Chiral Symmetry in Supersymmetric QED_3 and QED_4

by

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Statement of Originality

This thesis does not incorporate any material previously submitted for a degree or diploma at any university. To the best of my knowledge it does not contain any material previously published or written by another person, except where due reference has been made. This work is my own work or work done in collaboration with Dr. C.J. Burden.

Chapter 1 is largely a review of the literature required for this thesis. Chapters 2 to 4 explain the background material that this work is based on, all of which was previously known. The original work is largely contained in chapters 5 to 7, with additional material in appendices E to G. Chapter 8 gives a summary of the work done and some conclusions.

Michael Luke Walker

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My gratitude also to Prof. V. Miransky for useful discussions regarding the CJT effective potential.

I am in eternal debt to my parents for their lifelong love and support. Thankyou also to my sisters Anna and Briony for their love and encouragement over the years.

Last but not least, I wish to thank H.W. Taylor, V. Masters, C. Short and C. Leviston for contributing some of the quotes that appear in this thesis.

Abstract

We investigate chiral symmetry breaking in supersymmetric quantum electrodynamics (SQED). Our approach is to take a selection of nonperturbative tools from conventional quantum field theory (QFT) and adapt them to supersymmetry (SUSY). The methods we choose are the Dyson-Schwinger equations (DSEs) and the CJT effective potential.

In adapting these techniques to SUSY, we initially have difficulty interpreting the auxiliary fields f and g which belong to the electron's SUSY multiplet. We circumvent this difficulty with the development of a suitable notation. It is then a simple matter to write down the electron's DSE in the SUSY theory. It also follows simply that the effective potential in a SUSY theory is uniformly zero, thus extending a long known perturbative theorem into the nonperturbative region.

The DSE for the electron in SQED has more self-energy terms than for ordinary quantum electrodynamics (QED). The extra terms are due to the super-partners and their interactions. It was known from previous authors that it is possible to determine the unknown propagators of these extra particles in terms of the known ones by using SUSY Ward identities (SWIs), analogous to the Ward-Takahashi identities (WTIs). Obtaining the unknown propagators in this way, we solve the DSE numerically in the rainbow approximation, finding both an achiral and a chiral solution.

However an examination of the DSE for the scalar partners of the electron reveals that the rainbow approximation is far too restrictive to be very useful in a SUSY theory. We therefore set about to transcend the rainbow approximation. We find the dressed form of the vertices for the photon and its SUSY partner, the photino, by adapting the method used to find the unknown propagators, ie. by deriving the SWIs governing the vertices. We combine these identities with the WTIs to find the most general form possible for vertices in SQED.

We employ the general vertices to investigate previous claims based on analyses in the superfield notation that there is no achiral solution to the DSEs in SQED. Our analysis, which requires far less use of truncation than the superfield studies,

indicates that this is not true.

Contents

Appendices

Notation

Abbreviations Used

SUSY	Supersymmetry (n), supersymmetric (adj.)
WI, WTI, SWI	Ward identity, Ward-Takahashi identity, supersymmetric Ward identity.
BC, SBC	Ball-Chiu, supersymmetric Ball-Chiu.
WZ	Wess-Zumino.
CJT	Cornwall-Jackiw-Tomboulis.
QFT	Quantum field theory.
QED_3, QED_4	Quantum electrodynamics in $2+1$ and $3+1$ dimensions respectively.
QED	Quantum electrodynamics in either $2+1$ or $3+1$ dimensions.
$SQED_3$, $SQED_4$, SQED	Supersymmetric QED_3, QED_4, QED respectively.
QCD	Quantum chromodynamics.
DSE	Dyson-Schwinger equation.

Commonly Used Symbols

ϕ	Generic field (usually scalar).
$A_\mu, F_{\mu\nu}$	Gauge field, field strength tensor.
λ	Photino.
Φ	Chiral multiplet/superfield.
V	General multiplet/superfield.
$Q, \bar{Q}$	Supersymmetry operator.
$D, \bar{D}$	Supercovariant derivative.
η	Metric tensor.

ε	Levi-Civita symbol.
δ_S	Supersymmetry transformation.
δ_G	$U(1)$ gauge transformation.
δ_{WZ}	Gauge transformation to restore WZ gauge after δ_S.
δ	Functional derivative, $\delta_S + \delta_{WZ}$ (according to context).
S	Electron propagator.
D_{aa}	a propagator.
D_{bb}	b propagator.
$D_{af} = D_{fa}$	a to f, f to a propagator.
$D_{bg} = D_{gb}$	b to g, g to b propagator.
D_{ff}	f propagator.
D_{gg}	g propagator.
$[a]$	$\begin{pmatrix} a \\ f \end{pmatrix}$.
$[b]$	$\begin{pmatrix} b \\ g \end{pmatrix}$.
$[D]$	$[a], [b]$ propagator.
$D_{\mu\nu}$	Photon propagator.
S_λ	Photino propagator.
D_D	D propagator.
A, B	Vector part of inverse electron propagator, scalar part of inverse electron propagator.
$\mathcal{M}$	Effective electron mass ($\mathcal{M} = \frac{B}{A}$).
$\mathcal{Z}$	Wavefunction renormalisation A^{-1}, generating functional of Green's functions.
$\mathcal{S}$	Action.
$\mathcal{L}$	Lagrangian.
H	Hamiltonian.
$\mathcal{W}$	Generating functional of connected Green's functions.
Γ	Effective action.
J, η	Boson field source, fermion field source.
ϕ_{cl}	Field expectation value.
Φ_{cl}	Multiplet expectation value.
J_Φ	Chiral multiplet source.

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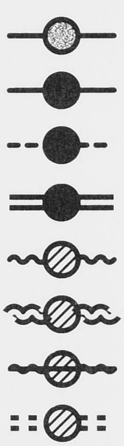
G, G_C	Green's function, connected Green's function.
$\Gamma_{X...Z}$	Proper function.
$\mathcal{C}$	Charge conjugation operator.
C	Charge conjugation matrix.
$\mathcal{P}$	Parity operator.
Π	Parity matrix.
m	Bare fermion mass.
ξ	Gauge fixing parameter.
p, q, k	Momenta.

Propagator Graphics

We use the following conventions for graphically representing the various full propagators. The bare propagators differ only in that the blobs are absent and the bare electron propagator has an arrow.



generic particle (usually scalar): $D = \frac{\delta^2 \mathcal{W}}{\delta J \delta J}$.

electron: $S = \frac{\delta^2 \mathcal{W}}{\delta \bar{\eta}_\psi \delta \eta_\psi}$.

a, b: $D_{(a,b)} = \frac{\delta^2 \mathcal{W}}{\delta J^*_{(a,b)} \delta J_{(a,b)}}$

f, g,: $D_{(f,g)} = \frac{\delta^2 \mathcal{W}}{\delta J^*_{(f,g)} \delta J_{(f,g)}}$.

photon: $D_{\mu\nu} = \frac{\delta^2 \mathcal{W}}{\delta J^\mu_A \delta J^\nu_A}$.

K: $D_K = \frac{\delta^2 \mathcal{W}}{\delta J_K \delta J_K}$.

photino: $S_\lambda = \frac{\delta^2 \mathcal{W}}{\delta \bar{\eta}_\lambda \delta \eta_\lambda}$.

D: $D_D = \frac{\delta^2 \mathcal{W}}{\delta J_D \delta J_D}$.

We mainly work in the quenched approximation in which the propagators of the photon, and its superpartners the photino and D, are taken to be bare.

Fourier Transform Conventions

The Fourier transform of $\mathcal{F}(x, y, z)$ into momentum space is given (in $3+1$ dimensions) by

$$(2\pi)^4 \delta^4(p - q - k)\mathcal{F}(p, q) = \int d^4x d^4y d^4z \mathrm{e}^{-i(p \cdot x - q \cdot y - k \cdot z)} \mathcal{F}(x, y, z).$$

Equivalently,

$$\mathcal{F}(x,y,z) = \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}(2\pi)^4\delta^4(p-q-k)\mathrm{e}^{i(p\cdot x-q\cdot y-k\cdot z)}\mathcal{F}(p,q).$$

We then have

$$\gamma\cdot\partial^x\mathcal{F}(x,y,z) \xrightarrow{\text{F.T.}} i\gamma\cdot p\mathcal{F}(p,q),$$
$$\gamma\cdot\partial^y\mathcal{F}(x,y,z) \xrightarrow{\text{F.T.}} -i\gamma\cdot q\mathcal{F}(p,q),$$

"Why is it I get my best ideas while shaving?"

- Albert Einstein.

CHAPTER 1

Introduction

A major goal of physics for much of this century has been the unification of forces. This was initiated, in its modern form, by Einstein who spent his later years in an unsuccessful attempt to unite gravity with electromagnetism. With the identification of the nuclear forces we are now in a much better position than Einstein to achieve unification, and a major step toward this goal was achieved by Weinberg, Salam and Glashow in 1967 with the unification of electromagnetism with the weak nuclear force into the electroweak force [1].

At about the same time, physicists were beginning to suspect that the inclusion of gravity in any unification scheme was an impossibility. The suspicion appeared confirmed with the publication of the Coleman-Mandula theorem [2] which essentially stated that a physically realistic theory can not include the symmetries of spacetime (Poincaré symmetries) within a bosonic Lie group. Any attempt to do so must predict unphysical behaviour such as discrete scattering angles.

The escape from this 'no-go' theorem is to generalise from Lie algebras to graded Lie algebras [3, 4], which include fermionic generators. Being of half-integral spin, these generators alter the statistics of whatever state they act upon and so the symmetry they generate is a *fermi-bose* symmetry, otherwise known as supersymmetry (SUSY). That SUSY will be manifest in any successful unification scheme seems certain. Indeed a consistent formulation of superstring theory, widely touted as being the most likely candidate for a theory of everything before its absorption into the more general M-theory, requires SUSY [5]. Such being the case, we are clearly going to need a well-stocked set of analytic tools with which to understand SUSY theories. Perturbative tools are so well developed that perturbative calculations can be made more easily in SUSY theories than in conventional ones [6]. The same alas cannot be said of nonperturbative techniques. This represents a serious deficiency in our analytic apparatus. Several important phenomena, such as chiral symmetry breaking and bound state phenomenology, are inherently nonperturbative.

Chiral symmetry breaking occurs in several interesting theories, including quantum chromodynamics [7] (QCD), the theory of the strong nuclear force. This is an important result because the Goldstone boson generated by chiral symmetry breaking is the pion, responsible for the interaction of protons and neutrons. Chiral symmetry breaking occurs in other theories like quantum electrodynamics (QED) in 2+1 dimensions [8, 9] (QED_3) which has a confining logarithmic potential [10]. This is of great use to theorists for whom QED_3 is a toy model, used to develop nonperturbative techniques since it is an Abelian theory free of the complications that can make analysing a non-Abelian theory a daunting task. It also transpires that QED in $3+1$ dimensions (QED_4) shows chiral symmetry breaking if its coupling constant is sufficiently high [11, 12, 13, 14]. We use the SUSY version of QED_3 ($SQED_3$) for numerical work because its lower dimensionality makes it super-renormalisable.

Nonperturbative analyses generally use one of two complimentary methods, either lattice gauge theory in which the theory is modeled on a lattice of points instead of a continuum, or Dyson-Schwinger equations (DSEs). DSEs form an infinite tower of equations which encode all information about the quantum corrections of a theory. Unfortunately, in addition to there being an infinite number of them, they are also nonlinear, so that solving them exactly is an unattainable goal. What can be done is to make physically reasonable truncations which allow us to find useful, approximate solutions. For example, in QED_3, in the study of chiral symmetry breaking, useful results are obtained by considering only the DSE for the electron propagator and choosing an *ansatz* for the electron-photon interaction. A significant effort (eg. [15, 16, 17]) has gone into the development of suitable vertex *ansätze*. Early studies used the rainbow approximation in which all dressing to the vertex is ignored. While relatively easy to use, the rainbow approximation is generally considered a poor one since it seriously violates gauge invariance. Improved *ansätze* were found, such as the Ball-Chiu [15] (BC) and Curtis-Pennington [16] vertices, and used in subsequent studies [8, 11, 14].

The application of this approach to SUSY theories is a lot less advanced. This is due to a number of difficulties specific to such theories. As we explain in more detail in chapter 4, every fermion in a SUSY theory has bosonic partners and *vice versa*. This is to be expected since SUSY is a *fermi-bose* symmetry. These extra particles and the additional interactions they induce are an obvious source of difficulties. For example, the electron in SQED interacts not only with the photon as in ordinary QED, but also with the photon's SUSY partner, the photino. We see immediately that this will complexify any attempt to dress the vertices. For even if we have a suitable *ansatz* for the photon vertex, we need a corresponding *ansatz* for the photino vertex before we can calculate anything. Some very simple attempts have been made [18] to dress the vertices so that their gauge covariance is improved at zero momentum transfer but the approximations used are far from satisfying. We explain this in more detail in Sec. 7.1.

It transpires that the electron, apart from interacting with the photino, also interacts with its own super-partners. This means that the electron's DSE cannot be solved unless those for its super-partners are solved simultaneously. A way was found around this difficulty by Iliopoulos and Zumino [19] who demonstrated that the propagators of the electron's super-partners can be expressed in terms of that for the electron by using equations derived from SUSY called SUSY Ward identities (SWIs). We explain this approach more fully in Sec. 5.1.

In fact the idea of Ward identities (WIs), ie. identities derived from a theory's symmetries, did not originate with SUSY but with gauge symmetry. In Sec. 3.4 we outline the derivation of the Ward-Takahashi identities (WTIs) which are derived from gauge symmetry.

In addition to giving us the propagators of the electron's super-partners, SWIs can also give us the form of the dressed photino vertices from those of the photon, where the form of the photon vertices is restricted by the WTIs. The set of SWIs constraining the vertices in SQED is presented in Sec. 7.2 and the form that those vertices must conform to is presented in Sec. 7.3.

A second difficulty with SUSY theories, in the component formalism, is the existence of auxiliary fields. Auxiliary fields are those with no derivatives in the Lagrangian. This allows them to be expressed in terms of the other fields by the equations of motion so that, classically at least, they contain no new information. This leaves us with questions such as "Do their propagators feature in the DSE?", "Do they interact with the electron at a nonperturbative level when they don't classically?", and "How do they contribute to the effective potential?". These questions and more are answered in Sec. 5.3.

There has been disagreement in the literature regarding the existence of a non-renormalisation theorem in SQED which forbids chiral symmetry breaking. Clark and Love [20] reached this result by examining the DSE of $3+1$ dimensional SQED

($SQED_4$) in the superfield formalism, explained in Sec. 4.7. They found that the dynamically generated mass contains a prefactor of $\xi - 1$ which vanishes in Feynman gauge and argue that the result must hold in all gauges if it holds in one. This result was criticised by Kaiser and Selipsky [21], chiefly on the grounds that the $\xi - 1$ might be cancelled in the limit that $\xi \to 1$.

Nonrenormalisation theorems have long been a feature of SUSY theories. It has long been a result that the mass in a SUSY theory receives no corrections to any order in perturbation theory [22, 23, 19, 4]. This result holds also for certain coupling constants in some theories. What was new about the Clark and Love result was that they claimed this nonrenormalisation theorem holds at the nonperturbative level. A recent paper [24] has claimed to overcome the objections of Kaiser and Selipsky in $SQED_3$. However their approach depends on the existence of a compactification scale generated by dimensional reduction from $SQED_4$ so their result cannot be applied to the higher dimensional theory.

Our own research has something to say on this issue. Using component fields instead of superfields, we substitute the general form of the vertices into the DSE and find no evidence that the effective mass vanishes in any gauge, thus challenging the validity of the nonrenormalisation theorem, as does an earlier study by Pisarski [25] which found that chiral was broken in the many-flavour limit. We attribute the discrepancy between our results and those of [24] to the extensive approximations used in that paper. Our analysis, by contrast, is relatively free of approximations, with the exception of ignoring the dressing on the photon and photino propagators (quenched approximation).

A nonrenormalisation theorem also applies to the effective potential of SUSY theories. The ground state of a SUSY theory has exactly zero energy unless SUSY is spontaneously broken [23, 26, 25]. It is a long standing result that the ground state remains at zero energy to all orders in perturbation theory. We investigate the nonperturbative properties of the effective potential to see if it can be used to select the dynamically favoured solution to the DSEs. We find that this nonrenormalisation theorem does continue into the nonperturbative region.

We describe the fundamentals of quantum field theory (QFT) in chapter 2 and outline the nonperturbative methods we intend to use in chapter 3. Chapter 4 is an introduction to SUSY, describing the motivations for it and its mathematical basis. We present the standard SUSY multiplets and explain how they are manipulated to construct a SUSY theory. After constructing SQED, we conclude this chapter with a description of superfields.

With the necessary introductions done we spend chapter 5 adapting the methods of chapter 3 to SUSY theories. We start by demonstrating how the propagators of SUSY partners are related to known propagators by SWIs. The difficulties induced by the auxiliary fields are described and solved. The chapter ends by finding the electron's DSE in SQED before deriving a nonperturbative nonrenormalisation theorem for the effective action which renders it useless for distinguishing between solutions of the DSE.

In chapter 6 we present the results of a numerical study of $SQED_3$ in the rainbow approximation after adapting the SUSY multiplets to incorporate extra degrees of freedom in its Clifford algebra. We then go on in chapter 7 to describe the inconsistency between the rainbow approximation and SUSY, which is in addition to that between it and $U(1)$ gauge symmetry. Chapter 7 goes on to derive the SWIs relating the various vertices of SQED before presenting their solution which we use to examine the possibility of a nonrenormalisation theorem for SQED.

The thesis ends with a summary of our conclusions and some useful technical appendices.

"Most of the world's most important conversations never end up happening, just because people won't put themselves on the line."

- "Headgames", Nick Earls.

CHAPTER 2

Basic Concepts in QFT

2.1 The Generating Functional of Green's Functions

QFT is an attempt to understand the workings of nature at a fundamental level. Postulating the existence of particular elementary particles, we ask QFT for the probability that a system will evolve from a given initial state to a particular final state. The relevant probability amplitude is a weighted sum over all possible paths, where the weight of each path is its own probability amplitude, given by

$$e^{\frac{i}{\hbar}\mathcal{S}[\phi(x)]}, \tag{2.1.1}$$

where $\mathcal{S}$ is the action and $\phi(x)$ represents any field.

That the probability amplitude is the exponential of the action should not be a surprise. In classical mechanics with coordinate $q(t)$, conjugate momentum $p(t)$, and Hamiltonian $H(q,p,t)$, so that

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \tag{2.1.2}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \tag{2.1.3}$$

the action $\mathcal{S}[q]$, given by

$$\frac{d\mathcal{S}[q]}{dt} = -H(p,q,t) + p\dot{q}, \tag{2.1.4}$$

can be shown [27] to be the generating functional of the canonical transformation taking the coordinates q and p from time t_1 to time t_2. By analogy, in QFT we expect the action of the fields to determine their transition in state space. Notice that in the classical limit, as $\hbar \to 0$, Eq. (2.1.1) oscillates more rapidly with changes in $\mathcal{S}$ and all paths except the classical one where $\mathcal{S}$ is stationary are cancelled. This is in accordance with the correspondence principle. We shall henceforth use units such that $\hbar = c = 1$.

We therefore use the path integral

$$\int \mathcal{D}\phi e^{i\int d^dx \mathcal{L}(\phi(x),\partial_\mu\phi(x))}, \tag{2.1.5}$$

where $\mathcal{L}$ is the Lagrangian. (The rest of this chapter is based principally on Refs. [27, 28].) In principle, we need only substitute in the initial and final states and start integrating. However we can only do this if we know *a priori* what the physical states of the theory are. Since this is what we are trying to find, we assume a) the existence of a vacuum state, and b) that the system begins and ends in this vacuum. Eq. (2.1.5) therefore is the probability amplitude that a system beginning in the vacuum state at initial time $t = -\infty$ will end in a vacuum state at $t = +\infty$, and is normalised to one.

Since observed transitions are not from vacuum to vacuum, we add the source term,

$$J(x)\phi(x), \tag{2.1.6}$$

to $\mathcal{L}$, where J acts as a source and sink for the particles we wish to scatter. We now define the generating functional

$$\mathcal{Z}[J] = \int \mathcal{D}\phi e^{\frac{i}{\hbar}\int d^dx\{\mathcal{L}(\phi(x),\partial_\mu\phi(x))+J(x)\phi(x)\}}, \tag{2.1.7}$$

with the Taylor expansion

$$\mathcal{Z}[J] = \mathcal{Z}[0]\sum_{N=0}^{\infty}\frac{1}{N!}\int d^dx_1\cdots d^dx_N J(x_1)\cdots J(x_N)G^{(N)}(x_1,\ldots,x_N), \tag{2.1.8}$$

where

$$G^{(N)}(x_1,\ldots,x_N) = \frac{1}{\mathcal{Z}[J]}\frac{\delta^N\mathcal{Z}[J]}{\delta J(x_1)\cdots\delta J(x_N)}|_{J_1=\cdots=J_N=0}, \tag{2.1.9}$$

the N-point Green's functions, are the probability amplitudes for the corresponding N-point scattering process. To connect with canonical quantisation,

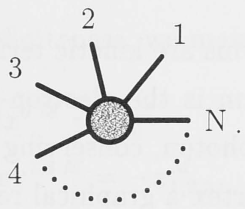
$$G^{(N)}(x_1,\ldots,x_N) = \langle 0|T[\phi(x_1)\cdots\phi(x_N)]|0\rangle. \tag{2.1.10}$$

It is useful to define the functional

$$\mathcal{W}[J] = -i\ln\mathcal{Z}[J], \tag{2.1.11}$$

and the connected Green's functions

$$G_C^{(N)}(x_1,\ldots,x_N) = \frac{\delta^N\mathcal{W}[J]}{\delta J(x_1)\cdots\delta J(x_N)}|_{J_1=\cdots=J_N=0}, \tag{2.1.12}$$

which are represented graphically by 

A simple calculation shows that

$$G^{(2)}(x_1, x_2) =$$  $$\cdot \tag{2.1.13}$$

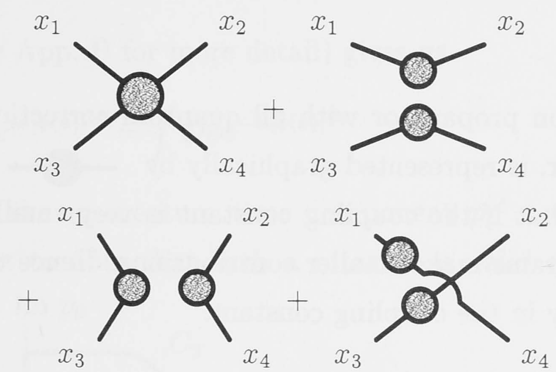
However the one-point contributions do not correspond to anything physical and are usually removed with a suitable field redefinition so that

$$G^{(2)}(x_1, x_2) = G_C^{(2)}(x_1, x_2), \tag{2.1.14}$$

and

$$\begin{aligned} G^{(4)}(x_1, x_2, x_3, x_4) &= G_C^{(4)}(x_1, x_2, x_3, x_4) + G_C^{(2)}(x_1, x_2)G_C^{(2)}(x_3, x_4) \\ &\quad + G_C^{(2)}(x_1, x_3)G_C^{(2)}(x_2, x_4) + G_C^{(2)}(x_1, x_4)G_C^{(2)}(x_2, x_3), \end{aligned} \tag{2.1.15}$$

shown graphically by

$$G^{(4)}(x_1, x_2, x_3, x_4) =$$ 

We could proceed from here, given the Lagrangian of any theory, to calculate the Green's functions from functional derivatives of $\mathcal{Z}$. However doing so quickly generates a combinatorial nightmare, even at low N. What we need is a methodical procedure of generating the Green's functions order by order in some small parameter.

An efficient and very intuitive method is to draw Feynman diagrams. Let us use QED as an example. QED has the Lagrangian

$$\mathcal{L} = -\bar{\psi}(\gamma \cdot \partial + im)\psi - ieA^\mu \bar{\psi}\gamma_\mu \psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{2.1.16}$$

where the first and last terms are kinetic terms for the electron and photon respectively, and the middle term is the electron-photon interaction where an electron either emits or absorbs a photon, conserving momentum in the process. We assign to each propagator and vertex a graphical representation, called a 'Feynman rule'. The Feynman rules for QED are as follows:

1. An electron propagator is shown as  ,
2. a photon propagator is shown as , and
3. an electron-photon vertex as  .

We now draw every relevant process contributing to the Green's function we wish to calculate. For example, the two-point function $G^{(2)}$ contains not only the bare propagator, but higher order corrections such as those shown in Fig. (2.1). The

Figure 2.1: Two of the self-interaction terms which contribute to the propagator.

full electron propagator with all quantum corrections, also known as the dressed propagator, is represented graphically by  .

Note that if the coupling constant is very small, as it is in QED, then higher order diagrams make smaller contributions. Hence the $G^{(N)}$ can be calculated perturbatively in the coupling constant.

2.2 The Wick Rotation

The sharp-eyed reader will have noticed that the integrand of Eq. (2.1.5) is oscillatory. The path integral therefore is not well-defined. There are two ways to remedy this. The first is to put a damping term into the Lagrangian of the form $i\epsilon\phi^2$, where ϵ is infinitesimal and positive, and take the limit $\epsilon \to 0$ at the end of the calculation. The second is to re-express the Minkowski space Lagrangian in Euclidean coordinates. A complete set of rules for converting from one convention to the other is given in App. B. For now we give a brief outline.

Working in 3 + 1 dimensions for definiteness, we make the replacement

$$\begin{aligned} x_4^E &= ix^{0\,M} = ix_0^M \\ x_{1,2,3}^E &= x^{1,2,3\,M} = -x_{1,2,3}^M, \end{aligned} \tag{2.2.1}$$

where the superscripts E, M denote Euclidean and Minkowski space coordinates respectively. The corresponding transformation for momenta, derivatives and vector potentials is

$$\begin{aligned} P_4^E &= -iP^{0\,M} = -iP_0^M \\ P_{1,2,3}^E &= -P^{1,2,3\,M} = P_{1,2,3}^M, \end{aligned} \tag{2.2.2}$$

while Dirac matrices transform as

$$\begin{aligned} \gamma_4^E &= \gamma^{0\,M} = \gamma_0^M \\ \gamma_{1,2,3}^E &= -i\gamma^{1,2,3\,M} = i\gamma_{1,2,3}^M. \end{aligned} \tag{2.2.3}$$

The rule for transforming the volume element is easily seen from Eq. (2.2.1) to be,

$$(d^4x)^E = i(d^4x)^M = i(dx^0dx^1dx^2dx^3)^M. \tag{2.2.4}$$

Making these substitutions (see App. B for more detail) gives us

$$\int \mathcal{D}\phi e^{iS[\phi(x)]^M} = \int \mathcal{D}\phi e^{-S[\phi(x)]^E}. \tag{2.2.5}$$

This is equivalent to rotating the momentum time component p_0^M by $\frac{\pi}{2}$ in the complex plane. It is important to note that such a rotation is valid only if the

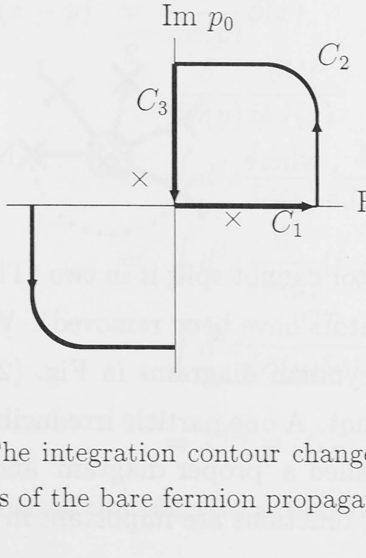


Figure 2.2: The integration contour change due to the Wick rotation and the singularities of the bare fermion propagator.

momentum space propagator is free of poles in both the upper right and lower left quadrants of the complex plane. Fig. (2.2) illustrates why. The integral around the contour C is given by

$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = \sum \text{Residues contained inside } C. \tag{2.2.6}$$

If we are to have

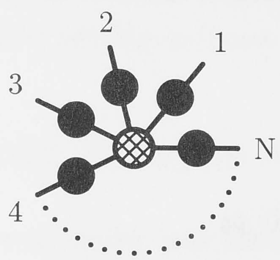
$$\int_{C_1} = -\int_{C_3}, \tag{2.2.7}$$

as required, then since the integrand of $\int_{C_2}$ is generally assumed to vanish in the ultraviolet limit, the region bounded by C must be free of singularities. While the literature abounds with solutions to the DSEs where this is not the case, the condition does hold for perturbative QED and we assume it to hold henceforth.

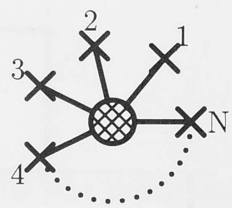
Converting to Euclidean space coordinates is also advantageous because it is easier to work with the positive-definite Euclidean metric than the Minkowski metric, especially when performing angular integrations, which we shall do in Secs. 6.3 and 7.4. The approach we have taken is to work in Minkowski space until are we ready to solve the DSEs, and then convert our propagators and vertices to Euclidean space using the prescription in App. B for ease of calculation. The reader may therefore assume that all work is in Minkowski space unless specifically told otherwise.

2.3 The Effective Action

Let us consider the higher order contributions to the connected Green's functions $G_C^{(N)}$. We see that any contribution to a connected Green's function is of the general form



, where



has the property that removing a single bare propagator cannot split it in two (The crosses on the external legs indicate that the propagators have been removed). We call this property 'one particle irreducibility'. The Feynman diagrams in Fig. (2.1) are one particle irreducible, but that in Fig. (2.3) is not. A one particle irreducible diagram with the external propagators truncated is called a 'proper diagram' and the corresponding function a 'proper function'. Proper functions are important in our later discussion

Figure 2.3: This diagram is not one-particle irreducible because removing the centre propagator turns it into two separate diagrams.

of the DSEs, as well as in QFT generally, so we present their generating functional Γ, also called the 'effective action', defined by the Legendre transformation

$$\Gamma[\phi_{cl}] = \mathcal{W}[J] - i\int d^dx J\phi_{cl}, \tag{2.3.1}$$

where ϕ_{cl} is defined by

$$\phi_{cl} = \frac{\delta}{i\delta J}\mathcal{W}. \tag{2.3.2}$$

It is easy to demonstrate that Γ is independent of J. Γ has the Taylor expansion

$$\Gamma[\phi_{cl}] = \sum_{N=0}^{\infty}\frac{1}{N!}\int d^dx_1\cdots d^dx_N\phi_{cl}(x_1)\cdots\phi_{cl}(x_N)\Gamma^{(N)}(x_1,\ldots,x_N), \tag{2.3.3}$$

where

$$\Gamma^{(N)}(x_1,\ldots,x_N) = \frac{\delta^N\Gamma[\phi_{cl}]}{\delta\phi_{cl}(x_1)\cdots\delta\phi_{cl}(x_N)}|_{\phi_{cl1}=\cdots=\phi_{clN}=0}, \tag{2.3.4}$$

are the N-point proper functions. Future reference to proper functions will not include the suffix $_{cl}$. Instead, it shall be implied by context.

The expression for the 2-point proper function is not clear from the above description so we derive it [28] rigorously.

$$\begin{aligned}
\delta^d(x-y) &= \frac{\delta}{\delta\phi(y)}\phi(x) \\
&= \frac{\delta}{\delta\phi(y)}\frac{\delta\mathcal{W}}{i\delta J(x)} \\
&= \int d^dz\frac{\delta J(z)}{\delta\phi(y)}\frac{\delta}{\delta J(z)}\frac{\delta\mathcal{W}}{i\delta J(x)} \\
&= \int d^dz\frac{\delta}{\delta\phi(y)}(i\frac{\delta\Gamma}{\delta\phi(z)})\frac{\delta^2\mathcal{W}}{i\delta J(z)\delta J(x)} \\
&= \int d^dz\frac{\delta^2\Gamma}{\delta\phi(y)\delta\phi(z)}\frac{\delta^2\mathcal{W}}{\delta J(z)\delta J(x)} \\
&\Rightarrow \Gamma^{(2)} = (G_C^{(2)})^{-1}.
\end{aligned} \tag{2.3.5}$$

We make extensive use of the effective action throughout much of this thesis.

