# Algebro-geometric analysis of bisectors of two algebraic plane curves *

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**Abstract**

In this paper, a general theoretical study, from the perspective of the algebraic geometry, of the untrimmed bisector of two real algebraic plane curves is presented. The curves are considered in $\mathbb{C}^2$, and the real bisector is obtained by restriction to $\mathbb{R}^2$. If the implicit equations of the curves are given, the equation of the bisector is obtained by projection from a variety contained in $\mathbb{C}^7$, called the incidence variety, into $\mathbb{C}^2$. It is proved that all the components of the bisector have dimension 1. A similar method is used when the curves are given by parametrizations, but in this case, the incidence variety is in $\mathbb{C}^5$. In addition, a parametric representation of the bisector is introduced, as well as a method for its computation.

## 1 Introduction

Given two geometric objects, their bisector is often defined as the geometric locus of the points which are equidistant from both objects. Examples of bisectors in the Euclidean plane are the perpendicular bisector of two points (or a segment), the angle bisector (the equidistant half-line from the sides of the angle), and the parabola, which is the equidistant curve between a straight line and a point external to the line. Subjects of particular interest are the study of the bisector of two curves, in the plane or in 3-dimensional space, and the bisector of two surfaces. The bisector of two curves is sometimes called the equidistant curve. The untrimmed bisector is the locus of the centers of all the circles which are tangent to both curves. The untrimmed bisector

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contains the bisector as defined above, and a trimming method is a procedure to eliminate from it the parts that are not contained in the bisector.

Bisectors have been studied in the context of Computational Geometry because they play an important role in the construction of Voronoi diagrams (see [4], [3], [6]). Various papers on bisectors of algebraic curves have been written in the context of CAGD, starting with the articles [8] and [9] where the notion of untrimmed bisector is considered, for pairs of regular polynomial or rational curves, and a trimming procedure is presented. In [12] a system of equations for the untrimmed bisector is proposed, together with the elimination of certain extraneous components. The authors of [7] consider $C^1$-continuous plane rational curves, and present a method of elimination to obtain a representation of the bisector in terms of the parameters of the initial curves. Some geometric and algebraic properties of the bisector of two curves, a curve and a surface, and two surfaces, are studied in [15]. In the thesis of Adamou [1], a method for the parametrization of bisectors of rational curves is presented. Several approximate or interpolation methods for the computation of bisectors have been proposed (see, for example [10], [14] or [13]).

In this paper, a general theoretical treatment, from the perspective of the algebraic geometry, of the untrimmed bisector of two real algebraic plane curves is presented. Similar analyses to other geometric objects, as offsets or conchoids, can be found in [2], [18] and [19]. The curves are considered in $\mathbb{C}^2$, and the equation of the bisector is obtained by projection from a variety $\mathcal{A}$ contained in $\mathbb{C}^7$, called the incidence variety, into $\mathbb{C}^2$. Each element of $\mathcal{A}$ is composed by one point (in complex coordinates) from each curve, one point in the bisector and an auxiliary variable. They must obey suitable equations. It is proved that all the bisector components have dimesion 1. If the coordinates are restricted to $\mathbb{R}^2$, the real bisector is obtained. A similar method is applied to the case where both curves, or one of them, are given parametrically. In this case, the incidence varieties are contained in $\mathbb{C}^5$ or $\mathbb{C}^6$, respectively. The bisector of two curves, although being a curve, turns to be a more complicated object. For instance, there is an explosion of the degree and the genus (see e.g. Examples 2.7, 2.8). From the point of view of applications, this is a serious obstacle since the implicit equation can be huge, and hence hard to manage. On the hand, as pointed above, the genus is not invariant under the bisector operation, and thus the component of the bisector usually have positive genus. Therefore, in general, there do not exist rational parametrizations. In [8], an alternative representation for irreducible bisectors, based on the parameters space, is introduced. They assume the curves C1 and C2 to be rational, regular and C1-continuous. In this paper, we formally extend this representation to the general case.

The structure of the manuscript is the following. In section 2, the definition of untrimmed bisector using an incidence variety is presented, for the case of implicit curves, and some related theorems are proved. The characterization of the bisector as the intersection of offset curves at variable distance is analyzed in this context. In section 3, the incidence variety is introduced for the case of two parametric curves, assuming the parametrizations are normal, which is not much restrictive. The combination of one implicit and one parametric curves is also considered. A method to get a parametric representation of the bisector is presented in section 4. Several examples are presented in the three sections. In the last section some conclusions are stated, and directions in which to extend this research in the near future are presented.

