

## LINEAR BOUND IN TERMS OF MAXMAXFLOW FOR THE CHROMATIC ROOTS OF SERIES-PARALLEL GRAPHS\*

GORDON F. ROYLE<sup>†</sup> AND ALAN D. SOKAL<sup>‡</sup>

**Abstract.** We prove that the (real or complex) chromatic roots of a series-parallel graph with maxmaxflow  $\Lambda$  lie in the disc  $|q - 1| < (\Lambda - 1)/\log 2$ . More generally, the same bound holds for the (real or complex) roots of the multivariate Tutte polynomial when the edge weights lie in the “real antiferromagnetic regime”  $-1 \leq v_e \leq 0$ . For each  $\Lambda \geq 3$ , we exhibit a family of graphs, namely, the “leaf-joined trees”, with maxmaxflow  $\Lambda$  and chromatic roots accumulating densely on the circle  $|q - 1| = \Lambda - 1$ , thereby showing that our result is within a factor  $1/\log 2 \approx 1.442695$  of being sharp.

**Key words.** chromatic polynomial, multivariate Tutte polynomial, antiferromagnetic Potts model, chromatic roots, maxmaxflow, series-parallel graph

**AMS subject classifications.** Primary, 05C31; Secondary, 05A20, 05C15, 05C99, 05E99, 30C15, 37F10, 37F45, 82B20

**DOI.** 10.1137/130930133

**1. Introduction.** The roots of the chromatic polynomial of a graph, and their location in the complex plane, have been extensively studied both by combinatorial mathematicians and by statistical physicists [16, 32, 25, 23]. Combinatorial mathematicians were originally motivated by attempts (thus far unsuccessful) to use analytic techniques to prove the four-color theorem [3, p. 357], while statistical physicists are motivated by the deep connections to the partition function of the  $q$ -state Potts model and the Yang–Lee theory of phase transitions [32].

For both groups of researchers, one of the fundamental questions that arises is to find bounds on the location of chromatic roots in terms of graph structure or parameters of the graph. Early conjectures that there might be *absolute* bounds on the location of chromatic roots, such as being restricted to the right half-plane [13], were disproved by the following strong result.

**THEOREM 1.1** (see [31, Theorems 1.1–1.4]). *Chromatic roots are dense in the whole complex plane. Indeed, even the chromatic roots of the generalized theta graphs  $\Theta^{(s,p)}$  are dense in the whole complex plane with the possible exception of the disc  $|q - 1| < 1$ .*<sup>1</sup>

Biggs, Damerell, and Sands [2] were the first to suggest, in the early 1970s, that the degree (i.e., valency) of a regular graph might be relevant to the location of its chromatic roots. They conjectured (on rather limited evidence) the existence of a function  $f$  such that the chromatic roots of a regular graph of degree  $r$  lie in the disc  $|q| \leq f(r)$ . Two decades later, Brenti, Royle, and Wagner [9] extended this conjecture

---

\*Received by the editors July 22, 2013; accepted for publication (in revised form) July 22, 2015; published electronically October 29, 2015. This research was supported in part by U.S. National Science Foundation grant PHY-0424082, and by Australian Research Council Discovery Project DP110101596.

<http://www.siam.org/journals/sidma/29-4/93013.html>

<sup>†</sup>School of Mathematics and Statistics, University of Western Australia, 35 Stirling Highway, Nedlands, WA 6009, Australia ([gordon.royle@uwa.edu.au](mailto:gordon.royle@uwa.edu.au)).

<sup>‡</sup>Department of Physics, New York University, New York, NY 10003, and Department of Mathematics, University College London, London WC1E 6BT, UK ([sokal@nyu.edu](mailto:sokal@nyu.edu)).

<sup>1</sup>The generalized theta graph  $\Theta^{(s,p)}$  consists of a pair of end vertices joined by  $p$  internally disjoint paths, each path consisting of  $s$  edges. The letters  $s$  and  $p$  are chosen to indicate “series” and “parallel,” respectively.

to not-necessarily-regular graphs of maximum degree  $r$ .<sup>2</sup> This latter conjecture was finally confirmed by one of us, who used cluster-expansion techniques from statistical physics to show that taking  $f(r) \approx 8r$  would suffice.

THEOREM 1.2 (see [30, Corollary 5.3 and Proposition 5.4]). *The chromatic roots of a graph of maximum degree  $\Delta$  lie in the disc  $|q| \leq 7.963907\Delta$ .*

Moreover, almost the same bound holds when the largest degree  $\Delta$  is replaced by the second-largest degree  $\Delta_2$ , namely, all the chromatic roots lie in the disc  $|q| \leq 7.963907\Delta_2 + 1$  [30, Corollary 6.4].<sup>3</sup> The constant 7.963907 (see also [5]) is an artifact of the proof and is not likely to be close to the true value. Fernández and Procacci [14] have recently improved the constant in Theorem 1.2 to 6.907652 (see also [17]), but this is probably still far from best possible. Of course, the linear dependence on  $\Delta$  is indeed best possible, since the complete graph  $K_{\Delta+1}$  has chromatic roots  $0, 1, 2, \dots, \Delta$ .<sup>4</sup>

The parameters  $\Delta$  and  $\Delta_2$  are, however, unsatisfactory in various ways. For example,  $\Delta$  and  $\Delta_2$  can be made arbitrarily large by gluing together blocks at a cut vertex, yet this operation does not alter the chromatic roots. The underlying reason for this discrepancy is that the chromatic polynomial is essentially a property of the cycle matroid of the graph, but vertex degrees are not. Therefore it would be of great interest to find a matroidal parameter that could play the role of maximum degree (or second-largest degree) in results of this type. Motivated by some remarks of Shrock and Tsai [28, 29], a few years ago Sokal [30, section 7] and Jackson and Sokal [19] suggested considering a graph parameter that they called *maxmaxflow*, defined as follows: If  $x$  and  $y$  are distinct vertices in a graph  $G$ , then let  $\lambda_G(x, y)$  denote the maximum flow from  $x$  to  $y$ :

$$(1.1a) \quad \lambda_G(x, y) = \max. \text{ number of edge disjoint paths from } x \text{ to } y$$

$$(1.1b) \quad = \min. \text{ number of edges separating } x \text{ from } y.$$

Then define the maxmaxflow  $\Lambda(G)$  to be the maximum of these values over all pairs of distinct vertices:

$$(1.2) \quad \Lambda(G) = \max_{x \neq y} \lambda_G(x, y).$$

Although this definition appears to use the nonmatroidal concept of a “vertex” in

<sup>2</sup>This latter conjecture actually follows from the former one, as indicated by Thomassen [34, p. 505]: If  $r$  is odd, then there is a graph  $H$  with all vertices but one having degree  $r$  and the remaining vertex having degree 1. Then, given any graph  $G$  of maximum degree  $r - 1$  or  $r$ , we glue enough copies of this “gadget”  $H$  (using its vertex of degree 1) to the vertices of degree less than  $r$  in  $G$ , thereby yielding a regular graph of degree  $r$  whose chromatic roots are the union of the chromatic roots of the original graph  $G$  and those of  $H$ .

<sup>3</sup>Note that it is *not* possible to go farther and obtain a bound in terms of the third-largest degree  $\Delta_3$ , as the chromatic roots of the generalized theta graphs  $\Theta^{(s,p)}$ —which have  $\Delta = \Delta_2 = p$  but  $\Delta_3 = 2$ —are dense in the whole complex plane with the possible exception of the disc  $|q - 1| < 1$ .

<sup>4</sup>Perhaps surprisingly, the complete graph  $K_{\Delta+1}$  is *not* the extremal graph for this problem (except presumably for  $\Delta = 1, 2, 3$ ), and a bound  $|q| \leq \Delta$  is *not* possible. In fact, a nonrigorous (but probably rigorizable) asymptotic analysis, confirmed by numerical calculations, shows [27] that the complete bipartite graph  $K_{\Delta, \Delta}$  has a chromatic root  $\alpha\Delta + o(\Delta)$ , where  $\alpha = -2/W(-2/e) \approx 0.678345 + 1.447937i$ ; here  $W$  denotes the principal branch of the Lambert  $W$  function (the inverse function of  $w \mapsto we^w$ ) [11]. So the constant in Theorem 1.2 cannot be better than  $|\alpha| \approx 1.598960$ . One of us has conjectured [25, Conjecture 6.6] that, for  $\Delta \geq 4$ , the complete bipartite graph  $K_{\Delta, \Delta}$  has the chromatic root of largest modulus (and also largest imaginary part) among all graphs of maximum degree  $\Delta$ . Furthermore, it seems empirically true that the largest modulus of a chromatic root of  $K_{\Delta, \Delta}$ , divided by  $\Delta$ , is an increasing function of  $\Delta$ . If these conjectures are correct, then the optimal constant in Theorem 1.2 would be  $|\alpha|$ .

a fundamental way, Jackson and Sokal [19] proved that maxmaxflow has a “dual” formulation in terms of cocycle bases, namely,

$$(1.3) \quad \Lambda(G) = \min_{\mathcal{B}} \max_{C \in \mathcal{B}} |C|,$$

where  $M(G)$  is the cycle matroid of the graph  $G$ , the min runs over all bases  $\mathcal{B}$  of the cocycle space of  $M(G)$  [over  $GF(2)$ ], and the max runs over all cocycles in the basis  $\mathcal{B}$ . Thus, by taking (1.3) as the *definition* of  $\Lambda(M)$  for an arbitrary binary matroid  $M$ , we obtain a matroidal parameter that specializes to maxmaxflow for a graphic matroid. Furthermore, for graphs,  $\Lambda(G)$  behaves exactly as one would wish with respect to gluing together blocks at a cut vertex, namely, the maxmaxflow of a graph is the maximum of the maxmaxflows of its blocks. Furthermore, from either description it is immediate that

$$(1.4) \quad \Lambda(G) \leq \Delta_2(G).$$

It is therefore natural to make the following conjecture [30, 19], which if true would extend the known bound on chromatic roots in terms of second-largest degree.

CONJECTURE 1.3 (see [19, Conjecture 1.1]). *There exist universal constants  $C(\Lambda) < \infty$  such that all the chromatic roots (real or complex) of all loopless graphs of maxmaxflow  $\Lambda$  lie in the disc  $|q| \leq C(\Lambda)$ . Indeed, it is conjectured that  $C(\Lambda)$  can be taken to be linear in  $\Lambda$ .*

However, there are some serious difficulties in modifying the existing cluster-expansion proof of Theorem 1.2 to get an analogous bound in terms of  $\Lambda$ ; and although some progress has been made in this direction [19], a number of obstacles remain.<sup>5</sup>

In this paper, we restrict our attention to *series-parallel graphs* and use an entirely different approach to prove the following main result.

THEOREM 1.4. *Fix an integer  $\Lambda \geq 2$ , and let  $G$  be a loopless series-parallel graph of maxmaxflow at most  $\Lambda$ . Then all the roots (real or complex) of the chromatic polynomial  $P_G(q)$  lie in the disc  $|q - 1| < (\Lambda - 1) / \log 2 \approx 1.442695(\Lambda - 1)$ .*

Since there are series-parallel graphs of maxmaxflow  $\Lambda$  having chromatic roots arbitrarily close to every point of the circle  $|q - 1| = \Lambda - 1$  (see [26, Appendix B]), the constant in Theorem 1.4 is nonsharp by at most a factor  $1 / \log 2 \approx 1.442695$ . Moreover, a bound  $|q - 1| \leq \Lambda - 1$  cannot hold in general, since at least for  $\Lambda = 3$  we can exhibit a 94-vertex series-parallel graph with a chromatic root at  $|q - 1| \approx 2.009462$  (see section 6).

Let us also remark that in this paper we use only the definition (1.2) of maxmaxflow; we do not use the result (1.3).

The essence of our approach is to view the chromatic polynomial  $P_G(q)$  as a special case of the multivariate Tutte polynomial  $Z_G(q, \mathbf{v})$  of a graph equipped with edge weights  $\mathbf{v} = \{v_e\}_{e \in E}$ , namely, the case in which all the edge weights take the special value  $v_e = -1$ .<sup>6</sup> By working within the more flexible framework of the multivariate Tutte polynomial, we can use the rules for series and parallel reduction [32, sections 4.4 and 4.5] to transform a graph  $G$  into a smaller graph with different edge weights and the same (or closely related) multivariate Tutte polynomial. In particular, a series-parallel graph can be transformed into a one-edge graph with a complicated weight (a messy rational function of  $q$  and  $\{v_e\}$ ) on its single edge. Although this

<sup>5</sup>See [32, section 9.2] for a brief discussion.

<sup>6</sup>See [32] for a review on the multivariate Tutte polynomial (which is also known in statistical physics as the partition function of the  $q$ -state Potts model in the Fortuin–Kasteleyn representation).

weight is complicated, we are able in certain circumstances to bound where it lies in the complex plane and thereby to ensure that the multivariate Tutte polynomial is nonvanishing. After some fairly straightforward real and complex analysis, we can prove Theorem 1.4.

We shall actually prove a result that is slightly stronger than Theorem 1.4 in two ways: First of all, the chromatic roots will be shown to lie in a disc  $|q - 1| < Q_\Lambda^*$ , where  $Q_\Lambda^*$  is the solution of a particular polynomial equation of degree  $2\Lambda - 3$  and satisfies  $Q_\Lambda^* < (\Lambda - \frac{3}{2} \log 2) / \log 2 < (\Lambda - 1) / \log 2$ . [Note that  $\frac{3}{2} \log 2 \approx 1.039721$ .] Second, the chromatic polynomial  $P_G(q)$  can be replaced by the multivariate Tutte polynomial  $Z_G(q, \mathbf{v})$  where the edge weights  $\mathbf{v} = \{v_e\}_{e \in E}$  lie in a suitable set. See Theorem 5.1 for details.

At this point the reader might well wonder: Since series-parallel graphs form a tiny subset of planar graphs, which in turn form a tiny subset of all graphs, what is the interest of a result restricted to the former? The answer is that Theorem 1.1 already shows that even series-parallel graphs can exhibit “wild” behavior in their chromatic roots. If one wishes to bound those roots, then *some* additional parameter is clearly needed. It is a nontrivial fact that maxmaxflow is such a parameter. Theorem 1.4 gives the first nontrivial family of graphs whose chromatic roots are bounded in terms of their maxmaxflow. Whether or not this is good evidence for the truth of the more general Conjecture 1.3 remains to be seen.

The techniques used in proving Theorem 1.4 lend themselves to a number of direct extensions. For example, one fairly easy extension is to permit the original graph to have edge weights throughout the “real antiferromagnetic regime,” i.e., taking  $v_e \in [-1, 0]$  independently for each edge  $e$ . It turns out that exactly the same bound holds.

**THEOREM 1.5.** *Fix an integer  $\Lambda \geq 2$ . Let  $G = (V, E)$  be a loopless series-parallel graph of maxmaxflow at most  $\Lambda$ , and let the edge weights  $\mathbf{v} = \{v_e\}_{e \in E}$  satisfy  $v_e \in [-1, 0]$  for all  $e$ . Then all the roots (real or complex) of the multivariate Tutte polynomial  $Z_G(q, \mathbf{v})$  lie in the disc  $|q - 1| < (\Lambda - 1) / \log 2 \approx 1.442695(\Lambda - 1)$ .*

Once again, we shall actually prove a slightly stronger result, in which the chromatic roots are shown to lie in the disc  $|q - 1| < Q_\Lambda^*$ , and in which the edge weights  $\mathbf{v} = \{v_e\}_{e \in E}$  are allowed to lie in a set that is somewhat larger than  $[-1, 0]$ . See Theorem 7.1.

A second extension is to consider graphs that are not series-parallel but are nevertheless built up by using series and parallel compositions from a fixed “starting set” of graphs. For instance, we can prove the following.

**THEOREM 1.6.** *Let  $G = (G, s, t)$  be a 2-terminal graph that can be obtained from  $K_2$  and the Wheatstone bridge  $W$  by successive series and parallel compositions.<sup>7</sup> If  $G$  has maxmaxflow at most  $\Lambda$  (where  $\Lambda \geq 3$ ), then all the roots (real or complex) of the chromatic polynomial  $P_G(q)$  lie in the disc  $|q - 1| < (\Lambda - \log 2) / \log 2$ .*

Once again, we shall actually prove a slightly stronger result: See Theorem 8.2 and Corollary 8.4.

The plan of this paper is as follows: In section 2 we review the properties of the multivariate Tutte polynomial, with emphasis on its behavior under series and parallel composition. In section 3 we discuss series-parallel graphs and decomposition trees for 2-terminal graphs. In section 4 we state and prove an abstract result that gives a sufficient condition for the multivariate Tutte polynomial to be nonzero, involving sets

<sup>7</sup>The *Wheatstone bridge* is the 2-terminal graph  $W = (W, s, t)$  obtained from  $W = K_4 - e$  by taking the two vertices of degree 2 to be the terminals  $s$  and  $t$ . See section 8.2.

$S_1 \subseteq S_2 \subseteq \dots \subseteq S_{\Lambda-1}$  in the complex plane satisfying certain conditions. In section 5 we prove Theorem 1.4 (and the stronger Theorem 5.1) by constructing suitable sets  $S_1 \subseteq S_2 \subseteq \dots \subseteq S_{\Lambda-1}$ . In section 6 we give a slightly sharper result for the case  $\Lambda = 3$ . In section 7 we prove Theorem 1.5 (and the stronger Theorem 7.1) by a slight generalization of our previous construction. In section 8 we prove Theorem 1.6 (and the stronger Theorem 8.2 and Corollary 8.4).

In the arXiv version of this paper [26] we include two appendices that are being omitted from this journal version due to space constraints: Appendix A in which we define parallel and series connection for edge weights lying in the Riemann sphere, and Appendix B in which we prove Theorem 3.11 on the chromatic roots of leaf-joined trees by using methods from the theory of holomorphic dynamics.

**2. The multivariate Tutte polynomial.** In this section we begin by reviewing the definition and elementary properties of the multivariate Tutte polynomial (section 2.1). We then discuss the technical tools that will play a central role in this paper: parallel and series reduction of edges (section 2.2), the partial multivariate Tutte polynomials and “effective weights”  $v_{\text{eff}}$  for 2-terminal graphs (section 2.3), and the parallel and series composition of 2-terminal graphs (section 2.4).

**2.1. Definition and elementary properties.** Let  $G = (V, E)$  be a finite undirected graph (which may have loops and/or multiple edges). Then the *multivariate Tutte polynomial* of  $G$  is the polynomial

$$(2.1) \quad Z_G(q, \mathbf{v}) = \sum_{A \subseteq E} q^{k(A)} \prod_{e \in A} v_e ,$$

where  $q$  and  $\mathbf{v} = \{v_e\}_{e \in E}$  are commuting indeterminates and  $k(A)$  is the number of connected components in the subgraph  $(V, A)$ . See [32] for a review on the multivariate Tutte polynomial. In this paper we shall sometimes consider  $Z_G(q, \mathbf{v})$  algebraically as a polynomial belonging to the polynomial ring  $\mathbb{Z}[q, \mathbf{v}]$  or  $\mathbb{C}[q, \mathbf{v}]$ , but we shall most often take an analytic point of view and consider  $Z_G(q, \mathbf{v})$  to be a polynomial function of the complex variables  $q$  and  $\{v_e\}$ .

If  $q$  is a positive integer, then the multivariate Tutte polynomial is equal to the partition function of the  $q$ -state Potts model in statistical mechanics, which is defined by

$$(2.2) \quad Z_G^{\text{Potts}}(q, \mathbf{v}) = \sum_{\sigma: V \rightarrow \{1, 2, \dots, q\}} \prod_{e \in E} \left[ 1 + v_e \delta(\sigma(x_1(e)), \sigma(x_2(e))) \right],$$

where the sum runs over all maps  $\sigma: V \rightarrow \{1, 2, \dots, q\}$ , the  $\delta$  is the Kronecker delta

$$(2.3) \quad \delta(a, b) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b, \end{cases}$$

and  $x_1(e), x_2(e) \in V$  are the two endpoints of the edge  $e$  (in arbitrary order). More precisely, we have the following.

**THEOREM 2.1** (Fortuin–Kasteleyn representation of the Potts model). *For integer  $q \geq 1$ ,*

$$(2.4) \quad Z_G^{\text{Potts}}(q, \mathbf{v}) = Z_G(q, \mathbf{v}) .$$

*That is, the Potts-model partition function is simply the specialization of the multivariate Tutte polynomial to  $q \in \mathbb{Z}_+$ .*

See, e.g., [32, section 2.2] for an easy proof.

We shall adopt the terminology from statistical mechanics to designate various sets of values for the edge weights  $v_e$ . In particular, we shall say that a real weight  $v_e$  is *ferromagnetic* if  $v_e \geq 0$  and *antiferromagnetic* if  $-1 \leq v_e \leq 0$ . We shall also sometimes say that a complex weight  $v_e$  is *complex ferromagnetic* if  $|1 + v_e| \geq 1$  and *complex antiferromagnetic* if  $|1 + v_e| \leq 1$ . Finally, we shall say that a set of weights  $\mathbf{v} = \{v_e\}_{e \in E}$  is ferromagnetic or antiferromagnetic if *all* of the  $v_e$  are.

The *zero-temperature limit of the antiferromagnetic Potts model* arises when  $v_e = -1$  for all edges  $e$ ; then  $Z_G^{\text{Potts}}$  gives weight 1 to each proper coloring and weight 0 to each improper coloring, and so counts the proper colorings. It follows from Theorem 2.1 that the number of proper  $q$ -colorings of  $G$  is, in fact, the restriction to  $q \in \mathbb{Z}_+$  of a polynomial in  $q$ , namely, the *chromatic polynomial*

$$(2.5) \quad P_G(q) = Z_G(q, \{-1\}).$$

The multivariate Tutte polynomial factorizes in a simple way over connected components and blocks. If  $G$  is the disjoint union of  $G_1$  and  $G_2$  then, trivially,

$$(2.6) \quad Z_G(q, \mathbf{v}) = Z_{G_1}(q, \mathbf{v}) Z_{G_2}(q, \mathbf{v}).$$

If  $G$  consists of subgraphs  $G_1$  and  $G_2$  joined at a single cut vertex, then it is not hard to see [32, section 4.1] that

$$(2.7) \quad Z_G(q, \mathbf{v}) = \frac{Z_{G_1}(q, \mathbf{v}) Z_{G_2}(q, \mathbf{v})}{q}.$$

Therefore, when studying the multivariate Tutte polynomial, it suffices to restrict attention to *nonseparable* graphs  $G$ .<sup>8</sup>

Note also that a loop  $e$  contributes a trivial prefactor  $1 + v_e$  to  $Z_G(q, \mathbf{v})$ . If  $v_e = -1$  (as it is, e.g., for the chromatic polynomial), this causes  $Z_G$  to be identically zero as a polynomial in  $q$ ; if  $v_e \neq -1$ , the loop does not affect the roots of  $Z_G$  at all. Since in this paper we want to allow  $v_e = -1$ , we shall assume in our main theorems that the graph  $G$  is loopless.