2.4 Chiral Symmetry

Much of the work in this thesis is concerned with chiral symmetry, a mathematical symmetry exhibited by massless fermions. A chiral transformation is defined by

$$\psi \rightarrow e^{i\theta\gamma_5}\psi, \tag{2.4.1}$$

where ψ is a Dirac spinor, usually the electron, θ is a constant, and the operator $\frac{1}{2}(1 \pm \gamma_5)$ projects out the right- and left-handed components of ψ. The electron kinetic term is invariant to this operation;

$$\begin{aligned}
\bar{\psi}\gamma \cdot \partial\psi &= \psi^\dagger\gamma^0\gamma \cdot \partial\psi \\
&\longrightarrow \psi^\dagger e^{-i\theta\gamma_5}\gamma^0\gamma \cdot \partial e^{i\theta\gamma_5}\psi \\
&= \psi^\dagger\gamma^0 e^{i\theta\gamma_5}\gamma \cdot \partial e^{i\theta\gamma_5}\psi \\
&= \psi^\dagger\gamma^0\gamma \cdot \partial e^{i\theta\gamma_5}e^{-i\theta\gamma_5}\gamma \cdot \partial\psi \\
&= \bar{\psi}\gamma \cdot \partial\psi.
\end{aligned} \tag{2.4.2}$$

That the electron-photon interaction term $ieA^\mu\bar{\psi}\gamma_\mu\psi$ is also invariant under a chiral transformation is shown similarly. The mass term however transforms as

$$-im\bar{\psi}\psi \longrightarrow -im\bar{\psi}e^{i2\theta\gamma_5}\psi, \tag{2.4.3}$$

and is not invariant unless $m = 0$.

There is a physical picture for this. A massless particle moves at the speed of light for all observers. Its chirality is either right-handed for all observers or left-handed for all observers according to whether its spin is parallel or anti-parallel with its momentum. A massive particle on the other hand does not have the same chirality in all reference frames since the direction of its spin is invariant to Lorentz transformations whereas its velocity is reversed by a suitable boost. So in some reference frames it is right-handed, in others it is left-handed and in its rest frame the chirality is not defined. A massive particle therefore cannot have chiral symmetry.

It follows that an initially massless theory spontaneously generating mass *via* its interactions is equivalent to an initially chiral theory spontaneously breaking chiral symmetry. The terms "chiral symmetry breaking" and "spontaneous mass generation" can therefore be used interchangably when discussing fermions.

"Enough Euripides! What's going on?"

- Women's chorus, "Lysistrata", Aristophenes.

CHAPTER 3

Nonperturbative Techniques

3.1 The Need for Nonperturbative Techniques

The perturbative expansion outlined in Sec. 2.1 is a powerful technique allowing straightforward if lengthy calculations of probability amplitudes. Indeed, all physically measurable phenomena in QED can be found in perturbation theory if sufficiently many diagrams are calculated. However perturbation theory has its limitations. For one, it requires the coupling constant to be very small, as it is in QED. Since the perturbative calculation is an expansion in powers of the coupling constant, it becomes inaccurate as the coupling constant becomes large. This is a problem for theories such as QCD in which the coupling constant is significantly enhanced in the infrared, although perturbation theory can be used in the ultraviolet region where the theory is asymptotically free.

Another shortcoming of the perturbative approach is that some phenomena, such as bound states and chiral symmetry breaking, can not be detected perturbatively. As an example, consider the spontaneous breaking of chiral symmetry, known to take place in QCD [7, 14] and QED_3 [25, 29]. Corrections to the fermion (ie. electron/quark) propagator in QED_3/QCD are due to self-interaction via the emission and reabsorption of bosons (ie. photons/gluons). Perturbatively, we add corrections to the fermion propagator by adding bosons to it, as shown in Fig. (2.1). The fermion propagator is given by

$$S(p) = -i\frac{\mathcal{Z}(p^2)}{\gamma \cdot p + \mathcal{M}(p^2)} = -i\mathcal{Z}(p^2)\frac{\gamma \cdot p - \mathcal{M}(p^2)}{p^2 - \mathcal{M}^2(p^2)}, \tag{3.1.1}$$

where $\mathcal{Z}$ and $\mathcal{M}$ are scalar functions giving the renormalisation and the effective mass respectively. It follows from the form of Eq. (3.1.1) that any correction to the mass term must have an even number of γ matrices. However the addition of bosons to the self-energy adds two fermion-boson interactions, each contributing one γ matrix. So if the fermion has no bare mass, then its bare propagator is a purely odd γ matrix expression and will remain so to all orders in perturbation theory, since each correction adds two γ matrices. (The corresponding argument for spontaneous mass generation with scalars is less obvious but the result still applies.) Yet experiment tells us that chiral symmetry breaking does occur. Indeed, the pion is a Goldstone boson resulting from dynamical chiral symmetry breaking in QCD.

We clearly have need of nonperturbative methods. The method chosen for this thesis is the DSE although other techniques such as lattice gauge theory are also in common use. We draw the material in this chapter from Refs. [27, 28, 30].

3.2 The DSE

Let $-\Sigma(p)$ be the sum of all proper self-energy diagrams, represented graphically by . The full propagator is of the form

 (3.2.2)

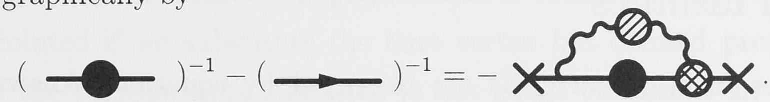
or equivalently,

$$\begin{aligned} S(p) &= S_0(p) + S_0(p)(-\Sigma(p))S_0(p) + S_0(p)(-\Sigma(p))S_0(p)(-\Sigma(p))S_0(p) + \cdots \\ &= S_0(p)(1+\Sigma(p)S_0(p))^{-1}, \\ &\Rightarrow S^{-1}(p) - S_0^{-1}(p) = \Sigma(p), \end{aligned} \tag{3.2.3}$$

where $S(p)$ is the full propagator and $S_0(p)$ is the bare one. Let us now consider the structure of Σ. The simplest contribution to Σ is the single emission and absorption of a photon. Higher contributions can be packaged as dressings on one of the internal electron propagator, the photon propagator or the electron-photon vertex. To see this, consider that once the photon is emitted, both the photon and the electron interact in all ways consistent with the eventual reabsorption of the photon. From our construction of the proper functions in Sec. 2.3 we see that those corrections which do not apply to either the electron or photon propagators are part of the three-point proper function, also called the dressed vertex, at which the electron is reabsorbed. The DSE in QED is therefore

$$S^{-1}(p) - S_0^{-1}(p) = -\int \frac{d^4p}{(2\pi)^4} D_{\mu\nu}(p-q)\gamma^\mu S(q)\Gamma^\nu_{\bar\psi A_\mu \psi}(q,p), \tag{3.2.4}$$

represented graphically by



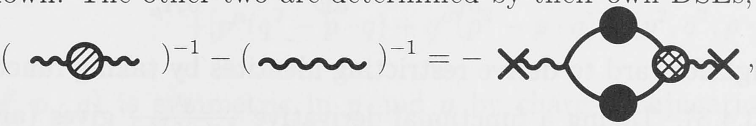
(3.2.5)

Note that only one vertex in the self-energy diagram is dressed, as to dress them both is to double count the vertex corrections.

We have given the above perturbation theory derivation of the DSE because we feel it is more intuitive. However the DSE, like any nonperturbative instrument, can be derived rigorously without using perturbation theory. Such a derivation, based on functional derivatives of the generating functional $\mathcal{Z}$, is given in Ref. [19].

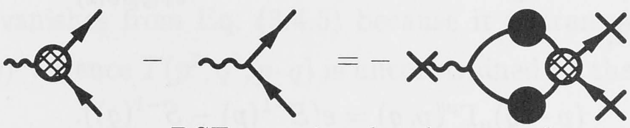
3.3 Truncating the DSE

The DSE expresses the dressed electron propagator in terms of the bare electron propagator, the dressed photon propagator and the dressed vertex. Of these, only the first is known. The other two are determined by their own DSEs,



(3.3.6)

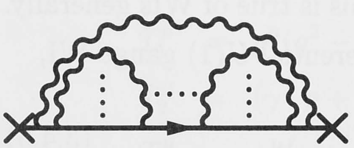
and



(3.3.7)

respectively. The propagator DSEs require the electron-photon vertex whose DSE requires the four-point proper function (called the 'kernel') and the series continues. The complete set of DSEs is an intractable, infinite tower of nonlinear equations which we truncate with carefully chosen *ansätze*.

Our first approximation is the 'quenched' approximation in which we ignore corrections to the photon propagator. The quenched approximation will apply for the remainder of this thesis but we will discuss various *ansätze* for the vertices. A simple choice is the rainbow approximation in which we also ignore the vertex corrections, so-called because the self-energy contains only diagrams of the form



The rainbow approximation is useful because it is (relatively) easy to use and allows mass generating solutions. However it is poor in terms of quantitative accuracy for reasons for explained below.

3.4 Ward Identites

The proper functions of QED are restricted by equations, known as Ward-Takahashi identities [31] (WTIs) which derive from gauge symmetry. We illustrate the derivation here because in Secs. 5.1 and 7.2 we will find similar identities resulting from SUSY. (Identities derived from a symmetry of a theory are known generally as Ward identities (WIs).) QED is invariant under gauge transformations δ_G, given by

$$\delta_G\psi = -i\theta\psi,\ \delta_G\bar{\psi} = i\theta\bar{\psi},\ \delta_G A_\mu = \frac{1}{e}\partial_\mu\theta. \tag{3.4.1}$$

Let Γ be the effective action defined in Sec. 2.3. Then

$$\delta_G\Gamma = 0. \tag{3.4.2}$$

We can expand this equation in terms of fields,

$$\delta_G\Gamma = \delta_G\psi\frac{\delta\Gamma}{\delta\psi} + \delta_G\bar{\psi}\frac{\delta\Gamma}{\delta\bar{\psi}} + \delta_G A_\mu\frac{\delta\Gamma}{\delta A_\mu}. \tag{3.4.3}$$

It is now straightforward to derive restricting identites by taking functional derivatives of Eq. (3.4.3). Taking a functional derivative $\frac{\delta^2}{\delta\psi(y)\delta\bar{\psi}(x)}$ gives (after a Fourier transform) the WTI,

$$(p-q)_\mu\Gamma^\mu(p,q) = e(S^{-1}(p) - S^{-1}(q)). \tag{3.4.4}$$

For later reference we present the WTI for scalar QED, derived similarly:

$$(p-q)_\mu\Gamma^\mu_{scalar}(p,q) = e(\Gamma^{(2)}(p^2) - \Gamma^{(2)}(q^2)), \tag{3.4.5}$$

This equation is normally written with the two-point proper vertices expressed as inverse propagators, since the two are usually equal. It will transpire in later chapters however that they are not always equal when auxiliary fields (described later) are used, which they must be in SUSY theories. The WTIs are satisfied at all levels of perturbation theory, including the bare level. This is true of WIs generally.

The first WI was found by Ward [32]. It was the differential $U(1)$ gauge WI,

$$\Gamma^\mu(p,p) = e\frac{\partial}{\partial p_\mu}S^{-1}(p), \tag{3.4.6}$$

of which the WTI is a generalisation. The WTI reduces to the original differential WI in the limit $q \to p$, or equivalently, at zero momentum transfer.

We see now why the rainbow approximation is considered a poor one, for the WTIs are violated if we substitute the bare vertex but dressed propagators into them. Thus the rainbow approximation is not gauge covariant, an unacceptable result in a theory whose power derives from gauge symmetry.

The methods and concepts used in this section will be used again later to overcome the difficulties of improving upon the rainbow approximation in SUSY theories.

3.5 The Ball-Chiu Vertex

The construction of vertices compliant with the WTIs and free of kinematic singularites was achieved by Ball and Chiu in 1980 [15]. They found the scalar-photon vertex to be of the form

$$\begin{aligned}\Gamma^{\mu}_{scalar}(p,q) &= (\Gamma^{(2)}(p^2)-\Gamma^{(2)}(q^2))\frac{(p+q)^{\mu}}{(p^2-q^2)} \\ &\quad +[p^{\mu}(q^2-p\cdot q)+q^{\mu}(p^2-p\cdot q)]T(p^2,q^2,p\cdot q), \end{aligned} \tag{3.5.1}$$

where $T(p^2,q^2,p\cdot q)$ is symmetric in p and q by charge conjugation invariance. The latter term vanishes from Eq. (3.4.5) because it is transverse to the photon momentum $(p-q)^{\mu}$. Hence $T(p^2,q^2,p\cdot q)$ is unconstrained by the WTI and requires an *ansatz*.

This is also true of the electron-photon Ball-Chiu vertex. The component which 'solves' the WTI is

$$\begin{aligned}\Gamma^{\mu}_{BC}(p,q) &= \frac{1}{2}\frac{ie}{p^2-q^2}(\gamma\cdot p+\gamma\cdot q)(A(p^2)-A(q^2))(p+q)^{\mu} \\ &\quad +ie\frac{1}{2}(A(p^2)+A(q^2))\gamma^{\mu}+\frac{ie}{p^2-q^2}(B(p^2)-B(q^2))(p+q)^{\mu}, \end{aligned} \tag{3.5.2}$$

and the transverse component is a linear combination of the following tensors:

$$T_1^{\mu} = p^{\mu}(q^2-p\cdot q)+q^{\mu}(p^2-p\cdot q), \tag{3.5.3}$$
$$T_2^{\mu} = (\gamma\cdot p+\gamma\cdot q)T_1^{\mu}, \tag{3.5.4}$$
$$T_3^{\mu} = \gamma^{\mu}(p-q)^2-(\gamma\cdot p-\gamma\cdot q)(p-q)^{\mu}], \tag{3.5.5}$$
$$T_4^{\mu} = [p^{\mu}(q^2-p\cdot q)+q^{\mu}(p^2-p\cdot q)]\sigma^{\mu\nu}q_{\mu}p_{\nu}, \tag{3.5.6}$$
$$T_5^{\mu} = \sigma^{\mu\nu}(p-q)_{\nu}, \tag{3.5.7}$$

$$
\begin{aligned}
T_6^\mu &= \gamma^\mu(p^2-q^2)-(\gamma\cdot p-\gamma\cdot q)(p+q)^\mu, &(3.5.8)\\
T_7^\mu &= \frac{p^2-q^2}{2}[\gamma^\mu(\gamma\cdot p+\gamma\cdot q)-p^\mu-q^\mu]+(p+q)^\mu\sigma^{\mu\nu}q_\mu p_\nu, &(3.5.9)\\
T_8^\mu &= \frac{1}{2}(\gamma\cdot p\gamma\cdot q\gamma^\mu-\gamma^\mu\gamma\cdot q\gamma\cdot p). &(3.5.10)
\end{aligned}
$$

We now have a vertex *ansatz* which is compliant with the WTI to use in the DSE. Several authors have used it in both QED_3 [8] and QED_4 [11]. That it leads to better results can be demonstrated with an examination of the chiral condensate, given by

$$\langle\bar\psi\psi\rangle = \mathrm{tr}S(x=0)=\frac{2}{\pi^2}\int_0^\infty dp\frac{p^2B(p^2)}{p^2A^2(p^2)+B^2(p^2)}. \tag{3.5.11}$$

This quantity is gauge invariant, hence we can test our solution to the DSE by finding the corresponding chiral condensate over a range of gauges. Fig. (6.3) in Sec. 6.3 convincingly demonstrates that the BC vertex in QED_3 is more compliant with gauge symmetry than the bare one.

3.6 The CJT Effective Potential

The Legendre transform performed in Eq. (2.3.1) removed the one particle reducible Feynman diagrams from the connected Green's functions. We will attempt to use an effective action which leaves out two particle reducible diagrams [33]. First we define the generating functional for Green's functions of nonlocal, composite fields:

$$\mathcal{Z}(J,K)=\int\mathcal{D}\phi e^{i(S(\phi)+\int d^dx\phi(x)J(x)+\frac{1}{2}\int d^dxd^dy\phi(x)K(x,y)\phi(x))}. \tag{3.6.1}$$

Defining

$$\mathcal{W}(J,K)=\frac{1}{i}\ln\mathcal{Z}(J,K), \tag{3.6.2}$$

puts us in a position to define the CJT effective action [33]

$$
\begin{aligned}
\Gamma(\phi,G) &= \mathcal{W}(J,K)-\int d^dx\phi(x)J(x)-\frac{1}{2}\int d^dxd^dy\phi(x)K(x,y)\phi(x)\\
&\quad -\frac{1}{2}\int d^dxd^dyG(x,y)K(y,x), &(3.6.3)
\end{aligned}
$$

using the definitions

$$
\begin{aligned}
\frac{\delta\mathcal{W}(J,K)}{\delta J(x)} &= \phi(x), &(3.6.4)\\
\frac{\delta\mathcal{W}(J,K)}{\delta K(x,y)} &= \frac{1}{2}(\phi(x)\phi(y)+G(x,y)). &(3.6.5)
\end{aligned}
$$

It follows that

$$\frac{\delta\Gamma(\phi,G)}{\delta\phi(x)} = -J(x) - \int d^dyK(x,y)\phi(y), \tag{3.6.6}$$

$$\frac{\delta\Gamma(\phi,G)}{\delta G(x,y)} = -\frac{1}{2}K(x,y). \tag{3.6.7}$$

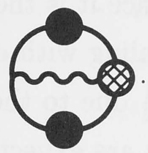
A further (and much more difficult) calculation reveals [33] that $G(x,y)$ is the propagator. This is a significant result as we can set $\phi(x)$ to be constant and take $G(x,y)$ to be a function only of $x-y$. We then define the effective *potential* by

$$V(\phi,G)\int d^dx = -\Gamma(\phi,G)|_{\text{translation invariant}}, \tag{3.6.8}$$

giving us that

$$\begin{aligned} V(\phi,G) &= -\frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\ln\det G_0(p)G^{-1}(p) \\ &\quad -\frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\mathrm{Tr}[G_0^{-1}(p)G(p)-1] + V_2(\phi,G), \end{aligned} \tag{3.6.9}$$

where $G_0(p)$ signifies the bare (classical) propagator and $V_2(\phi,G)$ is the sum of all two particle irreducible vacuum diagrams, which in the case of QED is

$$V_2(S,D_{\mu\nu}) =$$  . (3.6.10)

Setting $\frac{\delta V}{\delta G}=0$ gives us

$$0 = \frac{1}{2}G^{-1}(p) - \frac{1}{2}G_0^{-1}(p) + \frac{\delta V_2}{\delta G}, \tag{3.6.11}$$

where

$$\frac{\delta V_2}{\delta G} = -\frac{1}{2}\Sigma(\phi,G), \tag{3.6.12}$$

and Σ is the self energy. A quick inspection verifies that Eq. (3.6.11) is the DSE! Therefore, solutions to the DSE are stationary points of the CJT effective potential!

Typically we find two solutions when solving the DSE, one of them chirally symmetric, the other not so. Our problem is to find the dynamically favoured one. To demonstrate the application of the CJT effective potential to this task we imagine it has the generic form given in Fig. (3.1) (In reality the functional is multidimensional but the figure is intended only for illustration). There are two stationary points in this figure, labelled a and b. a is a minimum of the functional whereas b is a maximum. Since it is minima that are stable, the dynamically

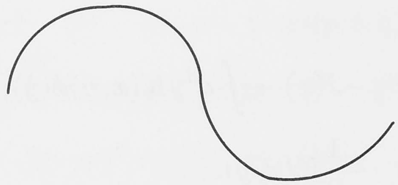


Figure 3.1: A generic CJT effective potential (vertical axis) as a functional of the propagator G. In reality the horizontal axis is multidimensional. a represents the stable solution to the DSE. b is unstable.

preferred solution to the DSE is the one whose effective potential is lower. Taking V_{A} and V_{S} to be the values of the effective potential corresponding to the chirally asymmetric and chirally symmetric solutions to the DSE respectively, the idea is to find $V_{\mathrm{A}}[S] - V_{\mathrm{S}}[S]$. The chirally symmetric solution is preferred if this difference is positive and *vice versa.*

We can substitute the DSE into Eq. (3.6.9) to simplify it but only at the stationary points [4]. From Eq. (3.6.12) this gives us $V_2 = -\frac{1}{2}\int \frac{d^dp}{(2\pi)^d}\mathrm{Tr}[\Sigma(p)S(p)]$.

We find $V(S)$ for QED since it is the SUSY form of this theory that we intend to use it for. We are now dealing with charged fermions instead of neutral bosons so a factor of $+1$ replaces $-\frac{1}{2}$ due to the change in statistics. We also neglect the ψ dependence since the fields are expected to have a zero expectation value in the absence of sources. So

$$V[S] = \int \frac{d^dp}{(2\pi)^d}(\mathrm{Tr}\ln[S_0^{-1}(p)S(p)] + \frac{1}{2}\mathrm{Tr}[1 - S_0^{-1}(p)S(p)]). \qquad (3.6.13)$$

Strictly speaking, there should also be a bosonic term giving dependence on the photon propagator, but since we use the quenched approximation throughout this thesis, the photon contribution to the effective action is exactly zero.

We give this derivation here to illustrate the principles behind the CJT effective potential. We will examine it in the context of SQED after an explanation of SUSY.

"Is this the real life? Is this just fantasy?"

- "Bohemian Rhapsody", Queen.

CHAPTER 4

Basic Concepts of SUSY

4.1 The Motivation for SUSY: The Coleman-Mandula Theorem

SUSY developed out of attempts during the 1960s to unify gravity with the gauge forces. After a decade of unsuccessful attempts, it was proved by Coleman and Mandula [2], with a small number of reasonable assumptions, that it is not possible to unite gravity with gauge forces in a physically realistic theory. Their "no-go" theorem stated that within the context of Lie algebras, the Lie group of symmetries of the S-matrix in any physically realistic theory is a direct product of the Poincaré symmetry and the internal symmetries. Theories whose Lie symmetry groups are not of this form display unphysical behaviour such as discrete scattering angles whereas scattering angles in the real world are observed to be continuous. This was best summarised by Witten [4] who observed that additional spacetime symmetries to energy, momentum and angular momentum overconstrain the elastic scattering angles.

Far from the being the end of attempts to unify gauge forces and gravity, the Coleman-Mandula theorem demonstrated the way forward. Haag, Sohnius and J.T. Lopuszański exploited a loophole in the no-go theorem [3, 4]. By adding fermionic generators, generalising the Lie algebra of the symmetry generators to a graded Lie algebra (described in Sec. 4.2) and considering the constraints imposed by representation theory, they constructed the SUSY algebra, consisting of the Poincaré, internal and SUSY generators. The significance of the SUSY algebra is that it is not a direct product of the Poincaré algebra, or of the gauge group algebra, with the rest of the symmetry group, yet it places no unphysical constraints on scattering angles. The Coleman-Mandula theorem is therefore circumvented.

4.2 The Algebraic Basis of SUSY

As mentioned in the previous section, the symmetry group in a SUSY theory contains fermionic generators in addition to bosonic ones, and the Lie algebra is generalised to a graded Lie algebra, also called a superalgebra. If B represents any bosonic generator and Q any fermionic (SUSY) generator, then the generalised Jacobi identities that define a graded Lie algebra are

$$[[B_1, B_2], B_3] + [[B_3, B_1], B_2] + [[B_2, B_3], B_1] = 0, \tag{4.2.1}$$

$$[[B_1, B_2], Q_3] + [[Q_3, B_1], B_2] + [[B_2, Q_3], B_1] = 0, \tag{4.2.2}$$

$$\{[B_1, Q_2], Q_3\} + \{[Q_3, B_1], Q_2\} + [\{Q_2, Q_3\}, B_1] = 0, \tag{4.2.3}$$

$$[\{Q_1, Q_2\}, Q_3] + [\{Q_3, Q_1\}, Q_2] + [\{Q_2, Q_3\}, Q_1] = 0, \tag{4.2.4}$$

where $[,]$ represents the commutator and $\{,\}$ the anticommutator. The SUSY generators are represented by $Q^i_\alpha, \bar{Q}_{i\dot{\alpha}}$ where the Greek indices are spinor indices. The SUSY generators have spin half, and therefore change the statistics of whatever field they act upon. Consequently, fermions are mapped by SUSY transformations into bosons and *vice versa*. Each particle in a SUSY theory is therefore part of a 'multiplet' containing both bosons and fermions. The full SUSY algebra is presented in App. C. For now we are content to discuss a couple of the more important (anti-)commutation relations. Arguably the most important is

$$\{Q^i_\alpha, \bar{Q}_{j\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}}\delta^i_j P_\mu, \tag{4.2.5}$$

where P_μ is the translation operator and $i, j = 1, 2, ..., N$ for some positive integer N. SUSY with $N > 1$ is known as 'extended' SUSY.

The other important equation in the SUSY algebra we wish to mention is

$$[Q^i_\alpha, P_\mu] = [\bar{Q}_{j\dot{\beta}}, P_\mu] = 0. \tag{4.2.6}$$

This tells us that a SUSY transformation alters neither the energy nor the momentum of any state it acts upon, so all members of a SUSY multiplet must have the same mass.

This thesis deals exclusively with the special case $N = 1$ and the lower case Latin indices will henceforth be dropped. We will also be using a four-component representation† given by

$$Q = \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot{\beta}} \end{pmatrix}, \tag{4.2.7}$$

where Q is Majorana.

Eq. (4.2.5) gives a natural inroad to gravity [4, 6] since local translations are generated when the SUSY transformations are governed by local parameters. This is known as 'supergravity' [6, 34]. Our work does not address supergravity and the parameters of SUSY transformations will henceforth be global.

A single SUSY transformation is an injective mapping whereas two successive transformations, being a translation (see Eq. (4.2.5)), form a bijective map. This is only possible if the number of fermionic states in a multiplet is equal to the number of bosonic states, the *fermions = bosons* rule. We remark that there are representations of the SUSY algebra for which the *fermions = bosons* rule does not apply. The adjoint representation, where it is the generators themselves which represent the algebra, is immune to this rule since the translation operator is zero. The other class of exceptions are non-linear representations, which do not make use of vector spaces of fields. This thesis uses the standard linear representation of SUSY and so the *fermions = bosons* rule applies throughout.

4.3 Spontaneous SUSY Breaking

Particles in a SUSY theory come in multiplets of the same mass and charge because of Eq. (4.2.6). Since we do not observe this in nature it follows that SUSY is a broken symmetry.

The condition that SUSY be spontaneously broken is that the ground state has positive energy. To see that this is so, set $\beta = \alpha$ in Eq. (4.2.5) and sum. This gives us

$$\sum_{\alpha=1}^{2} \{Q^i_\alpha, \bar{Q}_{j\dot{\alpha}}\} = 4\delta^i_j E, \tag{4.3.1}$$

where E is energy, since all the σ matrices have trace equal to zero except for σ^0.

†The numerical work in this thesis is in four-component $SQED_3$. Because the irreducible representation of the Clifford algebra in 2+1 dimensions has two-component spinors, many authors (for example see [18, 24]) consider a SUSY theory in this lower dimensional spacetime with one four-component generator to have two, two-component generators and therefore refer to it as an $N = 2$ theory.

As a consequence, if we denote the vacuum state by $|0\rangle$ then

$$E|0\rangle = 0 \Leftrightarrow Q|0\rangle = 0 \text{ and } \bar{Q}|0\rangle = 0 \quad \text{for all } Q. \tag{4.3.2}$$

It follows from this that *the ground state energy in a theory in which SUSY holds is exactly zero.* Equivalently, *SUSY is spontaneously broken if and only if the ground state energy is not exactly zero.* This will be of consequence in our discussion of the CJT effective potential. Another consequence of Eq. (4.3.1) is that E can only have nonnegative values since $\{Q^i_\alpha, \bar{Q}_{j\dot{\beta}}\}$ is positive definite, so if the ground state energy is not zero then it must be positive.

Witten [35] used the *fermions = bosons* rule to show that if the quantity

$$\text{zero energy fermionic states} - \text{zero energy bosonic states}, \tag{4.3.3}$$

known as the 'Witten index', is nonzero, then the theory must have a zero energy ground state so SUSY is not dynamically broken. After refining his argument to deal with gauge symmetry, he then went on to show that SUSY is not broken in SQED. His arguments can be avoided by certain classes of non-Abelian theories such as SUSY QCD when the number of flavours is less than the number of colours [?, ?]. However this is a non-Abelian effect and does not apply to SQED.

4.4 Other Incentives for SUSY

While circumventing the Coleman-Mandula theorem was the original motivation for SUSY, SUSY has other attractive features leading theorists to believe it is physical.

The first of these is that SUSY theories have much improved ultraviolet behaviour. So much so that theorists initially hoped that SUSY theories would be super-renormalisable. This was excessively optimistic, although physical parameters such as mass are not renormalised. In ordinary QFT notation (component formalism), this was seen to result from 'miraculous cancellations' between bosonic and fermionic loops. With the advent of 'superfields', described in Sec. 4.7, this result was proved rigorously and is now refered to as the 'nonrenormalisation theorem'. There have been attempts [20] to extend the nonrenormalisation theorem to eliminate the possibility of dynamical mass generation altogether in both $SQED_3$ and $SQED_4$. These have involved some controversy and we will discuss them in further detail in later chapters.

Another attractive feature is that extended SUSY offers a possible solution to the hierarchy problem. It is known that SUSY must be spontaneously broken and while either all SUSY generators are broken or none are in a theory with only global SUSY, there is reason to believe that a heirarchy of SUSY breakdowns could occur in an extended supergravity theory, which could justify the enormous gap between the electroweak scale of 100 GeV and the Planck scale of 10^{19} GeV.

4.5 SUSY Multiplets: The Construction of SUSY Theories

For the duration of this chapter I will present results in $3+1$ dimensions. The reduction to $2+1$ dimensions is complicated by subtleties involving the charge conjugation and parity matrices and will be dealt with in Sec. 6.2.

The simplest multiplet, and the one usually shown first in any SUSY textbook, is the 'chiral' multiplet, given by

$$\begin{aligned}
\delta_S a &= -i\bar{\zeta}\psi \\
\delta_S b &= \bar{\zeta}\gamma_5\psi \\
\delta_S \psi &= (f+i\gamma_5 g)\zeta + i\gamma\cdot\partial(a+i\gamma_5 b)\zeta \\
\delta_S f &= \bar{\zeta}\gamma\cdot\partial\psi \\
\delta_S g &= i\bar{\zeta}\gamma_5\gamma\cdot\partial\psi,
\end{aligned} \tag{4.5.1}$$

where a, b, f and g are real, ζ and ψ are Majorana and $\delta_S = \bar{\zeta}Q$. SUSY transformations obey commutator, and not anticommutator equations, since ζ is fermionic. The commutator of two SUSY transformations is

$$[\delta_1, \delta_2]X = 2\bar{\zeta}_2\gamma^\mu\zeta_1\partial_\mu X, \tag{4.5.2}$$

where X is any component of the multiplet.

The chiral multiplet is a special case of the "general multiplet",

$$\begin{aligned}
\delta_S C &= \bar{\zeta}\gamma_5\chi \\
\delta_S \chi &= (M+i\gamma_5 N)\zeta + i\gamma^\mu(A_\mu + i\gamma_5\partial_\mu C)\zeta \\
\delta_S M &= \bar{\zeta}(\partial\!\!\!/\chi + i\lambda) \\
\delta_S N &= i\bar{\zeta}\gamma_5(\partial\!\!\!/\chi + i\lambda) \\
\delta_S A_\mu &= \bar{\zeta}\gamma_\mu\lambda - i\bar{\zeta}\partial_\mu\chi \\
\delta_S \lambda &= \frac{1}{2}(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu)\partial_\mu A_\nu\zeta + i\gamma_5 D\zeta \\
\delta_S D &= i\bar{\zeta}\gamma_5\partial\!\!\!/\lambda.
\end{aligned} \tag{4.5.3}$$

To recover the chiral multiplet, set $\lambda = D = 0$ and $A_\mu = \partial_\mu a$.

