Matchings and Representation Theory

by

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This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Statement of Contribution

The results in Chapter 2 are based on joint work with Radu Curticapean and Jesper Nederlof that appeared in the Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms. The results in Chapter 5 are based on joint work in progress with Gary Au and Levent Tunçel.

Abstract

In this thesis we investigate the algebraic properties of matchings via representation theory. We identify three scenarios in different areas of combinatorial mathematics where the algebraic structure of matchings gives keen insight into the combinatorial problem at hand. In particular, we prove tight conditional lower bounds on the computational complexity of counting Hamiltonian cycles, resolve an asymptotic version of a conjecture of Godsil and Meagher in Erdős-Ko-Rado combinatorics, and shed light on the algebraic structure of symmetric semidefinite relaxations of the perfect matching problem.

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I would like to thank my advisors Joseph Cheryian and Chris Godsil for their support during my stay at Waterloo. Their honesty and wisdom put me back on the right track at pivotal moments throughout my degree, and they have made me a better mathematician. It has been a privilege.

I am also grateful to my committee members, Nantel Bergeron, Ian Goulden, Lap Chi Lau, and Levent Tunçel for their thorough comments and suggestions, which have greatly improved the quality of this thesis.

Contents

List of Symbols

$\ell(\lambda)$ the length of an integer partition λ. 10

λ' the transpose of an integer partition λ. 10, 45

$\lambda \vdash n$ an integer partition of n. 9

$\mathbb{C}[G/K]$ the G/K-coset algebra. 57

$\mathbb{C}[G]$ the group algebra of G. 57

$\mathbb{C}[K\backslash G/K]$ the $(K\backslash G/K)$-double coset algebra. 57

$\mathbb{C}[X]$ the space of complex-valued functions over a set X. 38

$\mathbb{C}[[\overline{x}]]$ the ring of formal power series. 11

$\mathcal{A}_n$ the perfect matchings association scheme of K_{2n}. 10

$\mathcal{A}'_n$ the perfect matchings association scheme of $K_{n,n}$. 10

$\mathcal{D}_n$ the perfect matching derangement graph. 40

$\mathcal{F}\downarrow_{ij}$ the restriction of a family $\mathcal{F}$ to the edge ij. 36

$\mathcal{F}_T$ the canonically t-intersecting family corresponding to T. 53

$\mathcal{F}_{ij}$ the canonically intersecting family corresponding to the edge ij. 34

$\mathcal{M}_{2n}$ the collection of perfect matchings of K_{2n}. 8

$\mathcal{M}_{n,n}$ the collection of perfect matchings of $K_{n,n}$. 8

$\phi\downarrow^G_H$ the restriction of the G-representation ϕ to subgroup H. 39

$\phi\uparrow^G_H$ the induction of the H-representation ϕ to G. 59

$\theta_{k,\lambda}$ the λ-eigenvalue of $M_{k,n}$. 16

$\theta'_{k,\lambda}$ the λ-eigenvalue of $M'_{k,n}$. 16

$c_\alpha(\lambda)$ the product of the α-content of a shape λ. 14

$d()$ the cycle type function. 9

e_λ the elementary symmetric function corresponding to $\lambda \vdash n$. 12

Chapter 1

Introduction

Matchings need no introduction. They have remained at the forefront of combinatorics since the beginning and their remarkable structure has blossomed into a mature, far-reaching combinatorial theory.

The point of this thesis is not to extend their combinatorial lore per se, but to explore matchings algebraically through the lens of *representation theory*. To this end, we identify three scenarios in different areas of combinatorial mathematics where the algebraic structure of matchings gives keen insight into the combinatorial problem at hand.

Our first instance concerns the graph Hamiltonicity problem — deciding if a graph has a cycle containing all of its vertices. Like many problems of this nature, it is widely believed that there is no subexponential-time algorithm for solving this problem, which has led to the design and analysis of ever faster exponential-time algorithms for Hamiltonicity and counting the number of Hamiltonian cycles over various fields [7, 8, 15].

The centerpiece of Cygan et al. [15] is a matrix defined over perfect matchings that they call *the matchings connectivity matrix*. They give a combinatorial analysis of its rank over the two-element field that allowed them to determine the complexity of Hamiltonicity and counting Hamiltonian cycles modulo 2 assuming the Strong Exponential-Time Hypothesis (SETH); however, their combinatorial approach could not be extended to fields of odd prime characteristic or characteristic zero. Decades earlier, a bipartite incarnation of their matchings connectivity matrix appeared in work of Raz and Spieker [61] that was central to disproving a well-known conjecture in complexity theory.

We introduce a family of matrices indexed by perfect matchings that is a common generalization of the matrices in [15, 61], and we use Jack symmetric functions with a little representation theory to give a unified analysis of their eigenvalues. As a consequence, we obtain an exact formula for the rank of the matching connectivity matrix over the reals which we use in joint work with Radu Curticapean and Jesper Nederlof to determine the complexity (up to polynomial factors) of counting Hamiltonian cycles assuming SETH.

Next, we investigate some open problems in *Erdős-Ko-Rado combinatorics*, a branch of extremal combinatorics that investigates how large families of objects can be subject to the restriction that any two elements of the family "intersect". The prototypical example that gave the field its namesake is *the Erdős-Ko-Rado Theorem*, which states that for all $n \geq 2k$ and $t \leq k$, if $\mathcal{F}$ is a family of k-element subsets of $\{1, 2, \cdots, n\}$ such that $|S \cap T| \geq t$ for any $S, T \in \mathcal{F}$, then

$$|\mathcal{F}| \leq \binom{n-t}{k-t},$$

and equality holds if and only if all the members of $\mathcal{F}$ have a fixed set of t elements in common. Several different combinatorial proofs of this result and variations thereof appeared shortly thereafter, but a radically different algebraic proof of this upper bound was eventually found by Lovász [48] and Schrijver [67] that led to a development of algebraic methods for Erdős-Ko-Rado-type problems. The recent book of Godsil and Meagher [30] gives a detailed account this algebraic theory wherein they pose many interesting open questions.

We address a conjecture of theirs concerning *t-intersecting families of perfect matchings* — families such that any two members share t or more edges. Such a family is *canonically t-intersecting* if it is composed of all the perfect matchings that contain some fixed set of t disjoint edges.

Conjecture (Godsil, Meagher [30]) For all $n \geq 3t/2 + 1$, if $\mathcal{F}$ is a t-intersecting family of perfect matchings of the complete graph K_{2n}, then

$$|\mathcal{F}| \leq (2(n-t) - 1)!!,$$

and equality holds if and only if $\mathcal{F}$ is a canonically t-intersecting family.

Meagher and Moura [56] gave a combinatorial proof of this conjecture for $t = 1$, which was followed by an algebraic proof due to Lindzey [45]. Curiously, the $t \geq 2$ case has remained impervious to the many combinatorial methods in this area, which has even resulted in a few incorrect proofs of this conjecture.

We give an algebraic proof that Godsil and Meagher's conjecture holds for constant t and sufficiently large n depending on t. The proof is a melange of representation theory and symmetric functions paired with classical algebraic methods and recent stability techniques of Ellis [21]. An important stepping stone towards showing the case of equality was a stability result for 1-intersecting families of perfect matchings of K_{2n} — that "large" 1-intersecting families are contained in canonically 1-intersecting families for sufficiently large n [46]. We include a treatment of this result as an aperitif to the main course.

Our final example brings us to the genesis of combinatorial optimization, namely, the problem of finding a perfect matching in a graph. Great strides have been made in recent years towards understanding polyhedral formulations of this problem [54, 63], but our knowledge of spectahedral formulations and relaxations leaves much to be desired.

In this vein, Au [3] studied a variety of so-called Lift-and-Project semidefinite relaxations of the perfect matching problem. He observed that the eigenspaces of matrices related to some of these relaxations had many intriguing combinatorial properties and asked whether these connections could be understood in a more unifying way. We use representation theory to answer this question and make progress on a few conjectures posed in his dissertation [3, Ch. 8]. Proving these conjectures in full is still joint work in progress with Gary Au and Levent Tunçel, but we present some interesting partial results that we have obtained thus far.

We believe this work demonstrates that representation theory can be an effective way of understanding and approaching combinatorial questions about matchings of a linear-algebraic nature. It seems likely there are other problems related to matchings that can benefit from this point of view and the theory presented in this thesis. More generally, we hope this offering inspires fellow combinatorialists and theoretical computer scientists to consider representation theory for solving other combinatorial problems.

Chapter 2

Perfect Matchings and Communication Matrices

Let us begin with a game: each of two players gets a perfect matching of the complete bipartite graph $K_{n,n}$, and their goal is to decide whether or not the union of the two matchings forms a Hamiltonian cycle by sharing as little information as possible. We may model the game's input as a $n! \times n!$ *communication matrix* M'_n indexed by perfect matchings of $K_{n,n}$ such that

$$[M'_n]_{i,j} := \begin{cases} 1 & \text{if } i \cup j \text{ is a Hamiltonian cycle;} \\ 0 & \text{otherwise.} \end{cases}$$

The *communication complexity* of this matrix is the fewest number of shared bits needed for them to decide if the (i,j)-entry of M'_n is one or zero. In [61], Raz and Spieker used this game to show that the communication complexity of a game's input matrix and the logarithm of the rank of the game's input matrix may differ by non-constant factors, which then refuted a stronger formulation of the *Log-Rank Conjecture* in the field of communication complexity [52].

A crucial part of their argument was proving (with the help of Conway and Lovász) that the rank of M'_n over the reals is $\binom{2n-2}{n-1}$, which is considerably less than $n!$. By insisting that the players be given perfect matchings of $K_{n,n}$, equivalently, elements of the symmetric group S_n, the group representation theory of S_n was at their command for deriving this result; however, one may just as well define a non-bipartite version of their communication game where each of the two players gets a perfect matching of K_{2n}, and their goal is to again decide if their union is a Hamiltonian cycle as efficiently as possible. Its matrix M_n is indexed by perfect matchings of K_{2n} and is defined similarly.

In [15], Cygan, Kratsch, and Nederlof consider the structure of M_n, which they call the *matchings connectivity matrix*, for designing exact algorithms for deciding Hamiltonicity and counting Hamiltonian cycles modulo 2. Using combinatorial methods, they showed that the 2-rank of M_n is precisely 2^{n-1} by giving a rank factorization $M_n = X^\top X$ over the two-element field. This factorization allowed them to determine the complexity (up to polynomial factors) of deciding Hamiltonicity and counting Hamiltonian cycles modulo 2 assuming the Strong Exponential-Time Hypothesis.

In light of the applicability of these matrices in theoretical computer science, we offer a broader class of "matchings connectivity matrices". Let $M_{k,n}$ be the matrix indexed by perfect matchings of K_{2n} defined such that

$$[M_{k,n}]_{i,j} := \begin{cases} 1 & \text{if } i \cup j \text{ has exactly } k \text{ connected components;} \\ 0 & \text{otherwise,} \end{cases}$$

and define the matrix $M'_{k,n}$ indexed by perfect matchings of $K_{n,n}$ similarly.

In the next section, we briefly introduce the theory of *association schemes* and show that $M'_{k,n}$ and $M_{k,n}$ both belong to association schemes defined over perfect matchings of $K_{n,n}$ and K_{2n} respectively. The well-versed reader may be aware that this fact furnishes us with off-the-shelf formulas for the eigenvalues of $M'_{k,n}$ and $M_{k,n}$ in terms of spherical functions and characters of the symmetric group, but these expressions are too unwieldy to be useful in practice.

We instead use specializations of Jack symmetric functions to give a unified analysis of the eigenvalues of $M'_{k,n}$ and $M_{k,n}$ for all $k \leq n$, providing simpler and more explicit expressions for these eigenvalues. The expressions we derive also reveal much information about the ranks of $M'_{k,n}$ and $M_{k,n}$. In particular, we obtain an exact formula for the rank of M_n which can be seen as the non-bipartite version of Raz and Spieker's result. More generally, we show the ranks of $M'_{k,n}$ and $M_{k,n}$ become vanishingly small provided $k = o(n^\epsilon)$ for all $\epsilon > 0$.

The last contribution of this chapter is an overview of an application of our non-bipartite analogue of Raz and Spieker's result to complexity theory. In joint work with Radu Curticapean and Jesper Nederlof, we determine the complexity (up to polynomial factors) of counting Hamiltonian cycles assuming that the Strong Exponential-Time Hypothesis is true.

2.1 Association Schemes I

The following material can be found in more detail in Bannai and Ito's text [4]. For a treatment of association schemes geared more towards the symmetric group and combinatorial applications, see Godsil and Meagher's manuscript [30].

A *symmetric association scheme* is a collection of $m+1$ binary $|X| \times |X|$ matrices $A_0, A_1, \cdots, A_m$ over a set X that satisfy the following axioms:

1. A_i is symmetric for all $0 \leq i \leq m$,
2. $A_0 = I$ where I is the identity matrix,
3. $\sum_{i=0}^{m} A_i = J$ where J is the all-ones matrix, and
4. $A_i A_j = A_j A_i \in \mathrm{Span}\{A_0, A_1, \cdots, A_m\}$ for all $0 \leq i, j \leq m$.

From a combinatorial point of view, the *associates* $A_1, \cdots, A_m$ are adjacency matrices of regular spanning subgraphs of $K_{|X|}$ that partition $E(K_{|X|})$ subject to other regularity conditions that need not concern us in this thesis. The *valency* v_i is the degree of the graph corresponding to the ith associate, equivalently, the largest eigenvalue of A_i.

The matrix algebra generated by the identity matrix and its associates is the association scheme's *Bose-Mesner algebra*, and these matrices form a basis for the algebra. This is a commutative matrix algebra, and so its matrices are *simultaneously diagonalizable*, i.e., they share a common orthonormal system of eigenvectors. A consequence of this fact is that Bose-Mesner algebras afford a canonical dual basis of *primitive idempotents*, positive semi-definite matrices $E_0, E_1, \cdots, E_m$ that are pairwise-orthogonal and satisfy $\sum_{i=0}^{m} E_i = I$. In particular, the ith primitive idempotent E_i is the projector of the ith eigenspace of any matrix that belongs to the association scheme's Bose-Mesner algebra.

The foregoing is hardly a complete treatment of the theory of association schemes, but it will suffice for this chapter. More of the theory and terminology will be introduced as we go. Before we define our association schemes over the perfect matchings of K_{2n} and $K_{n,n}$, we recall some basic facts and terminology surrounding the combinatorics of perfect matchings and integer partitions.

2.1.1 Perfect Matchings and Integer Partitions

Let $\mathcal{M}_{n,n} \cong S_n$ be the set of perfect matchings of the complete bipartite graph $K_{n,n}$, and let $\mathcal{M}_{2n}$ be the set of perfect matchings of K_{2n}, the complete graph on an even number of vertices. Let P be a partition of $\{1, 2, \cdots, n\}$ into k parts of size ℓ, that is,

$$P = \{P_1, P_2, \cdots, P_k\} \text{ and } P_i = \{P_{i_1}, P_{i_2}, \cdots, P_{i_\ell}\}.$$

Using a short-hand notation, we may write this partition as

$$P = P_{1_1}\ P_{1_2}\ \cdots\ P_{1_\ell}|P_{2_1}\ P_{2_2}\ \cdots\ P_{2_\ell}|\cdots|P_{k_1}\ P_{k_2}\ \cdots\ P_{k_\ell}.$$

Since $\mathcal{M}_{2n}$ is in bijection with partitions of $[2n] := \{1, 2, \cdots, 2n\}$ into parts of size two, we may write any perfect matching as a partition

$$m = m_1\ m_2|m_3\ m_4|\cdots|m_{2n-1}\ m_{2n} \text{ such that } m_i \in [2n].$$

Let $m^* := 1\ 2|3\ 4|\cdots|2n\text{-}1\ 2n$ be the *identity perfect matching*. The *symmetric group* S_{2n} on $2n$ symbols acts transitively on $\mathcal{M}_{2n}$ under the following action:

$$\sigma m = \sigma(m_1)\ \sigma(m_2)\ |\ \sigma(m_3)\ \sigma(m_4)\ |\ \cdots\ |\ \sigma(m_{2n-1})\ \sigma(m_{2n}).$$

The *hyperoctahedral group* $H_n := S_2 \wr S_n$ has order $(2n)!! := 2^n n!$ and is isomorphic to the stabilizer of m^* under this action. Since perfect matchings are in one-to-one correspondence with cosets of the quotient S_{2n}/H_n, we see that

$$|\mathcal{M}_{2n}| = (2n-1)!! := 1 \times 3 \times 5 \times \cdots \times (2n-3) \times (2n-1).$$

The following proposition can be shown using Stirling's formula.

2.1.1 Proposition. [5] *For all $n \in \mathbb{N}$, we have $(2n-1)!! < (2n)!!/\sqrt{\pi n}$.*

For any two perfect matchings m and m' of an arbitrary graph, let $\Gamma(m, m')$ be the multigraph whose edge multiset is the multiset union $m \cup m'$. It is clear that $\Gamma(m, m') \cong \Gamma(m', m)$ is composed of disjoint cycles of even parity. Let k denote the number of disjoint cycles and let $2\lambda_i$ denote the length of an even cycle. If we order the cycles from longest to shortest and divide

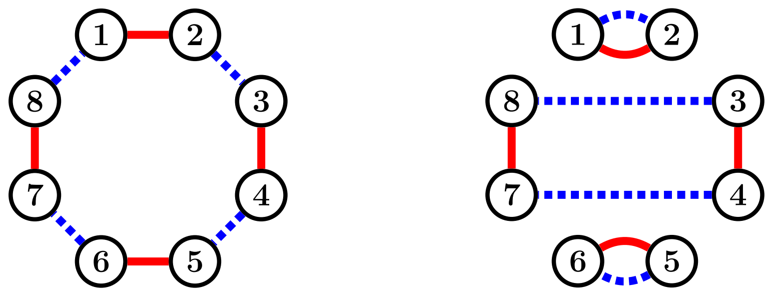
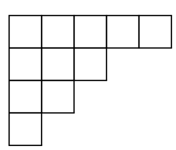


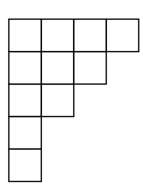
Figure 2.1: The perfect matching 2 3|4 5|6 7|1 8 on the left has cycle type $(n) \vdash n$ whereas the perfect matching 1 2|3 8|4 7|5 6 on the right has cycle type $(2, 1^{n-2}) \vdash n$ for $n = 4$.

each of their lengths by two, we see that each graph corresponds to an *integer partition* $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \vdash n$. For any $\lambda \vdash n$, if there are k parts that all have the same size λ_i we use λ_i^k to denote the multiplicity. Let $d(m, m')$ denote the integer partition corresponding to $\Gamma(m, m')$ which we shall call the *cycle type of m' with respect to m*, and we say that $d(m^*, m)$ is *the cycle type of m*. Since $\Gamma(x, y) \cong \Gamma(x', y')$ if and only if $d(x, y) = d(x', y')$, let the graph Γ_λ be a distinct representative from the isomorphism class $\lambda \vdash n$. Illustrations of the graphs $\Gamma_{(n)}$ and $\Gamma_{(2,1^{n-2})}$ are provided in Figure 2.1.

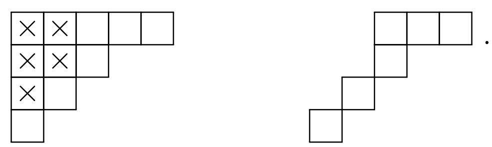
Often we shall view an integer partition $\lambda = (\lambda_1, \cdots, \lambda_\ell) \vdash n$ as a *Ferrers diagram*, a left-justified table of cells such that the ith row has λ_i cells. When appealing to this view, we call λ a *shape*, and we let $|\lambda|$ denote the number of cells that compose the shape. For example, the Ferrers diagram below has shape $(5, 3, 2, 1) \vdash 11$:

 .

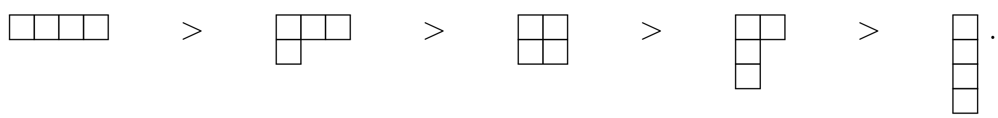
Let $\lambda' \vdash n$ denote the *transpose* of λ, that is, the partition obtained by interchanging the columns and the rows of the corresponding Ferrers diagram of λ. The transpose of $(5, 3, 2, 1) \vdash 11$ is $(4, 3, 2, 1, 1) \vdash 11$, as illustrated below:

 .

Let $\ell(\lambda)$ denote the *length* of λ, that is, the number of rows in its Ferrers diagram. Since a shape λ is composed of a set of cells, we use standard set notation to reference subsets of cells within λ and the notation $(i, j) \in \lambda$ to refer to the cell that resides at the ith row and jth column of λ. We say that λ *covers* μ if $\mu_i \leq \lambda_i$ for each $i \in [\ell(\mu)]$. If λ and μ are two partitions such that λ covers μ, then we obtain the *skew shape* λ/μ by removing the cells corresponding to μ from λ. For instance, the shape $(5, 3, 2, 1)$ covers $(2, 2, 1)$, so we may consider the skew shape $(5, 3, 2, 1)/(2, 2, 1)$:



The only order on the set of partitions $\lambda(n)$ that we consider is the *lexicographical order* $(\lambda(n), \leq)$, defined such that $\mu \leq \lambda$ if and only if $\mu_j < \lambda_j$ where j is the first index where μ and λ differ, or $\mu = \lambda$. For $n = 4$, we have



2.1.2 Perfect Matching Association Schemes

For each $\lambda \vdash n$, let A_λ be the following $(2n-1)!! \times (2n-1)!!$ matrix:

$$(A_\lambda)_{i,j} = \begin{cases} 1, & \text{if } d(i,j) = \lambda \\ 0, & \text{otherwise,} \end{cases}$$

where $i, j \in \mathcal{M}_{2n}$. We call the collection of matrices $\mathcal{A}_n := \{A_\lambda\}_{\lambda \vdash n}$ *the perfect matching association scheme* of K_{2n}, and we may define the perfect matching association scheme $\mathcal{A}'_n$ of $K_{n,n}$ analogously. The latter association scheme is more commonly known as the *conjugacy-class association scheme of the symmetric group*, since permutations on the symbols $[n] := \{1, 2, \cdots, n\}$ can be identified as perfect matchings of $K_{n,n}$ and conjugacy classes of S_n are in one-to-one correspondence with cycle types of permutations. For proofs that $\mathcal{A}_n$ and $\mathcal{A}'_n$ are indeed association schemes, see [30].

The eigenspaces of matrices in the Bose-Mesner algebras of these association schemes are also parameterized by integer partitions of n. For example, when $n = 4$ we have $\mathcal{A}_4 = \{A_{(4)}, A_{(3,1)}, A_{(2,2)}, A_{(2,1^2)}, A_{(1^4)}\}$, and we may record their eigenvalues in a 5×5 matrix P with columns indexed by associates and rows indexed by eigenspaces in reverse-lexicographical order:

$$P = \begin{array}{c} \\ E_{(4)} \\ E_{(3,1)} \\ E_{(2,2)} \\ E_{(2,1^2)} \\ E_{(1^4)} \end{array} \begin{array}{c} \begin{array}{ccccc} A_{(4)} & A_{(3,1)} & A_{(2,2)} & A_{(2,1^2)} & A_{(1^4)} \end{array} \\ \begin{bmatrix} 48 & 32 & 12 & 12 & 1 \\ -8 & 4 & -2 & 5 & 1 \\ -2 & -8 & 7 & 2 & 1 \\ 4 & -2 & -2 & -1 & 1 \\ -6 & 8 & 3 & -6 & 1 \end{bmatrix} \end{array}.$$

Note that the first row is composed of the valencies $v_{(4)}, \cdots, v_{(1^4)}$ of the associates. More generally, the *P-matrix* or *character table* of an association scheme is a $(m+1) \times (m+1)$ matrix P defined such that its (i, j)-entry is the eigenvalue associated to the ith eigenspace of the jth associate.

It is well-known that the character table of $\mathcal{A}'_n$ is a normalization of the group character table of S_n, and as their name suggests, the character tables of association schemes can be seen as a generalization of the character tables of finite groups (see [4]).

It is now easy to verify that the generalized matchings connectivity matrix $M_{k,n}$ is a sum of associates of $\mathcal{A}_n$, since

$$M_{k,n} = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) = k}} A_\lambda,$$

and similarly for $M'_{k,n}$ with respect to $\mathcal{A}'_n$. In the next section we define a family of symmetric functions that describes the character tables of both $\mathcal{A}_n$ and $\mathcal{A}'_n$.

2.2 Symmetric Functions I

The entirety of this section can be found in Macdonald's text on the subject [53]. Let $\overline{x} := x_1, x_2, \cdots$ be an infinite set of indeterminates and let $\mathbb{C}[[\overline{x}]]$ be the ring of formal power series with complex coefficients.

For any integer partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$, we define the *monomial symmetric function* $m_\lambda \in \mathbb{C}[[\overline{x}]]$ to be

$$m_\lambda = m_\lambda(\overline{x}) := \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_\ell}^{\lambda_\ell}$$

where the sum ranges over all the monomials that have exponents $\lambda_1, \lambda_2, \cdots, \lambda_\ell$. A function $f(\overline{x}) \in \mathbb{C}[[\overline{x}]]$ is *degree-n homogeneous* if the sum of the exponents of each monomial in $f(\overline{x})$ equals n. Note that if λ is an integer partition of n, then m_λ is degree-n homogeneous.

Let $\Lambda = \Lambda(\overline{x}) := \mathrm{Span}\{m_\lambda\}$ be the ring spanned by the m_λ's for all integer partitions λ. We say a function $f(\overline{x}) \in \mathbb{C}[[\overline{x}]]$ is *symmetric* if $f(\overline{x}) \in \Lambda$, and we call Λ the *ring of symmetric functions*. This ring may be decomposed as

$$\Lambda = \bigoplus_{n\geq 0} \Lambda^n,$$

where Λ^n is the vector space of *degree-n homogeneous symmetric functions* which is spanned by $\{m_\lambda\}_{\lambda\vdash n}$. One can check that the m_λ's are all independent, and so $\{m_\lambda\}_{\lambda\vdash n}$ is in fact a basis for Λ^n.

We now recall a few other well-known bases for the vector space of degree-n homogeneous symmetric functions. Let the *kth elementary symmetric function* $e_k = e_k(\overline{x}) \in \mathbb{C}[[\overline{x}]]$ be defined as

$$e_k(\overline{x}) := \sum_{i_1<i_2<\cdots<i_k} x_{i_1}x_{i_2}\cdots x_{i_k},$$

and we define the *kth power sum symmetric function* $p_k = p_k(\overline{x}) \in \mathbb{C}[[\overline{x}]]$ to be

$$p_k(\overline{x}) := x_1^k + x_2^k + \cdots .$$

For any $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$, let $p_\lambda = p_\lambda(\overline{x}) \in \mathbb{C}[[\overline{x}]]$ and $e_\lambda = e_\lambda(\overline{x}) \in \mathbb{C}[[\overline{x}]]$ be

$$p_\lambda(\overline{x}) := p_{\lambda_1}p_{\lambda_2}\cdots p_{\lambda_\ell}, \quad e_\lambda(\overline{x}) := e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_\ell}.$$

It is well-known that the *power sum symmetric functions* $\{p_\lambda\}_{\lambda\vdash n}$ and the *elementary symmetric functions* $\{e_\lambda\}_{\lambda\vdash n}$ both form bases for Λ^n. Let

$$z_\lambda := 1^{i_1}2^{i_2}\cdots i_1!i_2!\cdots$$

where the partition λ has i_j parts equal to $j \geq 1$. We now define $\langle\cdot,\cdot\rangle$ to be the unique inner product over Λ^n that satisfies

$$\langle p_\lambda, p_\mu\rangle = 1^{\ell(\lambda)} z_\lambda \delta_{\lambda,\mu}$$

where $\delta_{\lambda,\mu} = 1$ if $\lambda = \mu$; otherwise, $\delta_{\lambda,\mu} = 0$.

Another basis of Λ^n of unparalleled importance in the theory of symmetric functions is the family of *Schur symmetric functions* $\{s_\lambda\}_{\lambda \vdash n}$. They admit an elegant combinatorial definition, but it behooves us to define them algebraically as the unique basis of Λ^n satisfying $\langle s_\lambda, s_\mu \rangle = 0$ for all $\mu \neq \lambda$ and

$$s_\lambda = s_\lambda(\overline{x}) := m_\lambda + \sum_{\mu < \lambda} K_{\lambda,\mu} m_\mu,$$

where the sum ranges over all $\mu \vdash n$ that are lexicographically smaller than $\lambda \vdash n$. It turns out that $K_{\lambda,\mu} \in \mathbb{N}$ and in due time we shall provide a combinatorial interpretation of these coefficients. Arranging these coefficients as a matrix shows that $(K_{\lambda,\mu})_{\lambda,\mu \vdash n}$ is upper-unitrangular when indexed in reverse-lexicographical order, that is, the *transition matrix* from the Schur basis to the monomial basis is upper-unitriangular. Following Macdonald, for any bases $\{a_\lambda\}_{\lambda \vdash n}$ and $\{b_\lambda\}_{\lambda \vdash n}$ of Λ^n, let $M(a,b)$ be the transition matrix in reverse-lexicographical order from $\{a_\lambda\}_{\lambda \vdash n}$ to $\{b_\lambda\}_{\lambda \vdash n}$.

It turns out that another well-known basis of Λ^n, the *normalized zonal polynomials* $\{Z'_\lambda\}_{\lambda \vdash n}$, can be defined similarly as the unique basis such that $\langle Z'_\lambda, Z'_\mu \rangle_2 = 0$ with respect to $\langle \cdot, \cdot \rangle_2$ defined such that

$$\langle p_\lambda, p_\mu \rangle_2 = 2^{\ell(\lambda)} z_\lambda \delta_{\lambda,\mu},$$

and the transition matrix $M(Z', m)$ is upper-unitriangular:

$$Z'_\lambda := m_\lambda + \sum_{\mu < \lambda} K^{(2)}_{\lambda,\mu} m_\mu.$$

Continuing in this manner, Jack [39] showed that for any $\alpha \in \mathbb{R}$, there is a unique basis $\{P^{(\alpha)}_\lambda\}_{\lambda \vdash n}$ for Λ^n such that $\langle P^{(\alpha)}_\lambda, P^{(\alpha)}_\mu \rangle_\alpha = 0$ with respect to the inner product $\langle \cdot, \cdot \rangle_\alpha$ satisfying

$$\langle p_\lambda, p_\mu \rangle_\alpha = \alpha^{\ell(\lambda)} z_\lambda \delta_{\lambda,\mu}$$

and the transition matrix $M(P^{(\alpha)}, m)$ is upper-unitriangular:

$$P^{(\alpha)}_\lambda := m_\lambda + \sum_{\mu < \lambda} K^{(\alpha)}_{\lambda,\mu} m_\mu.$$

We call $\{P^{(\alpha)}_\lambda\}_{\lambda \vdash n}$ the *normalized Jack polynomials*.

We now define what turns out to be a more convenient basis $\{J_\lambda^{(\alpha)}\}_{\lambda\vdash n}$ that we call *the Jack symmetric functions*:

$$J_\lambda^{(\alpha)} := c_\alpha(\lambda)P_\lambda, \quad c_\alpha(\lambda) := \prod_{s\in\lambda}(\alpha a_\lambda(s) + l_\lambda(s) + 1),$$

where the *arm length* $a_\lambda(s)$ in the number of cells to the right of s in the same row and the *leg length* $l_\lambda(s)$ is the number of cells below s in the same column. If we specialize the Jack symmetric functions at $\alpha = 1$, we obtain a normalization of the Schur symmetric functions that we denote as $\{S_\lambda\}_{\lambda\vdash n}$. Specializing at $\alpha = 2$ recovers *the zonal polynomials* $\{Z_\lambda\}_{\lambda\vdash n}$.

We close this section with an unpublished result of Stanley, that the coefficients of $\{S_\lambda\}_{\lambda\vdash n}$ and $\{Z_\lambda\}_{\lambda\vdash n}$ when expressed in the power sum basis are precisely the entries of the character tables P' and P of $\mathcal{A}'_n$ and $\mathcal{A}_n$.

2.2.1 Theorem. [53, 57] *Let P be the character table of $\mathcal{A}_n$ and let P' be the character table of $\mathcal{A}'_n$. Then $P' = M(S,p)$ and $P = M(Z,p)$.*

Without further ado, we give a unified analysis of the eigenvalues of $M'_{k,n}$ and $M_{k,n}$ using Jack symmetric functions.

2.3 Eigenvalues and Jack Symmetric Functions

Let us start by expressing the Jack symmetric functions in the power sum basis:

$$J_\lambda^{(\alpha)}(\overline{x}) = \sum_{\mu\vdash n} \psi_\lambda^\mu p_\mu(\overline{x}) \quad \text{ for all } \lambda\vdash n.$$

By Theorem 2.2.1, if $\alpha = 1$, then ψ_λ^μ is the λ-eigenvalue of the μ-associate of $\mathcal{A}'_n$, and if $\alpha = 2$, then ψ_λ^μ is the λ-eigenvalue of the μ-associate of $\mathcal{A}_n$. If we evaluate $J_\lambda^{(\alpha)}(\overline{x})$ at

$$\overline{x} = \underbrace{1, 1, \cdots, 1}_{n \text{ times}}, 0, 0, \cdots =: 1^n,$$

$\mathbf{0}$	1	2	3	4	5	$\cdots$
-1	$\mathbf{0}$	1	2	3	4	$\cdots$
-2	-1	$\mathbf{0}$	1	2	3	$\cdots$
-3	-2	-1	$\mathbf{0}$	1	2	$\cdots$
-4	-3	-2	-1	$\mathbf{0}$	1	$\cdots$
-5	-4	-3	-2	-1	$\mathbf{0}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$\mathbf{0}$	2	4	6	8	10	12	$\cdots$
-1	1	3	5	7	9	11	$\cdots$
-2	$\mathbf{0}$	2	4	6	8	10	$\cdots$
-3	-1	1	3	5	7	9	$\cdots$
-4	-2	$\mathbf{0}$	2	4	6	8	$\cdots$
-5	-3	-1	1	3	5	7	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Figure 2.2: The grids above illustrate α-contents for $\alpha = 1, 2$ and offers a combinatorial interpretation of the factors that arise in Equation (2.3.1). The cells in bold lettering belong to the ray $R^{(\alpha)}$.

then due to the fact that $p_\mu(1^n) = n^{\ell(\mu)}$, we obtain a polynomial in n:

$$\begin{aligned} J_\lambda^{(\alpha)}(1^n) &= \sum_{\mu \vdash n} \psi_\lambda^\mu n^{\ell(\mu)} \\ &= \sum_{i=1}^{n} n^i \left(\sum_{\substack{\mu \vdash n \\ \ell(\mu)=i}} \psi_\lambda^\mu \right). \end{aligned}$$

Since $M_{k,n}$ and $M'_{k,n}$ lie in the Bose-Mesner algebras of $\mathcal{A}_n$ and $\mathcal{A}'_n$ respectively, the λ-eigenvalue of $M_{k,n}$ and $M'_{k,n}$ equals the sum of the λ-eigenvalues of its constituent associates; therefore, the coefficient of n^i on the right-hand side is the precisely the λ-eigenvalue of the $M'_{i,n}$ and $M_{i,n}$ for $\alpha = 1$ and $\alpha = 2$. A result of Stanley gives another expression for the 1^n specialization.

2.3.1 Theorem. [70] *Let $\lambda \vdash n$ and $\alpha \in \mathbb{R}$. Then*

$$J_\lambda^{(\alpha)}(1^n) = \prod_{(i,j)\in\lambda} (n - (i-1) + \alpha(j-1))$$

where $(i, j) \in \lambda$ is the cell in the ith row and jth column of λ.

Since both expressions of $J_\lambda^{(\alpha)}(1^n)$ are polynomials in n, equating coefficients of n^k gives

$$\sum_{\substack{S\subseteq\lambda \\ |S|=k}} \prod_{(i,j)\in\lambda\setminus S} (\alpha(j-1) - (i-1)) = \sum_{\substack{\mu\vdash n \\ \ell(\mu)=k}} \psi_\lambda^\mu. \tag{2.3.1}$$

For any cell (i,j) of a Ferrers diagram, the number $\alpha(j-i)-(i-1)$ is the *α-content* of (i,j) (see Figure 2.2). For any shape $\lambda \vdash n$ and $\alpha \in \mathbb{Z}$, we may label its Ferrers diagram such that the cell (i,j) is assigned the integer $\alpha(j-i)-(i-1)$, which we call the *α-content tableau* of λ.

Equation (2.3.1) already gives a combinatorial way of computing the λ-eigenvalue of $M'_{k,n}$ and $M_{k,n}$ if we set $\alpha = 1$ and $\alpha = 2$. In particular, the sum ranges over all ways of removing k cells from the α-content tableau of λ, and each term is the product of the α-content tableau of λ excluding those k cells.

