Performance analysis of maximally improper signaling for multiple-antenna systems

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***Abstract*—The transmission of improper Gaussian signals, instead of the conventional proper ones, has been shown to improve the performance in interference-limited networks. In this work we analyze the performance of a multiple-antenna user that transmits maximally improper signals and whose transmit covariance matrix satisfies a set of constraints that limit the harmfulness of the interference caused by this user. As opposed to the single-antenna case, there are different possible improper spatial signatures, which provide different performance. We first obtain new results for maximally improper random vectors based on majorization theory. We then apply these results to derive the improper spatial signatures that either maximize or minimize the performance. Numerical examples show that the performance difference between these two extreme cases can be surprisingly large.**

***Index Terms*—Improper signaling, transmitter optimization, MIMO, majorization theory, underlay cognitive radio.**

I. Introduction

In wireless communications it is typically assumed that the transmitted signals are distributed as proper Gaussian random vectors, which are uncorrelated with their complex conjugate [1]. This is because such random vectors maximize entropy and therefore achieve capacity in additive white Gaussian noise (AWGN) channels. When several users share the same wireless channel, it has been shown that transmitting *improper* Gaussian signals (which are correlated with their complex conjugate) significantly improves the performance because improper interference is less harmful than proper interference.

Improper Gaussian signaling was first shown to outperform its proper counterpart in [2] in a degrees-of-freedom (DoF) study of the interference channel. Other works have focused on the general performance improvement attained by improper signaling. Following these lines, the design of improper signaling schemes has been considered for different interference networks, such as the interference channel [3]–[7], the Z-interference channel [8]–[10], and cognitive radio (CR) networks [11], [12].

In our previous works [10], [11], we analyzed the performance of improper signaling for single-antenna users and derived the optimal transmission parameters for some specific interference-limited scenarios. When users are equipped with multiple antennas, the problem becomes much more difficult. This is not only due to the spatial structure of the transmit signals, but also because of different possible improper structures, i.e., the different ways of constructing an improper signal. The multiple-input multiple-output (MIMO) Z-interference channel is considered in [9], and some insights about the impact of different improper structures are drawn from numerical experiments. Additionally, the design proposed in [9] requires no cooperation between users, which is always desirable from a practical viewpoint.

In this paper we aim at deriving insights into the behavior of improper signaling in MIMO interference-limited networks, which will also be useful as design guidelines for underlay CR and other multiuser scenarios with limited cooperation. To this end, we consider a multiple-antenna user who is constrained to transmit maximally improper signals (which are *perfectly* correlated with their complex conjugate) and at the same time satisfies a set of constraints on its transmit covariance matrix. For instance, in an underlay CR scenario these constraints protect the primary user from the interference by the secondary user. For this setting, we derive the improper spatial signatures that lead to the best and worst performance. A surprising finding is that there can be substantial performance differences between different improper signaling schemes, even in the case where all of them are maximally improper. This work is connected to our previous work [13], where we considered a similar approach from the point of view of a user that is affected by maximally improper interference.

II. Preliminaries

A. Majorization theory

We start by introducing the basic definitions of majorization and Schur-convex/concave functions, which will be used to derive our results. We refer the interested reader to [14] for a broad treatment of the topic.

Definition 1 ([14]): Let $\mathbf{x} = [x_1, \ldots, x_N]^T$ be a real vector and $[x_{[1]}, \ldots, x_{[N]}]^T$ a rearrangement of $\mathbf{x}$ such that its elements are in decreasing order. Let $\mathbf{y}$ be a vector the same length N. We say that $\mathbf{x}$ is majorized by $\mathbf{y}$ and express it as $\mathbf{x} \prec \mathbf{y}$ if

$$\begin{cases} \sum_{n=1}^{k} x_{[n]} \leq \sum_{n=1}^{k} y_{[n]} \ , & k = 1, \ldots, N-1 \ , \\ \sum_{n=1}^{N} x_{[n]} = \sum_{n=1}^{N} y_{[n]} \ . \end{cases} \tag{1}$$

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Intuitively, if $\mathbf{x} \prec \mathbf{y}$, the components of $\mathbf{x}$ are less spread out or "more equal" than the components of $\mathbf{y}$. Majorization introduces a preordering on $\mathbb{R}^N$, and functions that preserve this preordering are called Schur-convex.

Definition 2 ([14]): We say that a real-valued function $\phi : \mathbb{R}^N \to \mathbb{R}$ is Schur-convex if $\mathbf{x} \prec \mathbf{y} \Rightarrow \phi(\mathbf{x}) \leq \phi(\mathbf{y})$. We say that ϕ is Schur-concave if $\mathbf{x} \prec \mathbf{y} \Rightarrow \phi(\mathbf{x}) \geq \phi(\mathbf{y})$.

B. Improper random vectors

In this section we present some definitions and properties of improper complex random vectors that will be used throughout the paper. We refer the reader to [1] for a comprehensive treatment of the subject.

