q)$ for some $p, q \in [1, \infty]$, $\mu, \mu_1 \in [0, \pi/2)$ and operator A iff

$$\|f_z(A)|\mathcal{L}(L_p(\mathbb{R}^n), L_q(\mathbb{R}^n))\| \leq C|z|^{\frac{n}{q}-\frac{n}{p}}, \; z \in S_{\mu_1} \qquad (*)$$

We say that function family $\{f_z\}_{z\in S_{\mu_1}} \subset H_\infty(S_\mu)$ is in $\bar{D}(A, S_{\mu_1}, p, q, N)$ for some $p, q \in [1, \infty]$, $N > 0$ and operator A iff

$$\|f_z(A)|\mathcal{L}(L_p(F), L_q(F_1))\| \leq C|z|^{\frac{n}{q}-\frac{n}{p}}\left(\frac{\rho}{|z|}\right)^{-N}, z \in S_{\mu_1}, \; \rho = \operatorname{dist}(F, F_1). \qquad (**)$$

for any closed subsets F, F_1 of $\mathbb{R}^n$ and $t > 0$.

The next designations is used extensively:

$$\underline{R(z)} := (1-z)^{-1};\ \Gamma_\nu^\pm := \{te^{\pm i\nu}\}_{t>0}, \Gamma_\nu = \Gamma_\nu^+ \cup \Gamma_\nu^-.$$

We designate $\underline{\Delta^k f(z)} := \sum_{l=0}^k (-1)^l \binom{k}{l} f(lz)$, $k \in \mathbb{N}$ for any (operator-valued) function $f(z)$.

DEFINITION 2.2. Assume Γ is linear differential operator with $\mathbb{C}^{m\times m}$ matrix-valued coefficients, B is operator of pointwise multiplication on matrix $B(x) \in \mathbb{C}^{m\times m}$.
General condition is

$$\|B(\cdot)|L_\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})\|, \|B^{-1}(\cdot)|L_\infty(\mathbb{R}^n, \mathbb{C}^{m\times m})\| < C. \qquad \mathbf{G}$$

Accretivity condition of order $\omega \in [0, \pi/2)$ means inequality $|\arg(f, \mathrm{B}f)| \le \omega$ for $f \in L_2 \setminus \{0\}$. As to the ellipticity condition, we shall refer to

$$\Pi = \Gamma + \Gamma^*, \quad \|\Pi f|L_2\| \asymp \|\nabla f|L_2\|. \qquad \mathbf{E}$$

DEFINITION 2.3. For Γ as above, we designate $\underline{\Gamma_B^*} = \mathrm{B}^{-1}\Gamma^*\mathrm{B}$, $\underline{\Pi_B} = \Gamma + \Gamma_B^*$, $\underline{\Delta_B} = \Pi_B^2$.

REMARK 1. Note that, if $\Gamma^2 = 0$, then one has

$$\Gamma_B^{*2} = 0,\ (\Gamma_B^*\Pi_B^{-1})^2 = \Pi_B^{-1}\Gamma = \Gamma_B^*\Pi_B^{-1},\ (\Gamma\Pi_B^{-1})^2 = \Gamma\Pi_B^{-1} = \Pi_B^{-1}\Gamma_B^*. \qquad \mathbf{I}$$

DEFINITION 2.4. Property $\mathbf{P}(q)$, $q \in [1, 2]$ means the boundedness of projectors

$$\Gamma\Pi_B^{-1}, \Gamma_B^*\Pi_B^{-1} \in \mathcal{L}(L_q), \text{ where } \Gamma^2 = 0. \qquad \mathbf{P}(q)$$

REMARK 2. It was proved in [**93**] that accretivity condition implies $\mathbf{P}(2)$.

DEFINITION 2.5. For $q \in (1, \infty]$, designation $\underline{\mathrm{F}(q) = \mathrm{F}(A, q, \Omega)}$ means that operator A possesses $H_\infty(\Omega)$-functional calculus on L_q, i.e. $\|f(A)|\mathcal{L}(L_q)\| \le C\|f|H_\infty(\Omega)\|$.

For $0 \le k \le n$, let Λ_k^n be the $\binom{n}{k}$-dimensional space of all antisymmetric linear forms of kth order, $\Lambda^n := \prod_{k=0}^n \Lambda_k^n$; $f_{\Lambda_k^n}$ be Λ_k^n-valued component of Λ^n-valued function f and operator $d = \nabla \wedge f$ for such f possessing the Sobolev generalized derivatives of the first order.

3. Main results

3.1. Functional calculus. Here we state the results regarding the relations between the existence of the functional calculus ($\mathbf{P}(q)$) and the other basic properties defined in §2.

THEOREM 3.1. *Assume* G, P(q), $(q_s, 2) \in \mathcal{S}_1$, F$(2)$ *for some* $q, q_s \in (1,2)$. *Then one has* F(p) *for* $p \in (p_0, 2)$, $p_0 = \max(q, q_s)$.

The proof of Theorem 3.1. Let us begin with the observation that the theorem will follow from Theorem 5.1 if we check the validness of the conditions $(1,2)$ and $T \in \mathcal{L}(L_2)$ of that theorem for $T = f(\Pi_B)$, $f \in H_\infty(S_\mu)$, $A_t = R(t\Pi_B)$ and sufficiently big $l > N + n/2 - n/p > n(3/2 - 1/p)$. The last condition is insured by the assumption F(2). Then Theorems 4.6, 4.7 and 4.2, $b)$ imply

$$b^{kn} t^{\frac{n}{p}-\frac{n}{2}} \|\{\|f(\Pi_B)\Delta^l R(t\Pi_B)|\mathcal{L}(L_p(Q_t(z)), L_2(Q_{b^{k+1}t\delta}(z) \setminus Q_{b^k t\delta}(z)))\| \leq C\|f|H_\infty\| b^{k(n-N)},$$

where one can choose $N > n$. Whence, we see that condition (1) of Theorem 5.1 satisfied too. In the same manner, the validness of condition (2) of that theorem is a consequence of the following estimates provided by Theorem 4.7:

$$b^{kn} t^{\frac{n}{p}-\frac{n}{2}} \|\{\|R(t\Pi_B)|\mathcal{L}(L_p(Q_t(z)), L_2(Q_{b^{k+1}t\delta}(z) \setminus Q_{b^k t\delta}(z)))\| \leq C\|f|H_\infty\| b^{k(n-N_1)}$$

for some $N_1 > n$ and $t^{\frac{n}{p}-\frac{n}{2}} \|\{\|R(t\Pi_B)|\mathcal{L}(L_p(Q_t(z)), L_2(Q_{t\delta}(z)))\| < \infty$. $\mathcal{Q.E.D.}$