We can further simplify Equation (2.3.1) by considering the ray

$$R^{(\alpha)} := \{(i,j) \in \mathbb{N}_+ \times \mathbb{N}_+ : \alpha(j-1)-(i-1) = 0\}$$

as illustrated in Figure 2.2. Since $j = 1 + (i-1)/\alpha$ and $j \in \mathbb{N}_+$, we have

$$R^{(\alpha)} = \{(i, 1+(i-1)/\alpha) : i = 1, 1+\alpha, 1+2\alpha, \cdots\}.$$

Let $R_k^{(\alpha)} := \{(i,j) \in R^{(\alpha)} : j \leq k\}$. We can now write our expression as

$$\begin{aligned} J_\lambda^{(\alpha)}(1^n) &= \prod_{(i,j)\in R_k^{(\alpha)}\cap\lambda} (n-(i-1)+\alpha(j-1)) \prod_{(i,j)\in\lambda\setminus R_k^{(\alpha)}} (n-(i-1)+\alpha(j-1)) \\ &= n^{|R_k^{(\alpha)}\cap\lambda|} \prod_{(i,j)\in\lambda\setminus R_k^{(\alpha)}} (n-(i-1)+\alpha(j-1)). \end{aligned}$$

For any $\alpha \in \mathbb{N}_+$, define the *α-trace* of a shape λ to be $\mathrm{tr}_\alpha(\lambda) := |R_k^{(\alpha)} \cap \lambda|$, and henceforth we let $\ell := k - \mathrm{tr}_\alpha(\lambda)$. Equating coefficients gives us

$$[n^k]\ J_\lambda^{(\alpha)}(1^n) = \sum_{\substack{S\subseteq\lambda\setminus R_k^{(\alpha)} \\ |S|=\ell}} \prod_{(i,j)\in\lambda\setminus(R_k^{(\alpha)}\cup S)} (\alpha(j-1)-(i-1)). \tag{2.3.2}$$

This combinatorial expression is nice enough to do nontrivial calculations with some ease, as demonstrated in Figure 2.3. In particular, for $\ell = 0$ the specializations $\alpha = 1, 2$ give rather elegant expressions for the λ-eigenvalue of $M'_{k,n}$ and $M_{k,n}$, which we state as theorems.

2.3.2 Theorem. *For any $k \leq \sqrt{n}$, let $\lambda \vdash n$ be a shape such that $(k^k) \subseteq \lambda$. Then the eigenvalue $\theta'_{k,\lambda}$ corresponding to the λ-eigenspace of $M'_{k,n}$ is*

$$\theta'_{k,\lambda} = \prod_{\substack{(i,j)\in\lambda \\ (i,j)\neq(1,1),(2,2),\cdots,(k,k)}} (j-i).$$

Moreover, if $(k+1)^{k+1} \subseteq \lambda$*, then* $\theta'_{k,\lambda} = 0$.

2.3.3 Theorem. *For any* k *such that* $2k^2 - k \leq n$*, let* $\lambda \vdash n$ *be a partition such that* $(k^{2k-1}) \subseteq \lambda$*. Then the eigenvalue* $\theta_{k,\lambda}$ *corresponding to the* λ*-eigenspace of* $M_{k,n}$ *is*

$$\theta_{k,\lambda} = \prod_{\substack{(i,j)\in\lambda \\ (i,j)\neq(1,1),(3,2),\cdots,(2k-1,k)}} (2j-i-1).$$

Moreover, if $(k+1)^{2k+1} \subseteq \lambda$*, then* $\theta_{k,\lambda} = 0$.

The case where $k = 1$ is especially important due to vacuous properties of integer partitions, namely, that every nonempty shape λ contains the cell $(1,1)$ and the only $\lambda \vdash n$ with precisely one part is (n). These facts imply a simple formula for the eigenvalues of the (n)-associate of $\mathcal{A}'_n$ and $\mathcal{A}_n$. The former is a folklore result of the representation theory of the symmetric group and the latter happens to be an unpublished result of Diaconis and Lander.

2.3.4 Corollary. *The eigenvalue corresponding to the* λ*-eigenspace of* $M'_{1,n}$ *can be written as*

$$\theta'_{1,\lambda} = \prod_{\substack{(i,j)\in\lambda \\ (i,j)\neq(1,1)}} (j-i).$$

2.3.5 Corollary (Diaconis and Lander [53])**.** *The eigenvalue corresponding to the* λ*-eigenspace of* $M_{1,n}$ *can be written as*

$$\theta_{1,\lambda} = \prod_{\substack{(i,j)\in\lambda \\ (i,j)\neq(1,1)}} (2j-i-1).$$

For $k \geq 2$ there are of course still many shapes λ such that $\mathrm{tr}_\alpha(\lambda) \neq k$, i.e., $(k^k) \not\subseteq \lambda$ and $(k^{2k-1}) \not\subseteq \lambda$ for $\alpha = 1$ and $\alpha = 2$. We now derive a recursive expression for determining eigenvalues corresponding to these shapes.

Let us first consider the partitions (n) and (1^n), which will give us a close encounter with the *Stirling numbers of the first kind* $(-1)^{n-k}\left[{n \atop k}\right]$, where the *unsigned Stirling number of the first kind* $\left[{n \atop k}\right]$ denotes the number of permutations of S_n that can be factored into k disjoint cycles. In a more graph-theoretic language, the unsigned Stirling number $\left[{n \atop k}\right]$ counts the number of perfect matchings $m \in \mathcal{M}_{n,n}$ such that $m' \cup m$ has precisely k connected

	1	2	3	4	5	⋯
−1		1	2	3	4	⋯
−2	−1		1	2	3	⋯
−3	−2	−1	**0**	1	2	⋯
−4	−3	−2	−1	**0**	1	⋯
−5	−4	−3	−2	−1	**0**	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋱

	2	4	6	8	10	12	⋯
−1	1	3	5	7	9	11	⋯
−2		2	4	6	8	10	⋯
−3	−1	1	3	5	7	9	⋯
−4	−2	**0**	2	4	6	8	⋯
−5	−3	−1	1	3	5	7	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Figure 2.3: Since $\lambda = (6,5,4,3,2,1) \vdash 21$ has $\mathrm{tr}_1(\lambda) = 3$ and $\mathrm{tr}_2(\lambda) = 2$, the λ-eigenvalue of $M'_{3,21}$ is $-(5!)^2 \cdot (3!)^2$. The λ-eigenvalue of $M'_{2,21}$ is 0 since $\mathrm{tr}_1(\lambda) > 2$, whereas the λ-eigenvalue of $M_{2,21}$ is $-5! \cdot 10!! \cdot 7!! \cdot 2^2 \cdot 4$.

components for some fixed $m' \in \mathcal{M}_{n,n}$. The rows of $M'_{k,n}$ sum to $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ by definition. For any fixed perfect matching $m' \in \mathcal{M}_{2n}$, let $\left[\left[\begin{smallmatrix} 2n \\ k \end{smallmatrix}\right]\right]$ be the number of perfect matchings $m \in \mathcal{M}_{2n}$ such that $m' \cup m$ has exactly k connected components. The rows of $M_{k,n}$ sum to $\left[\left[\begin{smallmatrix} 2n \\ k \end{smallmatrix}\right]\right]$ by definition.

2.3.6 Proposition. *For all $n \in \mathbb{N}$, we have*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \theta'_{k,(n)} = (n-1)!e_{k-1}(1, 1/2, \cdots, 1/(n-1), 0, 0, \cdots), \text{ and}$$

$$\left[\begin{bmatrix} 2n \\ k \end{bmatrix}\right] = \theta_{k,(n)} = (2n-2)!!e_{k-1}(1/2, 1/4, \cdots, 1/(2n-2), 0, 0, \cdots).$$

Proof. By Equation 2.3.2 and setting $\alpha = 1, 2$, we have

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix} = \theta'_{k,(n)} &= \sum_{\substack{S \subseteq \lambda \setminus \{(1,1)\} \\ |S| = k-1}} \prod_{(i,j) \in \lambda \setminus S} (j-1) \\ &= (n-1)!e_{k-1}(1, 1/2, \cdots, 1/(n-1), 0, 0, \cdots), \text{ and} \end{aligned}$$

$$\begin{aligned} \left[\begin{bmatrix} 2n \\ k \end{bmatrix}\right] = \theta_{k,(n)} &= \sum_{\substack{S \subseteq \lambda \setminus \{(1,1)\} \\ |S| = k-1}} \prod_{(i,j) \in \lambda \setminus S} 2(j-1) \\ &= (2n-2)!!\ e_{k-1}(1/2, 1/4, \cdots, 1/(2n-2), 0, 0, \cdots). \end{aligned}$$

□

For arbitrary k these numbers do not appear in the OEIS database [69], but they are possibly known. Using these counts along with Proposition 2.1.1, we can compute some interesting probabilities. For example, if we fix a perfect matching $m' \in \mathcal{M}_{2n}$, and draw another perfect matching $m \in \mathcal{M}_{2n}$ uniformly at random, then the probability that $m' \cup m$ is a Hamiltonian cycle is

$$\frac{\left[\left[{2n \atop 1}\right]\right]}{(2n-1)!!} \approx \frac{1}{\sqrt{\pi n}}.$$

One can generalize these results by taking $\alpha \geq 3$, but we are unaware of any combinatorial interpretation.

2.3.7 Proposition.

$$\theta'_{k,(1^n)} = (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} = \theta_{k,(1^n)}.$$

Proof. By Equation 2.3.2 and setting $\alpha = 1, 2$, we have

$$\begin{aligned} \theta'_{k,(1^n)} &= \sum_{\substack{S \subseteq \lambda \backslash \{(1,1)\} \\ |S| = k-1}} \prod_{(i,j) \in \lambda \backslash S} -(i-1) \\ &= (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}, \text{ and} \\ \theta_{k,(1^n)} &= \sum_{\substack{S \subseteq \lambda \backslash \{(1,1)\} \\ |S| = k-1}} \prod_{(i,j) \in \lambda \backslash S} -(i-1) \\ &= (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}. \end{aligned}$$

□

Let us now consider a slightly more general class of shapes of the form $\lambda = (n-l, 1^l) \vdash n$ such that $0 \leq l \leq n$. These shapes are known as *hooks* and they too admit simple content tableaux, as seen below with $l' := n - l - 1$:

	α	2α	$\cdots$	$l'\alpha$
-1				
-2				
$\vdots$				
$-l$				

.

2.3.8 Lemma. *If* $\lambda = (n-l, 1^l) \vdash n$, *then*

$$\theta'_{k,\lambda} = \sum_{i=1}^{k} (-1)^{l+1-i} \begin{bmatrix} l+1 \\ i \end{bmatrix} \begin{bmatrix} n-l \\ k+1-i \end{bmatrix}.$$

Proof. Let $\hat{\lambda} = \lambda \setminus \{(1,1)\}$ and set $\alpha = 1$. Since λ is a hook, by Equation (2.3.2) we have

$$\begin{aligned} \theta'_{k,\lambda} &= \sum_{\substack{S_1 \subseteq \hat{\lambda}_1, S_2 \subseteq \hat{\lambda} \setminus \hat{\lambda}_1 \\ |S_1|+|S_2|=k-1}} \prod_{(i,j)\in \lambda_1 \setminus S_1} (j-i) \prod_{(i,j)\in \lambda \setminus (\lambda_1 \cup S_2)} (j-i) \\ &= \sum_{k'=0}^{k-1} \left(\sum_{\substack{S_1 \subseteq \hat{\lambda}_1 \\ |S_1| = k'}} \prod_{(i,j)\in \lambda_1 \setminus S_1} (j-i) \sum_{\substack{S_2 \subseteq \hat{\lambda} \setminus \hat{\lambda}_1 \\ |S_2| = k-1-k'}} \prod_{(i,j)\in \lambda \setminus (\lambda_1 \cup S_2)} (j-i) \right) \\ &= \sum_{i=1}^{k} (-1)^{l+1-i} \begin{bmatrix} l+1 \\ i \end{bmatrix} \begin{bmatrix} n-l \\ k+1-i \end{bmatrix} \end{aligned}$$

where the last equality holds by Proposition 2.3.7 and Proposition 2.3.6. □

This lemma will serve as the base case of our recursive formula for $\theta'_{k,\lambda}$. Because the α-contents for $\alpha = 2$ do not have transpose symmetry (see Figure 2.2), we will also need to consider the so-called *near-hooks* of the form $(n-m-l, m, 1^l) \vdash n$ in order to arrive at a recursive expression for $\theta_{k,\lambda}$. The proof of the lemma below is similar to the one given above.

2.3.9 Lemma. *If* $\lambda = (n-l, 1^l) \vdash n$ *and* $\mu = (n-m-l, m, 1^l) \vdash n$, *then*

$$\theta_{k,\lambda} = \sum_{i=1}^{k} (-1)^{l+1-i} \begin{bmatrix} l+1 \\ i \end{bmatrix} \left[\begin{bmatrix} 2n-2l \\ k+1-i \end{bmatrix}\right], \ \textit{and}$$

$$\theta_{k,\mu} = \sum_{j=0}^{k} \theta_{k-j,(n-m-l,1^{l+1})} (2m-3)!! e_j(1, 1/3, 1/5, \cdots, 1/(2m-3), 0, 0, \cdots).$$

We are now in a position to give a recursive formula for the eigenvalues of $M'_{k,n}$ and $M_{k,n}$. For any shape λ, let ζ be the largest hook shape that is contained in λ, equivalently, the union of the cells in its first row and first column. A *borderstrip* is a connected skew shape that contains no 2×2 square, and let ξ be the largest borderstrip contained in λ. In the illustration below, the cells in bold lettering compose $\zeta = (5, 1^5)$ whereas the yellow cells compose ξ for the shape $\lambda = (5, 4, 4, 2, 1, 1)$:

0	**1**	**2**	**3**	**4**
−1	0	1	2	
−2	−1	0	1	
−3	−2			
−4				
−5				

.

2.3.10 Theorem. *Let $\theta'_{k,\lambda}$ denote the eigenvalue corresponding to the λ-eigenspace of $M'_{k,|\lambda|}$. Then*

$$\theta'_{k,\lambda} = \sum_{j=1}^{k} \theta'_{j,\zeta}\ \theta'_{k-j+1,\lambda\setminus\xi}.$$

Proof. We proceed by induction on $\mathrm{tr}_1(\lambda)$. If λ is a hook, then $\mathrm{tr}_1 = 1$ and the claim holds by Lemma 2.3.8. Assume the claim holds for all shapes $\lambda \vdash n$ such that $\mathrm{tr}_1(\lambda) < k'$.

Let ζ be the largest hook shape contained in λ and let ξ be the largest borderstrip contained in λ. Note that $c_1(\lambda/\zeta) = c_1(\lambda \setminus \xi)$ since ζ and ξ are composed of the same number of cells and their cells compose monotonically increasing paths of the Ferrers diagram of λ.

Let $\hat{\lambda} = \lambda \setminus \{(1,1)\}$. By Equation (2.3.2) we have

$$\begin{aligned}
\theta'_{k,\lambda} &= \sum_{\substack{S_1 \subseteq \xi, S_2 \subseteq \hat{\lambda}\setminus\xi \\ |S_1|+|S_2|=k-1}} \prod_{(i,j)\in\xi\setminus S_1} (j-i) \prod_{(i,j)\in\lambda\setminus(\xi\cup S_2)} (j-i) \\
&= \sum_{k'=0}^{k-1} \left(\sum_{\substack{S_1\subseteq\xi \\ |S_1|=k'}} \prod_{(i,j)\in\xi\setminus S_1} (j-i) \sum_{\substack{S_2\subseteq\hat{\lambda}\setminus\xi \\ |S_2|=k-1-k'}} \prod_{(i,j)\in\lambda\setminus(\xi\cup S_2)} (j-i) \right) \\
&= \sum_{j=1}^{k} \theta'_{j,\zeta}\ \theta'_{k-j+1,\lambda\setminus\xi},
\end{aligned}$$

where the last equality follows by Lemma 2.3.8 and induction. $\square$

When $\alpha = 2$, the content tableaux do not have transpose symmetry, so we must remove shapes of the form $(n-m-l, m, 1^l)$ to arrive at a recursive formula. Arguing similarly as above using Lemma 2.3.9 and Lemma 2.3.8 as the base case, we have the following.

2.3.11 Theorem. *Let $\theta_{k,\lambda}$ denote the eigenvalue corresponding to the λ-eigenspace of $M_{k,|\lambda|}$. Let ν be the largest shape of the form $(n-m-l, m, 1^l)$ that is contained in λ. Then*

$$\theta_{k,\lambda} = \sum_{j=1}^{k} \theta_{j,\nu}\ \theta_{k-j+1,\mu}$$

where μ is the shape obtained by shifting each cell of λ/ν to the left by one cell, then shifting up by two cells.

We say that a family of perfect matchings of K_{2n} is *non-Hamiltonian* if the union of any two of its members does not form a Hamiltonian cycle. Such families are independent sets of vertices in the graph of $M_{1,n}$, and in [45] it was shown that the largest non-Hamiltonian families are those of the form

$$\mathcal{F}_{ij} = \{m \in \mathcal{M}_{2n} : ij \in m\} \text{ for some } ij \in E(K_{2n}).$$

These families are often called *the canonically intersecting families of perfect matchings*, and we shall revisit them in Chapter 3. A key part of the argument was determining the minimum eigenvalue of $M_{1,n}$, which was done using the theory presented in this section. We believe that our eigenvalue formulas for $M'_{k,n}$ and $M_{k,n}$ are explicit enough to determine the minimum eigenvalue and other such statistics for $k \geq 2$, should the need to do so arise.

In their study of Brauer's centralizer algebras, Hanlon and Wales [35] determined the eigenvalues of matrices of the form

$$T_n(x) := \sum_{i=1}^{n} x^i M_{i,n}$$

remarkably without Jack symmetric functions. Their results do not seem to imply ours or vice versa since the x^i's play a significant role in their arguments.

Finally, we have only scratched the surface of the algebraic combinatorics surrounding the Stirling numbers of the first kind (see [60] for a comprehensive account). It is plausible that some of our formulas can be further simplified and may perhaps be connected to other results in the vast literature on the subject.

2.4 Ranks of Matchings Connectivity Matrices

We have observed that the eigenspaces of $M'_{k,n}$ and $M_{k,n}$ are parameterized by integer partitions of n, but we have avoided the issue of their dimension, equivalently, the ranks of the primitive idempotents of $\mathcal{A}'_n$ and $\mathcal{A}_n$. In the language of association schemes, the rank of the ith primitive idempotent E_i is the *multiplicity* of i, which we denote as m_i.

There are general formulas for deducing the multiplicities of association schemes from their character tables [4]; however, our association schemes are defined over S_n and cosets of S_{2n}, so the simplest way of determining the multiplicities m'_λ and m_λ of $\mathcal{A}'_n$ and $\mathcal{A}_n$ is by appealing to the representation theory of the symmetric group. Our proper overview of this theory is deferred to Section 4.2.1, but the following sneak preview below will suffice for now.

A *standard Young tableau* of shape $\lambda \vdash n$ is a Ferrers diagram of shape λ such that the cells are assigned distinct labels from $[n]$ that are strictly increasing along rows and strictly increasing along columns. For example, below is a standard Young tableau of shape $(5, 3, 2, 1) \vdash 11$:

1	2	5	8	9
3	6	7		
4	10			
11				

.

Recall that

$$c_1(\lambda) := \prod_{s\in\lambda}(a_\lambda(s) + l_\lambda(s) + 1) \text{ and } c_2(\lambda) := \prod_{s\in\lambda}(2a_\lambda(s) + l_\lambda(s) + 1).$$

We let f^λ denote the number of standard Young tableaux of shape λ, and a classic result in combinatorics is that $f^\lambda = n!/c_1(\lambda)$. This fact is the tip of the iceberg that f^λ is the dimension of the irreducible representation of S_n associated to $\lambda \vdash n$. In the next chapter, we will see that the eigenspaces of $\mathcal{A}'_n$ and $\mathcal{A}_n$ can be written in terms of irreducible representations of the symmetric group and that their dimensions are

$$m'_\lambda = (f^\lambda)^2 = \left(\frac{n!}{c_1(\lambda)}\right)^2 \quad \text{and} \quad m_\lambda = f^{2\lambda} = \frac{(2n)!}{c_2(\lambda)}$$

where $2\lambda := (2\lambda_1, 2\lambda_2, \cdots, 2\lambda_\ell) \vdash 2n$ for any $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell) \vdash n$.

With this combinatorial formula for the multiplicities of $\mathcal{A}'_n$ and $\mathcal{A}_n$ in hand, our expressions for the eigenvalues of $M'_{k,n}$ and $M_{k,n}$ now reveal a great deal of information about their ranks. For example, a consequence of Theorem 2.3.2 is the aforementioned result of Raz and Spieker, stated below.

2.4.1 Theorem. [61] *The rank of $M'_n = M'_{1,n}$ is*

$$\operatorname{rank} M'_n = \sum_{\substack{\lambda \vdash n \\ (2,2) \not\subseteq \lambda}} (f^\lambda)^2 = \binom{2n-2}{n-1}.$$

The summation above ranges over all the hook shapes of size n. Using the hook formula, it is easy to see that the number of standard Young tableaux of shape $(n-k, 1^k)$ is $\binom{n-1}{k}$, and so the second equality is a simple exercise.

Similarly, by Theorem 2.3.3, we now arrive at the non-bipartite analogue of Raz and Spieker's result.

2.4.2 Theorem. [14] *The rank of M_n is*

$$\operatorname{rank} M_n = \sum_{\substack{\lambda \vdash n \\ (2,2,2) \not\subseteq \lambda}} f^{2\lambda}.$$

We are not as fortunate in the non-bipartite case, as the number of standard Young tableau of shape $2\lambda \vdash 2n$ satisfying $(2,2,2) \not\subseteq \lambda$ does not seem to admit a simple combinatorial description.

Fortunately, we can still find an upper bound and a lower bound on the rank of M_n that are tight up to polynomial factors. To make this latter notion more formal, for any positive functions $f(n)$ and $g(n)$, we write $f(n) = \Theta^*(g(n))$ if there exists a rational function $h(n)$ and a positive constant $c \in \mathbb{R}$ such that

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} h(n) = c,$$

and $f(n) = O^*(g(n))$ if there exists a polynomial $h(n)$ such that the same holds. While on the subject of asymptotic notation, we say that $f(n)$ is $\omega(g(n))$ if $g(n) = o(f(n))$.

By Theorem 2.4.1, an upper bound of

$$\operatorname{rank} M_n \leq \frac{1}{2}\binom{2n}{n}\binom{2n-2}{n-1}$$

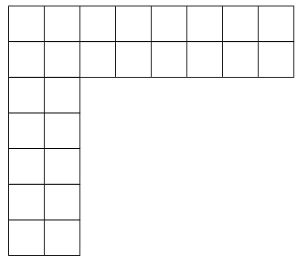


Figure 2.4: The Ferrers diagram of the domino hook $2(4,4,1^5) \vdash 26$

is immediate, as there are $\frac{1}{2}\binom{2n}{n}$ ways to partition the vertices of K_{2n} into two parts of size n. For the lower bound, we consider the following family of even shapes $2\lambda \vdash 2n$ satisfying $(2,2,2) \not\subseteq \lambda$.

We say a partition $2\lambda \vdash 2n$ is a *domino hook* if $\lambda = (k,k,1^{n-2k})$ for some $0 \le k \le n$ (see Figure 2.4). Using the Wilf-Zeilberger method [59], Regev [2] showed that the sum of the number of standard Young tableaux of domino hook shape admits an elegant count.

2.4.3 Theorem. [2] *Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ be the nth Catalan number. Then*

$$C_{n-1}C_n = \sum_{\substack{2\lambda \vdash 2n \\ 2\lambda \text{ is a domino hook}}} f^{2\lambda}.$$

This can be seen as a generalization of the well-known identity $C_n = f^{(n,n)}$ [71]. Combining inequalities, we have that

$$C_{n-1}C_n \quad \le \quad \text{rank } M_n \quad \le \quad \frac{1}{2}\binom{2n}{n}\binom{2n-2}{n-1}.$$

It is well-known that the Catalan numbers can be estimated asymptotically as

$$\lim_{n\to\infty} \frac{4^n/\sqrt{\pi}n^{3/2}}{C_n} = 1,$$

and it will be convenient to let $N := 2n$ for the remainder of this chapter. Ignoring polynomial factors in the bounds above gives us the following.

2.4.4 Theorem. [14] *The rank of $M_n = M_{1,n}$ is $\Theta^*(4^N)$.*

We were unable to find nice singly-exponential Θ^* approximations of the ranks of $M'_{k,n}$ and $M_{k,n}$ for constant k, but in lieu of these estimates, we give a simple proof that these matrices at least have *low rank*, that is, their ranks become vanishingly small as $n \to \infty$ provided $k = o(n^\epsilon)$ for any $\epsilon > 0$.

First, we observe that there are $\binom{n}{\lambda_1}$ ways of choosing which numbers get assigned to the first row of a standard Young tableau of shape $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_d)$ and at most $f^{(\lambda_2, \cdots, \lambda_d)}$ ways of completing it to a standard Young tableau, thus

$$f^\lambda \leq \binom{n}{\lambda_1} f^{(\lambda_2, \cdots, \lambda_d)} \leq 2^{dn}.$$

If $\mathrm{tr}_1(\lambda) = d$, then the foregoing shows rather crudely that $f^\lambda \leq 4^{dn}$.

Now let D be the number of shapes $\lambda \vdash n$ such that $\mathrm{tr}_1(\lambda) = d$. To see that

$$D \leq n^{2d},$$

consider the shape below the first d rows of λ. There are no more than n^d ways of choosing the columns of this shape, and the same argument applies if we consider the shape to the right of the first d columns of λ. We deduce that the number of shapes $\lambda \vdash n$ such that $\mathrm{tr}_1(\lambda) \leq d$ is $O(n^{2d})$.

If $k = o(n^\epsilon)$ for any $\epsilon > 0$, then we have

$$\text{rank } M'_{k,n} \leq \sum_{\substack{\lambda \vdash n \\ \mathrm{tr}_1(\lambda) \leq k}} m'_\lambda \leq \sum_{\substack{\lambda \vdash n \\ \mathrm{tr}_1(\lambda) \leq k}} 16^{kn} = O(n^{2k}) 16^{kn} = o(n!),$$

Finally, one can argue similarly that the number of shapes of size n such that $\mathrm{tr}_2(\lambda) \leq d$ is $O(n^{3d})$ and that

$$\text{rank } M_{k,n} \leq \sum_{\substack{\lambda \vdash n \\ \mathrm{tr}_2(\lambda) \leq k}} m_\lambda = \sum_{\substack{\lambda \vdash n \\ \mathrm{tr}_2(\lambda) \leq k}} f^{2\lambda} = o((2n-1)!!),$$

which proves that

$$\lim_{n \to \infty} \frac{\text{rank } M'_{k,n}}{n!} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{\text{rank } M_{k,n}}{(2n-1)!!} = 0$$

provided $k = o(n^\epsilon)$ for all $\epsilon > 0$, as desired.

We have given a unified proof that rank $M'_n = \Theta^*(4^{N/2})$ and rank $M_n = \Theta^*(4^N)$, which shows that the jump from bipartite to non-bipartite affords roughly a quadratic increase in rank. A surprising consequence of this quadratic blowup is a lower bound on the complexity of counting Hamiltonian cycles that is *tight* (up to polynomial factors) assuming the Strong Exponential-Time Hypothesis. In the next section, we briefly survey this result, which is joint work with Radu Curticapean and Jesper Nederlof that appeared in the *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms.*

2.5 Counting Hamiltonian Cycles

The design and analysis of polynomial-time algorithms held an important role in the early years of theoretical computer science, as it painted a monochromatic picture of the complexity landscape in strokes of tractability and intractability. As one might guess, the field of *fine-grained complexity* takes a more nuanced view of complexity, i.e., distinguishing which classes of problems in P admit quadratic vs. cubic algorithms, finding faster exponential algorithms for hard problems, and showing conditional lower bounds for problems assuming widely-believed but difficult-to-prove conjectures in complexity theory.

One such conjecture is the *Strong Exponential-Time Hypothesis* (SETH), which asserts that for all $\epsilon > 0$ there exists a k such that k-CNF-SAT cannot be solved in $O^*((2-\epsilon)^n)$ time. It has been shown that many known exact algorithms for NP-hard optimization problems and #P-hard counting problems are optimal assuming SETH [15, 47]. Along these lines, in joint work with Radu Curticapean and Jesper Nederlof, we settle an open question on the parameterized complexity of counting Hamiltonian cycles assuming SETH.

2.5.1 Theorem. [14] *Assuming* SETH, *for any* $\epsilon > 0$, *there is no algorithm for counting the number of Hamiltonian cycles of a graph* G *in time* $O^*((6-\epsilon)^{\mathrm{tw}})$ *where* tw *denotes the treewidth of* G.

This bound is tight due to a $O^*(6^{\mathrm{tw}})$-time algorithm of Bodlaender et al [9]. Assuming SETH, it was recently observed that all of the NP-complete problems whose Θ^* complexity was known had the same Θ^* complexity for its corresponding #P counting problem, which prompted the following question of Holger Dell [16]: if we assume SETH, is deciding just as hard as counting for NP-complete problems? Theorem 2.5.1 gives a negative answer to Dell's question, as an $O^*((2+\sqrt{2})^{\mathrm{tw}})$-time algorithm exists for deciding Hamiltonicity [15].

The proof of Theorem 2.5.1 proceeds by a long and technical reduction, and a complete description of the reduction would take us too far astray from the algebraic combinatorics of matchings. Instead, we offer a sketch that highlights the main points and illustrates how our estimate on the rank of M_n plays a central role, referring the interested reader to [14] for a complete proof.

2.5.1 A Sketch of the Reduction

For a precise definition of *treewidth* we refer the reader to [19], but it is essentially the size of a hierarchy of vertex-separators of size k, a so-called *tree-decomposition*, that allows problems to be solved more efficiently through the use of dynamic programming. This problem-solving paradigm is no stranger to Hamiltonicity [36, 6, 9], and it worthwhile to understand the efficacy of dynamic programming approaches for finding and counting Hamiltonian cycles under various hypotheses in complexity.

For any $k \in \mathbb{N}$, let a *k-boundaried graph* be a simple labeled graph with k distinguished *boundary vertices* $B \subseteq V$ that are labeled $1, \cdots, k$. We say that a *fingerprint* of a k-boundaried graph is a pair (d, m), where $d \in \{0, 1, 2\}^k$ assigns 0, 1 or 2 to each boundary vertex, and m is a perfect matching on the boundary vertices to which d assigns 1. Here, we consider $\emptyset$ to be a perfect matching of K_0 and we define $M_0 := 1$.

Fingerprints are essentially the states one would use in the natural dynamic programming routine for counting Hamiltonian cycles using a tree-decomposition; they describe the behavior of a Hamiltonian cycle on a given side of a separation. We say that a pair of fingerprints (d, m) and (d', m') on B *combines* if $d_v + d'_v = 2$ for every $v \in B$ and either $m \cup m'$ forms a single cycle or $m \cup m'$ is empty. The *fingerprint matrix* H_k is a binary matrix indexed by fingerprints such that $[H_k]_{f,f'} = 1$ if fingerprints f and f' combine; otherwise, $[H_k]_{f,f'} = 0$. Low-rank fingerprints matrices have been exploited to speed up the standard treewidth-based dynamic programming routine for a variety of problems, producing several algorithms for solving NP-hard problems that are singly-exponential in treewidth [9].

If (d, m) is a fingerprint that combines with another fingerprint (d', m'), then $d + d' = (2, 2, \cdots, 2)$ which implies that

$$d'_i = \begin{cases} d_i + 2 & \text{if } d_i = 0, \\ d_i & \text{if } d_i = 1, \text{ and} \\ d_i - 2 & \text{if } d_i = 2 \end{cases}$$

for all $1 \leq i \leq k$. This gives a bijection on the set of vectors $\{0, 1, 2\}^k$ that have an even number of ones, and so we may write H_k as a block-anti-diagonal matrix where the blocks correspond to ordered pairs of vectors $d, d' \in \{0, 1, 2\}^k$ that have an even number of ones, and are sorted in lexicographical order:

$$H_k = \begin{bmatrix} 0 & 0 & \cdots & 0 & H_{\substack{00\cdots00 \\ 22\cdots22}} \\ 0 & 0 & \cdots & H_{\substack{00\cdots11 \\ 22\cdots11}} & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & H_{\substack{22\cdots11 \\ 00\cdots11}} & \cdots & 0 & 0 \\ H_{\substack{22\cdots22 \\ 00\cdots00}} & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

By our choice of indexing, each block is indexed by the set of all perfect matchings on the boundary vertices v such that $d_v = 1$. If H is a nonzero block indexed by d, d' such that d and d' each have i ones, then $H \cong M_i$. There are $\binom{k}{i}$ nonzero blocks such that d and d' each have i ones there are 2^{k-i} ways of assigning zeros and twos to the remaining vertices. Since H_k is block-anti-diagonal, we have

$$\text{rank } H_k = \sum_{i \text{ even}}^{k} \binom{k}{i} 2^{k-i} \text{ rank } M_i. \tag{2.5.1}$$

Applying Theorem 2.4.2 to Equation (2.5.1) gives an exact formula for the rank of H_k, which after a straightforward application of Theorem 2.4.4 and the Binomial Theorem gives us

$$\text{rank } H_k = \Theta^*(6^k).$$

At this point, the matchings connectivity matrix leaves the scene and the reduction begins, which we now briefly sketch.

A *CNF-SAT formula* φ is a Boolean expression with n variables and m clauses $C_1, C_2, \cdots, C_m$ of the form $\wedge_{i=0}^{m} C_m$ where each clause C_i is a disjunction of literals x_j or negated literals $\neg x_j$. Given a CNF-SAT formula φ, the reduction produces a graph G with treewidth $\mathrm{tw} \leq n/\log_2(6)$ that "encodes" φ. This encoding is done so that the 2^n assignments of the variables of φ are represented by fingerprints that are linearly independent in the matrix H_k, and G is constructed so that the number of partial solutions associated with a given fingerprint is zero if the fingerprint encodes an assignment not satisfying φ, and a fixed positive quantity (depending on the fingerprint) otherwise. As with most reductions, the construction of G involves a substantial amount of both standard and novel gadgetry, but it also depends on the rank of the fingerprints matrix. The fact that its rank is essentially 6^k accounts for the $c/\log_2(6)$ upperbound on the treewidth of G.

The crux of the reduction is that these graphs which encode CNF-SAT formulas have treewidth no greater than $n/\log_2(6)$, so a $O^*((6-\epsilon)^{\mathrm{tw}})$-time algorithm for counting Hamiltonian cycles would refute SETH, since there would be an $\epsilon > 0$ such that k-CNF-SAT can be solved in $O^*((2-\epsilon)^n)$ time for all k.

We note that there are other SETH hardness results for #P counting problems that involve reductions similar to the one sketched here [47, 15], but what is most novel about this reduction is that it relies only on the rank of a fingerprints matrix rather than certain combinatorial properties and explicit factorizations of fingerprints matrices. This dependence on a number rather than a structure was crucial since we were not able to find such a rank factorization of M_n over the reals that had the right combinatorial properties.

2.5.2 Connection Matrices

As a brief aside, we note that the matrices M_n and H_n are also closely related to a class of so-called *connection matrices* introduced by Freedman, Lovász, and Schrijver [29]. These matrices too are indexed by k-boundaried graphs, but we do not assume they have a fixed size. More precisely, the index set of these matrices is the infinite set $\mathcal{I}$ of all ordered pairs (G, B) where $G = (V, E)$ is a labeled graph and $B \subseteq V$ is a distinguished set of $k \leq |V|$ boundary vertices. Two k-boundaried graphs G and G' can be glued together, yielding a multigraph $G \oplus G'$ by taking the disjoint union of G and G' and identifying vertices with the same label.

The *kth connection matrix* C_k with respect to some graph parameter function $f : G \to \mathbb{R}$ is an $\mathcal{I} \times \mathcal{I}$ matrix defined such that $[C_k]_{G,G'} = f(G \oplus G')$. Let C_k^{ham} be the kth connection matrix obtained by setting f to be the graph parameter function that counts the number of Hamiltonian cycles in $G \oplus G'$. In [14], it was shown that C_k^{ham}, although infinite, has the same rank as H_k.

The ranks of connection matrices are closely related to graph-theoretic, algorithmic, and model-theoretic properties of graph parameters [29, 51]. Lovász was able to determine the exact rank of many important classes of connection matrices [49], but he was unable to find an exact expression for the rank of C_k^{ham} [49, Ex. 2.6]. We speculate this difficulty is related to our inability to find a nice formula for the rank of M_n. Equation (2.5.1) along with Theorem 2.4.2 solves Lovász's problem, though it is not as clean as one would hope (see [14] for more discussion).

2.6 Concluding Remarks and Open Questions

We showed that the ranks of $M'_{k,n}$ and $M_{k,n}$ become vanishing small provided $k = o(n^\epsilon)$ for all $\epsilon > 0$, but it would be interesting to refine this result to an approximation of the rank that is tight up to polynomial factors. By modifying the reduction of [14], such approximations in all likelihood would give SETH hardness results for counting types of *restricted 2-matchings* [66], in our case, spanning subgraphs of maximum degree 2 with exactly k connected components.

Another interesting feature of the reduction in [14] is that it can also show SETH-based lower bounds for counting Hamiltonian cycles modulo p provided the p-rank of the fingerprints matrix is known. We may then ask, assuming SETH, what is the optimal constant c_p such that counting Hamiltonian cycles modulo p can be solved in time $O^*(c_p^{\text{tw}})$ but not $O^*((c_p - \epsilon)^{\text{tw}})$? This leads one to study the p-rank of M_n for constant primes $p \geq 3$. As mentioned before, the 2-rank of M_n was determined in [15] by combinatorial means, and in [14] it is shown that the 2-rank of M_n is less than the 3-rank of M_n, but not much else is known. What would be interesting is if the modular representation theory of the symmetric group could be utilized to determine the ranks of these matrices over fields of odd prime characteristic.