Finally, if  $G$  consists of a single vertex and no edges (i.e.,  $G = K_1$ ), then  $Z_G(q, \mathbf{v}) = q$ . So we can assume without loss of generality that  $G$  is loopless, nonseparable, and contains at least one edge.

There are several reasons why it can be advantageous to consider the multivariate Tutte polynomial, even when the ultimate goal is to obtain results on the chromatic polynomial. The first reason is that  $Z_G(q, \mathbf{v})$  is *multiaffine* in the variables  $\mathbf{v}$  (i.e., of degree 1 in each  $v_e$  separately), and often a multiaffine polynomial in many variables is easier to handle than a general polynomial in a single variable (e.g., it may permit simple proofs by induction on the number of variables). Second, allowing unequal edge weights  $v_e$  permits more flexibility in inductive proofs; indeed, in some cases the stronger result is much *easier* to prove. In particular, local operations on graphs can be reflected in local changes to the edge weights of the affected edges, which is impossible if all edge weights are constrained to be equal.<sup>9</sup> In this context, two of the most important such “local operations” are parallel and series reductions, to be discussed in the next subsection.

<sup>8</sup>See section 3 for a precise definition of “nonseparable” for graphs that may contain loops.

<sup>9</sup>One striking example of this phenomenon is the three-line proof of the multivariate Brown–Colbourn property for series-parallel graphs [30, Remark 3 in section 4.1] [24, Theorem 5.6(c)  $\implies$  (a)], which contrasts with the 20-page proof of the corresponding univariate result [36]. See [18] for several further instances in which results on the chromatic polynomial can be proven more easily by working within the more general framework of the multivariate Tutte polynomial.



**2.2. Parallel and series reduction.** We say that edges  $e, f \in E$  are *in parallel* if they connect the same pair of distinct vertices  $x$  and  $y$ . In this case they can be replaced, without changing the value of the multivariate Tutte polynomial, by a single new edge with “effective weight”

$$(2.8) \quad v_{\text{eff}} = (1 + v_e)(1 + v_f) - 1 .$$

This operation of replacing two parallel edges by a single edge is called *parallel reduction*, and we write  $v_e \parallel v_f$  as a shorthand for  $(1 + v_e)(1 + v_f) - 1$ .

We say that edges  $e, f \in E$  are *in series (in the narrow sense)*<sup>10</sup> if there are vertices  $x, y, z \in V$  with  $x \neq y$  and  $y \neq z$  such that  $e$  connects  $x$  and  $y$ ,  $f$  connects  $y$  and  $z$ , and  $y$  has degree 2. In this case, replacing the edges  $e$  and  $f$  with a single edge of effective weight

$$(2.9) \quad v_{\text{eff}} = \frac{v_e v_f}{q + v_e + v_f}$$

yields a graph whose multivariate Tutte polynomial—when multiplied by the prefactor  $q + v_e + v_f$ —is the same as that of the original graph, provided that  $q + v_e + v_f \neq 0$ . More formally, we can consider the new graph to be obtained from  $G$  by contracting  $f$ , and we can write

$$(2.10) \quad Z_G(q, \mathbf{v}_{\neq e, f}, v_e, v_f) = (q + v_e + v_f) Z_{G/f}(q, \mathbf{v}_{\neq e, f}, v_e v_f / (q + v_e + v_f)) .$$

See [32, section 4.5] for an easy proof. Naturally this operation is called *series reduction*, and we write

$$(2.11) \quad v_e \bowtie_q v_f = \begin{cases} \frac{v_e v_f}{q + v_e + v_f} & \text{if } q + v_e + v_f \neq 0 , \\ \text{undefined} & \text{if } q + v_e + v_f = 0 , \end{cases}$$

where “undefined” is a special value (not a complex number). We furthermore declare that any  $\parallel$  or  $\bowtie_q$  operation in which one or both of the inputs is undefined yields an output that is also undefined. The operators  $\parallel$  and  $\bowtie_q$  are thus maps  $\widehat{\mathbb{C}} \times \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ , where  $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\text{undefined}\}$ .<sup>11</sup>

There are other ways to parametrize the edge weights occurring in the multivariate Tutte polynomial, and there are often advantages in using the variables that give the simplest expression for the immediate task at hand. In particular, in this paper we will use *three* sets of variables, namely, the edge weights  $\{v_e\}$ , the *transmissivities*  $\{t_e\}$  defined by

$$(2.12) \quad t_e = \frac{v_e}{q + v_e}, \quad v_e = \frac{q t_e}{1 - t_e} ,$$

and a third set of variables  $\{y_e\}$  given by

$$(2.13) \quad y_e = 1 + v_e, \quad v_e = y_e - 1 .$$

<sup>10</sup>Note that this definition of “edges in series” is more restrictive than the matroidal definition of elements in series, but the distinction is not important in our context. See [32, section 4.5] for further discussion.

<sup>11</sup>However, see the Remark at the end of this subsection, as well as [26, Appendix A].

There are two main reasons for using these different sets of variables. The first reason is that the variables  $\{t_e\}$  and  $\{y_e\}$  each make one of the operations of series and parallel reduction trivial. More precisely, let  $\parallel^V$ ,  $\parallel^T$ , and  $\parallel^Y$  denote the parallel-reduction operation expressed in the  $v$ ,  $t$ , and  $y$  variables, respectively, and similarly for  $\bowtie^V$ ,  $\bowtie^T$ , and  $\bowtie^Y$ . Then we have

$$(2.14) \quad v_e \parallel^V v_f = (1 + v_e)(1 + v_f) - 1 ,$$

$$(2.15) \quad v_e \bowtie_q^V v_f = \frac{v_e v_f}{q + v_e + v_f} ,$$

$$(2.16) \quad t_e \parallel_q^T t_f = \frac{t_e + t_f + (q - 2)t_e t_f}{1 + (q - 1)t_e t_f} ,$$

$$(2.17) \quad t_e \bowtie^T t_f = t_e t_f ,$$

$$(2.18) \quad y_e \parallel^Y y_f = y_e y_f ,$$

$$(2.19) \quad y_e \bowtie_q^Y y_f = \frac{q - 1 + y_e y_f}{q - 2 + y_e + y_f} ,$$

where it is understood in (2.15), (2.16), (2.19) that the result is declared to be undefined whenever the denominator vanishes, as in (2.11). We have given the operators a  $q$ -subscript whenever the corresponding expression depends on  $q$ . Note that series reduction is particularly easy in the  $t$ -variables, while parallel reduction is particularly easy in the  $y$ -variables. We shall also use the obvious notation

$$(2.20) \quad A \parallel^V B = \{a \parallel^V b : a \in A, b \in B\} ,$$

$$(2.21) \quad A \bowtie_q^V B = \{a \bowtie_q^V b : a \in A, b \in B\} ,$$

when  $A$  and  $B$  are subsets of the complex plane, and analogously for the other variables.

The second reason for introducing these different sets of variables is that the regions we are attempting to bound have different shapes in the complex  $v$ -plane,  $t$ -plane, and  $y$ -plane, and we will ultimately choose the variables in which the regions are the easiest to effectively bound. Of course, since the maps (2.12) and (2.13) are Möbius transformations, discs in any one of these planes always map to discs (or their exteriors) in any other one of these planes; but discs centered at the origin do not in general map to discs centered at the origin, and concentric discs do not in general map to concentric discs. It is convenient, as we shall see, to choose variables in which we can use discs centered at the origin.

To avoid notational overload, we will normally specify the variables being used in each section of the paper and use the convention that  $\parallel$  and  $\bowtie$  with no superscript refer to the expressions applicable to the current choice.

*Remark.* The definitions given in this section concerning the use of the value undefined are convenient for the main purposes of this paper, where we will be dealing with regions that belong to the finite plane simultaneously in both the  $v$ - and  $t$ -variables, but they are somewhat unnatural because the conditions for being undefined in (2.15), (2.19), and (2.16) do *not* correspond:  $q + v_e + v_f = 0$  is *not* equivalent to  $1 + (q - 1)t_e t_f = 0$ . A more natural approach is to define the operations  $\parallel$  and  $\bowtie$  on the Riemann sphere  $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  in such a way that the conditions for being undefined are the same no matter which variables are used. This approach is outlined and used in the appendices of [26].



**2.3. Partial multivariate Tutte polynomials and  $v_{\text{eff}}$  for 2-terminal graphs.** A 2-terminal graph  $G = (G, s, t)$  is a graph  $G$  with two distinguished vertices  $s$  and  $t$  ( $s \neq t$ ), called the *terminals*. (We do not insist here that  $G$  be connected but, in practice, it always will be.) Given a 2-terminal graph  $(G, s, t)$ , we define the *partial multivariate Tutte polynomials*

$$(2.22) \quad Z_G^{(s \not\leftrightarrow t)}(q, \mathbf{v}) = \sum_{\substack{E' \subseteq E \\ E' \text{ does not connect } s \text{ to } t}} q^{k(E')} \prod_{e \in E'} v_e,$$

$$(2.23) \quad Z_G^{(s \leftrightarrow t)}(q, \mathbf{v}) = \sum_{\substack{E' \subseteq E \\ E' \text{ connects } s \text{ to } t}} q^{k(E')} \prod_{e \in E'} v_e.$$

From (2.1) we have trivially

$$(2.24) \quad Z_G(q, \mathbf{v}) = Z_G^{(s \not\leftrightarrow t)}(q, \mathbf{v}) + Z_G^{(s \leftrightarrow t)}(q, \mathbf{v}).$$

Since clearly  $k(E') \geq 2$  (resp., 1) whenever  $E'$  does not connect (resp., connects)  $s$  to  $t$ , it is convenient to define

$$(2.25) \quad A_{G,s,t}(q, \mathbf{v}) = q^{-2} Z_G^{(s \not\leftrightarrow t)}(q, \mathbf{v}),$$

$$(2.26) \quad B_{G,s,t}(q, \mathbf{v}) = q^{-1} Z_G^{(s \leftrightarrow t)}(q, \mathbf{v}).$$

$A_{G,s,t}(q, \mathbf{v})$  and  $B_{G,s,t}(q, \mathbf{v})$  are thus defined by sums like (2.22), (2.23) but in which only those connected components *not containing one or both of the terminals  $s, t$*  receive a factor  $q$ . We also define the “effective weight”

$$(2.27) \quad v_{\text{eff}}(G, s, t) \equiv \frac{B_{G,s,t}(q, \mathbf{v})}{A_{G,s,t}(q, \mathbf{v})} = \frac{q Z_G^{(s \leftrightarrow t)}(q, \mathbf{v})}{Z_G^{(s \not\leftrightarrow t)}(q, \mathbf{v})},$$

which is a rational function of  $q$  and  $\{v_e\}$ . [Note that the polynomial  $Z_G^{(s \not\leftrightarrow t)}(q, \mathbf{v})$  cannot vanish identically, because the term  $E' = \emptyset$  in (2.22) contributes  $q^{|V(G)|}$ .] More precisely, we have the following.

LEMMA 2.1. *Let  $(G, s, t)$  be a 2-terminal graph.*

- (a) *If  $G$  contains an  $st$ -path, then  $v_{\text{eff}}(G, s, t)$  is a rational function of  $q$  and  $\{v_e\}$  that depends nontrivially on  $\{v_e\}$ .*
- (b) *If  $G$  does not contain an  $st$ -path, then  $v_{\text{eff}}(G, s, t) \equiv 0$ .*

*Proof.*

- (a) If  $G$  contains an  $st$ -path, then  $B_{G,s,t}(q, \mathbf{v}) \neq 0$ , and every monomial in  $B_{G,s,t}(q, \mathbf{v})$  contains at least one factor  $v_e$ . On the other hand,  $A_{G,s,t}(q, \mathbf{v})$  contains a monomial  $q^{|V(G)|-2}$  (coming from  $E' = \emptyset$ ) that contains no factors  $v_e$ . Therefore, it cannot happen that  $B_{G,s,t}(q, \mathbf{v}) = f(q) A_{G,s,t}(q, \mathbf{v})$ .
- (b) The proof is trivial.  $\square$

*Remarks.*

1. The “effective transmissivity”  $t_{\text{eff}} \equiv v_{\text{eff}}/(q + v_{\text{eff}})$  is given by the simple formula

$$(2.28) \quad t_{\text{eff}}(G, s, t) = \frac{Z_G^{(s \leftrightarrow t)}(q, \mathbf{v})}{Z_G(q, \mathbf{v})}$$

and thus represents the “probability” that  $s$  is connected to  $t$ . In fact, when  $\mathbf{v} \geq 0$  this is a true probability in the random-cluster model [15].

2. If  $G$  is a graph and  $s, t$  are distinct vertices of  $G$ , we define  $G/st$  to be the graph in which  $s$  and  $t$  are contracted to a single vertex. (N.B. If  $G$  contains one or more edges  $st$ , then these edges are *not* deleted, but become loops in  $G/st$ .) It is then easy to see that

$$(2.29) \quad Z_{G/st}(q, \mathbf{v}) = Z_G^{(s \leftrightarrow t)}(q, \mathbf{v}) + q^{-1} Z_G^{(s \not\leftrightarrow t)}(q, \mathbf{v}).$$

One convenient way of calculating  $Z_G^{(s \leftrightarrow t)}$  and  $Z_G^{(s \not\leftrightarrow t)}$  is to first calculate  $Z_G$  and  $Z_{G/st}$  (for instance, by deletion contraction) and then solve (2.24), (2.29) for  $Z_G^{(s \leftrightarrow t)}$  and  $Z_G^{(s \not\leftrightarrow t)}$ . See [32, section 4.6] for more information on the partial multivariate Tutte polynomials.  $\square$

Let us now justify the name  $v_{\text{eff}}$  by showing that when  $(G, s, t)$  is inserted inside a larger graph, it acts essentially (modulo a prefactor) as a single edge with effective weight  $v_{\text{eff}}(G, s, t)$ . The precise construction is as follows: Let  $H$  be a graph, and let  $e_\star$  be an edge of  $H$  with endpoints  $a$  and  $b$ .<sup>12</sup> Let us denote by  $H[(e_\star, a, b) \rightarrow (G, s, t)]$  the graph obtained from the disjoint union of  $H \setminus e_\star$  and  $G$  by identifying  $s$  with  $a$  and  $t$  with  $b$ . So the edge set of  $H[(e_\star, a, b) \rightarrow (G, s, t)]$  can be identified with  $(E(H) \setminus \{e_\star\}) \cup E(G)$ . Now put weights  $\mathbf{v} = \{v_e\}_{e \in E(H)}$  on the edges of  $H$  and weights  $\mathbf{w} = \{w_e\}_{e \in E(G)}$  on the edges of  $G$ , so that  $v_{\text{eff}}(G, s, t) = B_{G, s, t}(q, \mathbf{w})/A_{G, s, t}(q, \mathbf{w})$  is a rational function of  $q$  and  $\mathbf{w}$ . We use the notation  $\mathbf{v}_{\neq e_\star} = \{v_e\}_{e \in E(H) \setminus \{e_\star\}}$  and hence  $Z_H(q, \mathbf{v}) = Z_H(q, \mathbf{v}_{\neq e_\star}, v_{e_\star})$ . We then have the following.

**PROPOSITION 2.2.** *When a 2-terminal graph  $(G, s, t)$  is inserted into a graph  $H$  as above,*

$$(2.30) \quad Z_{H[(e_\star, a, b) \rightarrow (G, s, t)]}(q, \mathbf{v}_{\neq e_\star}, \mathbf{w}) = A_{G, s, t}(q, \mathbf{w}) Z_H(q, \mathbf{v}_{\neq e_\star}, v_{\text{eff}}(G, s, t)).$$

*Proof.* The sets  $A \subseteq (E(H) \setminus \{e_\star\}) \cup E(G)$  contributing to the multivariate Tutte polynomial (2.1) of  $H[(e_\star, a, b) \rightarrow (G, s, t)]$  can be classified according to whether  $a$  is or is not connected to  $b$  via edges in  $E(G)$ . Those that do not connect  $a$  to  $b$  give a factor  $A_{G, s, t}(q, \mathbf{w})$  and correspond to the sets  $A' \not\ni e_\star$  contributing to the multivariate Tutte polynomial (2.1) of  $H$ , while those that connect  $a$  to  $b$  give a factor  $B_{G, s, t}(q, \mathbf{w})$  and correspond to the sets  $A' \ni e_\star$  contributing to the multivariate Tutte polynomial (2.1) of  $H$ . Since  $v_{\text{eff}}(G, s, t) = B_{G, s, t}(q, \mathbf{w})/A_{G, s, t}(q, \mathbf{w})$ , the formula (2.30) is an immediate consequence of this correspondence.  $\square$

*Remarks.*

1. The graphical construction of inserting  $(G, s, t)$  inside  $H$  depends on the chosen order of endpoints for the edge  $e_\star$ , but the resulting multivariate Tutte polynomial does not. That is,  $H[(e_\star, a, b) \rightarrow (G, s, t)]$  and  $H[(e_\star, b, a) \rightarrow (G, s, t)]$  are in general nonisomorphic as graphs, but Proposition 2.2 shows that they have the same multivariate Tutte polynomial.
2. The formula (2.10) for series reduction is a special case of (2.30), in which the inserted graph  $(G, s, t)$  is a two-edge path.

**2.4. Parallel and series composition of 2-terminal graphs.** If  $G_1 = (G_1, s_1, t_1)$  and  $G_2 = (G_2, s_2, t_2)$  are 2-terminal graphs on disjoint vertex sets, then

<sup>12</sup>The result of Proposition 2.2 below is valid even when  $a = b$  (i.e.,  $e_\star$  is a loop), although we will never use it in this situation.

their *parallel composition* is the 2-terminal graph

$$(2.31) \quad G_1 \parallel G_2 = (H, s_1, t_1),$$

where  $H$  is obtained from  $G_1 \cup G_2$  by identifying  $s_2$  with  $s_1$  and  $t_2$  with  $t_1$ , and their *series composition* is the 2-terminal graph

$$(2.32) \quad G_1 \bowtie G_2 = (H, s_1, t_2),$$

where  $H$  is obtained from  $G_1 \cup G_2$  by identifying  $t_1$  with  $s_2$ . For future use (see section 3.3), let us say that a 2-terminal graph is *prime* if it cannot be written as the parallel or series composition of two strictly smaller 2-terminal graphs.<sup>13</sup>

*Remark.* We trivially have  $G_1 \parallel G_2 = G_2 \parallel G_1$ . On the other hand,  $G_1 \bowtie G_2 \neq G_2 \bowtie G_1$ , if only because the terminals are different in the two cases; moreover, even the graphs underlying  $G_1 \bowtie G_2$  and  $G_2 \bowtie G_1$  (ignoring the terminals) need not be isomorphic, as can be seen by simple examples. But this subtlety will play no role in this paper, because  $G_1 \bowtie G_2$  and  $G_2 \bowtie G_1$  will have the same multivariate Tutte polynomial; indeed, they will have the same partial multivariate Tutte polynomials (2.22), (2.23) and hence also the same  $v_{\text{eff}}$ . This is a reflection of the fact that the multivariate Tutte polynomial of a graph  $G$  depends only on the graphic matroid  $M(G)$  [except for an overall prefactor  $q^{|V(G)|}$ ] and that series connection of *matroids* does not depend on any orientation.  $\square$

Let us now show how the partial multivariate Tutte polynomials  $Z_G^{(s \leftrightarrow t)}$  and  $Z_G^{(s \nleftrightarrow t)}$  of a parallel or series composition of 2-terminal graphs  $(G_1, s_1, t_1)$  and  $(G_2, s_2, t_2)$  can be computed from the partial multivariate Tutte polynomials of the two input graphs. It is convenient to use the modified partial multivariate Tutte polynomials  $A_{G,s,t}$  and  $B_{G,s,t}$  defined in (2.25), (2.26).

PROPOSITION 2.3.

- (a) Consider a parallel composition  $(G, s, t) = (G_1, s_1, t_1) \parallel (G_2, s_2, t_2)$ . Writing  $A = A_{G,s,t}$  and  $A_i = A_{G_i,s_i,t_i}$  for  $i = 1, 2$  and likewise for  $B$ , we have

$$(2.33) \quad A = A_1 A_2,$$

$$(2.34) \quad B = A_1 B_2 + A_2 B_1 + B_1 B_2,$$

and, in particular,

$$(2.35) \quad A + B = (A_1 + B_1)(A_2 + B_2)$$

and

$$(2.36) \quad v_{\text{eff}}(G, s, t) = v_{\text{eff}}(G_1, s_1, t_1) \parallel v_{\text{eff}}(G_2, s_2, t_2).$$

- (b) Consider a series composition  $(G, s, t) = (G_1, s_1, t_1) \bowtie (G_2, s_2, t_2)$ . Writing  $A = A_{G,s,t}$  and  $A_i = A_{G_i,s_i,t_i}$  for  $i = 1, 2$  and likewise for  $B$ , we have

$$(2.37) \quad A = A_1 B_2 + A_2 B_1 + q A_1 A_2,$$

$$(2.38) \quad B = B_1 B_2,$$

<sup>13</sup>We say “strictly smaller” because every 2-terminal graph  $G$  can be written as  $G = G \parallel \bar{K}_2$  where  $\bar{K}_2$  is the graph with two vertices (the terminals) and no edges. It is to exclude this trivial type of parallel composition that we write “each have at least one edge” in Lemmas 3.1(d) and 3.2(c). In section 3.3 and thereafter, this trivial case will be excluded by requiring that all graphs appearing in a decomposition tree be connected. We could avoid all these technicalities by requiring connectedness from the start, but we refrain from doing so because connectedness plays no role in the formulas of the present section.

and, in particular,

$$(2.39) \quad qA + B = (qA_1 + B_1)(qA_2 + B_2)$$

and

$$(2.40) \quad v_{\text{eff}}(G, s, t) = v_{\text{eff}}(G_1, s_1, t_1) \bowtie_q v_{\text{eff}}(G_2, s_2, t_2).$$

*Proof.* We recall that  $A_{G,s,t}(q, \mathbf{v})$  and  $B_{G,s,t}(q, \mathbf{v})$  are defined by sums like (2.22), (2.23) but in which only those connected components not containing one or both of the terminals  $s, t$  (let us call these “nonterminal components”) receive a factor  $q$ .

For a parallel composition,  $s$  is connected to  $t$  in a spanning subgraph of  $G$  if and only if it is connected in the corresponding spanning subgraph of  $G_1$  or  $G_2$  or both; and the number of nonterminal components in  $G$  is the sum of those in  $G_1$  and  $G_2$ . This proves (2.33), (2.34); then (2.35) and (2.36) are an immediate consequence.