The task now is to construct theories from products of these multiplets whose actions are SUSY invariant. Chiral multiplets have various products. Consider two chiral multiplets, $\Phi_1 = (a_1, b_1; \psi_1; f_1, g_1)$ and $\Phi_2 = (a_2, b_2; \psi_2; f_2, g_2)$, say. Their simplest product is the dot product, $\Phi_3 = \Phi_1 \cdot \Phi_2$, which is commutative.

$$
\begin{aligned}
a_3 &= a_1 a_2 - b_1 b_2 \\
b_3 &= a_1 b_2 + a_2 b_1 \\
\psi_3 &= (a_1 - i\gamma_5 b_1)\psi_2 + (1 \longleftrightarrow 2) \\
f_3 &= a_1 f_2 + b_1 g_2 + a_2 f_1 + b_2 g_1 + i\bar{\psi}_1 \psi_2 \\
g_3 &= a_1 g_2 - b_1 f_2 + a_2 g_1 - b_2 f_1 + \bar{\psi}_1 \gamma_5 \psi_2 .
\end{aligned}
\tag{4.5.4}
$$

The "cross" product of Φ_1 and Φ_2, written $\Phi_1 \times \Phi_2$, is a general multiplet. This operation is also commutative.

$$
\begin{aligned}
C &= a_1 a_2 + b_1 b_2 \\
\chi &= (b_1 - i\gamma_5 a_1)\psi_2 + (1 \longleftrightarrow 2) \\
M &= b_1 f_2 + a_1 g_2 + (1 \longleftrightarrow 2) \\
N &= b_1 g_2 - a_1 f_2 + (1 \longleftrightarrow 2) \\
A_\mu &= b_1 \overleftrightarrow{\partial}_\mu a_2 + b_2 \overleftrightarrow{\partial}_\mu a_1 + i\bar{\psi}_1 \gamma_\mu \gamma_5 \psi_2 \\
\lambda &= -i\gamma_5 f_2 \psi_1 - g_2 \psi_1 + i\gamma \cdot \partial b_2 \psi_1 - \gamma_5 \gamma \cdot \partial a_2 \psi_1 + (1 \longleftrightarrow 2) \\
D &= -2 f_1 f_2 - 2 g_1 g_2 - 2\partial_\mu a_1 \partial^\mu a_2 - 2\partial_\mu b_1 \partial^\mu b_2 + \bar{\psi}_1 \, \gamma \cdot \overleftrightarrow{\partial} \, \psi_2 .
\end{aligned}
\tag{4.5.5}
$$

The remaining product of chiral superfields is the "wedge" product, $\Phi_1 \wedge \Phi_2$. This also produces a general multiplet but this operation anti-commutes.

$$
\begin{aligned}
C &= a_2 b_1 - a_1 b_2 \\
\chi &= (a_2 + i\gamma_5 b_2)\psi_1 - (1 \longleftrightarrow 2) \\
M &= a_2 f_1 - b_2 g_1 - (1 \longleftrightarrow 2) \\
N &= a_2 g_1 + b_2 f_1 - (1 \longleftrightarrow 2) \\
A_\mu &= a_2 \overleftrightarrow{\partial}_\mu a_1 + b_2 \overleftrightarrow{\partial}_\mu b_1 - \bar{\psi}_1 \gamma_\mu \psi_2 \\
\lambda &= (f_2 - i\gamma_5 g_2)\psi_1 + \gamma \cdot \partial (i a_2 - \gamma_5 b_2)\psi_1 \; - (1 \longleftrightarrow 2) \\
D &= 2 f_2 g_1 - 2 f_1 g_2 + 2\partial_\mu b_2 \partial^\mu a_1 - 2\partial_\mu a_2 \partial^\mu b_1 \\
&\quad + i\bar{\psi}_1 \gamma_5 \gamma \cdot \partial \psi_2 - i\bar{\psi}_2 \gamma_5 \gamma \cdot \partial \psi_1 .
\end{aligned}
\tag{4.5.6}
$$

Finally, there is the product of general multiplets. Consider the general multiplets $V_1 = (C_1; \chi_1; M_1, N_1, A_1^\mu; \lambda_1; D_1)$ and $V_2 = (C_2; \chi_2; M_2, N_2, A_2^\mu; \lambda_2; D_2)$. Then $V_1 \cdot V_2$ is given by

$$\begin{aligned}
C_3 &= C_1C_2 \\
\chi_3 &= C_1\chi_2 + C_2\chi_1 \\
M_3 &= C_1M_2 + C_2M_1 - \frac{1}{2}\bar{\chi}_1\gamma_5\chi_2 \\
N_3 &= C_1N_2 + C_2N_1 + \frac{1}{2}i\bar{\chi}_1\chi_2 \\
A_3^\mu &= C_1A_2^\mu + C_2A_1^\mu + \frac{1}{2}i\bar{\chi}_1\gamma^\mu\gamma_5\chi_2 \qquad (4.5.7)\\
\lambda_3 &= C_1\lambda_2 - \frac{1}{2}(i\gamma_5M_2\chi_1 - N_2\chi_1 - \gamma_5\gamma\cdot A_2\chi_1) + i\gamma\cdot\partial C_2\chi_1 + (1 \longleftrightarrow 2) \\
D_3 &= C_1D_2 + C_2D_1 - M_1M_2 - N_1N_2 - A_1^\mu A_{2\mu} \\
&\quad -\partial_\mu C_1\partial^\mu C_2 + i\bar{\lambda}_2\chi_1 + i\bar{\lambda}_1\chi_2 - \frac{1}{2}\partial_\mu\bar{\chi}_2\gamma^\mu\chi_1 - \frac{1}{2}\partial_\mu\bar{\chi}_1\gamma^\mu\chi_2.
\end{aligned}$$

This operation, like the corresponding one for chiral multiplets, is commutative and all the above multiplet products are associative.

How is a Lagrangian extracted from a multiplet? The answer is simple. A Lagrangian must be either the f component of a chiral multiplet or the D component of a general multiplet because these transform as pure divergences under SUSY, leaving the action invariant. Indeed, the D component of $\Phi\times\Phi$ (written $[\Phi\times\Phi]_D$) taken from Eq. (4.5.5) suggestively contains the kinetic terms of both scalars and fermions. The most general renormalisable Lagrangian for a single chiral multiplet is the Wess-Zumino (WZ) Lagrangian,

$$\mathcal{L} = -[\frac{1}{2}\Phi\times\Phi]_D - [\frac{1}{2}m\Phi\cdot\Phi + \frac{e^3}{3}\Phi\cdot\Phi\cdot\Phi]_f = \mathcal{L}_0 + \mathcal{L}_m + \mathcal{L}_e, \qquad (4.5.8)$$

where

$$\begin{aligned}
\mathcal{L}_0 &= \frac{1}{2}(|f|^2 + |g|^2 + |\partial_\mu a|^2 + |\partial_\mu b|^2 - \bar{\psi}\gamma\cdot\partial\psi) \\
\mathcal{L}_m &= -m(af + bg + i\frac{1}{2}\bar{\psi}\psi) \\
\mathcal{L}_e &= -e((a^2-b^2)f + 2abg + i\bar{\psi}_1(a - i\gamma_5 b)\psi_2). \qquad (4.5.9)
\end{aligned}$$

The theory needs source terms if we are to obtain its Green's functions. The source terms for particles in a chiral multiplet Φ also form a chiral multiplet, J_Φ, defined by

$$\delta_S J_f = -i\bar{\zeta}\eta$$

$$\begin{aligned}\delta_S J_g &= \bar{\zeta}\gamma_5\eta \\ \delta_S\eta &= (f+i\gamma_5 g)+i\gamma\cdot\partial(a+i\gamma_5 b)\zeta \\ \delta_S J_a &= \bar{\zeta}\gamma\cdot\partial\eta \\ \delta_S J_b &= i\bar{\zeta}\gamma_5\gamma\cdot\partial\eta.\end{aligned} \tag{4.5.10}$$

We may now add source terms to $\mathcal{L}$ in Eq. (4.5.8) by adding $[\Phi\cdot J_\Phi]_f$. Similarly, if $\mathcal{W}[J_\Phi]$ is the generating functional of connected Green's functions, defined in Sec. 2.1, then the generating functional for proper functions is defined by the Legendre transform

$$\Gamma[\Phi_{cl}] = \mathcal{W}[J_\Phi] - i\int d^dx[J_\Phi\cdot\Phi_{cl}]_f, \tag{4.5.11}$$

where $\Phi_{cl} = (a_{cl}, b_{cl}; \psi_{cl}; f_{cl}, g_{cl})$.

The observant reader will have noticed that the 'kinetic' terms for f and g contain no derivatives. f and g are known as auxiliary fields and are given by equations of motion in terms of the other scalars. For example, in the WZ model just given, the Euler-Lagrange equations give f and g as

$$\begin{aligned} f &= -ma + e(a^2-b^2) \\ g &= -mb + 2eab. \end{aligned} \tag{4.5.12}$$

They therefore carry no information of their own when the classical equations of motion hold, ie. the fields are on mass-shell. Indeed, as long as the fields are on mass-shell, f and g can be removed from Eq. (4.5.1) and the SUSY algebra will still hold. However if the fields are off mass-shell, then the SUSY algebra is violated when the auxiliary fields are not present. This is explained in more detail in App. C. A common approach to constructing SUSY theories in higher dimensions or with multiple SUSY generators is to find the multiplets on mass-shell and then add the auxiliary fields as needed to complete the SUSY algebra off mass-shell. Finding the auxiliary fields is often difficult but is unfortunately necessary as Feynman diagrams containing loops do not obey the classical equations of motion. The D field in Eq. (4.5.3) is also an auxiliary field.

4.6 SQED4

The construction of a SUSY, $U(1)$ gauge invariant theory was achieved by Wess and Zumino [22] in 1974. We summarise their method here because we use it to construct $SQED_3$, given in Sec. 6.3.

A $U(1)$ gauge symmetric theory requires complex/Dirac fields whereas the multiplets presented in Eqs. (4.5.1) and (4.5.3) are real/Majorana. We therefore take our chiral multiplet to be $\Phi = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2)$. and seek a combination of Φ_1, Φ_2 and the general multiplet V which is gauge invariant.

An ordinary gauge transformation is effected by

$$\delta_G A_\mu = \frac{1}{e}\partial_\mu \theta, \quad \delta_G X_1 = \theta X_2, \quad \delta_G X_2 = -\theta X_1, \tag{4.6.1}$$

where δ_G is an infinitesimal $U(1)$ gauge transformation, θ a real function and $X_{1,2}$ stands for any member of the multiplet $\Phi_{1,2}$. We observe that this form does not commute with the SUSY transformations. The antidote for this affliction is to replace the single function θ with a chiral multiplet $\Lambda = (a_\Lambda, b_\Lambda; \psi_\Lambda; f_\Lambda, g_\Lambda)$. The super-covariant extension of (4.6.1) is

$$\delta_G V = \frac{1}{e}\partial\Lambda, \quad \delta_G \Phi_1 = \Lambda \cdot \Phi_2, \quad \delta_G \Phi_2 = -\Lambda \cdot \Phi_1, \tag{4.6.2}$$

where ∂ is the operator

$$\partial\Lambda = (b_\Lambda; \psi_\Lambda; f_\Lambda, g_\Lambda, \partial_\mu a_\Lambda; 0; 0). \tag{4.6.3}$$

A simple consistency check is that the form of Eq. (4.6.1) is the special case of Eq. (4.6.2) when all elements of Λ are zero except a. Still following Wess and Zumino we define

$$V_I = \frac{1}{2}(\Phi_1 \times \Phi_1 + \Phi_2 \times \Phi_2), \tag{4.6.4}$$
$$V_{II} = \Phi_1 \wedge \Phi_2, \tag{4.6.5}$$
$$V_a = V_I + V_{II}, \tag{4.6.6}$$
$$V_b = V_I - V_{II}, \tag{4.6.7}$$

whose gauge transformations are

$$\delta_G V_I = 2V_{II} \cdot \partial\Lambda, \tag{4.6.8}$$
$$\delta_G V_{II} = 2V_I \cdot \partial\Lambda, \tag{4.6.9}$$
$$\delta_G V_a = 2V_a \cdot \partial\Lambda, \tag{4.6.10}$$
$$\delta_G V_b = -2V_b \cdot \partial\Lambda. \tag{4.6.11}$$

We construct from these the gauge invariant expressions

$$V_a \cdot \mathrm{e}^{-2eV}, \tag{4.6.12}$$
$$V_b \cdot \mathrm{e}^{2eV}. \tag{4.6.13}$$

The problem of constructing a suitable combination of Φ_1, Φ_2 and V is reduced to that of finding a suitable combination of Eqs. (4.6.12),(4.6.13). The expression

$$\frac{1}{4}[V_a \cdot \mathrm{e}^{2eV} + V_b \cdot \mathrm{e}^{-2eV}]_D \tag{4.6.14}$$

has the desirable property of reducing to the free (massless) Lagrangian when the gauge coupling e is set to zero, and is therefore chosen.

We have one last step to take before writing the Lagrangian explicitly in terms of its component fields. Eq. (4.6.14) contains exponentials of V so the Lagrangian has infinitely many terms. To make the Lagrangian polynomial, and with the added benefit of having fewer particles to deal with, we take the WZ gauge in which all fields in the general mutiplet are taken to zero except for the photon A_μ, the photino λ, and the D. That this can be done by choosing $\Lambda = (0, -C; -\chi; -M, -N)$ is evident from Eq. (4.6.2). This reduces V to the 'vector' multiplet $(A_\mu; \lambda; D)$. The exponential expansion now terminates at $V \cdot V = V^2$.

Eq. (4.6.14) does not include the mass terms for the fields in the chiral multiplet. These are given by

$$-m[\Phi_1 \cdot \Phi_1 + \Phi_2 \cdot \Phi_2]_f. \tag{4.6.15}$$

It also fails to include the kinetic terms of the members of the vector multiplet. These are derived from a special submultiplet of the general multiplet V, known as the 'curl' multiplet,

$$dV = (\lambda; F_{\mu\nu}, D). \tag{4.6.16}$$

Eq. (4.6.16) is a multiplet only if $F_{\mu\nu}$ is a curl, which is true when $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The transformation laws, which follow easily since dV is a submultiplet of V, are

$$\begin{aligned} \delta_S \lambda &= \frac{1}{2}\sigma^{\nu\mu} F_{\mu\nu}\zeta + i\gamma_5 D\zeta \\ \delta_S F_{\mu\nu} &= \bar{\zeta}(\gamma_\nu \partial_\mu - \gamma_\mu \partial_\nu)\lambda \\ \delta_S D &= i\bar{\zeta}\gamma_5 \partial\!\!\!/ \lambda. \end{aligned} \tag{4.6.17}$$

It is straightforward to show using these transformations that the combination

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\bar{\lambda}\gamma \cdot \partial\lambda + \frac{1}{2}D^2, \tag{4.6.18}$$

is invariant under SUSY up to a divergence.

Finally, the Lagrangian of SQED is

$$\mathcal{L} = |f|^2 + |g|^2 + |\partial_\mu a|^2 + |\partial_\mu b|^2 - \bar{\psi}\gamma \cdot \partial\psi$$

$$-m(a^*f + af^* + b^*g + bg^* + i\bar{\psi}\psi)$$

$$-ieA^\mu(a^* \overleftrightarrow{\partial}_\mu a + b^* \overleftrightarrow{\partial}_\mu b + \bar{\psi}\gamma_\mu\psi)$$

$$-e[\bar{\lambda}(a^* + i\gamma_5 b^*)\psi - \bar{\psi}(a + i\gamma_5 b)\lambda]$$

$$+ieD(a^*b - ab^*) + e^2 A_\mu A^\mu(|a|^2 + |b|^2)$$

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\bar{\lambda}\gamma \cdot \partial\lambda + \frac{1}{2}D^2, \tag{4.6.19}$$

where $a = \frac{1}{\sqrt{2}}(a_1 + ia_2)$ etc.

It is necessary to mention before moving on that the WZ gauge is not SUSY covariant. Thus while SQED is SUSY by construction, the SUSY is not explicit. This means the Lagrangian (4.6.19) is no longer invariant under a SUSY transformation (even up to a divergence!). It is invariant however under a SUSY transformation *followed by a special gauge transformation* δ_{WZ} *that restores the WZ gauge.* The gauge chiral multiplet that does this is $\Lambda_{WZ} = (0, 0; -i\gamma \cdot A\zeta; -i\bar{\zeta}\lambda, \bar{\zeta}\gamma_5\lambda)$, so the set of transformations leaving Eq. (4.6.19) invariant, $\delta = \delta_S + \delta_{WZ}$, is given by

$$\begin{aligned}
\delta a &= -i\bar{\zeta}\psi \\
\delta b &= \bar{\zeta}\gamma_5\psi \\
\delta\psi &= [f + i\gamma_5 g + i\gamma \cdot \partial(a + i\gamma_5 b) - e\gamma \cdot A(a - i\gamma_5 b)]\zeta \\
\delta f &= \bar{\zeta}[\gamma \cdot \partial\psi + e[-a\lambda - ib\gamma_5\lambda + i\gamma \cdot A\psi]] \\
\delta g &= i\bar{\zeta}[\gamma_5\gamma \cdot \partial\psi + e[-\gamma_5\lambda - ib\lambda - i\gamma \cdot A\gamma_5\psi]]
\end{aligned} \tag{4.6.20}$$

$$\begin{aligned}
\delta A_\mu &= \bar{\zeta}\gamma_\mu\lambda \\
\delta\lambda &= \sigma^{\nu\mu}\partial_\mu A_\nu\zeta + i\gamma_5 D\zeta \\
\delta D &= i\bar{\zeta}\gamma_5\gamma \cdot \partial\lambda.
\end{aligned}$$

We will use Eq. (4.6.20) to derive the SUSY WIs in Secs. 5.1 and 7.2.

4.7 Superfields and Superspace

An introduction to SUSY is not complete without at least a brief description of superspace and superfields. Although not used in this thesis for reasons outlined below, the superfield notation is extremely elegant. Many perturbative calculations are more easily done with superfields than with ordinary fields, even with only one field present, especially in perturbation theory. SUSY remains explicit in all calculations, and many results, such as the perturbative nonrenormalisation theorem, acquire simple, elegant proofs.

A superfield is a polynomial in Grassman coordinates that represents the SUSY algebra. Superfields therefore constitute an alternative notation to multiplets.

The defining feature of Grassman numbers [36] is that they anticommute,

$$\theta_1\theta_2 = -\theta_2\theta_1. \tag{4.7.1}$$

It follows that

$$\theta_1\theta_1 = 0. \tag{4.7.2}$$

The Grassman derivative is defined as we näively expect,

$$\frac{\partial}{\partial\theta}\theta = 1, \tag{4.7.3}$$

but θ must be immediately to the right of the derivative operator, ie.

$$\frac{\partial}{\partial\theta_1}\theta_2\theta_1 = -\frac{\partial}{\partial\theta_1}\theta_1\theta_2 = -\theta_2. \tag{4.7.4}$$

Grassman integration is not so intuitive. It is given by

$$\int d\theta \;=\; 0, \tag{4.7.5}$$

$$\int d\theta\theta \;=\; 1. \tag{4.7.6}$$

In fact, with Grassman numbers, integration and differentiation are formally the same operation!

Most treatments of superspace take two-component spinors as their Grassman variables, and that is our convention here. The general superfield is given by

$$\begin{aligned} V \;=\; & C + \theta\chi + \bar{\chi}\bar{\theta} + \frac{1}{2}\theta\theta(M + iN) + \frac{1}{2}\bar{\theta}\bar{\theta}(M - iN) \\ & + \frac{1}{2}i\bar{\theta}\sigma^\mu\theta A_\mu + \theta^2\bar{\theta}(i\bar{\lambda} + \frac{1}{2}\sigma\cdot\partial\bar{\chi}) \\ & + \bar{\theta}^2\theta(i\lambda + \frac{1}{2}\sigma\cdot\partial\chi) - \frac{1}{2}\bar{\theta}^2\theta^2(D + \frac{1}{2}\Box C). \end{aligned} \tag{4.7.7}$$

We draw attention to the presence of fields from the general multiplet, given in Eq. (4.5.3). Note that it is not possible to add terms with greater powers of the Grassman coordinates. Since the Grassman variables θ and $\bar{\theta}$ each have two components, a product of more than two powers of θ or $\bar{\theta}$ is automatically zero by Eq. (4.7.2).

A SUSY transformation is effected by the operator

$$Q = \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot{\alpha}} \end{pmatrix}, \tag{4.7.8}$$

where

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - (\sigma^\mu\bar{\theta})_\alpha\partial_\mu,\ \bar{Q}^{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} - (\theta\sigma^\mu)^{\dot{\alpha}}\partial_\mu, \tag{4.7.9}$$

and the SUSY transformation laws are given by

$$\delta V = (\bar{\zeta}Q)\, V. \tag{4.7.10}$$

It is a straightforward though tiresome task to verify that the action of Eq. (4.7.9) on Eq. (4.7.7) is equivalent to the transformations given in Eq. (4.5.3).

We will also have use for super-covariant derivatives (analogous to gauge covariant derivatives), which anti-commute with the SUSY operator, given by

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + (\sigma^\mu\bar{\theta})_\alpha\partial_\mu,\ \bar{D}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + (\sigma^\mu\theta)_{\dot{\alpha}}\partial_\mu, \tag{4.7.11}$$

The most common use for these derivatives is to impose super-covariant restraints on the general superfield V. Imposing the condition

$$\bar{D}_{\dot{\alpha}}V = 0, \tag{4.7.12}$$

produces the chiral superfield

$$\Phi(x,\theta,\bar{\theta}) = \exp\left(\theta\sigma^\mu\bar{\theta}\partial_\mu\right)\Phi(x,\theta), \tag{4.7.13}$$

where

$$\Phi(x,\theta) = (A - iB) - i\theta\psi - \frac{1}{2}i\theta\theta(F + iG), \tag{4.7.14}$$

is the chiral superfield. In a similar way

$$D_\alpha V = 0, \tag{4.7.15}$$

generates the antichiral superfield

$$\bar{\Phi}(x,\theta,\bar{\theta}) = \exp\left(-\theta\sigma^\mu\bar{\theta}\partial_\mu\right)\bar{\Phi}(x,\bar{\theta}), \tag{4.7.16}$$

where

$$\bar{\Phi}(x,\bar{\theta}) = (A+iB) + i\bar{\psi}\bar{\theta} + \frac{1}{2}i\bar{\theta}\bar{\theta}(F-iG). \quad (4.7.17)$$

It is natural to expect that every multiplet product given in Sec. 4.5 has a corresponding superfield product. Superfields multiply like polynomials so the product of superfields is another superfield. The corresponding superfield combinations for each multiplet product are

$$\text{Dot Product} \quad \Phi_3 = \Phi_1\Phi_2, \quad (4.7.18)$$

$$\text{Cross Product} \quad V = \frac{1}{2}(\Phi_1\bar{\Phi}_2 + \bar{\Phi}_1\Phi_2), \quad (4.7.19)$$

$$\text{Wedge Product} \quad V = \frac{1}{2}(\Phi_1\bar{\Phi}_2 - \bar{\Phi}_1\Phi_2), \quad (4.7.20)$$

$$\text{Product of General Multiplets} \quad V_3 = V_1V_2. \quad (4.7.21)$$

The superfield is a function not only of the spacetime coordinate x^μ (via its component fields), but also of the Grassman coordinate θ. It therefore lives in an extension of Minkowski space known as 'superspace' where the usual spacetime coordinates are joined by four Grassman coordinates, forming an eight dimensional space. A SUSY transformation then has the interpretation of a translation in the Grassman coordinates of superspace. The superspace action is an integral over both Minkowski and Grassman coordinates of the superfield Lagrangian,

$$\mathcal{S} = \int d^4x\, d^2\bar{\theta}\, d^2\theta L_V, \quad (4.7.22)$$

where $\mathcal{L}_V$ is a general superfield, and

$$\mathcal{S} = \int d^4x\, d^2\theta \mathcal{L}_\Phi, \quad \mathcal{S} = \int d^4x\, d^2\bar{\theta}\mathcal{L}_{\bar{\Phi}}, \quad (4.7.23)$$

where $\mathcal{L}_{(\Phi,\bar{\Phi})}$ is a (anti)chiral superfield. Note that both of these superspace integrals will naturally eliminate all components of the Lagrangian except for that with the greatest power of the Grassman coordinate due to Eqs. (4.7.5),(4.7.6). This justifies our convenient choice of taking the Lagrangian from the D or f component of a multiplet.

We can now go on to define super-Green's functions, super-propagators, super-Feynman diagrams etc. It can be shown [6] that any non-zero super-Feynman diagram must generate an integral of the form given in Eq. (4.7.22). However mass and many other interaction terms, such as the cubic coupling in the WZ model, are of the form Eq. (4.7.23). It follows that no perturbative expansion can generate corrections, infinite or otherwise, to the mass or coupling constants, even if they are not zero to start with. This is known as the nonrenormalisation theorem. Its proof requires working knowledge of super-diagrams and the many identites of the super-covariant derivatives, and the time taken to describe them can not be justified since our work does not make use of them. The interested reader is referred to the many textbook explanations available (eg. [6, 34]).

As described here, the nonrenormalisation theorem is a perturbative result. In Sec. 7.4 we describe attempts to demonstrate a nonperturbative version, followed by our own analysis which is at variance with them.

For the sake of completeness we give the action of SQED in the superfield language. First we define the chiral superfields Φ_+, Φ_- to have the $U(1)$ gauge transformations

$$\Phi_+ \to e^{-i\Lambda}\Phi_+, \qquad \Phi_- \to e^{i\Lambda}\Phi_-. \tag{4.7.24}$$

The curl multiplet of Eq. (4.6.16) is equivalent to the chiral multiplet

$$W_\alpha = \frac{1}{4}\bar{D}^2 D_\alpha V. \tag{4.7.25}$$

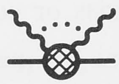
SQED can now be expressed as

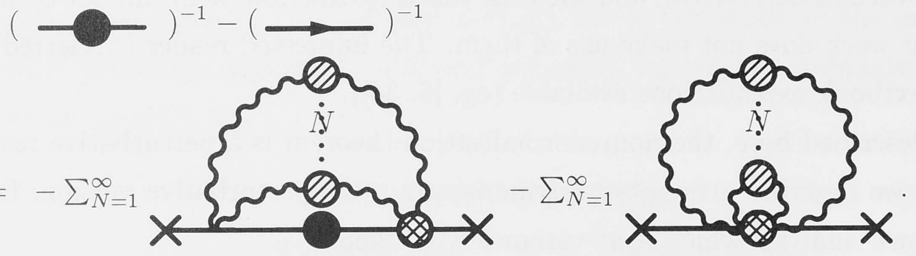
$$\begin{aligned} \mathcal{S} &= \frac{1}{8}\int d^4x d^2\bar{\theta} d^2\theta (\bar{\Phi}_+ e^{2V}\Phi_+ + \bar{\Phi}_- e^{-2V}\Phi_-) + \frac{1}{16}\int d^4x (d\theta)^2 W^2 \\ &\quad -m\int d^4x d^2\theta \Phi_+\Phi_- - m\int d^4x d^2\bar{\theta}\bar{\Phi}_+\bar{\Phi}_-. \end{aligned} \tag{4.7.26}$$

4.8 Components *vs.* Superfields

We have two different notations for describing SUSY theories, component notation and superfield notation, each with its own advantages and drawbacks.

The superfield approach is manifestly SUSY whereas the manifest SUSY of the component notation is spoiled by the choice of WZ gauge. Superfields also have the advantage that the whole superfield is treated as a single object whereas in the component notation, each field in the multiplet must be treated separately. This greatly reduces the number of Feyman diagrams to be calculated and we mentioned in Sec. 4.7 that the super-Feynman diagrams are also often easier to calculate.

However, the superfield approach can be somewhat awkward in nonperturbative applications. For example, the presence of exponentials in the superspace Lagrangian for SQED means that there are infinitely many chiral superfield-general superfield vertices of the form  . It follows that there are infinitely many terms in the self-energy so that the superfield DSE is given by



(4.8.1)

Since it is generally not possible to solve for infinitely many terms without further constraint, many authors [20, 24] truncate Eq. (4.8.1) at an arbitrary point, usually by dropping all two particle irreducible diagrams.

The other disadvantage of the superfield notation is that it is plagued by spurious infrared divergences. Some authors get around this problem [20] by introducing a mass which softly breaks SUSY, others [24] avoid them with carefully chosen approximations.

We elected, after careful consideration, to work in the component formalism. Similar work to that done in this thesis was done simultaneously using superfields by Campbell-Smith and Mavromatos [24] so our choice was a fortunate one.

"I saw two shooting stars last night,
I wished on them, but they were only satellites,
It's wrong to wish on space hardware,
And I wish, I wish...."

- "A New England", Billy Bragg.

CHAPTER 5

Nonperturbative methods in SQED

5.1 2-Point SWIs

We will study the electron DSE of SQED in the upcoming chapters and find that SUSY has added a few terms. These additional terms contain propagators for the super-partners of the particles already present in QED. The propagators of the photon's partners remain bare in the quenched approximation but those of the electron's partners do not. To leave them bare would be a flagrant and unnecessary violation of SUSY. While the partners do have their own DSEs, to solve them simultaneously with that of the electron is a near-intractable problem and a numerical nightmare.

Fortunately there exists a much easier way. Sec. 3.4 gave the derivation of WTIs from gauge symmetry in ordinary QED. The same reasoning can be applied to SQED to derive SWIs which relate the additional two-point Green's and proper functions to those of the electron.

Let us consider the propagators, derived from the generator, $\mathcal{W}$, of connected Green's functions. $\mathcal{W}$, of course, is invariant under SUSY. By expanding its variation in terms of the sources we obtain [19]

$$\begin{aligned}\delta\mathcal{W} &= (\delta_S+\delta_{WZ})\mathcal{W} = \delta J_f\frac{\delta\mathcal{W}}{\delta J_f}+\delta J_g\frac{\delta\mathcal{W}}{\delta J_g}\\ &+\delta J_a\frac{\delta\mathcal{W}}{\delta J_a}+\delta J_b\frac{\delta\mathcal{W}}{\delta J_b}+\delta\bar{\eta}_\psi\frac{\delta\mathcal{W}}{\delta\bar{\eta}_\psi}\\ &+\delta J_f^*\frac{\delta\mathcal{W}}{\delta J_f^*}+\delta J_g^*\frac{\delta\mathcal{W}}{\delta J_g^*}\end{aligned}$$

$$+\delta J_a^* \frac{\delta \mathcal{W}}{\delta J_a^*} + \delta J_b^* \frac{\delta \mathcal{W}}{\delta J_b^*} + \delta \eta \frac{\delta \mathcal{W}}{\delta \eta_\psi}$$
$$+\delta J_{A^\mu} \frac{\delta \mathcal{W}}{\delta J_{A^\mu}} + \delta \eta_\lambda \frac{\delta \mathcal{W}}{\delta \eta_\lambda} + \delta J_D \frac{\delta \mathcal{W}}{\delta J_D}. \tag{5.1.1}$$

We are now in a position to place some very useful restrictions on the scalar propagators. Following the path of Iliopoulos and Zumino [19], we first take the equation

$$\frac{\delta^2}{\delta J_a^* \delta \eta_\psi} \delta \mathcal{W} = 0, \tag{5.1.2}$$

and then set the sources to zero. This produces (after a Fourier transform)

$$S(p) = iD_{af}(p^2) - i\gamma \cdot p D_{aa}(p^2). \tag{5.1.3}$$

Similarly, from

$$\frac{\delta^2}{\delta J_f^* \delta \eta_\psi} \delta \mathcal{W} = 0, \tag{5.1.4}$$

we obtain

$$\gamma \cdot p S(p) = -iD_{ff}(p^2) + i\gamma \cdot p D_{af}(p^2). \tag{5.1.5}$$

Substituting in the fermion propagator

$$S(p) \equiv \langle \psi \bar{\psi} \rangle = \frac{-i}{\gamma \cdot p A(p^2) + B(p^2)}, \tag{5.1.6}$$

gives the scalar propagators

$$D_{aa}(p^2) = \frac{A(p^2)}{p^2 A^2(p^2) - B^2(p^2)}, \tag{5.1.7}$$

$$D_{a^* f}(p^2) = D_{b^* g}(p^2) = \frac{B(p^2)}{p^2 A^2(p^2) - B^2(p^2)} = \mathcal{M}(p^2) D_{aa}(p^2), \tag{5.1.8}$$

and

$$D_{f^* f}(p^2) = D_{g^* g}(p^2) = \frac{p^2 A(p^2)}{p^2 A^2(p^2) - B^2(p^2)}, \tag{5.1.9}$$

which may then be substituted directly into the DSE, Eq. (3.2.4).

The above SWIs and scalar propagators were first found by Iliopoulos and Zumino [19] in 1974. We find something unexpected when we derive the two-point proper functions from the effective action Γ, described in Sec. 2.3.

We define $\Gamma_{X..Z} \equiv \frac{\delta^n \Gamma}{\delta X \ldots \delta Z}$. Expanding $\delta\Gamma$ in terms of its component fields produces

$$\delta\Gamma = (\delta_S + \delta_{WZ})\Gamma = \delta f \frac{\delta \Gamma}{\delta f} + \delta g \frac{\delta \Gamma}{\delta g}$$

$$
\begin{aligned}
&+\delta a\frac{\delta\Gamma}{\delta a}+\delta b\frac{\delta\Gamma}{\delta b}+\delta\bar{\psi}\frac{\delta\Gamma}{\delta\bar{\psi}}\\
&+\delta f^*\frac{\delta\Gamma}{\delta f^*}+\delta g^*\frac{\delta\Gamma}{\delta g^*}\\
&+\delta a^*\frac{\delta\Gamma}{\delta a^*}+\delta b^*\frac{\delta\Gamma}{\delta b^*}+\delta\psi\frac{\delta\Gamma}{\delta\psi}\\
&+\delta A^\mu\frac{\delta\Gamma}{\delta A^\mu}+\delta\lambda\frac{\delta\Gamma}{\delta\lambda}+\delta D\frac{\delta\Gamma}{\delta D},
\end{aligned}
\tag{5.1.10}
$$

from which we obtain SWIs constraining the proper functions by taking appropriate derivatives.