Complexity aside, it is natural to wonder whether other integer specializations of α exist for which the evaluation $J_\lambda^{(\alpha)}(1^n)$ lists the eigenvalues of a graph. Such graphs are unlikely to stem from group-based association schemes, since with the exception of $\alpha = 1/2, 1, 2$ no specialization of α is known to have a representation-theoretic model [34, 70], but we entertain this curiosity for $n = 3$. In this case, such a graph would have three eigenvalues, and would therefore be *strongly regular* [32]. Because no two partitions of three have the same number of parts, the evaluation $J_\lambda^{(\alpha)}(1^3)$ recovers $J_\lambda^{(\alpha)}$ expressed in the power sum basis for all $\lambda \vdash 3$. The corresponding transition matrix can be written as

$$P^{(\alpha)} = \begin{bmatrix} 2\alpha^2 & 3\alpha & 1 \\ -\alpha & (\alpha-1) & 1 \\ 2 & -3 & 1 \end{bmatrix}.$$

A computational experiment shows that the only known strongly regular graphs in the strongly regular graph database of SAGE [72] whose set of eigenvalues is a column of $P^{(\alpha)}$ are

1. $\alpha = 1$: the complete bipartite graph $K_{3,3}$,
2. $\alpha = 2$: *the generalized quadrangle* $GQ(2,2)$,
3. $\alpha = 4$: *the generalized quadrangle* $GQ(4,2)$,
4. $\alpha = 10$: *the Cameron Graph*,

and of course their complements. It would be interesting to know if this numerology is evidence of a deeper connection between the Jack symmetric functions at $\alpha = 4, 10$ and the last two strongly regular graphs listed above.

Chapter 3

Stability for Intersecting Families of Perfect Matchings

In extremal combinatorics, *stability* theorems are qualitative results which show that the structure of “large” objects satisfying a given property must be “close” in structure to the extremal objects satisfying that property. The quintessential example of this phenomenon begins with a well-known theorem of Turán, that the K_{r+1}-free graph on n vertices with the most edges is the graph $T_{n,r}$ obtained by partitioning a set of n vertices into r parts (making their sizes as equal as possible), then joining two vertices if they belong to different parts. A stability version of this theorem would show that a K_{r+1}-free graph having almost the same number of edges as $T_{n,r}$ must be a graph whose structure is somehow close to $T_{n,r}$. Such a result was given by Erdos, Stone, and Simonovits, who showed that any K_{r+1}-free graph having nearly the same number of edges as $T_{n,r}$ must have a large r-partite subgraph. Results of this nature have their place in extremal combinatorics as they provide a more complete understanding of the structure of objects that satisfy a given combinatorial property.

In the subfield of Erdős-Ko-Rado combinatorics, stability results show that large intersecting families are similar in structure to the largest intersecting families, which are often the so-called *trivially intersecting* or *canonically intersecting families* [30]. The classic example in this setting dates back to Hilton and Milner [37], who gave a stability version of the original Erdős-Ko-Rado theorem for $t = 1$. In particular, they showed that for all $n \geq 2k$, any intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$ of size greater than $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$ is contained in a canonically intersecting family

$$\mathcal{F}_i = \left\{ S \in \binom{[n]}{k} : i \in S \right\} \text{ for some } i \in [n].$$

This result too implies that the canonically intersecting families are the extremal families for $n \geq 2k$, but in a stronger sense. Like the Erdős-Ko-Rado theorem, Hilton and Milner's result has since been generalized to other classes of combinatorial objects, see [30] for a survey of these results. In this chapter, we offer an analogue of Hilton and Milner's stability result for perfect matchings of K_{2n}.

A family of perfect matchings $\mathcal{F} \subseteq \mathcal{M}_{2n}$ is *intersecting* if $m \cap m' \neq \emptyset$ for any $m, m' \in \mathcal{F}$. Recall from the previous chapter that the canonically intersecting families of $\mathcal{M}_{2n}$ are of the form

$$\mathcal{F}_{ij} = \{m \in \mathcal{M}_{2n} : ij \in m\} \text{ for some } ij \in E(K_{2n}).$$

It is well-known that the largest intersecting families of $\mathcal{M}_{2n}$ are the canonically intersecting families.

3.0.1 Theorem. [30, 45, 56] *If $\mathcal{F} \subseteq \mathcal{M}_{2n}$ is an intersecting family, then*

$$|\mathcal{F}| \leq (2n-3)!!.$$

Moreover, equality holds if and only if $\mathcal{F}$ is a canonically intersecting family.

The main result of this chapter is that the extremal families in the theorem above are *stable* for sufficiently large n.

3.0.2 Theorem. *For any $\epsilon \in (0, 1/\sqrt{e})$ and $n > n(\epsilon)$, any intersecting family of perfect matchings of size greater than $(1-1/\sqrt{e}+\epsilon)(2n-3)!!$ is contained in a canonically intersecting family.*

To see that this bound is best possible, consider the following intersecting family:

$$\mathcal{H}_{1,2} = \{m \in \mathcal{F}_{1,2} : m \text{ intersects } (1\ 3)m^*\} \cup \{(1\ 3)m^*, (1\ 4)m^*\}$$

where $(1\ 3)m^*$ and $(1\ 4)m^*$ are the perfect matchings obtained by letting the transpositions $(1\ 3) \in S_{2n}$ and $(1\ 4) \in S_{2n}$ act on $m^* = \{\{1,2\},\{3,4\},\cdots,\{2n-$

$1, 2n\}\}$. This family is not contained in any canonically intersecting family, and for every member $m \in \mathcal{H}_{1,2} \setminus \{(1\ 3)m^*, (1\ 4)m^*\}$, we have that

$$\{1,4\}, \{2,3\}, \{2,4\}, \{1,3\} \notin m \text{ and } m \cap \{\{5,6\}, \{7,8\}, \cdots, \{2n-1, 2n\}\} \neq \emptyset.$$

Recall that $d(m^*, m)$ is the cycle type of m. For any $\lambda \vdash n$, let $\mathrm{fp}(\lambda)$ denote the number of singleton parts of λ (i.e., fixed points), and for any $m \in \mathcal{M}_{2n}$, define $\mathrm{fp}(m) := \mathrm{fp}(d(m^*, m))$. A *derangement* of $\mathcal{M}_{2n}$ is a perfect matching $m \in \mathcal{M}_{2n}$ such that $\mathrm{fp}(m) = 0$. It is well-known (see [30]) that the number of derangements of $\mathcal{M}_{2n}$, denoted as D_{2n}, can be counted using a recurrence quite similar to the classic one for counting derangements of permutations:

$$D_{2n} = 2(n-1)(D_{2(n-1)} + D_{2(n-2)}),$$

where $D_0 = 1$ and $D_2 = 0$. Alternatively, via the principle of inclusion-exclusion we have

$$D_{2n} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2(n-k)-1)!! = (2n-1)!! \sum_{k=0}^{n} (-1)^k \frac{(n)_k}{k!((2n-1))_k},$$

where $((2n-1))_t := (2n-1) \times (2(n-1)-1) \times \cdots \times (2(n-t+1)-1)$ is the odd double factorial analogue of the *falling factorial* $(n)_t := n!/(n-t)!$. After taking limits, we see that

$$D_{2n} = (2n-1)!!\ (1/\sqrt{e} + o(1)) \quad \text{as} \quad n \to \infty.$$

The number of perfect matchings $m \in \mathcal{M}_{2n}$ such that $m \cap m^* = \{\{1,2\}\}$ is $D_{2(n-1)}$. Similarly, the number of perfect matchings such that

$$m \cap m^* = \{\{1,2\}, \{3,4\}\}$$

is $D_{2(n-2)}$. Since $|\mathcal{F}_{1,2}| = (2n-3)!!$, we deduce that the number of perfect matchings containing $\{1,2\}$ and an edge of $\{\{5,6\}, \{7,8\}, \cdots, \{2n-1, 2n\}\}$ is

$$|\mathcal{H}_{1,2}| - 2 = (2n-3)!! - D_{2(n-1)} - D_{2(n-2)} = (1 - 1/\sqrt{e} + o(1))(2n-3)!!.$$

Note that relabeling vertices of K_{2n} gives isomorphic families $\mathcal{H}_{i,j}$ for any ij.

For any intersecting family $\mathcal{F} \subseteq \mathcal{M}_{2n}$, we define the *restriction* $\mathcal{F} \downarrow_{ij} \subseteq \mathcal{F}$ to be the subfamily of members that all contain the edge ij, formally,

$$\mathcal{F} \downarrow_{ij} := \{m \in \mathcal{F} : ij \in m\}.$$

To show Theorem 3.0.2, we prove the following key stability lemma.

3.0.3 Lemma (Key Stability Lemma)**.** *For any $c \in (0,1)$, there exists a $C > 0$ such that the following holds. If $\mathcal{F} \subset \mathcal{M}_{2n}$ is an intersecting family such that $|\mathcal{F}| \geq c(2n-3)!!$, then there exist an edge ij such that*

$$|\mathcal{F} \setminus \mathcal{F} \downarrow_{ij}| \leq C(2n-5)!!.$$

The following argument shows that the key stability lemma implies Theorem 3.0.2.

Let $\mathcal{F} \subseteq \mathcal{M}_{2n}$ be an intersecting family of perfect matchings such that $|\mathcal{F}| \geq c(2n-3)!!$ and $c \in (1 - 1/\sqrt{e}, 1)$. Assuming the key stability lemma, there is an edge $ij \in E(K_{2n})$ such that $|\mathcal{F} \setminus \mathcal{F} \downarrow_{ij}| = O((2n-5)!!)$. This implies

$$|\mathcal{F} \downarrow_{ij}| \geq (c - O(1/n))(2n-3)!!. \tag{3.0.1}$$

For sake of contradiction, suppose there exists an $m \in \mathcal{F}$ such that $ij \notin m$. Since any member of $\mathcal{F} \downarrow_{ij}$ must share an edge with m, we have that

$$|\mathcal{F} \downarrow_{ij}| \leq (2n-3)!! - D_{2(n-1)} - D_{2(n-2)} = (1 - 1/\sqrt{e} - o(1))(2n-3)!!.$$

This contradicts Equation (3.0.1) for n sufficiently large depending on c, which completes the proof of Theorem 3.0.2.

In light of this, we spend the remainder of this chapter proving the key stability lemma. Before we start down this path, we provide some backstory to the proof technique that we use and its role in Erdős-Ko-Rado combinatorics.

Our method of proof was originally used by Ellis [22] to prove the bipartite version of Theorem 3.0.2 which was conjectured by Cameron and Ku [11]. In the sequel [21], he showed this method can be extended to show stability results for *t-intersecting families* of perfect matchings of $K_{n,n}$, that is, families such that any two members share t edges. The latter stability result [21] implies the case of equality in the following seminal result of Ellis, Friedgut, and Pilpel.

3.0.4 Theorem. [20, 21] *Let $t \in \mathbb{N}$. If $\mathcal{F}$ is a t-intersecting family of perfect matchings of $K_{n,n}$, then for sufficiently large n, we have*

$$|\mathcal{F}| \leq (n-t)!.$$

Moreover, equality holds if and only if $\mathcal{F}$ is a canonically t-intersecting family, that is, every member of $\mathcal{F}$ contains a fixed set of t disjoint edges of $K_{n,n}$.

For the $t = 1$ case in the theorem above, one can obtain a stronger characterization of the largest 1-intersecting families that holds for all n using polyhedral techniques [30]. It was believed that these polyhedral techniques could be extended to the problem of characterizing the extremal t-intersecting families of perfect matchings of $K_{n,n}$ [20, Theorem 27], but Filmus [28] recently showed the proof is incorrect. This refutation has sparked renewed interest in Ellis' method, since it currently provides the simplest proof of the case of equality in Theorem 3.0.4 (see [23, pg. 37] for more discussion).

In the next chapter, we adopt the same strategy: we extend the proof of Theorem 3.0.2 to a stability result for t-intersecting families as a means to characterize the largest t-intersecting families of $\mathcal{M}_{2n}$ for sufficiently large n. The material of this chapter is of didactical importance, as the proof of $t \geq 2$ case follows along the same lines as the $t = 1$ case presented here, but there are several technical challenges that must be overcome. Not as many obstacles arise in the $t = 1$ case which makes for a cleaner exposition of Ellis' method.

We have gone as far as we can without making an introduction to the theory of finite group representations. The following is short primer on the subject which we will build upon later.

3.1 Finite Group Representation Theory I

Our work draws upon the fundamentals of the ordinary representation theory of finite groups and their Fourier analysis. Statisticians [13, 18] have given a few treatments of group representation theory from a Fourier-analytical point of view, to which we refer the reader for more details. Throughout this section, let $H, K \leq G$ be subgroups of a finite group G, and V be a finite dimensional vector space over $\mathbb{C}$.

For any set X, let $\mathbb{C}[X]$ denote the vector space of dimension $|X|$ of complex-valued functions over X, equipped with the inner product

$$\langle f, g\rangle_X := \sum_{x\in X} f(x)\overline{g(x)}.$$

A *representation* (ϕ, V) of G is a homomorphism $\phi : G \to GL(V)$ where $GL(V)$ is the *general linear group*, that is, the group of $(\dim V)\times(\dim V)$ invertible matrices. It is customary to be less formal and denote the representation (ϕ, V) simply as ϕ when V is understood, or as V when ϕ is understood. For any representation ϕ, we define its *dimension* or *degree* to be $\dim\phi := \dim V$. When working concretely with a representation ϕ, we abuse terminology and let $\phi(g)$ refer to a $(\dim\phi)\times(\dim\phi)$ matrix realization of ϕ. Two representations ρ, ϕ are *equivalent* if there exists a square matrix P such that $P^{-1}\rho(g)P = \phi(g)$ for all $g \in G$ (i.e., they are similar).

Let (ϕ, V) be a representation of G, and let $W \leq V$ be a *G-invariant* subspace, that is, $\phi(g)w \in W$ for all $w \in W$ and for all $g \in G$. We say that $(\phi|_W, W)$ is a *subrepresentation* of ϕ where $\phi|_W$ is the restriction of ϕ to the subspace W. A representation (ϕ, V) is an *irreducible representation* (or simply, an *irreducible*) if it has no proper subrepresentations. The *trivial representation* $(1, \mathbb{C})$ defined such that $1 : g \to 1$ for all $g \in G$ is clearly an irreducible of dimension one for any group G.

It is well-known that there is a bijection between the set of inequivalent irreducibles of G and its conjugacy classes $\mathcal{C}$, and that any representation V of G decomposes uniquely as a direct sum of inequivalent irreducibles V_i of G:

$$V \cong \bigoplus_{i=1}^{|\mathcal{C}|} m_i V_i$$

where m_i is the *multiplicity* of V_i, that is, the number of times that V_i occurs in the decomposition.

Since two elements are conjugate in S_n if and only if they have the same cycle type, each irreducible identifies with an integer partition of n. Throughout this work, we frequently abuse notation by letting λ refer to irreducible associated to λ, which should not result in any confusion.

A natural way to find representations of groups is to let them act on sets. In particular, for any group G acting on a set X, let $(\phi, \mathbb{C}[X])$ be the *permutation representation* of G on X defined such that

$$\phi(g)[f(x)] = f(g^{-1}x)$$

for all $g \in G$, $f \in \mathbb{C}[X]$, and $x \in X$. For instance, if we let G act on itself by left multiplication, then we obtain the *regular representation*, which admits the following decomposition into irreducibles:

$$\mathbb{C}[G] \cong \bigoplus_{i=1}^{|\mathcal{C}|} (\dim V_i)\ V_i$$

where V_i is the ith irreducible of G. For the $G \cong S_n$, this becomes

$$\mathbb{C}[S_n] \cong \bigoplus_{\lambda \vdash n} f^\lambda\ \lambda$$

where f^λ is the number of standard Young tableaux of shape $\lambda \vdash n$. In the next chapter, we revisit this decomposition in more detail.

It is well-known that any representation ϕ of G is isomorphic to a direct sum of irreducibles of any subgroup $H \leq G$. This representation $(\phi \downarrow_H^G, V \downarrow_H^G)$ is called the *restriction* of ϕ to H, and is obtained simply by restricting the domain of ϕ to H. Even if ϕ is an irreducible of G, the restricted representation is typically not an irreducible of H.

If λ is an irreducible of S_n, then there is a particularly elegant combinatorial rule for determining the multiplicities of irreducibles in the restricted representation $\lambda \downarrow_{S_m}^{S_n}$ for any $m < n$.

3.1.1 Theorem (The Branching Rule [64])**.** *For any irreducible representation λ of S_n, we have*

$$\lambda \downarrow_{S_{n-1}}^{S_n} \cong \bigoplus_{\lambda^-} \lambda^-$$

where λ^- ranges over all shapes obtainable from λ by removing a cell $s \in \lambda$ such that $a_\lambda(s) = 0$ and $l_\lambda(s) = 0$.

For any irreducible λ of S_n, repeated application of the branching rule

$$\lambda \downarrow_{S_{n-1}}^{S_n} \downarrow_{S_{n-2}}^{S_{n-1}} \cdots \downarrow_{S_1}^{S_2}$$

breaks λ into f^λ 1-dimensional orthogonal subspaces, proving that $\dim \lambda = f^\lambda$.

We say that an irreducible of S_{2n} is an *even irreducible* if the integer partition associated to the irreducible is of the form $2\lambda = (2\lambda_1, 2\lambda_2, \cdots, 2\lambda_\ell)$ for some $\lambda \vdash n$. Another consequence of the branching rule is the following.

3.1.2 Corollary. *For any irreducible μ of S_m and $2 \leq i < m$ such that $\mu \neq (m)$ or (1^m), the representation $\mu \downarrow^{S_m}_{S_{m-i}}$ is reducible. Moreover, if μ is an even irreducible of S_{2m} and $1 \leq i < m$, then the representation $\mu \downarrow^{S_{2m}}_{S_{2(m-1)}}$ contains at least two even irreducibles of $S_{2(m-1)}$ unless μ has rectangular shape $(a^b) \vdash 2n$.*

Finally, a result of Thrall [73] shows that the permutation representation of S_{2n} acting on $\mathcal{M}_{2n}$ admits the following decomposition into irreducibles of S_{2n}.

3.1.3 Theorem. *Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \vdash n$ and let 2λ denote the irreducible of S_{2n} corresponding to the partition $2\lambda = (2\lambda_1, 2\lambda_2, \cdots, 2\lambda_k) \vdash 2n$. Then*

$$\mathbb{R}[\mathcal{M}_{2n}] \cong \bigoplus_{\lambda \vdash n} 2\lambda.$$

These irreducibles in the decomposition above are the eigenspaces of matrices in the Bose-Mesner algebra of $\mathcal{A}_n$, i.e., the range of the projector $E_\lambda \in \mathcal{A}_n$ is isomorphic to the irreducible 2λ of S_{2n}. In the next section, we use properties of the dimensions of these eigenspaces to determine spectral information about an important graph that lives in the Bose-Mesner algebra of $\mathcal{A}_n$.

3.2 Eigenvalues of the Derangement Graph

The *derangement graph* is the graph $\mathcal{D}_n$ over $\mathcal{M}_{2n}$ such that $m, m' \in \mathcal{M}_{2n}$ are adjacent if $m \cap m' = \emptyset$, or in the language of association schemes:

$$\mathcal{D}_n = \sum_{\substack{\lambda \vdash n \\ \mathrm{fp}(\lambda)=0}} A_\lambda$$

where $A_\lambda \in \mathcal{A}_n$. If we draw the associates instead from $\mathcal{A}'_n$, then we obtain its bipartite cousin, the so-called *permutation derangement graph*, whose algebraic properties have been the subject of several papers [62, 41, 42]. By definition, non-adjacent perfect matchings in $\mathcal{D}_n$ are intersecting, thus its independent sets are intersecting families of $\mathcal{M}_{2n}$. Let $\{\eta_\lambda\}_{\lambda \vdash n}$ be the eigenvalues of $\mathcal{D}_n$.

We begin with a short proof that the least eigenvalue of the perfect matching derangement graph is

$$\eta_{(n-1,1)} = -D_{2n}/2(n-1)$$

and the magnitudes of its eigenvalues, aside from the least and greatest, are $O((2n-5)!!)$. Godsil and Meagher [31] first showed the former, but it seems that slightly stronger arguments are needed to deduce the latter result, which will indeed be an essential ingredient in our proof of Theorem 3.0.2.

A technique of James and Kerber [40] based on the branching rule allows us to obtain lower bounds on the degrees of even irreducibles of S_{2n} that are not too small in reverse-lexicographical order.

3.2.1 Lemma. *For $n \geq 8$, the only even shapes λ of S_{2n} such that*

$$f^\lambda < \binom{2n-4}{4} - \binom{2n-4}{3}$$

are $(2n)$ and $(2n-2,2)$.

Proof. We proceed by induction on $n \geq 8$. Suppose the claim is true for $S_{2(n-1)}$ but not S_{2n}. Let $\lambda \vdash 2n$ be an even partition such that $f^\lambda < \binom{2n-4}{4} - \binom{2n-4}{3}$.

If $\lambda \downarrow^{S_{2n}}_{S_{2(n-1)}}$ contains $(2n-2)$ or $(2n-4,2)$ as an irreducible representation, then by the branching rule, the only possibilities for λ are

$$(2n), (2n-2,2), (2n-4,4), \text{ and } (2n-4,2^2),$$

as illustrated below:

$$\begin{tikzcd}
(2n) \arrow[d, dash] & (2n-2,2) \arrow[d, dash] \arrow[dr, dash] & (2n-4,4) \arrow[dr, dash] & (2n-4,2^2) \arrow[dr, dash] & \\
(2n-1) \arrow[d, dash] & (2n-2,1) \arrow[dl, dash] & (2n-3,2) \arrow[d, dash] & (2n-4,3) \arrow[dl, dash] & (2n-4,2,1) \arrow[dll, dash] \\
(2n-2) & & (2n-4,2) & &
\end{tikzcd}$$

.

Recalling that $f^\lambda = (2n)!/c_1(\lambda)$, we have

$$f^\lambda < \binom{2n-4}{4} - \binom{2n-4}{3} = f^{(2n-4,4)} < f^{(2n-4,2^2)},$$

which rules out $(2n-4,4)$ and $(2n-4,2^2)$. We conclude that $(2n-2)$ and $(2n-4,2)$ do not appear in the irreducible decomposition of $\lambda \downarrow^{S_{2n}}_{S_{2(n-1)}}$.

By the induction hypothesis, all other even irreducibles $\mu < (2n-4,2)$ of $S_{2(n-1)}$ have

$$f^{\mu} \geq \binom{2(n-1)-4}{4} - \binom{2(n-1)-4}{3}.$$

Moreover, for $n \geq 8$ we have

$$2\left(\binom{2(n-1)-4}{4} - \binom{2(n-1)-4}{3}\right) \geq \binom{2n-4}{4} - \binom{2n-4}{3}.$$

The second part of Corollary 3.1.2 implies that $\lambda = (a^b)$ for some a, b such that $ab = 2n$. Without loss of generality we may assume that $b \leq a$. We have

$$f^{(a^b)} = \frac{(2n)!(b-1)!}{(a+b-1)!\cdots a!} \geq f^{(n^2)} = \frac{1}{n+1}\binom{2n}{n} \geq \binom{2n-4}{4} - \binom{2n-4}{3}$$

for all $n \geq 8$ and $b \neq 1$. This implies that $b = 1$, i.e., $\lambda = (2n)$. We conclude that the claim holds for S_{2n}, a contradiction. □

We now use the following folklore result to upper bound $|\eta_\lambda|$ such that $\lambda \neq (n)$ or $(n-1,1)$.

3.2.2 Lemma (The Trace Bound)**.** *Let $\Gamma = (V, E)$ be a graph on N vertices and let $\{\eta_i\}_{i=1}^N$ be the eigenvalues of its adjacency matrix. Then $\sum_{i=1}^N \eta_i^2 = \mathrm{Tr}(\Gamma^2) = 2|E|$.*

3.2.3 Lemma. *For all $\lambda \neq (n)$ or $(n-1,1)$, we have $|\eta_\lambda| = O((2n-5)!!)$.*

Proof. Since $((2n-1)!!)^2(1/\sqrt{e} + o(1))$ equals twice the number of edges of the derangement graph, Lemma 3.2.2 implies that

$$\sum_{\lambda \vdash n} (\sqrt{\dim 2\lambda}\ \eta_\lambda)^2 = ((2n-1)!!)^2(1/\sqrt{e} + o(1)).$$

By the non-negativity of the terms on the left-hand side, we have

$$\begin{aligned} |\eta_\lambda| &\leq \sqrt{\frac{(2n-1)!!^2(1/\sqrt{e}+o(1))}{\dim 2\lambda}} \\ &= \frac{(2n-1)!!}{\sqrt{\dim 2\lambda}}\sqrt{1/\sqrt{e}+o(1)} \\ &= O((2n-5)!!), \end{aligned}$$

where the last equality follows from Lemma 3.2.1. □

3.2.4 Lemma. [53, Ch. VII] *Let P be the character table of $\mathcal{A}_n$ and let v_λ be the valency of the associate $A_\lambda \in \mathcal{A}_n$. Then*

$$P_{(n-1,1),\lambda} = v_\lambda \left(\frac{(2n-1)\mathrm{fp}(\lambda) - n}{2n(n-1)} \right).$$

We are now in a position to give a short proof of the following.

3.2.5 Theorem (Godsil, Meagher [31])**.** *The least eigenvalue of $\mathcal{D}_n$ is*

$$\eta_{(n-1,1)} = -\frac{D_{2n}}{2(n-1)}.$$

Proof. By Lemma 3.2.3, only $\eta_{(n)} = D_{2n}$ and $|\eta_{(n-1,1)}|$ are $\omega((2n-5)!!)$. For any $\lambda \vdash n$ with no singleton parts, Lemma 3.2.4 implies that

$$P_{(n-1,1),\lambda} = -\frac{v_\lambda}{2(n-1)}$$

for any $\lambda \vdash n$ with no singleton parts. Since $\mathcal{D}_n$ is the sum of all associates A_λ such that $\lambda \vdash n$ is a derangement, we have $\eta_{(n-1,1)} = -D_{2n}/2(n-1)$, as desired. □

With the largest and least eigenvalue in hand, the so-called *ratio bound* of Delsarte and Hoffman and a little bit of arithmetic makes short work of the first part of Theorem 3.0.1.

3.2.6 Theorem (Ratio Bound [17])**.** *Let $\Gamma = (V, E)$ be a d-regular graph with eigenvalues $d = \eta_1 \geq \eta_2 \geq \cdots \geq \eta_{\min}$ and corresponding eigenvectors $v_1, v_2 \cdots, v_{\min}$. If $S \subseteq V$ is an independent set of Γ, then*

$$|S| \leq |V| \frac{-\eta_{\min}}{d - \eta_{\min}}.$$

If equality holds, then $1_S \in \mathrm{Span}\left(\{v_1\} \cup \{v_i : \eta_i = \eta_{\min}\}\right)$ where 1_S is the characteristic vector of S.

The importance of the ratio bound in Erdős-Ko-Rado combinatorics cannot be overstated (see [30]). Indeed, several generalizations and analogues of this result will be central to a few of our results yet to come.

3.3 Cross-Intersecting Families

We say two families $\mathcal{F}, \mathcal{G} \subseteq \mathcal{M}_{2n}$ are *cross-intersecting* if $m \cap m' \neq \emptyset$ for all $m \in \mathcal{F}$ and $m' \in \mathcal{G}$. Using the *cross-ratio bound*, we easily obtain Theorem 3.3.2, which is a "cross-independent" version of the first part of Theorem 3.0.1.

3.3.1 Theorem (Cross-Ratio Bound [1])**.** *Let $\Gamma = (V, E)$ be a d-regular graph on N vertices with eigenvalues $d = |\eta_1| \geq |\eta_2| \geq \cdots \geq |\eta_N|$. If $S, T \subseteq V$ are vertices such that there are no edges between S and T, then*

$$\sqrt{\frac{|S||T|}{|V|^2}} \leq \frac{|\eta_2|}{d + |\eta_2|}.$$

3.3.2 Theorem. *If $\mathcal{F}, \mathcal{G} \subseteq \mathcal{M}_{2n}$ are cross-intersecting families, then*

$$|\mathcal{F}| \cdot |\mathcal{G}| \leq ((2n-3)!!)^2.$$

Let $\mathcal{H}$ be the graph over $\mathcal{M}_{2n}$ such that m, m' are adjacent if and only if $m \cup m'$ is a Hamiltonian cycle of K_{2n}. Similarly, let $\mathcal{H}'$ be the graph over $\mathcal{M}_{2n-1}$ such that m, m' are adjacent if and only if $m \cup m'$ is a Hamiltonian path of K_{2n-1}. Observe that any maximum matching of K_{2n-1} can be extended to a unique perfect matching of K_{2n} by matching the unmatched vertex of K_{2n-1} to the vertex labeled $2n$, and vice versa. This gives a bijection between Hamiltonian paths of K_{2n-1} and Hamiltonian cycles of K_{2n}, and shows that $\mathcal{H} \cong \mathcal{H}'$. Since the second largest eigenvalue (in magnitude) of the adjacency matrix of $\mathcal{H}$ is $-2^{n-2}(n-2)!$ (see [45, Corollary 5.2] for a proof), the following lemma is immediate.

3.3.3 Lemma. *The second largest eigenvalue of $\mathcal{H}'$ in magnitude is $-|H_{n-2}| = -2^{n-2}(n-2)!$.*

3.3.4 Lemma. *If $\mathcal{F}, \mathcal{G} \subseteq \mathcal{M}_{2n-1}$ are cross-intersecting families, then*

$$|\mathcal{F}| \cdot |\mathcal{G}| \leq ((2n-3)!!)^2.$$

Proof. Note that $\mathcal{H}'$ is a subgraph of the maximum matching derangement graph (two maximum matchings of K_{2n-1} adjacent if and only if they share no edges). It follows that any pair of cross-intersecting families of maximum matchings of K_{2n-1} are cross-independent sets in $\mathcal{H}'$. Lemma 3.3.3 gives the second largest eigenvalue of $\mathcal{H}'$ in magnitude, so plugging this value into the cross-ratio bound and some arithmetic gives the result. □

3.3.5 Lemma. *Let $\mathcal{F} \subseteq \mathcal{M}_{2n}$ be an intersecting family. Then for all i, j and k with $j \neq k$, we have*

$$|\mathcal{F}\downarrow_{ij}| \cdot |\mathcal{F}\downarrow_{ik}| \leq ((2n-5)!!)^2.$$

Proof. Without loss of generality, assume $i = 1$, $j = 2$, and $k = 3$. Note that $\mathcal{F}\downarrow_{12} \cap \mathcal{F}\downarrow_{13} = \emptyset$. Assume both restrictions are nonempty; otherwise, the claim is trivial. Since $\mathcal{F}$ is an intersecting family, any two $m \in \mathcal{F}\downarrow_{12}$ and $m' \in \mathcal{F}\downarrow_{13}$ must share an edge of $E(K_{2n} \setminus \{1,2,3\})$. In other words, $\mathcal{F}\downarrow_{12}$ and $\mathcal{F}\downarrow_{13}$ are isomorphic to two families $\mathcal{G}$ and $\mathcal{G}'$ of $\mathcal{M}_{2n-3}$ that are cross-intersecting. The result now follows from Lemma 3.3.4. □

3.4 Isoperimetry of the Transposition Graph

The *transposition graph* is the graph $\mathcal{T}_n$ over $\mathcal{M}_{2n}$ such that $m, m' \in \mathcal{M}_{2n}$ are adjacent if there exists a transposition $(i\ j) \in S_{2n}$ such that $(i\ j)m = m'$, or equivalently, $\mathcal{T}_n$ is the $(2, 1^{n-2})$-associate of $\mathcal{A}_n$.

The isoperimetric properties of the transposition graph play a pivotal role in the proof of our key stability lemma. The *h-neighborhood* of a set $X \subseteq V$ is the set of vertices $N_h(X) := \{v \in V : \text{dist}(v, X) \leq h\}$ where $\text{dist}(v, X)$ is the length of a shortest path from v to any vertex of X. It is instructive to think of these neighborhoods in the transposition graph as balls of radius h in a discrete metric space, as perfect matchings in a ball of small radius around some point in the transposition graph are structurally similar, i.e., they share many edges.

Like the permutation transposition graph $A_{(2,1^{n-2})} \in \mathcal{A}'_n$, our transposition graph also has a nice recursive form. The following is not too hard to show.

3.4.1 Proposition. *The transposition graph $\mathcal{T}_n$ can be written as the following $(2n-1) \times (2n-1)$ block matrix*

$$\mathcal{T}_n \cong \begin{bmatrix} \mathcal{T}_{n-1} & & & * \\ & \mathcal{T}_{n-1} & & \\ * & & \ddots & \\ & & & \mathcal{T}_{n-1} \end{bmatrix}$$

where any off-diagonal block in the $*$ region is a $(2n-3)!! \times (2n-3)!!$ permutation matrix. Furthermore, $\mathcal{T}_n$ has diameter $n-1$.

Proof. Order the adjacency matrix of $\mathcal{T}_n$ such that the rows and columns of the kth diagonal block are indexed by perfect matchings that contain the edge $\{1, k+1\}$.

Let m be a perfect matching that contains the edge $\{1, i\}$ and let m' be a perfect matching that contains the edge $\{1, j\}$. If m and m' are adjacent in the transposition graph, then $(i\ j)m = m'$. This implies that $m(i) = m'(j)$ where $m(i)$ is the other endpoint of the edge of m incident to i, and $m'(j)$ is the other endpoint of the edge of m' incident to j. The remaining edges of m not incident to $1, i, j$, or $m(i)$ are fixed points of $(i\ j)$ and also belong to m'. It is clear that no other transposition sends the edge $\{1, i\}$ to the edge $\{1, j\}$; therefore, there is a precisely a single 1 in each row of the off-diagonal (i, j)-block and a single 1 in each column of the off diagonal (i, j)-block, and so each off-diagonal block is a permutation matrix.

Finally, that the diameter of $\mathcal{T}_n$ equals $n-1$ follows by induction on n. □

A *partition sequence* of a graph $\Gamma = (V, E)$ is a sequence $\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_m$ of increasingly refined partitions of V where $\mathcal{P}_0 = V$ is the trivial partition, $\mathcal{P}_m$ is the discrete partition into singleton blocks, along with a sequence of numbers $c_0, c_1, \cdots, c_m$ with the following property: for each $i \in \{1, 2, \cdots, m\}$, whenever $A, B \in \mathcal{P}_i$, and $A, B \subseteq C \in \mathcal{P}_{i-1}$ for some C, then there is a bijection $\varphi : A \to B$ with $d_\Gamma(x, \varphi(x)) \leq c_i$ for all $x \in A$. We say that a partition sequence is *nice* if $m = \text{diameter}(\Gamma)$ and $c_i \leq 1$ for all $i \in \{1, 2, \cdots, m\}$.

3.4.2 Theorem (McDiarmid's Bound [55])**.** *Let $\Gamma = (V, E)$ be a graph that admits a partition sequence $\{\mathcal{P}_i\}_{i=0}^m, \{c_i\}_{i=0}^m$, and let $X \subset V$ such that $|X| \geq a|V|$ for some $a \in (0, 1)$. Then for any $h \in \mathbb{N}$ such that*

$$h > h_0 = \sqrt{\frac{1}{2}\sum_{i=0}^{m} c_i^2 \ln(1/a)},$$

the following holds:

$$N_h(X) \geq \left(1 - \exp\left(\frac{-2(h-h_0)^2}{\sum_{i=0}^m c_i^2}\right)\right)|V|.$$

We claim that the transposition graph admits a nice partition sequence. The sequence $\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_{m-1}$ is given by recursively applying the block decomposition of the vertices of $\mathcal{T}_n$ stated in Proposition 3.4.1, and the bijection $\varphi : A \to B$ $(A, B \in \mathcal{P}_i)$ is given by the permutation matrix of the (A, B)

off-diagonal matrix of $\mathcal{T}_{n-i}$ where $d_\Gamma(m, \varphi(m)) \leq c_i = 1$ for all $m \in A$. By McDiarmid's bound, we obtain the following.

3.4.3 Proposition. *Let $X \subset \mathcal{M}_{2n}$ such that $|X| \geq a(2n-1)!!$ for some $a \in (0,1)$. Then for any $h \in \mathbb{N}$ such that*

$$h > h_0 = \sqrt{\frac{n}{2}\ln(1/a)},$$

the following holds:

$$N_h(X) \geq \left(1 - \exp\left(\frac{-2(h-h_0)^2}{n}\right)\right)(2n-1)!!.$$

3.5 Stability Preliminaries

A few more preliminary results are required before embarking on the proof of the key stability lemma. First in this list is a generalization of the ratio bound.