The complementary covariance matrix of a zero-mean complex random vector $\mathbf{x}$ is defined as $\tilde{\mathbf{R}}_{xx} = \mathrm{E}\{\mathbf{x}\mathbf{x}^T\}$, where $\mathrm{E}\{\cdot\}$ denotes expectation. If $\tilde{\mathbf{R}}_{xx} = \mathbf{0}$, we call $\mathbf{x}$ proper, otherwise improper. Without loss of generality, the complementary covariance matrix can be expressed as [1, Section 3.2.3][1]

$$\tilde{\mathbf{R}}_{xx} = \mathbf{R}_{xx}^{1/2}\mathbf{F}_x\mathbf{C}_x\mathbf{F}_x^T\mathbf{R}_{xx}^{T/2} , \tag{2}$$

where $\mathbf{R}_{xx} = \mathrm{E}\{\mathbf{x}\mathbf{x}^H\}$ is the covariance matrix, $\mathbf{F}_x$ is a complex-valued unitary matrix, which we call the *improper spatial signature*, and $\mathbf{C}_x$ is a diagonal matrix containing the circularity coefficients, which measure the degree of impropriety and belong to the range $[0, 1]$. If $\mathbf{C}_x = \mathbf{I}$, we call $\mathbf{x}$ maximally improper. Finally, it is often useful to collect the second-order statistics of $\mathbf{x}$ in the augmented covariance matrix, which is defined as

$$\underline{\mathbf{R}}_{xx} = \mathrm{E}\{\underline{\mathbf{x}}\underline{\mathbf{x}}^H\} = \begin{bmatrix} \mathbf{R}_{xx} & \tilde{\mathbf{R}}_{xx} \\ \tilde{\mathbf{R}}_{xx}^* & \mathbf{R}_{xx}^* \end{bmatrix} , \tag{3}$$

where $\underline{\mathbf{x}} = [\mathbf{x}^T \ \mathbf{x}^H]^T$ is the augmented vector.

Remark: An improper Gaussian random vector $\mathbf{x}$ with covariance matrix $\mathbf{R}_{xx}$ and complementary covariance matrix $\tilde{\mathbf{R}}_{xx}$ can be generated by a *widely linear* transformation of a proper Gaussian random vector $\mathbf{s}$ with covariance matrix $\mathbf{I}$. Specifically, $\mathbf{x} = \mathbf{T}_1\mathbf{s} + \mathbf{T}_2\mathbf{s}^*$, where $\mathbf{T}_1$ and $\mathbf{T}_2$ are functions of $\mathbf{R}_{xx}$ and $\tilde{\mathbf{R}}_{xx}$. We refer the interested reader to [4], where the procedure of generating $\mathbf{x}$ from $\mathbf{s}$ is described in detail.

III. New results on maximally improper random vectors

This section presents new results for maximally improper random vectors. As we will show in Section IV, these results have applications in multi-user communication problems. We start with the following lemma, which characterizes the eigenvalues of the augmented covariance matrix of a maximally improper random vector.

Lemma 1: Let $\mathbf{x} \in \mathbb{C}^{N\times 1}$ be a maximally improper random vector with covariance matrix $\mathbf{R}_{xx}$ and complementary covariance matrix $\tilde{\mathbf{R}}_{xx} = \mathbf{R}_{xx}^{1/2}\mathbf{F}_x\mathbf{F}_x^T\mathbf{R}_{xx}^{T/2}$. The eigenvalues of $\underline{\mathbf{R}}_{xx}$ are then given by

$$\boldsymbol{\lambda}(\underline{\mathbf{R}}_{xx}) = \begin{bmatrix} \boldsymbol{\lambda}(\mathbf{R}_{xx} + \mathbf{F}_x\mathbf{F}_x^T\mathbf{R}_{xx}^*\mathbf{F}_x^*\mathbf{F}_x^H) \\ \mathbf{0} \end{bmatrix} . \tag{4}$$

Proof: Since $\underline{\mathbf{R}}_{xx}$ is positive semidefinite, it admits the decomposition

$$\underline{\mathbf{R}}_{xx} = \mathbf{X}^H\mathbf{X} = \begin{bmatrix} \mathbf{X}_1^H \\ \mathbf{X}_2^H \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 \end{bmatrix} , \tag{5}$$

with matrices $\mathbf{X}_1$ and $\mathbf{X}_2$ of dimension $N \times N$. This is because the rank of $\underline{\mathbf{R}}_{xx}$ is at most N, as $\mathbf{x}$ is maximally improper. It can be easily checked that $\mathbf{X}_1 = \mathbf{F}_x^H\mathbf{R}_{xx}^{1/2}$ and $\mathbf{X}_2 = \mathbf{F}_x^T\mathbf{R}_{xx}^{*/2}$. Notice that the non-zero eigenvalues of $\underline{\mathbf{R}}_{xx}$ are equal to the eigenvalues of

$$\mathbf{X}\mathbf{X}^H = \mathbf{X}_1\mathbf{X}_1^H + \mathbf{X}_2\mathbf{X}_2^H = \mathbf{F}_x^H\mathbf{R}_{xx}\mathbf{F}_x + \mathbf{F}_x^T\mathbf{R}_{xx}^*\mathbf{F}_x^* . \tag{6}$$