For a series composition,  $s$  is connected to  $t$  in a spanning subgraph of  $G$  if and only if  $s_i$  is connected to  $t_i$  in the corresponding spanning subgraph of  $G_i$  for *both*  $i = 1$  and  $i = 2$ ; and the number of nonterminal components in  $G$  is the sum of those in  $G_1$  and  $G_2$  *except that* there is an extra nonterminal component containing the “inner terminal”  $s_2 = t_1$  whenever  $s_i$  is disconnected from  $t_i$  in  $G_i$  for *both*  $i = 1$  and  $i = 2$  [this explains the factor  $q$  in front of  $A_1 A_2$  in (2.37)]. This proves (2.37), (2.38); then (2.39) and (2.40) are an immediate consequence.  $\square$

Of course, it is no accident that  $v_{\text{eff}}$  satisfies (2.36) and (2.40) under parallel and series composition: By Proposition 2.2,  $v_{\text{eff}}$  *must* behave under parallel and series composition exactly like the parallel and series connection of single edges. Indeed, this argument gives an alternate way of proving (2.36) and (2.40).

**3. Series-parallel graphs and decomposition trees.** In this section we begin by making some further remarks on series and parallel composition of 2-terminal graphs (section 3.1); we then discuss series-parallel graphs (section 3.2), decomposition trees for 2-terminal graphs (section 3.3), and the use of decomposition trees to compute the multivariate Tutte polynomial (section 3.4). Finally, we introduce an important family of example graphs, the leaf-joined trees (section 3.5).

Before starting, however, we need to clarify our usage of the term “nonseparable” as concerns graphs with loops. So let us call a graph *separable* if it is either disconnected or can be obtained by gluing at a vertex two graphs that each have at least one edge; otherwise we call it *nonseparable*. Equivalently, a graph is nonseparable if it is either a single vertex with no edges, a single vertex with a single loop, a pair of vertices connected by one or more edges, or a 2-connected graph. Note in particular that, in our definition, a nonseparable graph must be loopless unless it consists of a single vertex with a single loop. (By contrast, the usual definition of “separable” for connected graphs—namely, a graph with a cut vertex—deems a single vertex with multiple loops to be nonseparable. This definition has the disadvantage of not being invariant under planar duality.) Our definition of “nonseparable” agrees with the usual definition when restricted to loopless graphs.

**3.1. Nice 2-terminal graphs.** As preparation for a more detailed study of series and parallel composition of 2-terminal graphs, we wish to single out a class of 2-terminal graphs that are “well-behaved” in the sense that they connect the terminals without containing “dangling ends.” More precisely, let us say that a 2-terminal graph  $(G, s, t)$  is *nice* if  $G$  is connected and  $G + st$  is nonseparable. (Here  $G + st$  denotes the graph obtained from  $G$  by adding a new edge from  $s$  to  $t$ , irrespective of whether

or not such an edge was already present.) Equivalently,  $(G, s, t)$  is nice if either  $G$  is nonseparable or else  $G$  is a block path (with more than one block) in which  $s$  lies in one endblock and  $t$  in the other and neither of them is a cut vertex. In the latter case  $(G, s, t)$  can be written uniquely as a series composition  $H_1 \bowtie H_2 \bowtie \cdots \bowtie H_k$ , where  $k \geq 2$  and all the  $H_i$  are nonseparable.<sup>14</sup> Conversely, if  $(G, s, t)$  is nice and *not* the series composition of two smaller 2-terminal graphs, then  $G$  must be nonseparable.

The following facts are easily verified.

LEMMA 3.1. *Let  $G_1 = (G_1, s_1, t_1)$  and  $G_2 = (G_2, s_2, t_2)$  be 2-terminal graphs.*

*Then we have the following:*

- (a) *The series composition  $G_1 \bowtie G_2$  is always separable.*
- (b) *The series composition  $G_1 \bowtie G_2$  is nice if and only if both  $G_1$  and  $G_2$  are nice.*
- (c) *If  $G_1$  and  $G_2$  are nice, then the parallel composition  $G_1 \parallel G_2$  is nonseparable (and hence also nice).*
- (d) *Conversely, if  $G_1$  and  $G_2$  each have at least one edge and the parallel composition  $G_1 \parallel G_2$  is nice, then  $G_1$  and  $G_2$  are both nice (and hence  $G_1 \parallel G_2$  is nonseparable).*

*In particular, any 2-terminal graph formed by successive series and parallel compositions of nice 2-terminal graphs is nice.*

**3.2. Series-parallel graphs.** In the literature one can find two slightly different concepts of “series-parallel graph”: one applying to graphs, and the other applying to 2-terminal graphs. In this paper we shall need to use *both* of these concepts. We therefore begin by reviewing the two definitions and the theorems relating them.

In section 2.4 we defined the parallel and series composition of 2-terminal graphs. We now define a *2-terminal series-parallel graph* to be a 2-terminal graph that is either  $K_2$  (with the two vertices as terminals) or else the parallel or series composition of two smaller 2-terminal series-parallel graphs. Note that a 2-terminal series-parallel graph is always loopless. Note also that if  $(G, s, t)$  is 2-terminal series-parallel, then it is nice, i.e.,  $G$  is connected and  $G + st$  is nonseparable; this is an immediate consequence of Lemma 3.1 and the fact that  $K_2$  is nice.

For 2-terminal series-parallel graphs we have the following analogue of Lemma 3.1.

LEMMA 3.2. *Let  $G_1 = (G_1, s_1, t_1)$  and  $G_2 = (G_2, s_2, t_2)$  be 2-terminal graphs.*

*Then we have the following.*

- (a) *The series composition  $G_1 \bowtie G_2$  is 2-terminal series-parallel if and only if both  $G_1$  and  $G_2$  are 2-terminal series-parallel.*
- (b) *If  $G_1$  and  $G_2$  are 2-terminal series-parallel, then the parallel composition  $G_1 \parallel G_2$  is 2-terminal series-parallel.*
- (c) *Conversely, if  $G_1$  and  $G_2$  each have at least one edge and the parallel composition  $G_1 \parallel G_2$  is 2-terminal series-parallel, then  $G_1$  and  $G_2$  are 2-terminal series-parallel.*

*Proof.* The “if” part of (a) is obvious. For the “only if,” we observe that if  $G = G_1 \bowtie G_2$  is 2-terminal series-parallel, then it is nice and separable and hence can be written uniquely as  $H_1 \bowtie H_2 \bowtie \cdots \bowtie H_k$  with  $k \geq 2$  and all the  $H_i$  nonseparable; moreover, we must have  $G_1 = H_1 \bowtie \cdots \bowtie H_\ell$  and  $G_2 = H_{\ell+1} \bowtie \cdots \bowtie H_k$  for some  $\ell$ . We now claim that all the  $H_i$  are 2-terminal series-parallel (so that  $G_1$  and  $G_2$  are as well), and we shall prove this by induction on  $k$ . If  $k = 2$ , the last operation in the series-parallel construction of  $G$  must have been the series connection of  $H_1$  with  $H_2$ , so

<sup>14</sup>Saying “ $H_i$  is nonseparable” is a convenient shorthand for the more precise but pedantic statement “ $H_i = (H_i, s_i, t_i)$  with  $H_i$  nonseparable”. In what follows we shall repeatedly use this shorthand in order to avoid ponderous locutions.

$H_1$  and  $H_2$  must be 2-terminal series-parallel. If  $k > 2$ , then the last operation in the series-parallel construction of  $G$  must have been the series connection of  $H_1 \boxtimes \cdots \boxtimes H_m$  with  $H_{m+1} \boxtimes \cdots \boxtimes H_k$  for some  $m$ , so both of these must be 2-terminal series-parallel; and since they each have less than  $k$  blocks, the inductive hypothesis implies that all the  $H_i$  are 2-terminal series-parallel.

(b) The proof is obvious. For (c), we observe that  $G = G_1 \parallel G_2$  is nice, and hence by Lemma 3.1(d) it is nonseparable. Now any nonseparable 2-terminal graph  $G$  can be written uniquely (modulo ordering) as  $G = H_1 \parallel H_2 \parallel \cdots \parallel H_k$ , where none of the  $H_i$  can be further decomposed as a nontrivial parallel composition.<sup>15</sup> (The summands  $H_i$  are the *st*-bridges in  $G$ .) An argument essentially identical to the one used in part (a) shows that if  $G$  is 2-terminal series-parallel, then all the  $H_i$  are 2-terminal series-parallel, and moreover  $G_1$  and  $G_2$  are obtained by parallel composition of some (complementary) nonempty subsets of the  $H_i$ .  $\square$

Let us now turn to the definition of “series-parallel graph” tout court. Unfortunately, there seems to be no completely standard definition of “series-parallel graph”; a plethora of slightly different definitions can be found in the literature [12, 10, 20, 21, 7, 24]. So let us be completely precise about our own usage: We shall call a loopless graph *series-parallel* if it can be obtained from a forest by a finite (possibly empty) sequence of series and parallel extensions of edges (i.e., replacing an edge by two edges in series or two edges in parallel). We shall call a general graph (allowing loops) series-parallel if its underlying loopless graph is series-parallel. Some authors write “obtained from a tree,” “obtained from  $K_2$ ,” or “obtained from  $C_2$ ” in place of “obtained from a forest”; in our terminology these definitions yield, respectively, all *connected* series-parallel graphs, all connected series-parallel graphs whose blocks form a path, or all *nonseparable* series-parallel graphs with the exception of  $K_2$ . See [7, section 11.2] for a more extensive bibliography.

The precise relationship between the 2-terminal and pure-graph definitions of “series-parallel” is given by the following theorem, which follows from results of Duffin [12] (see also Oxley [20]).

**THEOREM 3.3.** *If  $G$  is a loopless nonseparable graph with at least one edge, then the following are equivalent:*

- (1)  $G$  is series-parallel.
- (2)  $(G, s, t)$  is 2-terminal series-parallel for some pair of vertices  $s, t$ .
- (3)  $(G, s, t)$  is 2-terminal series-parallel for every pair of adjacent vertices  $s, t$ .

One useful consequence of Theorem 3.3 is the following.

**COROLLARY 3.4.** *Let  $(G, s, t)$  be a 2-terminal graph, where  $G$  is loopless and has at least one edge, and  $G + st$  is nonseparable. Then the following are equivalent:*

- (1)  $(G, s, t)$  is 2-terminal series-parallel.
- (2)  $(G + st, s, t)$  is 2-terminal series-parallel.
- (3)  $G + st$  is series-parallel.

*Proof.* Applying Theorem 3.3 to  $G + st$  proves the equivalence of (2) and (3). Moreover, (1)  $\implies$  (2) is trivial, and (2)  $\implies$  (1) is a special case of Lemma 3.2(c).  $\square$

The reason for using the 2-terminal notion of series-parallel graph in this paper is that, although we are unable to precisely control the maxmaxflow of a series-parallel graph, we *can* control the flow between its terminals via the following trivial fact.

<sup>15</sup>We say that a parallel composition is *nontrivial* if each of the graphs occurring in it has at least one edge. See footnote 13 above.



LEMMA 3.5. Let  $(G_1, s_1, t_1)$  and  $(G_2, s_2, t_2)$  be 2-terminal graphs (not necessarily series-parallel). Then

$$(3.1) \quad \lambda_{G_1 \bowtie G_2}(s, t) = \min[\lambda_{G_1}(s_1, t_1), \lambda_{G_2}(s_2, t_2)] ,$$

$$(3.2) \quad \lambda_{G_1 \parallel G_2}(s, t) = \lambda_{G_1}(s_1, t_1) + \lambda_{G_2}(s_2, t_2) ,$$

where  $s$  and  $t$  denote the terminals of  $G_1 \bowtie G_2$  and  $G_1 \parallel G_2$ , respectively. [Recall that  $\lambda_G(x, y)$  denotes the maximum flow in  $G$  from  $x$  to  $y$ , as defined in (1.1b).]

**3.3. Decomposition trees.** Let  $G = (G, s, t)$  be a 2-terminal graph, where we now assume that  $G$  is connected and loopless. A *decomposition tree* for  $(G, s, t)$  is a rooted binary tree with three types of nodes—called *s-nodes*, *p-nodes*, and *leaf nodes*—in which the children of each *s-node* are ordered, and each node is a *connected* 2-terminal graph (whose underlying graph is a subgraph of  $G$ ), as follows: The root node is  $G$ ; if  $H$  is an *s-node* and  $H_1$  and  $H_2$  are its children (in order), then  $H = H_1 \bowtie H_2$ ; if  $H$  is a *p-node* and  $H_1$  and  $H_2$  are its children (in either order), then  $H = H_1 \parallel H_2$ ; and if  $H$  is a *leaf node*, then it has no children.<sup>16</sup> If  $G$  is edge weighted, then the graph at each node is also edge weighted with the weights inherited from its parent. The graphs that appear as nodes in this decomposition tree are called the *constituents* of  $G$  (with respect to the particular decomposition tree), and a constituent is *proper* if it is not equal to  $G$ .

A given 2-terminal graph  $G = (G, s, t)$  can have many distinct decomposition trees, and this for two separate reasons. First, one is free to stop the decomposition at any stage. Indeed, in the extreme case the decomposition tree can consist of the single node  $G$  (which is then a leaf node); we call this the *trivial* decomposition tree. At the other extreme, we say that a decomposition tree is *maximal* if each leaf node corresponds to a *prime* 2-terminal graph. Second, if  $G$  or one of its constituents is formed by placing *three* or more 2-terminal graphs in series or in parallel, then these may be paired up in various ways. (This nonuniqueness arises from our insistence that a decomposition tree is a *binary* tree.)

*Remarks.*

1. The order of the children at an *s-node* is important to reconstructing the graph (since  $G_1 \bowtie G_2 \neq G_2 \bowtie G_1$ ) but is irrelevant to the multivariate Tutte polynomial.
2. We have insisted here that the decomposition tree be a *binary* tree: this means that we need only consider parallel or series composition of *pairs* of 2-terminal subgraphs, but it also means that the maximal decomposition tree is nonunique whenever  $G$  or one of its constituents is formed by placing *three* or more 2-terminal graphs in series or in parallel, since these may be paired up in various ways. Alternatively, we could allow the decomposition tree to be a

<sup>16</sup>The concept of a decomposition tree for a 2-terminal graph is very natural and has been used sporadically in the literature, albeit with no standard definition. Brandstädt, Le, and Sprinrad [7, section 11.2] define decomposition trees essentially as we do, but only for series-parallel graphs. Bodlaender and van Antwerpen-de Fluiter [4] likewise define decomposition trees for series-parallel graphs, with a definition that differs slightly from ours by allowing nonbinary trees (see Remark 2 below). Bern, Lawler, and Wong [1] and Borie, Parker, and Tovey [6] define decomposition trees in the more general setting of  $k$ -terminal graphs for any fixed  $k$ ; their definitions specialized to  $k = 2$  are almost the same as ours. (Borie, Parker, and Tovey require the graphs at leaf nodes to have no nonterminal vertices—something we do not wish to do, as it would restrict us to series-parallel graphs only—but they immediately add [6, p. 558] that “this could be generalized to permit additional base graphs.”) See also Spinrad [33, section 11.3] for a brief description of this latter work.

general rooted tree; then the maximal decomposition tree would be unique, but we would need to consider parallel and series composition of an arbitrary number of 2-terminal subgraphs: six of one, half dozen of the other.

3. Many authors have defined and applied decomposition trees for 2-terminal *series-parallel* graphs (see footnote 16 above), and most of the present paper is indeed concerned with this special case (Theorems 1.4 and 1.5). But the technique set forth here is more general, and applies to graphs that are not series-parallel but are nevertheless built up by using series and parallel compositions from a fixed starting set of 2-terminal “base graphs” (see section 8). A simple example of such a result is Theorem 1.6, where the set of base graphs is taken to be  $K_2$  and the Wheatstone bridge. It is for this reason that we have developed the theory of decomposition trees for 2-terminal graphs that are not necessarily series-parallel.  $\square$

We have the following basic facts concerning the structure of decomposition trees.

LEMMA 3.6. *Let  $(G, s, t)$  be a 2-terminal graph, with  $G$  connected and loopless, and fix a decomposition tree for it. Then the following are equivalent:*

- (1) *The root node  $(G, s, t)$  is nice.*
- (2) *Every node is nice.*
- (3) *Every leaf node is nice.*

Moreover, when these equivalent conditions hold, every  $p$ -node is nonseparable and every  $s$ -node is separable.

*Proof.* This is an immediate consequence of Lemma 3.1.  $\square$

An analogous result holds for 2-terminal series-parallel graphs.

LEMMA 3.7. *Let  $(G, s, t)$  be a 2-terminal graph, with  $G$  connected and loopless, and fix a decomposition tree for it. Then the following are equivalent:*

- (1)  *$(G, s, t)$  is 2-terminal series-parallel.*
- (2) *Every node is 2-terminal series-parallel.*
- (3) *Every leaf node is 2-terminal series-parallel.*

*Proof.* This is an immediate consequence of Lemma 3.2.  $\square$

Among 2-terminal graphs, the series-parallel ones can be characterized as follows.

LEMMA 3.8. *Let  $(G, s, t)$  be a 2-terminal graph. Then the following are equivalent:*

- (1)  *$(G, s, t)$  is 2-terminal series-parallel.*
- (2)  *$(G, s, t)$  has a decomposition tree in which all leaf nodes are single edges.*
- (3) *In every maximal decomposition tree for  $(G, s, t)$ , all leaf nodes are single edges.*

*Proof.* (1)  $\iff$  (2) follows directly from the definition of 2-terminal series-parallel. Furthermore, (3)  $\implies$  (2) is trivial because every 2-terminal graph does possess a maximal decomposition tree. Finally, to show (1)  $\implies$  (3), we observe from Lemma 3.7 that every leaf node is 2-terminal series-parallel; so if a leaf node is not a single edge, then it must be either a series or parallel composition of two smaller 2-terminal series-parallel graphs, contradicting the hypothesis that the decomposition tree is maximal.  $\square$

Let us now note a simple but important fact that will play a key role in the remainder of this paper.

LEMMA 3.9. *Let  $G = (G, s, t)$  be a 2-terminal graph, and consider a decomposition tree for  $G$  in which the root is a  $p$ -node. [If  $G$  is nonseparable, then every decomposition tree other than the trivial one has this property.] If  $G$  has  $\text{maxflow}$   $\Lambda$ , then all its proper constituents  $(H, a, b)$  have between-terminals flow  $\lambda_H(a, b)$  at most  $\Lambda - 1$ .*

*Proof.* Suppose that there is a proper constituent  $(H, a, b)$  such that  $\lambda_H(a, b) \geq \Lambda$ . Let  $(F, c, d)$  be the first ancestor of  $(H, a, b)$  that is a  $p$ -node (such a node must exist

since the root is a  $p$ -node). Then one of the children of  $(F, c, d)$  is a connected series extension  $(F_1, c, d)$  of  $(H, a, b)$  [possibly  $(H, a, b)$  itself], while the other child  $(F_2, c, d)$  is connected and has no edges in common with  $H$ . Therefore, by concatenating a  $cd$ -path from  $F_2$  with  $ac$ - and  $bd$ -paths from  $F_1 \setminus H$  (these paths will degenerate to empty paths if  $a = c$  or  $b = d$ , respectively), we obtain an  $ab$ -path in  $F$  that uses only edges not in  $H$ . Therefore, in  $F$  (and hence also in  $G$ ) there are at least  $\Lambda + 1$  edge-disjoint paths between  $a$  and  $b$ , which contradicts the hypothesis that  $G$  has maxmaxflow  $\Lambda$ .  $\square$

In less formal terms, the key point of this lemma is that if a 2-terminal series-parallel graph of maxmaxflow  $\Lambda$  is constructed via a sequence of series and parallel compositions, with the last stage being a parallel composition, then every intermediate graph has between-terminals flow at most  $\Lambda - 1$  (as well as, of course, having maxmaxflow at most  $\Lambda$ ).

**3.4. Computing the multivariate Tutte polynomial using a decomposition tree.** Let  $(G, s, t)$  be a 2-terminal graph, with  $G$  connected and loopless, and fix a decomposition tree for it. We will now describe a simple algorithm for computing the partial multivariate Tutte polynomials  $A_{G,s,t}(q, \mathbf{v})$  and  $B_{G,s,t}(q, \mathbf{v})$ —and more generally the partial multivariate Tutte polynomials  $A_{H,a,b}(q, \mathbf{v})$  and  $B_{H,a,b}(q, \mathbf{v})$  for each node  $(H, a, b)$  in the decomposition tree—given the partial multivariate Tutte polynomials of all the *leaf* nodes. In particular, we will be able to compute the multivariate Tutte polynomial

$$(3.3) \quad Z_G(q, \mathbf{v}) = q^2 A_{G,s,t}(q, \mathbf{v}) + q B_{G,s,t}(q, \mathbf{v}).$$

Before stating the algorithm, however, let us remark briefly on the different ways that it can be interpreted. Since  $A_{G,s,t}(q, \mathbf{v})$ ,  $B_{G,s,t}(q, \mathbf{v})$ , and  $Z_G(q, \mathbf{v})$  are polynomials with *integer* coefficients—i.e., they belong to the polynomial ring  $\mathbb{Z}[q, \mathbf{v}]$ —they induce well-defined polynomial *functions* on every commutative ring  $R$ , i.e.,  $A_{G,s,t}: R \times R^E \rightarrow R$  and likewise for the other two. Therefore, if  $R$  is an arbitrary commutative ring and  $q$  and  $\{v_e\}$  are given specified values in  $R$ , then it makes sense to compute the value (which again lies in  $R$ ) of the polynomial functions  $A_{G,s,t}(q, \mathbf{v})$ ,  $B_{G,s,t}(q, \mathbf{v})$ , and  $Z_G(q, \mathbf{v})$ . This is what our algorithm will do, using only addition and multiplication in the ring  $R$ ; it thus works, without any modification, for an *arbitrary* choice of the commutative ring  $R$ . The two most interesting choices for our purposes are the following:

- $R = \mathbb{Z}[q, \mathbf{v}]$  with  $q$  and  $\{v_e\}$  taken to be indeterminates. This allows us to compute *symbolically* the various multivariate Tutte polynomials.
- $R = \mathbb{C}$  (or  $\mathbb{R}$  or  $\mathbb{Q}$  or  $\mathbb{Z}$ ) with  $q$  and  $\{v_e\}$  given specified numerical values. This allows us to compute the *numerical values* of the various multivariate Tutte polynomials.

Let us now state the algorithm, which is in fact a trivial application of Proposition 2.3.