The two-point proper vertices are constrained by

$$
\begin{aligned}
\Gamma_{\bar{\psi}\psi}(p) &\equiv S^{-1}(p)\\
&= -i\Gamma_{f^*a}(p^2)+i\gamma\cdot p\Gamma_{f^*f}(p^2)\\
&= -i\Gamma_{g^*b}(p^2)+i\gamma\cdot p\Gamma_{g^*g}(p^2),
\end{aligned}
\tag{5.1.11}
$$

$$
\gamma\cdot p\Gamma_{\bar{\psi}\psi}(p)=i\Gamma_{a^*a}(p^2)-i\gamma\cdot p\Gamma_{a^*f}(p^2)=i\Gamma_{b^*b}(p^2)-i\gamma\cdot p\Gamma_{b^*g}(p^2),
\tag{5.1.12}
$$

to be

$$
\Gamma_{a^*a}(p^2)=\Gamma_{b^*b}(p^2)=p^2A(p^2),
\tag{5.1.13}
$$

$$
\Gamma_{a^*f}(p^2)=\Gamma_{f^*a}(p^2)=\Gamma_{b^*g}(p^2)=\Gamma_{g^*b}(p^2)=-B(p^2),
\tag{5.1.14}
$$

$$
\Gamma_{f^*f}(p^2)=\Gamma_{g^*g}(p^2)=A(p^2).
\tag{5.1.15}
$$

Contrary to Eq. (2.3.5), the two-point proper vertices are not inverse to their corresponding propagators. This can be attributed to the presence of the auxiliary fields f and g which spoil the diagonal quadratic form of the Lagrangian when the interactions are turned off. It is difficult to see how to include such fields in non-perturbative calculations. We discuss these difficulties in greater detail in the next section.

5.2 The Difficulty with Auxiliary fields

Our next step forward is to adapt the techniques of chapter 3 to SQED. Näively this seems as simple as adding the self-energy terms generated by the electron's super-partners to the DSE and including the scalar propagator contribution to the CJT effective potential. However this näive approach is rendered erroneous by the auxiliary fields.

As mentioned in Sec. 4.5, the auxiliary fields f and g are given in terms of the ordinary scalars a and b at the bare level by the Euler-Lagrange equations. They derive this property from their non-diagonal quadratic terms in the free (ie. non-interacting) Lagrangian. These terms generate the unusual propagators $D_{af}(p^2), D_{bg}(p^2), D_{ff}(p^2)$ and $D_{gg}(p^2)$ in addition to the standard $D_{aa}(p^2)$, $D_{bb}(p^2)$. The form of these propagators, given by Eqs. (5.1.7),(5.1.8),(5.1.9), leads us to (correctly) anticipate that there is significant redundancy between them. It follows that we cannot blindly substitute all extra propagators and vertices into our nonperturbative equations.

Although they are necessitated in SUSY theories by the requirements of the SUSY algebra, auxiliary fields can also be included in non-SUSY theories. This is obvious for non-SUSY scalar QED. It is also possible to define auxiliary spinors for non-SUSY fermionic QED. The standard action for a free Dirac spinor is

$$\mathcal{S} = -\int d^4x(\bar{\psi}\ \partial\!\!\!/\psi + im\bar{\psi}\psi)). \tag{5.2.1}$$

An auxiliary Dirac spinor ϵ is brought in by,

$$\mathcal{S} = -\int d^4x(\bar{\psi}\ \partial\!\!\!/\psi - i\bar{\epsilon}\epsilon - im^{\frac{1}{2}}(\bar{\psi}\epsilon + \bar{\epsilon}\psi)), \tag{5.2.2}$$

and determined by the Euler-Lagrange equations,

$$\epsilon = -m^{\frac{1}{2}}\psi. \tag{5.2.3}$$

We see that auxiliary fields are not an exclusively SUSY phenomena, and since $D_{aa}(p^2) = D_{bb}(p^2)$ are of the form $\frac{A(p^2)}{p^2A^2(p^2)-B^2(p^2)}$ regardless of whether scalar auxiliary fields are used or not, it would *seem* that f and g have no new information to contribute to either the DSE or the CJT effective potential. It is intuitive then to leave $D_{af}(p^2), D_{bg}(p^2), D_{ff}(p^2)$ and $D_{gg}(p^2)$ out of the nonperturbative equations altogether and assume that all relevant information is contained in the fields a and b. This overly simplistic approach was used in our rainbow approximation analysis of SQED_3. This is fine for the DSE since the f and g contributions vanish in the rainbow approximation but it does lead to an incorrect expression for the CJT effective potential. We now describe the proper handling of auxiliary fields.

5.3 Handling the Proper Functions of Auxiliary Fields

A difficulty of the component notation in SQED is dealing with the auxiliary fields f, g and D. The first two are particularly troublesome as they contribute the off-diagonal quadratic terms which give the scalar propagators their unfamiliar form. To make the free field theory manifestly Gaussian we define,

$$[a] \equiv \begin{pmatrix} a \\ f \end{pmatrix}, \tag{5.3.1}$$

$$[b] \equiv \begin{pmatrix} b \\ g \end{pmatrix}, \tag{5.3.2}$$

$$[a]^\dagger \equiv \begin{pmatrix} a^* & f^* \end{pmatrix}, \tag{5.3.3}$$

$$[b]^\dagger \equiv \begin{pmatrix} b^* & g^* \end{pmatrix}. \tag{5.3.4}$$

The Lagrangian becomes

$$\begin{aligned}
\mathcal{L} \;=\; & [a]^\dagger \begin{bmatrix} -\partial^2 & -m \\ -m & 1 \end{bmatrix} [a] + [b]^\dagger \begin{bmatrix} -\partial^2 & -m \\ -m & 1 \end{bmatrix} [b] - \bar{\psi}(\not{\partial} + im)\psi \\
& -ieA^\mu \left([a]^\dagger \begin{bmatrix} \overleftrightarrow{\partial}_\mu & 0 \\ 0 & 0 \end{bmatrix} [a] + [b]^\dagger \begin{bmatrix} \overleftrightarrow{\partial}_\mu & 0 \\ 0 & 0 \end{bmatrix} [b] + \bar{\psi}\gamma_\mu\psi \right) \\
& -e \left[\bar{\lambda}([a]^\dagger + i\gamma_5[b]^\dagger) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \psi - \bar{\psi}[\; 1 \;\; 0 \;]([a] + i\gamma_5[b])\lambda \right] \\
& +ieD \left([a]^\dagger \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} [b] - [b]^\dagger \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} [a] \right) \\
& +e^2 A_\mu A^\mu \left([a]^\dagger \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} [a] + [b]^\dagger \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} [b] \right) \\
& -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2}\bar{\lambda}\, \not{\partial}\lambda + \frac{1}{2} D^2,
\end{aligned} \tag{5.3.5}$$

and the problem of "interpreting" auxiliary fields is therefore side-stepped.

We shall denote the propagators or proper vertices involving $[a]$ or $[b]$ by enclosing them in square brackets to distinguish them from the propagators or vertices of the single component fields a, b, f and g. Thus the $[a]$ and $[b]$ propagators are

$$[D(p^2)] \equiv \begin{bmatrix} D_{aa}(p^2) & D_{af}(p^2) \\ D_{fa}(p^2) & D_{ff}(p^2) \end{bmatrix} = \begin{bmatrix} D_{bb}(p^2) & D_{bg}(p^2) \\ D_{gb}(p^2) & D_{gg}(p^2) \end{bmatrix}; \tag{5.3.6}$$

their photon interaction is

$$[\Gamma_{(a,b)^* A_\mu (a,b)}](p,q) \equiv \begin{bmatrix} \Gamma_{(a,b)^* A_\mu (a,b)}(p,q) & \Gamma_{(a,b)^* A_\mu (f,g)}(p,q) \\ \Gamma_{(f,g)^* A_\mu (a,b)}(p,q) & \Gamma_{(f,g)^* A_\mu (f,g)}(p,q) \end{bmatrix}; \tag{5.3.7}$$

the photino interactions are

$$[\Gamma_{\bar{\lambda}(a,b)^*\psi}](p,q) \equiv [\ \Gamma_{\bar{\lambda}(a,b)^*\psi}(p,q) \quad \Gamma_{\bar{\lambda}(f,g)^*\psi}(p,q)\ ], \tag{5.3.8}$$

and

$$[\Gamma_{\bar{\psi}(a,b)\lambda}](p,q) \equiv \begin{bmatrix} \Gamma_{\bar{\psi}(a,b)\lambda}(p,q) \\ \Gamma_{\bar{\psi}(f,g)\lambda}(p,q) \end{bmatrix}; \tag{5.3.9}$$

and their D interactions are

$$[\Gamma_{(a,b)^*D(b,a)}](p,q) \equiv \begin{bmatrix} \Gamma_{(a,b)^*D(b,a)}(p,q) & \Gamma_{(a,b)^*A_\mu(g,f)}(p,q) \\ \Gamma_{(f,g)^*A_\mu(b,a)}(p,q) & \Gamma_{(f,g)^*A_\mu(g,f)}(p,q) \end{bmatrix}. \tag{5.3.10}$$

One easily verifies that Eqs. (5.1.13) to (5.1.15) are consistent with

$$[\Gamma_{(a,b)^*(a,b)}](p) \equiv \begin{bmatrix} \Gamma_{(a,b)^*(a,b)}(p) & \Gamma_{(a,b)^*(f,g)}(p) \\ \Gamma_{(f,g)^*(a,b)}(p) & \Gamma_{(f,g)^*(f,g)}(p) \end{bmatrix} = [D(p^2)]^{-1}, \tag{5.3.11}$$

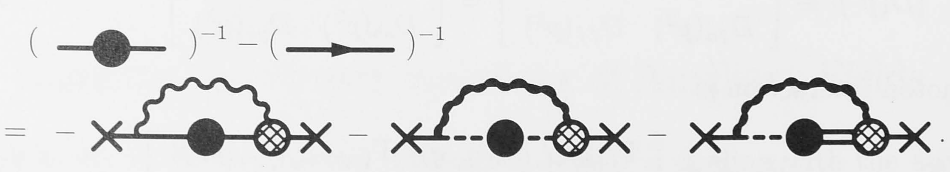
as required by Eq. (2.3.5).

5.4 Nonperturbative methods in SQED

With the Lagrangian in its familiar form and our notation established, it is now straightforward to adapt the nonperturbative methods described in chapter 3 to SUSY. Including the photino interaction in the DSE now gives us

$$\begin{aligned} & S^{-1}(p) - S_0^{-1}(p) \\ &= -\int \frac{d^4p}{(2\pi)^4}\{D_{\mu\nu}(p-q)\gamma^\mu S(q)\Gamma^\nu_{\bar{\psi}A_\mu\psi}(q,p) \\ &\qquad +S_\lambda(p-q)[\ 1 \quad 0\ ][D(q)][\Gamma_{\bar{\lambda}a^*\psi}](q,p)\} \\ &= -\int \frac{d^4p}{(2\pi)^4}\{D_{\mu\nu}(p-q)\gamma^\mu S(q)\Gamma^\nu_{\bar{\psi}A_\mu\psi}(q,p) + S_\lambda(p-q)D_{aa}(q)\Gamma_{\bar{\lambda}a^*\psi}(q,p) \\ &\qquad +S_\lambda(p-q)D_{af}(q)\Gamma_{\bar{\lambda}f^*\psi}(q,p)\}, \end{aligned} \tag{5.4.12}$$

represented graphically in Eq. (5.4.13).



(5.4.13)

With the scalar propagators given by the SWIs in Sec. 5.1, we can solve the DSE for the electron by reducing it to two coupled integral equations with the two unknown functions $A(p^2)$ and $B(p^2)$, and we do so for $SQED_3$ in Sec. 6.3.

As in the non-SUSY case, the DSE has two solutions, one of them chirally symmetric and the other not. We had hoped to use the CJT effective potential for $SQED_3$ to see which is dynamically preferred. Ignoring the photon, photino and D propagators due to the quenched approximation, the CJT effective potential for SQED is given by

$$\begin{aligned} V[S,[D]_a,[D]_b] &= \int \frac{d^dp}{(2\pi)^d}(\mathrm{Tr}\ln[S_0^{-1}(p)S(p)] + \frac{1}{2}\mathrm{Tr}[1 - S_0^{-1}(p)S(p)]) \\ &\quad -2\int \frac{d^dp}{(2\pi)^d}(\mathrm{Tr}\ln[[D(p^2)]_0^{-1}[D(p^2)]] \\ &\quad +\frac{1}{2}\mathrm{Tr}[1 - [D(p^2)]_0^{-1}[D(p^2)]]). \end{aligned} \tag{5.4.14}$$

It is a long-standing result in perturbation theory [23, 26] that the effective potential is exactly zero to all orders in a SUSY theory. (Pisarski has adapted these proofs to the many flavour limit in the nonperturbative theory [25].) We also saw in Sec. 4.2 that the effective potential must be zero unless SUSY is broken, which it isn't in SQED. However we are dealing with nonperturbative phenomena (with one flavour) and while the favoured solution must have a potential of exactly zero, it is reasonable to ask if the unfavoured one does not.

Substituting $S(p)$ and $[D(p^2)]$ into Eq. (5.4.14) unfortunately reveals that the CJT effective potential is zero at its extrema, and therefore uniformly zero. It follows that Pisarski's result is not restricted to the many flavour limit, ie. the non-renormalisation theorem for the effective potential also applies nonperturbatively. It is possible that it may be spoilt by the vacuum polarisation when the quenched approximation is abandoned but there is no evidence to support this. It is trivial to extend our result to arbitrary numbers of flavours.

"Don't fret about copping life's grand awards. Enjoy its tiny delights. There are plenty for all of us."

- Anonymous.

CHAPTER 6

The Rainbow Approximation: A Numerical Study of QED_3

6.1 Introduction

In our first analysis of the DSE [37], we choose a very simple *ansatz* for the vertices, the rainbow approximation. Our numerical analysis is done in QED_3 as it is super-renormalisable, unlike its $3+1$ dimensional counterpart.

We consider the four-component fermion version of $SQED_3$ first proposed by Pisarski [25] who obtained the model by dimensional reduction from $SQED_4$. However we develop it using the Wess-Zumino construction [22] described in Sec. 4.6. Our construction produces SUSY multiplets which differ slightly from those of Pisarski due to a degree of freedom in the charge conjugation matrix not present in $3+1$ dimensions. Pisarski's approach was an analytic one based on a $1/N_{\text{flavour}}$ expansion. His analysis indicates the existence of a dynamical mass generating solution in the large N_{flavour} limit.

6.2 The Algebra of $SQED_3$

In $2+1$ dimensions there are two inequivalent, irreducible representations of the Clifford algebra, given in terms of 2×2 matrices. These two representations differ by a minus sign [4], and have the undesirable property that either one leads to a version of QED_3 which is parity non-invariant. To circumvent this property, it is common to consider a four-component version of QED_3 incorporating Dirac matrices which are a direct sum of the two inequivalent representations [25]. The Dirac matrix algebra we employ here is constructed as follows [38]:

The 4×4 matrices γ_μ satisfy $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$, $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1)$ where μ takes the values 0,1 and 2. We take the complete set of 16 matrices

$$\{\gamma_A\} = \{I, \gamma_4, \gamma_5, \gamma_{45}, \gamma_\mu, \gamma_{\mu 4}, \gamma_{\mu 5}, \gamma_{\mu 45}\},$$

$$\gamma_0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \ \gamma_{1,2} = -i \begin{pmatrix} \sigma_{1,2} & 0 \\ 0 & -\sigma_{1,2} \end{pmatrix},$$

$$\gamma_4 = \gamma^4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \ \gamma_5 = \gamma^5 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \ \gamma_{45} = \gamma^{45} = -i\gamma_4\gamma_5,$$

$$\gamma_{\mu 4} = i\gamma_\mu\gamma_4, \ \gamma_{\mu 5} = i\gamma_\mu\gamma_5, \ \gamma_{\mu 45} = \gamma_\mu\gamma_{45}.$$

The matrices $I, \gamma_4, \gamma_5, \gamma_{45}$ are the Pauli matrices in block form and as such generate a $U(2)$ algebra. In the non-SUSY chiral theory, spontaneous mass generation leads to a nonperturbative breaking of this symmetry down to $U_I(1) \times U_{45}(1)$. The parity and charge conjugation rules for four-component Dirac spinors are given by

$$\begin{aligned} \mathcal{P}\psi(x)\mathcal{P}^{-1} &= \Pi\psi(x^0, -x^1, x^2), \qquad \mathcal{P}\bar{\psi}(x)\mathcal{P}^{-1} = \bar{\psi}(x^0, -x^1, x^2)\Pi^{-1}, \\ \mathcal{C}\psi\mathcal{C}^{-1} &= C\bar{\psi}^T, \qquad \mathcal{C}\bar{\psi}\mathcal{C}^{-1} = -\psi^T C^{-1}, \end{aligned} \tag{6.2.1}$$

respectively where

$$\Pi = \gamma_{14}e^{i\phi_P\gamma_{45}}, \ C = \gamma_2 e^{i\phi_C\gamma_{45}}, \tag{6.2.2}$$

and $(0 \le \phi_P, \phi_C < 2\pi)$. The arbitrary phases ϕ_C and ϕ_P are important for classifying the bound states in QED_3 [38].

We have

$$\begin{aligned} C^{-1}\begin{pmatrix} \gamma_4 \\ \gamma_5 \end{pmatrix} C &= R_C \begin{pmatrix} \gamma_4^T \\ \gamma_5^T \end{pmatrix}, \\ \Pi^{-1}\begin{pmatrix} \gamma_4 \\ \gamma_5 \end{pmatrix} \Pi &= R_P \begin{pmatrix} \gamma_4 \\ \gamma_5 \end{pmatrix}, \end{aligned} \tag{6.2.3}$$

where

$$R_P = \begin{pmatrix} -\cos 2\phi_P & -\sin 2\phi_P \\ -\sin 2\phi_P & \cos 2\phi_P \end{pmatrix}, \quad R_C = \begin{pmatrix} -\cos 2\phi_C & \sin 2\phi_C \\ \sin 2\phi_C & \cos 2\phi_C \end{pmatrix}. \tag{6.2.4}$$

Notice that if the chiral multiplet from $3+1$ dimensions (Eq. (4.5.1)) is used in $2+1$ dimensions then the SUSY algebra no longer holds. For example, in the case of a we have

$$[\,\delta_1, \delta_2\,]a = \bar{\zeta}_2\gamma^\mu\zeta_1\partial_\mu a - i\bar{\zeta}_2\gamma_5\gamma^\mu\zeta_1\partial_\mu b - (1 \longleftrightarrow 2). \tag{6.2.5}$$

Now, since ζ is Majorana

$$\bar{\zeta}_1\gamma^\mu\zeta_2 \;=\; -\bar{\zeta}_2\gamma^\mu\zeta_1, \tag{6.2.6}$$

as required, but using Eqs. (6.2.3),(6.2.4)

$$\bar{\zeta}_1\gamma_5\gamma^\mu\zeta_2 \;=\; \bar{\zeta}_2(\gamma_5\cos 2\phi_C + \gamma_4\sin 2\phi_C)\gamma^\mu\zeta_1, \tag{6.2.7}$$

which does not cancel $\bar{\zeta}_2\gamma_5\gamma^\mu\zeta_1$. A similar situation arises if a is replaced by any other member of the multiplet. To stop this 'blowing out' of terms we could simply set $\phi_C = 0$. This would be unfortunate though as the angular freedom ϕ_C in the matrix C would be lost. As stated earlier, this angle is important for classifying the bound states in QED$_3$ so we would like to preserve it if possible to see what effects it has (if any) in the SUSY theory. To this end we define the rotated Dirac matrices. This can be done by making the substitution

$$\begin{pmatrix}\gamma_4\\ \gamma_5\end{pmatrix} \longrightarrow \begin{pmatrix}\gamma_P\\ \gamma_W\end{pmatrix} = \begin{pmatrix}\cos\phi_C & -\sin\phi_C\\ \sin\phi_C & \cos\phi_C\end{pmatrix}\begin{pmatrix}\gamma_4\\ \gamma_5\end{pmatrix} = M\begin{pmatrix}\gamma_4\\ \gamma_5\end{pmatrix}, \tag{6.2.8}$$

in the Clifford algebra. (Note that $-i\gamma_P\gamma_W = \gamma_{45}$ so the matrices $I, \gamma_P, \gamma_W, \gamma_{45}$ again generate an $U(2)$ algebra.) Then

$$C^{-1}\begin{pmatrix}\gamma_P\\ \gamma_W\end{pmatrix}C = \begin{pmatrix}-\gamma_P^T\\ \gamma_W^T\end{pmatrix}, \tag{6.2.9}$$

since

$$MR_CM^{-1} = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}. \tag{6.2.10}$$

The rotated matrices can be used to define a SUSY transformation consistent with Eq. (4.5.2). In order to do this, all terms except $\bar{\zeta}_1\gamma^\mu\zeta_2$ generated by $\delta_1\delta_2X$ must be symmetric under interchange of ζ_1, ζ_2, i.e.

$$\bar{\zeta}_2\gamma_W\gamma^\mu\zeta_1 = \bar{\zeta}_1\gamma_W\gamma^\mu\zeta_2. \tag{6.2.11}$$

In this sense γ_W is well-behaved but γ_P and γ_{45} are problem matrices because

$$\bar{\zeta}_2(\gamma_P, \gamma_{45})\gamma^\mu\zeta_1 = -\bar{\zeta}_1(\gamma_P, \gamma_{45})\gamma^\mu\zeta_2. \tag{6.2.12}$$

Making the substitution (6.2.8) gives the 2 + 1 dimensional chiral multiplet

$$\begin{aligned}
\delta_S a &= -i\bar{\zeta}\psi\\
\delta_S b &= \bar{\zeta}\gamma_W\psi\\
\delta_S \psi &= (f + i\gamma_W g) + i\gamma^\mu\partial_\mu(a + i\gamma_W b)\zeta\\
\delta_S f &= \bar{\zeta}\;\partial\!\!\!/\psi\\
\delta_S g &= i\bar{\zeta}\gamma_W\;\partial\!\!\!/\psi,
\end{aligned} \tag{6.2.13}$$

which is the analogue of the standard chiral multiplet in $3+1$ dimensions.

The relative difference between the arbitrary phases ϕ_P and ϕ_C is fixed by the imposition of SUSY. Indeed, from Eqs. (6.2.3),(6.2.8) we have that

$$\Pi^{-1}\begin{pmatrix}\gamma_P\\ \gamma_W\end{pmatrix}\Pi = MR_PM^{-1}\begin{pmatrix}\gamma_P\\ \gamma_W\end{pmatrix}$$

where

$$MR_PM^{-1} = \begin{pmatrix} -\cos 2(\phi_P+\phi_C) & -\sin 2(\phi_P+\phi_C) \\ -\sin 2(\phi_P+\phi_C) & \cos 2(\phi_P+\phi_C)\end{pmatrix}. \tag{6.2.14}$$

If the form of the chiral multiplet transformation Eq. (6.2.13) is to be maintained under parity transformations, the off-diagonal terms in this matrix must be set to zero. ϕ_P must therefore be set to one of

$$\phi_P = -\phi_C, \frac{\pi}{2}-\phi_C, \pi-\phi_C, \frac{3\pi}{2}-\phi_C, \tag{6.2.15}$$

and we choose the first of these. The Clifford algebra to be used is now

$$\begin{aligned}\gamma_A &= \{I, \gamma_P, \gamma_W, \gamma_{45}, \gamma_\mu, \gamma_{\mu P}\gamma_{\mu W}, \gamma_{\mu 45}\},\\ C &= \gamma_2 e^{i\phi_C\gamma_{45}},\ \Pi = \gamma_1\gamma_P.\end{aligned} \tag{6.2.16}$$

With the Clifford algebra and chiral multiplets established we now look for a general multiplet. Our $2+1$ dimensional general multiplet V is defined by the following fields and transformations:

$$\begin{aligned}\delta_S C &= \bar\zeta\gamma_W\chi\\ \delta_S\chi &= (M+i\gamma_W N)\zeta + i\gamma^\mu(A_\mu + i\gamma_W\partial_\mu C)\zeta - \gamma_P K\zeta\\ \delta_S M &= \bar\zeta(\partial\!\!\!/\chi + i\lambda)\\ \delta_S N &= i\bar\zeta\gamma_W(\partial\!\!\!/\chi + i\lambda)\\ \delta_S A_\mu &= \bar\zeta\gamma_\mu\lambda - i\bar\zeta\partial_\mu\chi\\ \delta_S K &= -i\bar\zeta\gamma_P\lambda\\ \delta_S\lambda &= \tfrac{1}{2}(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu)\partial_\mu A_\nu\zeta + i\gamma_W D\zeta + i\gamma_P\ \partial\!\!\!/ K\zeta\\ \delta_S D &= i\bar\zeta\gamma_W\partial\!\!\!/\lambda.\end{aligned} \tag{6.2.17}$$

The chief difference between the $2+1$ dimensional general multiplet and the $3+1$ dimensional general multiplet, apart from the different Dirac matrices, is the extra particle K in $2+1$ dimensions. K is a scalar produced by the dimensional reduction of the photon field. That the SUSY algebra should require it is a manifestation of the *fermions = bosons* rule.

6.3 Solving the DSEs of SQED$_3$

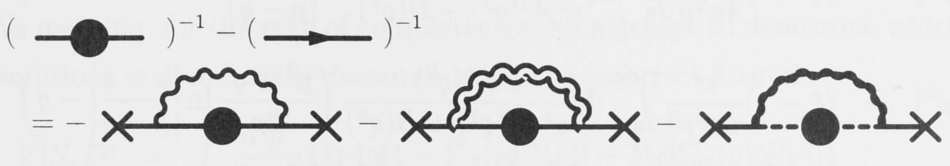
Repeating the method of Wess and Zumino [22] yields the SQED$_3$ Lagrangian

$$
\begin{aligned}
\mathcal{L} \;=\; & |f|^2 + |g|^2 + |\partial_\mu a|^2 + |\partial_\mu b|^2 - \bar{\psi}\, \partial\!\!\!/ \psi \\
& - m(a^* f + a f^* + b^* g + b g^* + i\bar{\psi}\psi) \\
& - ieA^\mu(a^* \overleftrightarrow{\partial}_\mu a + b^* \overleftrightarrow{\partial}_\mu b + \bar{\psi}\gamma_\mu\psi) + eK\bar{\psi}\gamma_P\psi \\
& - e[\bar{\lambda}(a^* + i\gamma_W b^*)\psi - \bar{\psi}(a + i\gamma_W b)\lambda] \\
& + ieD(a^* b - a b^*) - e^2(K^2 - A_\mu A^\mu)(|a|^2 + |b|^2) \\
& - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\bar{\lambda}\, \partial\!\!\!/ \lambda + \frac{1}{2}\partial^\mu K \partial_\mu K + \frac{1}{2}D^2, \qquad (6.3.1)
\end{aligned}
$$

which becomes that found by Pisarski [25] by dimensional reduction of SQED$_4$ when ϕ_C is set to zero and the scalar fields are trivially redefined.

The chiral limit is defined by taking $m \to 0$. In this limit the bare Lagrangian is invariant with respect to a global $U(2)$ symmetry generated by I, γ_P, γ_W and γ_{45}.

The electron DSE in SQED$_3$ in rainbow approximation is given by



(6.3.2)

The changed terms require explanation. The double wiggly line represents the propagator of the K particle mentioned above. The missing term with the D_{af} propagator has vanished because the bare vertex between the photino, the f and the electron is zero. That we lose a bosonic degree of freedom in the DSE without losing a corresponding fermionic one is our first indicator that the rainbow approximation might not be compatible with SUSY.

We substitute the propagators from Sec. 5.1 into Eq. (6.3.2) after converting to Euclidean space as described in Sec. 2.2 and App. B.

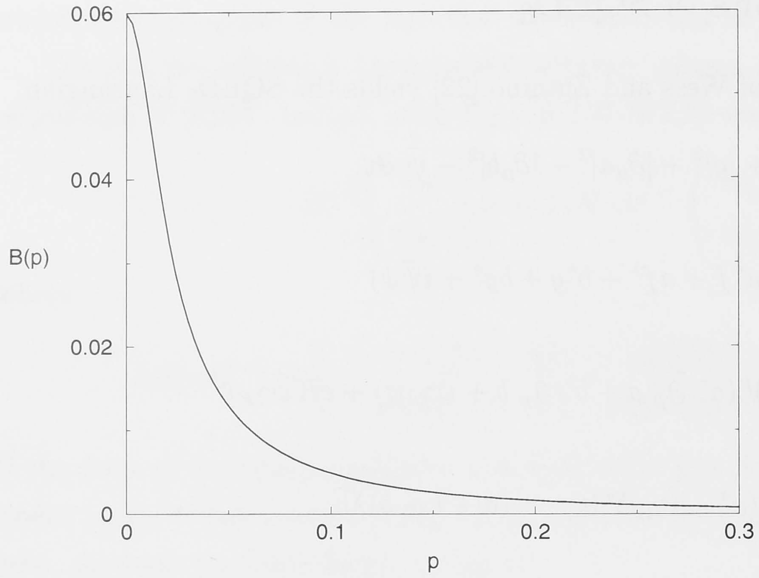


Figure 6.1: The chirally asymmetric solution for the propagator function $B(p^2)$ in Feynman gauge, $\xi = 1$. If chiral symmetry is not broken, B is identically zero.

We are left, after angular integration, with the following coupled integral equations for $B(p^2)$ and $A(p^2)$:

$$B(p^2) = (\xi+3)\frac{e^2}{4\pi^2 p}\int_0^\infty dq \frac{qB(q^2)}{q^2A^2(q^2)+B(q^2)}\ln\left|\frac{p+q}{p-q}\right| \tag{6.3.3}$$

$$\begin{aligned} A(p^2) &= (\xi-1)\frac{e^2}{4\pi^2 p^2}\int_0^\infty dq \frac{qA(q^2)}{q^2A^2(q^2)+B(q^2)}\left(\frac{p^2+q^2}{2p}\ln\left|\frac{p+q}{p-q}\right| - q\right) \\ &+ \frac{e^2}{2\pi^2 p}\int_0^\infty dq \frac{qA(q^2)}{q^2A^2(q^2)+B(q^2)}\ln\left|\frac{p+q}{p-q}\right| + 1 \end{aligned} \tag{6.3.4}$$

We solve Eqs. (6.3.3),(6.3.4) numerically using the standard iterative procedure introduced by Applequist *et al.* [39]. The functions $A(p^2)$ and $B(p^2)$ are defined on a non-uniform grid of fifty-one points concentrated at small momenta where the function varies more rapidly. The integrand is interpolated using a cubic spline with an ultraviolet cut-off of $p = 1000e^2$.

We show in Figs. (6.1),(6.2) both the chirally symmetric and asymmetric solutions to the massless ($m = 0$), Feynman gauge ($\xi = 1$) electron DSE of Eq. (6.3.2). The results are plotted in units with $e^2 = 1$. It is apparent in Fig. (6.1) that mass

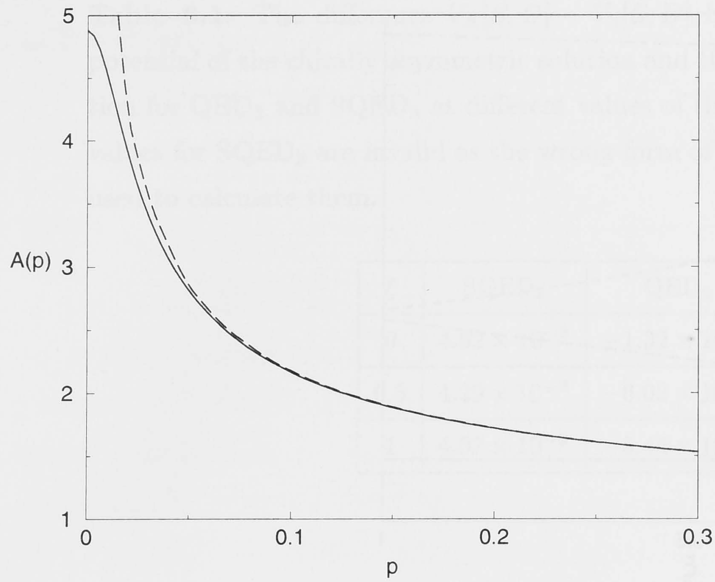


Figure 6.2: The chirally asymmetric (solid curve) and symmetric (dashed curve) solutions for the propagator function $A(p^2)$ in Feynman gauge, $\xi = 1$.

generation is remarkably suppressed and both graphs descend steeply to the values assumed when there is no dressing. We have tested the convergence of our numerical iteration procedure by varying the initial guess for A and B and the ultra-violet cutoff. These changes had no significant effect on the solution obtained.