Finally, since $\mathbf{F}_x$ is unitary, the non-zero eigenvalues of $\underline{\mathbf{R}}_{xx}$ can be obtained as those of $\mathbf{F}_x\mathbf{X}\mathbf{X}^H\mathbf{F}_x^H$, where $\mathbf{X}\mathbf{X}^H$ is given by (6), which yields (4). ■

Corollary 1: For a maximally improper random vector $\mathbf{x} \in \mathbb{C}^{N\times 1}$ with covariance matrix $\mathbf{R}_{xx}$, there exists the majorization preordering

$$\begin{bmatrix} \boldsymbol{\lambda}^{\downarrow}(\mathbf{R}_{xx}) + \boldsymbol{\lambda}^{\uparrow}(\mathbf{R}_{xx}) \\ \mathbf{0} \end{bmatrix} \prec \boldsymbol{\lambda}(\underline{\mathbf{R}}_{xx}) \prec \begin{bmatrix} 2\boldsymbol{\lambda}(\mathbf{R}_{xx}) \\ \mathbf{0} \end{bmatrix} , \tag{7}$$

where $(\cdot)^{\downarrow}$ and $(\cdot)^{\uparrow}$ indicate a rearrangement of the elements in decreasing and increasing order, respectively. Furthermore, the improper signature matrices leading to the right- and left-hand sides of (7) are $\mathbf{F}_x = \mathbf{U}$ and $\mathbf{F}_x = \mathbf{U}\mathbf{J}^{1/2}$, respectively, where $\mathbf{U}$ contains the eigenvectors of $\mathbf{R}_{xx}$ whose corresponding eigenvalues are arranged in decreasing order, and $\mathbf{J}$ is the exchange matrix.[2]

Proof: Consider the following majorization preordering [15]

$$\boldsymbol{\lambda}^{\downarrow}(\mathbf{A}) + \boldsymbol{\lambda}^{\uparrow}(\mathbf{B}) \prec \boldsymbol{\lambda}(\mathbf{A} + \mathbf{B}) \prec \boldsymbol{\lambda}^{\downarrow}(\mathbf{A}) + \boldsymbol{\lambda}^{\downarrow}(\mathbf{B}) . \tag{8}$$

Combining (4) and (8), along with the fact that $\boldsymbol{\lambda}(\mathbf{F}_x\mathbf{F}_x^T\mathbf{R}_{xx}^*\mathbf{F}_x^*\mathbf{F}_x^H) = \boldsymbol{\lambda}(\mathbf{R}_{xx})$ (since $\mathbf{F}_x$ is unitary), yields (7). Because of (4), it is then clear that $\mathbf{F}_x = \mathbf{U}$ yields the right-hand side of (7), whereas the left-hand side is obtained by setting $\mathbf{F}_x = \mathbf{U}\mathbf{J}^{1/2}$. ■

While the right-hand side of (7) was already shown in [16], the left-hand side is a new result. As Schur-concave functions preserve the majorization preordering, Corollary 1 implies that any Schur-concave function of the eigenvalues of $\underline{\mathbf{R}}_{xx}$ is maximized (minimized) if the eigenvalues are least (most) spread-out, which is controlled by the improper signature matrix $\mathbf{F}_x$. We express this more formally in the following lemma.

Lemma 2: Let $\phi : \mathbb{R}^{2N} \to \mathbb{R}$ be an arbitrary Schur-concave function. Then we have

$$\phi(\boldsymbol{\lambda}(\underline{\mathbf{R}}_{xx}^{\mathrm{lb}})) \leq \phi(\boldsymbol{\lambda}(\underline{\mathbf{R}}_{xx})) \leq \phi(\boldsymbol{\lambda}(\underline{\mathbf{R}}_{xx}^{\mathrm{ub}})) , \tag{9}$$

where $\underline{\mathbf{R}}_{xx}^{\mathrm{lb}}$ is the augmented covariance matrix for $\mathbf{F}_x = \mathbf{F}_x^{\mathrm{lb}} = \mathbf{U}$, and $\underline{\mathbf{R}}_{xx}^{\mathrm{ub}}$ is the augmented covariance matrix for $\mathbf{F}_x = \mathbf{F}_x^{\mathrm{ub}} = \mathbf{U}\mathbf{J}^{1/2}$; with $\mathbf{U}$ being the matrix of (ordered) eigenvectors of $\mathbf{R}_{xx}$.

Proof: This lemma is a direct consequence of Corollary 1 and the Schur-concavity of ϕ. ■

[1]We use in this paper the unique positive-semidefinite square root for all matrix square roots.

[2]The exchange matrix is a square matrix with ones on the counterdiagonal and zeros elsewhere.