3.5.1 Theorem (Stability Version of Ratio Bound [22])**.** *Let $\Gamma = (V, E)$ be a d-regular graph on N vertices with eigenvalues $\eta_{\min}, \cdots, \eta_{\max} = d$ ordered from least to greatest, and corresponding orthonormal eigenvectors $v_{\min}, \cdots, v_{\max}$. Define $\mu := \min\{\eta_i : \eta_i \neq \eta_{\min}\}$. Let $X \subseteq V$ be a set of vertices of measure $\alpha := |X|/N$ and let ℓ denote the number of edges of the subgraph induced by X. Let D be the Euclidean distance from the characteristic function f of X to the subspace $U = \operatorname{Span}(\{v_{\max}\} \cup \{v_i : \eta_i = \eta_{\min}\})$. Then*

$$D^2 \leq \alpha \frac{(1-\alpha)|\eta_{\min}| - d\alpha}{|\eta_{\min}| - |\mu|} + 2\ell.$$

Theorem 3.5.1 together with our spectral information on $\mathcal{D}_n$ provides us with bounds on how far any family (intersecting or not) is from U. Recall that equality is met when we apply the ratio bound to $\mathcal{D}_n$, which implies that

$$1_{\mathcal{F}_{ij}} \in U \cong (2n) \oplus (2n-2, 2).$$

We are concerned with how far a "large" intersecting family $\mathcal{F}$ is from U where "large" means having size $c(2n-3)!!$ for some $c \in (0,1)$. The Euclidean distance D from $1_{\mathcal{F}}$ to U can be written as $D = \|P_{U^\perp} 1_{\mathcal{F}}\|_2$ where P_V denotes the projection onto any subspace $V \leq \mathbb{R}[\mathcal{M}_{2n}]$. Since $(2n)$ is the space of constant functions, the projection of any characteristic function $1_{\mathcal{F}} \in \mathbb{R}[\mathcal{M}_{2n}]$ onto $(2n)$ is just $(|\mathcal{F}|/(2n-1)!!)1_{\mathcal{M}_{2n}}$. More generally, we have the following.

3.5.2 Proposition. [45, 13] *Let $E_\mu : \mathbb{R}[\mathcal{M}_{2n}] \to 2\mu$ denote the orthogonal projection onto 2μ where $\mu \vdash n$. Then*

$$[E_\mu f](m) = \frac{\dim 2\mu}{(2n-1)!!} \sum_{\lambda \vdash n} \frac{1}{v_\lambda} \left(\sum_{\substack{m' \in \mathcal{M}_{2n} \\ d(m,m')=\lambda}} f(m') P_{\mu,\lambda} \right).$$

3.5.3 Lemma. *The orthogonal projection $E_{(n-1,1)} : \mathbb{R}[\mathcal{M}_{2n}] \to 2(n-1,1)$ of the characteristic function $f \in \mathbb{R}[\mathcal{M}_{2n}]$ of a family $\mathcal{F} \subseteq \mathcal{M}_{2n}$ can be written as*

$$[E_{(n-1,1)} f](m) = \frac{1}{(2n-5)!! \cdot 2(n-1)} \left(\sum_{ij \in m} |\mathcal{F} \downarrow_{ij}| \right) - \frac{|\mathcal{F}|}{2(n-1)}$$

for all $m \in \mathcal{M}_{2n}$.

Proof. Applying Proposition 3.5.2 and Lemma 3.2.4 gives us,

$$\begin{aligned}
[E_{(n-1,1)} f](m) &= \frac{\dim 2(n-1,1)}{(2n-1)!!} \sum_{\lambda \vdash n} \frac{1}{v_\lambda} \left(\sum_{\substack{m' \in \mathcal{M}_{2n} \\ d(m,m')=\lambda}} f(m') P_{(n-1,1),\lambda} \right) \\
&= \frac{\dim 2(n-1,1)}{(2n-1)!!} \sum_{\lambda \vdash n} \frac{1}{v_\lambda} \sum_{\substack{m' \in \mathcal{F} \\ d(m,m')=\lambda}} P_{(n-1,1),\lambda} \\
&= \frac{\dim 2(n-1,1)}{(2n-1)!!} \sum_{\lambda \vdash n} \sum_{\substack{m' \in \mathcal{F} \\ d(m,m')=\lambda}} \frac{(2n-1)\mathrm{fp}(\lambda) - n}{2n(n-1)} \\
&= \frac{1}{(2n-5)!! \cdot 2(n-1)} \left(\sum_{\lambda \vdash n} \sum_{\substack{m' \in \mathcal{F} \\ d(m,m')=\lambda}} \mathrm{fp}(\lambda) \right) - \frac{n|\mathcal{F}|}{2n(n-1)} \\
&= \frac{1}{(2n-5)!! \cdot 2(n-1)} \left(\sum_{ij \in m} |\mathcal{F} \downarrow_{ij}| \right) - \frac{|\mathcal{F}|}{2(n-1)}
\end{aligned}$$

where the last equality follows by a double-counting argument. □

3.6 Proof of the Key Stability Lemma

We now begin the proof of the key stability lemma (Lemma 3.0.3), which says that for any $c \in (0,1)$, if $\mathcal{F} \subset \mathcal{M}_{2n}$ is an intersecting family such that $|\mathcal{F}| \geq c(2n-3)!!$, then there exists a $C > 0$ and an edge $ij \in E(K_{2n})$ such that $|\mathcal{F} \setminus \mathcal{F}\downarrow_{ij}| \leq C(2n-5)!!$. Due to similarities in the asymptotics of perfect matchings and permutations, a few steps follow from Ellis [22] *mutatis mutandis*. In these places we have made an attempt to keep our notation consistent with [22].

Let $\mathcal{F}$ be an intersecting family such that $|\mathcal{F}| \geq c(2n-3)!!$ and $c \in (0,1)$. Let f be the characteristic function of $\mathcal{F}$, and let $\alpha = |\mathcal{F}|/(2n-1)!!$. Let D be the Euclidean distance from f to U. By Theorem 4.7.3, we have

$$\begin{aligned} D^2 &\leq \alpha \frac{(1-\alpha)D_{2n}/2(n-1) - D_{2n}\alpha}{D_{2n}/2(n-1) - |\mu|} \\ &= \frac{|\mathcal{F}|}{(2n-1)!!} \; \frac{1-\alpha-2(n-1)\alpha}{1-2(n-1)|\mu|/D_{2n}} \\ &= \frac{|\mathcal{F}|}{(2n-1)!!} \; \frac{1-(2n-1)\alpha}{1-O(1/n)} \\ &\leq \frac{|\mathcal{F}|}{(2n-1)!!} \; (1-(2n-1)\alpha)(1+O(1/n)), \end{aligned}$$

where the penultimate equality uses the fact that $|\mu| = o((2n-3)!!)$ from Lemma 3.2.3. Now pick $\delta < 1$ so that $|\mathcal{F}| \leq (1-\delta)(2n-3)!!$. We have

$$\|P_{U^\perp} f\|_2^2 = \|f - P_U f\|_2^2 = D^2 \leq \delta(1+O(1/n))\frac{|\mathcal{F}|}{(2n-1)!!},$$

which tends to zero as $n \to \infty$. This already shows that f is "close" to being a linear combination of canonically intersecting families, but we now seek a combinatorial explanation for this proximity.

By Lemma 3.5.3, the projection $P_m := [E_{(n)}f + E_{(n-1,1)}f](m)$ of $f(m)$ onto the subspace U for any $m \in \mathcal{M}_{2n}$ is

$$P_m = \frac{1}{(2n-5)!! \cdot 2(n-1)} \left(\sum_{ij \in m} |\mathcal{F}\downarrow_{ij}| \right) - \frac{|\mathcal{F}|}{2(n-1)} + \frac{|\mathcal{F}|}{(2n-1)!!}. \quad (3.6.1)$$

Note that

$$\|f - P_U f\|_2^2 = \frac{1}{(2n-1)!!}\left(\sum_{m\in\mathcal{F}}(1-P_m)^2 + \sum_{m\notin\mathcal{F}} P_m^2\right) \leq \frac{|\mathcal{F}|\delta(1+O(1/n))}{(2n-1)!!},$$

which gives us

$$\sum_{m\in\mathcal{F}}(1-P_m)^2 + \sum_{m\notin\mathcal{F}} P_m^2 \leq |\mathcal{F}|\delta(1+O(1/n)).$$

Following Ellis, pick $C > 0$ large enough so that

$$\sum_{m\in\mathcal{F}}(1-P_m)^2 + \sum_{m\notin\mathcal{F}} P_m^2 \leq |\mathcal{F}|\delta(1+O(1/n)) \leq |\mathcal{F}|(1-1/n)\delta(1+C/n).$$

By the non-negativity of each term on the left-hand side of (3.6.1), at least $|\mathcal{F}|/n$ members of $\mathcal{F}$ satisfy $(1-P_m)^2 < \delta(1+C/n)$; therefore, there is a set

$$\mathcal{F}_1 = \{m \in \mathcal{F} : (1-P_m)^2 < \delta(1+C/n)\}$$

such that $|\mathcal{F}_1| \geq |\mathcal{F}|/n$.

Similarly, suppose there are more than

$$(2n-1)|\mathcal{F}|(1+O(1/n))/2 \geq (1-\delta)(2n-1)!!(1+O(1/n))/2$$

perfect matchings outside of $\mathcal{F}$ having $P_m^2 \geq 2\delta/(2n-1)$. Then

$$\sum_{m\notin\mathcal{F}} P_m^2 > \frac{2\delta}{(2n-1)}(1-\delta)(2n-1)!!(1+O(1/n))/2 \geq |\mathcal{F}|\delta(1+O(1/n))$$

a contradiction; thus there also exists a set

$$\mathcal{F}_0 = \{m \notin \mathcal{F} : P_m^2 < 2\delta/(2n-1)\}$$

such that

$$|\mathcal{F}_0| \geq (2n-1)!! - (1-\delta)(2n-1)!!(1+O(1/n))/2 - (1-\delta)(2n-3)!!.$$

The projections of the elements of $\mathcal{F}_0$ and $\mathcal{F}_1$ are close to 0 and 1 respectively. We now show that there exists an $m_1 \in \mathcal{F}_1$ and $m_0 \in \mathcal{F}_0$ that are close together in the transposition graph, which implies that the two share many edges.

To this end, we claim that there is a path p connecting m_0 and m_1 in $\mathcal{T}_n$ of length at most $2\sqrt{n/2\log n}$. To see this, take $a := 1/n^4$ and $h := 2h_0$ in McDiarmid's bound. Since

$$|\mathcal{F}_1| \geq c(2n-3)!!/n \geq (2n-1)!!/n^4,$$

McDiarmid's bound gives us

$$|N_h(\mathcal{F}_1)| \geq \left(1 - \frac{1}{n^4}\right)(2n-1)!!.$$

Since $|\mathcal{F}_0| > (2n-1)!!/n^4$, we have $|\mathcal{F}_0 \cap N_h(\mathcal{F}_1)| \neq \emptyset$, thus there exists a path p in $\mathcal{T}_n$ of length no more than $2\sqrt{n/2\log n}$, as desired.

The foregoing shows there exist two perfect matchings $m_1 \in \mathcal{F}$, $m_0 \notin \mathcal{F}$ that are structurally quite similar, differing only in $O(\sqrt{n\log(n)})$ partner swaps, yet

$$1 - \sqrt{\delta(1 + C/n)} < P_{m_1} \text{ and } P_{m_0} < \sqrt{2\delta/n}.$$

Combining inequalities reveals that

$$P_{m_1} - P_{m_0} > (1 - \sqrt{\delta} - O(1/\sqrt{n})).$$

By Equation (3.6.1), this implies that m_1 has many more edges in common with members of $\mathcal{F}$ than m_0 does, more formally,

$$\left(\sum_{ij \in m_1} |\mathcal{F}\downarrow_{ij}|\right) - \left(\sum_{ij \in m_0} |\mathcal{F}\downarrow_{ij}|\right) \geq (2n-5)!! \cdot 2(n-1)(1 - \sqrt{\delta} - O(1/\sqrt{n})).$$

For any $m \in \mathcal{M}_{2n}$, let $m(v)$ denote the partner of $v \in V(K_{2n})$. Let $V(p)$ denote the vertices of p. Let $I \subseteq V(K_{2n})$ denote the set of vertices whose partner left them somewhere along the way, less dramatically,

$$I := \{v \in V(K_{2n}) : m(v) \neq m'(v) \text{ for some } m, m' \in V(p)\}.$$

Clearly $|I| \leq 4\ell$, where ℓ is the length of p, and for any $v \notin I$, we have $m(v) = m'(v)$ for all $m, m' \in V(p)$. We now have

$$\left(\sum_{\substack{ij \in m_1 \\ i \in I}} |\mathcal{F}\downarrow_{ij}|\right) - \left(\sum_{\substack{ij \in m_0 \\ i \in I}} |\mathcal{F}\downarrow_{ij}|\right) \geq (2n-5)!! \cdot 2(n-1)(1 - \sqrt{\delta} - O(1/\sqrt{n})).$$

This of course implies that

$$\sum_{\substack{ij \in m_1 \\ i \in I}} |\mathcal{F}\downarrow_{ij}| \geq (2n-5)!! \cdot 2(n-1)(1-\sqrt{\delta}-O(1/\sqrt{n})).$$

Averaging gives us

$$|\mathcal{F}\downarrow_{ij}| \geq \frac{(2n-5)!! \cdot 2(n-1)}{4\ell}(1-\sqrt{\delta}-O(1/\sqrt{n}))$$

for some $i \in I$. Now we have

$$|\mathcal{F}\downarrow_{ij}| \geq \frac{(2n-5)!! \cdot 2(n-1)}{4\sqrt{n/2\log(n)}}(1-\sqrt{1-c}-O(1/\sqrt{n})) = \omega((2n-5)!!).$$

Lemma 3.3.5 implies that $|\mathcal{F}\downarrow_{ik}| = o((2n-5)!!)$ for all $k \neq j$. Summing over all $k \neq j$, we have

$$|\mathcal{F} \setminus \mathcal{F}\downarrow_{ij}| = \sum_{k \neq j} |\mathcal{F}\downarrow_{ik}| = o((2n-3)!!).$$

This gives us

$$|\mathcal{F}\downarrow_{ij}| = |\mathcal{F}| - |\mathcal{F} \setminus \mathcal{F}\downarrow_{ij}| = (c-o(1))(2n-3)!!.$$

Since $|\mathcal{F}\downarrow_{ij}| = O((2n-3)!!)$, Lemma 3.3.5 again implies

$$|\mathcal{F}\downarrow_{ik}| = O((2n-7)!!)$$

for all $k \neq j$. Summing over all $k \neq j$ again gives

$$|\mathcal{F} \setminus \mathcal{F}\downarrow_{ij}| = \sum_{k \neq j} |\mathcal{F}\downarrow_{ik}| = O((2n-5)!!),$$

which completes the proof of the key lemma. $\square$

Chapter 4

On a Conjecture of Godsil and Meagher

In this chapter, we investigate families of perfect matchings $\mathcal{F} \subseteq \mathcal{M}_{2n}$ that are *t-intersecting*, that is, $|m \cap m'| \geq t$ for any $m, m' \in \mathcal{F}$. In particular, we seek a characterization of the largest t-intersecting families of perfect matchings. The first candidates that come to mind are those families whose members all share a fixed set $T \subseteq E(K_{2n})$ of t disjoint edges:

$$\mathcal{F}_T := \{m \in \mathcal{M}_{2n} : T \subseteq m\},$$

which we call a *canonically t-intersecting family*. In their recent book on Erdős-Ko-Rado combinatorics, Godsil and Meagher posed the following conjecture on t-intersecting families of $\mathcal{M}_{2n}$.

Conjecture (Godsil, Meagher [30]) For all $n \geq 3t/2 + 1$, if $\mathcal{F}$ is a t-intersecting family of perfect matchings of the complete graph K_{2n}, then

$$|\mathcal{F}| \leq (2(n-t)-1)!!,$$

and equality holds if and only if $\mathcal{F}$ is a canonically t-intersecting family.

This chapter's main result is that their conjecture holds for sufficiently large n.

4.0.1 Theorem. *For any $t \in \mathbb{N}$, if $\mathcal{F} \subseteq \mathcal{M}_{2n}$ is a t-intersecting family, then*

$$|\mathcal{F}| \leq (2(n-t)-1)!!$$

for sufficiently large n depending on t. Moreover, equality holds if and only if $\mathcal{F}$ is a canonically t-intersecting family.

Our proof is similar in spirit to a few algebraic proofs of t-intersecting Erdős-Ko-Rado results [30], including the somewhat recent proof of Deza and Frankl's conjecture on *t-intersecting families of permutations*, equivalently, perfect matchings of the bipartite graph $K_{n,n}$.

4.0.2 Theorem. [20, 21] *For any $t \in \mathbb{N}$, if $\mathcal{F}$ is a t-intersecting family of perfect matchings of $K_{n,n}$, then*

$$|\mathcal{F}| \leq (n-t)!.$$

for sufficiently large n depending on t. Moreover, equality holds if and only if $\mathcal{F}$ is a canonically t-intersecting family.

One may interpret our result as the non-bipartite analogue of Theorem 4.0.2. Viewing it as such, the most significant point of departure from the bipartite case is that $\mathcal{M}_{2n}$ does not afford a group structure. Due to this fact, significantly more algebraic overhead and arguments are needed than in the bipartite case.

Before we begin, we first cover some preliminary material needed in order to map out the first part of our main result.

4.1 Preliminaries and a Proof Sketch

A *pseudo-adjacency matrix* of a graph $\Gamma = (V, E)$ is a symmetric $|V| \times |V|$ matrix $\widetilde{A}(\Gamma)$ with constant row sum such that $\widetilde{A}(\Gamma)_{uv} \neq 0$ only if $uv \in E(\Gamma)$. We let η_i denote the eigenvalue associated to the ith eigenspace of a given pseudo-adjacency matrix, and we let $\eta_{\min} := \min_i \eta_i$ denote its least eigenvalue. For any subgraph Γ' of a graph $\Gamma = (V, E)$, let $V(\Gamma') \subseteq V$ be the vertices of Γ'.

It is well-known that the ratio bound of Delsarte and Hoffman encountered in the previous chapter holds for pseudo-adjacency matrices of graphs (see [30]), which will the centerpiece of the first part of our main result.

4.1.1 Theorem (Ratio Bound)**.** *Let $\widetilde{A}(\Gamma)$ be a pseudo-adjacency matrix of $\Gamma = (V, E)$ with eigenvalues $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_{\min}$ and corresponding* *orthonormal eigenvectors* $v_1, v_2 \cdots, v_{\min}$. *If* $S \subseteq V$ *is an independent set of* Γ, *then*

$$|S| \leq |V| \frac{-\eta_{\min}}{\eta_1 - \eta_{\min}}.$$

Moreover, if equality holds, then

$$1_S \in \operatorname{Span}\left(\{v_1\} \cup \{v_i : \eta_i = \eta_{\min}\}\right).$$

After writing $f := 1_S = \sum_{i=1}^{|V|} a_i v_i$ in the basis of orthonormal eigenvectors and setting $\alpha := |S|/|V|$, the ratio bound is easy to see once one observes (4.1.1), which is a consequence of Parseval's identity:

$$0 = f^\top \widetilde{A}(\Gamma) f = \sum_{i=1}^{|V|} \eta_i a_i^2 \geq \eta_1 \alpha^2 + \eta_{\min} \sum_{i=2}^{|V|} a_i^2 = \eta_1 \alpha^2 + \eta_{\min}(\alpha - \alpha^2). \tag{4.1.1}$$

Note that the first equality in the equation above holds due to the fact that f is the characteristic vector of an independent set.

A *t-derangement* of $\mathcal{M}_{2n}$ is a perfect matching $m \in \mathcal{M}_{2n}$ whose cycle type has fewer than t parts of size 1. Their number, denoted as $D_2(n,t)$, can again be counted via a recurrence akin to the classic one for permutation t-derangements:

$$D_2(0,1) = 1; D_2(1,1) = 0;$$
$$D_2(n,1) = 2(n-1)(D_2(n-1,1) + D_2(n-2,1));$$
$$D_2(n,t) = \sum_{i=1}^{t} \binom{2n}{2i} (2i-1)!! D_2(n-i,1).$$

Let us also recall from the previous chapter that

$$D_2(n,1) = (2n-1)!!(1/\sqrt{e} + o(1)) \text{ as } n \to \infty. \tag{4.1.2}$$

The graph that we will apply to the ratio bound is the *perfect matching t-derangement graph* Γ_t defined such that two perfect matchings $m, m' \in \mathcal{M}_{2n}$ are adjacent if they have less than t edges in common, equivalently,

$$\Gamma_t = \sum_{\substack{\lambda \vdash n \\ \operatorname{fp}(\lambda) < t}} A_\lambda$$

where $A_\lambda \in \mathcal{A}_n$. For $t = 1$, we recover the perfect matching derangement graph, which has received a fair amount attention in recent years [30, 31, 43, 45]. It is easy to see that the independent sets of Γ_t are in one-to-one correspondence with t-intersecting families of $\mathcal{M}_{2n}$; therefore, the ratio bound gives an upper bound on the size of a t-intersecting family of $\mathcal{M}_{2n}$.

As we have seen, the eigenspaces of Γ_t and pseudo-adjacency matrices $\widetilde{A}(\Gamma_t)$ belonging to the Bose-Mesner algebra of $\mathcal{A}_n$ are isomorphic to even irreducibles λ of S_{2n}, that is,

$$\lambda = 2\mu = (2\mu_1, 2\mu_2, \cdots, 2\mu_k)$$

for some $\mu \vdash n$. Of these eigenspaces, the ones corresponding to *fat even partitions*, $2\mu \vdash 2n$ such that $2\mu_1 \geq 2(n-t)$, will be of utmost importance.

We are now in a position to outline the proof of the bound of Theorem 4.0.1.

Proof Sketch I

Our goal is to show there exists a pseudo-adjacency matrix $\widetilde{A}(\Gamma_t)$ with eigenvalues $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_{\min}$ satisfying

$$(2(n-t)-1)!! = (2n-1)!!\frac{-\eta_{\min}}{\eta_1 - \eta_{\min}}.$$

We construct this pseudo-adjacency matrix $\widetilde{A}(\Gamma_t)$ by solving a particular a system of linear equations $Mx = b$ such that M is an appropriately defined leading principal minor of the character table P of the perfect matching association scheme $\mathcal{A}_n$. In Section 4.5 we show that M is invertible with entries uniformly bounded by a function of t, the latter of which will allow us to bound the magnitudes of non-fat eigenvalues.

By the ratio bound, such a matrix would imply that any canonically t-intersecting family $\mathcal{F}$ is a maximum independent set of Γ_t and that any maximum independent set S satisfies

$$1_S \in \mathrm{Span}\left(\{v_1\} \cup \{v_i : \eta_i = \eta_{\min}\}\right)$$

where v_i is an eigenvector corresponding to eigenvalue η_i. These two consequences imply

$$\mathrm{Span}\left\{\{v_1\} \cup \{v_i : \eta_i = \eta_{\min}\}\right\} \geq \mathrm{Span}\{1_{\mathcal{F}} : \mathcal{F} \text{ is canonically } t\text{-intersecting}\}.$$

We would like these two spaces to coincide, but the way the right-hand side is defined makes it particularly hard to determine which eigenspaces of our pseudo-adjacency matrix should correspond to the least eigenvalue. It turns out for $t < n/2$, that the span of characteristic functions of canonically t-intersecting families is a subspace of the eigenspaces corresponding to fat even partitions, which we show in Section 4.4.

In light of this, our pseudo-adjacency matrix $\widetilde{A}(\Gamma_t)$ will have $\eta_\lambda = \eta_{\min}$ for all fat even partitions 2λ except for $2\lambda = (2n)$. In Section 4.6, we show this is possible for sufficiently large n by constructing a pseudo-adjacency matrix $\widetilde{A}(\Gamma_t)$ satisfying the foregoing such that $|\eta_\mu| = o(|\eta_\lambda|)$ for all even partitions 2μ that are not fat, which will conclude the proof of the upper bound of our main result. Our proof sketch of the characterization of the extremal families is deferred to Section 4.7.

The strategy outlined above lies atop a somewhat baroque algebraic foundation, which we spend the next few sections developing.

4.2 Finite Group Representation Theory II

For any finite group G, if we let $e_g e_h = e_{gh}$ over the standard basis $\{e_g\}_{g\in G}$ of $\mathbb{C}[G]$, we see that $\mathbb{C}[G]$ is an algebra, the so-called *group algebra* of G. For any $K \leq G$, there is a chain of subalgebras $\mathbb{C}[K\backslash G/K] \leq \mathbb{C}[G/K] \leq \mathbb{C}[G]$ where

$$\mathbb{C}[G/K] = \{f \in \mathbb{C}[G] : f(g) = f(gk)\ \forall g \in G,\ \forall k \in K\}$$

is the algebra of functions that are constant on the right cosets G/K, and

$$\mathbb{C}[K\backslash G/K] = \{f \in \mathbb{C}[G] : f(g) = f(kgk')\ \forall g \in G,\ \forall k, k' \in K\}$$

is the algebra of functions that are constant on the double cosets. Recall that the double cosets $\{Kg_iK\}_{i=1}^{|K\backslash G/K|}$ where $Kg_iK = \{kg_ik' : k, k' \in K\}$ and $g_i \in G$ partition the cosets G/K. The following chain

$$\mathbb{R}[H_n\backslash S_{2n}/H_n] \leq \mathbb{R}[S_{2n}/H_n] \cong \mathbb{R}[\mathcal{M}_{2n}] \leq \mathbb{R}[S_{2n}]$$

will be most relevant to us in this work.

For any (irreducible) representation ϕ of G, the *(irreducible) character* χ_ϕ of ϕ is the map $\chi_\phi : G \to \mathbb{C}$ such that $\chi_\phi(g) := \mathrm{Tr}(\phi(g))$. Similar matrices have the same trace, thus characters of representations are *class functions*, that is, they are constant on conjugacy classes. Furthermore, the characters of the set of all irreducible representations of a group G form an orthonormal basis for the space of all class functions of $\mathbb{C}[G]$.

From these basic properties of characters, it is not hard to show the following.

4.2.1 Lemma. [18] *Let (ρ, V') be an irreducible of G and let (ϕ, V) be a representation of G such that*

$$V \cong \bigoplus_{i=1}^{|\mathcal{C}|} m_i V_i.$$

Then the number of irreducibles V_i equivalent to V' equals $\langle \chi_\phi, \chi_\rho \rangle = m_i$.

Lemma 4.2.3 is a generalization of the familiar fact that the sum of all of primitive nth roots of unity is zero. Its proof is essentially a corollary of Schur's lemma, and it will be helpful for simplifying sums of representations.

4.2.2 Lemma (Schur's Lemma)**.** *Let (φ, V) and (ψ, W) be representations of G, and let $T : V \to W$ be a linear transformation. If $T\varphi(g) = \psi(g)T$ for all $g \in G$, then T is either the zero map or an isomorphism. In particular, if (ϕ, V) is an irreducible of G, then the only linear operators of V that commute with ϕ are scalar multiples of the identity.*

4.2.3 Lemma. *If ρ is a non-trivial irreducible of G, then*

$$\sum_{g \in G} \rho(g) = 0.$$

Proof. For any $k \in G$, we have

$$\begin{aligned} \rho(k)\big(\sum_{g \in G} \rho(g)\big) = \sum_{g \in G} \rho(k)\rho(g) &= \sum_{g' \in kG} \rho(g') \\ &= \sum_{g' \in Gk} \rho(g') \\ &= \big(\sum_{g \in G} \rho(g)\big)\rho(k), \end{aligned}$$

implying that $\sum_{g \in G} \rho(g) = cI$ for some constant c by Schur's Lemma. Since $\rho \neq 1$ is irreducible, it follows that $\langle \chi_\rho, \chi_1 \rangle = 0$, thus $c = 0$. We deduce that $\sum_{g \in G} \rho(g) = 0$, as desired. □

Lemma 4.2.4 shows that direct products of groups and their irreducibles behave as expected.

4.2.4 Lemma. [68] *Let (τ, V) and (ρ, W) be irreducibles of finite groups G and G' respectively. Then $V \otimes W$ is an irreducible of $G \times G'$, and any irreducible of $G \times G'$ is of this form.*

Any representation of $H \leq G$ is isomorphic to a direct sum of irreducibles of G, which we describe below. Let (ρ, V) be a representation of H and $\hat{g}_1, \hat{g}_2, \cdots, \hat{g}_k$ be a system of distinct representatives (SDR) of G/H where $k := [G : H]$. Define

$$V \uparrow_H^G := \bigoplus_{i=1}^{k} \hat{\mathbf{g}}_\mathbf{i} V$$

where $\hat{\mathbf{g}}_\mathbf{i} V \cong V$ is a copy of V associated to the coset $\hat{g}_i H$. For any $g \in G$, there exists an $h_i \in H$ and $\hat{g}_{j(i)} \in G$ where $j(i) \in [k]$ such that $g^{-1}\hat{g}_i = \hat{g}_{j(i)} h_i$. Define $\rho \uparrow_H^G$ to act on $V \uparrow_H^G$ as follows:

$$\left(\rho \uparrow_H^G (g)\right) \sum_{i=1}^{k} \hat{\mathbf{g}}_\mathbf{i} v_i = \sum_{i=1}^{k} \hat{\mathbf{g}}_{\mathbf{j(i)}} \rho(h_i) v_i$$

such that $\hat{\mathbf{g}}_\mathbf{i} v_i \in \hat{\mathbf{g}}_\mathbf{i} V$ and $\hat{\mathbf{g}}_{\mathbf{j(i)}} \rho(h_i) v_i \in \hat{\mathbf{g}}_{\mathbf{j(i)}} V$ for all $g \in G$. We say that $(\rho \uparrow_H^G, V \uparrow_H^G)$ is the *induced representation* of ρ. It is easy to see that $\dim\left(V \uparrow_H^G\right) = k \cdot (\dim V)$, and we may compute the character of $\rho \uparrow_H^G$ as follows:

$$\chi_{\rho\uparrow_H^G}(g) = \sum_{x \in S} \chi_\rho(x^{-1}gx),$$

where S is an SDR for G/H and $\chi_\rho(x^{-1}gx) = 0$ if $x^{-1}gx \notin H$. Notice that $1 \uparrow_K^G$ is equivalent to the permutation representation of G on G/K

$$1 \uparrow_K^G (g) \sum_{i=1}^{k} e_{g_i} = \sum_{i=1}^{k} e_{g_{j(i)}},$$

and so it follows that $1 \uparrow_K^G$ and $\mathbb{C}[G/K]$ are isomorphic.

It is clear that restriction is transitive, that is, $(\rho \downarrow_H^G) \downarrow_K^H \cong \rho \downarrow_K^G$ for any representation ρ of G such that $K \leq H \leq G$. A less trivial fact is that the same is true of induction.

4.2.5 Theorem (Transitivity of Induction [68]). *Let ρ be a representation of K such that $K \leq H \leq G$. Then*

$$(\rho \uparrow_K^H) \uparrow_H^G \cong \rho \uparrow_K^G .$$

A discussion of group representations and characters would not be complete without at least mentioning the following theorem of Frobenius and its corollary.

4.2.6 Theorem (Frobenius Reciprocity for Characters [68]). *For any representation ϕ of G and representation ρ of H, we have*

$$\langle \chi_{\rho\uparrow_H^G}, \chi_\phi \rangle = \langle \chi_{\phi\downarrow_H^G}, \chi_\rho \rangle.$$

4.2.7 Corollary. [68] *Let ϕ be an irreducible of G and ρ be an irreducible of H. Then the the multiplicity of ϕ in $\rho \uparrow_H^G$ equals the multiplicity of ρ in $\phi \downarrow_H^G$.*

The bare essentials of Fourier analysis will be needed for some calculations. Let $f \in \mathbb{C}[G]$ and ϕ be an irreducible of G. The *Fourier transform* of f is a matrix-valued function $\hat{f}$ on irreducible representations

$$\hat{f}(\phi) = \frac{1}{|G|} \sum_{g \in G} f(g)\phi(g).$$

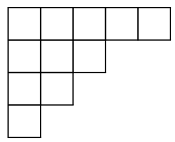
Letting $\hat{G}$ denote the set of irreducibles of G, we may write any $f \in \mathbb{C}[G]$ as

$$f(g) = \frac{1}{|G|} \sum_{\phi \in \hat{G}} (\dim \phi) \; \mathrm{Tr}[\phi(g^{-1})\hat{f}(\phi)].$$

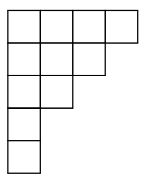
Doing calculations in the Fourier basis for arbitrary non-Abelian groups is usually quite difficult, as it requires a very concrete understanding of the group's irreducibles, which the Reverend Alfred Young divined for the symmetric group at the turn of the last century. The next section provides no more than an illustrated glossary of his theory, and we refer the reader to [53, 71] for first-rate introductions to the subject.

4.2.1 Representation Theory of the Symmetric Group

Recall that we may visualize any λ as a Ferrers diagram, and when referencing a Ferrers diagram, we alias λ as the *shape*. For example, the Ferrers diagram below has shape $(5, 3, 2, 1) \vdash 11$:

 .

Let $\lambda' \vdash n$ denote the *transpose* of λ, that is, the partition obtained by interchanging the columns and the rows of the corresponding Ferrers diagram of λ. The transpose of $(5, 3, 2, 1) \vdash 11$ is $(4, 3, 2, 1, 1) \vdash 11$, as illustrated below:

 .

The following is a picture show exhibition of *Young tableaux*, which we simply refer to as *tableaux*.

A *generalized* λ*-tableau* T is a set of n cells arranged in k left-justified ordered rows such that ordered row i has λ_i cells and each cell is given a number of $[n]$.

1	9	7	3	1
8	6	6		
5	2			
2				

A λ*-tableau* T is a set of n cells arranged in k left-justified ordered rows such that ordered row i has λ_i cells and each cell is assigned a unique number of $[n]$.

1	2	7	8	9
4	10	5		
3	6			
11				

A *standard λ-tableau* T_λ is a λ-tableau with entries strictly increasing along rows and strictly increasing along columns.

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 5 & 8 & 9 \\ \hline 3 & 6 & 7 \\ \cline{1-3} 4 & 10 \\ \cline{1-2} 11 \\ \cline{1-1}\end{array}$$

A *semistandard λ-tableau* T_λ is a generalized λ-tableau where the numbers are weakly increasing along the rows and strictly increasing along the columns. The multiplicity of each number is its *weight*, and the weights of $1, 2, \cdots, n$ are recorded as a k-tuple $\mu = (\mu_1, \mu_2, \cdots, \mu_k)$ where μ_i is the number of times that the number i occurs. For example, we have below a semistandard $(5, 3, 2, 1)$-tableau with weight $(1, 2, 4, 3, 1)$.

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 3 & 3 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 4 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

A *semistandard λ/μ-tableau* is a generalized tableau of skew shape λ/μ weakly increasing along rows and strictly increasing along columns.

Let $K_{\lambda,\mu}$ be the number of semistandard tableaux of shape λ and weight μ. The $K_{\lambda,\mu}$'s are called the *Kostka numbers* and we will meet a generalization of them in Section 4.5.

It turns out that the *tabloids* are actually a reliable source of information for the representation theory of symmetric group. A λ-*tabloid* $\{T\}$ is a collection of n cells, arranged in k left-justified unordered rows, such that unordered row i has λ_i cells and each of the n cells is labeled by a unique number of $[n]$. To emphasize the lack of order along the rows, we draw them like so

$$\begin{array}{ccccc}\hline 1 & 2 & 7 & 8 & 9 \\ \hline 4 & 10 & 5 \\ \cline{1-3} 3 & 6 \\ \cline{1-2} 11 \\ \cline{1-1}\end{array}\ .$$

The notation $\{T\}$ suggests that each λ-tabloid is an equivalence class of λ-tableaux, which is indeed the case. For any λ-tableau T, let $\{T\}$ be the tabloid that T belongs to. Let $\mathbb{T}_\lambda$ denote the set of all λ-tabloids. Note that

S_n acts on the entries of the cells of λ-tableaux and λ-tabloids in the obvious way

For any λ-tableau T, let $C_T \leq S_n$ be the *column-stabilizer* of T, i.e., the permutation group that fixes the columns of T setwise. In particular, if T has shape λ, then we may write the column-stabilizer as

$$C_T \cong S_{(\lambda')_1} \times S_{(\lambda')_2} \times \cdots \times S_{(\lambda')_{\lambda_1}}$$

where $S_{(\lambda')_j}$ acts on the row indices of the jth column of T, that is,

$$\sigma t_{i,j} = t_{\sigma(i),j}$$

for all $\sigma \in S_{(\lambda')_j}$ and cells $t_{i,j} \in T$ in the jth column of T. Having C_T act on the indices of cells rather than their entries, which is traditionally the case, will be useful for a few calculations in that arise in the proof of the key stability lemma in Section 4.9.

Let $(\phi, \mathbb{R}[\mathbb{T}_\lambda])$ be the permutation representation of S_n acting on $\mathbb{R}[\mathbb{T}_\lambda]$ with the standard basis $\{e_{\{T\}} : \{T\} \in \mathbb{T}_\lambda\}$ in the natural way. We briefly discuss how this representation decomposes into irreducibles, as several objects therein will resurface in Section 4.8.

For each λ-tableau T, define the *T-polytabloid* to be the following (± 1)-linear combination of λ-tabloids

$$e_T := \sum_{\pi \in C_T} \operatorname{sign}(\pi) e_{\{\pi T\}}.$$

To give an example, if we let

$$T = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 6 & 7 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}\,,$$

then its polytabloid is just

$$e_T = \begin{array}{ccccc}\hline 1 & 2 & 3 & 6 & 7 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} - \begin{array}{ccccc}\hline 4 & 2 & 3 & 6 & 7 \\ \hline 1 & 5 \\ \cline{1-2}\end{array} - \begin{array}{ccccc}\hline 1 & 5 & 3 & 6 & 7 \\ \hline 4 & 2 \\ \cline{1-2}\end{array} + \begin{array}{ccccc}\hline 4 & 5 & 3 & 6 & 7 \\ \hline 1 & 2 \\ \cline{1-2}\end{array}\,.$$

It is clear that for each $\lambda \vdash n$ the permutation representation of S_n acting on the set

$$\{e_T : T \text{ is a } \lambda\text{-tableau}\}$$

is a subrepresentation of $(\phi, \mathbb{R}[\mathbb{T}_\lambda])$. Specht showed this is in fact an irreducible of S_n and that $\{e_T : T \text{ is a standard } \lambda\text{-tableau}\}$ forms a basis for

$$S^\lambda := \mathrm{Span}\{e_T : T \text{ is a } \lambda\text{-tableau}\}.$$

This implies S^λ has dimension f^λ, which gives us another proof of the following.