We mention, for the sake of completeness, an attempt to determine which of the two solutions is dynamically favoured using the incorrect formula

$$\begin{aligned} V[S,D] &= \int \frac{d^3p}{(2\pi)^3} \left\{ \text{tr}\ \ln[1 - \Sigma_S(p)S(p)] + \tfrac{1}{2}\text{tr}[\Sigma_S(p)S(p)] \right\} \\ &\quad -2\int \frac{d^3p}{(2\pi)^3} \left\{ \text{tr}\ \ln[1 - \Sigma_D(p)D_{aa}(p)] + \tfrac{1}{2}\text{tr}[\Sigma_D(p)D_{aa}(p)] \right\}, \end{aligned} \tag{6.3.5}$$

for the CJT effective potential where $\Sigma_S(p) = S(p)^{-1} - S^{-1}_{\text{bare}}(p)$ and $\Sigma_D(p) = D(p)^{-1} - D^{-1}_{\text{bare}}(p)$. This formula is incorrect because, as we discussed in the last chapter, the scalar propagator is $[D(p^2)]$ and not $D_{aa}(p^2)$. The result of using this formula, shown in in Tab. 6.1, gives a misleading indication that chiral symmetry is not broken in $SQED_3$. However, as we said in Sec. 5.4, the effective potential is exactly zero for both solutions if SUSY holds and cannot be used to decide which

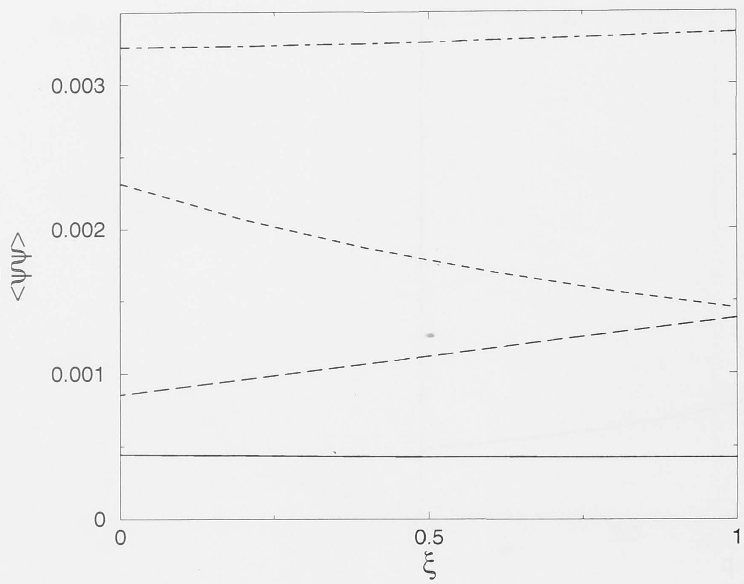


Figure 6.3: The chiral condensate of quenched $SQED_3$ (solid curve) and QED_3 (short dashes) in the bare vertex approximation. Also plotted is the chiral condensate of quenched QED_3 using the minimal Ball-Chiu *ansatz* for the fermion-photon vertex (dashed-dot curve) and the chiral condensate of quenched $SQED_3$ using the partial approximation to the Ball-Chiu *ansatz* described in the text (long dashes).

is dynamically favoured. However we can confirm the lattice result [29, 40] that the achiral solution is dynamically favoured in the *non*-SUSY case using Eq. (3.6.13).

We described in Sec. 3.5 how the rainbow approximation violates $U(1)$ gauge symmetry and how the gauge covariance of an *ansatz* could be tested by calculating the chiral condensate, given by

$$\langle\bar{\psi}\psi\rangle = \mathrm{tr}S(x=0) = \frac{2}{\pi^2}\int_0^\infty dp\frac{p^2B(p^2)}{p^2A^2(p^2)+B^2(p^2)}, \tag{6.3.6}$$

calculated in the asymmetric phase. In Fig. (6.3) we plot the chiral condensate for quenched rainbow $SQED_3$ and QED_3. While the calculated condensate for $SQED_3$ is surprisingly insensitive to the choice of gauge, the condensate for QED_3 is, as expected, strongly gauge dependent.

For QED_3 the invariance of the chiral condensate can be considerably improved by replacing the bare vertex with the minimal Ball-Chiu vertex *ansatz* [15], which is specifically designed to respect the $U(1)$ WTI and be free of kinematic singu-

Table 6.1: The difference $V_A[S,D] - V_S[S,D]$ between the CJT effective potential of the chirally asymmetric solution and the chirally symmetric solution for QED$_3$ and SQED$_3$ at different values of the gauge parameter ξ. The values for SQED$_3$ are invalid as the wrong form of the scalar propagator was used to calculate them.

ξ	SQED$_3$	QED$_3$
0	4.62×10^{-3}	-1.32×10^{-5}
0.5	4.29×10^{-3}	-6.02×10^{-6}
1	4.07×10^{-3}	-3.44×10^{-6}

larities. For comparison, the QED$_3$ condensate obtained in this way in Ref. [8] is also plotted in Fig. (6.3). Here we attempt a similar substitution for SQED$_3$. Note that the photon's SUSY partners K and λ are completely invariant under a gauge transformation so their vertices are not constrained by a WTI. Indeed, repeating the expansion in Eq. (5.1.1) for gauge symmetry in SQED$_3$ produces

$$\begin{aligned}
\delta_G\Gamma &= \delta_G f\frac{\delta\Gamma}{\delta f} + \delta_G g\frac{\delta\Gamma}{\delta g} \\
&\quad +\delta_G a\frac{\delta\Gamma}{\delta a} + \delta_G b\frac{\delta\Gamma}{\delta b} + \delta_G\bar{\psi}\frac{\delta\Gamma}{\delta\bar{\psi}} \\
&\quad +\delta_G f^*\frac{\delta\Gamma}{\delta f^*} + \delta_G g^*\frac{\delta\Gamma}{\delta g^*} \\
&\quad +\delta_G a^*\frac{\delta\Gamma}{\delta a^*} + \delta_G b^*\frac{\delta\Gamma}{\delta b^*} + \delta_G\psi\frac{\delta\Gamma}{\delta\psi} \\
&\quad +\delta_G A^\mu\frac{\delta\Gamma}{\delta A^\mu} + \delta_G K\frac{\delta\Gamma}{\delta K} + \delta_G\lambda\frac{\delta\Gamma}{\delta\lambda} + \delta_G D\frac{\delta\Gamma}{\delta D} \\
&= \delta_G f\frac{\delta\Gamma}{\delta f} + \delta_G g\frac{\delta\Gamma}{\delta g} \\
&\quad +\delta_G a\frac{\delta\Gamma}{\delta a} + \delta_G b\frac{\delta\Gamma}{\delta b} + \delta_G\bar{\psi}\frac{\delta\Gamma}{\delta\bar{\psi}} \\
&\quad +\delta_G f^*\frac{\delta\Gamma}{\delta f^*} + \delta_G g^*\frac{\delta\Gamma}{\delta g^*} \\
&\quad +\delta_G a^*\frac{\delta\Gamma}{\delta a^*} + \delta_G b^*\frac{\delta\Gamma}{\delta b^*} + \delta_G\psi\frac{\delta\Gamma}{\delta\psi} + \delta_G A^\mu\frac{\delta\Gamma}{\delta A^\mu}.
\end{aligned} \tag{6.3.7}$$

Since $\delta_G K = \delta_G \lambda = \delta_G D = 0$, there are no terms in Eq. (6.3.7) proportional to $\frac{\delta\Gamma}{\delta K}, \frac{\delta\Gamma}{\delta\lambda}$ or $\frac{\delta\Gamma}{\delta D}$ so $\frac{\delta^2\Gamma}{\delta\bar{\psi}\delta\psi}$ will not produce a vertex for any of K, λ or D. If we try to derive an identity with something like $\frac{\delta^3\Gamma}{\delta\bar{\psi}\delta a\delta\lambda}$, we obtain

$$\frac{\delta^3(\delta_G\Gamma)}{\delta\bar{\psi}\delta a\delta\lambda} = -i\theta\frac{\delta^3\Gamma}{\delta\bar{\psi}\delta a\delta\lambda} + i\theta\frac{\delta^3\Gamma}{\delta\bar{\psi}\delta a\delta\lambda} = 0. \tag{6.3.8}$$

Compliance with the WTI can therefore be achieved by replacing the bare photon-fermion vertex with the minimal Ball-Chiu *ansatz* while the remaining vertices are kept bare. This method incurs the penalty of further breaking SUSY. The resulting chiral condensate is plotted in Fig. (6.3). Surprisingly, the variation of the condensate with respect to the gauge parameter was found to be an order of magnitude greater than in the bare case. We attribute this to the violation of SUSY and conclude that any attempt to improve the vertex must remain SUSY. The development of a vertex *ansatz* which respects both the WTIs and SWIs is described in the next chapter.

"I haven't chosen a quote for this chapter."

- Michael Luke Walker.

CHAPTER 7

Beyond the Rainbow Approximation in SQED

7.1 Why Go "Over the Rainbow"?

Having investigated the rainbow approximation, our next step is to find a better *ansatz*. This has generally been the approach in non-SUSY field theory. The rainbow approximation is used for initial studies on account of its simplicitly and the fact that it allows for chiral symmetry breaking.

There are several incentives for transcending the rainbow approximation. The first one was mentioned in Sec. 3.4, namely that the rainbow approximation violates the WTI. We will see in the next section that it also violates some of the SWIs. So not only does it violate $U(1)$ gauge invariance but it also violates SUSY.

In fact, the incompatability of SUSY with the rainbow approximation is a severe one. Recall that the scalars $\begin{pmatrix} a \\ f \end{pmatrix}$ and $\begin{pmatrix} b \\ g \end{pmatrix}$ each have their own DSE. In the rainbow approximation and after rotating to Euclidean space, this is given by

$$\begin{aligned}
&\begin{bmatrix} p^2A(p^2) & -B(p^2) \\ -B(p^2) & A(p^2) \end{bmatrix} - \begin{bmatrix} p^2 & -m(=0) \\ -m(=0) & 1 \end{bmatrix} \\
&= -\int_0^\infty \frac{d^dq}{(2\pi)^d} \begin{bmatrix} (p+q)_\mu & 0 \\ 0 & 0 \end{bmatrix} [D(q^2)] \begin{bmatrix} (p+q)_\nu & 0 \\ 0 & 0 \end{bmatrix} D_{\mu\nu}(p-q) \\
&\quad -\int_0^\infty \frac{d^dq}{(2\pi)^d}\frac{d^dk}{(2\pi)^d} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} [D(q^2)] \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} D_{\mu\nu}(p-q-k)D_{\mu\nu}(k) \\
&\quad -\int_0^\infty \frac{d^dq}{(2\pi)^d} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} D_{\mu\mu}(p-q)
\end{aligned}$$

$$-\int_0^\infty \frac{d^dq}{(2\pi)^d}\begin{bmatrix}-1\\0\end{bmatrix}S(q)D_\lambda(p-q)[\ 1\quad 0\ ]$$
$$-\int_0^\infty \frac{d^dq}{(2\pi)^d}\begin{bmatrix}-i & 0\\0 & 0\end{bmatrix}[D(q^2)]\begin{bmatrix}i & 0\\0 & 0\end{bmatrix}D_D((p-q)^2). \tag{7.1.1}$$

It is immediately obvious in this equation that its right hand side is a two by two matrix of the form $\begin{bmatrix}? & 0\\0 & 0\end{bmatrix}$, where every entry except the top left hand corner is zero. Equating this with the right hand side of the equation we see from the other elements that $B(p^2)$ is constrained to be zero and $A(p^2)$ is constrained to be one, ie. the propagator is constrained to remain bare, regardless of gauge! This is forbidden by the Landau-Khalatnikov [41] transformations. It also seems unlikely that the top left hand corner of the right hand side would be exactly zero, although we haven't calculated it. What is certain is that even without considering the electron DSE, the scalar DSE has no physical solution in the rainbow approximation. The system is overconstrained.

The conflict seems to arise from the auxiliary fields which are inherent to any SUSY theory. Indeed, if the matrix propagators and vertices are replaced with standard scalar propagators and vertices then the problem does not arise. We conclude that the rainbow approximation is inherently unsuitable for applications in SQED and probably other SUSY theories as well. This is reminiscient of the Haeri vertex [42] being unsuitable for use in non-SUSY QED because it favours the chiral solution to the DSE [43] when the theory is known to be achiral [8, 9, 40]. Our work in the last section therefore, while useful for establishing methodology, should not have its results taken too seriously.

Improving upon the rainbow approximation in SUSY theories is a difficult task and few studies have attempted it. Those that do have largely used the superfield formalism. However some attempts to dress the vertices have been made. For example, Koopmans and Steringa [18] sought to be consistent with the differential $U(1)$ gauge WI in their component formalism analysis of SQED$_3$ with two-component fermions. To this end they multiplied the bare vertices by $A(q^2)$ where the electron propagator is given by $S^{-1}(q) = i(\gamma\cdot qA(q^2)+B(q^2))$. This approach is questionable as it implicitly approximates the functions $A(p^2)$ and $B(p^2)$ as being flat. While this approximation is reasonable over most of the momentum range, it is not valid in the low momentum limit where the dynamics are largely determined.

That dressing the vertex should be difficult in SQED using the component formalism is not surprising. Not only must the gauge particle vertices be dressed but the photino vertices also. We have already seen that substituting the minimal Ball and Chiu vertex for photon interactions in $SQED_3$ while leaving the other vertices bare exacerbates the DSE's gauge violating properties [37]. The problem of going beyond the rainbow approximation in SUSY theories is the problem of finding the photino vertices corresponding to the improved photon vertex. Photino vertices are not constrained by the WTI since the photino is invariant to gauge transformations. However they are related to the photon vertices by SWIs. Our next logical step therefore is to find these SWIs and solve them [44].

7.2 The Three-Point SWIs

In Sec. 5.1 we found the SWIs that constrain the various propagators. We now derive the SWIs for the vertices, found by taking third order functional derivatives of $\delta\Gamma = (\delta_S + \delta_{WZ})\Gamma = 0$. The functional derivatives corresponding to the following

Table 7.1: Each SWI is derived from a functional derivative of $\delta\Gamma = 0$. The functional derivative leading to each SWI (indicated by its equation number) is given in this table.

Functional Derivative of $\delta\Gamma = 0$	SWI	Functional Derivative of $\delta\Gamma = 0$	SWI
$\delta^3/(\delta a(y)\delta a^*(x)\delta\bar{\lambda}(z))$	7.2.1	$\delta^3/(\delta\psi(y)\delta D(z)\delta a^*(x))$	7.2.14
$\delta^3/(\delta b(y)\delta b^*(x)\delta\bar{\lambda}(z))$	7.2.2	$\delta^3/(\delta\psi(y)\delta D(z)\delta b^*(x))$	7.2.15
$\delta^3/(\delta f(y)\delta a^*(x)\delta\bar{\lambda}(z))$	7.2.3	$\delta^3/(\delta\psi(y)\delta D(z)\delta f^*(x))$	7.2.16
$\delta^3/(\delta g(y)\delta b^*(x)\delta\bar{\lambda}(z))$	7.2.4	$\delta^3/(\delta\psi(y)\delta D(z)\delta g^*(x))$	7.2.17
$\delta^3/(\delta f(y)\delta f^*(x)\delta\bar{\lambda}(z))$	7.2.5	$\delta^3/(\delta b(y)\delta D(z)\delta a^*(x))$	7.2.18
$\delta^3/(\delta g(y)\delta g^*(x)\delta\bar{\lambda}(z))$	7.2.6	$\delta^3/(\delta a(y)\delta\lambda(z)\delta b^*(x))$	7.2.19
$\delta^3/(\delta\psi(y)\delta A_\mu(z)\delta f^*(x))$	7.2.7	$\delta^3/(\delta g(y)\delta\lambda(z)\delta a^*(x))$	7.2.20
$\delta^3/(\delta\psi(y)\delta A_\mu(z)\delta g^*(x))$	7.2.8	$\delta^3/(\delta f(y)\delta\lambda(z)\delta b^*(x))$	7.2.21
$\delta^3/(\delta\psi(y)\delta A_\mu(z)\delta a^*(x))$	7.2.9	$\delta^3(\delta a(y)\delta\lambda(z)\delta g^*(x))$	7.2.22
$\delta^3/(\delta\psi(y)\delta A_\mu(z)\delta b^*(x))$	7.2.10	$\delta^3/(\delta b(y)\delta\lambda(z)\delta f^*(x))$	7.2.23
$\delta^3/(\delta\psi_\alpha(y)\delta\bar{\psi}^\beta(x)\delta\lambda_\kappa(z))$	7.2.11	$\delta^3/(\delta g(y)\delta\lambda(z)\delta f^*(x))$	7.2.24
		$\delta^3/(\delta f(y)\delta\lambda(z)\delta g^*(x))$	7.2.25

SWIs are given in table 7.1:

$$\begin{aligned}
&\gamma_\mu \Gamma^\mu_{a^* A_\mu a}(p,q) \\
&= \Gamma_{\bar{\lambda} a^* \psi}(p,q)\gamma\cdot q + e(B(p^2) - B(q^2)) + \Gamma_{\bar{\lambda} a^* \psi}(-q,-p)\gamma\cdot p, \qquad (7.2.1) \\
&\gamma_\mu \Gamma^\mu_{b^* A_\mu b}(p,q) \\
&= -i\Gamma_{\bar{\lambda} b^* \psi}(p,q)\gamma_5\gamma\cdot q - e(B(p^2) - B(q^2)) - i\Gamma_{\bar{\lambda} b^* \psi}(-q,-p)\gamma_5\gamma\cdot p, \qquad (7.2.2)
\end{aligned}$$

$$\begin{aligned}
\gamma_\mu \Gamma^\mu_{f^* A_\mu a}(p,q) + eA(p^2) &= \Gamma_{\bar{\lambda} a^* \psi}(-q,-p) + \Gamma_{\bar{\lambda} f^* \psi}(p,q)\gamma\cdot q, \qquad (7.2.3) \\
\gamma_\mu \Gamma^\mu_{g^* A_\mu b}(p,q) - eA(p^2) &= i\Gamma_{\bar{\lambda} b^* \psi}(-q,-p)\gamma_5 + i\Gamma_{\bar{\lambda} g^* \psi}(p,q)\gamma\cdot q\gamma_5, \qquad (7.2.4)
\end{aligned}$$

$$\begin{aligned}
\gamma_\mu \Gamma^\mu_{f^* A_\mu f}(p,q) &= \Gamma_{\bar{\lambda} f^* \psi}(-q,-p) - \Gamma_{\bar{\lambda} f^* \psi}(p,q), \qquad (7.2.5) \\
\gamma_\mu \Gamma^\mu_{g^* A_\mu g}(p,q) &= i\Gamma_{\bar{\lambda} g^* \psi}(-q,-p)\gamma_5 - i\Gamma_{\bar{\lambda} g^* \psi}(p,q)\gamma_5, \qquad (7.2.6)
\end{aligned}$$

$$\begin{aligned}
&i\sigma^{\mu\nu}(p-q)_\nu \Gamma_{\bar{\lambda} f^* \psi}(p,q) \\
&= \Gamma^\mu_{\bar{\psi} A_\mu \psi}(p,q) - i\gamma\cdot q\Gamma^\mu_{f^* A_\mu f}(p,q) + i\Gamma^\mu_{f^* A_\mu a}(p,q) - ie\gamma^\mu A(p^2), \qquad (7.2.7) \\
&i\sigma^{\mu\nu}(p-q)_\nu \Gamma_{\bar{\lambda} g^* \psi}(p,q) \\
&= i\gamma_5\Gamma^\mu_{\bar{\psi} A_\mu \psi}(p,q) + \gamma_5\gamma\cdot q\Gamma^\mu_{g^* A_\mu g}(p,q) - \gamma_5\Gamma^\mu_{g^* A_\mu b}(p,q) + e\gamma_5\gamma^\mu A(p^2), \qquad (7.2.8)
\end{aligned}$$

$$\begin{aligned}
&i\sigma^{\mu\nu}(p-q)_\nu \Gamma_{\bar{\lambda} a^* \psi}(p,q) \\
&= i\Gamma^\mu_{a^* A_\mu a}(p,q) - i\gamma\cdot q\Gamma^\mu_{a^* A_\mu f}(p,q) - e\gamma^\mu S^{-1}(p) - \gamma\cdot p\Gamma^\mu_{\bar{\psi} A_\mu \psi}(p,q) \\
&\quad + ie\gamma^\mu B(p^2), \qquad (7.2.9) \\
&i\sigma^{\mu\nu}(p-q)_\nu \Gamma_{\bar{\lambda} b^* \psi}(p,q) \\
&= -\gamma_5\Gamma^\mu_{b^* A_\mu b}(p,q) + \gamma_5\gamma\cdot q\Gamma^\mu_{b^* A_\mu g}(p,q) - i\gamma_5 e\gamma^\mu S^{-1}(p) \\
&\quad - i\gamma_5\gamma\cdot p\Gamma^\mu_{\bar{\psi} A_\mu \psi}(p,q) - e\gamma_5\gamma^\mu B(p^2). \qquad (7.2.10)
\end{aligned}$$

It follows from both (7.2.9) and (7.2.10) that the rainbow approximation violates SUSY in the same way that it violates $U(1)$ gauge invariance.

From

$$\begin{aligned} 0 &= -i(\gamma\cdot q)_\sigma{}^\alpha(\Gamma_{\bar\psi f\lambda}(p,q))_\beta{}^\kappa + (\gamma_5\gamma\cdot q)_\sigma{}^\alpha(\Gamma_{\bar\psi g\lambda}(p,q))_\beta{}^\kappa \\ &\quad -i(\gamma\cdot pC)_{\beta\sigma}\frac{\delta^2}{\delta\psi_\alpha(q)\delta f^*(p)}(\frac{\delta\Gamma}{\delta\lambda(p-q)}C^{-1})^\kappa \\ &\quad -(\gamma_5\gamma\cdot pC)_{\beta\sigma}\frac{\delta^2}{\delta\psi_\alpha(q)\delta g^*(p)}(\frac{\delta\Gamma}{\delta\lambda(p-q)}C^{-1})^\kappa \\ &\quad -(\gamma_5(\gamma\cdot p-\gamma\cdot q))_\sigma{}^\kappa(\Gamma_{\bar\psi D\psi}(p,q))_\beta{}^\alpha, \end{aligned} \tag{7.2.11}$$

where C is the charge conjugation matrix, we obtain

$$\begin{aligned} 0 &= (\gamma\cdot p-\gamma\cdot q)\gamma_5\mathrm{Tr}(\Gamma_{\bar\psi D\psi}(p,q)) + \gamma_\mu\mathrm{Tr}(\Gamma^\mu_{\bar\psi A_\mu}(p,q)) + i\Gamma_{\bar\psi a\lambda}(p,q) \\ &\quad -\gamma_5\Gamma_{\bar\psi b\lambda}(p,q) - i\Gamma_{\bar\psi a\lambda\psi}(-q,-p) + \gamma_5\Gamma_{\bar\psi b\lambda}(-q,-p) \\ &\quad -i\gamma\cdot q\Gamma_{\bar\psi f\lambda}(p,q) + \gamma_5\gamma\cdot q\Gamma_{\bar\psi g\lambda}(p,q) - i\gamma\cdot p\Gamma_{\bar\psi f\lambda}(-q,-p) \\ &\quad +\gamma_5\gamma\cdot p\Gamma_{\bar\psi g\lambda}(-q,-p), \end{aligned} \tag{7.2.12}$$

by setting $\beta=\alpha$ and summing, and

$$\begin{aligned} 0 &= i\mathrm{Tr}(\Gamma_{\bar\psi a\lambda}(p,q)) - \gamma_5\mathrm{Tr}(\Gamma_{\bar\psi b\lambda}(p,q)) - i\gamma\cdot q\mathrm{Tr}(\Gamma_{\bar\psi f\lambda}(p,q)) \\ &\quad +\gamma_5\gamma\cdot q\mathrm{Tr}(\Gamma_{\bar\psi g\lambda}(p,q)) - i\Gamma_{\bar\lambda a^*\psi}(p,q) + \gamma_5\Gamma_{\bar\lambda b^*\psi}(p,q) - i\gamma\cdot p\Gamma_{\bar\lambda f^*\psi}(p,q) \\ &\quad -\gamma\cdot p\gamma_5\Gamma_{\bar\lambda g^*\psi}(p,q) + \gamma_\mu\Gamma^\mu_{\bar\psi A_\mu\psi}(p,q) - \gamma_5(\gamma\cdot p-\gamma\cdot q)\Gamma_{\bar\psi D\psi}(p,q), \end{aligned} \tag{7.2.13}$$

by setting $\beta=\kappa$ and summing.

Finally there are the SWIs governing the vertices of the D particle;

$$\begin{aligned} & i\gamma_5\Gamma_{\bar\lambda a^*\psi}(p,q) \\ &\quad = \gamma\cdot p\Gamma_{\bar\psi D\psi}(p,q) + \gamma_5\Gamma_{a^*Db}(p,q) - \gamma_5\gamma\cdot q\Gamma_{a^*Dg}(p,q), \end{aligned} \tag{7.2.14}$$

$$\begin{aligned} & i\gamma_5\Gamma_{\bar\lambda b^*\psi}(p,q) \\ &\quad = i\gamma_5\gamma\cdot p\Gamma_{\bar\psi D\psi}(p,q) - i\Gamma_{b^*Da}(p,q) + i\gamma\cdot q\Gamma_{b^*Df}(p,q), \end{aligned} \tag{7.2.15}$$

$$\begin{aligned} & \gamma_5\Gamma_{f^*Db}(p,q) \\ &\quad = i\gamma_5\Gamma_{\bar\lambda f^*\psi}(p,q) + \gamma_5\gamma\cdot q\Gamma_{f^*Dg}(p,q) + \Gamma_{\bar\psi D\psi}(p,q), \end{aligned} \tag{7.2.16}$$

$$\begin{aligned} & \gamma_5\Gamma_{g^*Da}(p,q) \\ &\quad = -\Gamma_{\bar\lambda g^*\psi}(p,q) + \gamma_5\gamma\cdot q\Gamma_{g^*Df}(p,q) - \Gamma_{\bar\psi D\psi}(p,q), \end{aligned} \tag{7.2.17}$$

$$\begin{aligned}&\gamma_5(\gamma\cdot p-\gamma\cdot q)\Gamma_{a^*Db}(p,q)\\&=\Gamma_{\bar\lambda b^*\psi}(-q,-p)\gamma\cdot p+i\Gamma_{\bar\lambda a^*\psi}(p,q)\gamma\cdot q\gamma_5+ie\gamma_5(B(p^2)-B(q^2)),\end{aligned}\tag{7.2.18}$$

$$\begin{aligned}&\gamma_5(\gamma\cdot p-\gamma\cdot q)\Gamma_{b^*Da}(p,q)\\&=i\Gamma_{\bar\lambda a^*\psi}(-q,-p)\gamma\cdot p\gamma_5+\Gamma_{\bar\lambda b^*\psi}(p,q)\gamma\cdot q+ie\gamma_5(B(p^2)-B(q^2)),\end{aligned}\tag{7.2.19}$$

$$\begin{aligned}&\gamma_5(\gamma\cdot p-\gamma\cdot q)\Gamma_{a^*Dg}(p,q)\\&=\Gamma_{\bar\lambda g^*\psi}(-q,-p)\gamma\cdot p-i\Gamma_{\bar\lambda a^*\psi}(p,q)\gamma_5+ie\gamma_5A(q^2),\end{aligned}\tag{7.2.20}$$

$$\begin{aligned}&\gamma_5(\gamma\cdot p-\gamma\cdot q)\Gamma_{b^*Df}(p,q)\\&=i\Gamma_{\bar\lambda f^*\psi}(-q,-p)\gamma\cdot p\gamma_5-\Gamma_{\bar\lambda b^*\psi}(p,q)+ie\gamma_5A(q^2),\end{aligned}\tag{7.2.21}$$

$$\begin{aligned}&\gamma_5(\gamma\cdot p-\gamma\cdot q)\Gamma_{g^*Da}(p,q)\\&=\Gamma_{\bar\lambda g^*\psi}(p,q)\gamma\cdot q+i\Gamma_{\bar\lambda a^*\psi}(-q,-p)\gamma_5-ie\gamma_5A(p^2),\end{aligned}\tag{7.2.22}$$

$$\begin{aligned}&\gamma_5(\gamma\cdot p-\gamma\cdot q)\Gamma_{f^*Db}(p,q)\\&=i\Gamma_{\bar\lambda f^*\psi}(p,q)\gamma\cdot q\gamma_5+\Gamma_{\bar\lambda b^*\psi}(-q,-p)-ie\gamma_5A(p^2),\end{aligned}\tag{7.2.23}$$

$$\gamma_5(\gamma\cdot p-\gamma\cdot q)\Gamma_{f^*Dg}(p,q)=\Gamma_{\bar\lambda g^*\psi}(-q,-p)-i\Gamma_{\bar\lambda f^*\psi}(p,q)\gamma_5,\tag{7.2.24}$$

$$\gamma_5(\gamma\cdot p-\gamma\cdot q)\Gamma_{g^*Df}(p,q)=i\Gamma_{\bar\lambda f^*\psi}(-q,-p)\gamma_5-\Gamma_{\bar\lambda g^*\psi}(p,q).\tag{7.2.25}$$

These make up the entire set of SWIs containing only three-or-fewer point proper functions, modulo charge conjugation. A suitable vertex *ansatz* must also be consistent with the WTIs and we also have from charge conjugation invariance that

$$\begin{aligned}[\Gamma_{\bar\psi(a,b)\lambda}](p,q)&=-C[\Gamma_{\bar\lambda(a^*,b^*)\psi}](-q,-p)^TC^{-1}\\ [\Gamma_{(a^*,b^*)D(b,a)}](p,q)&=-[\Gamma_{(b^*,a^*)D(a,b)}](-q,-p).\end{aligned}\tag{7.2.26}$$

The general form of all suitable *ansätze* is given in the next section.

7.3 The General Form of the Vertices in SQED

Below is a solution for the SWIs and WTIs. It is the most general set of vertices consistent with both the WTIs and the SWIs and free of kinematic singularities if one assumes charge conjugation invariance and

$$[\Gamma_{a^*A_\mu a}]^\mu(p,q)=[\Gamma_{b^*A_\mu b}]^\mu(p,q).\tag{7.3.1}$$

Proof of this is presented in Appendix F. The assumption of Eq. (7.3.1) is true to all orders in perturbation theory, and any nonperturbative violations of this assumption are restricted by the WTIs to lie completely within their transverse components.