COROLLARY 3.2. *Assume* $m = 2^n$, $p_0 = \max(q, q_s)$, $\Gamma f = d \wedge f$, $(q_s, 2) \in \mathcal{S}_1$ *for some* $q, q_s \in (1, 2)$. *Then the following implications take place:*

$a)$ F$(q) \Rightarrow \|(\Pi_B^2)^{1/2} f|L_q(\mathbb{R}^n, \Lambda^n)\| \asymp \|\Pi_B f|L_q(\mathbb{R}^n, \Lambda^n)\|$;

$b)$ $\|(\Pi_B^2)^{1/2} f|L_q(\mathbb{R}^n, \Lambda^n)\| \asymp \|\Pi_B f|L_q(\mathbb{R}^n, \Lambda^n)\| \Rightarrow$ P(q);

$c)$ P$(q) \Rightarrow$ F(p) for $p \in (p_0, 2)$.

The proof of Corollary 3.2. Implication F$(q) \Rightarrow$ P(q) has been obtained in [**93**], and its proof contains both $a)$ and $b)$. The same source contains the proof that properties G and F(2) are fulfilled for $\Gamma = d$. Part $c)$ is a particular case of Theorem 3.1. $\mathcal{Q.E.D.}$

3.2. Generalized embedding theorems. In this subsection, we study the boundedness properties of powers of the operator Π_B from Besov, or Lizorkin-Triebel spaces into Lorentz, or Lebesgue spaces correspondingly.

THEOREM 3.3. *Under the conditions of Theorem* 3.1 *and* $r \in (0,p]$, $\theta \in (0,\infty]$, $\beta \in (0,s]$ *with* $(s-\beta)/n = r^{-1} - p^{-1}$, *assume that* B *is constant and either* $s \in (0,1)$ *and* $Y_r^s = l_{r,\theta}^s$, *or* $s=1$ *and* $Y_r^s = w_r^1$. *Then we have*

$$\Pi_B^\beta \in \mathcal{L}(Y_r^s, L_p).$$

Using the real interpolation, we obtain

COROLLARY 3.4. *Under the conditions of Theorem* 3.3, *one also has*

$$\Pi_B^\beta \in \mathcal{L}(b_{r,\theta}^s, L_{p,\theta}).$$

The proof of Theorem 3.3 Let us choose holomorphic in $\mathbb{C}$ function $\zeta \in \Xi(S_\mu)$ with $\xi(0)=0$. For Π_B^β, let us use the representation

$$z^\beta = \int_0^\infty z^\beta \zeta^2(tz)\frac{dt}{t}. \tag{1}$$

Without loss of generality, we can assume also that

$$\xi(z) := \int_1^\infty (tz)^\beta \zeta^2(tz)\frac{dt}{t} \in H^\infty(S_\mu). \tag{2}$$

Then Theorem 3.1 shows that

$$\xi(\Pi_B) \in \mathcal{L}(L_p, L_p) \tag{3}$$

due to F(p) and, therefore, involving embedding properties of Sobolev and Lizorkin-Triebel spaces into Lebesgue spaces, we see also that

$$\xi(\Pi_B) \in \mathcal{L}(Y_r^s, L_p). \tag{4}$$

Identity

$$(\xi(\Pi_{B(\tau\cdot)})f)(\cdot/\tau) = \xi(\tau\Pi_B)f(\cdot/\tau) \tag{5}$$

and the homogeneity of Y_r^s show the uniform boundedness

$$\|\tau^{-\beta}\xi(\tau\Pi_B)|\mathcal{L}(Y_r^s, L_p)\| = \|\xi(\Pi_B)|\mathcal{L}(Y_r^s, L_p)\| \text{ for every } \tau > 0. \tag{6}$$

Thus, Banach-Steinhause theorem finishes the proof with the aid of the identity

$$\Pi_B^\beta = \int_0^\infty \Pi_B^\beta \zeta^2(t\Pi_B)\frac{dt}{t} =$$

$$= \lim_{\tau\to+0}\int_\tau^\infty \Pi_B^\beta \zeta^2(t\Pi_B)\frac{dt}{t} = \lim_{\tau\to+0}\tau^{-\beta}\xi(\tau\Pi_B). \mathcal{Q}.\mathcal{E}.\mathcal{D}.$$

4. Off-diagonal estimates

We begin this section with a simple identity suggesting a high order counterpart of the Hilbert identity, and, then, derive the key results on off-diagonal estimates.

LEMMA 4.1.

$$\Delta^k R(z) = \frac{z^k}{\prod_{l=1}^k (l^{-1} - z)}, \quad k \in \mathbb{N}.$$

This simple identity suggests generalizations of the Hilbert identity for the differences of arbitrary high order in the form of lemma 4.8.

THEOREM 4.2. *Assume $\mu \in (0, \pi/2)$, $\nu \in (0, \mu)$, $\nu_1 < \min(\nu, \mu - \nu)$, $\|f|H_\infty(S_\mu)\| \le 1$, $k \in \mathbb{N}$, $p, q \in [1, \infty]$. If two families $\{R(ze^{\pm i\nu}\cdot)\}_{z \in S_0} \in \bar{D}(A, S_0, p, q)$, or $\{R(ze^{\pm i\nu}\cdot)\}_{z \in S_0} \in \bar{D}(A, S_0, p, q, N)$, then one has, correspondingly,*

$$a)\{f(\cdot)\Delta^k R(z\cdot)\}_{z \in S_{\nu_1}} \in \bar{D}(A, S_{\nu_1}, p, q) \text{ for } k > \frac{n}{q} - \frac{n}{p} > 0, \text{ or}$$

$$b)\{f(\cdot)\Delta^k R(z\cdot)\}_{z \in S_{\nu_1}} \in \bar{D}(A, S_{\nu_1}, p, q, N) \text{ for } k > N + \frac{n}{q} - \frac{n}{p} > 0.$$