IV. Performance bounds of maximally improper signaling

In this section, we will use Lemma 2 to derive bounds on the performance of transmitter-receiver pairs that employ a maximally improper signaling scheme. We will proceed along the following lines.

We measure performance with a utility function in terms of the eigenvalues of the augmented mean-square error (MSE) matrix $\underline{\mathbf{E}}$ (to be defined in (10)). Utility functions are typically Schur-convex/concave, so that the worst and best performance is obtained by the cases where the eigenvalues of $\underline{\mathbf{E}}$ are either the most spread-out or least spread-out, which is measured by majorization. This idea was used in [17] to compare the performance of linear and widely linear processing.

The eigenvalue spread of $\underline{\mathbf{E}}$ is controlled by the improper signature matrix $\mathbf{F}$ of the transmit signal. We derive the $\mathbf{F}$ matrices that lead to these extreme cases. We also look at the performance obtained in these extreme cases, which can differ significantly.

A. Problem formulation

Let us consider a MIMO transmitter-receiver pair with M transmit and N receive antennas. We denote the channel matrix by $\mathbf{H} \in \mathbb{C}^{N\times M}$ and the transmit covariance matrix by $\mathbf{Q} \in \mathbb{S}_+^M$, where $\mathbb{S}_+^M$ denotes the set of $M \times M$ positive semidefinite Hermitian matrices. We assume that the transmit signal is improper with complementary covariance matrix $\tilde{\mathbf{Q}}$. For the sake of exposition and tractability, we assume that the receiver is corrupted by a proper interference-plus-noise signal with covariance matrix $\mathbf{K} \in \mathbb{S}_+^N$. For convenience, we define the equivalent channel as $\mathbf{G} = \mathbf{K}^{-1/2}\mathbf{H}$. Treating the interference as noise and assuming optimal widely linear decoding, the MSE matrix of the augmented model can be derived as (see [1, Section 5.4] and [9])

$$\underline{\mathbf{E}} = \left(\mathbf{I} + \underline{\mathbf{Q}}^{H/2}\underline{\mathbf{G}}^H\underline{\mathbf{G}}\underline{\mathbf{Q}}^{1/2}\right)^{-1} , \qquad (10)$$

where $\underline{\mathbf{Q}} \in \mathbb{S}_+^{2M}$ is the augmented covariance matrix of the transmit signal, and

$$\underline{\mathbf{G}} = \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}^* \end{bmatrix} . \qquad (11)$$

A problem of interest is the maximization of a function $f(\boldsymbol{\lambda}(\underline{\mathbf{E}}^{-1}))$ subject to a set of constraints on $\mathbf{Q}$ and $\tilde{\mathbf{Q}}$. Specifically,

$$\mathcal{P}: \quad \max_{\substack{\mathbf{Q}\in\mathbb{Q} \\ \tilde{\mathbf{Q}}\in\tilde{\mathbb{Q}}}} f(\boldsymbol{\lambda}(\underline{\mathbf{E}}^{-1})) ,$$

where $\mathbb{Q}$ and $\tilde{\mathbb{Q}}$ are the constraint sets of $\mathbf{Q}$ and $\tilde{\mathbf{Q}}$, respectively. The function $f(\mathbf{x})$ can be any Schur-concave utility function. For example, letting $f(\mathbf{x}) = -\sum_i \frac{1}{x_i}$, $\mathcal{P}$ turns into the minimization of the MSE, whereas setting $f(\mathbf{x}) = \sum_i \frac{1}{2}\log(x_i)$ yields the maximum achievable rate [9] (note that the sum of concave functions is Schur-concave [14, 3.C.1]).

Problem $\mathcal{P}$ may arise in the optimization of improper transceivers in interference-limited networks. As an illustrative example, we consider an underlay CR network comprised of a primary user (PU) and a secondary user (SU).[3] In this case, interference constraints must be imposed on the secondary transmission in order to protect the PU and ensure that it achieves a given prescribed performance. Thus, the set $\mathbb{Q}$ may include, e.g., a transmit power constraint, $\mathrm{Tr}(\mathbf{Q}) \leq p$ for some $p > 0$, an interference power constraint (also denoted interference temperature in this context), $\mathrm{Tr}(\mathbf{H}_{12}\mathbf{Q}\mathbf{H}_{12}^H) \leq t$ for some $t \geq 0$, where $\mathbf{H}_{12}$ is the SU-PU cross-channel matrix, and/or an interference shaping constraint, $\mathbf{H}_{12}\mathbf{Q}\mathbf{H}_{12}^H \preceq \mathbf{S}$ for some $\mathbf{S} \succeq \mathbf{0}$ [18]. Additionally, imposing constraints on the complementary covariance matrix by means of the set $\tilde{\mathbb{Q}}$ provides a more focused protection to the PU, which may help increase the performance of the SU, as we have shown in [11], [13].