Our general solution is as follows:
The scalar-photon vertices are

$$\begin{aligned}\Gamma^{\mu}_{a^*A_\mu a}(p,q) = \Gamma^{\mu}_{b^*A_\mu b}(p,q) &= \frac{e}{p^2-q^2}(p^2A(p^2)-q^2A(q^2))(p+q)^\mu \\ &+[p^\mu(q^2-p\cdot q)+q^\mu(p^2-p\cdot q)]T_{aa}(p^2,q^2,p\cdot q), \end{aligned} \tag{7.3.2}$$

$$\begin{aligned}\Gamma^{\mu}_{a^*A_\mu f}(p,q) &= \Gamma^{\mu}_{b^*A_\mu g}(p,q) = \Gamma^{\mu}_{f^*A_\mu a}(p,q) = \Gamma^{\mu}_{g^*A_\mu b}(p,q) \\ &= \frac{-e}{p^2-q^2}(B(p^2)-B(q^2))(p+q)^\mu \\ &+[p^\mu(q^2-p\cdot q)+q^\mu(p^2-p\cdot q)]T_{af}(p^2,q^2,p\cdot q), \end{aligned} \tag{7.3.3}$$

$$\begin{aligned}\Gamma^{\mu}_{f^*A_\mu f}(p,q) = \Gamma^{\mu}_{g^*A_\mu g}(p,q) &= \frac{e}{p^2-q^2}(A(p^2)-A(q^2))(p+q)^\mu \\ &+[p^\mu(q^2-p\cdot q)+q^\mu(p^2-p\cdot q)]T_{ff}(p^2,q^2,p\cdot q), \end{aligned} \tag{7.3.4}$$

where the three functions $T_{aa}(p^2,q^2,p\cdot q), T_{af}(p^2,q^2,p\cdot q)$ and $T_{ff}(p^2,q^2,p\cdot q)$, each satisfying $T(p^2,q^2,p\cdot q)=T(q^2,p^2,p\cdot q)$, are free of kinematic singularities and represent the only degrees of freedom inherent in the solution. The forms (7.3.2) to (7.3.4) are equivalent to that given by Ball and Chiu [15] in the context of non-SUSY scalar QED. The photino vertices are

$$\begin{aligned}\Gamma_{\bar\lambda a^*\psi}(p,q) &= \frac{e}{p^2-q^2}(p^2A(p^2)-q^2A(q^2)) + \frac{e}{p^2-q^2}(B(p^2)-B(q^2))\gamma\cdot q \\ &+\frac{1}{2}e(p^2-\gamma\cdot q\gamma\cdot p)T_{aa}(p^2,q^2,p\cdot q) \\ &+\frac{1}{2}e[\gamma\cdot p(p^2-q^2)-2\gamma\cdot q(p^2-p\cdot q)]T_{af}(p^2,q^2,p\cdot q), \\ &+\frac{1}{2}ep^2(q^2-\gamma\cdot p\gamma\cdot q)T_{ff}(p^2,q^2,p\cdot q), \end{aligned} \tag{7.3.5}$$

and

$$\begin{aligned}\Gamma_{\bar\lambda f^*\psi}(p,q) &= \frac{-e}{p^2-q^2}(A(p^2)-A(q^2))\gamma\cdot q - \frac{e}{p^2-q^2}(B(p^2)-B(q^2)) \\ &+\frac{1}{2}e(\gamma\cdot p-\gamma\cdot q)T_{aa}(p^2,q^2,p\cdot q) \\ &+\frac{1}{2}e(p-q)^2T_{af}(p^2,q^2,p\cdot q) \\ &-\frac{1}{2}e\gamma\cdot q(p^2-\gamma\cdot p\gamma\cdot q)T_{ff}(p^2,q^2,p\cdot q). \end{aligned} \tag{7.3.6}$$

The electron-photon vertex must be restricted at least to the form given by Ball and Chiu [15] for non-SUSY QED. For the SUSY case we find

$$\begin{aligned}\Gamma^{\mu}_{\bar{\psi}A_\mu\psi}(p,q) &= \Gamma^{\mu}_{BC}(p,q) + \frac{ie}{p^2-q^2}(A(p^2)-A(q^2))[\frac{1}{2}T_3^\mu - T_8^\mu] \\ &-\frac{ie}{p^2-q^2}(B(p^2)-B(q^2))T_5^\mu + \frac{1}{2}ieT_{aa}(p^2,q^2,p\cdot q)T_3^\mu \\ &+ieT_{af}(p^2,q^2,p\cdot q)[\frac{1}{2}(p-q)^2T_5^\mu - T_1^\mu] \\ &+\frac{1}{2}ieT_{ff}(p^2,q^2,p\cdot q)[T_2^\mu - p\cdot qT_3^\mu - (p-q)^2T_8^\mu], \qquad (7.3.7)\end{aligned}$$

where

$$\begin{aligned}\Gamma^{\mu}_{BC}(p,q) &= \frac{1}{2}\frac{ie}{p^2-q^2}(\gamma\cdot p+\gamma\cdot q)(A(p^2)-A(q^2))(p+q)^\mu \\ &+ie\frac{1}{2}(A(p^2)+A(q^2))\gamma^\mu + \frac{ie}{p^2-q^2}(B(p^2)-B(q^2))(p+q)^\mu, \qquad (7.3.8)\end{aligned}$$

$$T_1^\mu = p^\mu(q^2-p\cdot q)+q^\mu(p^2-p\cdot q), \qquad (7.3.9)$$

$$T_2^\mu = (\gamma\cdot p+\gamma\cdot q)T_1^\mu, \qquad (7.3.10)$$

$$T_3^\mu = \gamma^\mu(p-q)^2-(\gamma\cdot p-\gamma\cdot q)(p-q)^\mu], \qquad (7.3.11)$$

$$T_5^\mu = \sigma^{\mu\nu}(p-q)_\nu, \qquad (7.3.12)$$

$$T_8^\mu = \frac{1}{2}(\gamma\cdot p\gamma\cdot q\gamma^\mu - \gamma^\mu\gamma\cdot q\gamma\cdot p). \qquad (7.3.13)$$

Finally there are the vertices for the D-boson, namely,

$$\begin{aligned}&\Gamma_{a^*Db}(p,q) = -\Gamma_{b^*Da}(p,q) \\ &= \frac{ie}{p^2-q^2}(p^2A(p^2)-q^2A(q^2)) - iep\cdot qT_{a^*a}(p^2,q^2,p\cdot q) \\ &+\frac{1}{2}iep^2q^2T_{ff}(p^2,q^2,p\cdot q), \qquad (7.3.14)\end{aligned}$$

$$\begin{aligned}&\Gamma_{f^*Dg}(p,q) = -\Gamma_{g^*Df}(p,q) \\ &= \frac{ie}{p^2-q^2}(A(p^2)-A(q^2)) + ieT_{a^*a}(p^2,q^2,p\cdot q) \\ &-iep\cdot qT_{f^*f}(p^2,q^2,p\cdot q), \qquad (7.3.15)\end{aligned}$$

$$\Gamma_{g^*Da}(p,q) = \frac{ie}{p^2-q^2}(B(p^2)-B(q^2))$$

$$
\begin{aligned}
& -ie(q^2 - p\cdot q)T_{af}(p^2,q^2,p\cdot q), & (7.3.16)\\
\Gamma_{a^*Dg}(p,q) &= \frac{-ie}{p^2-q^2}(B(p^2)-B(q^2)) \\
& +ie(p^2 - p\cdot q)T_{af}(p^2,q^2,p\cdot q), & (7.3.17)\\
\Gamma_{f^*Db}(p,q) &= \frac{-ie}{p^2-q^2}(B(p^2)-B(q^2)) \\
& +ie(q^2 - p\cdot q)T_{af}(p^2,q^2,p\cdot q), & (7.3.18)\\
\Gamma_{b^*Df}(p,q) &= \frac{ie}{p^2-q^2}(B(p^2)-B(q^2)) \\
& -ie(p^2 - p\cdot q)T_{af}(p^2,q^2,p\cdot q), & (7.3.19)
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma_{\bar\psi D\psi}(p,q) &= \frac{1}{2}ie\gamma_5[(\gamma\cdot p + \gamma\cdot q)T_{a^*a}(p^2,q^2,p\cdot q) \\
& +(p^2-q^2)T_{af}(p^2,q^2,p\cdot q) \\
& -(\gamma\cdot qp^2 + \gamma\cdot pq^2)T_{ff}(p^2,q^2,p\cdot q)]. & (7.3.20)
\end{aligned}
$$

We observe that the vertices are specified up to three functions, T_{aa}, T_{af} and T_{ff} whose only constraint is that they must be symmetric in p and q due to charge conjugation invariance. As a matter of notation we refer to these functions as 'transverse' functions since they contribute only to the transverse components of the vertices. The vertices with the transverse functions set to zero we call the 'minimal' SUSY BC (SBC) vertices. The contribution to the DSE from the minimal SBC vertices will be referred to as the minimal contribution.

7.4 The Nonrenormalisation Theorem Revisited

In Sec. 4.7 we presented the nonrenormalisation theorem which stated that a SUSY theory could not generate perturbative corrections to particle masses. Several authors [20, 21, 24] have investigated the possibility of a nonperturbative nonrenormalisation theorem using the superfield formalism in both $SQED_3$ and $SQED_4$. The first was Clark and Love [20] who derived a differential $U(1)$ gauge WI for the superfields. After truncating diagrams containing seagull and higher order n-point vertices, they found that the effective mass $\mathcal{M}$ contains a prefactor $\xi - 1$ which vanishes in Feynman gauge. Reasoning that if the mass vanishes in one gauge then it must vanish in all gauges, they concluded that there can be no dynamic chiral symmetry breaking in SQED, even beyond the rainbow approximation.

The work of Clark and Love was criticized by Kaiser and Selipsky on two grounds [21]. Firstly they argue that the truncation of seagull diagrams is too severe as it ignores contributions even at the one-loop level. Secondly they point out that infinities arising from the infrared divergences which plague the superfield formalism can counter the vanishing prefactor and allow spontaneous mass generation.

Campbell-Smith and Mavromatos [24] have investigated chiral symmetry breaking in $SQED_3$ using superfields with both two- and four-component spinors. In the four-component theory [24] they also find a nonrenormalisation theorem. Their analysis dimensionally reduces $SQED_4$ to $SQED_3$, introducing a compactification scale in the process. After truncating all two-particle irreducible diagrams from the DSE, taking the limit that all momenta are small compared to the momentum scale of the compactification, and making several other approximations, they find the same prefactor in front of the effective mass as Clark and Love and claim that its cancellation by infrared divergences is subverted by the lack of a corresponding prefactor in the renormalisation factor $\mathcal{Z}$. Since their argument depends on dimensional reduction of $SQED_4$, it cannot be applied in $3+1$ dimensions.

The essence of the nonperturbative nonrenormalisation theorem is that the chiral solution to the DSE in SQED is not merely favoured but unique, ie. that there is no achiral solution. However our analysis in the component formalism finds no evidence for such a theorem. It is certainly the case that no vanishing gauge dependent prefactor emerges.

Performing the Wick rotation into Euclidean space and substituting the full vertices into the DSE gives us the following integral equations:

$$
\begin{aligned}
B(p^2) \;=\; & 2e^2\int\frac{d^dq}{(2\pi)^d}\frac{1}{(p-q)^2}\frac{1}{p^2-q^2}[D_{af}(q^2)p^2A(p^2) \\
& \qquad +(p^2-2q^2)D_{aa}(q^2)B(p^2)] \\
& +(\xi-1)e^2\int\frac{d^dq}{(2\pi)^d}\frac{1}{(p-q)^4}[D_{af}(q^2)p^2A(p^2)+D_{aa}(q^2)q^2B(p^2)] \\
& -(\xi-1)e^2\int\frac{d^dq}{(2\pi)^d}\frac{p\cdot q}{(p-q)^4}[D_{af}(q^2)A(p^2)+D_{aa}(q^2)B(p^2)] \\
& -\frac{1}{2}e^2\int\frac{d^dq}{(2\pi)^d}D_{af}(q^2)T_{aa}(p^2,q^2,p\cdot q) \\
& -e^2\int\frac{d^dq}{(2\pi)^d}D_{aa}(q^2)T_{af}(p^2,q^2,p\cdot q)[\frac{(p\cdot q)^2-p^2q^2}{(p-q)^2}+q^2-p\cdot q]
\end{aligned}
$$

$$+\frac{1}{2}e^2\int\frac{d^dq}{(2\pi)^d}D_{af}(q^2)T_{ff}(p^2,q^2,p\cdot q)(p^2\frac{q^2-p\cdot q}{(p-q)^2}+q^2\frac{p^2-p\cdot q}{(p-q)^2}), \tag{7.4.1}$$

$$\begin{aligned}
A(p^2)-1 \;=\; & 2e^2\int\frac{d^dq}{(2\pi)^d}\frac{1}{(p-q)^2}\frac{1}{p^2-q^2}D_{aa}(q^2)\{(p^2-2q^2)A(p^2)+q^2A(q^2)\} \\
& -2e^2\int\frac{d^dq}{(2\pi)^d}\frac{1}{(p-q)^2}\frac{1}{p^2-q^2}D_{af}(q^2)(B(p^2)-B(q^2)) \\
& +(\xi-1)e^2\int\frac{d^dq}{(2\pi)^d}\frac{1}{(p-q)^4}D_{aa}(q^2)q^2\{A(p^2)+A(q^2)\} \\
& -(\xi-1)e^2\int\frac{d^dq}{(2\pi)^d}\frac{1}{(p-q)^4}D_{af}(q^2)(B(p^2)-B(q^2)) \\
& +(\xi-1)\frac{e^2}{p^2}\int\frac{d^dq}{(2\pi)^d}\frac{p\cdot q}{(p-q)^4}D_{af}(q^2)(B(p^2)-B(q^2)) \\
& -(\xi-1)\frac{e^2}{p^2}\int\frac{d^dq}{(2\pi)^d}\frac{p\cdot q}{(p-q)^4}D_{aa}(q^2)[p^2A(p^2)+q^2A(q^2)] \\
& +\frac{e^2}{p^2}\int\frac{d^dq}{(2\pi)^d}D_{aa}(q^2)T_{aa}(p^2,q^2,p\cdot q) \\
& \qquad [\frac{3}{2}p\cdot q-p^2\frac{q^2-p\cdot q}{(p-q)^2}-p\cdot q\frac{p^2-p\cdot q}{(p-q)^2}] \\
& +\frac{1}{2}\frac{e^2}{p^2}\int\frac{d^dq}{(2\pi)^d}D_{af}(q^2)T_{af}(p^2,q^2,p\cdot q) \\
& \qquad [p\cdot q-3p^2+2(p^2-p\cdot q)\frac{q^2-p\cdot q}{(p-q)^2}] \\
& +\frac{3}{2}\frac{e^2}{p^2}\int\frac{d^dq}{(2\pi)^d}D_{aa}(q^2)T_{ff}(p^2,q^2,p\cdot q)q^2p^2,
\end{aligned} \tag{7.4.2}$$

where the dimensionality d can be either 3 or 4. Considering first the minimal SBC *ansatz* where the transverse functions are set to zero, we see immediately that there is no compelling reason why $B(p^2)$ should vanish. We can do the angular integration with the minimal *ansatz*. We get

$$\begin{aligned}
B(p^2) \;=\; & \frac{e^2}{8\pi^2p}\int dq\frac{q}{p^2-q^2}(\ln|p+q|-\ln|p-q|)[D_{af}(q^2)A(p^2)(3p^2+q^2+\xi(p^2-q^2)) \\
& +D_{aa}(q^2)B(p^2)(3p^2-q^2+\xi(p^2-q^2))] \\
& +(\xi-1)\frac{e^2}{4\pi^2}\int dq\frac{q^2}{p^2-q^2}[D_{af}(q^2)A(p^2)-D_{aa}(q^2)B(p^2)],
\end{aligned} \tag{7.4.3}$$

which shows no sign of vanishing in any gauge. We can also eliminate the possibility of the transverse functions inducing a nonrenormalisation theorem by cancelling off the minimal contribution. To see this, recall that the transverse functions are required to be symmetric in p and q by charge conjugation invariance and examine the coefficients of $D_{af}(q^2)$. Its coefficients in the minimal contribution are asymmetric in p, q because of the $A(p^2)$ factor whereas those in the transverse contribution are exactly symmetric. (A corresponding argument for terms in Eq. (7.4.1) proportional to $D_{aa}(q^2)$ cannot be made because the transverse contribution has an asymmetric component.) It follows that the integrand of $B(p^2)$ will not vanish, regardless of the choice of transverse functions. This result will still hold after angular integration as the transverse contribution will be a symmetric function multiplied by q^{d-1} whereas the minimal contribution will not. However we cannot eliminate the possibility that Eqs. (7.4.1),(7.4.2) may simply not be solvable unless $B(p^2)$ is set to zero.

This apparent contradiction between ourselves and Campbell-Smith *et al.* [24] requires explanation. If an achiral solution is forbidden in the superfield formalism then it must also be forbidden in the component formalism, and yet our analysis finds no evidence that an achiral solution cannot exist. It is conceivable that our choice of the quenched approximation has obscured a nonrenormalisation theorem since the Campbell-Smith and Mavromatos paper includes the effects of massless matter loops. We consider this unlikely however, as both their and Clark and Love's vanishing prefactor arises from the superspace integration without considering the form of the vacuum polarisation.

Since the quenched approximation is the only one used to derive Eqs. (7.4.1), (7.4.2), the most likely possibility is that the vanishing gauge dependent prefactor in superspace treatments [20, 24] is an artifact of the extensive approximations used. The approximations in [24] were generally chosen so as to have minimal impact in the infrared region where chiral symmetry breaking is largely determined, but to combine such approximations with a gauge dependent argument is dangerous. Consider, for example, the equivalent of Eq. (6.3.3) in non-SUSY QED_3 in the quenched rainbow approximation,

$$B(p^2) = (\xi + 2)\frac{e^2}{4\pi^2 p}\int_0^\infty dq \frac{qB(q^2)}{q^2A^2(q^2) + B^2(q^2)}(\ln|p+q| - \ln|p-q|). \qquad (7.4.4)$$

In the special gauge of $\xi = -2$ the right hand side of Eq. (7.4.4) vanishes unless $A(q^2)$ and $B(q^2)$ conspire to cancel this prefactor. It does not follow though that chiral symmetry is unbroken. In fact non-SUSY QED_3 is known to break chiral symmetry from lattice studies [29, 40]. The vanishing prefactor in Eq. (7.4.4) is an artifact of the rainbow approximation.

"One Ring to rule them all,
One Ring to find them,
One Ring to bring them all
and in the darkness bind them...."

- "The Lord of the Rings", J.R.R. Tolkein.

CHAPTER 8

Conclusions and Summary

We have investigated several nonpertubative tools from conventional (ie. non-SUSY) QFT to see how well they could be adapted to the study of chiral symmetry breaking in SUSY theories, SQED in particular. Some interesting limits on how the various concepts could be adapted were found. We have also investigated the possibility of a nonrenormalisation theorem in SQED that claim the DSE does not allow a solution with a spontaneously generated mass.

The use of DSEs in SQED immediately raises the issue of how to treat the propagators of the electron's super-partners. It is acceptable to take the photon and photino propagators as bare (quenched approximation) but to do this for the super-partners of the electron constitutes an unacceptable violation of SUSY. The hard way to deal with this is to simultaneously solve the DSE for the electron's partners. However this approach constitutes a possibly intractable numerical nightmare, and there is a much easier way. SWIs relating the scalar propagators to the electron propagator have been known for some time [19]. By substituting in the scalar propagators found in this way it becomes sufficient to solve the electron DSE on its own. This is only true strictly speaking if SUSY is not spontaneously broken, which is the case in SQED [25, 35].

Finding the form of both the electron DSE and the CJT effective potential was not a trivial exercise as some way of consistently handling the auxiliary fields f and g had to be found first. The difficulty lay in interpreting these fields, which are not fully fledged fields in their own right since they are proportional to a and b respectively on mass shell. However they cannot be ignored either as many quantum processes involve loop integrals which go off mass shell. They have these properties because

1. the auxiliary fields have no derivatives in the Lagrangian, and

2. they occur in off-diagonal quadratic terms like $a^* f$.

A revealing inconsistency of treating the auxiliary fields näively is that the two-point proper functions of scalars, calculated from SWIs, are not the inverses of their corresponding propagators. This tells us that the true propagators are not what we näively expect.

We need these fields in a Gaussian integral before we can apply our standard methods and interpretions to them, and we achieve this by replacing the fields a, f, b, g with the matrix fields $\begin{pmatrix} a \\ f \end{pmatrix}$ and $\begin{pmatrix} b \\ g \end{pmatrix}$. Substituting this notation into the Lagrangian gives it the standard quadratic form we require with the propagators of $\begin{pmatrix} a \\ f \end{pmatrix}$ and $\begin{pmatrix} b \\ g \end{pmatrix}$ being $\begin{pmatrix} D_{aa} & D_{af} \\ D_{fa} & D_{ff} \end{pmatrix}$ and $\begin{pmatrix} D_{bb} & D_{bg} \\ D_{gb} & D_{gg} \end{pmatrix}$ respectively. Furthermore, their two-point proper functions are the inverses of the propagators so we are now in a position to handle the scalars with some confidence.

Using these 'matrix' propagators for the electron's super-partners in the CJT effective potential we extend the perturbative result that the effective potential is uniformly zero [23, 26]. Since we are dealing with a nonperturbative phenomena it is not unreasonable to hope that the theorem might cease to apply, although we would expect the preferred solution to the DSE to have an effective potential of zero since SUSY is not broken. What we found was that the CJT effective potential is exactly zero regardless of whether the solution is chiral or achiral, and therefore zero everywhere since the solutions to the DSE are its turning points [33]. This is a generalisation of a result of Pisarski [25] who found the effective potential to be uniformly zero in the many-flavour limit. Hence the CJT effective potential cannot be used in the manner described in Sec. 3.6 to identify the preferred solution to the DSE. This result holds for any theory in which SUSY remains unbroken.

Substituting the matrix notation into the electron DSE gives us Eq. (5.4.13), which is what we would have näively guessed. (It is, of course, far preferable to derive it rigorously.) Applying a standard iterative procedure allows us to find both a chiral and an achiral solution in the rainbow approximation. We are not satisfied to rest there however, as the rainbow approximation is known to be highly deficient due to its inconsistency with the WTI. We found in this work that it is also incompatible with SUSY. One way to see this is to write down the DSE for the (matrix) scalar propagators and substitute in the rainbow approximation (See Eq. (7.1.1)). The equation is easily seen to be overconstrained, even when ignoring the further constraint of the electron DSE. Another way is to inspect the SWIs that govern the vertices. Some of them (Eqs. (7.2.9),(7.2.10)) contain the electron propagator and are therefore violated if the propagators are dressed while the vertices remain bare.

Deriving the SWIs that constrain the vertices was necessary for us to properly transcend the rainbow approximation in SQED. Other authors [18] have attempted to do so by multiplying the bare vertices by the renormalisation factor $A(p^2)$ to effect compliance with the WTI at zero momentum transfer. In fact they only achieve this within the approximation that $A(p^2)$ and $B(p^2)$ are flat everywhere. While this approximation is valid over most of the momentum range, it is not true in the infrared where the dynamics of chiral symmetry breaking are largely determined. This approximation is therefore a poor one and we sought to improve upon it.

An important point to remember when deriving SWIs is that the Lagrangian in the component formalism is in WZ gauge in which SUSY is no longer explicit. The action then is not invariant to SUSY transformations δ_S but to SUSY transformations followed by a gauge transformation, δ_{WZ}, which restores the WZ gauge. It is from this additional gauge transformation that the propagators enter the SWIs for the vertices. It must also be remembered that the f and g fields cannot be simply discarded. It is then a straightforward though lengthy set of calculations, to derive the many SWIs, given in Sec. 7.2, which the vertices must conform to.

The SWIs, combined with the WTIs, determine the vertices up to three unknown functions, the transverse components of the scalar-photon vertices. The general form of the vertices, which constitutes a SUSY generalisation of the BC vertex, is presented in Sec. 7.3.

The form of the equation for $B(p^2)$ when dressed vertices are used casts enormous doubt over the existence of a nonperturbative nonrenormalisation theorem in SQED. There has been some controversy over this theorem. The initial result by Clark and Love [20] found a prefactor of $\xi - 1$ in the effective mass which vanishes in Feynman gauge. This approach was criticised by Kaiser and Selipsky [21] who suggested the prefactor may be cancelled in the limit $\xi \to 1$. Campbell-Smith and Mavromatos [24] have recently entered the fray with an investigation into SQED_3 with four-component fermions (N=2 SUSY in their notation), which found that such a cancellation did not occur. This contradicts an earlier study by Pisarski [25] in the many-flavour limit, which not only derived a massive solution, but found it was dynamically preferred. Both papers finding a nonrenormalisation theorem used the superfield formalism and required enormous truncations to be made in order to eliminate both the spurious infrared divergences inherent to the superfield formalism and the infinite number of terms in the self energy. Our approach required no such truncations. Indeed our only approximation was the quenched approximation. Using the form of Eq. (7.4.1) and the symmetry of the transverse functions due to charge conjugation invariance, we show that the integrand of $B(p^2)$ cannot vanish, regardless of gauge, even after angular integration. While we cannot rule out the possibility of a nonrenormalisation theorem entirely, since we haven't actually succeeded in finding an achiral solution, we have made a strong case against it.

In brief, we have used the DSE to find both chiral and achiral propagators in the rainbow approximation of SQED, although we also find several difficiences of this approximation. We developed a useful notation for properly dealing with auxiliary fields and used it to prove the nonrenormalisation theorem for the effective potential at the nonpertubative level. We then found the complete set of SWIs restricting the many vertices in the theory and solved them to find the most general form these vertices can take. Finally, we investigate the possibility of a nonperturbative nonrenormalisation theorem in either $SQED_3$ or $SQED_4$ using the algebraic form of the effective mass and, finding the possibility to be remote, suggest that its appearance in previous works is an artifact of their approximations.

"My son, there is something else to watch out for. There is no end to the writing of books, and too much study will wear you out."

\- Ecclesiastes 12:12.

APPENDIX A

Spinor and Clifford Algebra Conventions

We state the conventions used in this paper for the spinor algebra and $3+1$ dimensional Clifford algebra in this appendix. The Clifford algebra we use in $2+1$ dimensions is described separately in Sec. 6.2 due to subtle complications involving the charge conjugation and parity matrices. Our conventions are based largely on those found in [4].

Two-spinor indices are raised and lowered by the anti-symmetric Levi-Civita tensors

$$\varepsilon_{12} = \varepsilon^{12} = -\varepsilon_{\dot{1}\dot{2}} = -\varepsilon^{\dot{1}\dot{2}} = +1. \tag{A1}$$

We then define the raising and lowering of spinor indices by

$$\psi^\alpha \equiv \varepsilon^{\alpha\beta}\psi_\beta,\ \bar{\psi}^{\dot{\alpha}} \equiv \bar{\psi}_{\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\alpha}},\ \psi_\alpha \equiv \psi^\beta\varepsilon_{\beta\alpha},\ \bar{\psi}_{\dot{\alpha}} \equiv \varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}}. \tag{A2}$$

The summation convention we employ is "ten to four" for undotted indices and "eight to two" for dotted ones, ie.

$$\psi\lambda = \psi^\alpha\lambda_\alpha, \quad \bar{\psi}\bar{\lambda} = \bar{\psi}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}. \tag{A3}$$

We define the matrices σ^μ by

$$\sigma^\mu = (I, \sigma^i), \tag{A4}$$

where σ^i are the Pauli matrices. It is also useful to define

$$\bar{\sigma}^\mu = (I, -\sigma^i). \tag{A5}$$

The Dirac matrices γ^μ are given by

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \tag{A6}$$

and fulfill

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \tag{A7}$$

where

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1), \tag{A8}$$

is the metric. The four-spinors on which they act are of the form

$$\psi \equiv \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}, \tag{A9}$$

It shall be understood from context whether a spinor has two or four components. The charge conjugation matrix is given by

$$C \equiv \begin{pmatrix} -\varepsilon_{\alpha\beta} & 0 \\ 0 & -\varepsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix}. \tag{A10}$$

Defining

$$\bar{\psi} \equiv \psi^\dagger \gamma^0, \tag{A11}$$

we say that a four-spinor is Majorana if

$$\psi = C\bar{\psi}^T \Leftrightarrow \bar{\psi} = -\psi^T C^\dagger. \tag{A12}$$

The left- and right-handed components of ψ are projected by the operator $\frac{1}{2}(1 \pm \gamma_5)$ where γ_5 is given by

$$\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{A13}$$

Finally, is be useful to define the notations

$$\sigma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu), \tag{A14}$$

and

$$\gamma_{\mu 5} = i\gamma_\mu\gamma_5. \tag{A15}$$

The Clifford algebra is then given by the 16 matrices

$$\gamma_A = \{I, \gamma_\mu, \sigma_{\mu\nu}, \gamma_5, \gamma_{\mu 5}\}. \tag{A16}$$

APPENDIX B

Euclidean Space Co-ordinates

In this appendix we give the Euclidean space conventions used in this thesis. The motivation for using Euclidean space is given in Sec. 2.2. Most of the work in this thesis is in Minkowski space although we perform the Wick rotation into Euclidean space before substituting our *ansätze* into the DSEs. The presentation here is in $3+1$ dimensions. Where the reduction to $2+1$ is not straightforward we deal with it explicitly.

Using the superscripts E, M to denote Euclidean and Minkowski space conventions respectively, the spacetime coordinates are related by

$$\begin{aligned} x_4^E &= ix^{0\,M} = ix_0^M \\ x_{1,2,3}^E &= x^{1,2,3\,M} = -x_{1,2,3}^M, \end{aligned} \tag{B1}$$

Momenta, derivatives and vector potentials are related by

$$\begin{aligned} P_4^E &= -iP^{0\,M} = -iP_0^M \\ P_{1,2,3}^E &= -P^{1,2,3\,M} = P_{1,2,3}^M, \end{aligned} \tag{B2}$$

while Dirac matrices transform as

$$\begin{aligned} \gamma_4^E &= \gamma^{0\,M} = \gamma_0^M \\ \gamma_{1,2,3}^E &= -i\gamma^{1,2,3\,M} = i\gamma_{1,2,3}^M, \end{aligned} \tag{B3}$$

while γ_5 remains unchanged. The relation between the volume elements is therefore

$$(d^4x)^E = i(d^4x)^M = i(dx^0dx^1dx^2dx^3)^M, \tag{B4}$$

from Eq. (B1). The relation between the path integrals is given by

$$\int \mathcal{D}\phi \mathrm{e}^{iS[\phi(x)]^M} = \int \mathcal{D}\phi \mathrm{e}^{-S[\phi(x)]^E}. \tag{B5}$$

We also have in $SQED_3$ the matrices γ_P, γ_W, and γ_{45}, which remain unchanged.

The prescription for converting a momentum space Green's or proper function from Minkowski space to Euclidean space is as follows:

1. Multiply every squared momentum, every dot product of two momenta, and every momentum with a free Lorentz index by -1 (from Eq. (B2)),

2. Multiply every dot product of a γ matrix with momentum, and every γ matrix with a free Lorentz index by i (from Eq. (B3)) and

3. Multiply by -1 overall (from Eqs. (B4),(B5)).

Applying this prescription to the propagators found in Sec. 5.1 gives us

$$S^E(p) = \frac{1}{\gamma \cdot p^E A(p^2) - iB(p^2)} \quad \text{(B6)}$$

$$D^E_{aa}(p^2) = \frac{A(p^2)}{p^2 A^2(p^2) + B^2(p^2)} \quad \text{(B7)}$$

$$D^E_{af}(p^2) = \frac{B(p^2)}{p^2 A^2(p^2) + B^2(p^2)} \quad \text{(B8)}$$

$$D^E_{ff}(p^2) = \frac{-p^2 A(p^2)}{p^2 A^2(p^2) + B^2(p^2)} \quad \text{(B9)}$$

The Euclidean space propagators for the photon and photino are

$$D^E_{\mu\nu}(p) = (\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2})\frac{1}{p^2} + \xi \frac{p_\mu p_\nu}{p^4}, \quad \text{(B10)}$$

and

$$S^E_\lambda(p) = \frac{1}{i\gamma \cdot p}, \quad \text{(B11)}$$

repectively. The Euclidean two point proper functions are

$$\Gamma^E_{\bar{\psi}\psi}(p) = (S^E(p))^{-1} \quad \text{(B12)}$$

$$\Gamma^E_{a^*a}(p^2) = \Gamma^E_{b^*b}(p^2) = p^2 A(p^2), \quad \text{(B13)}$$

$$\Gamma^E_{a^*f}(p^2) = \Gamma^E_{f^*a}(p^2) = \Gamma^E_{b^*g}(p^2) = \Gamma^E_{g^*b}(p^2) = B(p^2), \quad \text{(B14)}$$

$$\Gamma^E_{f^*f}(p^2) = \Gamma^E_{g^*g}(p^2) = -A(p^2). \quad \text{(B15)}$$

We convert the three-point proper functions calculated in Sec. 7.3 to Euclidean space.

The scalar-photon vertices are

$$\Gamma^{\mu E}_{a^* A_\mu a}(p,q) = \Gamma^{\mu E}_{b^* A_\mu b}(p,q) = \frac{e}{p^2-q^2}(p^2 A(p^2) - q^2 A(q^2))(p+q)^\mu$$
$$-[p^\mu(q^2 - p\cdot q) + q^\mu(p^2 - p\cdot q)]T^E_{aa}(p^2,q^2,p\cdot q), \qquad \text{(B16)}$$

$$\Gamma^{\mu E}_{a^* A_\mu f}(p,q) = \Gamma^{\mu E}_{b^* A_\mu g}(p,q) = \Gamma^{\mu E}_{f^* A_\mu a}(p,q) = \Gamma^{\mu E}_{g^* A_\mu b}(p,q)$$
$$= \frac{e}{p^2-q^2}(B(p^2) - B(q^2))(p+q)^\mu$$
$$-[p^\mu(q^2 - p\cdot q) + q^\mu(p^2 - p\cdot q)]T^E_{af}(p^2,q^2,p\cdot q), \qquad \text{(B17)}$$

$$\Gamma^{\mu E}_{f^* A_\mu f}(p,q) = \Gamma^{\mu E}_{g^* A_\mu g}(p,q) = \frac{-e}{p^2-q^2}(A(p^2) - A(q^2))(p+q)^\mu$$

$$-[p^\mu(q^2 - p\cdot q) + q^\mu(p^2 - p\cdot q)]T^E_{ff}(p^2,q^2,p\cdot q), \qquad \text{(B18)}$$

where the three functions $T^E_{aa}(p^2,q^2,p\cdot q)$, $T^E_{af}(p^2,q^2,p\cdot q)$ and $T^E_{ff}(p^2,q^2,p\cdot q)$, are free of kinematic singularities and symmetric in p and q but are otherwise unconstrained. The photino vertices in Euclidean space are

$$\begin{aligned}\Gamma^E_{\bar\lambda a^* \psi}(p,q) &= \frac{-e}{p^2-q^2}(p^2 A(p^2) - q^2 A(q^2)) + i\frac{e}{p^2-q^2}(B(p^2) - B(q^2))\gamma\cdot q \\ &+\frac{1}{2}e(p^2 - \gamma\cdot q\gamma\cdot p)T^E_{aa}(p^2,q^2,p\cdot q) \\ &+i\frac{1}{2}e[\gamma\cdot p(p^2-q^2) - 2\gamma\cdot q(p^2 - p\cdot q)]T^E_{af}(p^2,q^2,p\cdot q) \\ &-\frac{1}{2}ep^2(q^2 - \gamma\cdot p\gamma\cdot q)T^E_{ff}(p^2,q^2,p\cdot q), \qquad \text{(B19)}\end{aligned}$$

and

$$\begin{aligned}\Gamma^E_{\bar\lambda f^* \psi}(p,q) &= \frac{-ie}{p^2-q^2}(A(p^2) - A(q^2))\gamma\cdot q - \frac{e}{p^2-q^2}(B(p^2) - B(q^2)) \\ &-\frac{1}{2}ie(\gamma\cdot p - \gamma\cdot q)T^E_{aa}(p^2,q^2,p\cdot q) \\ &+\frac{1}{2}e(p-q)^2 T^E_{af}(p^2,q^2,p\cdot q) \\ &-\frac{1}{2}ie\gamma\cdot q(p^2 - \gamma\cdot p\gamma\cdot q)T^E_{ff}(p^2,q^2,p\cdot q). \qquad \text{(B20)}\end{aligned}$$

We present finally the Euclidean electron-photon vertex,

$$\Gamma^{\mu E}_{\bar\psi A_\mu \psi}(p,q) = \Gamma^{\mu E}_{BC}(p,q) + \frac{ie}{p^2-q^2}(A(p^2) - A(q^2))[\frac{1}{2}T^{\mu E}_3 - T^{\mu E}_8]$$

$$
\begin{aligned}
&-\frac{ie}{p^2-q^2}(B(p^2)-B(q^2))T_5^{\mu E}-\frac{1}{2}ieT_{aa}^E(p^2,q^2,p\cdot q)T_3^{\mu E}\\
&+ieT_{af}^E(p^2,q^2,p\cdot q)[\frac{1}{2}(p-q)^2T_5^{\mu E}+T_1^{\mu E}]\\
&-\frac{1}{2}ieT_{ff}(p^2,q^2,p\cdot q)[T_2^{\mu E}+p\cdot qT_3^{\mu E}+(p-q)^2T_8^{\mu E}],
\end{aligned}
\tag{B21}
$$

where

$$
\begin{aligned}
\Gamma_{BC}^{\mu E}(p,q) &= \frac{1}{2}\frac{ie}{p^2-q^2}(\gamma\cdot p+\gamma\cdot q)(A(p^2)-A(q^2))(p+q)^\mu\\
&+e\frac{1}{2}(A(p^2)+A(q^2))\gamma^\mu-\frac{ie}{p^2-q^2}(B(p^2)-B(q^2))(p+q)^\mu,
\end{aligned}
\tag{B22}
$$

$$T_1^{\mu E} = p^\mu(q^2-p\cdot q)+q^\mu(p^2-p\cdot q), \tag{B23}$$

$$T_2^{\mu E} = i(\gamma\cdot p+\gamma\cdot q)T_1^\mu, \tag{B24}$$

$$T_3^{\mu E} = i(\gamma\cdot p-\gamma\cdot q)(p-q)^\mu]-i\gamma^\mu(p-q)^2, \tag{B25}$$

$$T_5^{\mu E} = -\sigma^{\mu\nu}(p-q)_\nu, \tag{B26}$$

$$T_8^{\mu E} = -i\frac{1}{2}(\gamma\cdot p\gamma\cdot q\gamma^\mu-\gamma^\mu\gamma\cdot q\gamma\cdot p). \tag{B27}$$

The Euclidean space form of the D vertices are found similarly. Since they are not used we do not present them here.