ALGORITHM 1. Fix a commutative ring  $R$ , and fix values  $q \in R$  and  $\mathbf{v} = \{v_e\} \in R^E$ . We assume that the values of  $A_{H,a,b}(q, \mathbf{v})$  and  $B_{H,a,b}(q, \mathbf{v})$  are known for every *leaf* node  $(H, a, b)$ . Then we proceed inductively up the tree, computing  $A_{H,a,b}(q, \mathbf{v})$  and  $B_{H,a,b}(q, \mathbf{v})$  using Proposition 2.3:

- If  $(H, a, b)$  is a  $p$ -node whose children  $(H_1, s_1, t_1)$  and  $(H_2, s_2, t_2)$  have already been computed, we set

$$(3.4a) \quad A_{H,a,b} = A_{H_1,s_1,t_1} A_{H_2,s_2,t_2},$$

$$(3.4b) \quad B_{H,a,b} = A_{H_1,s_1,t_1} B_{H_2,s_2,t_2} + A_{H_2,s_2,t_2} B_{H_1,s_1,t_1} + B_{H_1,s_1,t_1} B_{H_2,s_2,t_2}.$$

- If  $(H, a, b)$  is an  $s$ -node whose children  $(H_1, s_1, t_1)$  and  $(H_2, s_2, t_2)$  have already been computed, we set

$$(3.5a) A_{H,a,b} = A_{H_1,s_1,t_1} B_{H_2,s_2,t_2} + A_{H_2,s_2,t_2} B_{H_1,s_1,t_1} + q A_{H_1,s_1,t_1} A_{H_2,s_2,t_2},$$

$$(3.5b) B_{H,a,b} = B_{H_1,s_1,t_1} B_{H_2,s_2,t_2}. \quad \square$$

The validity of this algorithm is an immediate consequence of Proposition 2.3.

If the ring  $R$  is in fact a *field*  $F$ , then this algorithm can be usefully rephrased in terms of the “effective weights”  $v_{\text{eff}}(H, a, b) = B_{H,a,b}(q, \mathbf{v})/A_{H,a,b}(q, \mathbf{v})$ , *provided that we are careful to avoid division by zero*. The two most interesting choices of the field  $F$  are the following:

- $F = \mathbb{Q}(q, \mathbf{v})$ , the field of rational functions with rational coefficients in the indeterminates  $q$  and  $\{v_e\}$ . This will allow us to compute *symbolically* the various multivariate Tutte polynomials.
- $F = \mathbb{C}$  (or  $\mathbb{R}$  or  $\mathbb{Q}$ ). This will allow us to compute the *numerical values* of the various multivariate Tutte polynomials, when  $q$  and  $\{v_e\}$  are given specified numerical values.

In this version, the algorithm works as follows [for simplicity we concentrate on computing  $Z_G(q, \mathbf{v})$  and  $v_{\text{eff}}(H, a, b)$ ]:

ALGORITHM 2. Fix a field  $F$  and fix values  $q \in F$  and  $\mathbf{v} = \{v_e\} \in F^E$ . We assume that the values of  $A_{H,a,b}(q, \mathbf{v})$  and  $B_{H,a,b}(q, \mathbf{v})$  are known for every *leaf* node  $(H, a, b)$ , with all the  $A_{H,a,b}(q, \mathbf{v})$  nonzero. We can therefore define  $v_{\text{eff}}(H, a, b) = B_{H,a,b}(q, \mathbf{v})/A_{H,a,b}(q, \mathbf{v})$  for each leaf node. We now proceed inductively up the tree:

- If  $(H, a, b)$  is a  $p$ -node whose children  $(H_1, s_1, t_1)$  and  $(H_2, s_2, t_2)$  have already been labeled with values  $v_{\text{eff}}(H_1, s_1, t_1)$  and  $v_{\text{eff}}(H_2, s_2, t_2)$ , we then label  $(H, a, b)$  with

$$v_{\text{eff}}(H, a, b) = v_{\text{eff}}(H_1, s_1, t_1) \parallel v_{\text{eff}}(H_2, s_2, t_2).$$

- If  $(H, a, b)$  is an  $s$ -node whose children  $(H_1, s_1, t_1)$  and  $(H_2, s_2, t_2)$  have already been labeled with values  $v_{\text{eff}}(H_1, s_1, t_1)$  and  $v_{\text{eff}}(H_2, s_2, t_2)$ , we then label  $(H, a, b)$  with

$$v_{\text{eff}}(H, a, b) = v_{\text{eff}}(H_1, s_1, t_1) \bowtie_q v_{\text{eff}}(H_2, s_2, t_2)$$

*provided that*  $q + v_{\text{eff}}(H_1, s_1, t_1) + v_{\text{eff}}(H_2, s_2, t_2) \neq 0$ ; otherwise, we give  $v_{\text{eff}}(H, a, b)$  the value undefined and terminate the algorithm. In the former case, we also mark the node  $(H, a, b)$  as carrying a prefactor  $q + v_{\text{eff}}(H_1, s_1, t_1) + v_{\text{eff}}(H_2, s_2, t_2)$ .

If the algorithm succeeds in labeling the entire decomposition tree (i.e., does not encounter any value undefined), we then set  $Z_G(q, \mathbf{v})$  equal to  $q[q + v_{\text{eff}}(G, s, t)]$  times the product of the prefactors associated with all the  $s$ -nodes times the product of the  $A_{H,a,b}(q, \mathbf{v})$  from all the leaf nodes.  $\square$

Since this algorithm for computing  $v_{\text{eff}}(H, a, b)$  and  $Z_G(q, \mathbf{v})$  is simply a rephrasing of Algorithm 1 combined with (3.3), its validity follows immediately.

Of course, Algorithm 2 is not really an algorithm (i.e., a process that is always guaranteed to give an answer) because it could fail by encountering an undefined value at some  $s$ -node. But we can say the following:

First of all, Algorithm 2 is guaranteed to succeed when it is carried out symbolically, i.e., over the field  $\mathbb{Q}(q, \mathbf{v})$  of rational functions in the indeterminates  $q$  and  $\{v_e\}$ . More precisely, we have the following.

PROPOSITION 3.10. *Let  $(G, s, t)$  be a 2-terminal graph, with  $G$  connected and loopless, and fix a decomposition tree for it. Then, for each node  $(H, a, b)$  in the de-*

composition tree, the quantities  $v_{\text{eff}}(H, a, b)$ , considered as elements of the field  $\mathbb{Q}(q, \mathbf{v})$  of rational functions in the indeterminates  $q$  and  $\{v_e\}_{e \in E(H)}$ , have the following properties:

- (a)  $v_{\text{eff}}(H, a, b)$  is a rational function of  $q$  and  $\{v_e\}_{e \in E(H)}$  that depends nontrivially on  $\{v_e\}_{e \in E(H)}$ .
- (b) At an  $s$ -node  $(H, a, b)$  with children  $(H_1, s_1, t_1)$  and  $(H_2, s_2, t_2)$ , one can never have  $q + v_{\text{eff}}(H_1, s_1, t_1) + v_{\text{eff}}(H_2, s_2, t_2) = 0$  in  $\mathbb{Q}(q, \mathbf{v})$ .

Therefore, Algorithm 2 never encounters an undefined value when it is carried out over the field  $\mathbb{Q}(q, \mathbf{v})$ .

*Proof.* Statement (a) is simply Lemma 2.1(a) since by hypothesis  $H$  is connected. Statement (b) holds because, by (a),  $v_{\text{eff}}(H_1, s_1, t_1)$  and  $v_{\text{eff}}(H_2, s_2, t_2)$  depend nontrivially on disjoint sets of indeterminates.  $\square$

On the other hand, Algorithm 2 *can fail* when it is carried out over  $\mathbb{C}$  (i.e., with numerical values of  $q$  and  $\{v_e\}$ ). Suppose, for instance, that  $(G, s, t)$  [or some constituent thereof] consists of an edge  $e$  in series with the parallel combination of edges  $f$  and  $g$ . Then the multivariate Tutte polynomial for this graph is unambiguously

$$(3.6) \quad Z_G(q, \mathbf{v}) = q(q + v_e)(q + v_f + v_g + v_f v_g).$$

But if we choose  $v_e = -q$  (where  $q$  is any complex number),  $v_f = -1/2$ , and  $v_g = 1$ , then Algorithm 2 first computes  $v_f \parallel v_g = 0$  and then tries to compute  $v_e \bowtie_q (v_f \parallel v_g) = -q \bowtie_q 0$ , yielding an undefined result of  $0/0$ .

It is nevertheless worth stressing once again that whenever Algorithm 2, carried out over  $\mathbb{C}$  (or any other field), does give an answer, that answer is guaranteed to be correct.

In the remainder of this paper, when we use Algorithm 2 over  $\mathbb{C}$ , we will do so in the context of additional hypotheses that guarantee that no intermediate answer is ever undefined.

**Some remarks concerning computational complexity.**

1. There exists a linear-time algorithm for taking a 2-terminal graph  $(G, s, t)$  and finding a maximal decomposition tree for it (see [35]). Then Lemma 3.8 tells us, in particular, that  $(G, s, t)$  is 2-terminal series-parallel if and only if all the leaves of this maximal decomposition tree are  $K_2$ 's.
2. Given a maximal decomposition tree for a 2-terminal series-parallel graph  $(G, s, t)$ , Algorithm 2 provides a linear-time algorithm for computing  $Z_G(q, \mathbf{v})$  as well as  $v_{\text{eff}}(H, a, b)$  for every constituent  $(H, a, b)$ , provided that we work in a computational model where each field operation (in  $\mathbb{C}$  or in  $\mathbb{Q}(q, \mathbf{v})$  as the case may be) is assumed to take a time of order 1, and provided we take into account the possibility of failure when we work over  $\mathbb{C}$ .

**3.5. Leaf-joined trees.** Given a positive integer  $r \geq 2$ , we can form a graph  $G_n^r$  by taking a complete  $r$ -ary rooted tree of height  $n \geq 1$  and then identifying all the leaves into a single vertex. As an example, Figure 1 shows the graphs  $G_3^2$  and  $G_3^3$ .

We consider  $G_n^r$  as a 2-terminal graph in which the terminals are the root and the identified-leaves vertex. It is easy to see that  $G_n^r$  is in fact 2-terminal series-parallel, as it can be defined recursively as follows:

$$(3.7a) \quad G_1^r = K_2^{(r)},$$

$$(3.7b) \quad G_{n+1}^r = (K_2 \bowtie G_n^r)^{\parallel r},$$

where  $K_2^{(r)}$  is the graph with two vertices connected by  $r$  parallel edges, and  $G^{\parallel r}$  denotes the parallel composition of  $r$  copies of  $G$ . It then follows that  $G_n^r$  has  $(r^n +$

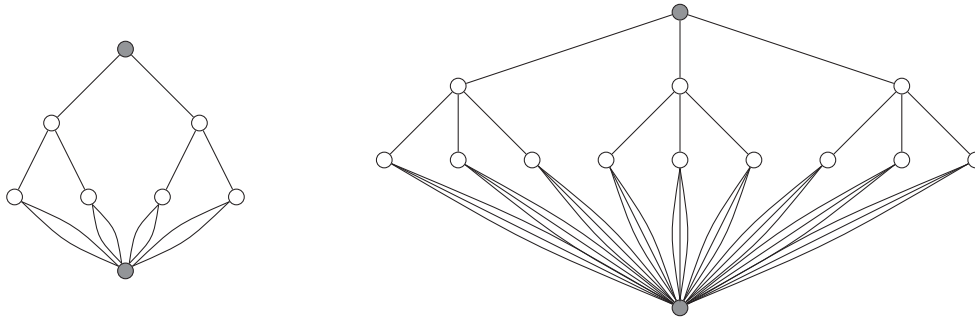


FIG. 1.  $G_3^2$  is a complete binary tree of height three with all leaves identified, and  $G_3^3$  is a complete ternary tree of height three with all leaves identified.

$r - 2)/(r - 1)$  vertices, that the flow between its terminals is  $r$  (for  $n \geq 1$ ), and that its maxflow is  $r + 1$  (for  $n \geq 2$ ).

In [26, Appendix B], we prove the following theorem.

**THEOREM 3.11.** *For fixed  $r \geq 2$ , every point of the circle  $|q - 1| = r$  is a limit point of chromatic roots for the family  $\{G_n^r\}_{n \geq 1}$  of leaf-joined trees of branching factor  $r$ . [More precisely, for every  $q_0$  satisfying  $|q_0 - 1| = r$  and every  $\epsilon > 0$ , there exists  $n_0 = n_0(q_0, \epsilon)$  such that for all  $n \geq n_0$  the graph  $G_n^r$  has a chromatic root  $q$  lying in the disc  $|q - q_0| < \epsilon$ .]*

**4. An abstract theorem on excluding roots.** The multivariate Tutte polynomial of the graph  $G = K_2$  having a single nonloop edge of weight  $v_e$  is  $Z_{K_2}(q, \mathbf{v}) = q(q + v_e)$ , which has roots at  $q = 0$  and  $q = -v_e$ . Given a 2-terminal series-parallel graph  $G$  with arbitrary complex edge weights  $\{v_e\}$  and a fixed complex number  $q$ , we can apply series and parallel reductions as in section 3.4 until  $G$  has been reduced to a single edge with some “effective weight”  $v_{\text{eff}} \in \mathbb{C} \cup \{\text{undefined}\}$ . If  $v_{\text{eff}} \neq \text{undefined}$ , then we can be sure that none of the prefactors of the form  $q + v_{e_1} + v_{e_2}$  generated during the series reductions were 0, and we can therefore conclude that  $Z_G(q, \mathbf{v}) = 0$  if and only if  $q = 0$  or  $v_{\text{eff}} = -q$ .

This observation then gives us a strategy for determining root-free regions for the multivariate Tutte polynomials of families of series-parallel graphs. For a fixed  $q \neq 0$  in the conjectured root-free region, we bound the regions of the (finite) complex  $v$ -plane where  $v_{\text{eff}}$  can lie for any graph in the family, and we show that these regions do not contain the point  $v_{\text{eff}} = -q$  that would correspond to a zero of  $Z_G(q, \mathbf{v})$ . If we can do this, then we have shown that  $Z_G(q, \mathbf{v}) \neq 0$ . The precise result is as follows.

**THEOREM 4.1.** *Let  $q \neq 0$  be a fixed complex number and let  $\Lambda \geq 2$  be a fixed integer. Let  $S_1 \subseteq S_2 \subseteq \dots \subseteq S_{\Lambda-1}$  be sets in the (finite) complex  $v$ -plane such that*

- (1)  $S_k \bowtie_q^V S_\ell \subseteq S_{\min(k, \ell)}$  for all  $k, \ell$ ,
- (2)  $S_k \parallel^V S_\ell \subseteq S_{k+\ell}$  for  $k + \ell \leq \Lambda - 1$ .

*Now consider any 2-terminal series-parallel graph  $(G, s, t)$  and any maximal decomposition tree for  $(G, s, t)$  in which all the proper constituents have between-terminals flow at most  $\Lambda - 1$ , and equip  $G$  with edge weights  $v_e \in S_1$ . Then, for every node  $(H, a, b)$  of the decomposition tree that has between-terminals flow  $\lambda_H(a, b) \leq \Lambda - 1$ , we have  $v_{\text{eff}}(H, a, b) \in S_{\lambda_H(a, b)}$ .*

*Now assume further that, in addition to (1) and (2), the following hypotheses hold:*

- (3)  $-q \notin S_{\Lambda-1}$ ,
- (4)  $-q \notin S_k \parallel^V S_\ell$  for  $k + \ell = \Lambda$ .



Then, for any loopless series-parallel graph  $G$  with maxmaxflow at most  $\Lambda$ , we have  $Z_G(q, \mathbf{v}) \neq 0$  whenever  $v_e \in S_1$  for all edges. (In particular, if  $-1 \in S_1$ , then  $q$  is not a chromatic root of  $G$ .)

*Remarks.*

1. It is implicit in condition (1) that the operation in question is always well-defined (i.e., does not take the value undefined) or, in other words, that  $q + v_1 + v_2 \neq 0$  whenever  $v_1 \in S_k$  and  $v_2 \in S_\ell$ .
2. When the root of the decomposition tree is a  $p$ -node (which occurs, in particular, whenever  $G$  is nonseparable and not  $K_2$ ) and  $G$  has maxmaxflow  $\Lambda$ , then Lemma 3.9 guarantees that every *proper* constituent has between-terminals flow at most  $\Lambda - 1$ . The root node  $(G, s, t)$ , by contrast, *might* have between-terminals flow as large as  $\Lambda$ .  $\square$

*Proof.* Let  $(G, s, t)$ , its maximal decomposition tree, and its edge weights be as specified. We want to prove that  $v_{\text{eff}}(H, a, b) \in S_{\lambda_H(a,b)}$  for all nodes  $(H, a, b)$  that satisfy  $\lambda_H(a, b) \leq \Lambda - 1$ . We shall prove this claim by induction upwards from the leaves of the decomposition tree. By Lemma 3.8, a leaf of the decomposition tree is an edge  $e$  of  $G$ , hence has between-terminals flow equal to 1, and  $v_{\text{eff}} = v_e \in S_1$  by hypothesis. So let  $(H, a, b)$  be a nonleaf node of the decomposition tree and suppose that the children of  $(H, a, b)$ , call them  $(H_1, a_1, b_1)$  and  $(H_2, a_2, b_2)$ , have between-terminals flow  $k$  and  $\ell$ , respectively. Since  $(H_1, a_1, b_1)$  and  $(H_2, a_2, b_2)$  are proper constituents, we have by hypothesis  $k, \ell \leq \Lambda - 1$ ; so by the inductive hypothesis, we have  $v_{\text{eff}}(H_1, a_1, b_1) \in S_k$  and  $v_{\text{eff}}(H_2, a_2, b_2) \in S_\ell$ . Using Lemma 3.5, it is clear that conditions (1) and (2) ensure that  $v_{\text{eff}}(H, a, b) \in S_{\lambda_H(a,b)}$  holds whenever  $\lambda_H(a, b) \leq \Lambda - 1$  (which holds for all proper constituents and might or might not hold for the root node). This proves the first half of the theorem.

As  $Z_G(q, \mathbf{v})$  is multiplicative over blocks, and the maxmaxflow of a separable graph is the maximum of the maxmaxflows of its blocks, it suffices to prove the second half of the theorem when  $G$  is a loopless *nonseparable* series-parallel graph of maxmaxflow at most  $\Lambda$ . Since the result holds trivially when  $G = K_1$ , we can assume that  $G$  has at least one edge. Therefore, by Theorem 3.3,  $G$  has a pair of vertices  $s, t$  such that  $(G, s, t)$  is a 2-terminal series-parallel graph and, hence, described by a maximal decomposition tree whose leaf nodes are single edges. Furthermore, by Lemma 3.9, all of the proper constituents of  $(G, s, t)$  have between-terminals flow at most  $\Lambda - 1$ . Therefore, if  $(H, a, b)$  is a proper constituent of  $(G, s, t)$  with between-terminals flow  $\lambda_H(a, b) = \lambda$ , we can apply the first half of the theorem to conclude that  $v_{\text{eff}}(H, a, b) \in S_\lambda$ .

By condition (3) [and the nesting  $S_i \subseteq S_{\Lambda-1}$ ], we have  $v_{\text{eff}}(H, a, b) \neq -q$  whenever  $(H, a, b)$  is a proper constituent of  $(G, s, t)$ . On the other hand, the final step (at the root of the decomposition tree) constructs  $(G, s, t)$  as the parallel composition of two proper constituents whose between-terminal flows sum to  $\lambda_G(s, t) \leq \Lambda$ , so conditions (4) and (2), (3) together ensure that  $v_{\text{eff}}(G, s, t) \neq -q$ . Therefore, by Algorithm 2 of section 3.4,  $Z_G(q, \mathbf{v})$  is equal to a nonzero prefactor—namely, the product over  $s$ -nodes of  $q + v_{\text{eff}}(G_1, s_1, t_1) + v_{\text{eff}}(G_2, s_2, t_2)$ , a quantity that is nonvanishing by virtue of Remark 1 preceding this proof—multiplied by  $q[q + v_{\text{eff}}(G, s, t)]$ , and is therefore nonzero as claimed.  $\square$

Of course, to *apply* this theorem, it is necessary to actually *identify* suitable sets  $S_1 \subseteq S_2 \subseteq \dots \subseteq S_{\Lambda-1}$ . In practice one usually starts from a specified set  $\mathcal{V} \subseteq \mathbb{C}$  of “allowed edge weights”—for instance,  $\mathcal{V} = \{-1\}$  for the chromatic polynomial—and one attempts to find sets  $S_1 \subseteq S_2 \subseteq \dots \subseteq S_{\Lambda-1}$  satisfying  $S_1 \supseteq \mathcal{V}$  along with the hypotheses (1)–(4) of Theorem 4.1. For any particular combination of  $q, \Lambda$ , and  $\mathcal{V}$ ,

there is always a collection of *minimal regions*  $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_{\Lambda-1}$ , where  $S_1 \supseteq \mathcal{V}$  and conditions (1) and (2) are satisfied. If one knows this collection of minimal regions, then conditions (3) and (4) become a “final check” certifying that  $q$  is not a root.

In practice, though, it is almost always impossible to describe the minimal regions even for specific values of  $q$  and  $\Lambda$ , let alone symbolically (but see section 5 for some computer-generated approximations). Therefore it is necessary to bound the optimal regions inside larger regions with shapes that are more amenable to analysis. But it is also important to fit the bounding regions as tightly as possible to the optimal regions, as conditions (1) and (2) cause any “unnecessary points” included in the approximation to a region to have a cascading effect on the approximations for the other regions, thereby incorporating still more possibly unnecessary points, and so on.

There are, in fact, two slightly different reasons why including unnecessary points in the regions  $S_i$  can lead to poor bounds. First, if we have chosen  $S_2, S_3, \dots$  to be much larger than they need to be, *for the given set*  $S_1$ , then the bounds one obtains from Theorem 4.1 may (not surprisingly) be much weaker than the truth. Second, it is important to observe that even if we are ultimately interested in proving  $Z_G(q, \mathbf{v}) \neq 0$  for weights  $v_e$  lying in a specified set  $\mathcal{V}$ , we will get from Theorem 4.1, whether we like it or not, the same result for all  $v_e \in S_1$ . Of course, if  $S_1$  is exactly the minimal region containing the given  $\mathcal{V}$  and satisfying conditions (1) and (2), then nothing is lost, as any bound valid for all series-parallel graphs of maxmaxflow  $\Lambda$  with weights in  $\mathcal{V}$  will also be valid for weights in  $S_1$  (since any  $v$  lying in the minimal region  $S_1$  is, in fact, the  $v_{\text{eff}}$  for a suitable 2-terminal series-parallel graph of maxmaxflow  $\Lambda$  and between-terminals flow 1, with edge weights in  $\mathcal{V}$ ). But if the chosen  $S_1$  is significantly larger than the minimal region, then even the best-possible bound for weights in  $S_1$  may be much weaker than the corresponding bound for weights in  $\mathcal{V}$ . In particular, if  $S_1$  extends much outside the “complex antiferromagnetic regime”  $|1 + v_e| \leq 1$ —where “much outside” means, roughly, more than a distance of order  $1/|q|$ —then one *expects* the  $q$ -plane roots of  $Z_G(q, \mathbf{v})$  to grow exponentially in  $\Lambda$  rather than linearly (see [17] for further discussion, and see also footnote 19 below).