In the single-antenna case there is a single circularity coefficient, and the performance of the SU is determined by the circularity coefficient alone, i.e., by the degree of impropriety. On the other hand, the multiple-antenna case is much more complicated. First, there are as many circularity coefficients as antennas. Second, the performance is also dependent on the improper signature matrix. Hence, even if the transmit signal is constrained to be maximally improper, the different maximally improper signaling schemes arising from different choices of $\mathbf{F}$ may provide different performance. In order to illuminate the effect of the improper signature matrix $\mathbf{F}$ on the performance, we analyze the range of solutions of problem $\mathcal{P}$ that can be achieved by varying $\mathbf{F}$ for the maximally improper case, i.e.,

$$\tilde{\mathbb{Q}} = \{\tilde{\mathbf{Q}} : \tilde{\mathbf{Q}} = \mathbf{Q}^{1/2}\mathbf{F}\mathbf{F}^T\mathbf{Q}^{T/2}\} . \qquad (12)$$

Specifically, we will use the results derived in Section III to obtain the choices of $\mathbf{F}$ that lead to the maximum and minimum value of the cost function of problem $\mathcal{P}$. These results will provide insights into how and how much the performance is affected by the choice of $\mathbf{F}$.

As we will show through numerical examples, the improper signature matrix tends to have the opposite effect on the user that is interfered with: when $\mathbf{F}$ is chosen to maximize the performance of the interfering user, the performance of the user that is interfered with is bound to decrease, and vice versa. This matrix then implies a trade-off between the performance of the interfering user and that of the users that are affected by this interference.

B. Performance bounds

In the following we apply Lemma 2 to determine the performance range of the problem under consideration. To this end, we first express the non-zero eigenvalues of $\underline{\mathbf{E}}^{-1}$ as

$$\boldsymbol{\lambda}(\underline{\mathbf{E}}^{-1}) = \mathbf{1} + \boldsymbol{\lambda}(\underline{\mathbf{G}}\underline{\mathbf{Q}}\underline{\mathbf{G}}^H) , \qquad (13)$$

[3] Note that the underlay CR model can be generalized to any interference network where a trade-off between cooperation overhead and performance is desired. Thus, by imposing interference constraints, the interference can be handled with minimum cooperation between users.

where we have used the fact that the non-zero eigenvalues of a matrix product $\mathbf{AB}$ are equal to those of $\mathbf{BA}$. Notice that $\underline{\mathbf{G}}\underline{\mathbf{Q}}\underline{\mathbf{G}}^H$ is actually the augmented covariance matrix of the equivalent received signal. Lemma 2 is then applicable by setting $\underline{\mathbf{R}}_{xx} = \underline{\mathbf{G}}\underline{\mathbf{Q}}\underline{\mathbf{G}}^H$ and $\phi(\mathbf{x}) = f(\mathbf{1} + \mathbf{x})$. Notice, however, that $\mathbf{F}_x \neq \mathbf{F}$, i.e., the improper signature matrix of the transmit signal is not equal to the improper signature matrix of the equivalent received signal. This is because the complementary covariance matrix of the equivalent received signal is $\tilde{\mathbf{R}}_{xx} = (\mathbf{GQG}^H)^{1/2}\mathbf{F}_x\mathbf{F}_x^T(\mathbf{GQG}^H)^{T/2}$. Since $\tilde{\mathbf{Q}} = \mathbf{Q}^{1/2}\mathbf{F}\mathbf{F}^T\mathbf{Q}^{T/2}$, we can also write $\tilde{\mathbf{R}}_{xx} = \mathbf{G}\mathbf{Q}^{1/2}\mathbf{F}\mathbf{F}^T\mathbf{Q}^{T/2}\mathbf{G}^T$. As $(\mathbf{GQG}^H)^{1/2} \neq \mathbf{G}\mathbf{Q}^{1/2}$, $\mathbf{F}_x \neq \mathbf{F}$ follows. Nevertheless, by simple matrix algebra we obtain $\mathbf{F} = \mathbf{V}\mathbf{U}^H\mathbf{F}_x$, where $\mathbf{U}$ and $\mathbf{V}$ are the left and right singular vectors, respectively, of $\mathbf{G}\mathbf{Q}^{1/2}$. With these considerations, we can apply Lemma 2 to obtain

$$\max_{\mathbf{Q}\in\mathbb{Q}} f(\boldsymbol{\lambda}(\underline{\mathbf{E}}_{\text{lb}}^{-1})) \leq \max_{\mathbf{Q}\in\mathbb{Q}} f(\boldsymbol{\lambda}(\underline{\mathbf{E}}^{-1})) \leq \max_{\mathbf{Q}\in\mathbb{Q}} f(\boldsymbol{\lambda}(\underline{\mathbf{E}}_{\text{ub}}^{-1})) \ , \tag{14}$$

where $\underline{\mathbf{E}}_{\text{lb}}^{-1}$ and $\underline{\mathbf{E}}_{\text{ub}}^{-1}$ are the MSE matrix for $\mathbf{F} = \mathbf{F}^{\text{lb}} = \mathbf{V}$ and $\mathbf{F} = \mathbf{F}^{\text{ub}} = \mathbf{V}\mathbf{J}^{1/2}$, respectively.