APPENDIX C

The SUSY Algebra

In Sec. 4.2 we discussed the essential (anti)commutation relations that make up the SUSY algebra. In this appendix we give the entire algebra and derive the more important relations. The material in this appendix is largely taken from [4] although use was made also of [6, 34]. As stated in Sec. 4.2, SUSY theories are based on a graded Lie algebra, or superalgebra, instead of an ordinary Lie algebra. We restate the graded Jacobi identites

$$[[B_1, B_2], B_3] + [[B_3, B_1], B_2] + [[B_2, B_3], B_1] = 0, \quad \text{(C1)}$$
$$[[B_1, B_2], Q_3] + [[Q_3, B_1], B_2] + [[B_2, Q_3], B_1] = 0, \quad \text{(C2)}$$
$$\{[B_1, Q_2], Q_3\} + \{[Q_3, B_1], Q_2\} + [\{Q_2, Q_3\}, B_1] = 0, \quad \text{(C3)}$$
$$[\{Q_1, Q_2\}, Q_3] + [\{Q_3, Q_1\}, Q_2] + [\{Q_2, Q_3\}, Q_1] = 0, \quad \text{(C4)}$$

where B represents any bosonic generator and Q any fermionic (SUSY) generator. Haag, Lopuszański and Sohnius used Eqs. (C1-C4) together with the properties of representations of the Lorentz group to show that the SUSY algebra must be given by the identities in Eqs. (C5) to (C18) if it is to avoid the Coleman-Mandula no-go theorem in a suitable way.

We first demonstrate that the fermionic generators Q can only have spin $\frac{1}{2}$. Eq. (C3) tells us that $\{Q, \bar{Q}\}$ must be bosonic. If Q is in the Lorentz representation (j, j'), then $\{Q, \bar{Q}\}$ is in the representation $(j+j', j+j')$. The only bosonic element in the superalgebra which meets this criterion is the translation operator P_μ, in the representation $(\frac{1}{2}, \frac{1}{2})$. This requires Q and $\bar{Q}$ to be spin $\frac{1}{2}$. It follows from this that

$$[Q_{\alpha i}, M_{\mu\nu}] = -\frac{1}{2} i (\sigma_{\mu\nu})_\alpha{}^\beta Q_{\beta i},$$
$$[\bar{Q}^i_{\dot{\alpha}}, M_{\mu\nu}] = \frac{1}{2} i (\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}} \bar{Q}^i_{\dot{\beta}},$$

where the $M_{\mu\nu}$ are the Lorentz group generators. Q can always be normalised so that

$$\{Q_{\alpha i}, \bar{Q}^j_{\dot{\beta}}\} = 2\delta^j_i(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu.$$

The remaining (anti)commutators can found by manipulating the graded Jacobi identites. We state the defining relations [4] of the SUSY algebra.

$$\begin{aligned}
[P_\mu, P_\nu] &= 0 && \text{(C5)}\\
[P_\mu, M_{\nu\rho}] &= i(\eta_{\mu\nu}P_\rho - \eta_{\mu\rho}P_\nu) && \text{(C6)}\\
[M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\mu\sigma}M_{\nu\rho}) && \text{(C7)}\\
[B_r, B_s] &= ic^t_{rs}B_t && \text{(C8)}\\
[B_r, P_\mu] &= [B_r, M_{\mu\nu}] = 0 && \text{(C9)}\\
[Q_{\alpha i}, P_\mu] &= [\bar{Q}^i_{\dot{\alpha}}, P_\mu] = 0 && \text{(C10)}\\
[Q_{\alpha i}, M_{\mu\nu}] &= -\frac{1}{2}i(\sigma_{\mu\nu})_\alpha{}^\beta Q_{\beta i} && \text{(C11)}\\
[\bar{Q}^i_{\dot{\alpha}}, M_{\mu\nu}] &= \frac{1}{2}i(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}\bar{Q}^i_{\dot{\beta}} && \text{(C12)}\\
[Q_{\alpha i}, B_r] &= (b_r)^j_i Q_{\alpha j} && \text{(C13)}\\
[\bar{Q}^i_{\dot{\alpha}}, B_r] &= -\bar{Q}^j_{\dot{\alpha}}(b_r)^i_j && \text{(C14)}\\
\{Q_{\alpha i}, \bar{Q}^j_{\dot{\beta}}\} &= 2\delta^j_i(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu\} && \text{(C15)}\\
\{Q_{\alpha i}, Q_{\beta j}\} &= 2\varepsilon_{\alpha\beta}Z_{ij} && \text{(C16)}\\
\{\bar{Q}^i_{\dot{\alpha}}, \bar{Q}^j_{\dot{\beta}}\} &= -2\varepsilon_{\dot{\alpha}\dot{\beta}}Z^{ij} \text{ where} Z^{ij} = Z^\dagger_{ij} && \text{(C17)}\\
[Z_{ij}, \text{anything}] &= 0. && \text{(C18)}
\end{aligned}$$

The terms Z_{ij} are linear combinations of the group generators B_r. They can be shown by combining various Jacobi identities to commute with each other. This implies that they form an invariant Abelian subalgebra of the gauge group. Hence Eq. (C18) holds and the Z_{ij} are called "central charges". It can also be shown that $Z_{ij} = -Z_{ji}$ so that in an $N = 1$ SUSY, such as we are using, the central charges must be zero.

APPENDIX D

Verifying that the Multiplets Represent the SUSY Algebra

In chapter 4 we presented the chiral and general multiplets and claimed without proof that they represent the SUSY algebra. We verify this claim here. We present the proof in $3+1$ dimensions but its reduction to $2+1$ dimensions is also valid. The charge conjugation properties of the γ matrices are crucial. We present them in table D1. The values in the third column follow from those in the second because

Table D1: The commutation properties of the γ matrices with the charge conjugation matrix C. ζ_1 and ζ_2 are Majorana.

γ_A	$C\gamma_A C^{-1}$	$\bar{\zeta}_2\gamma_A\zeta_1$
γ_μ	$-\gamma_\mu^T$	$-\bar{\zeta}_1\gamma_\mu\zeta_2$
γ_5	γ_5^T	$\bar{\zeta}_1\gamma_5\zeta_2$
$\gamma_{\mu 5}$	$\gamma_{\mu 5}^T$	$\bar{\zeta}_1\gamma_{\mu 5}\zeta_2$
$\sigma_{\mu\nu} = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$	$-\sigma_{\mu\nu}^T$	$-\bar{\zeta}_1\sigma_{\mu\nu}\zeta_2$

if $C\gamma_A C^{-1} = \pm\gamma_A^T$, then

$$\begin{aligned} \bar{\zeta}_2\gamma_A\zeta_1 &= -\zeta_2^T C^{-1}\gamma_A C\bar{\zeta}_1^T \\ &= \mp\zeta_2^T\gamma_A^T\bar{\zeta}_1^T \\ &= \pm\bar{\zeta}_1\gamma_A\zeta_2 \quad \text{taking transpose.} \end{aligned} \tag{D1}$$

We will also make use of the four-component Fierz identity. To prove the Fierz identity we make use of the following lemma:

Lemma:

$$\delta_\alpha{}^\lambda \delta_\beta{}^\sigma = \frac{1}{4}(\gamma_A)_\beta{}^\lambda(\gamma^A)_\alpha{}^\sigma, \tag{D2}$$

where we imply summation over the values of A where γ_A is one of $\{I, \gamma_\mu, \sigma_{\mu\nu}, \gamma_5, \gamma_{\mu 5}\}$.

Proof: Any 4×4 matrix X is given by

$$\begin{aligned} X &= \frac{1}{4}\gamma_A \mathrm{Tr}(X\gamma^A) \\ \Rightarrow (X)_\beta{}^\lambda &= \frac{1}{4}(\gamma_A)_\beta{}^\lambda(X)_\sigma{}^\alpha(\gamma^A)_\alpha{}^\sigma \\ &= (X)_\sigma{}^\alpha \delta_\alpha{}^\lambda \delta_\beta{}^\sigma, \end{aligned}$$

The lemma follows by dividing through by $(X)_\sigma{}^\alpha$.

Claim: If ζ, λ and ψ are four-component spinors then

$$(\bar{\zeta}\psi)\lambda = -\frac{1}{4}(\bar{\zeta}\gamma^A\lambda)\gamma_A\psi \quad \text{Fierz identity.} \tag{D3}$$

Proof:

$$\begin{aligned} (\bar{\zeta}\psi)\chi = \bar{\zeta}^\alpha\psi_\alpha\chi_\beta &= \delta_\beta{}^\sigma\delta_\alpha{}^\lambda\bar{\zeta}^\alpha\psi_\lambda\chi_\sigma \\ &= \frac{1}{4}(\gamma_A)_\beta{}^\lambda(\gamma^A)_\alpha{}^\sigma\bar{\zeta}^\alpha\psi_\lambda\chi_\sigma \quad \text{by the lemma} \\ &= -\frac{1}{4}(\bar{\zeta}\gamma^A\lambda)\gamma_A\psi \quad \text{fermions anticommute,} \ \square \end{aligned}$$

Claim: $[\delta_1, \delta_2](a, b) = 2\bar{\zeta}_2\gamma^\mu\zeta_1\partial_\mu(a, b)$.

Proof:

$$\begin{aligned} [\delta_1, \delta_2]a &= \delta_1\delta_2 a - \delta_2\delta_1 a \\ &= \delta_1(-i)\bar{\zeta}_2\psi - \delta_2(-i)\bar{\zeta}_1\psi \\ &= (-i)\bar{\zeta}_2(f + i\gamma_5 g)\zeta_1 + \bar{\zeta}_2\gamma\cdot\partial(a + i\gamma_5 b)\zeta_1 - (-i)\bar{\zeta}_1(f + i\gamma_5 g)\zeta_2 \\ &\quad -\bar{\zeta}_1\gamma\cdot\partial(a + i\gamma_5 b)\zeta_2 \\ &= -i\bar{\zeta}_2(f + i\gamma_5 g)\zeta_1 + \bar{\zeta}_2\gamma\cdot\partial(a + i\gamma_5 b)\zeta_1 \\ &\quad +i\bar{\zeta}_2(f + i\gamma_5 g)\zeta_1 + \bar{\zeta}_2\gamma\cdot\partial(a - i\gamma_5 b)\zeta_1 \\ &= 2\bar{\zeta}_2\gamma^\mu\zeta_1\partial_\mu a. \quad \square \end{aligned} \tag{D4}$$

The case for b is shown similarly.

Claim: $[\delta_1, \delta_2](f, g) = 2\bar{\zeta}_2\gamma^\mu\zeta_1\partial_\mu(f, g)$.

Proof:

$$\begin{aligned}
[\delta_1, \delta_2]f &= \delta_1\delta_2 f - \delta_2\delta_1 f \\
&= \delta_1\bar{\zeta}_2\gamma \cdot \partial\psi - \delta_2\bar{\zeta}_1\gamma \cdot \partial\psi \\
&= \bar{\zeta}_2\gamma \cdot \partial(f + i\gamma_5 g)\zeta_1 + \bar{\zeta}_2\gamma \cdot \partial i\gamma \cdot \partial(a - i\gamma_5 b)\zeta_1 \\
&\quad -\bar{\zeta}_1\gamma \cdot \partial(f + i\gamma_5 g)\zeta_2 - \bar{\zeta}_1\gamma \cdot \partial i\gamma \cdot \partial(a - i\gamma_5 b)\zeta_2 \\
&= \bar{\zeta}_2\gamma \cdot \partial(f + i\gamma_5 g)\zeta_1 + i\bar{\zeta}_2\Box(a - i\gamma_5 b)\zeta_1 \\
&\quad +\bar{\zeta}_2\gamma \cdot \partial(f - i\gamma_5 g)\zeta_1 - i\bar{\zeta}_1\Box(a - i\gamma_5 b)\zeta_2 \\
&= 2\bar{\zeta}_2\gamma \cdot \partial f\zeta_1 + i\bar{\zeta}_2\Box(a - i\gamma_5 b)\zeta_1 - i\bar{\zeta}_2\Box(a - i\gamma_5 b)\zeta_1 \\
&= 2\bar{\zeta}_2\gamma^\mu\zeta_1\partial_\mu f. \quad \Box
\end{aligned} \tag{D5}$$

The case for g is shown similarly.

Claim: $[\delta_1, \delta_2]\psi = 2\bar{\zeta}_2\gamma^\mu\zeta_1\partial_\mu f$.

Proof: We first prove the claim with the fields on mass-shell.

$$\begin{aligned}
[\delta_1, \delta_2]\psi &= \delta_1\delta_2\psi - \delta_2\delta_1\psi \\
&= \delta_1 i\gamma \cdot \partial(a + i\gamma_5 b)\zeta_2 - \delta_2 i\gamma \cdot \partial(a + i\gamma_5 b)\zeta_1 \\
&= (\bar{\zeta}_1\partial_\mu\psi)\gamma^\mu\zeta_2 - (\bar{\zeta}_1\gamma_5\partial_\mu\psi)\gamma^\mu\gamma_5\zeta_2 - (1 \longleftrightarrow 2) \\
&= -\frac{1}{4}(\bar{\zeta}_1\gamma^A\zeta_2)\gamma^\mu\gamma_A\partial_\mu\psi + \frac{1}{4}(\bar{\zeta}_1\gamma^A\zeta_2)\gamma^\mu\gamma_5\gamma_A\gamma_5\partial_\mu\psi - (1 \longleftrightarrow 2),
\end{aligned}$$

Subtracting the terms in $(1 \longleftrightarrow 2)$ will eliminate those γ_A for which $\bar{\zeta}_2\gamma_A\zeta_1 = \bar{\zeta}_1\gamma_A\zeta_2$ and reinforce those for which $\bar{\zeta}_2\gamma_A\zeta_1 = -\bar{\zeta}_1\gamma_A\zeta_2$. Denoting these latter γ_As by γ_B we have

$$[\delta_1, \delta_2]\psi = -\frac{1}{2}(\bar{\zeta}_1\gamma^B\zeta_2)\gamma^\mu\gamma_B\partial_\mu\psi + \frac{1}{2}(\bar{\zeta}_1\gamma^B\zeta_2)\gamma^\mu\gamma_5\gamma_B\gamma_5\partial_\mu\psi.$$

Those γ_B which commute with γ_5 are cancelled. Those that do not are reinforced and must be of the form γ_ν (see Table D1). Then

$$\begin{aligned}
[\delta_1, \delta_2]\psi &= -(\bar{\zeta}_1\gamma^\nu\zeta_2)\gamma^\mu\gamma_\nu\partial_\mu\psi \\
&= (\bar{\zeta}_2\gamma^\nu\zeta_1)\gamma^\mu\gamma_\nu\partial_\mu\psi,
\end{aligned}$$

and because we are on mass-shell $\gamma^\mu\partial_\mu\psi = 0$ so we have

$$\begin{aligned}
[\delta_1, \delta_2]\psi &= (\bar{\zeta}_2\gamma^\nu\zeta_1)\gamma^\mu\gamma_\nu\partial_\mu\psi + (\bar{\zeta}_2\gamma^\nu\zeta_1)\gamma_\nu\gamma^\mu\partial_\mu\psi \\
&= 2(\bar{\zeta}_2\gamma^\nu\zeta_1)\eta_{\nu\mu}\partial^\mu\psi \\
&= 2(\bar{\zeta}_2\gamma^\nu\zeta_1)\eta_{\nu\mu}\partial^\mu\psi.
\end{aligned}$$

All we need for the algebra to hold off mass-shell is for the contribution from the f and g fields to give us $(\bar{\zeta}_2\gamma^\nu\zeta_1)\gamma_\nu\gamma^\mu\partial_\mu\psi$, ie.

$$\begin{aligned}\delta_1(f+i\gamma_5 g)\zeta_2-\delta_2(f+i\gamma_5 g)\zeta_1 &= (\bar{\zeta}_1\gamma\cdot\partial\psi)\zeta_2-(\bar{\zeta}_1\gamma_5\gamma\cdot\partial\psi)\gamma_5\zeta_2-(1\longleftrightarrow 2)\\ &= -\frac{1}{4}(\bar{\zeta}_1\gamma^A\zeta_2)\gamma_A\gamma^\mu\partial_\mu\psi\\ &\quad+\frac{1}{4}(\bar{\zeta}_1\gamma^A\zeta_2)\gamma_5\gamma_A\gamma_5\gamma^\mu\partial_\mu\psi-(1\longleftrightarrow 2).\end{aligned}$$

As before, the terms $(1\longleftrightarrow 2)$ and the commutation with γ_5 eliminate all γ_A except for the γ_ν which are reinforced. Then

$$\begin{aligned}\delta_1(f+i\gamma_5 g)\zeta_2-\delta_2(f+i\gamma_5 g)\zeta_1 &= -(\bar{\zeta}_1\gamma^\nu\zeta_2)\gamma_\nu\gamma^\mu\partial_\mu\psi\\ &= (\bar{\zeta}_2\gamma^\nu\zeta_1)\gamma_\nu\gamma^\mu\partial_\mu\psi. \quad \square\end{aligned}$$

We now prove the same result for the general multiplet.

Claim: The general multiplet represents the SUSY algebra.

Proof:

$$\begin{aligned}[\delta_1,\delta_2]C &= \delta_1(\bar{\zeta}_2\gamma_5\chi)-\delta_2(\bar{\zeta}_1\gamma_5\chi)\\ &= \bar{\zeta}_2\gamma_5(M+i\gamma_5 N+i\gamma^\mu(A_\mu+i\gamma_5\partial_\mu C))\zeta_1-(1\longleftrightarrow 2)\\ &= \bar{\zeta}_2(\gamma_5 M+iN+i\gamma_5\gamma^\mu A_\mu+\gamma\cdot\partial C)\zeta_1-(1\longleftrightarrow 2)\end{aligned}$$

All of these terms are symmetric under exchange of ζ_1 and ζ_2 and vanish (see Table D1) except for $\gamma\cdot\partial C$, so

$$[\delta_1,\delta_2]C=2\bar{\zeta}_2\gamma^\mu\zeta_1\partial_\mu C, \tag{D6}$$

as required.

$$\begin{aligned}[\delta_1,\delta_2]\chi &= \delta_1(M+i\gamma_5 N+i\gamma^\mu A_\mu+\gamma_5\gamma\cdot\partial C)\zeta_2-(1\longleftrightarrow 2)\\ &= (\bar{\zeta}_1(\gamma\cdot\partial\chi+i\lambda))\zeta_2-(\bar{\zeta}_1\gamma_5(\gamma\cdot\partial\chi+i\lambda))\gamma_5\zeta_2+i(\bar{\zeta}_1\gamma_\mu\lambda)\gamma^\mu\zeta_2\\ &\quad+(\bar{\zeta}_1\partial_\mu\chi)\gamma^\mu\zeta_2+(\bar{\zeta}_1\gamma_5\partial_\mu\chi)\gamma_5\gamma^\mu\zeta_2-(1\longleftrightarrow 2)\\ &= -\frac{1}{4}\{(\bar{\zeta}_1\gamma^A\zeta_2)\gamma_A\gamma\cdot\partial\chi+i(\bar{\zeta}_1\gamma^A\zeta_2)\gamma_A\lambda-(\bar{\zeta}_1\gamma^A\zeta_2)\gamma_5\gamma_A\gamma_5\gamma\cdot\partial\chi\\ &\quad-i(\bar{\zeta}_1\gamma^A\zeta_2)\gamma_5\gamma_A\gamma_5\lambda+i(\bar{\zeta}_1\gamma^A\zeta_2)\gamma^\mu\gamma_A\gamma_\mu\lambda+(\bar{\zeta}_1\gamma^A\zeta_2)\gamma^\mu\gamma_A\partial_\mu\chi\\ &\quad+(\bar{\zeta}_1\gamma^A\zeta_2)\gamma_5\gamma^\mu\gamma_A\gamma_5\partial_\mu\chi\}-(1\longleftrightarrow 2)\end{aligned}$$

As in previous calculations, the only γ_As to survive are those that obey the equation

$$\bar{\zeta}_2\gamma^A\zeta_1=-\bar{\zeta}_1\gamma^A\zeta_2, \tag{D7}$$

ie. $\gamma_A = \sigma_{\nu\rho}, \gamma_\nu$. We check the λ terms first. Taking $\gamma_A = \gamma_\nu$,

$$\begin{aligned}
&-\frac{1}{2}i\{(\bar{\zeta}_1\gamma^\nu\zeta_2)\gamma_\nu\lambda - (\bar{\zeta}_1\gamma^\nu\zeta_2)\gamma_5\gamma_\nu\gamma_5\lambda + (\bar{\zeta}_1\gamma^\nu\zeta_2)\gamma^\mu\gamma_\nu\gamma_\mu\lambda\} \\
&= \frac{1}{2}i\{2(\bar{\zeta}_1\gamma^\nu\zeta_2)\gamma_\nu\lambda + (\bar{\zeta}_1\gamma^\nu\zeta_2)\gamma^\mu\{\gamma_\nu,\gamma_\mu\}\lambda\} - (\bar{\zeta}_1\gamma^\nu\zeta_2)\gamma^\mu\gamma_\mu\gamma_\nu\lambda\} \\
&= i\{(\bar{\zeta}_1\gamma^\nu\zeta_2)\gamma_\nu\lambda + (\bar{\zeta}_1\gamma^\nu\zeta_2)\gamma^\mu\eta_{\mu\nu}\lambda - 2(\bar{\zeta}_1\gamma^\nu\zeta_2)\gamma_\nu\lambda\} \\
&= 0.
\end{aligned}$$

Now we take $\gamma_A = \sigma_{\nu\rho}$ and note that for a given ν, ρ, any γ_μ is equally likely to commute as to anti-commute with $\sigma_{\nu\rho}$. It follows that

$$\gamma^\mu\sigma_{\nu\rho}\gamma_\mu = (2-2)\sigma_{\nu\rho} = 0. \tag{D8}$$

We then have

$$-\frac{1}{2}i\{(\bar{\zeta}_1\sigma^{\nu\rho}\zeta_2)\sigma_{\nu\rho}\lambda - (\bar{\zeta}_1\sigma^{\nu\rho}\zeta_2)\gamma_5\sigma_{\nu\rho}\gamma_5\lambda + (\bar{\zeta}_1\sigma^{\nu\rho}\zeta_2)\gamma^\mu\sigma_{\nu\rho}\gamma_\mu\lambda\} = 0. \tag{D9}$$

Now we calculate the $\gamma \cdot \partial\chi$ terms. We see immediately that γ_A must anti-commute with γ_5 as well as obeying Eq. (D7), ie. $\gamma_A = \gamma_\nu$. We find

$$\begin{aligned}
[\delta_1,\delta_2]\chi &= -(\bar{\zeta}_1\gamma^\nu\zeta_2)\gamma_\nu\gamma_\mu\partial^\mu\chi - (\bar{\zeta}_1\gamma^\nu\zeta_2)\gamma^\mu\gamma_\nu\partial_\mu\chi \\
&= -2(\bar{\zeta}_1\gamma^\nu\zeta_2)\eta_{\nu\mu}\partial^\mu\chi \\
&= 2(\bar{\zeta}_2\gamma^\nu\zeta_1)\partial_\nu\chi,
\end{aligned} \tag{D10}$$

as required. The same calculation for the scalar M is reasonably straightforward,

$$\begin{aligned}
[\delta_1,\delta_2]M &= \delta_1\bar{\zeta}_2(\gamma\cdot\partial\chi + i\lambda) - \delta_1\bar{\zeta}_2(\gamma\cdot\partial\chi + i\lambda) \\
&= \bar{\zeta}_2\gamma\cdot\partial(M + i\gamma_5 N + i\gamma^\nu(A_\nu + i\gamma_5\partial_\nu C))\zeta_1 \\
&\quad + i\bar{\zeta}_2(\sigma^{\nu\mu}\partial_\mu A_\nu + i\gamma_5 D)\zeta_1 - (1 \longleftrightarrow 2) \\
&= \bar{\zeta}_2(\gamma\cdot\partial M - i\gamma_5\gamma\cdot\partial N + i\gamma^\mu\gamma^\nu\partial_\mu A_\nu - \gamma_5\Box C \\
&\quad - i(\gamma^\mu\gamma^\nu\partial_\mu A_\nu - \partial_\mu A^\mu) + i\gamma_5 D)\zeta_1 - (1 \longleftrightarrow 2) \\
&= 2\bar{\zeta}_2(\gamma\cdot\partial M + i\gamma^\mu\gamma^\nu\partial_\mu A_\nu - i\gamma^\mu\gamma^\nu\partial_\mu A_\nu)\zeta_1 \\
&= 2(\bar{\zeta}_2\gamma^\mu\zeta_1)\partial_\mu M.
\end{aligned} \tag{D11}$$

The proof that $[\delta_1,\delta_2]N = (\bar{\zeta}_2\gamma^\mu\zeta_1)\partial_\mu N$ is almost identical to that for M so we do not show it explicitly. We now prove it for the photon field A_μ.

$$[\delta_1,\delta_2]A_\mu = \delta_1\bar{\zeta}_2(\gamma_\mu\lambda - i\partial_\mu\chi) - (1 \longleftrightarrow 2)$$

$$\begin{aligned}
&= \bar{\zeta}_2\gamma_\mu(\sigma^{\nu\rho}\partial_\rho A_\nu + i\gamma_5 D)\zeta_1 \\
&\quad -i\bar{\zeta}_2\partial_\mu(M + i\gamma_5 N + i\gamma^\nu A_\nu + \gamma_5\gamma\cdot\partial C)\zeta_1 - (1 \longleftrightarrow 2) \\
&= \bar{\zeta}_2(-\frac{1}{2}(\gamma^\nu\gamma_\mu\gamma^\rho - \gamma^\rho\gamma_\mu\gamma^\nu)\partial_\rho A_\nu + \gamma\cdot\partial A_\nu\eta^\nu_\mu - \eta^\rho_\mu\partial_\rho\gamma^\nu A_\nu + i\gamma_5 D)\zeta_1 \\
&\quad -i\bar{\zeta}_2\partial_\mu(M + i\gamma_5 N + i\gamma^\nu A_\nu + \gamma_5\gamma\cdot\partial C)\zeta_1 - (1 \longleftrightarrow 2) \\
&= 2\bar{\zeta}_2(\gamma\cdot\partial A_\mu - \partial_\mu\gamma^\nu A_\nu)\zeta_1 + 2\bar{\zeta}_2\partial_\mu\gamma^\nu A_\nu\zeta_1 \\
&= 2\bar{\zeta}_2\gamma^\nu\zeta_1\partial_\nu A_\mu. \qquad \text{(D12)}
\end{aligned}$$

We are nearly done. To prove that Eq. (4.5.2) holds for the photino we calculate

$$\begin{aligned}
[\delta_1,\delta_2]\lambda &= \sigma^{\nu\mu}\partial_\mu(\delta_1 A_\nu)\zeta_2 + i\gamma_5(\delta_1 D)\zeta_2 - (1 \longleftrightarrow 2) \\
&= \partial_\mu(\bar{\zeta}_1\gamma_\nu\lambda - i\bar{\zeta}_1\partial_\nu\chi)\sigma^{\nu\mu}\zeta_2 - (\bar{\zeta}_1\gamma_5\gamma\cdot\partial\lambda)\gamma_5\zeta_2 - (1 \longleftrightarrow 2) \\
&= -\frac{1}{4}\{(\bar{\zeta}_1\gamma^A\zeta_2)\sigma^{\nu\mu}\gamma_A\gamma_\nu\partial_\mu\lambda - (\bar{\zeta}_1\gamma^A\zeta_2)\gamma_5\gamma_A\gamma_5\gamma\cdot\partial\lambda\} - (1 \longleftrightarrow 2), \\
&= \frac{1}{4}\{(\bar{\zeta}_1\gamma^A\zeta_2)\sigma^{\mu\nu}\gamma_A\gamma_\nu\partial_\mu\lambda + (\bar{\zeta}_1\gamma^A\zeta_2)\gamma_5\gamma_A\gamma_5\gamma\cdot\partial\lambda\} - (1 \longleftrightarrow 2), \\
&= \frac{1}{4}\{(\bar{\zeta}_1\gamma^A\zeta_2)(\gamma^\mu\gamma^\nu - \eta^{\mu\nu})\gamma_A\gamma_\nu\partial_\mu\lambda + (\bar{\zeta}_1\gamma^A\zeta_2)\gamma_5\gamma_A\gamma_5\gamma\cdot\partial\lambda\} - (1 \longleftrightarrow 2),
\end{aligned}$$

and after noting that the antisymmetry with respect to ζ_1 and ζ_2 eliminates all γ_A except $\gamma_A = \sigma_{\rho\iota}, \gamma_\rho$ we have

$$\begin{aligned}
[\delta_1,\delta_2]\lambda &= \frac{1}{2}\{(\bar{\zeta}_1\sigma^{\rho\iota}\zeta_2)(\gamma^\mu\gamma^\nu - \eta^{\mu\nu})\sigma_{\rho\iota}\gamma_\nu\partial_\mu\lambda + (\bar{\zeta}_1\sigma^{\rho\iota}\zeta_2)\gamma_5\sigma_{\rho\iota}\gamma_5\gamma\cdot\partial\lambda \\
&\quad +(\bar{\zeta}_1\gamma^\rho\zeta_2)(\gamma^\mu\gamma^\nu - \eta^{\mu\nu})\gamma_\rho\gamma_\nu\partial_\mu\lambda + (\bar{\zeta}_1\gamma^\rho\zeta_2)\gamma_5\gamma_\rho\gamma_5\gamma\cdot\partial\lambda\} \\
&= -\frac{1}{2}(\bar{\zeta}_1\sigma^{\rho\iota}\zeta_2)\sigma_{\rho\iota}\gamma\cdot\partial\lambda + \frac{1}{2}(\bar{\zeta}_1\sigma^{\rho\iota}\zeta_2)\sigma_{\rho\iota}\gamma\cdot\partial\lambda - (\bar{\zeta}_1\gamma^\rho\zeta_2)\gamma^\mu\gamma_\rho\partial_\mu\lambda \\
&\quad -\frac{1}{2}(\bar{\zeta}_1\gamma^\rho\zeta_2)\gamma_\rho\gamma\cdot\partial\lambda - \frac{1}{2}(\bar{\zeta}_1\gamma^\rho\zeta_2)\gamma_\rho\gamma\cdot\partial\lambda\} \\
&= (\bar{\zeta}_2\gamma^\rho\zeta_1)(\gamma_\rho\gamma_\mu + \gamma_\mu\gamma_\rho)\partial^\mu\lambda\} \\
&= 2(\bar{\zeta}_2\gamma^\rho\zeta_1)\partial_\rho\lambda. \qquad \text{(D13)}
\end{aligned}$$

Finally,

$$\begin{aligned}
[\delta_1,\delta_2]D &= i\bar{\zeta}_2\gamma_5\gamma\cdot\partial(\delta_1\lambda) - (1 \longleftrightarrow 2), \\
&= i\bar{\zeta}_2\gamma_5\gamma\cdot\partial(\sigma^{\nu\mu}\partial_\mu A_\nu + i\gamma_5 D)\zeta_1 - (1 \longleftrightarrow 2), \\
&= \bar{\zeta}_2(i\gamma_5\gamma_\rho\partial^\rho\frac{1}{2}(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu)\partial_\mu A_\nu + \gamma\cdot\partial D)\zeta_1 - (1 \longleftrightarrow 2), \\
&= \bar{\zeta}_2(i\gamma_5\gamma_\mu(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu)\partial^\mu\partial_\mu A_\nu + \gamma\cdot\partial D)\zeta_1 - (1 \longleftrightarrow 2), \\
&= \bar{\zeta}_2(i\gamma_5(-2\gamma^\nu)\Box A_\nu + \gamma\cdot\partial D)\zeta_1 - (1 \longleftrightarrow 2), \\
&= 2(\bar{\zeta}_2\gamma_\mu\zeta_1)\partial^\mu D, \quad \Box \qquad \text{(D14)}
\end{aligned}$$

as required.