The proof of Theorem 4.2. Assuming $f \in \Xi(S_\mu)$ without loss of generality, we obtain the following Cauchy identities

$$f(A)\Delta^k R(zA) = \oint_{\Gamma_\nu} f(\zeta)\Delta^k R(z\zeta) R(\zeta^{-1}A)\frac{d\zeta}{\zeta}. \tag{1}$$

Therefore, we have, thanks to the triangle inequality,

$$\|f(A)\Delta^k R(zA)|\mathcal{L}(L_p, L_q)\| \le \int_{\Gamma_\nu, |z\zeta| \le 1} +$$
$$+ \int_{\Gamma_\nu, |z\zeta| > 1} |\Delta^k R(z\zeta)| \|R(\zeta^{-1}A)|\mathcal{L}(L_p, L_q)\| \frac{|d\zeta|}{|\zeta|} =: I_1^a + I_2^a, \tag{2}$$
$$\|f(A)\Delta^k R(zA)|\mathcal{L}(L_p(F), L_q(F_1))\| \le \int_{\Gamma_\nu, |z\zeta| \le 1} +$$
$$+ \int_{\Gamma_\nu, |z\zeta| > 1} |\Delta^k R(z\zeta)| \|R(\zeta^{-1}A)|\mathcal{L}(L_p(F), L_q(F_1))\| \frac{|d\zeta|}{|\zeta|} =$$
$$=: I_1^b + I_2^b. \tag{3}$$

Now Lemma 4.1 and the definitions of $\bar{D}$-classes imply

$$I_1^a \le C(A,k)k!(\sin|\nu - \nu_1|)^{-k}|z|^k \int_0^{|z|^{-1}} r^{k + \frac{n}{p} - \frac{n}{q} - 1} dr = C|z|^{\frac{n}{q} - \frac{n}{p}},$$

$$I_2^a \le C(A,k) \int_{|z|^{-1}}^{\infty} r^{\frac{n}{p} - \frac{n}{q} - 1} dr = C|z|^{\frac{n}{q} - \frac{n}{p}},$$

$$I_1^b \le C(A,k)k!(\sin|\nu-\nu_1|)^{-k}\rho^{-N}|z|^k \int_0^{|z|^{-1}} r^{k-N+\frac{n}{p}-\frac{n}{q}-1}dr = C(\frac{\rho}{|z|})^{-N}|z|^{\frac{n}{q}-\frac{n}{p}},$$

$$I_2^b \le C(A,k)\rho^{-N}\int_{|z|^{-1}}^{\infty} r^{\frac{n}{p}-\frac{n}{q}-N-1}dr = C(\frac{\rho}{|z|})^{-N}|z|^{\frac{n}{q}-\frac{n}{p}}.$$

These estimates finish the proof of the both parts of the theorem. $\mathcal{Q.E.D}$.

LEMMA 4.3. *For $0 \le \omega < \nu < \pi/2$, assume conditions $(q_s, 2) \in \mathcal{S}_1$, $\mathbf{G}$, ω-accretivity of B, $\mathbf{E}$, $\mathbf{P}(q)$ and inclusion $R(e^{\pm i\nu}\Pi_B) \in \mathcal{L}(L_2)$ for some $q \in (1,2)$. Then we have $\{R(ze^{\pm i\nu}\cdot)\}_{z\in S_0} \in \bar{D}(\Pi_B, S_0, p, 2)$ for $p \in [\max(q_s,q), 2)$.*

The proof of Lemma 4.3. Thanks to homogeneity considerations, it is sufficient to show the boundedness of operators $R(zA)$ for $|z| = 1$ only. Let us designate $q_0 = \max(q_s, q)$. If $q_s > q$, then we use Lemma 4.5 to obtain $\mathbf{P}_{q_s}$. Otherwise, one has $(q, 2) \in \mathcal{S}_1$. In both cases, $\mathbf{P}(q_0)$ is equivalent to $N(\Gamma) \oplus N(\Gamma_B^*)$, $N(\Gamma_B^*) = \mathrm{B}^{-1}N(\Gamma^*)$ in algebraic and topological senses. Therefore, we can prove the assertion of the lemma on each of these two subspaces separately. Relations (I) imply

$$(I - z\Pi_B)^{-1}g = ((I - z\Pi_B)^{-1}\Pi_B)(\Pi_B^{-1}\Gamma)\Pi^{-1}g \text{ for } g \in N(\Gamma), \text{ and for } g \in N(\Gamma_b^*), \quad (1)$$

$$(I-z\Pi_B)^{-1}h = ((I-z\Pi_B)^{-1}\Pi_B)\Pi_B^{-1}\mathrm{B}^{-1}\Gamma\Pi^{-1}\mathrm{B}h = ((I-z\Pi_B)^{-1}\Pi_B)(\Pi_B^{-1}\Gamma_B^*)(\mathrm{B}^{-1}\Pi^{-1}\mathrm{B})h \quad (2)$$

because of the identities $\Gamma\Pi^{-1}g = g$ and $\mathrm{B}^{-1}\Gamma\Pi^{-1}\mathrm{B}g = g$. Ellipticity property tells that operators $\Pi = \Pi^*$ are isomorphisms between w_2^1 and L_2. Whence, inclusions $\Pi_B^{-1}\Gamma, \Pi_B^{-1}\Gamma_B^* \in \mathcal{L}(L_2)$ reflecting $\mathbf{P}(2)$ (see remark 2.6) along with $(q_0, 2) \in \mathcal{S}_1$ imply

$$\Pi_B^{-1}\Gamma\Pi^{-1}, \Pi_B^{-1}\mathrm{B}^{-1}\Gamma\Pi^{-1}\mathrm{B} \in \mathcal{L}(L_{q_0}, L_2). \quad (3)$$

Eventually representation $(I - z\Pi_B)^{-1}\Pi_B = z^{-1}(R(z\Pi_B) - I)$, $|z| = 1$ and inclusion $R(e^{\pm i\nu}\Pi_B) \in \mathcal{L}(L_2)$ finish the proof. $\mathcal{Q.E.D}$.