4.2.8 Theorem (Dimension Formula [53])**.** $\dim S^\lambda = f^\lambda$.

We have been and will continue to abuse notation by letting λ refer to S^λ.

Another result that we require is a classical rule due to Pieri, which we state in its representation-theoretic form. A skew shape is a *horizontal strip* if each column has no more than one cell. For example, $(5,3,2,1)/(3,2,1)$ is a horizontal strip.

4.2.9 Theorem (Pieri's Rule [53])**.** *Let (m) be the trivial representation of S_m and let μ be any irreducible of S_n. Then*

$$((m) \otimes \mu) \uparrow_{S_m \times S_n}^{S_{m+n}} \cong \bigoplus_\lambda \lambda,$$

where the direct sum ranges over all partitions $\lambda \vdash (m+n)$ obtainable from μ by adding m cells to its Ferrers diagram, no two in the same column.

We end this section with some non-standard combinatorial terminology for tableaux and a few bounds on the dimensions of irreducibles of the symmetric group. Let $\lambda(n)$ denote the set of all partitions of n, and let $2\lambda(n)$ be the set of all even partitions of $\lambda(2n)$.

Following Ellis, Friedgut, and Pilpel [20], for any $t < n/2$, we say that $\lambda \vdash n$ is a *fat partition* if $\lambda \geq (n-t, 1^t)$, *tall partition* if $\lambda' \geq (n-1, 1^t)$, or *medium partition* otherwise. For any $t < n/2$, we say that an even partition $\lambda \vdash 2n$ is a *fat partition* if $\lambda \geq 2(n-t, 1^t)$, and is a *non-fat partition* otherwise. Observe that there are no tall even partitions for $t < n/2$.

4.2.10 Proposition. *For any $t < n/2$, there is no $\mu \in 2\lambda(n)$ such that μ' is fat.*

Another simple fact is that for any $t < n/2$, the number of fat partitions, which we denote as F_t, does not depend on n.

4.2.11 Proposition. *For any constant $t < n/2$, the number of fat partitions of $\lambda(n)$ is a constant F_t that depends only on t, and F_t is also equal to the number of fat partitions of $2\lambda(n)$.*

For example, if $t = 3$ and $n > 8$, then there are $F_3 = |\lambda(0)| + |\lambda(1)| + |\lambda(2)| + |\lambda(3)|$ partitions of $\lambda(n)$ that have no more than 3 cells below the first row, and they are

$$(n), (n-1,1), (n-2,2), (n-2,1,1), (n-3,3), (n-3,2,1), (n-3,1,1,1).$$

Lemma 4.2.12 is a lower bound on the dimension of non-fat irreducibles that follows immediately from the proof of [20, Lemma 2].[1]

4.2.12 Lemma. [20] *For any $t \in \mathbb{N}$, there exists a constant $C_t > 0$ depending only on t such that $\dim 2\lambda \geq C_t(2n)^{2(t+1)}$ for any non-fat even partition $2\lambda \vdash 2n$.*

Recall that for any $\lambda \vdash n$, rather crudely, we have

$$f^\lambda \leq \binom{n}{\lambda_1} f^{(\lambda_2,\lambda_3,\cdots,\lambda_\ell)}.$$

Since $\dim \lambda = f^\lambda$, we also have the representation-theoretic count

$$|S_n| = \sum_{\lambda \vdash n} (f^\lambda)^2,$$

which implies that $f^\lambda \leq \sqrt{n!}$. These two inequalities imply the following.

4.2.13 Theorem. *For any irreducible λ of S_n, we have $\dim \lambda \leq \binom{n}{\lambda_1}\sqrt{(n-\lambda_1)!}$.*

4.3 Association Schemes II

If a symmetric association scheme arises from a group action, like $\mathcal{A}_n$ and $\mathcal{A}'_n$, the entries of its character table can be described in terms of the spherical functions of a finite symmetric *Gelfand pair* [4].

[1] To avoid confusion, we note a typo in the statement of [20, Lemma 2]: "... greater than $n-k$ ..." should be "... less than $n-k$ ...".

4.3.1 Finite Gelfand Pairs

For a more detailed introduction to the theory of finite Gelfand pairs, see [13].

4.3.1 Theorem. [53] *Let $K \leq G$ be a group. Then the following are equivalent.*

1. *(G, K) is a Gelfand Pair;*
2. *The induced representation $1 \uparrow_K^G \cong \bigoplus_{i=1}^m V_i$ (equivalently, the permutation representation of G acting on G/K) is multiplicity-free;*
3. *The double-coset algebra $\mathbb{C}[K\backslash G/K]$ is commutative.*

Moreover, a Gelfand pair is symmetric if $KgK = Kg^{-1}K$ for all $g \in G$.

The symmetric Gelfand pair that we will be working with is (S_{2n}, H_n) [53, Ch. VII]. Let (G, K) be a Gelfand pair, $X := G/K$, and define χ_i to be the character of V_i as in the second statement of Theorem 4.3.1, with dimension $d_i := \chi_i(1)$. The functions $\omega^1, \omega^2, \cdots, \omega^m \in \mathbb{C}[X]$ defined such that

$$\omega^i(g) = \frac{1}{|K|} \sum_{k \in K} \chi_i(g^{-1}k) \quad \forall g \in G$$

are called the *spherical functions* and form an orthogonal basis for $\mathbb{C}[K\backslash G/K]$, equivalently, the algebra of (left) K-invariant functions of $\mathbb{C}[X]$. For any two indices i, j, define $\delta_{i,j} = 1$ if $i = j$; otherwise, $\delta_{i,j} = 0$.

4.3.2 Proposition. [13] *The spherical functions form a basis for the space of (left) K-invariant functions in $\mathbb{C}[X]$ and satisfy the orthogonality relations*

$$\langle \omega^i, \omega^j \rangle_X = \sum_{x \in X} \omega^i(x)\overline{\omega^j(x)} = \delta_{i,j} \frac{|X|}{d_i}.$$

The (left) K-orbits of X partition the cosets into $(K\backslash G/K)$-double cosets, or so-called *spheres* $\Omega_0, \Omega_1, \cdots, \Omega_m$. It is helpful to think of spheres and spherical functions as the spherical analogues of conjugacy classes and irreducible characters respectively. Indeed, the spherical functions are constant on spheres, and it can be shown that the number of distinct spherical functions equals the number of distinct irreducibles of $\mathbb{C}[X]$, equivalently, the number of spheres of X [13]. We write ω^i_j for the value of the spherical function ω^i corresponding to the ith irreducible on the double coset corresponding to Ω_j.

For any choice of $K \leq G$, a general procedure is given in [4] for constructing a (not necessarily commutative) association scheme whose Hecke algebra is isomorphic to $\mathbb{C}[K\backslash G/K]$. An association scheme $\mathcal{A}$ that arises from this construction will be called a $(K\backslash G/K)$-*association scheme*. In such a scheme, there is a natural bijection between the associates of $\mathcal{A}$ and the double cosets, and if $\mathcal{A}$ is a $(K\backslash G/K)$-association scheme, then $\mathcal{A}$ is symmetric if and only if (G,K) is a symmetric Gelfand pair [4].

The following theorem is a representation-theoretic characterization of the eigenvalues of any graph that belongs to the Bose-Mesner algebra of a commutative $(K\backslash G/K)$-association scheme.

4.3.3 Theorem. [45, 30] *Let $\Gamma = \sum_{j\in\Lambda} A_j$ be a sum of associates in a $(K\backslash G/K)$-association scheme such that (G,K) is a Gelfand pair and Λ is the index set of some subset of the associates. The eigenvalue η_i of Γ corresponding to the ith irreducible representation of $1\uparrow_K^G$ has multiplicity d_i and can be written as*

$$\eta_i = \sum_{j\in\Lambda} |\Omega_j|\omega^i_j.$$

4.3.4 Proposition. *Let $\lambda, \mu \vdash n$. Then $|\omega^\lambda_\mu| \leq 1$.*

Proof. Suppose there exists a λ, μ such that $|\omega^\lambda_\mu| > 1$. Then by Theorem 4.3.3, the μ-associate of $\mathcal{A}$ has an eigenvalue with magnitude is greater than its row sum $|\Omega_\mu|$, contradicting the Perron-Frobenius Theorem. □

The following lemma is a crude but useful upper bound on the magnitudes of the eigenvalues of such graphs.

4.3.5 Lemma. *Let $\Gamma = \sum_{j\in\Lambda} A_j$ be a sum of associates in a $(K\backslash G/K)$-association scheme such that (G,K) is a Gelfand pair, $X = G/K$, and Λ is the index set of some subset of the associates. Then for ith irreducible representation of $1\uparrow_K^G$ we have*

$$|\eta_i| \leq \sqrt{|X||\Omega_\Lambda|/d_i},$$

where $\Omega_\Lambda = \cup_{j\in\Lambda}\Omega_j$ is a disjoint union of spheres indexed by Λ.

Proof. Let $f \in \mathbb{C}[X]$ be the characteristic function of Ω_Λ. Theorem 4.3.3 implies that

$$\eta_i = \langle \omega^i, f \rangle_X,$$

By Proposition 4.3.2, we have $\langle \omega^i, \omega^i \rangle_X = |X|/d_i$ for any spherical function ω^i, so by the Cauchy-Schwarz inequality we see that

$$|\eta_i| = |\langle \omega^i, f \rangle_X| \leq \sqrt{\langle \omega^i, \omega^i \rangle_X \langle f, f \rangle_X} = \sqrt{|X||\Omega_\Lambda|/d_i},$$

as desired. □

Since (S_{2n}, H_n) is a symmetric Gelfand pair, the permutation representation of S_{2n} acting on $\mathcal{M}_{2n}$, equivalently $1 \uparrow_{H_n}^{S_{2n}}$, admits a multiplicity-free decomposition into irreducibles of S_{2n}, which in the previous chapter we saw to be

$$1 \uparrow_{H_n}^{S_{2n}} \cong \bigoplus_{\lambda \vdash n} 2\lambda.$$

The corresponding symmetric $(H_n \backslash S_{2n}/H_n)$-association scheme is the perfect matching association scheme $\mathcal{A}_n$. Note that the valencies v_λ of $\mathcal{A}_n$ are simply the sizes of the λ-*spheres*

$$\Omega_\lambda := \{m \in \mathcal{M}_{2n} : d(m^*, m) = \lambda\}$$

for all $\lambda \vdash n$. The λ-spheres partition $\mathcal{M}_{2n}$, and as mentioned before, play the role of conjugacy classes in our spherical setting. Indeed, the following proposition is reminiscent of the elementary formula for the number of permutations in a given conjugacy class.

4.3.6 Proposition. [53] *For any $\lambda \vdash n$, let m_i denote the number of parts of λ that equal i, and define $z_\lambda := \prod_{i \geq 1} i^{m_i} m_i!$. Then we have*

$$|\Omega_\lambda| = \frac{(2n)!!}{2^{\ell(\lambda)} z_\lambda}.$$

Lemma 4.3.7 essentially follows from [20, Lemma 5], but we include a proof for sake of completeness.

4.3.7 Lemma. [20] *Let $k < n/2$. If Ω_λ is a sphere such that λ has a part of size $n - k$, then*

$$\frac{2^n n!}{2(n-k)(2k)^k} \leq |\Omega_\lambda| \leq 2^{n+1}(n-1)!$$

Proof. Let $l := \ell(\lambda)$. Note that $\lambda_2 + \lambda_3 + \cdots + \lambda_l = k$, thus $l - 1 \leq k$. By Proposition 4.3.6, we have

$$|\Omega_\lambda| = \frac{2^n n!}{2(n-k)2^{l-1} z_{\lambda - \lambda_1}}.$$

Using the arithmetic mean-geometric mean inequality (AM/GM), we have

$$\begin{aligned}
n/2 < n - k < 2(n-k)&2^{l-1} z_{\lambda-\lambda_1} \\
&\leq 2(n-k)2^{l-1}(l-1)! \prod_{i=2}^{l} \lambda_i \\
&\leq 2(n-k)2^{l-1}(l-1)! \left(\frac{1}{(l-1)}\right)^{l-1} \qquad \text{(AM/GM)} \\
&\leq 2(n-k)(2k)^{l-1} \\
&\leq 2(n-k)(2k)^k,
\end{aligned}$$

and so we arrive at

$$\frac{2^n n!}{2(n-k)(2k)^k} \leq |\Omega_\lambda| \leq 2^{n+1}(n-1)!,$$

which completes the proof. □

4.4 The Support of Canonically t-Intersecting Families

The main result of this section is that the characteristic functions of canonically t-intersecting families of $\mathcal{M}_{2n}$ are supported on the "even low frequencies" of the Fourier spectrum of S_{2n}. More precisely, for any $t < n/2$, let U_t be the space of functions of $\mathbb{R}[\mathcal{M}_{2n}]$ supported on the fat even partitions, that is,

$$U_t = \{f \in \mathbb{R}[\mathcal{M}_{2n}] : \widehat{f}(\rho) = 0 \text{ for all } \rho < 2(n-t, 1^t)\}.$$

4.4.1 Theorem. *For any $t < n/2$, we have*

$$\text{Span}\{1_{\mathcal{F}} \in \mathbb{R}[\mathcal{M}_{2n}] : \mathcal{F} \subseteq \mathcal{M}_{2n} \text{ is canonically } t\text{-intersecting}\} \leq U_t.$$

Before we prove this theorem, some preliminaries are in order. Every $m \in \mathcal{M}_{2n}$ corresponds to a right coset σH_n for some $\sigma \in S_{2n}$, so there is a natural isomorphism between $\mathbb{R}[\mathcal{M}_{2n}]$ and the algebra of right H_n-invariant functions of $\mathbb{R}[S_{2n}]$, that is

$$\mathbb{R}[\mathcal{M}_{2n}] \cong \{f \in \mathbb{R}[S_{2n}] : f(\sigma) = f(\sigma h)\ \forall \sigma \in S_{2n},\ \forall h \in H_n\} \leq \mathbb{R}[S_{2n}].$$

Let $\widetilde{f} \in \mathbb{R}[S_{2n}]$ denote the H_n-invariant function of the group algebra of S_{2n} corresponding to $f \in \mathbb{R}[\mathcal{M}_{2n}]$ under this isomorphism, and for any set $\mathcal{F} \subseteq \mathcal{M}_{2n}$, let $\widetilde{\mathcal{F}} \subseteq S_{2n}$ denote the corresponding set of permutations of size $|\mathcal{F}|(2n)!!$ that is a union of right cosets. Under this isomorphism, it also follows that

$$\langle \widetilde{f}, \widetilde{f'} \rangle_{S_{2n}} = (2n)!! \langle f, f' \rangle_{\mathcal{M}_{2n}}$$

for any $f, f' \in \mathbb{R}[\mathcal{M}_{2n}]$. For any canonically t-intersecting family $\mathcal{F}_T$ we have

$$|\widetilde{\mathcal{F}_T}| = (2(n-t)-1)!!(2n)!! = (2(n-t))!2^t(n)_t.$$

Let $K := \mathrm{Stab}_{S_{2n}}(T) \cong S_{2(n-t)} \times H_t$ be the stabilizer of T with respect to S_{2n}. Recall that $T \subseteq m$ for each $m \in \mathcal{F}_T$, thus $K \subseteq \widetilde{\mathcal{F}_T}$; however, these are not the only permutations that keep the edges of T together. This can be seen by observing

$$\frac{|\widetilde{\mathcal{F}_T}|}{|K|} = \frac{(2(n-t))!2^t(n)_t}{(2(n-t))!2^t t!} = \binom{n}{t},$$

which suggests the following proposition that is not hard to see.

4.4.2 Proposition. *If $\mathcal{F}_T \subseteq \mathcal{M}_{2n}$ be canonically t-intersecting family, then its corresponding characteristic H_n-invariant function of $\mathbb{R}[S_{2n}]$ can be written as*

$$1_{\widetilde{\mathcal{F}_T}} = \sum_{s \in S} 1_{sK}$$

where S is a set of $\binom{n}{t}$ representatives of distinct cosets of S_{2n}/K and $1_{sK} \in \mathbb{R}[S_{2n}]$ is the characteristic function of the corresponding coset.

4.4.3 Lemma. *Let $\nu \vdash 2n$ be an irreducible of S_{2n} and let*

$$K := (S_{2(n-t)} \times H_t) \leq (S_{2(n-t)} \times S_{2t}) =: H \leq S_{2n}.$$

Then the multiplicity of the trivial representation 1_K in $\nu \downarrow_K^{S_{2n}}$ equals the number of partitions $\mu \vdash t$ such that the shape $\nu \vdash 2n$ can be obtained from the shape 2μ by adding $2(n-t)$ cells, no two in the same column.

Proof. For any group H, let 1_H denote the trivial representation of H. By Lemma 4.2.1, the multiplicity of 1_K in $\nu \downarrow_K^{S_{2n}}$ is $\langle \chi_{1_K}, \chi_{\nu\downarrow_K^{S_{2n}}} \rangle$. We have

$$\begin{aligned}
\langle \chi_{1_K}, \chi_{\nu\downarrow_K^{S_{2n}}} \rangle &= \langle \chi_{1_K \uparrow_K^{S_{2n}}}, \chi_\nu \rangle && \text{(Frobenius Reciprocity)} \\
&= \langle \chi_{1_{S_{2(n-t)}} \otimes 1_{H_t} \uparrow_K^H \uparrow_H^{S_{2n}}}, \chi_\nu \rangle && \text{(Lemma 4.2.4 \& Transitivity)} \\
&= \langle \chi_{(2(n-t)) \otimes 1_{H_t} \uparrow_{H_t}^{S_{2t}} \uparrow_H^{S_{2n}}}, \chi_\nu \rangle \\
&= \langle \chi_{\bigoplus_{\mu \vdash t} ((2(n-t)) \otimes (2\mu)) \uparrow_H^{S_{2n}}}, \chi_\nu \rangle && \text{(Theorem 3.1.3)} \\
&= \langle \chi_{\bigoplus_{\mu \vdash t} (((2(n-t)) \otimes (2\mu)) \uparrow_H^{S_{2n}})}, \chi_\nu \rangle && \text{(Linearity of } \uparrow) \\
&= \sum_{\mu \vdash t} \langle \sum_\lambda \chi_\lambda, \chi_\nu \rangle && \text{(Pieri's Rule)}
\end{aligned}$$

where $\sum_\lambda$ ranges over partitions $\lambda \vdash 2n$ obtainable from 2μ by adding $2(n-t)$ cells, no two in the same column. The result now follows from Lemma 4.2.1. □

4.4.4 Corollary. *Let $\nu \vdash 2n$ be a non-fat even partition. Then the trivial representation of K does not occur in $\nu \downarrow_K^{S_{2n}}$.*

Proof. Since ν is non-fat, we have $\nu_1 < 2(n-t)$ cells, but then ν cannot be obtained by adding $2(n-t)$ cells, no two in the same column, to any $\mu \vdash t$. □

Proof of Theorem 4.4.1: Let $f := 1_{\widetilde{\mathcal{F}_T}} \in \mathbb{R}[S_{2n}]$ be the characteristic function of a canonically t-intersecting family $\mathcal{F}_T \subseteq \mathcal{M}_{2n}$. Proposition 4.4.2 implies that $f = \sum_{s \in S} 1_{sK}$ where K is the stabilizer of T in S_{2n} and S is a set of $\binom{n}{t}$ coset representatives.

Let ρ be a non-fat even irreducible. Applying the Fourier transform gives us

$$\begin{aligned}
\widehat{f}(\rho) &= \frac{1}{(2n)!} \sum_{\sigma \in S_{2n}} f(\sigma)\rho(\sigma) \\
&= \frac{1}{(2n)!} \sum_{\sigma \in \widetilde{\mathcal{F}_T}} \rho(\sigma) \\
&= \frac{1}{(2n)!} \sum_{s \in S} \rho(s) \left(\sum_{k \in K} \rho(k) \right) \\
&= \frac{1}{(2n)!} \sum_{s \in S} \rho(s) \left(\sum_{k \in K} \rho \downarrow_K^{S_{2n}} (k) \right).
\end{aligned}$$

By Corollary 4.4.4, the trivial representation does not appear in $\rho \downarrow_K^{S_{2n}}$, so writing $\rho \downarrow_K^{S_{2n}}$ as a direct sum of irreducibles and applying Lemma 4.2.3 gives

$$= \frac{1}{(2n)!} \sum_{s \in S} \rho(s) \left(\sum_{k \in K} \rho \downarrow_K^{S_{2n}} (k) \right) = 0,$$

which completes the proof of Theorem 4.4.1.

The following lemma shows there is a canonically t-intersecting family whose characteristic function has non-zero Fourier weight on $(2(n-t, 1^t))$, showing that Theorem 4.4.1 is best possible in some sense.

4.4.5 Lemma. *Let $S = \{\{3,4\},\{5,6\},\cdots,\{2t+1,2t+2\}\}$. Then*

$$\widehat{1_{\mathcal{F}_S}}(2(n-t,1^t)) \neq 0.$$

Proof. Let T be the unique standard Young tableau of shape $2(n-t,1^t)$ such that the second row of T is $\{3,4\}$, the third row of T is $\{5,6\}$, and so on. Define $1_{\{T\}} \in \mathbb{R}[\mathcal{M}_{2n}]$ such that $1_{\{T\}}(m) = 1$ if the endpoints of each edge of m both exist in the same row of $\{T\}$; otherwise, $1_{\{T\}}(m) = 0$. Let

$$f_T = \sum_{\sigma \in C_T} \operatorname{sign}(\sigma) 1_{\{\sigma T\}},$$

which lives in the $2(n-t,1^t)$ irreducible subspace of $\mathbb{R}[\mathcal{M}_{2n}]$ (see [13, Ch.11] or Section 4.8 for a proof). For each $m \in \mathcal{F}_S$, we have

$$1_{\mathcal{F}_S}(m) \cdot \operatorname{sign}(\sigma) 1_{\{\sigma T\}}(m) \neq 0 \quad \text{ if and only if } \quad \sigma = \sigma_1 \sigma_2$$

where σ_1 and σ_2 are disjoint permutations that act on the cells of first and second columns respectively of T in the same way. Any such σ is even, so we have $\langle 1_{\mathcal{F}_S}, f_T \rangle_{\mathcal{M}_{2n}} > 0$. This implies that the projection of $1_{\mathcal{F}_S}$ onto the irreducible $2(n-t,1^t)$ is not zero, so we have $\widehat{1_{\mathcal{F}_S}}(2(n-t,1^t)) \neq 0$. □

4.5 Symmetric Functions II

We now resume our discussion of the character table P of the perfect matching association scheme $\mathcal{A}_n$ from the viewpoint of symmetric functions and their transition matrices, which will lead to compact proofs of a few results needed to show that a pseudo-adjacency matrix of the perfect matching t-derangement graph with the correct eigenvalues exists. The majority of the material in this section can be again found in Macdonald's text [53].

Recall that when the power sum symmetric functions $\{p_\lambda\}_{\lambda \vdash n}$ are expressed in terms of the Schur functions, we obtain the characters of S_n. Similarly, when the power sum symmetric functions are expressed in the monomial symmetric function basis, we get the so-called *permutation characters*:

$$p_\lambda = \sum_{\mu \vdash n} D_{\lambda,\mu} m_\mu,$$

where $D_{\lambda,\mu}$ is equal to the number of ordered partitions $\pi = (B_1, \cdots, B_{\ell(\mu)})$ of the set $\{1, 2, \cdots, \ell(\lambda)\}$ such that

$$\mu_j = \sum_{i \in B_j} \lambda_i$$

for all $1 \leq j \leq \ell(\mu)$, see [71] for a proof.

It will be instructive to first give a short proof of [20, Theorem 20] via symmetric functions. We begin by recalling a few well-known results.

4.5.1 Theorem. [53] *The matrix $M(p, m) = (D_{\lambda,\mu})$ is lower-triangular*

4.5.2 Theorem. *The matrix $M(m, s) = (K_{\lambda,\mu})^{-1}$ is upper-unitriangular.*

4.5.3 Theorem. [38] *An invertible matrix admits an LU-decomposition if and only if all its leading principal minors are nonsingular.*

These theorems provide an easy proof of the following.

4.5.4 Theorem. *Any leading principal minor of the character table of the symmetric group is invertible.*

Proof. The character table of S_n is a transition matrix, thus it is invertible. Its LU-decomposition is $L = (D_{\lambda,\mu})$ and $U = (K_{\lambda,\mu})^{-1}$. □

The leading principal minors relevant to us are the ones induced by all the fat partitions except for the skinniest fat partition $(n-t, 1^t)$. Define $F := F_t - 1$ where $t < n/2$. Macdonald observed that such minors exhibit a "stability" property (not to be confused with stability in extremal combinatorics). Let $D(n)$ and $K(n)$ be the transition matrices $M(p, m)$ and $M(s, m)$ indexed by partitions $\lambda \vdash n$ in reverse-lexicographic order. For the statements of the following results, we assume that $n > 2t$.

4.5.5 Lemma. [53] *The $F \times F$ leading principal minor of $K(n)$ (resp. $K(n)^{-1}$) equals the $F \times F$ leading principal minor of $K(n')$ (resp. $K(n')^{-1}$) for all $n' \geq n$.*

Essentially the same combinatorial argument described in [53, pg. 105] can show a similar result for $D(n)$, surely known to Macdonald, but also proven in [20].

4.5.6 Lemma. *The $F \times F$ leading principal minor of $D(n)$ (resp. $D(n)^{-1}$) equals the $F \times F$ leading principal minor of $D(n')$ (resp. $D(n')^{-1}$) for all $n' \geq n$.*

4.5.7 Corollary. *The $F \times F$ leading principal minor of S_n's character table equals the $F \times F$ leading principal minor of $S_{n'}$'s character table for all $n' \geq n$.*

The following result is now immediate.

4.5.8 Theorem. [20, Theorem 20] *The $F \times F$ leading principal minor of S_n's character table is invertible with entries uniformly bounded by a function of t.*

We seek a similar theorem for the zonal spherical analogue of characters, the so-called *zonal characters*. Define the *zonal character table* to be the $\lambda(n) \times \lambda(n)$ matrix (ω_ρ^λ) such that the (λ, ρ)-entry is given by ω_ρ^λ, i.e., evaluations of the spherical functions of the Gelfand pair (S_{2n}, H_n). Such characters arise naturally as coefficients of the normalized zonal polynomials Z'_λ, but first let us focus on the unnormalized zonal polynomials Z_λ.

Recall that the coefficients of Z_λ expressed in the power sum basis are the λ-eigenvalues of the associates $A_\rho \in \mathcal{A}$ (see [53, pg. 413]), which we may now compare to Theorem 4.3.3:

$$Z_\lambda = |H_n| \sum_{\rho \vdash n} z_{2\rho}^{-1} \omega_\rho^\lambda p_\rho = \sum_{\rho \vdash n} |\Omega_\rho| \omega_\rho^\lambda p_\rho.$$

If we invert, normalize, and define $Z'_\lambda := \frac{|H_n|}{c_2(\lambda)} Z_\lambda$, we obtain

$$p_\rho = \sum_{\lambda \vdash n} \frac{|H_n|}{c_2(\lambda)} \omega_\rho^\lambda Z_\lambda = \sum_{\lambda \vdash n} \omega_\rho^\lambda Z'_\lambda.$$

Since column and row normalization does not affect invertibility, we can easily deduce the following two results.

4.5.9 Theorem. *Every leading principal minor of the zonal character table* (ω_ρ^λ) *is invertible with each entry no greater than one.*

Proof. The transition matrix $M(m, Z)$ is upper-unitriangular [53, pg. 408]. The LU-decomposition of $M(p, Z)$ is $L = (D_{\lambda,\mu})$ and $U = M(m, Z)$. Since (ω_ρ^λ) is equivalent to $M(p, Z)$ up to normalization, it follows that (ω_ρ^λ) is an invertible matrix that admits an LU-decomposition. We deduce that the leading principal minors of (ω_ρ^λ) are invertible, and Proposition 4.3.4 shows the magnitudes of its entries are no greater than one.

4.5.10 Corollary. *Any leading principal minor of P is invertible.*

We now show the transition matrix $(K_{\lambda,\mu}^{(2)}) = M(Z', m)$ enjoys a "stability" property akin to Lemma 4.5.5. More generally, we show that the (i, j)-entry of the leading $F \times F$ principal minor of the transition matrix $(K_{\lambda,\mu}^{(\alpha)}) = M(J^{(\alpha)}, m)$ for $\alpha > 1$ is bounded above by the (i, j)-entry of the leading $F \times F$ principal minor of the matrix $(K_{\lambda,\mu}^{(1)})$. Macdonald [53] gives a combinatorial rule for computing the entries of these matrices, which we describe below.

For any cell $s \in \lambda$, we define

$$b_\lambda^{(\alpha)}(s) := \frac{\alpha a_\lambda(s) + l_\lambda(s) + 1}{\alpha a_\lambda(s) + l_\lambda(s) + \alpha},$$

which is less than 1 for all $\alpha > 1$. Let

$$\Psi_{\lambda/\mu}^{(\alpha)} := \prod_{s \in R_{\lambda/\mu} \setminus C_{\lambda/\mu}} \frac{b_\mu^{(\alpha)}(s)}{b_\lambda^{(\alpha)}(s)}$$

where $C_{\lambda/\mu}$ (resp. $R_{\lambda/\mu}$) is the union of the columns (resp. rows) that intersect λ/μ. Then let

$$\Psi_T^{(\alpha)} := \prod_{i=1}^{r} \Psi_{\lambda^{(i)}/\lambda^{(i-1)}}^{(\alpha)},$$

where $0 = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(r)}$ is the sequence of partitions determined by the tableau T so that each skew diagram $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal strip. Finally, we have

$$K_{\lambda,\mu}^{(\alpha)} = \sum_T \Psi_T^{(\alpha)}$$

where T ranges over semistandard tableaux of shape λ and weight μ. Observe that when $\alpha = 1$, these are simply the Kostka numbers. It is now a simple matter to deduce that the Kostka numbers bound the α-Kostka numbers from above.

4.5.11 Lemma. *If $\alpha \geq 1$, then $K_{\lambda,\mu} \geq K^{(\alpha)}_{\lambda,\mu}$ for all $\lambda, \mu \vdash n$.*

Proof. Since $K_{\lambda,\mu} = K^{(\alpha)}_{\lambda,\mu}$ for $\alpha = 1$, we may assume that $\alpha > 1$. It suffices to show that $\Psi^{(\alpha)}_{\lambda/\mu} < \Psi^{(1)}_{\lambda/\mu} = 1$. Since λ covers μ, we have $b^{(\alpha)}_\lambda(s) > b^{(\alpha)}_\mu(s)$, implying that

$$\Psi^{(\alpha)}_{\lambda/\mu} = \prod_{s \in R_{\lambda/\mu} \setminus C_{\lambda/\mu}} \frac{b^{(\alpha)}_\mu(s)}{b^{(\alpha)}_\lambda(s)} < 1,$$

as desired. □

Lemmas 4.5.11 and 4.5.5 now imply the following.

4.5.12 Corollary. *The (i,j)-entry of the $F \times F$ leading principal minor of $K^{(2)}(n)$ is less than or equal to the (i,j)-entry of the $F \times F$ leading principal minor of $K(n')$ for all $n' \geq n$ and $1 \leq i,j \leq F$. Moreover, the magnitudes of the entries of the $F \times F$ leading principal minor of $K^{(2)}(n)$ are bounded above by a function of t for all n.*

4.6 The Pseudo-Adjacency Matrix

Having completed the legwork, we now show that for sufficiently large n, there exists a pseudo-adjacency matrix of Γ_t that meets the ratio bound with equality, thereby proving the first part of the main result. Define

$$\zeta := -\frac{1}{((2n-1))_t - 1}.$$

It suffices to show there exists a pseudo-adjacency matrix $\widetilde{A}(\Gamma_t)$ whose eigenvalues satisfy

$$\eta_{\min}/\eta_1 = \zeta,$$

as this would imply via the ratio bound that $|S| \leq (2(n-t)-1)!!$ for any independent set S of Γ_t. To this end, we first show that for any $t \in \mathbb{N}$ there exists a pseudomatrix $\widetilde{A}(\Gamma_t)$ that has the desired eigenvalues on the fat partitions. We then show the magnitudes of the eigenvalues of $\widetilde{A}(\Gamma_t)$ corresponding to non-fat partitions are smaller than those of the fat partitions for sufficiently large n.

4.6.1 Theorem. *Let $A_1, A_2, \cdots, A_F$ be the first $F = F_t - 1$ associates of $\mathcal{A}$ in reverse-lexicographic order. Then there exists a constant $B_t > 0$ (depending only on t) and a linear combination of fat associates*

$$\widetilde{A}(\Gamma_t) := \sum_{j=1}^{F} x_j A_j$$

such that $\max_j |x_j| \leq B_t/(2n-2)!!$, with eigenvalues $\eta_1, \eta_2, \cdots, \eta_F$ satisfying

$$\eta_i = \sum_{j=1}^{F} x_j P_{i,j} = \begin{cases} 1 & \text{if } i = 1 \\ \zeta & \text{if } 1 < i \leq F. \end{cases}$$

Proof. For any matrix A indexed by integer partitions of n, let A_F denote its leading principal minor induced by all the fat partitions except for $(n-t, 1^t)$ The entries of $M := P_F$ correspond to eigenvalues of the fat eigenspaces excluding $(n-t, 1^t)$ of the first F associates. Let $x = (x_1, \cdots, x_F)$ and $b := (1, \zeta, \cdots, \zeta)$. By Theorem 4.5.9, we have that M is invertible, thus x is the unique solution to $Mx = b$. To bound x, observe that

$$\begin{aligned} x = M^{-1}b &= (M(p, Z')_F(\Omega_\lambda)_F)^{-1}b \\ &= (\Omega_\lambda)_F^{-1} M(p, Z')_F^{-1} b \\ &= (\Omega_\lambda)_F^{-1} M(Z', m)_F (D(n))_F^{-1} b. \end{aligned}$$

By Lemma 4.5.6, the magnitudes of the entries of $(D(n))_F^{-1}$ are bounded above by some function of t. By Corollary 4.5.12, the magnitudes of the entries of $M(Z', m)_F$ are bounded above by some function of t. By Lemma 4.3.7, we deduce that

$$|\Omega_\lambda|^{-1} \leq \frac{2(n-t)(2t)^t}{2^n n!} \leq \frac{B'_t}{(2n-2)!!},$$

where $B'_t > 0$ depends only on t. We deduce there exists a B_t such that

$$\max_j |x_j| \leq B_t/(2n-2)!!,$$

as desired. □

4.6.2 Theorem. *Let $\widetilde{A}(\Gamma_t)$ be as defined in Theorem 4.6.1, and let ρ be a non-fat partition with corresponding eigenvalue*

$$\eta_\rho = \sum_{j=1}^{F} x_j P_{\rho,j}.$$

Then there exists a constant $G_t > 0$ (depending only on t) such that

$$|\eta_\rho| \leq G_t|\zeta|/\sqrt{n} = o(|\zeta|).$$

Proof. Putting everything together, we have

$$\begin{aligned}
|\eta_\rho| &= \Big|\sum_{j=1}^{F_t} x_j P_{\rho,j}\Big| \\
&\leq F_t \ \max_j |x_j| \ \max_j |P_{\rho,j}| \\
&\leq F_t \frac{B_t}{(2n-2)!!} \ \max_j |P_{\rho,j}| && \text{(Theorem 4.6.1)} \\
&\leq F_t \frac{B_t}{(2n-2)!!} \sqrt{\frac{|\mathcal{M}_{2n}|2^{n+1}(n-1)!}{C_t(2n)^{2(t+1)}}} && \text{(Lemma 4.2.12 \& Lemma 4.3.5)} \\
&\leq F_t B_t D_t \frac{(2n-1)!!}{(2n-2)!!}\frac{|\zeta|}{2n} && \text{where } D_t > 0 \text{ depends only on } t \\
&= G_t|\zeta|/\sqrt{n} && \text{(Proposition 2.1.1)}
\end{aligned}$$

where $G_t > 0$ depends only on t, as desired. □

The only eigenvalue of $\widetilde{A}(\Gamma_t)$ not covered by Theorem 4.6.1 or Theorem 4.6.2 is $\eta_{F_t} = \eta_{(n-t,1^t)}$, which we now handle separately in the theorem below.