## 2 Untrimmed Bisectors: Implicit Case

We start this section analyzing the notion of bisector of two algebraic curves. Intuitively speaking, the (trimmed) bisector of two curves is the geometric locus of those points being at the same (Hausdorff) distance from the two curves. We recall that the distance from a point $P$ to a non-empty subset $A$ of a metric space, under a distance d, is

$$\mathrm{d}(P, A) = \inf\{\mathrm{d}(P, Q) \,|\, Q \in A\}.$$

In our case, $A$ will be a real algebraic affine plane curve, and hence a set with infinitely many points. In order to deal algebraically with this concept, we would like to somehow skip the infimum in our definition. This leads to the notion of (untrimmed) bisector that corresponds with the geometric locus of those points that, being on the normal lines to both curves, are at the same distance from the two footpoints in the intersection of each curve with the corresponding normal line. In other words, the points in the untrimmed bisector are the centers of the circles which are tangent to both curves. Note that the untrimmed bisector is a superset of the trimmed bisector. In Example 2.4 the untrimmed bisector of two concentric circles of radii 1 and 2 has two components: the circle with the same center and radius 3/2, which is the trimmed bisector, and the circle of radius 1/2 (see Figure 1).

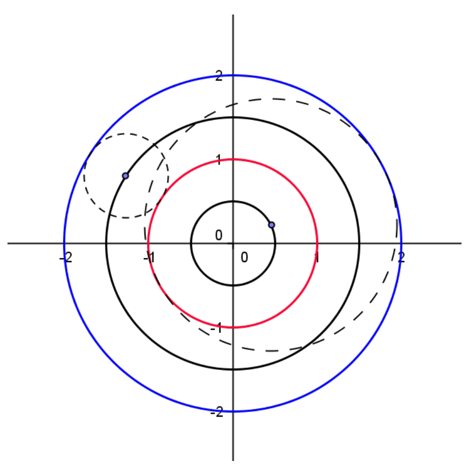


Figure 1: Two concentric circles of radii 1 and 2, and their bisector.

In the following we analyze the notion of untrimmed bisector. The idea, as stated above, is to define it as an algebraic set. When describing this algebraic set, and in some degenerated cases, extraneous components might be introduced. Our goal is to guarantee that none of these extraneous factors is the full plane since, in that case, the untrimmed bisector would be the plane and would provide no information (see Theorem 2.11). For this purpose, throughout this paper, let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two different real irreducible affine curves defined by $f_1(x, y)$ and $f_2(x, y)$, respectively. We use the notation $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2), \mathbf{z} = (z_1, z_2)$. Although we are interested in the case of real curves and real bisector, it is convenient to work with complex coordinates. Afterwards, $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ will be restricted to $\mathbb{R}^2$ to obtain the real bisector.

The idea of our formal definition consists in introducing an algebraic set $\mathcal{A}$, composed by elements of the form $(\mathbf{x}, \mathbf{y}, \mathbf{z}, W)$, where the element $\mathbf{x} \in \mathcal{C}_1, \mathbf{y} \in \mathcal{C}_2$, $\mathbf{z}$ belongs to the untrimmed bisector, and $W$ is an auxiliary variable; this variety $\mathcal{A}$ is called an incidence variety. Then, the untrimmed bisector will be the projection of the incidence variety on the set of $\mathbf{z}$ coordinates. As incidence variety, we consider the set

$$\mathcal{A} = \left\{ (\mathbf{x}, \mathbf{y}, \mathbf{z}, W) \in \mathbb{C}^7 \;\middle|\; \begin{array}{l} f_1(\mathbf{x}) = 0, \\ f_2(\mathbf{y}) = 0, \\ \operatorname{rank} \begin{pmatrix} \mathbf{z} - \mathbf{x} \\ \nabla f_1(\mathbf{x}) \end{pmatrix} = 1 = \operatorname{rank} \begin{pmatrix} \mathbf{z} - \mathbf{y} \\ \nabla f_2(\mathbf{y}) \end{pmatrix}, \\ \|\mathbf{x} - \mathbf{z}\|^2 = \|\mathbf{y} - \mathbf{z}\|^2, \\ \|\nabla f_1(\mathbf{x})\|^2 \|\nabla f_2(\mathbf{y})\|^2 W = 1 \end{array} \right\}. \tag{1}$$

In this situation, we consider the projection map

$$\begin{array}{rccc} \pi_{\mathbf{z}} : & \mathcal{A} \subset \mathbb{C}^7 & \to & \mathbb{C}^2 \\ & (\mathbf{x}, \mathbf{y}, \mathbf{z}, W) & \mapsto & \mathbf{z} \end{array}$$

Then, we have the following definition (for $S \subset \mathbb{C}^2$ we denote by $\overline{S}$ its Zariski closure).