Next two lemmas follow from interpolation theory.

LEMMA 4.4. ([**81, 94**]) Assume $p \in (0,2)$, $\theta \in (0,1)$, $p_\theta^{-1} = (1-\theta)p^{-1} + \theta 2^{-1}$, $\{R(ze^{\pm i\nu}\cdot)\}_{z\in S_0} \in \bar{D}(A, S_0, p, 2) \cap \bar{D}(A, S_0, q, 2, N)$. Then we have

$$\{R(ze^{\pm i\nu}\cdot)\}_{z\in S_0} \in \bar{D}(A, S_0, p_\theta, 2, \theta N).$$

LEMMA 4.5. $\mathbf{P}(p)$ *and* $\mathbf{P}(q) \Rightarrow \mathbf{P}(r)$, $r \in (p,q)$.

The next theorem was proved in [**93**].

THEOREM 4.6. *Assuming* **G** *and* ω*-accretivity of* B *for some* $0 \le \omega < \nu < \pi/2$, *one has* $\{R(ze^{\pm i\nu}\Pi_B)\}_{z\in S_0} \subset \mathcal{L}(L_2)$, $\{R(ze^{\pm i\nu}\cdot)\}_{z\in S_0} \in \bar{D}(\Pi_B, S_0, 2, 2, N)$ *for any* $N > 0$.

THEOREM 4.7. *Assuming* $(q_s, 2) \in \mathcal{S}_1$, $q_s, q \in (1, 2)$, **G**, ω*-accretivity of* B *for some* $0 \le \omega < \nu < \pi/2$, $\mathbf{P}(q)$ *and inclusion* $R(e^{\pm i\nu}\Pi_B) \in \mathcal{L}(L_2)$, *one has* $\{R(ze^{\pm i\nu}\cdot)\}_{z\in S_0} \in \bar{D}(\Pi_B, S_0, p, 2) \cap \bar{D}(\Pi_B, S_0, p, 2, N)$ *for any* $p \in (\max(q_s, q), 2)$, $N > 0$.

The proof of Theorem 4.7. Inclusion $\{R(ze^{\pm i\nu}\cdot)\}_{z\in S_0} \in \bar{D}(\Pi_B, S_0, p, 2)$ follows immediately from Lemma 4.3. It and Theorem 4.6 provide $\{R(ze^{\pm i\nu\cdot})\}_{z\in S_0} \in \bar{D}(\Pi_B, S_0, p, 2, N)$ thanks to Lemma 4.4. $\mathcal{Q.E.D.}$

LEMMA 4.8. *For* $h, z_0 \in \mathbb{C}$, $m \in \mathbb{N}$, *let* X *be a linear space, and* $A : X \to X$ *be a linear operator with existing inverse operators* $\{(A - (z_0 + ih)I)^{-1}\}_{i=0}^m$, *or* $\{R((z_0 + ih)A)\}_{i=0}^m$. *Then, for*

$$\Delta_h^m \psi(z_0) := \Delta_h^1 \Delta_h^{m-1} \psi(z_0),\ \Delta_h^1 \psi(z_0) := \psi(z_0 + h) - \psi(z_0),$$

one has, correspondingly:

$$a)\ \Delta_h^m \psi(z_0) := m! h^m \prod_{i=0}^m (A - (z_0 + ih)I)^{-1} \text{ for } \psi(z) := (A - zI)^{-1}, \text{ or}$$

$$b)\ \Delta_h^m \psi(z_0) := m!(hA)^m \prod_{i=0}^m (I - (z_0 + ih)A)^{-1} \text{ for } \psi(z) := R(zA).$$

The proof of Lemma 4.8. Let us prove part $a)$. The proof of $b)$ is essentially the same and, thus, is omitted. Using the induction by m, we see that, for $m = 1$, $a)$ is the Hilbert identity. Assuming the validity of $a)$ for $m = k - 1$, one sees

$$\Delta_h^k \psi(z_0) = (k-1)! h^{k-1} \left(\prod_{i=1}^k (A - (z_0 + ih)I)^{-1} - \prod_{i=0}^{k-1} (A - (z_0 + ih)I)^{-1} \right) =$$

$$= (k-1)! h^{k-1} \prod_{i=1}^k (A - (z_0 + ih)I)^{-1} \Delta_{kh}^1 \psi(z_0) = k! h^k \prod_{i=0}^k (A - (z_0 + ih)I)^{-1}.$$

$\mathcal{Q.E.D.}$

5. Extrapolation of operators

The next theorem extends celebrated results from [**78**].

THEOREM 5.1. *For* $l \in \mathbb{N}$, $b > 1$, $\delta > 0$, $p \in (1,2)$, *let operators* T *and* $\{A_t\}_{t\geq 0}$, $A_0 = I$ *to satisfy* $T \in \mathcal{L}(L_2)$,

$$\sup_{\substack{z\in\mathbb{R}^n \\ t>0}} \|\{\alpha_k\}_{k\in\mathbb{N}_0}|l_1\| = C_1 < \infty, \text{ where} \tag{1}$$

$$\alpha_k(t,z) := t^{\frac{n}{p}-\frac{n}{2}} b^{kn} \|T\Delta^l A_t|\mathcal{L}(L_p(Q_t(z)), L_2(Q_{b^{k+1}t\delta}(z) \setminus Q_{b^k t\delta}(z)))\|;$$

$$\sup_{\substack{z\in\mathbb{R}^n \\ t>0}} \left(t^{\frac{n}{p}-\frac{n}{2}}(\|A_t|\mathcal{L}(L_p(Q_t(z)), L_2(Q_{t\delta}(z))\| + \|\{\beta_k(t,z)\}_{k\in\mathbb{N}_0}|l_1\|\right) = C_2 < \infty, \text{ where} \tag{2}$$

$$\beta_k(t,z) := \{b^{kn}\|A_t|\mathcal{L}(L_p(Q_t(z)), L_2(Q_{b^{k+1}t\delta}(z) \setminus Q_{b^k t\delta}(z)))\|.$$

Then one has $\max(\|T|\mathcal{L}(L_p, L_{p,\infty})\|, \|T|\mathcal{L}(L_u, L_u)\| \leq C(C_1 + (1 + C_2\|T|\mathcal{L}(L_2)\|))$ *for* $u \in (p, 2)$.