The simplest types of regions to manipulate analytically are discs, especially discs centered at the origin, and so it is natural to try to bound the optimal regions inside suitable discs. If one insists on using discs centered at the origin, then it furthermore matters whether one uses the  $v$ -variables, the  $y$ -variables, or the  $t$ -variables. If one makes a poor choice—e.g., the optimal regions are either far from being discs, or far from being centered at the origin in the chosen variables—then one will obtain poor bounds, e.g., bounds that grow exponentially rather than linearly in  $\Lambda$ .

It turns out that the optimal regions are not too far from being discs centered at the origin *if we use the  $t$ -variables*, but are quite far from being discs centered at the origin if we use the  $v$ - or  $y$ -variables. We shall therefore use the  $t$ -variables in the remainder of this paper. Let us recall that the important points  $v = -1$ ,  $v = \infty$ , and  $v = -q$  correspond to  $t = 1/(1 - q)$ ,  $t = 1$  and  $t = \infty$ , respectively. We can therefore reexpress Theorem 4.1 in the language of transmissivities  $\{t_e\}$ . For simplicity we suppress the statements about  $v_{\text{eff}}$  (or  $t_{\text{eff}}$ ) and concentrate on the conclusion that  $Z_G(q, \mathbf{v}) \neq 0$ .

**THEOREM 4.2.** *Let  $q \neq 0$  be a fixed complex number and let  $\Lambda \geq 2$  be a fixed integer. Let  $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_{\Lambda-1}$  be sets in the (finite) complex  $t$ -plane such that*

- (1)  $S_k \bowtie^T S_\ell \subseteq S_{\min(k,\ell)}$  for all  $k, \ell$ ,
- (2)  $S_k \parallel_q^T S_\ell \subseteq S_{k+\ell}$  for  $k + \ell \leq \Lambda - 1$ ,
- (3')  $1 \notin S_{\Lambda-1}$ ,

(4)  $S_k \bowtie_q^T S_\ell \subseteq \mathbb{C}$  for  $k + \ell = \Lambda$  (i.e., does not ever take the value undefined).

Then, for any series-parallel graph  $G$  with *maxflow* at most  $\Lambda$ , we have  $Z_G(q, \mathbf{v}) \neq 0$  whenever  $v_e/(q + v_e) \in S_1$  for all edges.

In particular, to handle chromatic polynomials it suffices to arrange that  $1/(1 - q) \in S_1$ .

*Remark.* Condition (3) states merely that the set  $S_{\Lambda-1}$  avoids the point  $t = 1$ , but in practice we will always have  $S_{\Lambda-1} \subseteq \{|t| < 1\}$ . Indeed, if  $S_{\Lambda-1}$  contains any point with  $|t| = 1$  (resp.,  $|t| > 1$ ), then by condition (1) its closure  $\overline{S_{\Lambda-1}}$  must contain the point  $t = 1$  (resp.,  $t = \infty$ ); and while this is not explicitly forbidden, it is hard to see how one could satisfy all the hypotheses (1)–(4) in such a case.

*Proof of Theorem 4.2.* This is almost a direct translation of Theorem 4.1 into transmissivities. Indeed, conditions (1) and (2) here are direct translations of conditions (1) and (2) of Theorem 4.1. Condition (3') here is equivalent to the hypothesis in Theorem 4.1 that the sets lie in the *finite v*-plane, while condition (3) of Theorem 4.1 is equivalent to the hypothesis here that the sets lie in the *finite t*-plane. Finally, condition (4) here is a direct translation of condition (4) of Theorem 4.1.  $\square$

Since the regions  $S_i$  are assumed increasing, the condition (1) is most stringent for  $\ell = \Lambda - 1$ , and it reduces to

$$(1') \quad S_k \bowtie^T S_{\Lambda-1} \subseteq S_k \quad \text{for all } k.$$

Furthermore, there is a simple but very useful *sufficient* condition for condition (1), (1') to hold.

LEMMA 4.3. *If there exists  $r > 0$  such that*

$$(4.1) \quad D(r^2) \subseteq S_1 \subseteq S_2 \subseteq \dots \subseteq S_{\Lambda-1} \subseteq D(r),$$

where  $D(r) = \{t \in \mathbb{C} : |t| \leq r\}$ , then condition (1) of Theorem 4.2 holds.

*Proof.*  $S_k \bowtie^T S_\ell \subseteq D(r) \bowtie^T D(r) = D(r^2) \subseteq S_{\min(k,\ell)}$ .  $\square$

**5. Discs in the  $t$ -plane.** In this section we shall prove the following strengthening of Theorem 1.4.

THEOREM 5.1. *Fix an integer  $\Lambda \geq 2$ , and let  $G$  be a loopless series-parallel graph of *maxflow* at most  $\Lambda$ . Let  $\rho_\Lambda^*$  be the unique solution of*

$$(5.1) \quad (1 + \rho)^\Lambda = 2(1 + \rho^2)^{\Lambda-1}$$

in the interval  $(0, 1)$  when  $\Lambda \geq 3$ , and let  $\rho_2^* = 1$ . Then the multivariate Tutte polynomial  $Z_G(q, \mathbf{v})$  is nonvanishing whenever  $|q - 1| \geq 1/\rho_\Lambda^*$  (with  $\geq$  replaced by  $>$  when  $\Lambda = 2$ ) and the complex edge weights  $\mathbf{v} = \{v_e\}_{e \in E}$  satisfy

$$(5.2) \quad v_e = -1 \quad \text{or} \quad \left| \frac{v_e}{q + v_e} \right| \leq \rho \frac{X - 1}{1 - \rho X}$$

(again with strict inequality when  $\Lambda = 2$ ), where

$$(5.3) \quad \rho = \frac{1}{|q - 1|} \quad \text{and} \quad X = \left( \frac{2}{1 + \rho} \right)^{1/(\Lambda-1)}.$$

Furthermore we have  $\rho_\Lambda^* > (\log 2)/(\Lambda - \frac{3}{2} \log 2)$ , so that, in particular, all the roots (real or complex) of the chromatic polynomial  $P_G(q)$  lie in the disc  $|q - 1| < (\Lambda - \frac{3}{2} \log 2)/\log 2$ .

*Remark.* We shall see in Lemma 5.6 that under the hypothesis  $|q - 1| \geq 1/\rho_\Lambda^*$  (i.e.,  $\rho \leq \rho_\Lambda^*$ ) we have

$$(5.4) \quad \rho \frac{X - 1}{1 - \rho X} \geq \rho^2,$$

so that the conclusion of Theorem 5.1 holds under the more stringent but simpler condition

$$(5.5) \quad v_e = -1 \quad \text{or} \quad \left| \frac{v_e}{q + v_e} \right| \leq \frac{1}{|q - 1|^2}. \quad \square$$

We shall prove Theorem 5.1 by exhibiting regions  $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_{\Lambda-1}$  of the complex  $t$ -plane that satisfy the conditions of Theorem 4.2 when  $|q - 1| \geq 1/\rho_\Lambda^*$  and for which the set  $S_1$  corresponds precisely to (5.2). Since in this section we shall always be working in the  $t$ -plane, we shall henceforth drop the superscripts  $T$  from the operators  $\|_q^T$  and  $\bowtie^T$ . Let us also recall that, in the  $t$ -plane, series connection  $\bowtie$  is simply multiplication.

Before beginning this proof, it is instructive to engage in some informal motivation of our constructions.

If we want to handle the chromatic polynomial using Theorem 4.2, then we must certainly have  $1/(1 - q) \in S_1$ . The set of *minimal* regions  $S_i$  that contain the point  $1/(1 - q)$  and satisfy the first two conditions of Theorem 4.2 can be approximated by computer, because these conditions can be viewed as rules for constructing each  $S_i$  from certain others. By imposing a fine grid on the disc  $|t| < 1$  and “rounding” each complex number to the closest grid point, we can restrict our attention to a finite number of points. We start by marking  $t_0 = 1/(1 - q)$  as belonging to  $S_1$  (and hence to each  $S_i$ ); we then iteratively construct approximations to the regions  $S_1, S_2, \dots, S_{\Lambda-1}$  by using conditions (1) and (2) of Theorem 4.2 until the approximations are closed under further application of the rules.<sup>17</sup> If the resulting region  $S_{\Lambda-1}$  is contained in the open unit disc  $\{|t| < 1\}$ , then Theorem 4.2 implies that  $q$  is not a chromatic root for any graph of maxmaxflow  $\Lambda$ .

Repeating these experiments for a range of different values of  $q$  and moderate values of  $\Lambda$  suggests that although the minimal regions are generally complicated shapes, they are often loosely “disc-like” and can be bounded reasonably well by a disc in the  $t$ -plane centered at the origin. Some examples with  $\Lambda = 3$  are shown in Figure 2, and a more extensive set of plots is included with the preprint version of this paper at arXiv.org.<sup>18</sup>

In fact we need to be a bit more careful, because every region  $S_i$  must contain the point  $t_0 = 1/(1 - q)$ , but taking the smallest region  $S_1$  to be a disc of radius  $\rho = |t_0| = |1/(1 - q)|$  *cannot* give very good bounds. Indeed, with this choice of  $S_1$  there exist graphs  $G$  of maxmaxflow  $\Lambda$  having roots  $Z_G(q, v) = 0$  with  $v \in S_1$  and  $q$  growing *exponentially* in  $\Lambda$  (more precisely, like  $2^\Lambda$ ).<sup>19</sup>

<sup>17</sup>For instance, for  $\Lambda = 3$  the rules are simply  $S_1 \subseteq S_2$ ,  $S_1 \parallel S_1 \subseteq S_2$ ,  $S_1 S_2 \subseteq S_1$ , and  $S_2 S_2 \subseteq S_2$ .

<sup>18</sup>See the ancillary files S1S2\_2.2.pdf, S1S2\_2.4.pdf, and S1S2\_3.0.pdf. Each of these files shows  $S_1$  and  $S_2$  in the complex  $t$ -plane for  $\Lambda = 3$  and a set of values of  $q$  defined by  $q - 1 = Re^{i\theta}$ , where  $R$  takes the specified value (2.2, 2.4 or 3.0) and  $\theta = k\pi/180$  for  $k = 0, 5, 10, \dots, 180$ . These plots use the conventions explained in the caption of Figure 2.

<sup>19</sup>Just take  $G = K_2^{(k)}$  (i.e.,  $k$  edges in parallel), which has maxmaxflow  $k$ . Consider  $q < 0$ , and write  $q = -Q$  for simplicity. Then  $\rho = 1/(1 + Q)$ , and the point  $t = -\rho = -1/(1 + Q) \in S_1$  corresponds to  $v = Q/(Q + 2)$ . Then  $Z_{K_2^{(k)}}(q, v) = q + (1 + v)^k - 1$  vanishes when  $[(2Q + 2)/(Q + 2)]^k = Q + 1$ , which occurs for large  $k$  at  $Q = 2^k - k - 1 + O(k^2/2^k)$ .

What is going on here is that  $v = Q/(Q + 2)$  is *strongly ferromagnetic*: for  $Q \gg 1$  we have  $v \approx 1$ , hence  $y = 1 + v \approx 2$ ; so putting  $k$  such edges in parallel leads to a weight that grows like  $2^k$ . Similar behavior will occur whenever  $S_1$  contains any point having  $|1 + v|$  uniformly larger than 1. Indeed, we expect large roots in the  $q$ -plane whenever  $S_1$  contains any point having  $|1 + v| - 1 \gg 1/|q|$ .

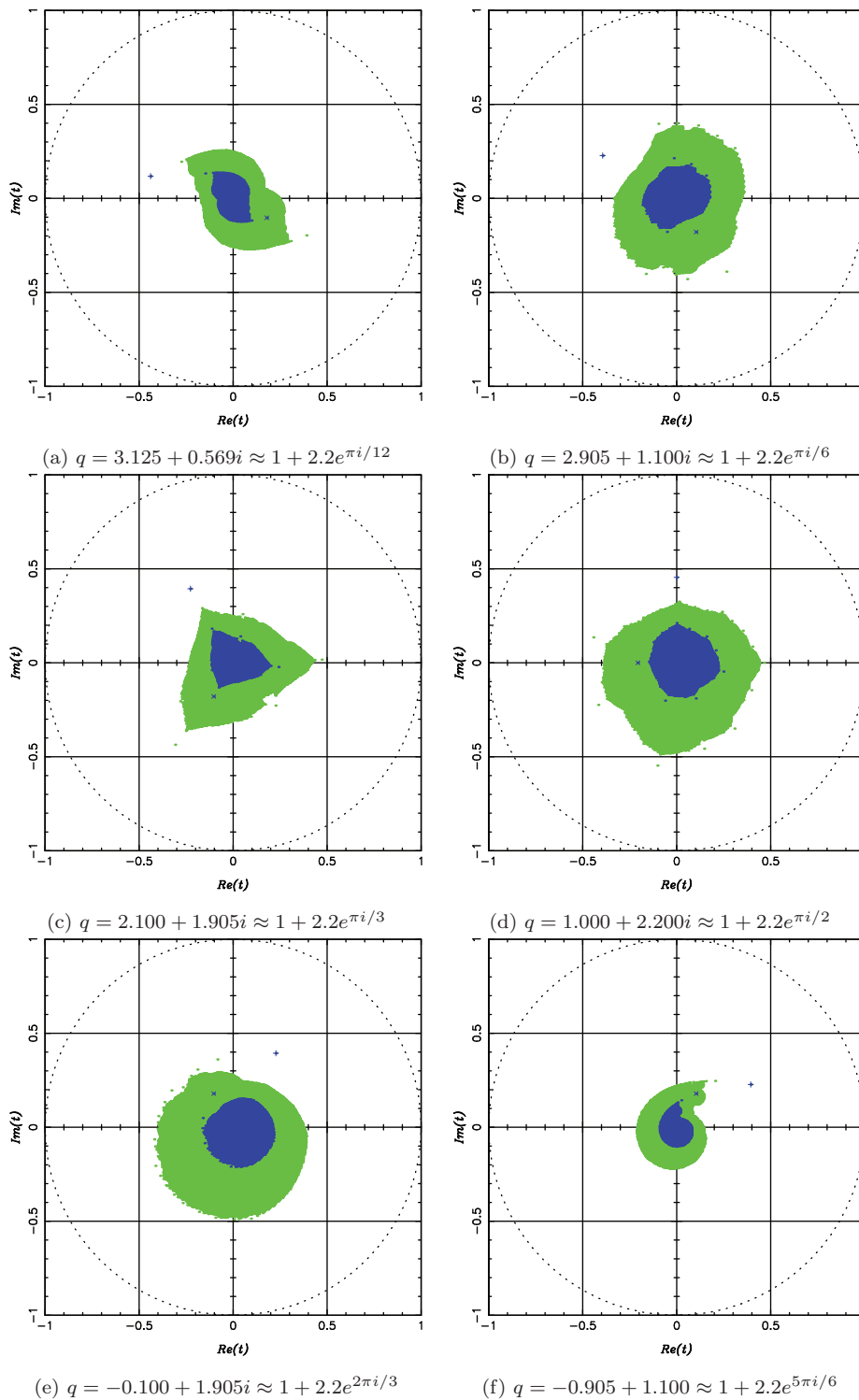


FIG. 2. Computer-generated approximations to  $S_1$  (dark blue) and  $S_2$  (light green) in the complex  $t$ -plane, for  $\Lambda = 3$  and selected values of  $q$ . Note that we always have  $S_1 \subseteq S_2$  and  $S_1 = \{t_0\} \cup t_0 S_2$ , where  $t_0 = 1/(1-q)$ . The points  $t_0$  and  $t_0 \bowtie t_0 = t_0^2$ , which both belong to  $S_1$ , are shown as dark blue  $+$  and  $\times$ , respectively. The circle  $|t| = 1$  is shown for reference in dashed black.

However, a slight modification works, namely, we take each region  $S_i$  to be a “point + disc”

$$(5.6) \quad S_i = \{1/(1-q)\} \cup D(r_i),$$

where  $D(r_i)$  is a closed disc of radius  $r_i$  centered at the origin. This choice results in a situation that is both amenable to analysis and also yields good bounds when the radii  $r_i$  are suitably chosen, as we will prove in this section.

The disc  $D(r_1)$  must have radius at least  $\rho^2$  because it must contain the point  $t_0 \bowtie t_0 = 1/(1-q)^2$ . So choose some  $r_1 \geq \rho^2$ ; this choice of  $r_1$  sets a lower bound on the possible values for  $r_2$  because  $S_1 \parallel_q S_1 \subseteq S_2$ . Continuing in this fashion,  $r_1$  and  $r_2$  determine the minimum allowable value for  $r_3$ ; then  $r_1, r_2$ , and  $r_3$  determine the minimum allowable value for  $r_4$ ; and so on. Ultimately this process determines a minimum allowable value for  $r_{\Lambda-1}$ ; and if  $r_{\Lambda-1} \leq \rho$ , then the set of radii  $r_1, r_2, \dots, r_{\Lambda-1}$  yields a set of regions  $S_i$  defined by (5.6) that satisfies the conditions of Theorem 4.2. We formalize this observation in the following proposition.

**PROPOSITION 5.2.** *Let  $\Lambda \geq 2$  be a fixed integer; then let  $q$  be a fixed complex number satisfying  $|q-1| > 1$ , and set  $t_0 = 1/(1-q)$  and  $\rho = |t_0|$ . If the real numbers  $r_1, r_2, \dots, r_{\Lambda-1}$  satisfy*

$$(5.7) \quad \rho^2 \leq r_1 \leq r_2 \leq \dots \leq r_{\Lambda-1} \leq \rho$$

and

$$(5.8) \quad r_s \geq \max\{|t_e \parallel_q t_f| : t_e \in D(r_k), t_f \in D(r_\ell), k + \ell = s\}$$

for  $2 \leq s \leq \Lambda - 1$ , then the set of regions  $S_1, S_2, \dots, S_{\Lambda-1}$  defined by

$$(5.9) \quad S_i = \{1/(1-q)\} \cup D(r_i)$$

satisfies the conditions of Theorem 4.2.

*Proof.* We need to show that the four conditions of Theorem 4.2 hold. Condition (1) holds by Lemma 4.3 with  $r = \rho$ . To check condition (2), we observe that

$$(5.10) \quad S_k \parallel_q S_\ell = (\{t_0\} \cup D(r_k)) \parallel_q (\{t_0\} \cup D(r_\ell)) = \{t_0\} \cup (D(r_k) \parallel_q D(r_\ell))$$

because  $t_0 \parallel_q t = t_0$  for every  $t$ . Therefore condition (5.8) on the radii is exactly what is needed to ensure that  $S_k \parallel_q S_\ell \subseteq S_{k+\ell}$ . Condition (3') holds because  $S_{\Lambda-1} \subseteq D(\rho)$  and  $\rho < 1$ . Finally, condition (4) fails only if there are  $t_e \in S_k$  and  $t_f \in S_\ell$  (with  $k + \ell = \Lambda$ , though we do not even need to use this constraint) such that  $t_e t_f = 1/(1-q)$ , but this is impossible because  $|t_e t_f| \leq \rho^2 < \rho = 1/|1-q|$ .  $\square$

To apply this theorem, we need to be able to bound the modulus of

$$(5.11) \quad t_e \parallel_q t_f = \frac{t_e + t_f + (q-2)t_e t_f}{1 + (q-1)t_e t_f}$$

when  $t_e \in D(r_k)$  and  $t_f \in D(r_\ell)$ . Since the maximum modulus of  $t_e \parallel_q t_f$  occurs when  $t_e$  and  $t_f$  are on the boundaries of their respective discs, let us define for  $x, y \in [0, \rho]$  the function

$$(5.12) \quad f_q(x, y) := \max\{|t_e \parallel_q t_f| : |t_e| = x, |t_f| = y\}.$$

If we bound (5.11) in the most naive way by replacing the numerator by an upper bound and the denominator by a lower bound, and we furthermore use  $|q-2| \leq$



$|q - 1| + 1$  to express the  $q$ -dependence in terms of the single number  $|q - 1|$ , then we get

$$(5.13) \quad f_q(x, y) \leq F_q(x, y) := \frac{x + y + (|q - 1| + 1)xy}{1 - |q - 1|xy} = \frac{x + y + (\rho^{-1} + 1)xy}{1 - \rho^{-1}xy} .$$

The condition  $x, y \in [0, \rho)$  ensures that the denominator of  $F_q(x, y)$  is strictly positive. Therefore, given the chosen value of  $r_1$ , we can define a sequence of radii  $r_2, r_3, \dots$  satisfying (5.8) using the iteration

$$(5.14) \quad r_s = \max\{F_q(r_k, r_\ell) : k + \ell = s\}$$

(stopping the iteration whenever a result  $r_s$  becomes  $\geq \rho$ ). It is immediate that  $r_1 \leq r_2 \leq \dots$ . If the iteration remains well-defined up to  $s = \Lambda - 1$  and satisfies  $r_{\Lambda-1} \leq \rho$ , then the radii satisfy the hypotheses of Proposition 5.2. (Henceforth let us write  $F$  in place of  $F_q$  to lighten the notation.)

At first sight, this seems rather unappealing for analysis because the max in (5.14) appears difficult to handle. However, this difficulty is illusory because it turns out that  $F$  is actually an *associative* function.

LEMMA 5.3. *Let  $G$  be a function of the form*

$$(5.15) \quad G(x, y) = \frac{x + y + Axy}{1 + Bxy} ,$$

where  $A, B$  are arbitrary constants. Then

$$(5.16) \quad G(x, G(y, z)) = G(y, G(x, z)) = G(z, G(x, y)) .$$

*Proof.* Direct calculation shows that

$$(5.17) \quad G(x, G(y, z)) = \frac{(x + y + z) + A(xy + yz + xz) + (A^2 + B)xyz}{1 + B(xy + xz + yz) + ABxyz} ,$$

which is clearly symmetric under all permutations of  $\{x, y, z\}$ .  $\square$

COROLLARY 5.4. *If  $F$  is given by (5.13) and  $r_2, \dots, r_{\Lambda-1}$  by (5.14), then*

$$(5.18) \quad F(r_k, r_\ell) = F(r_1, r_{k+\ell-1})$$

for all pairs  $k, \ell$  of positive integers such that  $k + \ell \leq \Lambda$ .