We now derive the dot product for chiral multiplets. Let $\Phi_1 = (a_1, b_1; \psi_1; f_1, g_1)$ and $\Phi_2 = (a_2, b_2; \psi_2; f_2, g_2)$. We begin our derivation of $\Phi_3 = \Phi_1 \cdot \Phi_2$ by postulating

$$a_3 = a_1 a_2 - b_1 b_2. \tag{D15}$$

The rest of Φ_3 is determined by SUSY.

$$\begin{aligned} \delta_S a_3 &= \delta_S a_1 a_2 + a_1 \delta_S a_2 - \delta_S b_1 b_2 - b_1 \delta_S b_2 \\ &= -i\bar{\zeta}\psi_1 a_2 - i a_1 \bar{\zeta}\psi_2 - \bar{\zeta}\gamma_5\psi_1 b_2 - b_1\bar{\zeta}\gamma_5\psi_2 \\ &= -i\bar{\zeta}(a_2 - i\gamma_5 b_2)\psi_1 - i\bar{\zeta}(a_2 - i\gamma_5 b_2)\psi_1. \end{aligned}$$

Requiring

$$\delta_S a_3 \equiv -i\bar{\zeta}\psi_3, \tag{D16}$$

we have

$$\psi_3 = (a_1 - i\gamma_5 b_1)\psi_2 + (a_2 - i\gamma_5 b_2)\psi_1. \tag{D17}$$

Calculating

$$\begin{aligned} \delta_S \psi_3 &= (\delta_S a_1 - i\gamma_5 \delta_S b_1)\psi_2 + (a_1 - i\gamma_5 b_1)\delta_S \psi_2 + (1 \longleftrightarrow 2) \\ &= (-i\bar{\zeta}\psi_1 - i\gamma_5(\bar{\zeta}\gamma_5\psi_1))\psi_2 + (a_1 - i\gamma_5 b_1)(f_2 + i\gamma_5 g_2)\zeta \\ &\quad + i(a_1 - i\gamma_5 b_1)\gamma\cdot\partial(a_2 + i\gamma_5 b_2)\zeta + (1 \longleftrightarrow 2) \\ &= (-i\bar{\psi}_1\zeta - i\gamma_5(\bar{\psi}_1\gamma_5\zeta))\psi_2 + (a_1 f_2 + b_1 g_2 + i\gamma_5(a_1 g_2 - b_1 f_2)) \\ &\quad + i(a_1\gamma\cdot\partial a_2 - b_1\gamma\cdot\partial b_2 - i\gamma_5(a_1\gamma\cdot\partial b_2 + b_1\gamma\cdot\partial a_2))\zeta \\ &\quad + (1 \longleftrightarrow 2). \end{aligned} \tag{D18}$$

We evalute the first term using the Fierz identity, given by Eq. (D3) and find,

$$\begin{aligned} &-i(\bar{\psi}_1\zeta + \gamma_5(\bar{\psi}_1\gamma_5\zeta))\psi_2 + (1 \longleftrightarrow 2) \\ &\quad = i\frac{1}{4}((\bar{\psi}_1\gamma^A\psi_2)\gamma_A\zeta + (\bar{\psi}_1\gamma^A\psi_2)\gamma_5\gamma_A\gamma_5\zeta) + (1 \longleftrightarrow 2). \end{aligned} \tag{D19}$$

We see by inspection that those γ_A that anticommute with γ_5 are eliminated. We then have

$$\begin{aligned} & i\frac{1}{2}((\bar{\psi}_1\psi_2) + (\bar{\psi}_1\gamma_5\psi_2)\gamma_5 + (\bar{\psi}_2\sigma^{\mu\nu}\psi_1)\sigma_{\mu\nu} + (\bar{\psi}_2\psi_1) \\ & + (\bar{\psi}_2\gamma_5\psi_1)\gamma_5 + (\bar{\psi}_2\sigma^{\mu\nu}\psi_1)\sigma_{\mu\nu})\zeta \\ = \; & i(\bar{\psi}_1\psi_2)\zeta + i(\bar{\psi}_1\gamma_5\psi_2)\gamma_5\zeta. \end{aligned} \tag{D20}$$

Requiring

$$\delta_S\psi_3 = (f_3 + i\gamma_5 g_3)\zeta + i\gamma\cdot\partial(a_3 + i\gamma_5 b_3)\zeta, \tag{D21}$$

and substituting in Eq. (D15), we find

$$\begin{aligned} b_3 &= a_1 b_2 + a_2 b_1, & \text{(D22)}\\ f_3 &= a_1 f_2 + b_1 g_2 + a_2 f_1 + b_2 g_1 + i\bar{\psi}_1\psi_2, & \text{(D23)}\\ g_3 &= a_1 g_2 - b_1 f_2 + a_2 f_1 - b_2 f_1 + \bar{\psi}_1\gamma_5\psi_2. & \text{(D24)} \end{aligned}$$

To verify that the form of b_3 is correct,

$$\begin{aligned} \delta_S b_3 &= a_1\delta_S b_2 + b_2\delta_S a_1 + (1\longleftrightarrow 2)\\ &= a_1\bar{\zeta}\gamma_5\psi_2 - ib_2\bar{\zeta}\psi_1 + (1\longleftrightarrow 2)\\ &= \bar{\zeta}\gamma_5(a_1 - i\gamma_5 b_1)\psi_2 + (1\longleftrightarrow 2)\\ &= \bar{\zeta}\gamma_5\psi_3. \end{aligned} \tag{D25}$$

The corresponding equation for f_3 is

$$\begin{aligned} \delta_S f_3 &= a_1\delta_S f_2 + f_2\delta_S a_1 + b_1\delta_S g_2 + g_2\delta_S b_1\\ &\quad +\frac{1}{2}i\delta_S(\bar{\psi}_1\psi) + (1\longleftrightarrow 2)\\ &= a_1\bar{\zeta}\gamma\cdot\partial\psi_2 - if_2\bar{\zeta}\psi_1 + ib_1\bar{\zeta}\gamma_5\gamma\cdot\partial\psi_2 + g_2\bar{\zeta}\gamma_5\psi_1 + i\bar{\psi}_1(f_2 + i\gamma_5 g_2)\zeta\\ &\quad -\bar{\psi}_1\gamma\cdot\partial(a_2 + i\gamma_5 b_2)\zeta + (1\longleftrightarrow 2)\\ &= \bar{\zeta}(a_1 + i\gamma_5 b_1)\gamma\cdot\partial\psi_2 - if_2\bar{\zeta}\psi_1 + g_2\bar{\zeta}\gamma_5\psi_1 + \bar{\zeta}(\gamma\cdot\partial(a_2 - i\gamma_5 b_2))\psi_1\\ &\quad +i\bar{\zeta}(f_2 + i\gamma_5 g_2)\psi_1 + (1\longleftrightarrow 2)\\ &= \bar{\zeta}\gamma\cdot\partial[(a_1 - i\gamma_5 b_1)\psi_2] + (1\longleftrightarrow 2)\\ &= \bar{\zeta}\gamma\cdot\partial\psi_3. \end{aligned} \tag{D26}$$

The form of g_3 is verified similarly.

The other multiplet products given in Sec. 4.5 are found by the same method. To derive the form of $V = \Phi_1 \times \Phi_2$ we start with

$$C = a_1 a_2 + b_1 b_2. \tag{D27}$$

For $V = \Phi_1 \wedge \Phi_2$ we start with

$$C = a_2 b_1 - a_1 b_2, \tag{D28}$$

and for $V_3 = V_1 \cdot V_2$ we start with

$$C_3 = C_1 C_2. \tag{D29}$$

APPENDIX E

Sample Derivation a SWI

In Sec. (7.2) we presented the 24 SWIs relating the various proper vertices of SQED while Table 7.1 listed the functional derivatives of $\delta\Gamma = (\delta_S + \delta_{WZ})\Gamma$ corresponding to each of them. The purpose of this appendix is to derive one of these identities so the reader has a clearer idea of how they were found.

We give the derivation for Eq. (7.2.1). From Table 7.1 we see that it comes from

$$\frac{\delta^3(\delta\Gamma)}{\delta a(y)\delta a^*(x)\delta\bar{\lambda}(z)} = 0. \tag{E1}$$

where $\delta\Gamma$ is given by Eq. (5.1.10). We derive Eq. (7.2.1) thus:

$$\begin{aligned}
0 &= \frac{\delta^3(\delta\Gamma)}{\delta a(y)\delta a^*(x)\delta\bar{\lambda}(z)} \\
&= \frac{\delta}{\delta a(y)}\frac{\delta}{\delta\bar{\lambda}(z)}\frac{\delta}{\delta a^*(x)}(-i(\bar{\zeta}\gamma\cdot\partial a^*(w)\frac{\delta\Gamma}{\delta\bar{\psi}(w)}) - i(\frac{\delta\Gamma}{\delta\psi(w)}\gamma\cdot\partial a(w)\zeta) \\
&\quad +ea^*(w)(\bar{\lambda}(w)\zeta)\frac{\delta\Gamma}{\delta f^*(w)} - ea(w)(\bar{\zeta}\lambda(w))\frac{\delta\Gamma}{\delta f(w)} - (\bar{\lambda}\gamma_\mu\zeta)\frac{\delta\Gamma}{\delta A_\mu(w)} \\
&\quad + \text{(terms with an expectation value of zero after the derivative is taken))} \\
&= i\frac{\delta}{\delta\bar{\lambda}(z)}\bar{\zeta}\delta(w-x)\gamma\cdot\partial\frac{\delta^2\Gamma}{\delta\bar{\psi}(w)\delta a(y)} + i\frac{\delta^3\Gamma}{\delta\bar{\lambda}(z)\delta a^*(x)\delta\psi(w)}\gamma\cdot\overleftarrow{\partial}\,\delta(w-y)\zeta \\
&\quad +e\delta(w-x)\delta(w-z)\zeta\frac{\delta^2\Gamma}{\delta f^*(w)\delta a(y)} - e\delta(w-y)\delta(w-z)\zeta\frac{\delta^2\Gamma}{\delta f(w)\delta a^*(x)} \\
&\quad -\delta(w-z)\gamma_\mu\zeta\frac{\delta^3\Gamma}{\delta a^*(x)\delta A_\mu(w)\delta a(y)} \\
&= -i\frac{\delta}{\delta\bar{\lambda}(z)}\frac{\delta^2\Gamma}{\delta a(y)\delta\bar{\psi}^T(x)}\gamma\cdot\overleftarrow{\partial}_x^T\,\bar{\zeta}^T + i\frac{\delta^3\Gamma}{\delta\bar{\lambda}(z)\delta a^*(x)\delta\psi(y)}\gamma\cdot\overleftarrow{\partial}_y\,\zeta \\
&\quad +e\delta(z-x)\zeta\frac{\delta^2\Gamma}{\delta f^*(z)\delta a(y)} - e\delta(z-y)\zeta\frac{\delta^2\Gamma}{\delta f(z)\delta a^*(x)} \\
&\quad -\gamma_\mu\zeta\frac{\delta^3\Gamma}{\delta a^*(x)\delta A_\mu(z)\delta a(y)} \text{ (integrating over w)}
\end{aligned}$$

$$
\begin{aligned}
&= iC\frac{\delta^3\Gamma}{\delta\lambda^T(z)\delta a(y)\delta\bar{\psi}^T(x)}\gamma\cdot \overleftarrow{\partial}_x^T C^{-1}\zeta + i\frac{\delta^3\Gamma}{\delta\bar{\lambda}(z)\delta a^*(x)\delta\psi(y)}\gamma\cdot \overleftarrow{\partial}_y \zeta \\
&\quad +e\delta(z-x)\zeta\frac{\delta^2\Gamma}{\delta f^*(z)\delta a(y)} - e\delta(z-y)\zeta\frac{\delta^2\Gamma}{\delta f(z)\delta a^*(x)} \\
&\quad -\gamma_\mu\zeta\frac{\delta^3\Gamma}{\delta a^*(x)\delta A_\mu(z)\delta a(y)} \\
&= iC\left(\frac{\delta^3\Gamma}{\delta\bar{\psi}(x)\delta a(y)\delta\lambda(z)}\right)^T C^{-1}\gamma\cdot \overleftarrow{\partial}_x \zeta + i\frac{\delta^3\Gamma}{\delta\bar{\lambda}(z)\delta a^*(x)\delta\psi(y)}\gamma\cdot \overleftarrow{\partial}_y \zeta \\
&\quad +e\delta(z-x)\zeta\frac{\delta^2\Gamma}{\delta f^*(z)\delta a(y)} - e\delta(z-y)\zeta\frac{\delta^2\Gamma}{\delta f(z)\delta a^*(x)} \\
&\quad -\gamma_\mu\zeta\frac{\delta^3\Gamma}{\delta a^*(x)\delta A_\mu(z)\delta a(y)} \\
&= -C\Gamma^T_{\bar{\psi}a\lambda}(p,q)C^{-1}\gamma\cdot p + \Gamma_{\bar{\lambda}a^*\psi}(p,q)\gamma\cdot q + e(B(p^2)-B(q^2)) \\
&\quad -\gamma_\mu\Gamma^\mu_{a^*A_\mu a}(p,q) \quad \text{(after a Fourier transform to momentum space).} \qquad \text{(E2)}
\end{aligned}
$$

Eq. (7.2.1) follows from charge conjugation invariance. The other SWIs are found similarly.

APPENDIX F

Derivation of the Nonperturbative Vertices

We prove [44] in this appendix that the proper vertices given in Sec. 7.3 constitute the most general possible solution to the SWIs and the WTIs, assuming freedom from kinematic singularities, charge conjugation invariance, and that

$$[\Gamma_{a^* A_\mu a}]^\mu(p,q) = [\Gamma_{b^* A_\mu b}]^\mu(p,q). \tag{F1}$$

The assumption of Eq. (F1) is true to all orders in perturbation theory, and any nonperturbative violations of this assumption are restricted by the WTIs to lie completely within their transverse components.

It is convenient to define the following notation:

The operator Ω performs the interchange $(p,q) \longleftrightarrow (-q,-p)$.

A function $F(p,q)$, invariant to Ω, is written as $F({}_{(}p,q_{)})$. If $F({}_{(}p,q_{)})$ is a scalar function $F(p^2,q^2,p\cdot q)$ then it is written as $F({}_{(}p^2,q^2_{)},p\cdot q)$. Alternately, a function $G(p,q)$ that changes sign under Ω is written as $G({}_{[}p,q_{]})$, or $G({}_{[}p^2,q^2_{]},p\cdot q)$ if it is scalar.

Eqs. (7.3.2),(7.3.3),(7.3.4) follow, by the reasoning of Ball and Chiu [15], from the WTI for $[a]$ and $[b]$ (See Eq. (3.4.4)).

Substituting Eq. (7.3.1) into Eq. (7.2.10) and comparing to Eq. (7.2.9) gives

$$\Gamma_{\bar\lambda b^*\psi}(p,q) = i\gamma_5 \Gamma_{\bar\lambda a^*\psi}(p,q). \tag{F2}$$

Similarly, from Eqs. (7.2.7),(7.2.8),

$$\Gamma_{\bar\lambda g^*\psi}(p,q) = i\gamma_5 \Gamma_{\bar\lambda f^*\psi}(p,q). \tag{F3}$$

Any $\Gamma_{\bar{\lambda}f^*\psi}(p,q)$ consistent with Eq. (7.2.5) can be put in the general form

$$\begin{aligned}\Gamma_{\bar{\lambda}f^*\psi}(p,q) &= \frac{-e}{p^2-q^2}(A(p^2)-A(q^2))\gamma\cdot q + H(_{(}p,q_{)}) \\ &\quad -\frac{1}{2}e[\gamma\cdot p(q^2-p\cdot q)+\gamma\cdot q(p^2-p\cdot q)]T_{ff}(_{(}p^2,q^2_{)},p\cdot q).\end{aligned} \tag{F4}$$

Using Eq. (F2) to equate Eqs. (7.2.16),(7.2.17), we find that

$$\Gamma_{f^*Db}(p,q) = -\Gamma_{g^*Da}(p,q), \tag{F5}$$

$$\Gamma_{f^*Dg}(p,q) = -\Gamma_{g^*Df}(p,q). \tag{F6}$$

We obtain, by substituting Eqs. (F3),(F4) into Eq. (7.2.25),

$$\begin{aligned}&\gamma_5(\gamma\cdot p-\gamma\cdot q)\Gamma_{g^*Df}(p^2,q^2,p\cdot q) \\ &= \frac{-ie}{p^2-q^2}(A(p^2)-A(q^2))\gamma_5(\gamma\cdot p-\gamma\cdot q)+iH(_{(}p,q_{)})\gamma_5-i\gamma_5H(_{(}p,q_{)}).\end{aligned} \tag{F7}$$

Dividing $H(_{(}p,q_{)})$ into its odd-numbered and even-numbered γ-matrix components, $H^{\rm odd}(_{(}p,q_{)})$ and $H^{\rm even}(_{(}p,q_{)})$ respectively, we see from Eq. (F7) that $H^{\rm odd}(_{(}p,q_{)})$ is of the form

$$H^{\rm odd}(_{(}p,q_{)}) = (\gamma\cdot p-\gamma\cdot q)\hat{H}(_{(}p^2,q^2_{)},p\cdot q), \tag{F8}$$

due to its anti-commutation with γ_5 and its invariance under Ω. If we substitute Eqs. (F3),(F5),(7.2.26) into Eq. (7.2.20) we get

$$\begin{aligned}&\gamma_5(\gamma\cdot p-\gamma\cdot q)\Gamma_{b^*Df}(p^2,q^2,p\cdot q) \\ &= i\Gamma_{\bar{\lambda}a^*\psi}(p,q)\gamma_5 - ie\gamma_5A(q^2)-i\gamma_5\Gamma_{\bar{\lambda}f^*\psi}(-q,-p)\gamma\cdot p,\end{aligned} \tag{F9}$$

which, when added to Eq. (7.2.21), produces

$$\begin{aligned}&i\gamma_5\Gamma_{\bar{\lambda}a^*\psi}(p,q)+i\Gamma_{\bar{\lambda}a^*\psi}(p,q)\gamma_5 \\ &= 2ieA(q^2)+i\gamma_5\Gamma_{\bar{\lambda}f^*\psi}(-q,-p)\gamma\cdot p-i\Gamma_{\bar{\lambda}f^*\psi}(-q,-p)\gamma_5\gamma\cdot p.\end{aligned} \tag{F10}$$

Any $\Gamma_{\bar{\lambda}a^*\psi}(p,q)$ consistent with Eqs. (F4),(F8),(F10) must be of the form

$$\begin{aligned}\Gamma_{\bar{\lambda}a^*\psi}(p,q) &= \frac{e}{p^2-q^2}(p^2A(p^2)-q^2A(q^2)) \\ &\quad +\frac{1}{2}e[p^2(q^2-p\cdot q)+\gamma\cdot q\gamma\cdot p(p^2-p\cdot q)]T_{ff}(_{(}p^2,q^2_{)},p\cdot q) \\ &\quad +(p^2-\gamma\cdot q\gamma\cdot p)\hat{H}(_{(}p^2,q^2_{)},p\cdot q)+\Gamma^{\rm odd}_{\bar{\lambda}a^*\psi}(p,q),\end{aligned} \tag{F11}$$

where the superscript "odd" on the last term denotes that it is the component of $\Gamma_{\bar{\lambda}a^*\psi}(p,q)$ with only odd numbers of γ-matrices. $\Gamma^{\rm odd}_{\bar{\lambda}a^*\psi}(p,q)$ is unrestricted by Eq. (F10) due to its anti-commutation with γ_5.

Substituting Eqs. (7.3.2),(F11) into Eq. (7.2.1) tells us that

$$\hat{H}\,((p^2,q^2_),p\cdot q)=\frac{1}{2}e(T_{aa}((p^2,q^2_),p\cdot q)-p\cdot qT_{ff}((p^2,q^2_),p\cdot q)). \tag{F12}$$

The even γ-matrix component of $\Gamma_{\bar{\lambda}a^*\psi}(p,q)$ is therefore

$$\begin{aligned}\Gamma^{\rm even}_{\bar{\lambda}a^*\psi}(p,q) &= \frac{e}{p^2-q^2}(p^2A(p^2)-q^2A(q^2))\\ &\quad+\frac{1}{2}e(p^2-\gamma\cdot q\gamma\cdot p)T_{aa}((p^2,q^2_),p\cdot q)\\ &\quad+\frac{1}{2}ep^2(q^2-\gamma\cdot p\gamma\cdot q)T_{ff}((p^2,q^2_),p\cdot q),\end{aligned} \tag{F13}$$

and the odd γ-matrix component of $\Gamma_{\bar{\lambda}f^*\psi}(p,q)$ is

$$\begin{aligned}\Gamma^{\rm odd}_{\bar{\lambda}f^*\psi}(p,q) &= \frac{-e}{p^2-q^2}(A(p^2)-A(q^2))\gamma\cdot q\\ &\quad+\frac{1}{2}e(\gamma\cdot p-\gamma\cdot q)T_{aa}((p^2,q^2_),p\cdot q)\\ &\quad-\frac{1}{2}e\gamma\cdot q(p^2-\gamma\cdot p\gamma\cdot q)T_{ff}((p^2,q^2_),p\cdot q).\end{aligned} \tag{F14}$$

It now remains to find $\Gamma^{\rm odd}_{\bar{\lambda}a^*\psi}(p,q)$ and $H^{\rm even}((p,q_))$. Subtracting Eq. (F9) from Eq. (7.2.21) we get

$$(\gamma\cdot p-\gamma\cdot q)\Gamma_{b^*Df}(p^2,q^2,p\cdot q)=-i\Gamma^{\rm odd}_{\bar{\lambda}a^*\psi}(p,q)-iH^{\rm even}((p,q_))\gamma\cdot p. \tag{F15}$$

The result of substituting Eqs. (F13),(F14) into Eq. (7.2.3) and operating with Ω is

$$\begin{aligned}0 &= \Gamma^{\rm odd}_{\bar{\lambda}a^*\psi}(p,q)-H^{\rm even}((p,q_))\gamma\cdot p\\ &\quad-\frac{e}{p^2-q^2}(B(p^2)-B(q^2))(\gamma\cdot p+\gamma\cdot q)\\ &\quad+e[\gamma\cdot p(q^2-p\cdot q)+\gamma\cdot q(p^2-p\cdot q)]T_{af}((p^2,q^2_),p\cdot q).\end{aligned} \tag{F16}$$

Adding Eq. (F16) to $-i\times$\{Eq. (F15)\} produces

$$\begin{aligned}&-i(\gamma\cdot p-\gamma\cdot q)\Gamma_{b^*Df}(p^2,q^2,p\cdot q)\\ &= -2H^{\rm even}((p,q_))\gamma\cdot p-\frac{e}{p^2-q^2}(B(p^2)-B(q^2))(\gamma\cdot p+\gamma\cdot q)\\ &\quad+e[\gamma\cdot p(q^2-p\cdot q)+\gamma\cdot q(p^2-p\cdot q)]T_{af}((p^2,q^2_),p\cdot q).\end{aligned} \tag{F17}$$

$H^{\text{even}}((p,q))$ is of the general form,

$$\begin{aligned} H^{\text{even}}((p,q)) &= H^{\text{scalar}}((p^2,q^2),p\cdot q)+\gamma_5 H^5((p^2,q^2),p\cdot q) \\ &+\frac{1}{2}(\gamma\cdot p\gamma\cdot q-\gamma\cdot q\gamma\cdot p)H^\sigma([p^2,q^2],p\cdot q) \\ &+\frac{1}{2}\gamma_5(\gamma\cdot p\gamma\cdot q-\gamma\cdot q\gamma\cdot p)H^{5\sigma}([p^2,q^2],p\cdot q). \end{aligned} \tag{F18}$$

The symmetry properties of the scalar functions in Eq. (F18) follow from the invariance of $H((p,q))$ under Ω. Remembering that $\Gamma_{b^*Df}(p^2,q^2,p\cdot q)$ is scalar, and substituting Eq. (F18) into Eq. (F17), we find that

$$H^{5\sigma}([p^2,q^2],p\cdot q)=0=H^5((p^2,q^2),p\cdot q), \tag{F19}$$

$$H^\sigma([p^2,q^2],p\cdot q)=0, \tag{F20}$$

and

$$\begin{aligned} &H^{\text{scalar}}((p^2,q^2),p\cdot q) \\ &= \frac{1}{2}e(p-q)^2 T_{af}((p^2,q^2),p\cdot q)-\frac{e}{p^2-q^2}(B(p^2)-B(q^2)). \end{aligned} \tag{F21}$$

Finally, substituting Eqs. (F18) to (F21) into Eq. (F16),

$$\begin{aligned} \Gamma^{\text{odd}}_{\bar\lambda a^*\psi}(p,q) &= \frac{e}{p^2-q^2}(B(p^2)-B(q^2))\gamma\cdot q \\ &+\frac{1}{2}e[\gamma\cdot p(p^2-q^2)-2\gamma\cdot q(p^2-p\cdot q)]T_{af}((p^2,q^2),p\cdot q). \end{aligned} \tag{F22}$$

We now have the vertices $\Gamma_{\bar\lambda f^*\psi}(p,q)$, given by Eq. (7.3.6), and $\Gamma_{\bar\lambda a^*\psi}(p,q)$, found by summing Eqs. (F13),(F22) and given by Eq. (7.3.5). $\Gamma^\mu_{\bar\psi A_\mu\psi}(p,q)$ is now determined by any one of Eqs. (7.2.7) to (7.2.10), the scalar D-vertices are given by Eqs. (7.2.18) through to (7.2.25), and the vertex $\Gamma_{\bar\psi D\psi}(p,q)$ is given by any one of Eqs. (7.2.14) through to (7.2.17). It is simple to verify that the solution presented in Sec. 7.3 is not further constrained by the SWIs not used in this derivation.

APPENDIX G

Dimensional Reduction from $SQED_4$ to $SQED_3$

Much of the work in this thesis is done in $SQED_4$. $SQED_3$ is really only considered for the purposes of numerical calculation for which it is better suited than $SQED_4$ because it is super-renormalisable. Rather than repeat everything in the lower dimensional theory it is simpler to give a prescription for dimensional reduction, given below.

To dimensionally reduce a theory from $3+1$ to $2+1$ dimensions, we remove its dependence on the third spatial coordinate x_3. It follows from the Fourier transform that the third component of all momenta vanish. The third spatial component of the vector potential, A_3, becomes an ordinary scalar particle, denoted K in this thesis.

In non-SUSY QED, the representation of the Clifford algebra by 4×4 matrices suffers only minor changes. While the components of course change, since the representation is no longer irreducible, the algebraic relations between the matrices do not. $3+1$ dimensional γ_3 is replaced by $2+1$ dimensional $i\gamma_4$ and $\gamma_3\gamma_5$ is replaced by $-\gamma_{45} = i\gamma_4\gamma_5$. The other significant difference is that the charge conjugation and parity matrices, C and Π respectively, are determined only up to an arbitrary phase, ie.

$$\Pi = \gamma_{14} e^{i\phi_P \gamma_{45}}, \; C = \gamma_2 e^{i\phi_C \gamma_{45}}, \tag{G1}$$

where $(0 \leq \phi_P, \phi_C < 2\pi)$. This creates complications in the construction of $SQED_3$. For the SUSY algebra to hold we must either set ϕ_C to $0, \pi$ or make the replacement

$$\begin{pmatrix} \gamma_4 \\ \gamma_5 \end{pmatrix} \longrightarrow \begin{pmatrix} \gamma_P \\ \gamma_W \end{pmatrix} = \begin{pmatrix} \cos\phi_C & -\sin\phi_C \\ \sin\phi_C & \cos\phi_C \end{pmatrix} \begin{pmatrix} \gamma_4 \\ \gamma_5 \end{pmatrix} = M \begin{pmatrix} \gamma_4 \\ \gamma_5 \end{pmatrix}, \tag{G2}$$

in the Clifford algebra.

The prescription in SQED therefore is to replace $3+1$ dimensional γ_3 and γ_5 by $2+1$ dimensional $i\gamma_P$ and γ_W respectively.

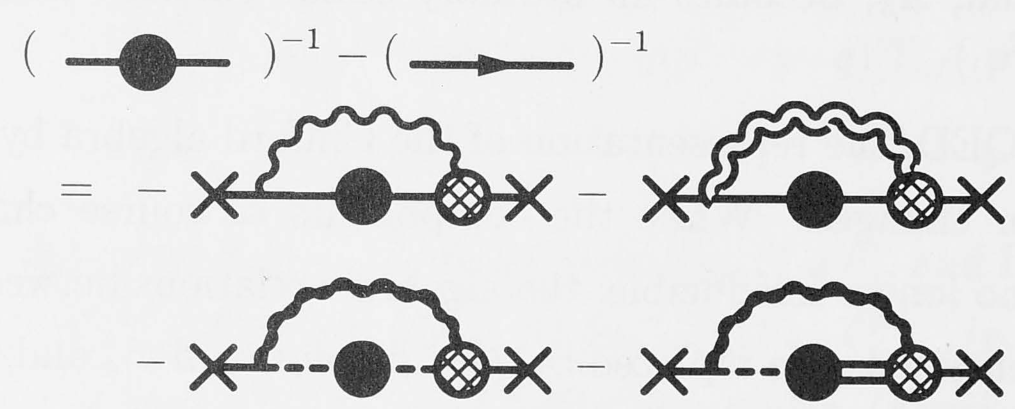
The propagator and proper vertices for the K are best deduced by dimensional reduction of those for the photon. Using the rules described above, the propagator of K is

$$D_K(k) = \frac{1}{k^2}. \tag{G3}$$

The three-point proper vertex of K with the scalars a, b, f and g is precisely zero but its interaction with the electron is given by

$$\begin{aligned}
\Gamma_K(p,q) &= -\frac{1}{2}e(A(p^2)+A(q^2))\gamma_P - \frac{e}{p^2-q^2}(B(p^2)-B(q^2))(\gamma\cdot p - \gamma\cdot q)\gamma_P \\
&\quad -\frac{1}{2}\frac{e}{p^2-q^2}(A(p^2)-A(q^2))[(p-q)^2 - (\gamma\cdot p\gamma\cdot q - \gamma\cdot q\gamma\cdot p)]\gamma_P \\
&\quad -\frac{1}{2}e(p-q)^2\gamma_P T_{aa}(p^2,q^2,p\cdot q) \\
&\quad +\frac{1}{2}e(\gamma\cdot p - \gamma\cdot q)(p-q)^2\gamma_P T_{af}(p^2,q^2,p\cdot q) \\
&\quad +\frac{1}{2}e(p-q)^2\gamma_P[p\cdot q + \frac{1}{2}(\gamma\cdot p\gamma\cdot q - \gamma\cdot q\gamma\cdot p)]T_{ff}(p^2,q^2,p\cdot q).
\end{aligned} \tag{G4}$$

The DSE in $SQED_3$ is given by



(G5)

Although we do so in chapter 6, it is not necessary to calculate the K contribution to the DSE explicitly. K is just the photon component lost by dimensional reduction and it follows from the form of its propagator and proper functions that the K contribution is included if we simply take the integrand of the $3+1$ dimensional self-energy and integrate it over $\int d^3q$ instead of $\int d^4q$. We use this notational shortcut in chapter 7.

"A facility for quotation often covers the absence of original thought."

- Dorothy Leigh Sayers.

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