4.6.3 Theorem. $\eta_{F_t} = \eta_{(n-t,1^t)} = \zeta$.

Proof. By Theorem 4.4.1, we can write the characteristic vector f of a canonically t-intersecting family as

$$f = \sum_{i=1}^{|\mathcal{M}_{2n}|} a_i v_i = \sum_{\lambda \vdash n} \sum_{i=1}^{\dim 2\lambda} a_{\lambda,i} v_{\lambda,i} = \sum_{\lambda \text{ fat}} \sum_{i=1}^{\dim 2\lambda} a_{\lambda,i} v_{\lambda,i},$$

where $\{v_{\lambda,i}\}_{i=1}^{\dim 2\lambda}$ is an orthonormal set of eigenvectors of $\widetilde{A}(\Gamma_t)$ that forms a basis for the irreducible 2λ. Let

$$w_\lambda := a_{\lambda,1}^2 + a_{\lambda,2}^2 + \cdots + a_{\lambda,\dim 2\lambda}^2.$$

By Lemma 4.4.5 we have $\hat{f}(2(n-t,1^t)) \neq 0$ for some f, which implies that $w_{(n-t,1^t)} \neq 0$. Let $\alpha = |\mathcal{F}_T|/(2n-1)!!$. Revisiting Equation (4.1.1) gives us

$$0 = f^\top \widetilde{A}(\Gamma_t) f = \sum_{\lambda \text{ fat}} \eta_\lambda w_\lambda = \alpha^2 + \zeta(\alpha - \alpha^2 - w_{(n-t,1^t)}) + \eta_{(n-t,1^t)} w_{(n-t,1^t)}.$$

By the definition of ζ, it follows that $\zeta(\alpha - \alpha^2) = -\alpha^2$. Since $w_{(n-t,1^t)} \neq 0$, we deduce that $\eta_{(n-t,1^t)} = \zeta$, as desired. □

Proof of the First Part of Theorem 4.0.1

By Theorem 4.6.1 and Theorem 4.6.3, there exists a pseudo-adjacency matrix of the perfect matching t-derangement graph $\widetilde{A}(\Gamma_t)$ with eigenvalues satisfying $\eta_1 = 1$ and $\eta_\lambda = \zeta$ for each non-trivial fat partition $\lambda \vdash 2n$. By Theorem 4.6.2, for each non-fat partition $\rho \vdash 2n$, we have $|\eta_\rho| < |\zeta|$ for sufficiently large n, thus ζ is the least eigenvalue of $\widetilde{A}(\Gamma_t)$ for sufficiently large n. By our choice of ζ and the ratio bound, we have that

$$|\mathcal{F}| \leq |V| \frac{-\eta_{\min}}{\eta_1 - \eta_{\min}} \leq (2n-1)!! \frac{-\zeta}{1-\zeta} \leq (2(n-t)-1)!!$$

for any t-intersecting family $\mathcal{F} \subseteq \mathcal{M}_{2n}$. This completes the proof of the first part of Theorem 4.0.1.

4.6.1 Cross-Intersecting Families: Redux

We say that two families $\mathcal{F}, \mathcal{G} \subseteq \mathcal{M}_{2n}$ are *t-cross-intersecting* if $|m \cap m'| \geq t$ for all $m \in \mathcal{F}, m' \in \mathcal{G}$. As a bonus, we have the following cross-intersecting variant of Theorem 4.0.1 that follows easily from the *cross-ratio bound* stated below.

4.6.4 Theorem (Cross-Ratio Bound [21]). *Let $\widetilde{A}(\Gamma)$ be a pseudo-adjacency matrix of a graph Γ with eigenvalues $|\eta_1| \geq |\eta_2| \geq \cdots \geq |\eta_n|$ and corresponding eigenvectors $v_1, v_2 \cdots, v_n$. Let $S, T \subseteq V$ be sets of vertices such that* *there are no edges between S and T. Then*

$$\sqrt{\frac{|S|\cdot|T|}{|V|^2}} \leq \frac{|\eta_2|}{\eta_1 + |\eta_2|}.$$

If equality holds, then

$$1_S, 1_T \in \mathrm{Span}(\{v_1\} \cup \{v_i : |\eta_i| = |\eta_2|\}).$$

4.6.5 Theorem. *For any $t \in \mathbb{N}$, if $\mathcal{F}, \mathcal{G} \subseteq \mathcal{M}_{2n}$ is t-cross-intersecting, then*

$$|\mathcal{F}| \cdot |\mathcal{G}| \leq ((2(n-t)-1)!!)^2$$

for sufficiently large n depending on t.

Proof. By Theorem 4.6.1 and Theorem 4.6.3, there exists a pseudo-adjacency matrix of the perfect matching t-derangement graph $\widetilde{A}(\Gamma_t)$ with eigenvalues satisfying $\eta_1 = 1$ and $\eta_\lambda = \zeta$ for each non-trivial fat even partition $\lambda \vdash 2n$. By Theorem 4.6.2, for each non-fat even partition $\rho \vdash 2n$, we have $|\eta_\rho| < |\zeta|$ for n sufficiently large, thus ζ is the second-largest eigenvalue in absolute value for sufficiently large n. By our choice of ζ and the cross-ratio bound, we have that

$$|\mathcal{F}| \cdot |\mathcal{G}| \leq ((2(n-t)-1)!!)^2,$$

for any t-cross-intersecting $\mathcal{F}, \mathcal{G} \subseteq \mathcal{M}_{2n}$. □

It remains to show that the largest t-intersecting families are precisely the canonically t-intersecting families for sufficiently large n. We shall do this, albeit a bit indirectly, by proving a stability theorem for t-intersecting families of $\mathcal{M}_{2n}$.

4.7 Stability and the Case of Equality

Our next result is a stability theorem for t-intersecting families of $\mathcal{M}_{2n}$.

4.7.1 Theorem. *For any $\epsilon \in (0, 1/\sqrt{e})$, $n > n(\epsilon)$, any t-intersecting family of $\mathcal{M}_{2n}$ larger than $(1-1/\sqrt{e}+\epsilon)(2(n-t)-1)!!$ is contained in a canonically t-intersecting family.*

It is clear that this theorem implies the characterization of the extremal families stated in Theorem 4.0.1. Note that we proved the $t = 1$ case of Theorem 4.7.1 in the previous chapter.

Most of our efforts will be spent towards proving a key lemma, which roughly asserts that almost all of the members of a large t-intersecting family have a set of t disjoint edges in common. For any set of disjoint edges $S \subseteq E(K_{2n})$, we define the set

$$\mathcal{F}{\downarrow_S} := \{m \in \mathcal{F} : S \subseteq m\}$$

to be the *restriction* of $\mathcal{F} \subseteq \mathcal{M}_{2n}$ with respect to S.

4.7.2 Lemma (Key Lemma)**.** *Let $c \in (0, 1)$. For any $t \in \mathbb{N}$, if $\mathcal{F} \subseteq \mathcal{M}_{2n}$ is a t-intersecting family such that $|\mathcal{F}| \geq c(2(n-t)-1)!!$, then there is a set of t disjoint edges $T \subseteq E(K_{2n})$ such that*

$$|\mathcal{F} \setminus \mathcal{F}{\downarrow_T}| = O((2(n-t-1)-1)!!)$$

for sufficiently large n depending on c and t.

For any set of disjoint edges m and vertex $u \in V(m)$, let $m(u)$ be the other endpoint of the edge incident to u in m, that is, the *partner* of u. We now show that the key lemma implies Theorem 4.7.1.

Let $\mathcal{F}$ be t-intersecting of size $c(2(n-t)-1)!!$ such that $c \in (1-1/\sqrt{e}, 1)$. Assuming the key lemma, there is a set of t disjoint edges $T \subseteq E(K_{2n})$ such that $|\mathcal{F} \setminus \mathcal{F}{\downarrow_T}| = O((2(n-t-1)-1)!!)$, which implies that

$$|\mathcal{F}{\downarrow_T}| \geq (c - O(1/n))(2(n-t)-1)!!. \tag{4.7.1}$$

For a contradiction, suppose there exists a perfect matching $m \in \mathcal{F}$ such that $T \not\subseteq m$, and let $s = |m \cap T|$. Since $\mathcal{F}$ is t-intersecting, any member of $\mathcal{F}{\downarrow_T}$ shares t edges with m, and therefore no member of $\mathcal{F}{\downarrow_T}$ can be a $(t-s)$-derangement with respect to m when restricted to the complete subgraph $K_{2n} \setminus V(T)$. This implies that

$$|\mathcal{F}{\downarrow_T}| \leq (2(n-t)-1)!! - D_2(n-t, t-s) \leq (2(n-t)-1)!! - D_2(n-t, 1).$$

Assume $t-s = 1$, and let ij be the edge of T that is not contained in m. Then $m(i), m(j) \notin V(T)$, and every member of $\mathcal{F}{\downarrow_T}$ that contains $\{m(i), m(j)\}$

that is also a derangement with respect to m when restricted to $K_{2n} \setminus V(T)$ cannot t-intersect m. This gives us

$$\begin{aligned}|\mathcal{F}{\downarrow_T}| &\leq (2(n-t)-1)!! - D_2(n-t, t-s) \\ &\leq (2(n-t)-1)!! - D_2(n-t,1) - D_2(n-t-1,1) \\ &= (1 - 1/\sqrt{e} - o(1))(2(n-t)-1)!!,\end{aligned}$$

where the equality follows from Equation (4.1.2). But this contradicts (4.7.1) for n sufficiently large depending on c and t, completing the proof.

By the argument above, it suffices to prove the key lemma. The core of its proof is a well-known generalization of the ratio bound. Recall that U_t is the space of real-valued functions on perfect matchings of K_{2n} that are supported on the fat even irreducibles as defined in Section 4.4.

4.7.3 Theorem (Stability Version of Ratio Bound [21])**.** *Let $\widetilde{A}(\Gamma)$ be the pseudo-adjacency matrix of a regular graph $\Gamma = (V, E)$ with eigenvalues $\eta_{\min} \leq \cdots \leq \eta_2 \leq \eta_1$ and corresponding orthonormal eigenvectors $v_{\min}, \cdots, v_2, v_1$. Let $\mu = \min_i\{\eta_i : \eta_i \neq \eta_{\min}\}$. Let $S \subseteq V$ be an independent set of vertices of measure $\alpha = |S|/|V|$. Let $D = \|P_{U_t^\perp}(1_S)\|$ be the Euclidean distance from the characteristic function of S to the subspace U_t. Then*

$$D^2 \leq \alpha \frac{(1-\alpha)|\eta_{\min}| - |\eta_1|\alpha}{|\eta_{\min}| - |\mu|}.$$

For any subset of vertices $V' \subseteq V(K_{2n})$, let $\Delta(V') \subseteq E(K_{2n})$ be the set of edges that have at least one endpoint in V'. The next lemmas follow from Theorem 4.6.5 and will be needed for a couple of combinatorial arguments. Let

$$T^* := \{\{1,2\},\{3,4\},\cdots,\{2t-1,2t\}\}.$$

4.7.4 Lemma. *Let $\mathcal{F} \subseteq \mathcal{M}_{2n}$ be a t-intersecting family. For any set of t disjoint edges $T \subseteq E(K_{2n})$ such that $T \cap T^* = \emptyset$, $|V(T) \cap \{2i-1, 2i\}| \geq 1$ for all $i \in [t]$, and $T \subseteq \Delta([2t])$, we have*

$$|\mathcal{F}{\downarrow_{T^*}}| \cdot |\mathcal{F}{\downarrow_T}| \leq ((2(n-2t)-1)!!)^2.$$

Proof. Note that because $\mathcal{F}$ is a t-intersecting family, we have that $\mathcal{F}{\downarrow_{T^*}}$ and $\mathcal{F}{\downarrow_T}$ are cross-t-intersecting on edges of $K_{2n} \setminus \{[2t] \cup V(T)\}$. By our choice of T, we have

$$V(T) \setminus [2t] = \{u_1, u_2, \cdots, u_k\} \quad \text{and} \quad [2t] \setminus V(T) = \{v_1, v_2, \cdots, v_k\}$$

for some $k \leq t$. Define the involution $\pi := (u_1\ v_1)(u_2\ v_2)\cdots(u_k\ v_k)$. For any two perfect matchings $m \in \mathcal{F}{\downarrow_{T^*}}$ and $m' \in \pi\,(\mathcal{F}{\downarrow_T})$, every edge of m and m' has either both of its endpoints in $[2t]$ or none of its endpoints in $[2t]$. Since π fixes every $v \notin [2t] \cup V(T)$, we also have that m, m' are cross-t-intersecting on edges of $K_{2n} \setminus \{[2t] \cup V(T)\}$. By deleting $[2t]$ we obtain $(\mathcal{F}{\downarrow_{T^*}})'$ and $(\pi\,(\mathcal{F}{\downarrow_T}))'$ which are t-cross-intersecting families of $\mathcal{M}_{2(n-t)}$. By Theorem 4.6.5, we have

$$\begin{aligned}|\mathcal{F}{\downarrow_{T^*}}| \cdot |\mathcal{F}{\downarrow_T}| = |\mathcal{F}{\downarrow_{T^*}}| \cdot |\pi\,(\mathcal{F}{\downarrow_T})| &= |\,(\mathcal{F}{\downarrow_{T^*}})'\,| \cdot |\,(\pi\,(\mathcal{F}{\downarrow_T}))'\,| \\ &\leq ((2(n-2t)-1)!!)^2,\end{aligned}$$

as desired. $\square$

A similar argument can be used to show the following.

4.7.5 Lemma. *If $\mathcal{F} \subseteq \mathcal{M}_{2n}$ is a t-intersecting family and $i, j, k \in V(K_{2n})$, then*

$$|\mathcal{F}{\downarrow_{ij}}| \cdot |\mathcal{F}{\downarrow_{ik}}| \leq ((2(n-t-1)-1)!!)^2.$$

We are now in a position to sketch a proof of the key lemma.

Proof Sketch II

Our starting point is the fact shown in Section 4.6, that for sufficiently large n the eigenvalues $\{\eta_\lambda\}_{\lambda \vdash n}$ of $\widetilde{A}(\Gamma_t)$ satisfy

$$|\eta_\mu| = o(|\zeta|) \quad \text{for all non-fat } \mu \vdash n.$$

With this in hand, we use the stability version of the ratio bound to show the characteristic function f of any large t-intersecting family $\mathcal{F}$ is close in Euclidean distance to U_t.

Following Ellis [21], we then use the fact that f is close to U_t along with Proposition 3.4.3 to find two perfect matchings $m_1 \in \mathcal{F}$ and $m_0 \notin \mathcal{F}$ that are structurally similar, i.e., sharing many edges, yet their orthogonal projections

$$[P_{U_t} f](m_1) \quad \text{and} \quad [P_{U_t} f](m_0)$$

onto the subspace U_t are close to 1 and 0 respectively. The projector P_{U_t} is the sum of primitive idempotents E_λ of $\mathcal{A}_n$ such that λ is fat:

$$P_{U_t} = E_{(n)} + E_{(n-1,1)} + \cdots + E_{(n-t,1^t)}.$$

This projector is quite difficult to work with in practice due to the fact that the character theory of $\mathcal{M}_{2n}$ is considerably less understood than the classical character theory of S_n. For example, no Murnaghan-Nakayama-type rule or Jacobi-Trudi-type determinantal identity is known for expressing the zonal characters, which are outstanding open questions in the theory of zonal and Jack symmetric functions [70]. In light of this, we must use some barebone combinatorial and representation-theoretical arguments to find crude estimates of these projections, which we cover in the next section.

Once we have boiled these projections down to their combinatorial essence, we use the fact that m_1 and m_0 share many edges to prove that $\mathcal{F}$ has a large restriction with respect to some set of t disjoint edges T; however, not large enough to deduce the key lemma. Following a bootstrapping argument of Ellis [21], we use our bounds on t-cross-intersecting families to show that almost every member of $\mathcal{F}$ has an edge in common with T. This fact, after an induction on t, leads to a proof of the key lemma.

The asymptotics of permutations and perfect matchings bear a strong resemblance, so there are points in the proof that closely follow Ellis [21]. We have adopted a notation that is consistent with his at these places.

4.8 Polytabloids and the Characters of $\mathcal{M}_{2n}$

Although the character theory of $\mathcal{M}_{2n}$ is determined by the zonal polynomials expressed in the power sum basis, arriving at tractable combinatorial expressions for these coefficients is considerably more difficult than it is for Schur functions. Instead, we work with combinatorial formulas for these quantities that stem from the fact that spherical functions of the Gelfand pair $(S_{2n}, S_2 \wr S_n)$ are the projections of characters of S_{2n} onto the space H_n-invariant functions. For a more detailed account of the material in this section, see [13, Ch. 11].

For any 2λ-tableau T, let $\mathrm{row}_T(i)$ be the index of the row of T that the cell labeled i belongs to, and let $\mathrm{col}_T(i)$ be the index of the column of T that the cell labeled i belongs to. We say that a 2λ-tabloid $\{T\}$ *covers* $m \in \mathcal{M}_{2n}$ if $\mathrm{row}_T(i) = \mathrm{row}_T(m(i))$ for all $i \in [2n]$. A 2λ-tableau T is *m-aligned* with respect to a perfect matching $m \in \mathcal{M}_{2n}$ if $\{T\}$ covers m and for any $i \in [2n]$ we have $\{\mathrm{col}_T(i), \mathrm{col}_T(m(i))\} = \{2j-1, 2j\}$ for some $j \in [n]$.

For example, the perfect matching

$$1\,7|2\,4|3\,8|5\,12|6\,11|9\,10|13\,14 \quad \text{is not covered by} \quad \begin{array}{l} \hline 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 11\ 12 \\ \hline 7\ \ 8\ \ 9\ \ 10 \\ \hline 13\ 14 \\ \hline \end{array},$$

but the perfect matching 1 12|2 3|4 5|6 11|7 9|8 10|13 14 is covered by this tabloid, as illustrated below:

$$\begin{array}{llllllll} 1 & 2 & 3 & 4 & 5 & 6 & 11 & 12 \\ 7 & 8 & 9 & 10 & & & & \\ 13 & 14 & & & & & & \end{array}.$$

Also note that the tableau shown above is a m^*-aligned. Let

$$\mathcal{P} := \frac{1}{|H_n|} \sum_{h \in H_n} e_h$$

denote the projection from $\mathbb{R}[S_{2n}/H_n]$ to its bi-H_n-invariant subalgebra $\mathbb{R}[H_n \backslash S_{2n}/H_n]$. Let $1_{\{T\}} \in \mathbb{R}[\mathcal{M}_{2n}]$ be the characteristic function of the set of perfect matchings that are covered by $\{T\}$, that is,

$$1_{\{T\}}(m) = \begin{cases} 1 & \text{if } \{T\} \text{ covers } m, \\ 0 & \text{otherwise} \end{cases}.$$

for all $m \in \mathcal{M}_{2n}$. For any $\lambda \vdash n$, define the map

$$\mathcal{I}'_\lambda : \{e_{\{T\}} \in \mathbb{R}[S_{2n}] : \{T\} \text{ is a } 2\lambda\text{-tabloid}\} \to \mathbb{R}[\mathcal{M}_{2n}]$$

such that

$$\mathcal{I}'_\lambda(e_{\{T\}}) = 1_{\{T\}}.$$

Recall from Section 4.2.1 that

$$\text{Span}\{e_{\{T\}} : T \text{ is a } 2\lambda\text{-tabloid}\} \cong \mathbb{R}[\mathbb{T}_{2\lambda}].$$

Let $\mathcal{I}_\lambda$ be the linear extension of $\mathcal{I}'_\lambda$, that is,

$$\mathcal{I}_\lambda : \mathbb{R}[\mathbb{T}_{2\lambda}] \to \mathbb{R}[\mathcal{M}_{2n}].$$

For any 2λ-tabloid $\{T\} \in \mathbb{T}_{2\lambda}$ and $\sigma \in S_{2n}$, we have that $m \in \mathcal{M}_{2n}$ is covered by $\{T\}$ if and only if σm is covered by $\sigma\{T\}$; therefore, $\mathcal{I}_\lambda$ *intertwines* the permutation representations $\mathbb{R}[\mathbb{T}_{2\lambda}]$ and $\mathbb{R}[\mathcal{M}_{2n}]$, in symbols:

$$\mathcal{I}_\lambda(\sigma e_{\{T\}}) = \sigma \mathcal{I}_\lambda(e_{\{T\}}) \quad \text{for all } \sigma \in S_{2n}.$$

By Schur's lemma, this implies for each irreducible 2λ that $\mathcal{I}_\lambda$ acts either as an isomorphism or as the zero map, and it is simple to show that the latter is not the case. In particular, this shows for any standard Young tableau T of shape 2λ and corresponding polytabloid e_T that

$$f_T := \mathcal{I}_\lambda e_T = \sum_{\sigma \in C_T} \operatorname{sign}(\sigma)\mathcal{I}_\lambda e_{\{\sigma T\}} = \sum_{\sigma \in C_T} \operatorname{sign}(\sigma) 1_{\{\sigma T\}}$$

lies in the 2λ irreducible subspace of $\mathbb{R}[\mathcal{M}_{2n}]$, and moreover, that

$$\{f_T \in \mathbb{R}[\mathcal{M}_{2n}] : T \text{ is a standard Young tableau of shape } 2\lambda\}$$

is a basis for the 2λ irreducible subspace of $\mathbb{R}[\mathcal{M}_{2n}]$.

For any $\lambda \vdash n$, the spherical function $\omega^\lambda \in \mathbb{R}[\mathcal{M}_{2n}]$ is the unique H_n-invariant function $\omega^\lambda \in \mathcal{I}_\lambda 2\lambda \leq \mathbb{R}[\mathcal{M}_{2n}]$, equivalently, bi-$H_n$-invariant function of $2\lambda \leq \mathbb{R}[S_{2n}]$, which satisfies $\omega^\lambda(m^*) = \omega^\lambda_{(1^n)} = 1$. For any $\pi \in S_{2n}$ and $m = \pi m^*$, let $\omega^\lambda_{d(m,\cdot)}$ denote the zonal spherical function translated by π so that $\omega^\lambda_{d(m,\cdot)}(m) = \omega^\lambda_{(1^n)} = 1$.

For any tableau T of even shape 2λ, let $C'_T \leq C_T$ be the subgroup of the column-stabilizer of T that stabilizes the odd-indexed columns of T and acts trivially on the even-indexed columns. More precisely, if we have

$$C_T \cong S_{(2\lambda')_1} \times S_{(2\lambda')_2} \times \cdots \times S_{(2\lambda')_{2\lambda_1}},$$

then $C'_T \cong S_{(\lambda')_1} \times S_{(\lambda')_2} \times \cdots \times S_{(\lambda')_{\lambda_1}}$. For any $m \in \mathcal{M}_{2n}$ and m-aligned 2λ-tableau T, we have

$$\omega^\lambda_{d(m,\cdot)} = \frac{1}{\lambda'_1! \cdots \lambda'_{\lambda_1}!} \mathcal{P}\mathcal{I}_\lambda \sum_{\sigma \in C_T} \operatorname{sign}(\sigma) e_{\{\sigma T\}} \tag{4.8.1}$$

$$= \frac{1}{\lambda'_1! \cdots \lambda'_{\lambda_1}! \, |H|} \sum_{h \in H} \sum_{\sigma \in C_T} \operatorname{sign}(\sigma) \mathcal{I}_\lambda e_{h\{\sigma T\}} \tag{4.8.2}$$

$$= \frac{1}{|H|} \sum_{h \in H} \sum_{\sigma \in C'_T} \operatorname{sign}(\sigma) 1_{\{h\sigma T\}}, \tag{4.8.3}$$

where H is a translate of H_n (see [13, Ch. 11]). The expression above together with the projection formula below gives us an explicit but complicated combinatorial formula for computing these projections.

4.8.1 Lemma. [45] *Let $E_\mu : \mathbb{R}[\mathcal{M}_{2n}] \to 2\mu$ denote the orthogonal projection onto 2μ. For any $f \in \mathbb{R}[\mathcal{M}_{2n}]$, we have*

$$[E_\mu f](m) = \frac{\dim 2\mu}{(2n-1)!!} \sum_{\lambda \vdash n} \left(\sum_{\substack{m' \in \mathcal{M}_{2n} \\ d(m,m')=\lambda}} f(m') \right) \omega_\lambda^\mu,$$

equivalently,

$$[E_\mu f](m) = \frac{\dim 2\mu}{(2n-1)!!} \sum_{m' \in \mathcal{M}_{2n}} f(m')\ \omega^\mu_{d(m,m')}.$$

Without further ado, we begin the proof of the key lemma.

4.9 Proof of the Key Lemma

Let $c \in (0,1)$ and let $\mathcal{F}$ be a t-intersecting family of perfect matchings such that

$$|\mathcal{F}| \geq c(2(n-t)-1)!!.$$

Recall the goal is to show there is a set of t disjoint edges $T \subseteq E(K_{2n})$ such that $|\mathcal{F} \setminus \mathcal{F}\downarrow_T| = O((2(n-t-1)-1)!!)$ for sufficiently large n depending on c, t.

Let f be the characteristic function of $\mathcal{F}$, $\alpha = |\mathcal{F}|/(2n-1)!! \geq c/((2n-1))_t$, and D be the Euclidean distance from f to U. Setting $S = \mathcal{F}$ and applying our pseudo-adjacency matrix $\widetilde{A}(\Gamma_t)$ to Theorem 4.7.3 gives us

$$\begin{aligned} D^2 \leq \alpha\frac{(1-\alpha)|\zeta| - \alpha}{|\zeta| - |\mu|} &= \alpha\frac{(1-\alpha) - \alpha/|\zeta|}{1 - |\mu|/|\zeta|} \\ &\leq \alpha\frac{(1-\alpha) - c}{1 - O(1/\sqrt{n})} \qquad \text{(Theorem 4.6.2)} \\ &\leq \alpha\frac{1-c}{1 - O(1/\sqrt{n})} \\ &\leq \delta(1 + O(1/\sqrt{n}))\frac{|\mathcal{F}|}{(2n-1)!!} \end{aligned}$$

where $\delta := 1 - c$. We have

$$\|P_{U_t^\perp} f\|_2^2 = \|f - P_{U_t} f\|_2^2 = D^2 \leq \delta(1 + O(1/\sqrt{n}))\frac{|\mathcal{F}|}{(2n-1)!!},$$

which tends to zero as $n \to \infty$. This already shows that f is close to U_t, but we now seek a combinatorial explanation for this proximity.

By Lemma 4.8.1, we may write

$$P_m := [P_{U_t} f](m) = \sum_{\mu \text{ fat}} \frac{\dim 2\mu}{(2n-1)!!} \sum_{m' \in \mathcal{M}_{2n}} f(m')\ \omega^{\mu}_{d(m,m')}.$$

Now we have

$$D^2 = \frac{1}{(2n-1)!!}\left(\sum_{m \in \mathcal{F}} (1 - P_m)^2 + \sum_{m \notin \mathcal{F}} P_m^2\right) \leq \delta(1 + O(1/\sqrt{n}))\frac{|\mathcal{F}|}{(2n-1)!!},$$

which gives us,

$$\sum_{m \in \mathcal{F}} (1 - P_m)^2 + \sum_{m \notin \mathcal{F}} P_m^2 \leq \delta(1 + O(1/\sqrt{n}))|\mathcal{F}|. \tag{4.9.1}$$

Following Ellis, pick $C > 0$ large enough so that

$$\sum_{m \in \mathcal{F}} (1 - P_m)^2 + \sum_{m \notin \mathcal{F}} P_m^2 \leq \delta(1 + O(1/\sqrt{n}))|\mathcal{F}| \leq |\mathcal{F}|(1 - 1/\sqrt{n})\delta(1 + C/\sqrt{n}).$$

By the nonnegativity of each term on the left-hand side of (4.9.1), at least $|\mathcal{F}|/\sqrt{n}$ members of $\mathcal{F}$ satisfy $(1 - P_m)^2 < \delta(1 + C/\sqrt{n})$; thus there is a

$$\mathcal{F}_1 := \{m \in \mathcal{F} : (1 - P_m)^2 < \delta(1 + C/\sqrt{n})\}$$

such that $|\mathcal{F}_1| \geq |\mathcal{F}|/\sqrt{n}$. The inequality (4.9.1) also implies that $P_m^2 < 2\delta/n$ for every $m \notin \mathcal{F}$ with the exception of at most $n|\mathcal{F}|(1 + O(1/\sqrt{n}))/2$ non-members, thus there is a set

$$\mathcal{F}_0 := \{m \notin \mathcal{F} : P_m^2 < 2\delta/n\}$$

such that

$$|\mathcal{F}_0| \geq (2n-1)!! - c(2(n-t)-1)!! - cn(2(n-t)-1)!!(1 + O(1/\sqrt{n}))/2.$$

Notice that the projections of the elements of $\mathcal{F}_0$ and $\mathcal{F}_1$ are close to 0 and 1.

We now show there exist $m_1 \in \mathcal{F}_1$ and $m_0 \in \mathcal{F}_0$ that are close together in the graph $\mathcal{T}_n$, differing by $O(\sqrt{n \log n})$ partner swaps, which implies that m_1 and m_0 share many edges.

We claim there is a path $m_1 p_2 p_3 \cdots p_{\ell-1} m_0$ in $\mathcal{T}_n$ of length at most

$$2\sqrt{(t+2)\frac{n}{2}\ln(2n)}.$$

Take $a = 1/(2n)^{t+2}$ and $h = 2h_0$ in Proposition 3.4.3. Since

$$|\mathcal{F}_1| \geq c(2(n-t)-1)!!/n \geq (2n-1)!!/(2n)^{t+2}$$

for sufficiently large n, Proposition 3.4.3 gives us

$$|N_h(\mathcal{F}_1)| \geq \left(1 - \frac{1}{n^{t+2}}\right)(2n-1)!!.$$

Since $|\mathcal{F}_0| > (2n-1)!!/(2n)^{t+2}$ for sufficiently large n, we have

$$|\mathcal{F}_0 \cap N_h(\mathcal{F}_1)| \neq \emptyset,$$

thus there is a path from m_1 to m_0 in $\mathcal{T}_n$ of length no more than

$$2\sqrt{(t+2)\frac{n}{2}\ln(2n)} = O(\sqrt{n \log n}).$$

The inequality (4.9.1) and the foregoing shows that

$$1 - \sqrt{\delta(1 + C/\sqrt{n})} < P_{m_1} \text{ and } P_{m_0} < \sqrt{2\delta/\sqrt{n}}.$$

Combining these inequalities reveals that

$$P_{m_1} - P_{m_0} \geq (1 - \sqrt{\delta} - O(1/\sqrt[4]{n})).$$

Rewriting using Lemma 4.8.1 gives us

$$\sum_{\mu \text{ fat}} \frac{\dim 2\mu}{(2n-1)!!}\left(\sum_{m \in \mathcal{F}} \omega^{\mu}_{d(m_1,m)} - \sum_{m \in \mathcal{F}} \omega^{\mu}_{d(m_0,m)}\right) \geq (1 - \sqrt{\delta} - O(1/\sqrt[4]{n})).$$

By averaging, there exists a fat $\mu \neq (n)$ such that

$$\frac{\dim 2\mu}{(2n-1)!!}\left(\sum_{m\in\mathcal{F}}\omega^{\mu}_{d(m_1,m)} - \sum_{m\in\mathcal{F}}\omega^{\mu}_{d(m_0,m)}\right) \geq \frac{(1-\sqrt{\delta}-O(1/\sqrt[4]{n}))}{F_t}.$$

Rearranging gives us

$$\sum_{m\in\mathcal{F}}\omega^{\mu}_{d(m_1,m)} - \sum_{m\in\mathcal{F}}\omega^{\mu}_{d(m_0,m)} \geq \frac{(1-\sqrt{\delta}-O(1/\sqrt[4]{n}))(2n-1)!!}{F_t\ \dim 2\mu}.$$

Without loss of generality, we may assume that $m_1 = m^*$ and $m_0 = \pi m^*$ such that $\pi \in S_{2n}$ is a product of $O(\sqrt{n \log n})$ transpositions. By (4.8.1) and interchanging summations, we have

$$\sum_{\sigma\in C'_T}\sum_{m\in\mathcal{F}} \operatorname{sign}(\sigma)\left(\mathcal{P}1_{\{\sigma T\}}(m) - \mathcal{P}1_{\pi\{\sigma T\}}(m)\right) \geq \frac{(1-\sqrt{\delta}-O(1/\sqrt[4]{n}))(2n-1)!!}{F_t\ \dim 2\mu}$$

where T is a m^*-aligned 2μ-tableau. By averaging, there is a $\sigma \in C'_T$ such that

$$\operatorname{sign}(\sigma)\left(\sum_{m\in\mathcal{F}}\mathcal{P}1_{\{\sigma T\}}(m) - \sum_{m\in\mathcal{F}}\mathcal{P}1_{\pi\{\sigma T\}}(m)\right) \geq \frac{(1-\sqrt{\delta}-O(1/\sqrt[4]{n}))(2n-1)!!}{F_t\ |C'_T|\ \dim 2\mu}.$$

Without loss of generality, we may assume

$$\left(\sum_{m\in\mathcal{F}}\mathcal{P}1_{\{\sigma T\}}(m) - \sum_{m\in\mathcal{F}}\mathcal{P}1_{\pi\{\sigma T\}}(m)\right) \geq \frac{(1-\sqrt{\delta}-O(1/\sqrt[4]{n}))(2n-1)!!}{F_t\ |C'_T|\ \dim 2\mu}.$$

By Equation (4.8.3), we have

$$\sum_{h\in H_n}\sum_{m\in\mathcal{F}}\left(1_{\{h\sigma T\}} - 1_{\pi\{h\sigma T\}}\right)(m) \geq \frac{(1-\sqrt{\delta}-O(1/\sqrt[4]{n}))(2n-1)!!\ |H_n|}{F_t\ |C'_T|\ \dim 2\mu}.$$

Note that if $\{h\sigma T\} = \pi\{h\sigma T\}$, then $1_{\{h\sigma T\}}(m) - 1_{\pi\{h\sigma T\}}(m) = 0$. Let

$$I := \{i \in V(K_{2n}) : \pi(i) \neq i\}$$

be the vertices moved by π, and for any tabloid $\{T\}$, let $\{\overline{T}\}$ be the subtabloid obtained by deleting the first row of $\{T\}$. After canceling some terms, we have

$$\sum_{\substack{h \in H_n \\ \exists i \in I : i \in \{\overline{h\sigma T}\}}} \sum_{m \in \mathcal{F}} \left(1_{\{h\sigma T\}} - 1_{\pi\{h\sigma T\}}\right)(m) \geq \frac{(1-\sqrt{\delta} - O(1/\sqrt[4]{n}))(2n-1)!!\ |H_n|}{F_t\ |C'_T|\ \dim 2\mu}.$$

Since m_1 and m_0 differ by only $O(\sqrt{n \log n})$ partner swaps, we have $|I| = o(n)$. The number of permutations $h \in H_n$ that send a vertex $i \in I$ to a row of $\{\overline{\sigma T}\}$ is $o(|H_n|)$. Since there are $o(|H_n|)$ terms in the outer summation, by averaging, there is a tabloid $\{h\sigma T\} =: \{S\}$ such that

$$\sum_{m \in \mathcal{F}} 1_{\{S\}}(m) - 1_{\pi\{S\}}(m) \geq \frac{(1-\sqrt{\delta} - O(1/\sqrt[4]{n}))(2n-1)!!\ \omega(1)}{F_t\ |C'_T|\ \dim 2\mu}.$$

Absorbing constants depending on c and t and dropping negative terms gives

$$\sum_{m \in \mathcal{F}} 1_{\{S\}}(m) \geq \frac{(2n-1)!!\ \omega(1)}{\dim 2\mu}.$$

Henceforth, we absorb constant factors on the right-hand side into $\omega(1)$. By the pigeonhole principle, there are $s := n - \mu_1 \leq t$ disjoint edges S' covered by $\{\overline{S}\}$ such that

$$|\mathcal{F}\downarrow_{S'}| \geq \frac{(2n-1)!!\ \omega(1)}{\dim 2\mu}.$$

Similarly, there are $t - s$ disjoint edges S'' disjoint from S' such that

$$|\mathcal{F}\downarrow_{S' \cup S''}| \geq \frac{(2n-1)!!\ \omega(1)}{\dim 2\mu\ (2n)_{2(t-s)}}.$$

By Theorem 4.2.13, we have

$$|\mathcal{F}\downarrow_{S' \cup S''}| \geq \omega((2(n-2t)-1)!!).$$

By relabeling the vertices of K_{2n}, we may assume without loss of generality that

$$|\mathcal{F}\downarrow_{T^*}| \geq \omega((2(n-2t)-1)!!).$$

Let $\mathcal{B}$ be the collection of all partitions of $[2t]$ into two parts $A = \{a_1, \cdots, a_t\}$ and $B = \{b_1, \cdots, b_t\}$. Crudely, the set of members of $\mathcal{F}$ with no edge in T^* can be written as

$$\mathcal{F} \setminus \bigcup_{i=1}^{t} \mathcal{F}{\downarrow_{\{2i-1,2i\}}} = \bigcup_{\substack{(A,B)\in\mathcal{B} \\ a_{i_1}v_{i_1},\cdots,a_{i_t}v_{i_t}\ :\ a_{i_j}\neq a_{i_k},\ v_{i_j}\neq v_{i_k}\forall j,k\in[t] \\ v_{i_j}\notin A,\ a_{i_j}v_{i_j}\notin T^*\ \ \forall j\in[t]}} \mathcal{F}{\downarrow_{\{a_{i_1}v_{i_1},\cdots,a_{i_t}v_{i_t}\}}}\ . \tag{4.9.2}$$

By Lemma 4.7.4, we have

$$|\mathcal{F}{\downarrow_{T^*}}| \cdot |\mathcal{F}{\downarrow_{\{a_{i_1}v_{i_1},\cdots,a_{i_t}v_{i_t}\}}}| \leq ((2(n-2t)-1)!!)^2 \tag{4.9.3}$$

for each term on the right-hand side of (4.9.2). Since $|\mathcal{F}{\downarrow_{T^*}}| \geq \omega((2(n-2t)-1)!!)$, the bound (4.9.3) implies that

$$|\mathcal{F}{\downarrow_{\{a_{i_1}v_{i_1},\cdots,a_{i_t}v_{i_t}\}}}| = o((2(n-2t)-1)!!)$$

for each term on the right-hand side of (4.9.2). Since the right-hand side of (4.9.2) has $O((2n)_t)$ terms, we have that

$$\begin{aligned} \left|\mathcal{F} \setminus \bigcup_{i=1}^{t} \mathcal{F}{\downarrow_{\{2i-1,2i\}}}\right| &\leq o((2(n-2t)-1)!!)O((2n)_t) \\ &= o((2(n-t)-1)!!). \end{aligned}$$

Recalling that $|\mathcal{F}| \geq c(2(n-t)-1)!!$, by averaging, there is an edge $ij \in T^*$ such that

$$|\mathcal{F}{\downarrow_{ij}}| \geq (c-o(1))(2(n-t)-1)!!/t.$$

The set of all members of $\mathcal{F}$ that do not contain the edge ij can be written as

$$\mathcal{F} \setminus \mathcal{F}{\downarrow_{ij}} = \bigcup_{k\neq j} \mathcal{F}{\downarrow_{ik}}\ . \tag{4.9.4}$$

By Lemma 4.7.5, we have $|\mathcal{F}{\downarrow_{ij}}| \cdot |\mathcal{F}{\downarrow_{ik}}| \leq ((2(n-t-1)-1)!!)^2$ for each term on the right-hand side of (4.9.4). Since $|\mathcal{F}{\downarrow_{ij}}| \geq \Omega((2(n-t)-1)!!)$, we deduce that $|\mathcal{F}{\downarrow_{ik}}| \leq O((2(n-t-2)-1)!!)$, which gives us

$$|\mathcal{F} \setminus \mathcal{F}_{ij}| = \sum_{k \neq j} |\mathcal{F}{\downarrow_{ik}}| \leq O((2(n-t-1)-1)!!).$$

At this point we have shown that any large t-intersecting family $\mathcal{F}$ is almost contained within a canonically intersecting family $\mathcal{F}_{ij}$. This may seem problematic, after all, the key lemma states that any large t-intersecting family is almost contained within a canonically t-intersecting family $\mathcal{F}_T$; however, we are in the homestretch, as a simple induction on t following Ellis [21] will take us the rest of the way.