*Proof.* We prove this by induction on  $s = k + \ell$ . The result clearly holds for  $s = 2$ . So suppose that the result is true for all  $k' + \ell' < k + \ell$ . Then

$$(5.19) \quad F(r_k, r_\ell) = F(F(r_1, r_{k-1}), r_\ell) = F(r_1, F(r_{k-1}, r_\ell)) = F(r_1, r_{k+\ell-1})$$

and the result holds.  $\square$

The key point of this lemma (which was used implicitly in the proof) is that all the terms in (5.14) are actually the same, and so we can arbitrarily choose any one of them to define  $r_s$ . So let us take  $r_{s+1} = F(r_1, r_s)$ , i.e.,

$$(5.20) \quad r_{s+1} = \frac{[1 + (\rho^{-1} + 1)r_1]r_s + r_1}{1 - \rho^{-1}r_1r_s} .$$

Since the map  $r_s \mapsto r_{s+1}$  is a Möbius transformation, we can obtain an explicit expression for  $r_k$ .

LEMMA 5.5. For fixed real numbers  $r_1$  and  $\rho \neq 1$ , define a sequence  $r_1, r_2, \dots \in \mathbb{R} \cup \{\infty\}$  by

$$(5.21) \quad r_{s+1} = \frac{[1 + (\rho^{-1} + 1)r_1]r_s + r_1}{1 - \rho^{-1}r_1r_s}.$$

Then

$$(5.22) \quad r_k = \rho \frac{(1 + r_1/\rho)^k - (1 + r_1)^k}{(1 + r_1)^k - \rho(1 + r_1/\rho)^k}.$$

*Proof.* The map  $r_s \mapsto r_{s+1}$  is a (real) Möbius transformation of the form

$$(5.23) \quad x \mapsto \frac{ax + b}{cx + d}$$

whose coefficients can be displayed in a suitable matrix

$$(5.24) \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 + (\rho^{-1} + 1)r_1 & r_1 \\ -\rho^{-1}r_1 & 1 \end{pmatrix}.$$

By standard results on Möbius transformations, the matrix  $M^k$  represents the  $k$ th iterate of this transformation. Now, the matrix  $M$  has eigenvalues  $1 + r_1/\rho$  and  $1 + r_1$ , and it can be diagonalized by  $M = QDQ^{-1}$ , where

$$(5.25) \quad D = \begin{pmatrix} 1 + r_1/\rho & 0 \\ 0 & 1 + r_1 \end{pmatrix},$$

$$(5.26) \quad Q = \frac{1}{1 - \rho} \begin{pmatrix} 1 & -\rho \\ -1 & 1 \end{pmatrix},$$

$$(5.27) \quad Q^{-1} = \begin{pmatrix} 1 & \rho \\ 1 & 1 \end{pmatrix}.$$

It follows immediately that  $M^k = QD^kQ^{-1}$  and so

$$(5.28) \quad M^k = \frac{1}{1 - \rho} \begin{pmatrix} (1 + r_1/\rho)^k - \rho(1 + r_1)^k & \rho[(1 + r_1/\rho)^k - (1 + r_1)^k] \\ (1 + r_1)^k - (1 + r_1/\rho)^k & (1 + r_1)^k - \rho(1 + r_1/\rho)^k \end{pmatrix}.$$

Treating this as a Möbius transformation and applying it to  $r_0 = 0$ , we get  $r_k = (M^k)_{12}/(M^k)_{22}$  and thus

$$(5.29) \quad r_k = \rho \frac{(1 + r_1/\rho)^k - (1 + r_1)^k}{(1 + r_1)^k - \rho(1 + r_1/\rho)^k}.$$

This also reproduces the correct value at  $k = 1$ .  $\square$

*Remarks.*

1. The formula (5.22), once we have it, can of course be proven by an easy induction on  $k$ . But we thought it preferable to give a more conceptual proof that shows where (5.22) comes from. Note also that we can rewrite (5.22) as

$$(5.30) \quad r_k = \rho \frac{X^k - 1}{1 - \rho X^k}, \quad \text{where } X = \frac{1 + r_1/\rho}{1 + r_1};$$

this will be useful later.

2. The reasoning in Lemma 5.3, Corollary 5.4, and Lemma 5.5 can be made even more explicit by observing that the associative function  $G(x, y) = (x + y + Axy)/(1 + Bxy)$  is actually conjugate to  $\widehat{G}(X, Y) = XY$ : it suffices to make the Möbius change of variables  $X = f(x) := (1 + \alpha x)/(1 + \beta x)$  with

$$(5.31a) \quad \alpha = \frac{A \pm \sqrt{A^2 + 4B}}{2},$$

$$(5.31b) \quad \beta = \frac{A \mp \sqrt{A^2 + 4B}}{2},$$

and we then have

$$(5.32) \quad f(G(f^{-1}(X), f^{-1}(Y))) = XY.$$

In our application we have  $A = 1 + \rho^{-1}$  and  $B = -\rho^{-1}$ , hence,  $\alpha = \rho^{-1}$  and  $\beta = 1$  (or the reverse). Therefore, defining  $R_k = f(r_k) := (1 + \rho^{-1}r_k)/(1 + r_k)$ , we have simply  $R_k = R_1^k$ , which is equivalent to (5.22). Further information on associative rational functions in two variables can be found in [8].  $\square$

The final step in proving Theorem 5.1 is to show that, for suitable  $q$ , we can choose  $r_1 \geq \rho^2$  and have  $r_k \leq \rho$  for  $1 \leq k \leq \Lambda - 1$ . Whenever this is the case, the radii  $r_1, r_2, \dots, r_{\Lambda-1}$  defined by (5.21), (5.22) will satisfy the conditions of Proposition 5.2, and hence the set of nested point + disc regions  $S_i$  will satisfy the conditions of Theorem 4.2, thereby certifying that  $Z_G(q, \mathbf{v}) \neq 0$  whenever  $G$  is a series-parallel graph of maxmaxflow at most  $\Lambda$  and  $v_e/(q + v_e) \in S_1$  for all edges  $e$ .

The simplest choice is to take  $r_1 = \rho^2$  exactly; then from (5.22) we have

$$(5.33) \quad r_k = \rho \frac{X^k - 1}{1 - \rho X^k}, \quad \text{where} \quad X = \frac{1 + \rho}{1 + \rho^2}.$$

When this choice works (i.e., satisfies  $r_k \leq \rho$  for  $1 \leq k \leq \Lambda - 1$ ), it yields the *minimal* regions  $S_i$  of the form (5.9) that satisfy the conditions of Proposition 5.2. However, a slightly better choice is to take  $r_{\Lambda-1} = \rho$  exactly; simple algebra using (5.22) then shows that

$$(5.34) \quad r_k = \rho \frac{X^k - 1}{1 - \rho X^k}, \quad \text{where} \quad X = \left( \frac{2}{1 + \rho} \right)^{1/(\Lambda-1)}.$$

When this choice works (i.e., satisfies  $\rho^2 \leq r_k \leq \rho$  for  $1 \leq k \leq \Lambda - 1$ ), it yields the *maximal* regions  $S_i$  of the form (5.9) that satisfy the conditions of Proposition 5.2, and, hence, the largest allowed set  $S_1$  of edge weights.<sup>20</sup> The following lemma shows that these two choices work in precisely the same set of circumstances, namely, when  $\rho \leq \rho_\Lambda^*$ , where  $\rho_\Lambda^*$  is defined by (5.1), (5.35). In the borderline case  $\rho = \rho_\Lambda^*$  both choices yield the same sequence, which satisfies *both*  $r_1 = \rho^2$  and  $r_{\Lambda-1} = \rho$ . But when  $\rho < \rho_\Lambda^*$  we get different sequences, and we prefer to use the second choice because it yields a larger region  $S_1$ .

LEMMA 5.6. *For  $\rho \in (0, 1)$  and integer  $\Lambda \geq 2$ , the following are equivalent:*

- (a) *There exist real numbers  $r_1, \dots, r_{\Lambda-1}$  satisfying (5.21) and  $\rho^2 \leq r_1 \leq \dots \leq r_{\Lambda-1} \leq \rho$ .*

<sup>20</sup>Of course, for people who care only about the chromatic polynomial, these two choices are equally good. They differ only in the allowed set of edge weights  $v_e \neq -1$ .

- (b) The sequence defined by (5.33) satisfies  $\rho^2 \leq r_k \leq \rho$  for  $1 \leq k \leq \Lambda - 1$ .
- (c) The sequence defined by (5.34) satisfies  $\rho^2 \leq r_k \leq \rho$  for  $1 \leq k \leq \Lambda - 1$ .
- (d)  $(1 + \rho)^\Lambda \leq 2(1 + \rho^2)^{\Lambda-1}$ .
- (e)  $\rho \leq \rho_\Lambda^*$ , where  $\rho_\Lambda^*$  is the unique solution of

$$(5.35) \quad (1 + \rho)^\Lambda = 2(1 + \rho^2)^{\Lambda-1}$$

in the interval  $(0, 1)$  when  $\Lambda \geq 3$ , and  $\rho_2^* = 1$ .

Let us remark that (5.35) has  $\rho = 1$  as a root, so that after division by  $\rho - 1$  it reduces to a polynomial equation of degree  $2\Lambda - 3$ .

*Proof of Lemma 5.6.* Fix  $\rho \in (0, 1)$  and  $r_1 > 0$  and define a sequence  $r_1, r_2, \dots, r_{\Lambda-1}$  by (5.21), (5.22) or, equivalently, fix  $\rho \in (0, 1)$  and  $X > 1$  and define  $r_1, r_2, \dots, r_{\Lambda-1}$  by (5.30). It is then easy to see that we have  $r_1 \leq r_2 \leq \dots \leq r_{\Lambda-1} < \infty$  if and only if  $X < \rho^{-1/(\Lambda-1)}$  [so that the denominator in the expression (5.30) for  $r_k$  is positive for all  $k \leq \Lambda - 1$ ] and each  $r_k$  is an increasing function of  $X$  (for fixed  $\rho$ ) in the region  $1 < X < \rho^{-1/(\Lambda-1)}$ . If we furthermore want to have  $r_1 \geq \rho^2$  and  $r_{\Lambda-1} \leq \rho$ , then we must have

$$(5.36) \quad \frac{1 + \rho}{1 + \rho^2} \leq X \leq \left( \frac{2}{1 + \rho} \right)^{1/(\Lambda-1)}$$

(note that  $[2/(1 + \rho)]^{1/(\Lambda-1)} < \rho^{-1/(\Lambda-1)}$ ), and by the just observed monotonicity in  $X$ , this condition is necessary and sufficient. This proves the equivalence of (a), (b), and (c). Moreover, there exists such an  $X$  if and only if

$$(5.37) \quad \frac{1 + \rho}{1 + \rho^2} \leq \left( \frac{2}{1 + \rho} \right)^{1/(\Lambda-1)},$$

which is equivalent to (d). So (a)–(d) are all equivalent.

Finally we shall prove the equivalence of (d) and (e). We do this in slightly greater generality than is claimed, namely, for all real  $\Lambda \geq 2$ . Consider the function

$$(5.38) \quad f_\Lambda(\rho) = \Lambda \log(1 + \rho) - (\Lambda - 1) \log(1 + \rho^2).$$

Clearly (d) holds if and only if  $f_\Lambda(\rho) \leq \log 2$ . Now the first two derivatives of  $f_\Lambda(\rho)$  are

$$(5.39a) \quad f'_\Lambda(\rho) = \frac{\Lambda}{1 + \rho} - \frac{2(\Lambda - 1)\rho}{1 + \rho^2},$$

$$(5.39b) \quad f''_\Lambda(\rho) = -\frac{4(1 + \rho + \rho^2 - \rho^3) + (\Lambda - 2)(3 + 4\rho + 2\rho^2 - 4\rho^3 - \rho^4)}{(1 + \rho)^2(1 + \rho^2)^2}.$$

For  $0 \leq \rho \leq 1$  we manifestly have  $1 + \rho + \rho^2 - \rho^3 \geq 1 + \rho \geq 1$  and  $3 + 4\rho + 2\rho^2 - 4\rho^3 - \rho^4 \geq 3 + \rho^2 \geq 3$ , so that  $f_\Lambda$  is strictly concave on  $[0, 1]$  whenever  $\Lambda \geq 2$ . We have  $f_\Lambda(0) = 0$ ,  $f'_\Lambda(0) = \Lambda > 0$ ,  $f_\Lambda(1) = \log 2$ , and  $f'_\Lambda(1) = -(\Lambda - 2)/2$ . Therefore, for  $\Lambda > 2$ , there is a unique  $\rho_\Lambda^* \in (0, 1)$  satisfying  $f_\Lambda(\rho_\Lambda^*) = \log 2$ ; and for  $\rho \in [0, 1]$  we have  $f_\Lambda(\rho) \leq \log 2$  if and only if  $\rho \leq \rho_\Lambda^*$ . This proves the equivalence of (d) and (e) for all real  $\Lambda > 2$ . When  $\Lambda = 2$ , (d) holds for all  $\rho \in [0, 1]$ , so (d) is again equivalent to (e) with  $\rho_2^* = 1$ .  $\square$

We have now completed the proof of the main part of Theorem 5.1. All that remains is to prove the final statement that  $\rho_\Lambda^* > (\log 2)/(\Lambda - \frac{3}{2} \log 2)$  for all integers

$\Lambda \geq 2$  or, equivalently (in view of Lemma 5.6d,e), that  $(1 + \rho)^\Lambda < 2(1 + \rho^2)^{\Lambda-1}$  when  $\rho = (\log 2)/(\Lambda - \frac{3}{2} \log 2)$ . We shall actually prove this for all *real*  $\Lambda > \frac{5}{2} \log 2 \approx 1.732868$  (this ensures that  $\rho < 1$ ). Taking the logarithm of  $2(1 + \rho^2)^{\Lambda-1}/(1 + \rho)^\Lambda$ , substituting for  $\Lambda$  in terms of  $\rho$ , and parametrizing by  $\rho \in (0, 1)$ , we see that this is equivalent to the following claim.

LEMMA 5.7. *The function*

$$g(\rho) = \rho \left[ \log 2 + \left( \frac{\log 2}{\rho} + \frac{3}{2} \log 2 - 1 \right) \log(1 + \rho^2) - \left( \frac{\log 2}{\rho} + \frac{3}{2} \log 2 \right) \log(1 + \rho) \right] \tag{5.40a}$$

$$= \rho \log \left( \frac{2}{1 + \rho} \right) - (\log 2) \log \left( \frac{1 + \rho}{1 + \rho^2} \right) - \left( \frac{3}{2} \log 2 - 1 \right) \rho \log \left( \frac{1 + \rho}{1 + \rho^2} \right) \tag{5.40b}$$

is strictly positive for  $0 < \rho < 1$ .

*Proof.* The second derivative of  $g$  is given by  $g''(\rho) = h(\rho) \times \rho / [(1 + \rho)^2(1 + \rho^2)^2]$ , where

$$h(\rho) = -(2 - \frac{3}{2} \log 2)\rho^4 - (4 - 2 \log 2)\rho^3 - (8 - 5 \log 2)\rho^2 - (12 - 14 \log 2)\rho + (\frac{23}{2} \log 2 - 6). \tag{5.41}$$

All the coefficients of  $h(\rho)$  are strictly negative except for the last (constant) term, so we have  $h'(\rho) < 0$  for all  $\rho \geq 0$ . Since  $h(0) = \frac{23}{2} \log 2 - 6 > 0$  and  $h(1) = 34 \log 2 - 32 < 0$  and  $h$  is strictly decreasing for  $\rho \geq 0$ , it follows that  $h(\rho)$  has exactly one positive real root  $\rho^*$  and that it lies between 0 and 1 (by computer  $\rho^* \approx 0.417876$ ). Therefore  $g$  is strictly convex on  $[0, \rho^*]$  and strictly concave on  $[\rho^*, \infty)$ . Since  $g(0) = g'(0) = 0$ , we have  $g(\rho) > 0$  for  $\rho \in (0, \rho^*]$ . Moreover, since  $g(\rho^*) > 0$  and  $g(1) = 0$  and  $g$  is strictly concave on  $[\rho^*, 1]$ , we have  $g(\rho) > 0$  for  $\rho \in [\rho^*, 1)$ . Hence  $g(\rho) > 0$  for all  $\rho \in (0, 1)$ , as claimed.  $\square$

*Remark.* A straightforward calculation shows that the large- $\Lambda$  asymptotic behavior of  $\rho_\Lambda^*$  is given by

$$\rho_\Lambda^* = (\log 2) \left[ \frac{1}{\Lambda - 1} + \frac{3 \log 2 - 2}{2(\Lambda - 1)^2} + \frac{25 \log^2 2 - 24 \log 2 + 6}{6(\Lambda - 1)^3} + \dots \right] \tag{5.42}$$

and hence

$$\frac{1}{\rho_\Lambda^*} = \frac{\Lambda - 1}{\log 2} - \frac{3 \log 2 - 2}{2 \log 2} - \frac{23 \log 2 - 12}{12(\Lambda - 1)} + \dots \tag{5.43}$$

So the inequality  $\rho_\Lambda^* > (\log 2)/(\Lambda - \frac{3}{2} \log 2)$  captures the first two terms of the large- $\Lambda$  asymptotic behavior.  $\square$

We have now completed the proof of Theorem 5.1.

**6. The case  $\Lambda = 3$ .** Theorem 1.4 is a strong result because it provides a linear bound for the chromatic roots of series-parallel graphs in terms of the maxmaxflow  $\Lambda$ , thereby achieving our main objective. Furthermore, the constant  $1/\log 2$  cannot be reduced below 1 (see [26, Appendix B]) and so it is reasonably close to optimal. However, the result applies uniformly for all  $\Lambda$ , its proof involves a number of steps where expressions are replaced by fairly naive upper bounds, and it only involves the *magnitude* of  $q - 1$ . Thus, for all these reasons, Theorem 1.4 does not give a very precise picture of the root-free region for any particular value of  $\Lambda$ .

In this section we consider how to get sharper results for the simplest nontrivial case, namely, for  $\Lambda = 3$ . In this case, the bound given by Theorem 1.4 is that chromatic roots for series-parallel graphs of maxmaxflow 3 are contained in the disk

$$(6.1) \quad |q - 1| \leq 2/(\log 2) \approx 2.8853900818.$$

An immediate improvement can be obtained from Theorem 5.1 by using the exact value of  $\rho_3^*$ , which gives the slightly better bound

$$(6.2) \quad |q - 1| \leq 1/\rho_3^* \approx 2.6589670819.$$

Both of these regions ultimately relied on the quantity  $F_q$  given by (5.13) as an upper bound for the true value  $f_q$ . We can do better by computing a numerical approximation to the *actual* value  $f_q(\rho^2, \rho^2)$ , and then imposing the condition  $f_q(\rho^2, \rho^2) \leq \rho$  that arises out of Proposition 5.2 with  $\Lambda = 3$ . Since  $f_q(\rho^2, \rho^2)$  depends on  $q$  and not just on  $|q-1|$ , this procedure will lead to a region with no simple analytic description. As  $t_e \parallel_q t_f$  is given by a ratio of symmetric multiaffine polynomials in  $t_e$  and  $t_f$  [cf. (5.11)] and  $D(\rho^2)$  is a circular region, the Grace–Walsh–Szegő coincidence theorem [22, Theorem 3.4.1b] implies that

$$(6.3) \quad \max_{t_e, t_f \in D(\rho^2)} |t_e \parallel_q t_f| = \max_{t \in D(\rho^2)} |t \parallel_q t|,$$

and so we can compute an approximation to  $f_q(\rho^2, \rho^2)$  by letting  $t$  range over the (discretized) boundary of  $D(\rho^2)$  and taking the maximum value of  $|t \parallel_q t|$  thus obtained. Then for each fixed angle  $\theta$  we can set  $1/(1-q) = \rho e^{i\theta}$  and use the bisection method to determine the maximum possible value of  $\rho$ . Figure 3 shows how this bound (shown as a green solid curve) compares with the circular regions (6.1) and (6.2). This bound is the optimal bound obtainable from Theorem 4.2 under the assumption that  $S_1$  is chosen to be a point + disk region  $S_1 = \{1/(1-q)\} \cup D(\rho^2)$ .

At this point it is natural to inquire: What is the best possible result? Otherwise put, can we describe exactly the closure of the set of *all* chromatic roots of *all* series-parallel graphs of maxmaxflow  $\Lambda = 3$  or, at least, the outer boundary of this set? In [25] one of the authors gave a computer approximation to this boundary, but he suspects that this approximation may become poor near the real axis, in part because this boundary is likely to be fractal-like rather than smooth.

As previously mentioned, we show in [26, Appendix B] that for each fixed  $r \geq 2$ , every point of the circle  $|q-1| = r$  is a limit point (as  $n \rightarrow \infty$ ) of chromatic roots of the family  $\{G_n^r\}_{n \geq 1}$  of leaf-joined trees of branching factor  $r$ , which have maxmaxflow  $\Lambda = r+1$ . Moreover, numerical calculations suggest (though we have no proof) that the chromatic roots of these leaf-joined trees always lie *inside* the circle  $|q-1| = r$ . This led us to conjecture that the exact answer to our question is the chromatic roots of series-parallel graphs of maxmaxflow  $\Lambda$  always lie inside the disc  $|q-1| < \Lambda-1$ , and this bound is sharp.

That would be neat, but it is *false!* In fact, a counterexample can be found by a simple modification of a leaf-joined tree. Let us first recall [32, Example 2.2] the multivariate Tutte polynomial of a cycle  $C$ :

$$(6.4) \quad Z_C(q, \mathbf{v}) = \prod_{e \in E(C)} (q + v_e) + (q-1) \prod_{e \in E(C)} v_e.$$

In particular, if we consider a cycle of  $N+1$  edges, where  $N$  edges carry weight  $v$  and the last edge carries weight  $-1$ , we have

$$(6.5) \quad Z_C(q, \mathbf{v}) = (q-1)[(q+v)^N - v^N],$$



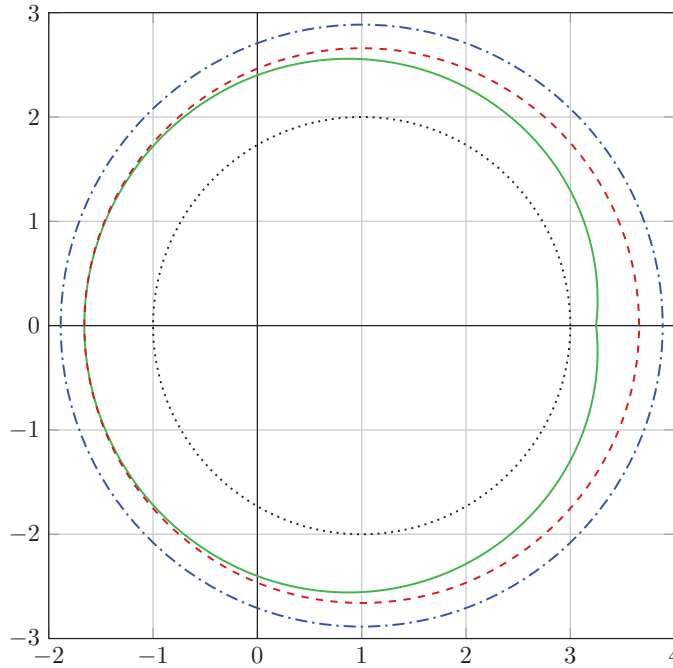


FIG. 3. Different bounds on the chromatic roots for  $\Lambda = 3$ : the bound based on  $f_q$  (green solid curve), the bound  $|q - 1| = 2.6589670819$  based on  $F_q$  (red dashed circle), and the bound  $|q - 1| = 2/\log 2 \approx 2.8853900818$  from Theorem 1.4 (blue dot-dashed outer circle). The inner circle  $|q - 1| = 2$  is also shown for reference (dotted gray).

which vanishes whenever  $t = v/(q + v)$  is an  $N$ th root of unity. It follows that if we consider any 2-terminal graph  $G = (G, s, t)$  and form the graph  $H$  consisting of  $N$  copies of  $G$  together with one  $K_2$  connected in a cycle, then  $H$  has a chromatic root whenever the effective transmissivity  $t_{\text{eff}}(G, s, t)$  is an  $N$ th root of unity.