The proof of Theorem 5.1. Without loss of generality, one can assume that $\delta > 2n^{1/2}l$. Thus, we see that Theorem 5.1 becomes a particular case of Theorem III.6.3, by taking $A = B = \mathbb{C}$, $\theta_0 = \theta_1 = \theta_2 = v_1 = w_1 = 2$, $v_0 = w_0 = p_0 = p_1 = p$, $t_1 = t_2 = 1$, $\lambda_0 = \lambda_1 = \nu_0 = \nu_1 = n$, $D_0 = \emptyset$ and $G(r,w) := I - \Delta^l A_r$ for any $Q_r(w) \subset \mathbb{R}^n$ in the conditions of the latter theorem. $\mathcal{Q.E.D.}$

6. Approach based on semigroups

In this section, we provide an example revealing a disadvantage of the approach based on the off-diagonal estimates for semigroups.

6.1. An example. In this subsection, we shall study off-diagonal properties of a particular example of an operator Γ suggested by McIntosh to the author:

$$\Gamma : \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \mapsto \begin{pmatrix} 0 & \mathrm{D} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} f_2' \\ 0 \end{pmatrix}, \ \Gamma^* = \begin{pmatrix} 0 & 0 \\ -\mathrm{D} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -f' & 0 \end{pmatrix}, \text{ where} \tag{1}$$

$f(x) = (f_1(x), f_2(x))$ is a function of one-dimensional variable $x \in \mathbb{R}$.

Let us assume that $f(x)$ is the restriction $f(x) = g(x + 0i)$ of an analytic function g satisfying Riemann formulas

$$g(x+iy) = \frac{\operatorname{sign}(y)}{2\Pi i} \int_{-\infty}^{\infty} \frac{f(t)dt}{t-x-iy} \text{ for } x \in \mathbb{R},\ y \in \mathbb{R} \setminus \{0\}. \tag{2}$$

Representing Π in the form

$$\Pi = \mathrm{D} \otimes J, \text{ where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3}$$

is a matrix imaginary unit of the 2×2 matrix representation of the complex numbers, we have, for $h \in \mathbb{C}$,

$$e^{h\Pi} = I\cos(h\Pi) + J\sin(h\Pi) = I\frac{e^{ih\Pi} + e^{-ih\Pi}}{2} + J\frac{e^{ih\Pi} - e^{-ih\Pi}}{2i}. \tag{4}$$

Let us note that assuming the convergence radius of the Taylor expansion of $g(z)$ at every z with $\mathrm{Im}z = 0$ to be greater than $|h|$, one has $e^{h\mathrm{D}}g(z) = g(z+h)$, i.e. $e^{h\mathrm{D}}$ is a shift operator. Therefore, we see that

$$e^{h\Pi}f(x) = I\frac{e^{ih\Pi}g(x) + e^{-ih\Pi}g(x)}{2} + J\frac{e^{ih\Pi}g(x) - e^{-ih\Pi}(x)}{2i} =$$
$$= I\frac{g(x+ih) + g(x-ih)}{2} + J\frac{g(x+ih) - g(x-ih)}{2i}. \tag{5}$$

Identity $J^2 = -I$ and (5) show the presence of the semigroup property

$$e^{h_1\Pi}e^{h_2\Pi} = e^{(h_1+h_2)\Pi}.$$

In particular, for $y > 0$, formulas $(2, 5)$ imply

$$e^{y\Pi}f(x) = (2\pi)^{-1}\left(If * \phi_y(x) + Jf * \psi_y\right) \text{ where } \phi_y = \frac{1}{y}\phi\left(\frac{\cdot}{y}\right), \ \psi_y = \frac{1}{y}\psi\left(\frac{\cdot}{y}\right), \tag{6}$$

and $\phi(x) := (1+x^2)^{-1}$, $\psi(x) := x\phi(x)$. Because $\phi \in L_1(\mathbb{R})$ and the convolution with ψ is a Calderón-Zygmund operator, the semigroup $e^{z\Pi} = e^{\mathrm{Re}z\Pi}e^{i\mathrm{Im}z\Pi}$ is well-defined on $L_p(\mathbb{R})$ for $p \in [1,\infty]$ and $\|e^{z\Pi}|\mathcal{L}(L_p)\| \leq C(p)$ for $p \in (1,\infty)$, $\mathrm{Re}z > 0$. Note that the semigroup property of $e^{z\Pi}$ is closely related with the identities

$$\phi_{t+s} = \phi_t * \phi_s - \psi_t * \psi_s, \ \psi_{t+s} = \phi_t * \psi_s + \psi_t * \phi_s, \ t, s > 0 \text{ reflecting, in turn,} \tag{7}$$

$$\varrho_{t+s} = \varrho_t * \varrho_s \text{ for } \varrho(x) := \phi(x) + i\psi(x) = (1-ix)^{-1}.$$

In the same time, we have, for $z = -iy$, $y \in \mathbb{R}$,

$$e^{-iy\Pi}f(x) = I\frac{f(x+y) + f(x-y)}{2} + J\frac{f(x+y) - f(x-y)}{2i}. \tag{8}$$

Thus, one cannot achieve off-diagonal estimates for this semigroup with an arbitrary N unless z is purely imaginary. In general, $N = 1$ implied by the convolution with ψ can only be gained.

Bibliography

[1] *Ajiev, S.S.*, On a uniform approach to singular integral operators // „International Conference on Harmonic Analysis and Related Topics" (Sydney, January 2002). Canberra: Austral. Nat. Univ., 2002. (Proc. Centre Math. Appl. ANU) 2003. V. 43. P. 1-16.

[2] *Calderon, A.P., Zygmund, A.*, On the existence of certain singular integrals // Acta Math. 1952. V. 88. P. 85–139.

[3] *Hörmander, L.*, Estimates for translation invariant operators on L^p spaces // Acta Math. 1960. V. 104. P. 93-139.

[4] *X.T. Duong and A. M^cIntosh*, Singular Integral Operators with Non-smooth Kernels on Irregular Domains // Rev. Mat. Iberoamericana 1999. V. 15(2). P. 233-265.