The foregoing result determines the achievable performance by maximally improper signaling in terms of the improper signature matrix $\mathbf{F}$, and it provides insights into how the performance is affected by this matrix. This is especially useful for the design of interference constraints for underlay CR or for reducing cooperation in other, more general, interference networks. Nevertheless, the specific choice of $\mathbf{F}$ also has an impact on the performance of the users that are affected by interference. As we showed in [13], the worst-case performance of a user affected by maximally improper interference with fixed power is best if its improper signature matrix is equal to the eigenvectors of its own signal covariance matrix. This corresponds to the same improper structure that achieves the lower bound in Lemma 2 but in a different signal space. As we will show in our upcoming journal paper [19], the worst-case performance (from the point of view of the user affected by interference) is attained when the interference has an improper signature matrix that exhibits the same structure as $\mathbf{F}^{\text{ub}}$. This inverse relation indicates a fundamental trade-off of improper signaling in MIMO systems. An improper signaling structure that is beneficial for the interfering user is generally detrimental for the user that is affected by this interference, and vice versa. Nevertheless, due to the different channel matrices, a particular improper structure at one receiver will not correspond, in general, to the same improper structure at other receivers.

Remark: At this point, we would like to comment on [9], where the impact of different improper signature matrices is also analyzed for the Z-interference channel. The authors propose to design the covariance and complementary covariance matrices as $\mathbf{WW}^H$ and $\mathbf{WZZ}^T\mathbf{W}^T$, respectively, for some optimized precoding matrix $\mathbf{W}$. They choose two different complementary covariance matrices as $\mathbf{ZZ}^T = \mathbf{I}$ and $\mathbf{ZZ}^T = \mathbf{J}$, which at first seems to resemble the improper structures that attain the upper and lower bounds stated in Lemma 2. Although it is stated in [9] that those choices provide opposite behaviors, this is actually not necessarily the case as they do not achieve, in general, the upper and lower bounds in Lemma 2.

V. Numerical examples

A. Simulation setup

In this section we illustrate the results presented in this paper with some numerical examples. To this end, we consider a two-user MIMO interference channel. We assume that the first user transmits proper Gaussian signals, whereas the second user is constrained to transmit maximally improper Gaussian signals. In this setting, we consider the problem of maximizing the rate of the second user subject to an interference power constraint on the first receiver. That is, we consider the particular instance of $\mathcal{P}$

$$\tilde{\mathcal{P}}: \quad \max_{\mathbf{Q}\in\mathbb{Q}} \frac{1}{2}\log_2 \det\left(\underline{\mathbf{E}}^{-1}\right) \ ,$$

with $\mathbb{Q} = \{\mathbf{Q} \succeq \mathbf{0} : \text{Tr}(\mathbf{Q}) \leq p_2, \text{Tr}(\mathbf{H}_{12}\mathbf{Q}\mathbf{H}_{12}^H) \leq t\}$, where $\mathbf{H}_{12}$ is the channel from the second transmitter to the first receiver. This scenario could be, e.g., an underlay CR network, where the second user would take the role of the SU. In such a case, the maximally improper constraint and the interference power constraint are placed to protect the first (primary) user from the SU, without requiring any cooperation between them. We will denote by Improper-LB the maximally improper scheme that chooses $\mathbf{F} = \mathbf{F}^{\text{lb}}$ and thus provides the second user with the lowest rate, and by Improper-UB the maximally improper scheme that provides the second user with the maximum rate ($\mathbf{F} = \mathbf{F}^{\text{ub}}$). Additionally, and unless stated otherwise, we consider each channel entry to be independently drawn from a proper complex distribution with zero mean and unit variance, and average the results over 1000 Monte-Carlo simulations. In order to numerically evaluate the derived bounds, which requires solving the optimization problems in the most left- and right-hand sides of (14), we proceed as described in the following section.

B. Computation of the bounds

Let us first consider the lower bound (most left-hand side of (14)), that is, we want to obtain the matrix $\mathbf{Q}$ that maximizes the minimum achievable rate, i.e., for $\mathbf{F} = \mathbf{F}^{\text{lb}}$. By Corollary 1, the optimization problem can be rewritten as

$$\mathcal{P}^{\text{lb}}: \quad \max_{\mathbf{Q}\in\mathbb{Q}} f(\mathbf{1} + 2\boldsymbol{\lambda}(\mathbf{GQG}^H)) \ .$$

Since the objective function is concave and the feasible set is convex, the above problem is a convex optimization problem and can then be solved using standard numerical methods [20].