We show in [26, Appendix B] that it is easy to compute the effective transmissivity for leaf-joined trees symbolically, as a function of  $q$ . We can then plot the curve in the complex  $q$ -plane, where  $|t_{\text{eff}}(G_n^r, s, t)| = 1$ . For  $r = 2$ , we find that this curve stays within the disc  $|q - 1| < 2$  when  $n \leq 4$ , but that it strays slightly outside this disc when  $n = 5$ . (On the circle  $q - 1 = 2e^{i\theta}$ ,  $|t_{\text{eff}}(G_5^2, s, t)|$  reaches a maximum value  $\approx 1.08448$  at  $\theta \approx \pm 0.679954\pi$ , corresponding to  $q \approx -0.071413 \pm 1.68881i$ .) If we now consider the graph  $H$  consisting of  $N = 3$  copies of  $G_5^2$  together with one  $K_2$  connected in a cycle—note that  $H$  has maxmaxflow 3 and has 94 vertices—we see that  $H$  has a chromatic root whenever  $t_{\text{eff}}(G_5^2, s, t)$  is a cube root of unity. Solving  $t_{\text{eff}}(G_5^2, s, t) = e^{\pm 2\pi i/3}$  for  $q$ , we find 31 roots, of which one ( $q \approx -0.144883 \mp 1.651418i$ ) has  $|q - 1| \approx 2.009462 > 2$ .

**7. The real antiferromagnetic regime.** The chromatic polynomial corresponds to the special case of the multivariate Tutte polynomial in which all the edge weights  $v_e$  take the value  $-1$ . However, it is often the case that results valid for this limiting case also hold throughout the “real antiferromagnetic regime” where edge weights  $v_e \in [-1, 0]$  are chosen independently for each edge. Expressed in transmissivities, we get  $t_e \in C_q$ , where  $C_q$  is the curve defined parametrically by

$$(7.1) \quad C_q = \left\{ \frac{v}{q + v} : v \in [-1, 0] \right\} .$$

In the complex  $t$ -plane,  $C_q$  traces out a circular arc that runs from the origin (when  $v = 0$ ) to the point  $1/(1 - q)$  [when  $v = -1$ ]. In this section we will show how we can handle this case by a minor modification of the argument given in section 5, thereby proving Theorem 1.5. In fact, we shall prove the following slight strengthening of Theorem 1.5, which is identical to Theorem 5.1 except that  $v_e = -1$  is replaced by  $-1 \leq v_e \leq 0$ .

**THEOREM 7.1.** *Fix an integer  $\Lambda \geq 2$ , and let  $G$  be a loopless series-parallel graph of maxflow at most  $\Lambda$ . Let  $\rho_\Lambda^*$  be the unique solution of*

$$(7.2) \quad (1 + \rho)^\Lambda = 2(1 + \rho^2)^{\Lambda-1}$$

*in the interval  $(0, 1)$  when  $\Lambda \geq 3$ , and let  $\rho_2^* = 1$ . Then the multivariate Tutte polynomial  $Z_G(q, \mathbf{v})$  is nonvanishing whenever  $|q - 1| \geq 1/\rho_\Lambda^*$  (with  $\geq$  replaced by  $>$  when  $\Lambda = 2$ ) and the complex edge weights  $\mathbf{v} = \{v_e\}_{e \in E}$  satisfy*

$$(7.3) \quad -1 \leq v_e \leq 0 \quad \text{or} \quad \left| \frac{v_e}{q + v_e} \right| \leq \rho \frac{X - 1}{1 - \rho X}$$

*(with strict inequality in the second expression when  $\Lambda = 2$ ), where*

$$(7.4) \quad \rho = \frac{1}{|q - 1|} \quad \text{and} \quad X = \left( \frac{2}{1 + \rho} \right)^{1/(\Lambda-1)}.$$

The first step in the proof of Theorem 7.1 is the following simple lemma, which shows how to combine a pair of families  $C_1 \subseteq C_2 \subseteq \cdots \subseteq C_{\Lambda-1}$  and  $D_1 \subseteq D_2 \subseteq \cdots \subseteq D_{\Lambda-1}$ , each of which satisfies the “parallel condition” (2) of Theorem 4.2, into a single family that also satisfies the “parallel condition.”

**LEMMA 7.2.** *Let  $C_1 \subseteq C_2 \subseteq \cdots \subseteq C_{\Lambda-1}$  and  $D_1 \subseteq D_2 \subseteq \cdots \subseteq D_{\Lambda-1}$  be subsets of the complex  $t$ -plane satisfying*

$$(7.5a) \quad C_k \parallel_q C_\ell \subseteq C_{k+\ell} \quad \text{whenever } k + \ell \leq \Lambda - 1,$$

$$(7.5b) \quad D_k \parallel_q D_\ell \subseteq D_{k+\ell} \quad \text{whenever } k + \ell \leq \Lambda - 1.$$

*Now define the sets  $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_{\Lambda-1}$  by*

$$(7.6) \quad S_k = \bigcup_{i=0}^k (C_i \parallel_q D_{k-i})$$

*with  $C_0 = D_0 = \{0\}$ . Then*

$$(7.7) \quad S_k \parallel_q S_\ell \subseteq S_{k+\ell} \quad \text{whenever } k + \ell \leq \Lambda - 1.$$

*Proof.* If  $k + l \leq \Lambda - 1$ , we have

$$\begin{aligned}
 S_k \parallel_q S_\ell &= \bigcup_{i=0}^k \bigcup_{j=0}^\ell (C_i \parallel_q D_{k-i}) \parallel_q (C_j \parallel_q D_{\ell-j}) \\
 &= \bigcup_{i=0}^k \bigcup_{j=0}^\ell (C_i \parallel_q C_j) \parallel_q (D_{k-i} \parallel_q D_{\ell-j}) \\
 &\subseteq \bigcup_{i=0}^k \bigcup_{j=0}^\ell (C_{i+j} \parallel_q D_{k+\ell-i-j}) \\
 &= \bigcup_{i=0}^{k+\ell} (C_i \parallel_q D_{k+\ell-i}) \\
 (7.8) \qquad &= S_{k+\ell} . \quad \square
 \end{aligned}$$

In section 5 we treated the chromatic-polynomial case by taking  $C_1 = \dots = C_{\Lambda-1} = \{t_0\}$ , where  $t_0 = 1/(1 - q)$ , and  $D_i = D(r_i)$ . The fact that  $t_0 \parallel_q t = t_0$  for all  $t$ —which is very special to chromatic polynomials—then ensures that all the terms  $1 \leq i \leq k$  in (7.6) equal  $\{t_0\}$ , while the term  $i = 0$  equals  $D(r_k)$ . So we indeed have  $S_k = \{t_0\} \cup D(r_k)$  as stated in Proposition 5.2, and the proof of the parallel condition (2) given as part of the proof of Proposition 5.2 is a special case of Lemma 7.2.

To treat the real antiferromagnetic regime, we will take  $C_1 = \dots = C_{\Lambda-1} = C_q$  and  $D_i = D(r_i)$  with  $r_1 \leq r_2 \leq \dots \leq r_{\Lambda-1}$ . The invariance of the real antiferromagnetic regime under parallel connection (which is most easily seen in the  $v$ -plane or  $y$ -plane) then guarantees that  $C_k \parallel_q C_\ell \subseteq C_{k+\ell}$ . It follows that

$$(7.9) \qquad S_k = (C_q \parallel_q D(r_{k-1})) \cup D(r_k),$$

where we have set  $r_0 = 0$  and hence  $D(r_0) = \{0\}$ . The sets  $S_k$  are no longer point + disc, but rather stalk + disc: for  $S_1$  the “stalk” is precisely the curve  $C_q$ , while for higher  $S_k$  the “stalk” gets increasingly “fattened out” by parallel connection with  $D(r_{k-1})$ . Figure 4 illustrates this situation for  $\Lambda = 3$  and  $q = -2 + 3i$ : the “stalk” for  $S_2$  is the cone-shaped region  $C_q \parallel_q D(r_1)$  that runs from  $D(r_1)$  to the point  $t_0 = 1/(1 - q)$ .

We can choose the radii  $\rho^2 = r_1 \leq r_2 \leq \dots \leq r_{\Lambda-1} \leq \rho$  exactly as in section 5, and this guarantees that  $D_k \parallel_q D_\ell \subseteq D_{k+\ell}$ .

To complete the proof of Theorem 7.1, it therefore suffices to verify the “series condition” (1) of Theorem 4.2. We shall do this, once again, by using Lemma 4.3 with  $r = \rho$ . Since  $\rho^2 = r_1 \leq r_2 \leq \dots \leq r_{\Lambda-1} \leq \rho$ , it suffices to verify that

$$(7.10) \qquad C_q \parallel_q D(r_{\Lambda-2}) \subseteq D(\rho) .$$

We shall prove the stronger statement that

$$(7.11) \qquad C_q \parallel_q D(\rho) \subseteq D(\rho) .$$

Indeed, we shall prove also a strong converse to this statement, although we shall not make use of this converse.

LEMMA 7.3. *Let  $q$  be a fixed complex number such that  $|q-1| > 1$ , let  $\rho = 1/|1-q|$ , and let  $C_q$  be defined by (7.1). Then a complex number  $t$  satisfies  $\{t\} \parallel_q D(\rho) \subseteq D(\rho)$*

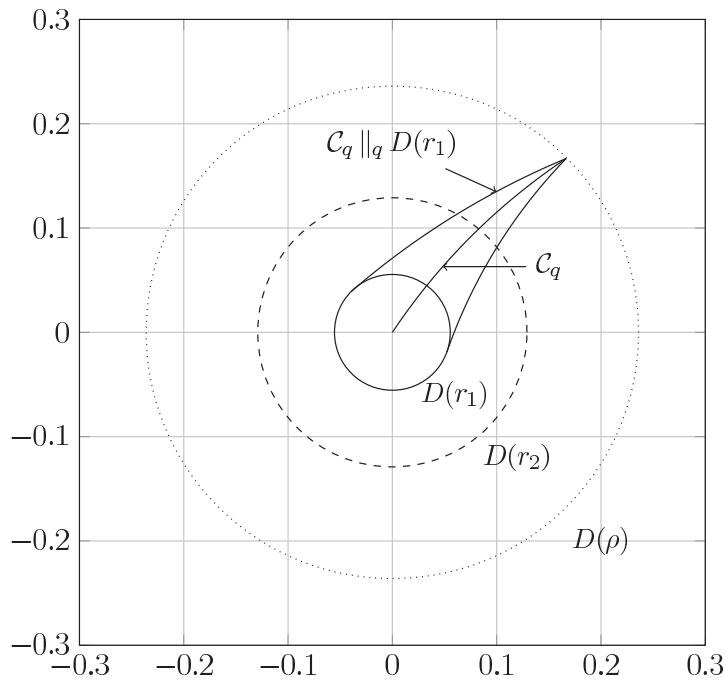


FIG. 4. The boundaries of the discs and stalks when  $q = -2 + 3i$ .

if and only if  $t \in \mathcal{C}_q$ . [Otherwise put, we have  $\mathcal{C}_q \parallel_q D(\rho) \subseteq D(\rho)$ , and  $\mathcal{C}_q$  is the largest set with this property.]

*Proof.* It is easiest to change variables once again and consider the situation in the complex  $y$ -plane, where parallel connection is simply multiplication,  $y_e \parallel^Y y_f = y_e y_f$ , and the curve  $\mathcal{C}_q$  corresponds to the segment  $[0, 1]$ . The relationship between  $y$  and  $t$  is given by the Möbius transformation

$$(7.12) \quad y = \frac{(q-1)t+1}{-t+1}.$$

Since  $D(\rho)$  is a closed disc in the complex  $t$ -plane having the point  $t_0 = 1/(1-q)$  on its boundary, the image of  $D(\rho)$  in the complex  $y$ -plane is a closed disc  $D^Y(\rho)$  having the origin  $y_0 = 0$  on its boundary.<sup>21</sup> (Here we have used  $|q-1| > 1$ . If  $|q-1| = 1$ , then  $D^Y(\rho)$  is a closed half-plane having the origin on its boundary, while if  $|q-1| < 1$ , then  $D^Y(\rho)$  is the closed exterior of a disc having the origin on its boundary.) Now, for any closed disc  $D$  having the origin on its boundary, it is easy to see that a complex number  $y$  satisfies  $yD \subseteq D$  if and only if  $y \in [0, 1]$ . Back in the  $t$ -plane, this says that  $t \in \mathcal{C}_q$ .  $\square$

Combining these results, we have the following.

*Proof of Theorem 7.1.* We choose the radii  $\rho^2 = r_1 \leq r_2 \leq \dots \leq r_{\Lambda-1} \leq \rho$  exactly as in section 5, which guarantees that  $D(r_k) \parallel_q D(r_\ell) \subseteq D(r_{k+\ell})$ . Then Lemmas 7.2, 7.3, and 4.3 guarantee that the sets  $S_k$  defined by (7.9) satisfy the conditions (1) and (2) of Theorem 4.2. Conditions (3') and (4) of Theorem 4.2 are verified exactly as in the proof of Proposition 5.2.  $\square$

<sup>21</sup>The center of the disc  $D^Y(\rho)$  is the point  $c = [(q-1)\rho^2 + 1]/(1-\rho^2)$ —and the radius is of course  $|c|$ —but we do not actually need this explicit formula for  $c$ .

The fact that Lemma 7.3 gives a *necessary and sufficient* condition suggests that there is something natural about the real antiferromagnetic regime  $v_e \in [-1, 0]$ . On the other hand, our strategy of proof does not really require us to prove the strong statement (7.11); it would suffice to prove the slightly weaker statement (7.10) [or perhaps even weaker bounds], and this might allow a somewhat larger set of weights  $v_e$ . In particular, the proof of Theorem 1.2 given in [30] works naturally for the complex antiferromagnetic regime  $|1 + v_e| \leq 1$  (see [17] for the changes when one goes beyond this), so it is reasonable to ask whether the bounds in terms of maxmaxflow can be extended to this case, possibly with a worse constant (but still growing only linearly in  $\Lambda$ ). We do not yet know the answer. As a warm-up, it might be helpful to study the intersection of the complex antiferromagnetic regime with the real axis, namely, the “extended real antiferromagnetic regime”  $v_e \in [-2, 0]$ .

One somewhat unsatisfying aspect of Theorem 7.1 is that one obtains the same bound on the zero-free region (namely,  $|q - 1| \geq 1/\rho_\Lambda^*$ ) no matter where in the regime  $-1 \leq v_e \leq 0$  the edge weights  $\mathbf{v} = \{v_e\}_{e \in E}$  lie. But we know, for any fixed graph  $G$ , that the chromatic roots shrink towards  $q = 0$  as  $\mathbf{v} \rightarrow 0$  [31, Lemma A.1], and it is natural to expect that this behavior occurs uniformly for all graphs (or at least all series-parallel graphs) of a given maxmaxflow  $\Lambda$ . So it would be desirable to find an improved version of Theorem 7.1 in which the bound would shrink (hopefully linearly) towards  $q = 0$  as  $\mathbf{v} \rightarrow 0$ . Indeed, for Theorem 1.2 there exists a variant bound [30, Corollary 5.5] of precisely this kind: it involves the *weighted maximum degree*

$$(7.13) \quad \Delta(G, \mathbf{v}) = \max_{x \in V} \sum_{e \ni x} |v_e|.$$

In a similar way one might hope that for Theorem 7.1 there exists a variant bound involving the *weighted maxmaxflow* in which  $|v_e|$  is interpreted as the capacity of the edge  $e$ , i.e.,

$$(7.14) \quad \Lambda(G, \mathbf{v}) = \max_{x \neq y} \lambda_G(x, y; \mathbf{v}),$$

where

$$(7.15a) \quad \lambda_G(x, y; \mathbf{v}) = \max \text{ flow from } x \text{ to } y \text{ with edge capacities } |\mathbf{v}|$$

$$(7.15b) \quad = \min \text{ cut between } x \text{ and } y \text{ with edge capacities } |\mathbf{v}|.$$

One approach to proving such a bound might be to replace the discrete sequence of sets  $S_1 \subseteq S_2 \subseteq \dots \subseteq S_{\Lambda-1}$  of Theorem 4.1 by a continuous nested family  $\{S_\lambda\}_{0 \leq \lambda \leq \Lambda}$ , but we have not yet succeeded in making this work. We therefore leave it as an open problem to find a suitable  $\mathbf{v}$ -dependent variant of Theorem 7.1.

**8. Generalization to non-series-parallel graphs.** In this section we show how our constructions can be generalized to handle graphs that are not series-parallel but are nevertheless built up by using series and parallel compositions from a fixed starting set of 2-terminal “base graphs.” We begin by stating an abstract theorem on excluding roots, which generalizes Theorems 4.1 and 4.2 to the non-series-parallel case (section 8.1). Then we apply this result to prove Theorem 1.6 (section 8.2).

**8.1. Generalized abstract theorem on excluding roots.** Here is Theorem 4.1 generalized to the non-series-parallel case.

**THEOREM 8.1.** *Let  $q \neq 0$  be a fixed complex number and let  $\Lambda \geq 2$  be a fixed integer. Let  $S_1 \subseteq S_2 \subseteq \dots \subseteq S_{\Lambda-1}$  be sets in the (finite) complex  $v$ -plane such that*

- (1)  $S_k \not\bowtie_q^V S_\ell \subseteq S_{\min(k,\ell)}$  for all  $k, \ell$ ,
- (2)  $S_k \parallel_q^V S_\ell \subseteq S_{k+\ell}$  for  $k + \ell \leq \Lambda - 1$ .

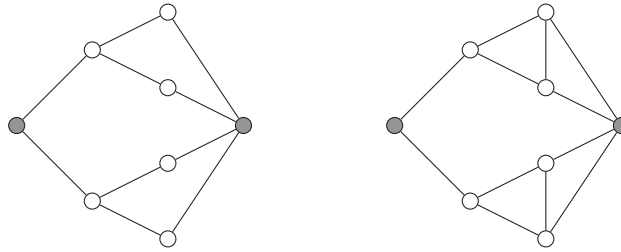


FIG. 5. The graphs  $(K_2 \boxtimes D) \parallel (K_2 \boxtimes D)$  and  $(K_2 \boxtimes W) \parallel (K_2 \boxtimes W)$ .

Now consider any (loopless connected) 2-terminal graph  $(G, s, t)$  and any nontrivial decomposition tree for  $(G, s, t)$  in which all the proper constituents have between-terminals flow at most  $\Lambda - 1$ . Suppose that we equip  $G$  with edge weights  $\{v_e\}$  such that for every leaf node  $(H, a, b)$  of the decomposition tree, we have  $v_{\text{eff}}(H, a, b) \in S_{\lambda_H(a,b)}$ . Then, for every node  $(H, a, b)$  of the decomposition tree that has between-terminals flow  $\lambda_H(a, b) \leq \Lambda - 1$ , we have  $v_{\text{eff}}(H, a, b) \in S_{\lambda_H(a,b)}$ .

Now assume further that, in addition to (1) and (2), the following hypotheses hold:

$$(3) \quad -q \notin S_{\Lambda-1},$$

$$(4) \quad -q \notin S_k \parallel^V S_\ell \text{ for } k + \ell = \Lambda.$$

Then, for any  $(G, s, t)$  and  $\{v_e\}$  as above, such that  $G$  has maxmaxflow at most  $\Lambda$ , we have  $Z_G(q, \mathbf{v}) \neq 0$ .

The proof of Theorem 8.1 is a minor modification of that of Theorem 4.1 and is left to the reader. There is also an obvious translation of Theorem 8.1 to the  $t$ -plane along the lines of Theorem 4.2, of which the statement and proof are again left to the reader.

**8.2. Wheatstone bridge.** The *Wheatstone bridge* is the 2-terminal graph  $W = (W, s, t)$  obtained from  $W = K_4 - e$  by taking the two vertices of degree 2 to be the terminals  $s$  and  $t$ . Note that although  $K_4 - e$  is a series-parallel graph,  $W$  is not a 2-terminal series-parallel graph (by Corollary 3.4, because  $W + st = K_4$  is not a series-parallel graph).

Now define the class  $\mathcal{W}$  of 2-terminal graphs to be the smallest class that contains both  $K_2$  (with the two vertices as terminals) and  $W$  and is closed under series and parallel composition. Figure 5 shows some graphs in  $\mathcal{W}$ : the first is a 2-terminal series-parallel graph, while the second has used  $W = (W, s, t)$  in place of the “diamond”  $D = (K_2 \boxtimes K_2) \parallel (K_2 \boxtimes K_2)$ .

For the Wheatstone bridge, simple calculations [e.g., using (2.24), (2.29)] show that if  $v_f = -1$  for every edge, then the partial Tutte polynomials (2.25), (2.26) are given by

$$(8.1a) \quad A_{W,s,t} = (q-2)(q-3),$$

$$(8.1b) \quad B_{W,s,t} = 2(q-2),$$

and, hence,

$$(8.2) \quad v_{\text{eff}}(W, s, t) = \frac{2}{q-3}.$$

Expressed in terms of transmissivities, this yields

$$(8.3) \quad t_{\text{eff}}(W, s, t) \equiv \frac{v_{\text{eff}}}{q + v_{\text{eff}}} = \frac{2}{(q-1)(q-2)}.$$



The maximum flow between the terminals of the Wheatstone bridge is equal to 2. Therefore, as far as the chromatic roots of graphs in  $\mathcal{W}$  are concerned, the Wheatstone bridge just behaves as a sort of “superedge” with capacity (in the flow-carrying sense) equal to 2 and effective transmissivity given by (8.3): that is the upshot of Theorem 8.1.