**Definition 2.1.** We define the (untrimmed) bisector of $\mathcal{C}_1, \mathcal{C}_2$ as the Zariski closure of $\pi_{\mathbf{z}}(\mathcal{A})$, and we represent it by $\operatorname{Bis}(\mathcal{C}_1, \mathcal{C}_2)$; i.e. $\operatorname{Bis}(\mathcal{C}_1, \mathcal{C}_2) = \overline{\pi_{\mathbf{z}}(\mathcal{A})}$.

Let us interpret the set $\mathcal{A}$. Suppose $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0, W_0) \in \mathcal{A}$. The first two equations imply that $\mathbf{x}_0 \in \mathcal{C}_1$ and $\mathbf{y}_0 \in \mathcal{C}_2$. The third equations (the rank conditions) ensure that $\mathbf{z}_0$ is on the normal line to $\mathcal{C}_1$ at $\mathbf{x}_0$, and on the normal line to $\mathcal{C}_2$ at $\mathbf{y}_0$. The fourth equation means that the distances (for $\mathbf{x}_0, \mathbf{y}_0$ real points) between $\mathbf{z}_0$ and $\mathbf{x}_0$ and between $\mathbf{z}_0$ and $\mathbf{y}_0$ are equal. The last equation implies that $\mathbf{x}_0$ and $\mathbf{y}_0$ are not singular points of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. But why do we need the last equation? If $\mathbf{x}_0 = \mathbf{y}_0 \in \mathcal{C}_1 \cap \mathcal{C}_2$ and it is a singular point on each curve, then $\|\mathbf{x}_0 - \mathbf{z}\|^2 = \|\mathbf{y}_0 - \mathbf{z}\|^2$ holds for all $\mathbf{z} \in \mathbb{C}^2$. Moreover, both rank conditions are trivial. So $\mathbf{x}_0, \mathbf{y}_0$ would generate in $\mathcal{A}$ a plane, namely $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z})$, and hence $\operatorname{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ would be the plane $\mathbb{C}^2$. Even if $\mathbf{x}_0 \neq \mathbf{y}_0$, extraneous components may appear because all points $\mathbf{z}_0$ satisfying $\|\mathbf{x}_0 - \mathbf{z}_0\|^2 = \|\mathbf{y}_0 - \mathbf{z}_0\|^2$ form a line which satisfies the equations. The last equation also guarantees that $\mathbf{x}_0, \mathbf{y}_0$ are not isotropic, what is used in the proof of Theorem 2.9

to relate bisectors and offsets; see also Remark 2.12. We recall that a point $P$ on a curve $g(x,y)=0$ is isotropic if

$$\frac{\partial g}{\partial x}(P)^2+\frac{\partial g}{\partial y}(P)^2=0.$$

Observe that not only singularities are isotropic; $(i,-1/2)$ is isotropic and regular on the parabola $y=x^2/2$.

**Remark 2.2.** Some authors (see e.g. [12]) choose to avoid the situations where there are infinitely many points in the bisector corresponding to the same footpoint. They happen when an element of $\mathcal{A}$ has $\mathbf{x}=\mathbf{y}$, (see the points $\mathbf{x}=(0,0)=\mathbf{y}$ in Examples 2.5 and 2.8). This sort of points could be avoided by replacing the last equation by $\|\nabla f_1(\mathbf{x})\|^2\|\nabla f_2(\mathbf{y})\|^2\left((x_1-y_1)^2+(x_2-y_2)^2\right)W=1$. However, we decided not to do so, because Theorem 2.9 below would not be true. On the other hand, the extraneous components arising from the cases $\mathbf{x}=\mathbf{y}$ can be removed in the trimming process.

Taking into account that the untrimmed bisector is a projection, it can be obtained as follows. Let $I$ be the ideal in $\mathbb{C}[\mathbf{x},\mathbf{y},\mathbf{z},W]$ generated by the polynomials defining $\mathcal{A}$. Then, by the Closure Theorem (see [5], p. 122), one has that the untrimmed bisector is the variety defined by $I\cap\mathbb{C}[\mathbf{z}]$. Hence elimination theory techniques, such as Gröbner bases, provide a method to compute the untrimmed bisector.

We illustrate the definition by means of some examples.

**Example 2.3.** We start with a simple example. Let $\mathcal{C}_1$ be the line $x_2=0$ and $\mathcal{C}_2$ the line $y_1=0$. Then, the incidence variety is defined by the polynomials

$$\{x_2,y_1,z_1-x_1,y_2-z_2,(z_1-x_1)^2+(z_2-x_2)^2-(z_1-y_1)^2-(z_2-y_2)^2,W-1\}.$$

Considering $W>x_1>x_2>y_1>y_2>z_1>z_2$, and computing a Gröbner basis w.r.t. the lex order, we get

$$\left\{z_1^2-z_2^2,-z_2+y_2,y_1,x_2,-z_1+x_1,W-1\right\},$$

and hence $\mathrm{Bis}(\mathcal{C}_1,\mathcal{C}_2)$ is defined by $z_1^2-z_2^2=0$, that is, the two lines $z_1=\pm z_2$.