[5] *Ajiev, S.S.* On the boundedness of singular integral operators from certain classes I. // Trudy Mat. Inst. Steklova. 2003. V. 243. P. 17–45. Engl. transl.: Proc. Steklov Math. Inst. 2003. V. 243. P. 11–38.

[6] *Besov, O.V., Il'in, V.P., Lizorkin, P.I.*, L_p-estimates of a class of anisotropically singular integrals. // Doklady Akad. Nauk SSSR. 1966. V. 169. P. 1250–1253. Engl. transl.: Soviet Math. Dokl. 1966. V. 7. P. 1065–1069.

[7] *Fefferman, C., Stein, E.M.*, H_p spaces of several variables // Acta Math. 1972. V. 129. P. 137-194.

[8] *Coifman, R.R.*, A real variable characterization of H_p. // Studia Math. 1974. V. 51. P. 269-274.

[9] *Álvarez, A. J., Milman, M.*, H^p continuity properties of Caldern-Zygmund-type operators. // J. Math. Anal. Appl. 1986 V. 118, no. 1. P. 63–79.

[10] *Álvarez, J.*, H_p and weak L_p continuity of Calderón-Zygmund type operators. // Fourier analysis (Orono, ME, 1992). P. 17-34. Lecture Notes in Pure and Appl. Math. V. 157. Dekker: New-York. 1994.

[11] *Fan, D. S.*, A weak estimate of Caldern-Zygmund operators. // Integral Equations Operator Theory. 1994. V. 20, no. 4. D. 383–394.

[12] *Fefferman R., Soria, F.*, The space Weak H_1. // Studia Math. 1987. V. 85. P. 1-16.

[13] *Liu, He Ping*, The weak H^p spaces on homogeneous groups. // Harmonic analysis (Tianjin, 1988). P. 113–118. Lecture Notes in Math. V. 1494. Springer: Berlin, 1991.

[14] *Schwartz, J.T.*, A remark on inequalities of Calderon-Zygmund type for vector-valued functions // Comm. Pure Appl. Math. 1961. V. 14. P. 785–799.

[15] *Triebel, H.*, Spaces of distributions of Besov type on Euclidean n-space. Duality, interpolation // Arkiv Mat. 1973. V. 11. P. 13-64.

[16] *Lizorkin, P.I.*, (L_p, L_q)-multipliers of Fourier integrals (Russian). // Dokl. Akad. Nauk SSSR. 1963. V. 152. P. 808–811.

[17] *Albrecht, D., Duong, X. and* M^cIntosh, *A.*, Operator Theory and Harmonic Analysis. //(Workshop in Analysis and Geometry, art III, C.M.A., A.N.U., Canberra, Jan.-Feb. 1995.) Proc. Centre Math. Appl. Austral. Nat. Univ. 1996. V. 34, P. 77-136.

[18] *Duong, X. T.*, H_∞ Functional Calculus of Elliptic Partial Differential Operators in L_p spaces. // PhD Thesis. Macquary University: Sydney, 1990.

[19] *Lizorkin, P.I.*, A Marcinkiewicz type theorem for H-valued functions. A continuous form of the Littlewood-Paley theorem. // engl. transl.: Math. USSR-Sb. 1972. V. 16, no. 2. P. 237–243.

[20] *Lizorkin, P.I.*, A Littlewood-Paley type theorem for multiple Fourier integrals (Russian). // Trudy Mat. Inst. Steklov. 1967. V. 89. P. 214–230.

[21] *Besov, O. V.*, The Littlewood-Paley theorem for a mixed norm, (Russian). Studies in the theory of differentiable functions of several variables and its applications. // Trudy Mat. Inst. Steklov. 1984. V. 170. P. 31–36, 274.

[22] *Ajiev, S.S.*, Characterizations of $B^s_{p,q}(G), L^s_{p,q}(G), W^s_p(G)$ and other function spaces. Applications. // Tr. Mat. Inst. Steklova. 1999. V. 227. P. 7–42. Engl. transl.: Proc. Steklov Inst. Math. 1999. V. 227. P. 1–36.

[23] *Fefferman, C., Rivière, N.M., Sagher, Y.*, Interpolation between H_p spaces: The real method. // Trans. Amer. Math. Soc. 1974. V. 191. P. 75-81.

[24] *Latter, R.*, A characterization of $H_p(\mathbb{R}^n)$ in terms of atoms. // Studia Math. 1978. V. 62. P. 93-101.

[25] *Sobolev, S. L.*, The Cauchy problem in function spaces. // Dokl. Akad. Nauk SSSR. 1935. V. 3. P. 291–294.

[26] *Sobolev, S. L.*, Méthode nouvelle à resoudre le problème de Cauchy pour les équations linéaires hyperboliques normales. // Matem. Sb. 1936. V. 1(43). P. 39–72.

[27] *Sobolev, S.L.*, Nekotorye primeneniya funkcional'nogo analiza v matematicheskoĭ fizike.-Leningrad: Izd. LGU. 1950. 3e pererab. izd. M.: Nauka. 1988. Engl. transl.: Applications of functional analysis in mathematical physics.-Providence: Amer. Math. Soc. 1963. 3d ed-n.: Some applications of functional analysis in mathematical physics.-Providence: Amer. Math. Soc. 1991.

[28] *Schwartz, L.*, Théories des distributions. Hermann & Cie.: Paris. 1950. V. I. P. 148. 1951. V. II. P. 169. (Publ. Inst. Math. Univ. Strasbourg 9,10.)

[29] *Gel'fand, I. M., Shilov, G. E.*, Generalized functions: vypusk 2, Spaces of fundamental and generalized functions. Moscow: Fizmatgiz, 1958. Engl. transl.: London- New York: Academic Press, Inc., 1968.

[30] *Nikol'skii, S. M.*, Theory of approximations of functions. 4. I. Functional analysis. Dnepropetrovsk State Univ. Press. 1947. P. 71.

[31] *Nikol'skii, S. M.*, Properties of some classes of functions of several variables on differentiable manifilds. // Matem. Sb. 1953. V. 33(75/2). P. 261–326.