Now suppose that  $S_1 \subseteq S_2 \subseteq \dots \subseteq S_{\Lambda-1}$  is a set of regions in the complex  $t$ -plane certifying (via Theorem 4.2) that a particular value of  $q$  is not the chromatic root of any series-parallel graph of maxmaxflow at most  $\Lambda$ . Then, by Theorem 8.1, the *same* set of regions will suffice for graphs in  $\mathcal{W}$  of maxmaxflow at most  $\Lambda$ , provided only that  $2/[(q-1)(q-2)] \in S_2$ .

So, under the same hypothesis  $|q-1| \geq 1/\rho_\Lambda^*$  as in Theorem 5.1, let us again choose point + disk regions (5.6) with radii  $r_k$  given by (5.34). We then have

$$(8.4) \quad r_2 = \rho \frac{X^2 - 1}{1 - \rho X^2}, \quad \text{where } \rho = \frac{1}{|q-1|} \text{ and } X = \left(\frac{2}{1+\rho}\right)^{1/(\Lambda-1)}.$$

Therefore, these regions suffice to show that  $q$  is not a chromatic root of any graph in  $\mathcal{W}$  whenever we have

$$(8.5) \quad \frac{2}{|q-2|} \leq \frac{X^2 - 1}{1 - \rho X^2}$$

in addition to the hypothesis  $|q-1| \geq 1/\rho_\Lambda^*$ . We have therefore proven the following.

**THEOREM 8.2.** *Fix an integer  $\Lambda \geq 3$ , and let  $G = (G, s, t)$  be a 2-terminal graph in the class  $\mathcal{W}$  such that  $G$  has maxmaxflow at most  $\Lambda$ . Then the chromatic polynomial  $P_G(q)$  is nonvanishing whenever*

$$(8.6) \quad |q-1| \geq 1/\rho_\Lambda^* \quad \text{and} \quad |q-2| \geq \frac{2(1 - \rho X^2)}{X^2 - 1},$$

where  $\rho_\Lambda^*$  is defined by (5.1) and  $\rho$  and  $X$  are defined by (8.4).

When  $\Lambda = 3$ , the condition (8.5) becomes particularly simple as it reduces to

$$(8.7) \quad |q-2| \geq 2.$$

The disk  $|q-2| < 2$  extends only slightly beyond the disk  $|q-1| < 1/\rho_3^*$  with the greatest protrusion  $4 - (1 + 1/\rho_3^*) \approx 0.34103\dots$  occurring on the positive real axis. Thus, the region guaranteed to contain the chromatic roots of the graphs in  $\mathcal{W}$  of maxmaxflow 3, given by the union of the discs  $|q-1| < 1/\rho_3^*$  and  $|q-2| < 2$  (the left-hand picture of Figure 6), is only slightly larger than the region  $|q-1| < 1/\rho_3^*$  guaranteed to contain the chromatic roots of series-parallel graphs of maxmaxflow 3.

For  $\Lambda > 3$ , the right-hand side of (8.5) is not independent of  $\rho$ , and so the corresponding region is not quite a circular disk (as it depends on the phase of  $q$ ), but rather a slightly squashed disk. Nevertheless, the corresponding region always extends slightly past the region  $|q-1| < 1/\rho_\Lambda^*$ , with maximum protrusion again on the positive real axis (see the right-hand picture of Figure 6 for  $\Lambda = 4$ ).

In order to obtain a simple *sufficient* condition depending only on  $|q-1|$ , we can use the trivial bound  $|q-2| \geq |q-1| - 1 = \rho^{-1} - 1$ . After some simple algebra we find that a *sufficient* condition on  $\rho$  for (8.5) to be satisfied is that

$$(8.8) \quad \left(\frac{2}{1+\rho}\right)^{2/(\Lambda-1)} \geq \frac{1+\rho}{1-\rho+2\rho^2}$$

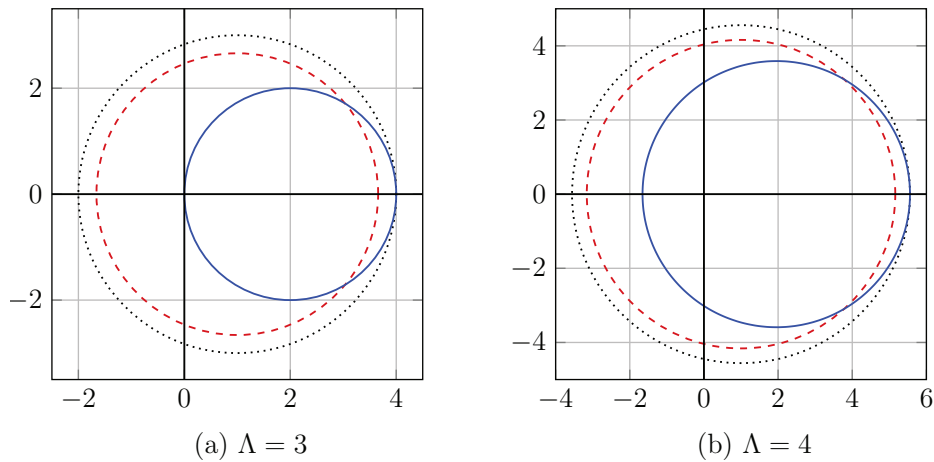


FIG. 6. Circles  $|q-1| = 1/\rho_\Lambda^*$  (red dashed) and  $|q-1| = 1/\rho_\Lambda^{**}$  (black dotted) and the boundary of the region (8.5) (blue solid curve) for (a)  $\Lambda = 3$  and (b)  $\Lambda = 4$ .

TABLE 1  
Values of  $\rho_\Lambda^*$ ,  $\rho_\Lambda^{**}$ ,  $1/\rho_\Lambda^*$ ,  $1/\rho_\Lambda^{**}$  for  $2 \leq \Lambda \leq 10$ .

| $\Lambda$ | $\rho_\Lambda^*$ | $\rho_\Lambda^{**}$ | $1/\rho_\Lambda^*$ | $1/\rho_\Lambda^{**}$ |
|-----------|------------------|---------------------|--------------------|-----------------------|
| 2         | 1                | 1                   | 1                  | 1                     |
| 3         | 0.376086         | 0.333333            | 2.658967           | 3                     |
| 4         | 0.240380         | 0.219471            | 4.160076           | 4.556417              |
| 5         | 0.177591         | 0.165204            | 5.630929           | 6.053134              |
| 6         | 0.141038         | 0.132841            | 7.090297           | 7.527812              |
| 7         | 0.117041         | 0.111213            | 8.544040           | 8.991750              |
| 8         | 0.100054         | 0.095697            | 9.994599           | 10.449611             |
| 9         | 0.087388         | 0.084008            | 11.443181          | 11.903688             |
| 10        | 0.077577         | 0.074877            | 12.890449          | 13.355246             |

or, equivalently, that

$$(8.9) \quad (1 + \rho)^{\Lambda+1} \leq 4(1 - \rho + 2\rho^2)^{\Lambda-1}.$$

This condition is handled by the following analogue of Lemma 5.6.

LEMMA 8.3. For  $\rho \in (0, 1)$  and real  $\Lambda > 2$ , the following are equivalent:

- (a)  $(1 + \rho)^{\Lambda+1} \leq 4(1 - \rho + 2\rho^2)^{\Lambda-1}$ .  
 (b)  $\rho \leq \rho_\Lambda^{**}$ , where  $\rho_\Lambda^{**}$  is the unique solution of

$$(8.10) \quad (1 + \rho)^{\Lambda+1} = 4(1 - \rho + 2\rho^2)^{\Lambda-1}$$

in the interval  $(0, 1)$ .

Deferring temporarily the proof of this lemma, let us observe that from (5.1) it follows easily that

$$(8.11) \quad \left( \frac{2}{1 + \rho_\Lambda^*} \right)^{1/(\Lambda-1)} = \frac{1 + \rho_\Lambda^*}{1 + \rho_\Lambda^{*2}} < \frac{1 + \rho_\Lambda^*}{1 - \rho_\Lambda^* + 2\rho_\Lambda^{*2}},$$

so that the condition (8.8) is false when  $\rho = \rho_\Lambda^*$  or, in other words, we have  $\rho_\Lambda^{**} < \rho_\Lambda^*$  whenever  $\Lambda > 2$  (see Table 1). We have therefore proven (subject to the proof of Lemma 8.3) the following corollary.

COROLLARY 8.4. Fix an integer  $\Lambda \geq 3$ , and let  $\mathbf{G} = (G, s, t)$  be a 2-terminal graph in the class  $\mathcal{W}$  such that  $G$  has maxflow at most  $\Lambda$ . Then the chromatic

polynomial  $P_G(q)$  is nonvanishing whenever

$$(8.12) \quad |q - 1| \geq 1/\rho_{\Lambda}^{**},$$

where  $\rho_{\Lambda}^{**}$  is the unique solution of

$$(8.13) \quad (1 + \rho)^{\Lambda+1} = 4(1 - \rho + 2\rho^2)^{\Lambda-1}$$

in the interval  $(0, 1)$ .

Since the bound  $|q - 2| \geq |q - 1| - 1$  holds as equality when  $q$  is real and  $q > 2$ , we see that the condition (8.8) is also *necessary* for (8.5) when  $q$  is real and positive or, in other words, the circle  $|q - 1| = 1/\rho_{\Lambda}^{**}$  coincides with the boundary of the region (8.5) on the positive real axis, but lies outside of it elsewhere. Figure 6 shows how these regions compare for  $\Lambda = 3, 4$ .

*Proof of Lemma 8.3.* Consider the function

$$(8.14) \quad f_{\Lambda}(\rho) = (\Lambda + 1) \log(1 + \rho) - (\Lambda - 1) \log(1 - \rho + 2\rho^2).$$

Its first two derivatives are

$$(8.15a) \quad f'_{\Lambda}(\rho) = \frac{\Lambda + 1}{1 + \rho} - \frac{(\Lambda - 1)(4\rho - 1)}{1 - \rho + 2\rho^2}$$

$$(8.15b) \quad f''_{\Lambda}(\rho) = -\frac{(6 + 4\rho + 18\rho^2 - 24\rho^3 + 4\rho^4) + (\Lambda - 2)(4 + 8\rho + 8\rho^2 - 16\rho^3 - 4\rho^4)}{(1 + \rho)^2(1 - \rho + 2\rho^2)^2}.$$

For  $0 \leq \rho \leq 1$  we manifestly have  $6 + 4\rho + 18\rho^2 - 24\rho^3 + 4\rho^4 \geq 4\rho - 4\rho^3 \geq 0$  and  $4 + 8\rho + 8\rho^2 - 16\rho^3 - 4\rho^4 \geq 4 - 4\rho^4 \geq 0$  with strict inequality when  $0 < \rho < 1$ , hence,  $f_{\Lambda}$  is strictly concave on  $[0, 1]$  whenever  $\Lambda \geq 2$ . We have  $f_{\Lambda}(0) = 0$ ,  $f'_{\Lambda}(0) = 2\Lambda > 0$ ,  $f_{\Lambda}(1) = \log 4$ , and  $f'_{\Lambda}(1) = -(\Lambda - 2)$ . Therefore, for  $\Lambda > 2$ , there is a unique  $\rho_{\Lambda}^{**} \in (0, 1)$  satisfying  $f_{\Lambda}(\rho_{\Lambda}^{**}) = \log 4$  and for  $\rho \in [0, 1)$  we have  $f_{\Lambda}(\rho) \leq \log 4$  if and only if  $\rho \leq \rho_{\Lambda}^{**}$ . This proves the equivalence of (a) and (b) for all real  $\Lambda > 2$ .  $\square$

Finally, let us deduce Theorem 1.6 as an immediate consequence of Corollary 8.4, by proving that  $\rho_{\Lambda}^{**} > (\log 2)/(\Lambda - \log 2)$  for all integers  $\Lambda \geq 3$  or, equivalently (in view of Lemma 8.3), that  $(1 + \rho)^{\Lambda+1} < 4(1 - \rho + 2\rho^2)^{\Lambda-1}$  when  $\rho = (\log 2)/(\Lambda - \log 2)$ . We shall actually prove this for all *real*  $\Lambda > 2 \log 2 \approx 1.386294$ . Taking logarithms and parametrizing by  $\rho \in (0, 1)$ , we see that this is equivalent to the following claim.

LEMMA 8.5. *The function*

$$(8.16a) \quad g(\rho) = \rho \left[ 2 \log 2 + \left( \frac{\log 2}{\rho} + \log 2 - 1 \right) \log(1 - \rho + 2\rho^2) - \left( \frac{\log 2}{\rho} + \log 2 + 1 \right) \log(1 + \rho) \right]$$

$$(8.16b) \quad = 2\rho \log \left( \frac{2}{1 + \rho} \right) - (\log 2) \log \left( \frac{1 + \rho}{1 - \rho + 2\rho^2} \right) - (\log 2 - 1)\rho \log \left( \frac{1 + \rho}{1 - \rho + 2\rho^2} \right)$$

is strictly positive for  $0 < \rho < 1$ .

*Proof.* The second derivative of  $g$  is given by  $g''(\rho) = h(\rho) \times \rho/(1 + \rho^2 + 2\rho^3)^2$ , where

$$(8.17) \quad h(\rho) = -(12 - 4 \log 2)\rho^4 - 12\rho^3 - (2 + 8 \log 2)\rho^2 - (24 - 16 \log 2)\rho + (20 \log 2 - 6).$$

All the coefficients of  $h(\rho)$  are strictly negative except for the last (constant) term, so we have  $h'(\rho) < 0$  for all  $\rho \geq 0$ . Since  $h(0) = 20 \log 2 - 6 > 0$  and  $h(1) = 32 \log 2 - 56 < 0$  and  $h$  is strictly decreasing for  $\rho \geq 0$ , it follows that  $h(\rho)$  has exactly one positive

real root  $\rho^*$  and that it lies between 0 and 1 (by computer  $\rho^* \approx 0.417655$ ). Therefore  $g$  is strictly convex on  $[0, \rho^*]$  and strictly concave on  $[\rho^*, \infty)$ . Since  $g(0) = g'(0) = 0$ , we have  $g(\rho) > 0$  for  $\rho \in (0, \rho^*]$ . Moreover, since  $g(\rho^*) > 0$  and  $g(1) = 0$  and  $g$  is strictly concave on  $[\rho^*, 1]$ , we have  $g(\rho) > 0$  for  $\rho \in [\rho^*, 1)$ . Hence  $g(\rho) > 0$  for all  $\rho \in (0, 1)$ , as claimed.  $\square$

*Remark.* A straightforward calculation shows that the large- $\Lambda$  asymptotic behavior of  $\rho_\Lambda^{**}$  is given by

$$(8.18) \quad \rho_\Lambda^{**} = (\log 2) \left[ \frac{1}{\Lambda - 1} + \frac{\log 2 - 1}{(\Lambda - 1)^2} + \frac{16 \log^2 2 - 15 \log 2 + 6}{6(\Lambda - 1)^3} + \dots \right]$$

and, hence,

$$(8.19) \quad \frac{1}{\rho_\Lambda^{**}} = \frac{\Lambda - 1}{\log 2} - \frac{\log 2 - 1}{\log 2} - \frac{10 \log 2 - 3}{6(\Lambda - 1)} + \dots$$

So the inequality  $\rho_\Lambda^{**} > (\log 2)/(\Lambda - \log 2)$  captures the first two terms of the large- $\Lambda$  asymptotic behavior.  $\square$

**Acknowledgments.** We are extremely grateful to the Isaac Newton Institute for Mathematical Sciences, University of Cambridge, for support during the programme on Combinatorics and Statistical Mechanics (January–June 2008), where this work was mostly carried out. We also thank the Laboratoire de Physique Théorique at the École Normale Supérieure for hospitality in April–June 2011 and April–July 2013.

#### REFERENCES

- [1] M.W. BERN, E.L. LAWLER, AND A.L. WONG, *Linear-time computation of optimal subgraphs of decomposable graphs*, J. Algorithms, 8 (1987), pp. 216–235.
- [2] N.L. BIGGS, R.M. DAMERELL, AND D.A. SANDS, *Recursive families of graphs*, J. Combin. Theory Ser. B, 12 (1972), pp. 123–131.
- [3] G.D. BIRKHOFF AND D.C. LEWIS, *Chromatic polynomials*, Trans. Amer. Math. Soc., 60 (1946), pp. 355–451.
- [4] H.L. BODLAENDER AND B. VAN ANTWERPEN-DE FLUITER, *Parallel algorithms for series parallel graphs and graphs with treewidth two*, Algorithmica, 29 (2001), pp. 534–559.
- [5] C. BORGS, *Absence of zeros for the chromatic polynomial of bounded degree graphs*, Combin. Probab. Comput., 15 (2006), pp. 63–74.
- [6] R.B. BORIE, R.G. PARKER, AND C.A. TOVEY, *Automatic generation of linear-time algorithms from predicate calculus descriptions of problems on recursively constructed graph families*, Algorithmica, 7 (1992), pp. 555–581.
- [7] A. BRANDSTÄDT, V.B. LE, AND J.P. SPINRAD, *Graph Classes: A Survey*, Discrete Math. Appl., SIAM, Philadelphia, 1999.
- [8] J.V. BRAWLEY, S. GAO, AND D. MILLS, *Associative rational functions in two variables*, in Finite Fields and Applications, D. Jungnickel and H. Niederreiter, eds., Springer-Verlag, Berlin, 2001, pp. 43–56.
- [9] F. BRENTI, G.F. ROYLE, AND D.G. WAGNER, *Location of zeros of chromatic and related polynomials of graphs*, Canad. J. Math., 46 (1994), pp. 55–80.
- [10] C.J. COLBURN, *The Combinatorics of Network Reliability*, Clarendon Press/Oxford University Press, New York, 1987.
- [11] R.M. CORLESS, G.H. GONNET, D.E.G. HARE, D.J. JEFFREY, AND D.E. KNUTH, *On the Lambert W function*, Adv. Comput. Math., 5 (1996), pp. 329–359.
- [12] R.J. DUFFIN, *Topology of series-parallel graphs*, J. Math. Anal. Appl., 10 (1965), pp. 303–318.
- [13] E.J. FARRELL, *Chromatic roots — some observations and conjectures*, Discrete Math., 29 (1980), pp. 161–167.
- [14] R. FERNÁNDEZ AND A. PROCACCI, *Regions without complex zeros for chromatic polynomials on graphs with bounded degree*, Combin. Probab. Comput., 17 (2008), pp. 225–238.
- [15] G. GRIMMETT, *The Random-Cluster Model*, Springer-Verlag, Berlin, 2006.

- [16] B. JACKSON, *Zeros of chromatic and flow polynomials of graphs*, J. Geom., 76 (2003), pp. 95–109.
- [17] B. JACKSON, A. PROCACCI, AND A.D. SOKAL, *Complex zero-free regions at large  $|q|$  for multivariate Tutte polynomials (alias Potts-model partition functions) with general complex edge weights*, J. Combin. Theory Ser. B, 103 (2013), pp. 21–45.
- [18] B. JACKSON AND A.D. SOKAL, *Zero-free regions for multivariate Tutte polynomials (alias Potts-model partition functions) of graphs and matroids*, J. Combin. Theory Ser. B, 99 (2009), pp. 869–903.
- [19] B. JACKSON AND A.D. SOKAL, *Maxmaxflow and counting subgraphs*, Electron. J. Combin., 17 (2010), R99.
- [20] J. OXLEY, *Graphs and series-parallel networks*, in Theory of Matroids, N. White, ed., Cambridge University Press, Cambridge, 1986, pp. 97–126.
- [21] J.G. OXLEY, *Matroid Theory*, Oxford University Press, New York, 1992.
- [22] Q.I. RAHMAN AND G. SCHMEISSER, *Analytic Theory of Polynomials*, Clarendon Press/Oxford University Press, Oxford, 2002.
- [23] R.C. READ AND G.F. ROYLE, *Chromatic roots of families of graphs*, in Graph Theory, Combinatorics, and Applications, Vol. 2, Y. Alavi, G. Chartrand, O.R. Oellermann, and A.J. Schwenk, eds., Wiley, New York, 1991, pp. 1009–1029.
- [24] G. ROYLE AND A.D. SOKAL, *The Brown–Colbourn conjecture on zeros of reliability polynomials is false*, J. Combin. Theory Ser. B, 91 (2004), pp. 345–360.
- [25] G.F. ROYLE, *Recent results on chromatic and flow roots of graphs and matroids*, in Surveys in Combinatorics, 2009, S. Huczynska, J.D. Mitchell, and C.M. Roney-Dougal, eds., Cambridge University Press, Cambridge, 2009, pp. 289–327.
- [26] G.F. ROYLE AND A.D. SOKAL, *Linear Bound in Terms of Maxmaxflow for the Chromatic Roots of Series-Parallel Graph*, expanded version, preprint, arXiv.1307.1721, 2013.
- [27] J. SALAS AND A.D. SOKAL, *Chromatic Roots of the Complete Bipartite Graphs*, manuscript, 2015.
- [28] R. SHROCK AND S.-H. TSAI, *Ground-state degeneracy of Potts antiferromagnets: Cases with noncompact  $W$  boundaries having multiple points at  $1/q = 0$* , J. Phys. A, 31 (1998), pp. 9641–9665.
- [29] R. SHROCK AND S.-H. TSAI, *Ground-state degeneracy of Potts antiferromagnets: Homeomorphic classes with noncompact  $W$  boundaries*, Phys. A, 265 (1999), pp. 186–223.
- [30] A.D. SOKAL, *Bounds on the complex zeros of (di)chromatic polynomials and Potts-model partition functions*, Combin. Probab. Comput., 10 (2001), pp. 41–77.
- [31] A.D. SOKAL, *Chromatic roots are dense in the whole complex plane*, Combin. Probab. Comput., 13 (2004), pp. 221–261.
- [32] A.D. SOKAL, *The multivariate Tutte polynomial (alias Potts model) for graphs and matroids*, in Surveys in Combinatorics, 2005, B.S. Webb, ed., Cambridge University Press, Cambridge, 2005, pp. 173–226.
- [33] J.P. SPINRAD, *Efficient Graph Representations*, AMS, Providence RI, 2003.
- [34] C. THOMASSEN, *The zero-free intervals for chromatic polynomials of graphs*, Combin. Probab. Comput., 6 (1997), pp. 497–506.
- [35] J. VALDES, R.E. TARJAN, AND E.L. LAWLER, *The recognition of series parallel digraphs*, SIAM J. Comput., 11 (1982), pp. 298–313.
- [36] D.G. WAGNER, *Zeros of reliability polynomials and  $f$ -vectors of matroids*, Combin. Probab. Comput., 9 (2000), pp. 167–190.