If $t = 1$, then we are done, so let us assume that the key lemma is true for $t-1$. Let $\mathcal{F} \subseteq \mathcal{M}_{2n}$ be a t-intersecting family of size at least $c(2(n-t)-1)!!$. We have shown there exists an edge ij such that

$$|\mathcal{F} \setminus \mathcal{F}{\downarrow_{ij}}| \leq O((2(n-t-1)-1)!!),$$

which implies that

$$|\mathcal{F}{\downarrow_{ij}}| \geq |\mathcal{F}| - O((2(n-t-1)-1)!!).$$

By removing the vertices i and j from each member of $\mathcal{F}{\downarrow_{ij}}$, we obtain a $(t-1)$-intersecting family $\mathcal{F}'$ of perfect matchings of $K_{2n} \setminus \{i,j\}$ such that

$$|\mathcal{F}'| \geq (c - O(1/n))(2(n-t)-1)!!.$$

For any $c' \in (0, c)$ we have

$$|\mathcal{F}'| \geq c'(2(n-t)-1)!!$$

for n is sufficiently large. By the induction hypothesis, there is a canonically $(t-1)$-intersecting family $\mathcal{F}'_{T'}$ of perfect matchings of $K_{2n} \setminus \{i,j\}$ such that

$$|\mathcal{F}' \setminus \mathcal{F}'_{T'}| \leq O((2(n-t-1)-1)!!).$$

Setting $T = T' \cup \{ij\}$, if we add the edge ij to each member of $\mathcal{F}'_{T'}$, then we obtain the canonically t-intersecting family $\mathcal{F}_T$ of perfect matchings of K_{2n}. This implies that

$$|\mathcal{F} \setminus \mathcal{F}_T| \leq O((2(n-t-1)-1)!!),$$

as desired.

This completes the proof of the key lemma and thus the proof of Theorem 4.0.1

4.10 Odds and Ends

It is natural to conjecture that similar results hold for so-called near-perfect matchings of K_{2n-1}. Without much extra effort, we can give an analogue of the first part of our main result for near-perfect matchings.

Let $\mathcal{M}_{2n-1}$ denote the collection of *near-perfect matchings* of K_{2n-1}, equivalently, maximum matchings of K_{2n-1}. We may identify them with the cosets of the quotient S_{2n-1}/H_{n-1}. The theorem below follows immediately from Pieri's rule and Theorem 3.1.3, where $\mathcal{O}(2n-1)$ denotes the irreducibles of S_{2n-1} that have precisely one odd part.

4.10.1 Theorem. *The space of real-valued functions over near-perfect matchings K_{2n-1} admits the following decomposition into irreducibles of S_{2n-1}:*

$$1 \uparrow_{H_{n-1}}^{S_{2n-1}} \cong \mathbb{R}[\mathcal{M}_{2n-1}] \cong \bigoplus_{\lambda \in \mathcal{O}(2n-1)} \lambda.$$

This implies that the permutation representation of S_{2n-1} acting on $\mathcal{M}_{2n-1}$ is multiplicity-free, so we have that (S_{2n-1}, H_{n-1}) is a symmetric Gelfand pair. We define the corresponding symmetric association scheme below.

For each $\lambda \in \mathcal{O}(2n-1)$, the λ-associate A_λ is the following matrix

$$(A_\lambda)_{i,j} = \begin{cases} 1, & \text{if } d'(i,j) = \lambda \\ 0, & \text{otherwise} \end{cases}$$

where $i, j \in \mathcal{M}_{2n-1}$ and d' is a cycle type function defined as follows. Recall that the multiset union of two near-perfect matchings m, m' is a collection of even cycles and precisely one path of even length. We may represent this multiset union again as a partition $d'(m, m') = \lambda \vdash (2n-1)$ such that λ has precisely one odd part. The odd part, say λ_i, represents the unique even path of length $\lambda_i - 1$. The collection of matrices $\{A_\lambda : \lambda \in \mathcal{O}(2n-1)\}$ forms the *near-perfect matching association scheme*.

Let Γ'_t be the near-perfect matching variant of the t-derangement graph, that is, $m, m' \in E(\Gamma'_t)$ if $d'(m, m')$ has less than t parts of size 2. Let Θ_t be the subgraph of Γ'_t whose adjacency matrix is the following sum of associates of the near-perfect matching association scheme

$$\Theta_t = \sum_{\lambda} A_\lambda$$

where λ ranges over all partitions of $\mathcal{O}(2n-1)$ that have less than t parts of size less than or equal to 2.

4.10.2 Proposition. $\Theta_t \cong \Gamma_t$

Proof. Identify the vertices of K_{2n-1} and K_{2n} with the sets $[2n-1]$ and $[2n]$ respectively. There is a natural map $\psi : \mathcal{M}_{2n-1} \to \mathcal{M}_{2n}$ defined such that $\psi(m') = m$ where $m \in \mathcal{M}_{2n}$ is the unique perfect matching that matches the vertex $2n \in V(K_{2n})$ with the unique unmatched vertex of m'. This map is a bijection such that

$$m_1', m_2' \in E(\Theta_t) \quad \text{if and only if} \quad \psi(m_1'), \psi(m_2') \in E(\Gamma_t)$$

for each pair $m_1', m_2' \in \mathcal{M}_{2n-1}$, which gives the desired isomorphism. $\square$

Since Θ_t is a subgraph of Γ_t', the canonically t-intersecting families of $\mathcal{M}_{2n-1}$, which have size $(2(n-t)-1)!!$, are also independent sets of Θ_t. Proposition 4.10.2 along with the results of Section 4.6 give us the following.

4.10.3 Theorem. *Let $t \in \mathbb{N}$. If $\mathcal{F} \subseteq \mathcal{M}_{2n-1}$ is t-intersecting, then*

$$|\mathcal{F}| \leq (2(n-t)-1)!!$$

for sufficiently large n depending on t.

4.10.4 Theorem. *Let $t \in \mathbb{N}$. If $\mathcal{F}, \mathcal{G} \subseteq \mathcal{M}_{2n-1}$ are cross-t-intersecting, then*

$$|\mathcal{F}| \cdot |\mathcal{G}| \leq ((2(n-t)-1)!!)^2$$

for sufficiently large n depending on t.

A similar characterization of the extremal t-intersecting families probably holds for near-perfect matchings, but we have not yet worked out these details. Furthermore, combinatorial ideas of Ellis [21] in all likelihood can be used to turn the stability results of this chapter into stronger Hilton-Milner-type results that characterize the largest t-intersecting families of perfect matchings that are not contained in any canonically t-intersecting family. We have worked out such a characterization for the $t = 1$ case in an unpublished note; but the proof is virtually identical to Ellis' [21].

We have not made an attempt to find a concrete $N_t \in \mathbb{N}$ for each $t \geq 2$ such that Theorem 4.0.1 holds for all $n \geq N_t$. This is due to the fact that the $\{N_t\}_{t=2}^{\infty}$ that one would obtain by walking through our proof would be quite far from the conjectured $N_t = 3t/2 + 1$ conjectured by Godsil and Meagher. For instance, the constant C_t^{-1} that arises in the proof of Theorem 4.6.1 is astronomical in t. We have also assumed that t is independent of n in several places, so it would seem that radically different techniques are needed to completely resolve Godsil and Meagher's conjecture.

We are confident that there are other symmetric association schemes where the "low-frequency eigenspaces" support the characteristic vectors of maximum independent sets of the union of its top associates. For this statement to make sense, such association schemes must have some sort of natural ordering on its associates and eigenspaces, which was the reverse-lexicographical ordering of $\lambda(n)$ for $\mathcal{A}_n$ and $\mathcal{A}'_n$ in our case.

An association scheme is *P-polynomial* if there exists an ordering $\{A_j\}_{j=0}^m$ such that for each $0 \leq j \leq m$, there is a degree-j polynomial p_j such that $P_{i,j} = p_j(\omega_i)$ where ω_i is the ith eigenvalue of A_1. Such an ordering of the associates is called a *P-polynomial ordering*, and we may write their character tables as follows:

$$P = \begin{pmatrix} p_m(\omega_m) & p_{m-1}(\omega_m) & \cdots & \omega_m & 1 \\ p_m(\omega_{m-1}) & p_{m-1}(\omega_{m-1}) & \cdots & \omega_{m-1} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_m(\omega_0) & p_{m-1}(\omega_0) & \cdots & \omega_0 & 1 \end{pmatrix}.$$

An association scheme is *Q-polynomial* if there exists an ordering $\{E_j\}_{j=0}^m$ such that for each $0 \leq j \leq m$, there is a degree-j polynomial p_j^* such that $Q_{i,j} = p_j^*(Q_{i,1})$ where $Q := P^{-1}/|X|$. Such an ordering of the primitive idempotents is called a *Q-polynomial ordering*.

Many of the classic algebraic proofs of t-intersecting Erdős-Ko-Rado results at a high level stem the fact that an association scheme is P-polynomial and Q-polynomial with orderings of the associates and eigenspaces such that

$$\left(\sum_{i=0}^{t} E_i\right) 1_S = 1_S$$

for any maximum independent set S of the graph $\sum_{i=0}^{t-1} A_{m-i}$. Notice that $\mathcal{A}_n$ and $\mathcal{A}'_n$ are neither P-polynomial or Q-polynomial association schemes, yet there was a canonical way to order the associates and eigenspaces that was essential for showing EKR results. Indeed, finding the right generalization of the P-polynomial and Q-polynomial property may be helpful for understanding when an association scheme in general has the right structure to derive t-intersecting EKR bounds as we have done in this chapter. We make no attempt here to formally define what such a "t-EKR property" should be for association schemes, but we plan on investigating this in future work.

Along these lines, it would be worthwhile to find other infinite families of association schemes $\{\mathcal{B}_i\}_{n=1}^{\infty}$, perhaps coming from groups, whose character tables subject to a suitable normalization have a "stability" property in Macdonald's sense: that for sufficiently large n depending on $k \in \mathbb{N}$, the entries of kth leading principal minor of its normalized character table are bounded by some function of k. Here, the number of conjugacy classes of the group should grow with the order of the group, but in this situation there is generally no canonical ordering of the conjugacy classes. Perhaps sorting, in descending order, the associates by the size of their respective conjugacy class, and sorting the eigenspaces by dimension in ascending order is the correct approach. Loosely speaking, if the rate at which the sizes of the conjugacy classes decay is proportional to the rate at which the dimensions of irreducibles grow, then one may be able obtain t-intersecting EKR-type upper bounds for other groups provided that the order of the group is sufficiently large.

Finally, the *raison d'être* of Godsil and Meagher's book on algebraic methods in Erdős-Ko-Rado combinatorics stems from their work on intersection problems involving k-uniform partitions of $[n]$, i.e., partitions of $[n]$ into k equal parts [30, 33]. For example, they ask how large a family of k-uniform partitions of $[k^2]$ can be subject to the restriction that no two of its members $P = \{P_i\}_{i=1}^k$ and $Q = \{Q_i\}_{i=1}^k$ satisfy $|P_i \cap Q_j| = 1$ for all $1 \leq i, j \leq k$. The natural "derangement graph" that models this problem is an orbital of $G = S_{k^2}$ acting on $(G/H) \times (G/H)$ where H is the stabilizer of any k-uniform partition of $[k^2]$. Unfortunately, the orbitals of this action do not form a commutative association scheme for $k \geq 4$ (i.e., the permutation representation of G acting on G/H is not multiplicity-free) and $|G/H|$ is prohibitively large for modest values of k.

An important question is whether algebraic techniques can be used in situations like this where the domain has no commutative association scheme. In current work, we have identified the domain of n-uniform partitions of $[3n]$ (i.e., partitions of the set $[3n]$ into n parts of size 3) as a promising testbed for future work in this direction, as its low-dimensional irreducibles are multiplicity-free, resembling the commutative setting.

Chapter 5

Harmonic Analysis on Matchings of K_n

We have seen how the algebraic structure of maximum matchings of K_n can be leveraged to solve combinatorial problems, so a natural next step would be to extend these methods to arbitrary matchings.

Naively, we may hope for the structure of a symmetric association scheme over matchings, but such structure is the exception rather than the rule. The space of matchings typically decomposes with multiplicities, which does not produce an association scheme, and multiplicities present a whole new array of challenges that do not arise in the multiplicity-free setting. For these reasons, such representations have had little contact with combinatorial problems, having earned the reputation of being too unwieldy to be useful in practice.

Finding some sort of tame harmonic-analytical theory for matchings is indeed an ambitious goal, but we take the first steps in this chapter. These first few steps allow us to make some progress on open questions and conjectures of Au concerning semidefinite relaxations of the perfect matching problem [3]. We also conclude with a few interesting partial results obtained in ongoing joint work with Gary Au and Levent Tunçel on semidefinite relaxations of the perfect matching problem.

5.1 Perfect Matching Juntas

In the area of theoretical computer science known as the analysis of Boolean functions [58], a function $f \in \mathbb{R}[\mathbb{Z}_2^n]$ is a *k-junta* if it depends on at most k of its input coordinates, more formally, there exist $i_1, \cdots, i_k \in [n]$ such that

$$f(x_1, x_2, \cdots, x_n) = g(x_{i_1}, \cdots, x_{i_k})$$

for some $g \in \mathbb{R}[\mathbb{Z}_2^k]$. The jargon is that a function is a *junta* if there exists a constant k for which the function is a k-junta, and that a function is a *dictator* if it is a 1-junta. Recall that the characters $\{\chi_S\}_{S\subseteq[n]}$ of the group $\mathbb{Z}_2^n$ are indexed by subsets of $[n]$ and form an orthonormal basis for $\mathbb{R}[\mathbb{Z}_2^n]$ [68]. A function $f \in \mathbb{R}[\mathbb{Z}_2^n]$ has *Fourier-degree* k if f can be written as

$$f = \sum_{\substack{S\subseteq[n] \\ |S|\leq k}} \hat{f}(S)\chi_S$$

where $\hat{f}(S) \in \mathbb{R}$ are the Fourier coefficients of f. A fundamental fact is that the juntas span the "low frequencies", i.e., a function $f \in \mathbb{R}[\mathbb{Z}_2^n]$ is a linear combination of k-juntas if and only if $f \in \mathbb{R}[\mathbb{Z}_2^n]$ has Fourier-degree k [58]. Since any finite Abelian group is isomorphic to a direct product of cyclic groups of prime-power order, the foregoing can be generalized to Abelian groups by letting the coordinates be the constituent factors of cyclic groups and the frequencies be the characters of the group.

Because juntas are of fundamental importance in the analysis of Boolean functions, there has been some interest towards obtaining "junta characterizations" for other domains (c.f., permutations [24, 25, 26], k-sets [27]) in order to generalize the theory. In this vein, the algebraic structure of perfect matchings was brought into question by Braun et al. [10] in their work on the symmetric semidefinite extension complexity of the perfect matching problem. There, it was observed that the main obstacle in adapting hypercube-based algebraic methods to perfect matchings is the non-trivial algebraic structure of $\mathcal{M}_{2n}$. They introduced the notion of a *perfect matching junta*, which was useful in their analysis; however, their techniques were not conducted in the "frequency domain" of matchings, so their description of a perfect matching junta lacked the usual Fourier-theoretic rationale. We now give a formal definition of a perfect matching junta and show that it is the proper analogue of the hypercube juntas, in particular, that they span the space of fat even irreducibles U_t (i.e., the "low frequencies").

Recall that $\Delta(U)$ is the set of edges of the complete graph $K_n = (V, E)$ that have an endpoint in $U \subseteq V$. A function $f \in \mathbb{R}[\mathcal{M}_{2n}]$ is a *(perfect matching) k-junta* if there exists a subset $K \subseteq V$ of k vertices such that

$$f(m) = g(\Delta(K) \cap m)$$

for some $g : \mathcal{K} \to \mathbb{R}$ where $\mathcal{K}$ is the collection of all k-matchings of the subgraph $\Delta(K)$. In other words, f is a k-junta if there exist k vertices $K \subseteq V$ such that $f(m)$ is only determined by the edges of m that touch K. We say that a function $f \in \mathbb{R}[\mathcal{M}_{2n}]$ has *Fourier-degree k* if $f \in U_k$.

5.1.1 Theorem (Junta Characterization of $\mathcal{M}_{2n}$)**.** *A real-valued function over $\mathcal{M}_{2n}$ has Fourier-degree k if and only if it is a real linear combination of k-juntas.*

Note that our k-junta definition is a bit different than the hypercube k-juntas since g is not a function over perfect matchings of a smaller complete subgraph, but rather a function over "cuts" of K_{2n}. Loosely speaking, this absence of self-similarity is due to the fact that there seems to be no so-called "product structure" over perfect matchings (i.e., no way to express S_{2n}/H_n algebraically as a product of smaller cosets S_{2k}/H_k) like there is for Abelian groups (e.g, $\mathbb{Z}_2^n \cong \mathbb{Z}_2 \otimes \cdots \otimes \mathbb{Z}_2$), which is problematic for translating hypercube-based methods over to perfect matchings [44].

It is not hard to see that the k-juntas can be written as a sum of characteristic functions of various canonically k'-intersecting families for $k/2 \leq k' \leq k$, and so it suffices to prove the theorem below.

5.1.2 Theorem. *The characteristic functions of canonically k-intersecting families span U_k.*

Proof. Theorem 4.4.1 shows that the span of the characteristic functions of canonically k-intersecting families is a subspace of U_k. Now let 2λ be a fat irreducible of S_{2n}. Recall from Section 4.8 that the functions

$$f_t = \sum_{\sigma \in C_t} \mathrm{sign}(\sigma) 1_{\{\sigma t\}}$$

for all standard tableaux t of shape 2λ form a basis for 2λ. It suffices to prove that $1_{\{\sigma t\}}$ can be written as a linear combination of characteristic functions of canonically k-intersecting families for any such t and $\sigma \in C_t$

Let s be the number of cells below the first row of λ. We can write any canonically s-intersecting family $\mathcal{F}_S \subseteq \mathcal{M}_{2n}$ as a linear combination of canonically k-intersecting families:

$$1_{\mathcal{F}_S} = \binom{n-s}{k-s}^{-1} \sum_{\substack{T \supseteq S \\ |T|=k}} 1_{\mathcal{F}_T}.$$

Now consider the set of s-matchings of $K_{2s} = (V, E)$ where V is the set of numbers that occur below the first row of the tableau σt. Of these s-matchings, let $\mathcal{S}$ be the subset that are covered by σt. Note that

$$\bigcap_{S \in \mathcal{S}} 1_{\mathcal{F}_S} = \emptyset \quad \text{and} \quad \bigcup_{S \in \mathcal{S}} 1_{\mathcal{F}_S}$$

is the collection of all $m \in \mathcal{M}_{2n}$ that are covered by σt. This shows that

$$1_{\{\sigma t\}} = \sum_{S \in \mathcal{S}} 1_{\mathcal{F}_S},$$

which finishes the proof. $\square$

In summary, we have shown that functions that describe "local" properties of perfect matchings have low Fourier complexity whereas functions that describe "global" properties of perfect matchings must have high Fourier complexity. To better illustrate this, let t be the standard tableau

1	2
3	4
$\vdots$	$\vdots$
$2n$-1	$2n$

and consider the function

$$f = \frac{1}{n!} f_t = \frac{1}{n!} \sum_{\sigma \in S_n \times S_n} \operatorname{sign}(\sigma) 1_{\{\sigma t\}}$$

that lives in the (2^n) irreducible of $\mathbb{R}[\mathcal{M}_{2n}]$, which is the highest Fourier frequency. Note that f is $\{-1, 0, 1\}$-valued function that is nonzero if and only if the input is a perfect matching of the complete bipartite subgraph induced by the odd and even numbered vertices. This function describes a global property of a perfect matching, as one must check $2(n-1)$ vertices in the worst case to determine whether the output is non-zero. On the other hand, the so-called *odd-set constraints*

$$c_S(m) = |\delta(S) \cap m| \quad \text{for all } m \in \mathcal{M}_{2n}$$

where $\delta(S)$ is the set of edges that have precisely one endpoint in a subset $S \subseteq V$ of size $2k+1$ are clearly $(2k+1)$-juntas.

5.2 The Space of k-Matchings

Recall that we may identify k-matchings $m \in \mathcal{M}_{n,k}$ with cosets of the quotient $S_n/(S_{n-2k} \times H_k)$, which gives the count

$$|\mathcal{M}_{n,k}| = \frac{n!}{(n-2k)!2^k k!} = \frac{1}{k!}\binom{n}{2}\binom{n-2}{2}\cdots\binom{n-2k}{2}.$$

Let $\Lambda(n)$ denote the set of partitions of n that minimize the number of odd parts. The following is just Thrall's result and Theorem 4.10.1 reformulated.

5.2.1 Theorem. *The decomposition of the space of real-valued functions over $\mathcal{M}_{n,\lfloor n/2 \rfloor}$ into irreducible representations of S_n is*

$$\mathbb{R}[\mathcal{M}_{n,\lfloor n/2 \rfloor}] \cong \bigoplus_{\lambda \in \Lambda(n)} \lambda.$$

By appealing to Frobenius Reciprocity at the level of representations rather than characters (see [68] for example), Lemma 4.4.3 implies a generalization of Theorem 5.2.1 to the space of k-matchings.

5.2.2 Theorem. *The unique decomposition of the space of real-valued functions over $\mathcal{M}_{n,k}$ into irreducible representations of S_n is*

$$\mathbb{R}[\mathcal{M}_{n,k}] \cong \bigoplus_{\lambda \vdash k} \bigoplus_{\mu} \mu$$

where μ ranges over all shapes such that $\mu/2\lambda$ is a horizontal strip of $n-2k$.

This implies an interesting representation-theoretic way of counting matchings.

5.2.3 Corollary. *The number of k-matchings of K_n admits the following count:*

$$|\mathcal{M}_{n,k}| = \sum_{\lambda \vdash k} \sum_{\mu} f^{\mu},$$

where μ ranges over all shapes such that $\mu/2\lambda$ is a horizontal strip of size $n-2k$.

For example, the 1485 2-matchings of K_{12} can be counted as follows

$$f^{(12)} + f^{(11,1)} + 2f^{(10,2)} + f^{(9,3)} + f^{(9,2,1)} + f^{(8,4)} + f^{(8,2,2)}.$$

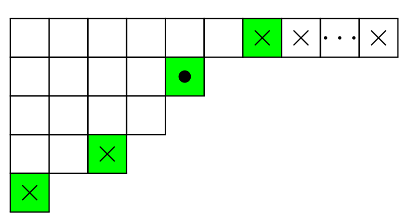
Another consequence of Theorem 5.2.2 and the combinatorics of Pieri's rule is the proposition below.

5.2.4 Proposition. *For all $n \in \mathbb{N}$, we have*

$$\mathbb{R}[E] \cong \mathbb{R}[\mathcal{M}_{n,1}] \leq \mathbb{R}[\mathcal{M}_{n,2}] \leq \cdots \leq \mathbb{R}[\mathcal{M}_{n,\lfloor n/4 \rfloor - 1}] \leq \mathbb{R}[\mathcal{M}_{n,\lfloor n/4 \rfloor}].$$

Moreover, if μ has multiplicity m_μ in $\mathbb{R}[\mathcal{M}_{n,k}]$ and $k < \lfloor n/4 \rfloor$, then μ has multiplicity at least m_μ in $\mathbb{R}[\mathcal{M}_{n,k+1}]$.

For the proof of our next result, it will be useful to introduce more combinatorial terminology. An *inner corner* with respect to the shape $\mu \vdash k$ is a cell $(i,j) \notin \mu$ such that $\mu \cup \{(i,j)\}$ is a valid shape of size $k+1$. A *marked shape* is a shape λ with marked cells $\times$ where the marked cells form a horizontal strip. For example, in the marked shape $\lambda = (n-12, 4^2, 3, 1) \vdash n$ illustrated below, the unmarked white cells form $2\mu = 2(3, 2^2, 1) \vdash 16$ and the colored cells are the inner corners of 2μ. The cells marked with $\times$ form the horizontal strip.



The *penultimate inner corner* of a shape is the second to last inner corner when considering them from left to right. In the example above, the penultimate inner corner is $\bullet$. The following gives a characterization of the isotypic components $m_\lambda \lambda \leq \mathbb{R}[\mathcal{M}_{n,k}]$ such that $m_\lambda = 1$, which we call the *multiplicity-1 irreducibles.*

5.2.5 Proposition. *For $k < \lfloor n/4 \rfloor$, the only non-trivial multiplicity-1 irreducibles $\lambda \leq \mathbb{R}[\mathcal{M}_{n,k}]$ are those shapes λ obtainable from $2\mu \vdash 2k$ by adding a horizontal strip h of $n - 2k$ cells that satisfies the following:*

1. *If there are two cells of h in the jth row of λ for some $2 \leq j \leq \ell(\mu) + 1$, then for each $1 \leq i \leq [\mu_1]$, the $(2i-1)$th column of λ contains a cell of h;*

2. *otherwise, each inner corner of 2μ is occupied by a cell of h, except possibly the penultimate inner corner.*

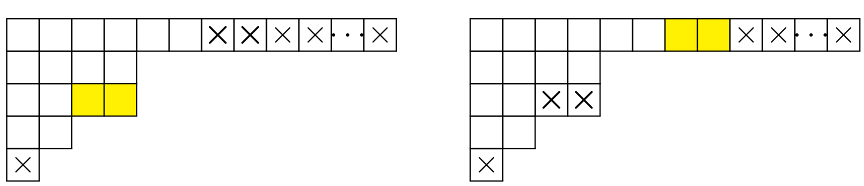
Proof. If there are two cells of h, say

$$(j, 2i' - 1), (j, 2i') \in \lambda \quad \text{for some} \quad 2 \leq j \leq \ell(\mu) + 1,$$

and an i such that the $(2i-1)$-th column of λ does not contain a cell of h, then the $(2i)$-th column of λ does not contain a cell of h. This implies there are two consecutive cells

$$(j', 2i - 1), (j', 2i) \in 2\mu$$

that do not have a cell of h or 2μ below them, but then we may obtain a different marked shape λ by interchanging the marked cells $(j', 2i-1), (j', 2i)$ with the unmarked cells $(j, 2i' - 1), (j, 2i')$, e.g.,


.

This gives more than one way to obtain λ from an even shape of size $2k$.

If no two cells of h are in the same row of $\lambda \backslash \lambda_1$ and there is an inner corner of 2μ, other than the penultimate and last, that is not occupied by a cell of h, then the cell above this inner corner (i, j) and the cell to its immediate right $(i, j + 1)$ are two cells of 2μ that do not have cells of h below them.

Arguing as above, we can now create a new marked diagram by swapping $(i, j), (i, j+1)$ with the first two cells of h that belong to the first row of λ, which again gives more than one way to obtain λ from an even shape of size $2k$. □

The reason why we have brought so much attention to the multiplicity-1 irreducibles of $\mathbb{R}[\mathcal{M}_{n,k}]$ is because we can actually construct a combinatorial basis for their respective isotypic components as follows. For any $\mu \vdash n$ and standard tableau t, define

$$f_t = \sum_{\sigma \in C_t} \operatorname{sign}(\sigma) 1_{\{\sigma t\}}$$

where $1_{\{\sigma t\}} \in \mathbb{R}[\mathcal{M}_{n,k}]$ is the characteristic function of the set of k-matchings covered by the tabloid $\{\sigma t\}$.

5.2.6 Theorem. *If $\mu \leq \mathbb{R}[\mathcal{M}_{n,k}]$ is irreducible, then $\{f_t\}_{t \in T_\mu}$ is a basis for μ.*

Proof. Let M^μ be the permutation representation of S_n acting on $\mathbb{T}_\mu$, the set of all μ-tabloids. Young proved that this representation decomposes as follows:

$$M^\mu \cong \left(\bigoplus_{\lambda > \mu} K_{\lambda,\mu} \lambda \right) \oplus \mu.$$

Consider the map $\phi : M^\mu \to \mathbb{R}[\mathcal{M}_{n,k}]$ defined such that $\phi : e_{\{t\}} \mapsto 1_{\{t\}}$, and let φ be its linear extension. For any μ-tabloid $\{t\}$ and $m \in \mathcal{M}_{n,k}$, we have that $\{t\}$ covers m if and only if $\{\pi t\}$ covers πm for all $\pi \in S_n$. This implies that φ intertwines the permutation representations

$$M^\mu \quad \text{and} \quad \mathbb{R}[\mathcal{M}_{n,k}] \cong 1 \uparrow_{S_{n-2k} \times S_2 \wr S_k}^{S_n} .$$

By Schur's lemma, for the irreducible μ of M^μ, we have that $\varphi|_\mu$ is either the zero map or an isomorphism. By our choice of μ, it follows that $\varphi|_\mu$ is an isomorphism.

Recall that the μ-polytabloids $\{e_t\}$ form a basis for $\mu \leq M^\mu$, thus

$$\varphi(e_t) = \sum_{\sigma \in C_t} \operatorname{sign}(\sigma) \varphi(e_{\{\sigma t\}}) = \sum_{\sigma \in C_t} \operatorname{sign}(\sigma) 1_{\{\sigma t\}} = f_t;$$

therefore, the $\{f_t\}$'s form a basis for $\mu \leq \mathbb{R}[\mathcal{M}_{n,k}]$, as desired. □

For $k = 1$ and $k = \lfloor n/2 \rfloor$, this gives a basis for $\mathbb{R}[\mathcal{M}_{n,k}]$, as their respective decompositions into irreducibles of S_n are multiplicity-free, which we saw in the previous chapter. The basis however is not orthogonal, so it is certainly not a "Fourier basis" per se.

For $2 \leq k \leq \lfloor n/2 \rfloor - 1$, the set of f_t's over all standard tableaux t of shape μ such that the μ-isotypic component in $\mathbb{R}[\mathcal{M}_{n,k}]$ is non-zero forms a basis for a somewhat large subspace of $\mathbb{R}[\mathcal{M}_{n,k}]$. The subspace is proper since $(n-2,2)$ occurs with multiplicity $m_{(n-2,2)} = 2$ for all such k. To extend this to a basis for all of $\mathbb{R}[\mathcal{M}_{n,k}]$, one would need to further decompose each of the μ-isotypic subspaces $m_\mu \mu \leq \mathbb{R}[\mathcal{M}_{n,k}]$ into m_μ mutually orthogonal subspaces:

$$m_\mu \mu \cong \operatorname{Span}\{f_t\} \oplus \mu^{(2)} \oplus \cdots \oplus \mu^{(m_\mu)},$$

and arriving at nice concrete bases for the remaining $\mu^{(i)}$'s that are orthogonal to the f_t's seems to be a difficult task. It is an important question whether there are nice bases for the isotypic components arising in $\mathbb{R}[\mathcal{M}_{n,k}]$. For other domains there has been some success finding such bases by inducing chains of multiplicity-free representations, so-called *Gelfand-Tsetlin bases* [13], which may also be useful in our setting.

5.2.1 The k-Linear Perfect Matching Space

For any maximum matching $M \in \mathcal{M}_{n,\lfloor n/2 \rfloor}$, define the *k-characteristic function* $1_M \in \mathbb{R}[\mathcal{M}_{2n}]$ to be

$$1_M(m) = \begin{cases} 1 & \text{if } m \subseteq M; \\ 0 & \text{otherwise} \end{cases}$$

for all $m \in \mathcal{M}_{n,k}$. For $k = 1$, then these functions are the well-known characteristic functions of maximum matchings of K_n. If n is even, then the span of these functions is known as *the perfect matching space* of K_n [66, Sec. 37.2]. There are several combinatorial proofs of the fact that the dimension of the perfect matching space of K_n is $\binom{n}{2} - n + 1$, but we offer a quick and more revealing algebraic proof of this fact that easily generalizes to higher dimensions.

Let A be the $|E| \times |\mathcal{M}_{n,\lfloor n/2 \rfloor}|$ matrix indexed by edges and perfect matchings respectively such that the row vector indexed by $e \in E$ is the characteristic vector of the canonically intersecting family $\mathcal{F}_e \subseteq \mathcal{M}_{n,\lfloor n/2 \rfloor}$. Since $A_{e,m} = A_{\sigma e, \sigma m}$ for all $\sigma \in S_n$, we have that A is a linear transformation that intertwines the S_n-representations $\mathbb{R}[E]$ and $\mathbb{R}[\mathcal{M}_{2n}]$. By Schur's Lemma, we have that A acts as the zero map or an isomorphism on the irreducibles of $\mathbb{R}[E]$ which are

$$\mathbb{R}[E] \cong (n) \oplus (n-1,1) \oplus (n-2,2).$$

Since $(n-1,1)$ is not an irreducible of $\mathbb{R}[\mathcal{M}_{2n}]$, we have $(n-1,1) \leq \ker A$. By Theorem 5.1.2, the row space of A is U_1, implying that

$$\text{rank } A = f^{(n)} + f^{(n-2,2)} = \binom{n}{2} - n + 1.$$

This implies that A must act as an isomorphism on the even irreducibles of $\mathbb{R}[E]$; therefore, the 1-characteristic functions of perfect matchings span $(n) \oplus (n-2,2)$.

It is easy to see that this argument generalizes to the following theorem.

5.2.7 Theorem. *The k-linear perfect matching space of K_n is $U_k \leq \mathbb{R}[\mathcal{M}_{n,k}]$.*

Similar results can be obtained for the *k-linear maximum matching space* of K_n for odd n. In particular, if n is odd, then the 1-linear maximum matching space is just $\mathbb{R}[E]$; however, for $k \geq 2$, the k-linear maximum matching space does not span $\mathbb{R}[\mathcal{M}_{n,k}]$.

5.2.2 The k-Linear Matching-Orthogonal Space

Let $\overline{x} := \{x_{uv}\}_{uv \in E}$, and assume that n is even. Another challenge noted in Braun et al. [10] is understanding when a homogeneous k-linear polynomial $p(\overline{x}) \in \mathbb{R}[\overline{x}]$ is identically-zero over $\mathcal{M}_{n,\lfloor n/2 \rfloor}$, i.e.,

$$p(\overline{x}) = \sum_{\substack{K \subseteq E \\ |K|=k}} c_K \prod_{ij \in K} x_{ij} \quad \text{and} \quad p(\overline{M}) = 0 \quad \text{for all } M \in \mathcal{M}_{n,\lfloor n/2 \rfloor}$$

where $\overline{M}$ is the input that assigns $x_{ij} = 1$ if $ij \in M$; otherwise, $x_{ij} = 0$. Note that it is a trivial matter to check if $p(\overline{x}) = 0$ for all Boolean inputs, since they are identically-zero precisely when $p = 0$, that is, $c_K = 0$ for all k-subsets $K \subseteq E$. The situation is quite different for perfect matchings, since even for linear functions ($k = 1$), the span of the characteristic vectors of perfect matchings in $\mathbb{R}[E]$ has dimension $\binom{n}{2} - n + 1 < |E|$. This implies there are non-trivial linear functions over perfect matchings that are identically zero, which we now review.