[32] *Nikol'skii, S. M.*, An imbedding theorem for functions with partial derivatives, considered in different metrics. // Izv. Akad. Nauk SSSR., ser. matem. 1958. V. 22. P. 321–336.

[33] *Nikol'skii, S. M.*, Constructive representation of null classes of differentiable functions of several variables. // Dokl. Akad. Nauk SSSR. 1966. V. 170. P. 542–545.

[34] *Nikol'skii, S. M.*, Integral representation and isomorphism of classes of differentiable functions of several variables. „Third mathematical summer school on the constructive theory of functions , held in Kaceveli, June-July 1965." Kiev: „Naukova Dumka". 1966. P. 135–238.

[35] *Nikol'skii, S. M.*, Approximation of functions of several variables and imbedding theorems. "Nauka": Moscow. 1969. P. 480; 2nd edition: "Nauka": Moscow. 1977. P. 455. Engl. transl.: Springer-Verlag: New York-Heidelberg. 1975. P. 418. (Die Grundlehren der Mathematischen Wissenschaften. Band 205.)

[36] *Slobodeckii, L. N.*, Spaces of S. L. Sobolev of fractional order and their application to boundary problems for partial differential equations. // Dokl. Akad. Nauk SSSR. 1958. V. 118. P. 243–246.

[37] *Besov, O. V.*, On a certain family of function spaces. Embedding and extension theorems. // Dokl. Akad. Nauk SSSR. 1959. V. 126. P. 1163–1165.

[38] *Kipriyanov, I. A.*, Fractional derivative and embedding theorems. // Dokl. Akad. Nauk SSSR. 1959. V. 126. P. 1187–1190.

[39] *Uspenskii, S. V.*, Properties of the classes $W_p^{(r)}$ with fractional derivatives on differentiable manifolds. // Dokl. Akad. Nauk SSSR. 1960. V. 132. P. 60–62.

[40] *Aronszain, N., Smith, K. T.*, Theory of Bessel potentials I. // Ann. Inst. Fourier. 1956. V. 11. P. 385–475.

[41] Stein E. M., The characterization of functions arising as potentials. // Bull. Amer. Math. Soc. I. 1961. V. 67. P. 102–104. II. 1962. V. 68. P. 577–584.

[42] *Lizorkin, P. I.*, Imbedding theorems for functions of the space L_p. // Dokl. Akad. Nauk SSSR. 1962. V. 143. P. 1042-1045.

[43] *Lizorkin, P. I.*, Generalized Liouville differentiation and the function spaces $L_p(\mathbb{R}^n)$. Imbedding theorems. // Matem. Sb. 1963. V. 60(102). P. 325–363.

[44] *Lizorkin, P. I.*, On the Fourier transform and Besov spaces. The null scale $B^0_{p\theta}$. // Dokl. Akad.Nauk SSSR. 1965. V. 163. P. 1318–1321.

[45] *Lizorkin, P. I.*, Nonisotropic Bessel potentials. Embedding theorems for the Sobolev spaces $L_p^{r_1,\dots,r_n}$ with fractional derivatives. // Dokl. Akad. Nauk SSSR. 1966. V. 170. P. 508–511.

[46] *Aronszain, N., Mulla, F., Szeptycki, P.*, On spaces of potentials connected with L_p classes. // Ann. Inst. Fourier. 1963. V. 13(2). P. 211–306.

[47] *Murray, M. A. M.*, Nonlinear operators and multilinear convolutions commuting with dilations. Yale Univ. Press. 1983. P. 1–162+xii. (PhD Dissertation.)

[48] *Murray, M. A. M.*, Commutators with fractional differentiation and BMO Sobolev spaces. // Indiana Univ. Math. J. 1985. V. 34(1). P. 205–215.

[49] *Kato, T., Ponce, G.*, Commutator estimates, and the Euler and Navier-Stokes equations. // Comm. Pure Appl. Math. 1988. V. 41. P. 891–907.

[50] *Hofmann, S.*, A characterization of commutators of parabolic singular integrals. „Fourier Analysis and Partial Differential Equations." Boca Raton: CRC Press. 1995. P. 195–210.

[51] *Hofmann, S.*, An off-diagonal $T1$-theorem and applications. // J. Funct. Anal. 1998. V. 160. P. 581–622.

[52] *Hofmann, S., Li, X., Yang, D.*, A generalized characterization of commutators of parabolic singular integrals. // Canad. Math. Bull. 1999. V. 42(4). P. 463–477.

[53] *Christ, F. M., Weinstein*, M. I. Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation. J. Funct. Anal. 1991. V. 100. P. 87–109.

[54] *Kenig, C. E., Ponce, G., Vega, L.*, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the Contraction principle. // Comm. Pure Appl. Math. 1993. V. XLVI. P. 527-620.

[55] *Segovia, C., Wheeden, R. L.*, Fractional differentiation of the commutator of the Hilbert transform. // J. Funct. Anal. 1971. V. 8. P. 341-359.

[56] *Cohen, J., Gosselin, J.*, A BMO-estimate for multilinear singular integral operator. // Illinois J. Math. 1986. V. 30. P. 445–464.

[57] *Hofmann, S.*, On certain nonstandard Calderón-Zygmund operators. // Studia Math. 1984. V. 109(2). P. 105–131.

[58] *Lu, Sh., Wu, Qi.*, Endpoint estimates for certain commutators of fractional and singular integrals. // Proc. Amer. Math. Soc. 2002. V. 131(2). P. 467-477.

[59] *Hu, G., Yang, D.*, A variant sharp estimate for multilinear singular integral operators. // Studia Math. 2000. V. 141(1). P. 25-42.

[60] *Lewis, J. L., Murray, M. A. M.*, The method of layer potentials for the heat equation in time-varying domains. // Mem. Amer. Math. Soc. 1995. V. 114(545).

[61] *Ajiev, S.S.*, Anisotropic supersingular integral operators and approximation formula. //

[62] *Triebel, H.*, Theory of function spaces. Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1983. P. 284. (Mathematik und ihre Anwendungen in Physik und Technik, 38.) Russian transl.: (with appendix due to P. I. Lizorkin).-Moskva: "Mir". 1986. P. 448.