Now we take $\mathbf{F} = \mathbf{F}^{\text{ub}}$ to compute the upper bound (most right-hand side of (14)). To this end we rewrite the optimization problem as (see Corollary 1)

$$\mathcal{P}^{\text{ub}}: \quad \max_{\mathbf{Q}\in\mathbb{Q}} f(\mathbf{1} + \boldsymbol{\lambda}^{\downarrow}(\mathbf{GQG}^H) + \boldsymbol{\lambda}^{\uparrow}(\mathbf{GQG}^H)) \ .$$

However, this problem is not convex in general due to the rearrangement of the eigenvalues. We propose an alternating

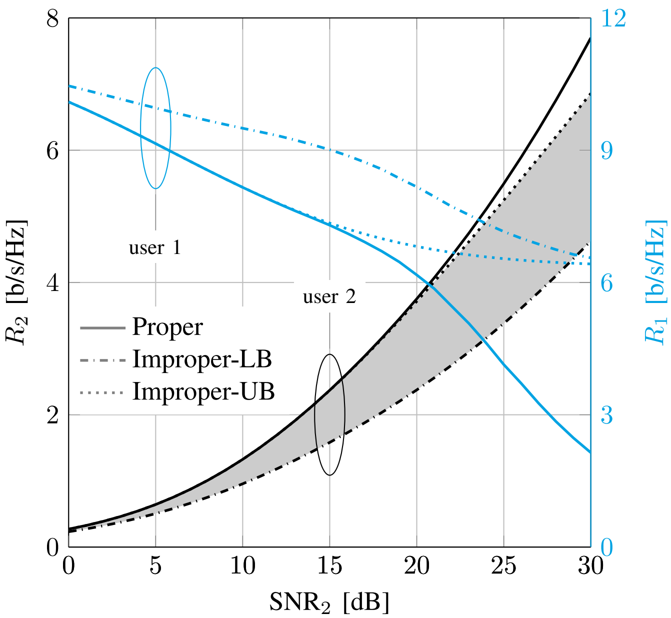

Fig. 1. Achievable rate of the second (black, left-hand axis) and first (blue, right-hand axis) user, both equipped with two antennas. In this example, the interference power constraint is not enforced.

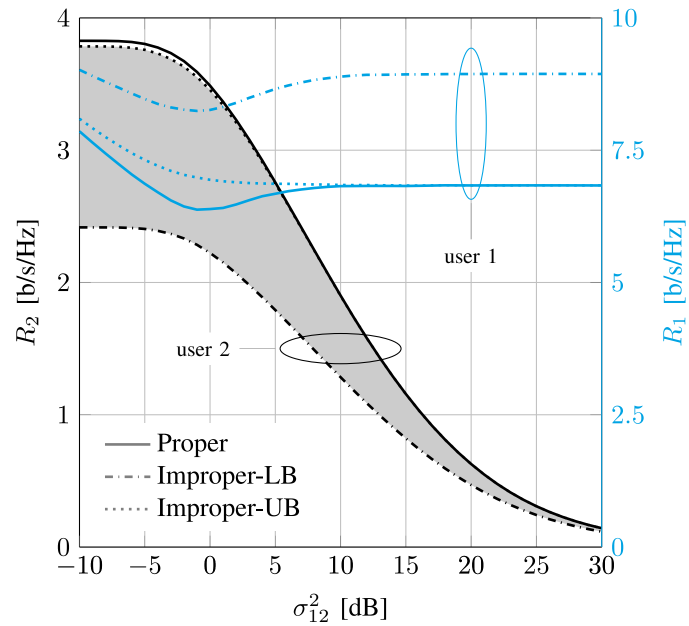

Fig. 2. Achievable rate of the second (black, left-hand axis) and first (blue, right-hand axis) user, both equipped with two antennas. In this example, the second user is constrained by a power budget and an interference power constraint.

optimization algorithm to find a stationary point as described next.

1) Set $i = 0$ and $\mathbf{F}_{(i)} = \mathbf{I}$.
2) Solve

$$\mathbf{Q}_{(i)} = \arg\max_{\mathbf{Q}\in\mathbb{Q}} f(\mathbf{1} + \boldsymbol{\lambda}(\mathbf{G}\mathbf{Q}\mathbf{G}^H + \mathbf{F}_{(i)}\mathbf{F}_{(i)}^T\mathbf{G}^*\mathbf{Q}^*\mathbf{G}^T\mathbf{F}_{(i)}^*\mathbf{F}_{(i)}^H)) \,. \tag{15}$$

3) Take $i = i{+}1$ and $\mathbf{F}_{(i)} = \mathbf{U}_{(i)}\mathbf{J}^{1/2}$, where $\mathbf{U}_{(i)}$ contains the eigenvectors of $\mathbf{G}\mathbf{Q}_{(i)}\mathbf{G}^H$.
4) If convergence criterion is met, go to the next step. Otherwise, go back to step 2.
5) Return the optimal solution as $\mathbf{Q} = \mathbf{Q}_{(i)}$ and $\mathbf{F}^{\mathrm{ub}} = \mathbf{V}_{(i)}\mathbf{J}^{1/2}$, where $\mathbf{V}_{(i)}$ contains the right singular vectors of $\mathbf{G}\mathbf{Q}_{(i)}^{1/2}$.