Following Lovász [50], a function $f \in \mathbb{R}[E]$ is *matching-orthogonal* if

$$\langle f, 1_M \rangle = 0 \quad \text{for all } M \in \mathcal{M}_{n,\lfloor n/2 \rfloor}.$$

The space of all such functions is called *the matching-orthogonal space* [50], as it is the orthogonal complement of the maximum matching space in $\mathbb{R}[E]$. As we saw in the previous section, the matching-orthogonal space is the irreducible $(n-1,1)$ when n is even, and $\{0\}$ otherwise. A basis for $(n-1,1)$ is easily obtained by considering the well-known *standard representation* of S_n

$$\mathbb{R}[V] \cong (n) \oplus (n-1,1)$$

(see [64]) then picking a basis for the orthogonal complement of the all-ones space (n). In fact, from the foregoing it is not difficult to observe a classic result in matching theory [66], that $f \in \mathbb{R}[E]$ is matching-orthogonal if and only if

$$f = \sum_{v \in V} a_v 1_{\delta(v)} \quad \text{and} \quad \sum_{v \in V} a_v = 0,$$

where $1_{\delta(v)} \in \mathbb{R}[E]$ is the characteristic function of the edges incident to v.

Notice that if $p(\overline{x})$ is a homogeneous k-linear polynomial with a term

$$q(\overline{x}) := c_K \prod_{ij \in K} x_{ij}$$

such that $K \notin \mathcal{M}_{n,k}$, then we have

$$(p-q)(\overline{M}) = p(\overline{M}) \quad \text{for all } M \in \mathcal{M}_{n,\lfloor n/2 \rfloor}$$

since $q(\overline{M}) = 0$ for all $M \in \mathcal{M}_{n,\lfloor n/2 \rfloor}$. We may assume that all the monomials of $p(\overline{x})$ correspond to k-matchings, which gives us the equivalence

$$p(\overline{M}) = 0 \quad \forall M \in \mathcal{M}_{n,k} \quad \text{if and only if} \quad \langle f_p, 1_M \rangle = 0 \quad \forall M \in \mathcal{M}_{n,k}$$

where $f_p \in \mathbb{R}[\mathcal{M}_{n,k}]$ is defined such that $f_p(m) := c_m$ for all $m \in \mathcal{M}_{n,k}$. By Theorem 5.2.7, the space of all functions $f \in \mathbb{R}[\mathcal{M}_{n,k}]$ that are orthogonal to the k-linear perfect matching space is simply

$$U_k^{\perp} \leq \mathbb{R}[\mathcal{M}_{n,k}]$$

which we call the *k-linear matching-orthogonal space*. To the best of our knowledge, no such characterization of the homogeneous k-linear polynomials over perfect matchings that are identically-zero was known for $k \geq 2$.

We saw there was an elegant combinatorial way of characterizing the space $U_1^\perp \cong (n-1,1)$ for even n, but it seems that a characterization of $U_k^\perp$ along these lines is much more involved. In particular, the irreducibles of $U_k^\perp$ typically have multiplicities greater than one, and we have not been able to find a permutation representation ρ of S_n such that

$$\rho \cong (n) \oplus U_k^\perp.$$

It is quite possible that ρ does not arise from the action of S_n on a set, which would make it more difficult to find a combinatorial characterization of this space.

The question of finding a nice characterization of $U_k^\perp$ is central to the work of Braun et al. [10]. Indeed, in their key technical result they construct for any homogeneous k-linear polynomial $p(\overline{x}) \in \mathbb{R}[\overline{x}]$ a homogeneous $(2k-1)$-linear polynomial $q(\overline{x})$ that is a product of low-degree identically-zero polynomials such that $(p-q)(\overline{M}) = 0$, providing a "degree-$(2k-1)$ certificate" that $p(\overline{x}) \in \mathbb{R}[\overline{x}]$ is identically-zero over $\mathcal{M}_{n,\lfloor n/2 \rfloor}$. We believe that there should be a way to use our representation-theoretic characterization of the k-linear matching-orthogonal space to characterize the k-linear polynomials that are identically-zero over $\mathcal{M}_{n,\lfloor n/2 \rfloor}$, as to provide optimal degree-k certificates.

5.3 Some Conjectures of Au

We now explore more connections of our results to semidefinite formulations and relaxations of the perfect matching problem. As mentioned before, the collection of k-matchings does not afford a symmetric association scheme; however, it does admit a "non-commutative association scheme" structure more commonly known as a homogeneous coherent configuration [12].

For any finite set X and index set $\mathcal{I}$, we say that $\mathcal{A} = \{A_i\}_{i\in\mathcal{I}}$ of $X \times X$ binary matrices is a *homogeneous coherent configuration* if the following axioms are satisfied:

1. $A_i = I$ for some $i \in \mathcal{I}$,
2. $B \in \mathcal{A} \Rightarrow B^T \in \mathcal{A}$,
3. $\sum_{i\in\mathcal{I}} A_i = J$ where J is the all-ones matrix, and
4. $A_iA_j = \sum_{k\in\mathcal{I}} p_{ij}^{(k)} A_k$ for all $i,j \in \mathcal{I}$ for some $p_{ij}^{(k)} \in \mathbb{N}$.

If we further have $A_iA_j = A_jA_i$ for all $i,j \in \mathcal{I}$, then the homogeneous coherent configuration is in fact an association scheme. A wide class of homogeneous coherent configurations arise from group actions. In particular, for any group G and subgroup $H \leq G$, consider the action of G on ordered pairs of cosets $G/H \times G/H$:

$$g \cdot (g_iH, g_jH) = ((gg_i)H, (gg_j)H) \quad \forall g \in G,$$

where $g_i, g_j \in G$ are coset representatives. The orbitals of this action partition the ordered pairs $G/H \times G/H$ and can be represented as binary matrices. It is routine to show that these matrices form an homogeneous coherent configuration [12], and that the eigenspaces of matrices in the coherent algebra $\mathrm{Span}\{A_i\}_{i\in\mathcal{I}}$ are direct sums of irreducibles of the acting group [4, 65].

Setting $G = S_n$ and $H = S_{n-2k} \times H_k$ gives us a homogeneous coherent configuration $\mathcal{A}_{n,k}$ defined over the k-matchings of K_n, which we call *the k-matchings coherent configuration*. It is not hard to show the following proposition that provides a combinatorial description of the orbitals of $\mathcal{A}_{n,k}$

5.3.1 Proposition. *The indices $\mathcal{I}$ of $\mathcal{A}_{n,k}$ are in bijection with the isomorphism classes of loopless properly 2-edge-colored multigraphs on $2k$ edges of maximum degree 2. In particular, for any isomorphism class representative $i \in \mathcal{I}$, the orbital $A_i \in \mathcal{A}_{n,k}$ is a $|\mathcal{M}_{n,k}| \times |\mathcal{M}_{n,k}|$ binary matrix indexed by k-matchings such that*

$$(A_i)[m,m'] = \begin{cases} 1 & \text{if } m \cup m' \cong i; \\ 0 & \text{otherwise} \end{cases}$$

for all $m, m' \in \mathcal{M}_{n,k}$.

In [3], a comprehensive treatment of semidefinite lift-and-project relaxations for the perfect matching problem was given, wherein the following family of matrices was introduced. For $1 \leq t \leq s \leq n$, let $Y_{n,s,t}$ be the $|\mathcal{M}_{n,s}| \times |\mathcal{M}_{n,t}|$ matrix indexed by $\mathcal{M}_{n,s}$ and $\mathcal{M}_{n,t}$ such that the (S,T)-entry is

$$Y_{n,s,t}[S,T] = \begin{cases} (n - 2|S \cup T| - 1)!! & \text{if } S \cup T \text{ is a matching of } K_n; \\ 0 & \text{otherwise} \end{cases}$$

for all $S \in \mathcal{M}_{n,s}$ and $T \in \mathcal{M}_{n,t}$. In particular, the $Y_{n,k,k}$ matrices were studied to establish their positive semidefiniteness in order to obtain lower bounds on the worst-case behavior of certain semidefinite lift-and-project operators with respect to the matching problem.

We now define a few variants of these $Y_{n,k,k}$ matrices also introduced in [3]. For any real vector y of dimension $k+1$ indexed by $\{0, 1, \cdots, k\}$, let

$$Y_{n,s,t}(y)[S,T] := \begin{cases} y_i & \text{if } S \cup T \text{ is a } (k+i)\text{-matching of } K_n; \\ 0 & \text{otherwise,} \end{cases}$$

for all $S \in \mathcal{M}_{n,s}$ and $T \in \mathcal{M}_{n,t}$. For any $0 \leq i \leq k$, it will be convenient to define

$$Y^{(i)}_{n,s,t}[S,T] := \begin{cases} 1 & \text{if } S \cup T \text{ is a } (k+i)\text{-matching of } K_n; \\ 0 & \text{otherwise.} \end{cases}$$

for all $S \in \mathcal{M}_{n,s}$ and $T \in \mathcal{M}_{n,t}$. If $s = t = k$, then clearly $Y^{(i)}_{n,k,k}$ is an orbital of $\mathcal{A}_{n,k}$ for $0 \leq i \leq k$. We refer to these orbitals as *the matching orbitals*.

5.3.2 Proposition. *For all $0 \leq k \leq n$, the eigenspaces of $Y_{n,k,k}(y)$ are direct sums of irreducibles of $\mathbb{R}[\mathcal{M}_{n,k}]$.*

Proof. Let $I \subseteq \mathcal{I}$ be the set of indices of associates of $\mathcal{A}_{n,k}$ that correspond to multiunion of two k-matchings where the union is a matching of K_n. We may write $Y_{n,k,k}(y)$ as a linear combination of the orbitals of $\mathcal{A}_{n,k}$

$$Y_{n,k,k} = \sum_{i \in I} y_{k-|E(i)|} A_i$$

where $|E(i)|$ denotes the number of edges in the multigraph i ignoring edge multiplicity. Since $Y_{n,k,k}(y)$ is in the coherent algebra generated by $\mathcal{A}_{n,k}$, its eigenspaces are isomorphic to direct sums of irreducibles in Theorem 5.2.2. $\square$

Since $Y_{n,k,j}(y)[S,T] = Y_{n,k,j}(y)[\sigma S, \sigma T]$ for all $\sigma \in S_n$, the following proposition is also immediate.

5.3.3 Proposition. *For all $0 \leq j \leq k \leq n$, the row space of $Y_{n,k,j}(y)$ is a direct sum of irreducibles of $\mathbb{R}[\mathcal{M}_{n,k}]$ and the column space of $Y_{n,k,j}(y)$ is a direct sum of irreducibles of $\mathbb{R}[\mathcal{M}_{n,j}]$.*

These propositions along with the theory presented in this chapter explains much of the combinatorial phenomena surrounding the eigenspaces of the $Y_{n,k,k}$ matrices noted by Au in his thesis (see [3, Ch. 8] for more discussion). For example, it allows us to make some progress on the following.

Conjecture (Au [3, Conjecture 82])
Let $k < \lfloor n/4 \rfloor$, and let $P(k)$ denote the set of integer partitions of size no greater than k. Then for every $\lambda = (\lambda_1, \cdots, \lambda_\ell) \in P(k)$, the matrix $Y_{n,k,k}$ is positive semidefinite and has a positive eigenspace of dimension $f^{(n-2|\lambda|, 2\lambda_1, \cdots, 2\lambda_\ell)}$. Moreover, these are all the positive eigenspaces of $Y_{n,k,k}$.

We now give a short proof of this fact for the case where n is even.

Proof. Let A be the $|\mathcal{M}_{n,k}| \times |\mathcal{M}_{n,\lfloor n/2 \rfloor}|$ matrix indexed by k-matchings and maximum matchings such that the row indexed by $m \in \mathcal{M}_{k,n}$ is the characteristic vector of the canonically k-intersecting family $\mathcal{F}_m$. The (m, m')-entry of $Y_{n,k,k}$ counts the number of maximum matchings M such that $m \subseteq M$ and $m' \subseteq M$. This shows that $Y_{n,k,k} = AA^T$, thus $Y_{n,k,k}$ is positive semidefinite.

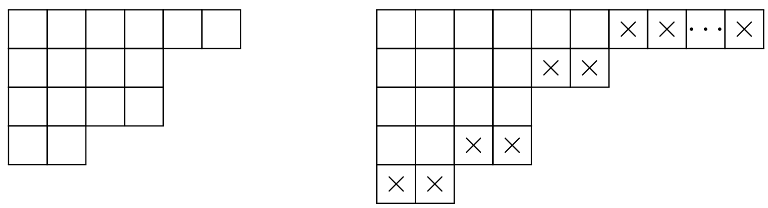
By Theorem 5.1.2, the rowspace of A is U_t. Since $\ker A^T = \ker AA^T$, it follows that each zero eigenspace of $Y_{n,k,k}$ is a subspace of $U_k^\perp$, and thus each positive eigenspace is a subspace of U_k. Each irreducible of U_k has multiplicity 1 and is of the form $(n-2|\lambda|, 2\lambda_1, \cdots, 2\lambda_\ell)$ for some $\lambda = (\lambda_1, \cdots, \lambda_\ell) \in P(k)$, as desired. □

This proof does not carry over to the case where n is odd due to the fact that the (m, m')-entry of $Y_{n,k,k}$ no longer counts something obvious. For small odd n and k, computations suggest the bewildering fact that the range of $Y_{n,k,k}$ is not a subspace of the k-linear maximum matching space. The algebraic structure of $Y_{n,k,k}$ for odd n is indeed tantalizing, and it is intriguing that its range has such an elegant Fourier support, but does not seem to lie in the span of k-characteristic functions of maximum matchings.

We now take a look at a less difficult conjecture of Au on a restricted class of eigenspaces.

Conjecture (Au [3, Conjecture 80])
Let $\lambda = (\lambda_1, \cdots, \lambda_\ell)$ be a partition of size k, and suppose $n \geq 2k + 2\lambda_1$. Let t be a standard tableau of shape $\overline{\lambda} = (n - 2|\lambda|, 2\lambda_1, \cdots, 2\lambda_\ell) \vdash n$. Then f_t is a positive eigenvector of $Y_{n,k,k}(y)$ for any $y \in \mathbb{R}^{k+1}$. Moreover, the set of all such f_t's forms a basis for the $\overline{\lambda}$-eigenspace of $Y_{n,k,k}(y)$.

By Proposition 5.2.5, the irreducibles $\overline{\lambda}$ are all multiplicity-1, as they are the unique tableau obtainable from $2\lambda \vdash 2k$ by laying a horizontal strip of size $n-2k$ across all of the columns. For example, if $\lambda = (3,2,2,1)$, $\overline{\lambda}$ is obtained as



In the original statement of this conjecture, Au proposed a different and more complicated family of eigenvectors [3, Ch. 8], but we suspect that they are essentially the f_t's as defined in Section 5.2. We refer the reader to [3, Ch. 8] for more discussion on this conjecture.

A straightforward way to prove this conjecture is to simply demonstrate that for any $y \in \mathbb{R}^{k+1}$ there exists a $\theta_{\overline{\lambda}} \in \mathbb{R}$ such that

$$[Y_{n,k,k}(y)]f_t = \theta_{\overline{\lambda}} f_t \qquad \text{for all standard tableau } t \text{ of shape } \overline{\lambda}.$$

We give a few partial results towards proving this conjecture along these lines.

For any tableau t of size n and k-matching m of K_n, our combinatorial reasoning will be aided by considering diagrams that superimpose m on t. For example, if t is the tableau below

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 19 & \cdots & n \\ 7 & 8 & 9 & 10 & 11 & 12 & & & \\ 13 & 14 & 15 & 16 & & & & & \\ 17 & 18 & & & & & & & \end{array}.$$

and m is the 6-matching $m = \{\{7,8\},\{9,10\},\{11,12\},\{13,14\},\{15,16\},\{17,18\}\}$, then we have the diagram

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 19 & \cdots & n \\ 7 \!-\!\!- & \!\!-\!\! 8 & 9 \!-\!\!- & \!\!-\!\! 10 & 11 \!-\!\!- & \!\!-\!\! 12 & & & \\ 13 \!-\!\!- & \!\!-\!\! 14 & 15 \!-\!\!- & \!\!-\!\! 16 & & & & & \\ 17 \!-\!\!- & \!\!-\!\! 18 & & & & & & & \end{array}.$$

A closer examination of the f_t's reveals that they are a relatively "sparse" basis for $\mu \leq \mathbb{R}[\mathcal{M}_{n,k}]$. For instance, if there is an edge $ij \in m$ such that both of its endpoints lie in the same column of t as shown below, then no element of C_t can send i or j different columns, thus $f_t(m) = 0$.

$$\begin{array}{lllllllll}
1 & 2 & 3 & 4 & 5 & 6 & 19 & \cdots & n \\
7 & 8 & 9 & 10 & 11 & 12 & & & \\
13 & 14 & 15 & 16 & & & & & \\
17 & 18 & & & & & & &
\end{array}$$

For any tableau t and k-matching m, a *hole* is a cell $t_{i,j}$ such that $t_{i,j} \notin V(m)$. Let us assume there exists a column with two holes, as illustrated below:

$$\begin{array}{lllllllll}
1 & 2 & 3 & 4 & 5 & 6 & 19 & \cdots & n \\
7 & 8 & 9 & 10 & 11 & 12 & & & \\
13 & 14 & 15 & 16 & & & & & \\
17 & 18 & & & & & & &
\end{array}.$$

Choose the holes $t_{i,\ell}, t_{j,\ell} \in t$ to be the westnorthern-most pair of two holes in the same column ℓ ($t_{1,1} = 1$ and $t_{2,1} = 7$ in the example above). Let $\pi \in C_t$ be a permutation such that πt covers m . Then $\pi t_{i,\ell}$ and $\pi t_{j,\ell}$ are two holes of column ℓ of πt. Define the map $\phi : C_t \to C_t$ such that $\phi(\pi) := (\pi t_{i,\ell}, \pi t_{j,\ell})\pi$. Since $\phi(\pi)t$ covers m if and only if πt covers m, and sgn $\pi = -$sgn $\phi(\pi)$, we have that ϕ is a sign-reversing involution of C_t. Let $A \sqcup \overline{A} = C_t$ be a bipartition such that $a \in A \Leftrightarrow \phi(a) \notin A$. We deduce that

$$\begin{aligned}
f_t(m) &= \sum_{\sigma \in C_t} \text{sign}(\sigma)\ 1_{\{\sigma t\}}(m) \\
&= \sum_{\pi \in A} \text{sign}(\pi)\ 1_{\{\pi t\}}(m) + \text{sign}(\phi(\pi))\ 1_{\{\phi(\pi) t\}}(m) \\
&= \sum_{\pi \in A} \text{sign}(\pi)\ 1_{\{\pi t\}}(m) - \text{sign}(\pi)\ 1_{\{\pi t\}}(m) \\
&= 0.
\end{aligned}$$

One can show there are standard tableaux t and k-matchings m such that $f_t(m) = 0$ even though there are no two holes in the same column and σt

covers m for some $\sigma \in C_t$. Indeed, for arbitrary standard tableaux t and matchings m, it seems rather difficult to determine when $f_t(m) = 0$, or find more revealing expressions for $f_t(m)$ that are not signed sums.

In light of this, we now focus our attention on standard tableaux t of shape $\overline{\lambda} \vdash n$ for some $\lambda \vdash k$ as defined in Au's conjecture. The next fact follows from the definition of $\overline{\lambda}$ and the pigeonhole principle.

5.3.4 Proposition. *For any standard tableau t of shape $\overline{\lambda}$ and $m \in \mathcal{M}_{n,k}$, if no column has two holes, then there is precisely one hole in each column.*

A consequence of this fact is that the cells that lie to the right of the $(2\lambda_1)$-th column in the first row must be holes.

5.3.5 Lemma. *Let t be a standard tableau of shape $\overline{\lambda}$ for some $\lambda \vdash k$, and let $m \in \mathcal{M}_{n,k}$ be a k-matching that has two holes in the same column or has an edge $ij \in m$ such that i and j are in the same column. Then for any $y \in \mathbb{R}^{k+1}$, we have that $[Y_{n,k,k}(y)f_t](m) = 0 = f_t(m)$.*

Proof. First, assume there is an edge $ij \in m$ such that i and j are in the same column. If p is a k-matching that does not belong to the set

$$N[m] := \{m' \in \mathcal{M}_{n,k} : m \cup m' \text{ is a matching}\},$$

then we have $Y_{n,k,k}(y)[m,p] = 0$; otherwise, we have $Y_{n,k,k}(y)[m,p] = y_{|m\cup p|-k}$. If $ij \in p \in N[m]$, then clearly $f_t(p) = 0$. If $ij \notin p \in N[m]$, then by the definition of $Y_{n,k,k}(y)$, we have $i, j \notin V(p)$. But then if we superimpose p on t, we have that i and j are two holes in the same column, thus $f_t(p) = 0$. It follows that

$$([Y_{n,k,k}(y)]f_t)(m) = \sum_{p \in N[m]} y_{|m\cup p|-k} f_t(p) = 0.$$

Now let i, j be the corresponding vertices of the westnorthern-most pair of holes in the same column, and let $p \in N[m]$. Again, if $ij \in p$, then $f_t(p) = 0$, and if $i, j \notin V(p)$, then i, j are two holes in the same column, thus $f_t(p) = 0$.

If at least one of i, j is not a hole, then $(i, j)p \in N[m]$. Let $N' \subseteq N[m]$ be the set of all $p \in N[m]$ such that at least one of i, j is not a hole in the diagram of t and p. Define the map $\phi : N' \to N'$ such that $\phi(p) = (i, j)p$. By the definition of $Y_{n,k,k}(y)$, we have $|m\cap p| = |m\cap\phi(p)|$ and $|f_t(p)| = |f_t(\phi(p))|$.

We also have that $\operatorname{sign}(f_t(p)) = -\operatorname{sign}(f_t(\phi(p)))$, thus the map ϕ is a sign-reversing and weight-preserving involution of N'. Let $A \sqcup \overline{A} = N'$ be a bipartition such that $a \in A \Leftrightarrow \phi(a) \notin A$. We deduce that

$$\begin{aligned}
([Y_{n,k,k}(y)]f_t)(m) &= \sum_{p \in N[m] \setminus N'} y_{|m \cup p|-k}\ f_t(p) + \sum_{p \in N'} y_{|m \cup p|-k}\ f_t(p) \\
&= \sum_{p \in N'} y_{|m \cup p|-k}\ f_t(p) \\
&= \sum_{p \in A} y_{|m \cup p|-k}\ f_t(p) + y_{|m \cup \phi(p)|-k}\ f_t(\phi(p)) \\
&= \sum_{p \in A} y_{|m \cup p|-k}\ f_t(p) - y_{|m \cup p|-k}\ f_t(p) \\
&= 0,
\end{aligned}$$

which finishes the proof. $\square$

For any standard tableau t of shape $\overline{\lambda}$, let us now consider k-matchings m such that $f_t(m) \neq 0$, for example,

$$\begin{array}{llllllll}
1 & 2 & 3 & 4 & 5 & 6 & 19 \cdots & n \\
7 \text{—} 8 & & 9 \text{—} 10 & & 11 \text{—} 12 & & & \\
13 \text{—} 14 & & 15 \text{—} 16 & & & & & \\
17 \text{—} 18 & & & & & & &
\end{array}.$$

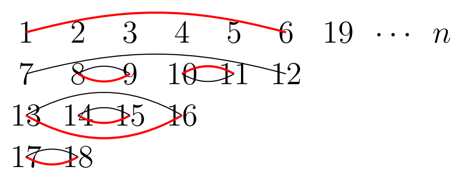
5.3.6 Proposition. *Let t be a standard tableau of shape $\overline{\lambda} \vdash n$ for any $\lambda \vdash k$. Then f_t is an eigenvector of $Y^{(1)}_{n,k,k}$ with eigenvalue k. Moreover, the f_t's form a basis for the $\overline{\lambda}$ eigenspace.*

Proof. Assume that no column has two holes. By Proposition 5.3.4, there is precisely one hole in each column. Let H be the set of holes that lie the first $2\lambda_1$ columns of $\overline{\lambda}$ and let

$$N[m] = \{m' \in \mathcal{M}_{n,k} : m \cup m' \text{ is a } (k+1)\text{-matching}\}.$$

By Proposition 5.3.4, any k-matching $p \in N[m]$ that contains an edge $ij \in p$ such that i or j lies in the first row past the $(2\lambda_i)$-th column of $\overline{\lambda}$, then $f_t(p) \neq 0$. It follows that if $f_t(p) \neq 0$, then we have $p \in N[m]$, the endpoints of $p \setminus m = \{ij\}$ belong to H, and there is an edge $i'j' \in m$ such that i, i'

belong to the same column of $\overline{\lambda}$ and j, j' belong to the same column of $\overline{\lambda}$ (see the figure below). Let $N' \subseteq N[m]$ be the k members of $N[m]$ that satisfy these properties.



.

The involution $(i,i')(j,j') \in C_t$ sends m to p, and since this involution is even, we have $f_t(m) = f_t(p)$. We now deduce that

$$\begin{aligned}([Y^{(1)}_{n,k,k}]f_t)(m) &= \sum_{p \in N[m]} f_t(p) \\ &= \sum_{p \in N'} f_t(p) \\ &= k f_t(m).\end{aligned}$$

By Lemma 5.3.5, if there are two holes in the same column, or an edge $ij \in m$ has both of its endpoints in the same column, then $([Y^{(1)}_{n,k,k}]f_t)(m) = 0 = k f_t(m)$. This implies that $Y^{(1)}_{n,k,k} f_t = k f_t$, which completes the proof. □

Extending this result to other matching orbitals $Y^{(i)}_{n,k,k}$ for $2 \le i \le k$ would allow us to resolve [3, Conjecture 82], as $Y_{n,k,k}$ is a linear combination of the matching orbitals; however, this is more involved, as the combinatorics of the k-matchings $p \in N[m]$ such that $f_t(p) \neq 0$ are more intricate and have varying values of $f_t(p)$.

Bibliography

[1] Noga Alon, Haim Kaplan, Michael Krivelevich, Dahlia Malkhi, and Julien Stern. Scalable secure storage when half the system is faulty. *Information and Computation*, 174(2):203 – 213, 2002.

[2] Regev Amitai. Identities for the number of standard Young tableaux in some (k, l)-hooks. *Séminaire Lotharingien de Combinatoire*, 63(3), 2010.

[3] Yu Hin Au. A comprehensive analysis of lift-and-project methods for combinatorial optimization, 2014. `https://uwspace.uwaterloo.ca/handle/10012/8662`.

[4] E. Bannai and T. Ito. *Algebraic Combinatorics I: Association Schemes*. Mathematics lecture note series. Benjamin/Cummings Pub. Co., 1984.

[5] F. L. Bauer. A remark on Stirling's formula and on approximations for the double factorial. *The Mathematical Intelligencer*, 29(2):10 – 14, 2007.

[6] Richard Bellman. Dynamic programming treatment of the Traveling Salesman problem. *J. ACM*, 9(1):61–63, January 1962.

[7] Andreas Björklund and Thore Husfeldt. The parity of directed Hamiltonian cycles. In *Proceedings of the IEEE 54th Annual Symposium on Foundations of Computer Science*, FOCS '13, pages 727–735, Washington, DC, USA, 2013. IEEE Computer Society.

[8] Andreas Björklund, Petteri Kaski, and Ioannis Koutis. Directed Hamiltonicity and out-branchings via generalized Laplacians. In *44th International Colloquium on Automata, Languages, and Programming, ICALP 2017, July 10-14, 2017, Warsaw, Poland*, pages 91:1–91:14, 2017.

[9] Hans L. Bodlaender, Marek Cygan, Stefan Kratsch, and Jesper Nederlof. Deterministic single exponential time algorithms for connectivity problems parameterized by treewidth. *Inf. Comput.*, 243:86–111, 2015.

[10] Gábor Braun, Jonah Brown-Cohen, Arefin Huq, Sebastian Pokutta, Prasad Raghavendra, Aurko Roy, Benjamin Weitz, and Daniel Zink. The matching problem has no small symmetric SDP. In *Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '16, pages 1067–1078, Philadelphia, PA, USA, 2016. Society for Industrial and Applied Mathematics.

[11] Peter J. Cameron and C.Y. Ku. Intersecting families of permutations. *European Journal of Combinatorics*, 24(7):881 – 890, 2003.

[12] P.J. Cameron, C.M. Series, and J.W. Bruce. *Permutation Groups*. London Mathematical Society. Cambridge University Press, 1999.

[13] T. Ceccherini-Silberstein, F. Scarabotti, and F. Tolli. *Harmonic Analysis on Finite Groups: Representation Theory, Gelfand Pairs and Markov Chains*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2008.

[14] Radu Curticapean, Nathan Lindzey, and Jesper Nederlof. A tight lower bound for counting Hamiltonian cycles via matrix rank. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '18, pages 1080–1099, Philadelphia, PA, USA, 2018. Society for Industrial and Applied Mathematics.

[15] Marek Cygan, Stefan Kratsch, and Jesper Nederlof. Fast Hamiltonicity checking via bases of perfect matchings. In *Symposium on Theory of Computing Conference, STOC'13, Palo Alto, CA, USA, June 1-4, 2013*, pages 301–310, 2013.

[16] Holger Dell. Fine-grained complexity classification of counting problems. Simons Institute, The Classification Program of Counting Complexity, 2016.

[17] P. Delsarte. *An Algebraic Approach to the Association Schemes of Coding Theory*. Philips research reports: Supplements. N.V. Philips' Gloeilampenfabrieken, 1973.

[18] Persi Diaconis. *Group Representations in Probability and Statistics*. Institute of Mathematical Statistics Lecture Notes—Monograph Series, 11. Institute of Mathematical Statistics, Hayward, CA, 1988.

[19] R. Diestel. *Graph Theory*. Electronic library of mathematics. Springer, 2006.

[20] D. Ellis, E. Friedgut, and H. Pilpel. Intersecting families of permutations. *J. Amer. Math. Soc.*, 24:649–682, 2011.

[21] David Ellis. Stability for t-intersecting families of permutations. *Journal of Combinatorial Theory, Series A*, 118(1):208 – 227, 2011.

[22] David Ellis. A proof of the Cameron-Ku conjecture. *J. London Math. Society*, 85(1):165–190, 2012.

[23] David Ellis, Yuval Filmus, and Ehud Friedgut. Low-degree Boolean functions on S_n, with an application to isoperimetry. *CoRR*, arXiv:1511.08694, 2015.

[24] David Ellis, Yuval Filmus, and Ehud Friedgut. A quasi-stability result for dictatorships in S_n. *Combinatorica*, 35(5):573–618, October 2015.

[25] David Ellis, Yuval Filmus, and Ehud Friedgut. A quasi-stability result for low-degree Boolean functions on S_n, 2015.

[26] David Ellis, Yuval Filmus, and Ehud Friedgut. A stability result for balanced dictatorships in S_n. *Random Struct. Algorithms*, 46(3):494–530, May 2015.

[27] Y. Filmus. Friedgut–Kalai–Naor theorem for slices of the Boolean cube. *ArXiv e-prints*, October 2014.

[28] Yuval Filmus. A comment on intersecting families of permutations. *CoRR*, arXiv:1706.10146, 2017.

[29] Michael Freedman, László Lovász, and Alexander Schrijver. Reflection positivity, rank connectivity, and homomorphism of graphs. *Journal of the American Mathematical Society*, 20(1):37–51, 1 2007.

[30] C. Godsil and K. Meagher. *Erdos-Ko-Rado Theorems: Algebraic Approaches*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2015.

[31] Chris Godsil and Karen Meagher. An algebraic proof of the Erdös-Ko-Rado theorem for intersecting families of perfect matchings. *ARS MATHEMATICA CONTEMPORANEA*, 12(2):205–217, 2016.

[32] Chris Godsil and Gordon Royle. *Algebraic Graph Theory*, volume 207 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.

[33] Chris D. Godsil and Karen Meagher. A new proof of the Erdös-Ko-Rado theorem for intersecting families of permutations. *Eur. J. Comb.*, 30(2):404–414, 2009.

[34] Ian Goulden. personal communication.

[35] Phil Hanlon and David Wales. On the decomposition of Brauer's centralizer algebras. *Journal of Algebra*, 121(2):409 – 445, 1989.

[36] Michael Held and Richard M. Karp. A dynamic programming approach to sequencing problems. In *Proceedings of the 1961 16th ACM National Meeting*, ACM '61, pages 71.201–71.204, New York, NY, USA, 1961. ACM.

[37] A.J.W. Hilton and E.C. Milner. *Some Intersection Theorems for Systems of Finite Sets*. Research paper. University of Calgary, Department of Mathematics, 1967.

[38] Roger A. Horn and Charles R. Johnson. *Matrix Analysis*. Cambridge University Press, New York, NY, USA, 1986.

[39] Henry Jack. A class of symmetric polynomials with a parameter. *Proc. Roy. Soc. Edinburgh Sect. A*, 69:1–18, 1970/1971.

[40] G.D. James and A. Kerber. *The Representation Theory of the Symmetric Group*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1984.

[41] Cheng Yeaw Ku and David B. Wales. Eigenvalues of the derangement graph. *Journal of Combinatorial Theory, Series A*, 117(3):289 – 312, 2010.

[42] Cheng Yeaw Ku and Kok Bin Wong. Solving the Ku-Wales conjecture on the eigenvalues of the derangement graph. *European Journal of Combinatorics*, 34(6):941 – 956, 2013.

[43] Cheng Yeaw Ku and Kok Bin Wong. Eigenvalues of the matching derangement graph. *Journal of Algebraic Combinatorics*, Dec 2017.

[44] James Lee. personal communication.

[45] Nathan Lindzey. Erdös-Ko-Rado for perfect matchings. *European Journal of Combinatorics*, 65:130 – 142, 2017.

[46] Nathan Lindzey. Stability for intersecting families of perfect matchings (submitted). *ArXiv e-prints*, August 2018.

[47] Daniel Lokshtanov, Dániel Marx, and Saket Saurabh. Known algorithms on graphs of bounded treewidth are probably optimal. In *Proceedings of the Twenty-second Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '11, pages 777–789, Philadelphia, PA, USA, 2011. Society for Industrial and Applied Mathematics.

[48] L. Lovász. On the Shannon capacity of a graph. *IEEE Trans. Inf. Theor.*, 25(1):1–7, January 1979.

[49] László Lovász. Connection matrices. `http://web.cs.elte.hu/~lovasz/welsh.pdf`.

[50] László Lovász. Matching structure and the matching lattice. *Journal of Combinatorial Theory, Series B*, 43(2):187 – 222, 1987.

[51] László Lovász. The rank of connection matrices and the dimension of graph algebras. *European Journal of Combinatorics*, 27(6):962 – 970, 2006.

[52] László Lovász and Michael E. Saks. Lattices, Möbius functions and communication complexity. In *29th Annual Symposium on Foundations of Computer Science, White Plains, New York, USA, 24-26 October 1988*, pages 81–90, 1988.

[53] I.G. Macdonald. *Symmetric Functions and Hall Polynomials.* Oxford mathematical monographs. Clarendon Press, 1995.

[54] Claire Mathieu and Alistair Sinclair. Sherali-Adams relaxations of the matching polytope. In *Proceedings of the Forty-first Annual ACM Symposium on Theory of Computing*, STOC '09, pages 293–302, New York, NY, USA, 2009. ACM.

[55] Colin McDiarmid. *On the method of bounded differences*, pages 148–188. London Mathematical Society Lecture Note Series. Cambridge University Press, 1989.

[56] Karen Meagher and Lucia Moura. Erdös-Ko-Rado theorems for uniform set-partition systems. *Electr. J. Comb.*, 12(1):Research Paper 40, 12 pp. (electronic), 2005.

[57] Mikhail Muzychuk. On association schemes of the symmetric group S_{2n} acting on partitions of type 2^n. In *Bayreuther Mathematische Schriften*, 1994.

[58] Ryan O'Donnell. *Analysis of Boolean Functions*. Cambridge University Press, 2014.

[59] M. Petkovsek, H.S. Wilf, and D. Zeilberger. *$A = B$*. A K Peters Series. Taylor & Francis, 1996.

[60] J. Quaintance. *Combinatorial Identities for Stirling Numbers: The Unpublished Notes of H W Gould*. World Scientific Publishing Company Pte Limited, 2015.

[61] Ran Raz and Boris Spieker. On the "log rank"-conjecture in communication complexity. *Combinatorica*, 15(4):567–588, 1995.

[62] Paul Renteln. On the spectrum of the derangement graph. *The Electronic Journal of Combinatorics*, 14(1):Research Paper 82, 17 pp. (electronic), 2007.

[63] Thomas Rothvoss. The matching polytope has exponential extension complexity. *J. ACM*, 64(6):41:1–41:19, September 2017.

[64] B. Sagan. *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*. Graduate Texts in Mathematics. Springer New York, 2001.

[65] Fabio Scarabotti and Filippo Tolli. Harmonic analysis on a finite homogeneous space. *Proceedings of the London Mathematical Society*, 100(2):348–376.

[66] A. Schrijver. *Combinatorial Optimization - Polyhedra and Efficiency*. Springer, 2003.

[67] Alexander Schrijver. Association schemes and the Shannon capacity: Eberlein-polynomials and the Erdös-Ko-Rado theorem. *Algebraic Methods in Graph Theory (L. Lovász and V.T. Sos, eds.)*, pages 671–688, January 1981.

[68] L.L. Scott and J.P. Serre. *Linear Representations of Finite Groups*. Graduate Texts in Mathematics. Springer New York, 1996.

[69] N. J. A. Sloane. The encyclopedia of integer sequences. `https://oeis.org/`.

[70] Richard P. Stanley. Some combinatorial properties of Jack symmetric functions. *Advances in Mathematics*, 77(1):76 – 115, 1989.

[71] R.P. Stanley. *Enumerative Combinatorics: Volume 2*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2001.

[72] W. A. Stein et al. *Sage Mathematics Software (Version 6.10)*. The Sage Development Team, 2016. `http://www.sagemath.org`.

[73] R. M. Thrall. On symmetrized Kronecker powers and the structure of the free Lie ring. *American Journal of Mathematics*, 64(1):pp. 371–388, 1942.

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