[63] *Coifman, R.R. and Meyer, Y.*, Wavelets: Calderón-Zygmund and multilinear operators. Cambridge University Press. 1997.

[64] *Il'in, V. P.*, The function spaces $\mathcal{L}^{\lambda,a;b_s}_{r,p,\theta}(G)$ // Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 1972. V. 30. P. 51–75.

[65] *Ajiev, S. S.*, in preparation.

[66] *Triebel, H.*, Local approximation spaces. // Z. Anal. Anwendungen. 1989. V. 8(3). P. 261–288.

[67] *DeVore, R. A., Sharpley, R.*, Maximal functions measuring smoothness. // Mem. Amer. Math. Soc. 1984. V. 47(293). P. 1–115.

[68] *S.S. Ajiev*, On the boundedness of singular integral operators from some classes II.

[69] *Rubio De Francia, J.L., Ruiz, F.J., Torrea, J.L.*, Calderón-Zygmund Theory for Operator-Valued Kernels // Adv. in Math. 1986. V. 62. P. 7–48.

[70] *Benedek, A., Calderón, A.-P., Panzone, R.*, Convolution operators on Banach space valued functions // Proc. Nat. Acad. Sci. U.S.A. 1962. V. 48. P. 356–365.

[71] *Bourgain, J.*, Extension of a result of Benedek, Caldern and Panzone // Ark. Mat. 1984. V. 22(1). P. 91–95.

[72] *Besov, O.V., Il'in, V.P., Nikol'skii, S.M.*, Integral representations of functions and embedding theorems.-Nauka: Moscow. 1975. 2nd ed. Nauka-Fizmatgiz: Moscow. 1996.

[73] *Nakai, E.; Yabuta, K.*, Singular integral operators on $L^{p,\Phi}$-spaces // Ann. Mat. Pura Appl. 1988/89. V. 153(4). P. 53–62.

[74] *Peetre, J.* On convolution operators leaving $L^{p,\lambda}$ spaces invariant // Ann. Mat. Pura Appl. (4). 1966. V. 72. P. 295–304.

[75] *Nakai, E.*, Singular integral operators on $L_k^{p,\Phi}$-spaces // Bull. Fac. Sci. Ibaraki Univ. Ser. A. 1987. V. 19 P. 71–78.

[76] *Yabuta, K.* Singular integral operators on Triebel-Lizorkin spaces // Bull. Fac. Sci. Ibaraki Univ. Ser. A 1988. V. 20. P. 9–17.

[77] *Yabuta, K.*, Singular integral operators on Besov spaces // Math. Nachr. 1988. V. 136. P. 163–175.

[78] *S. Blunck, P. Kunstmann*, Calderón-Zygmund theory for non-integral operators and the H^∞-functional calculus// To appear in Rev. Mat. Ibero.

[79] *W. Hebisch*, A multiplier theorem for Schrödinger operators // Colloq. Math. 1990. V. 60/61. P. 659–664.

[80] *X.T. Duong, D.W. Robinson*, Semigrop Kernels, Poisson Bounds, and Holomorphic Functional Calculus // J. Funct. Anal. 1996. V. 142. P. 89-128.

[81] *P. Auscher*, On L^p estimates for square roots of second order elliptic operators on $\mathbb{R}^n$ //To appear.

[82] *Stein, E.M., Zygmund, A.*, Boundedness of translation invariant operators on Hölder and L_p spaces// Ann. of Math. 1967. V. 85(2). P. 337-349.

[83] *Bastero, J., Milman, M., Ruiz, F.*, Reverse Hölder ihequalities and interpolation // ,,Function Spaces, Interpolation Spaces, and Related Topics". Providence (RI): Amer. Math. Soc., 1999. (Israel Math. Conf. Proc., Amer. Math. Soc. V. 13.)

[84] *S. Hofmann, A. M^c Intosh*, The solution of Kato problem in two dimensions. Proceedings of the conference on harmonic analysis and PDE (El Escurial), June 2000// Publ. Math. 2002. V. extra. P. 143-160.

[85] *P. Auscher, S. Hofmann, M. Lacey, A. M^c Intosh, Ph. Tchamitchian*, The solution of Kato square root problem for second order elliptic operators on $\mathbb{R}^n$ // Ann. Math. 2002. V. 156(2). P. 633-654.

[86] *P. Auscher, S. Hofmann, A. M^c Intosh, Ph. Tchamitchian*, The Kato square root problem for high order elliptic operators and systems on $\mathbb{R}^n$ //J. Evol. Equ. 2002. V. 1. P. 361-385.

[87] *A. M^c Intosh*, On representing closed accretive sesquilinear forms as $(A^{1/2}u,\ A^{*1/2}v)$. Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. III (Paris, 1980/1981) // Res. Notes in Math. 1982. V. 70, P. 252-267.

[88] *R. Coifman, A. M^c Intosh, Y. Meyer*, L'intégrale de Cauchy définit un opérateur borné sur $L_2(\mathbb{R}^n)$ pour les courbes lipschitziennes// Ann. Math. 1982. V. 116. P. 361-387.

[89] *P. Auscher, Ph. Tchamitchian*, Square root problem for divergence operators and related topics //Astérisque. Soc. Math. France. 1998.

[90] *A. M^c Intosh*, Operators which have an H_∞ functional calculus. // Proc. Centre Math. Anal. Austral. Nat. Univ. 1986. V. 14. P. 210–231.

[91] *D. Albrecht, X. Duong, A. M^c Intosh*, Operator theory and harmonic analysis. Instructional workshop on Analysis and Geometry, Part III (Canberra, 1995) //Proc. Cemtre Math. Appl. Austral. Nat. Univ. 1996. V. 34. P. 77-136.

[92] *M. Cowling, I. Doust, A. M^c Intosh, A. Yagi*, Banach space operators with a bounded H^∞-calculus //J. Aust. Math. Soc. (Series A). 1996. V. 60. P. 51-89.

[93] *A. Axelsson, S. Keith, A. M^c Intosh*, Quadratic estimates and functional calculi of perturbed Dirac operators. // In preparation.

[94] *E.B. Davies*, Uniformly elliptic operators with measurable coefficients// J. Funct. Anal. 1995. V. 132. P. 141-169.

Index of Function Spaces