The above procedure is a standard alternating optimization algorithm, whose convergence properties have been deeply studied (see, e.g., [21]). In particular, since we are solving each step optimally, the cost function does not decrease at each step. Additionally, the cost function is bounded above, hence convergence to a stationary point is guaranteed. Note also that, even though the above procedure may not converge to the global optimum, it still enables us to numerically evaluate the derived upper bound, and it thus permits insights into the performance range of the maximally improper schemes, as well as into their impact on the user that is interfered with.

C. Simulation results

We first consider $t = \infty$, i.e., there is no interference power constraint. We define the transmit signal-to-noise ratio (SNR) of the ith user as $\mathrm{SNR}_i = p_i/\sigma^2$, where p_i is its transmit power and σ^2 the noise variance, which we take equal to one without loss of generality. Each transmitter and each receiver is equipped with two antennas, and we set $\mathrm{SNR}_1 = 20\,\mathrm{dB}$. Figure 1 depicts the achievable rate of both users, R_1 and R_2, as a function of the transmit SNR of the second user for proper signaling and the two extreme maximally improper signaling cases. These two maximally improper signaling schemes provide the maximum and minimum rates achievable by maximally improper signaling in terms of the improper signature matrix.[4] Therefore, any other improper signature matrix will provide an achievable rate within the shaded area in Fig. 1. As can be observed, the two extreme improper signaling schemes lead to the opposite behavior in the user that is disturbed by interference (user 1). This clearly reflects the trade-off of the improper signature matrix: by selecting an improper signaling scheme that improves the rate of the interfering user, the rate of the user that is affected by interference is likely to be reduced.

For the next example we consider an interference power constraint $t = 100$, and we set $\mathrm{SNR}_1 = \mathrm{SNR}_2 = 20\,\mathrm{dB}$. We evaluate the achievable rate for different gains of the cross-channel $\mathbf{H}_{12}$, which we measure by the variance σ_{12}^2 of its entries. The results are depicted in Fig. 2 for the case of two transmit and receive antennas, and in Fig. 3 for the four-antenna case. This example is particularly interesting since we can observe the behavior of the achievable rates as the

[4] As we are computing the upper bound suboptimally, the maximum rate achieved by maximally improper signaling could be higher than the one depicted in the figures, but in any case below the curve corresponding to proper signaling.

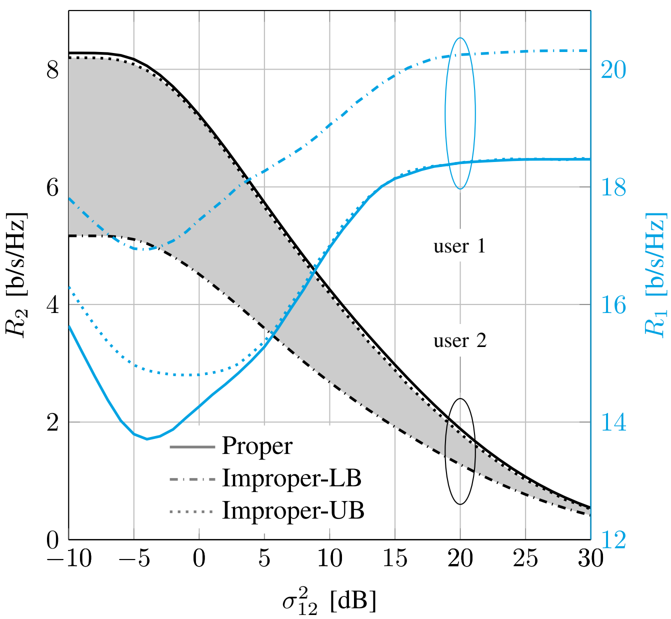

Fig. 3. Achievable rate of the second (black, left-hand axis) and first (blue, right-hand axis) user, both equipped with four antennas. In this example, the second user is constrained by a power budget and an interference power constraint.

interference becomes more dominant. The rate of user 1 does not significantly vary with σ^2_{12} since the interference power is constrained to be below t. This makes the rate of user 2 go down as σ^2_{12} increases, as its transmit power must be reduced to keep the interference power below the threshold. Thus, the range of rates achievable by improper signaling is significantly reduced when σ^2_{12} increases, while the different improper signaling schemes yield a substantial gap between the achievable rates of user 1 but, as expected, in the opposite direction. Hence, in the large σ^2_{12} regime, user 1 can obtain a notable rate increase by slightly reducing the rate of user 2. Comparing Fig. 2 and Fig. 3 we observe a similar behavior for higher number of antennas.

VI. Conclusions

In this paper we have analyzed the impact of the improper spatial signature on the rate achievable by a multiple-antenna user. We have derived the improper signature matrices that maximize and minimize the performance when the transmitted signals are maximally improper. We have shown with some numerical examples that the performance difference between these two extreme cases can be quite large. Moreover, an improper signaling structure that is beneficial for the interfering user is generally detrimental for the user affected by this interference. This reflects a trade-off introduced by the improper signature matrix between the performance of the interfering user and that of the user that is interfered with.

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