**Example 2.4.** Let $\mathcal{C}_1$ be the circle $x_1^2+x_2^2=4$ and $\mathcal{C}_2$ the circle $y_1^2+y_2^2=1$. Applying the ideas above, one gets that $\mathrm{Bis}(\mathcal{C}_1,\mathcal{C}_2)$ is the union of the two circles $4z_1^2+4z_2^2=1$ and $4z_1^2+4z_2^2=9$; see Fig. 1.

**Example 2.5.** Let $\mathcal{C}_1$ be the parabola $x_2=x_1^2$ and $\mathcal{C}_2$ the line $y_2=0$ (see Fig. 2, left). Then, the incidence variety is defined by the polynomials

$$\begin{aligned}\{-{x_1}^2+x_2,y_2,z_1-x_1+2\,(z_2-x_2)\,x_1,z_1-y_1,&\\ (z_1-x_1)^2+(z_2-x_2)^2-(z_1-y_1)^2-(z_2-y_2)^2,(4x_1^2+1)W-1\}\,.&\end{aligned}$$

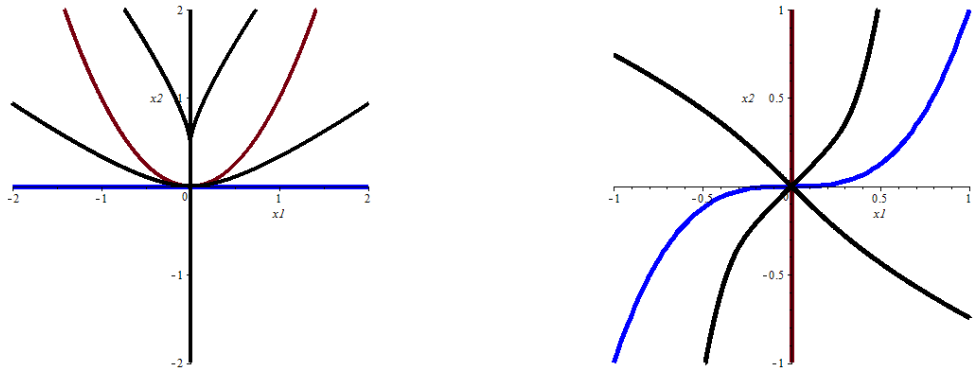


Figure 2: Left: Parabola $x_2 = x_1^2$, line $y_2 = 0$, and their bisector. Right: Line $x_1 = 0$, cubic $y_2 = y_1^3$, and their bisector.

Considering $W > x_1 > x_2 > y_1 > y_2 > z_1 > z_2$, and computing a Gröbner basis w.r.t. the lex order, we get that $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ is the quintic defined by

$$z_1 \left(16\, {z_1}^4 - 32\, {z_1}^2 {z_2}^2 + 16\, {z_2}^4 - 40\, {z_1}^2 z_2 - 24\, {z_2}^3 + {z_1}^2 + 12\, {z_2}^2 - 2\, z_2\right) .$$

We observe that, in this case, the genus of the quartic in $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ is 0. In this example, if one does not introduced the condition $\mathbf{x} \neq \mathbf{y}$, then the line $z_1 = 0$ appears (see Remark 2.2).

**Example 2.6.** Let $\mathcal{C}_1$ be the line $x_1 = 0$ and $\mathcal{C}_2$ the cubic $y_2 = y_1^3$ (see Fig. 2, right). Applying the method above, one gets that $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ is the 8th-degree curve defined by

$$5832 z_1^3 z_2^5 + 3125 z_1^6 - 3375 z_1^4 z_2^2 + 243 z_1^2 z_2^4 - 729 z_2^6 - 400 z_1^3 z_2 + 432 z_1 z_2^3 + 16 z_1^2 - 16 z_2^2.$$

We observe that, in this case, the genus of $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ is 1.

**Example 2.7.** Let $\mathcal{C}_1$ be the parabola $x_2^2 - x_1 = 0$ and $\mathcal{C}_2$ the parabola $-y_1^2 + y_2 = 0$ (see Fig. 3). Applying the ideas above, one gets that $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ is a curve of degree 15 and its defining polynomial has 114 nonzero terms. $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ factors into the line $z_1 = z_2$ and a 14th-degree curve of genus 4.

**Example 2.8.** Let $\mathcal{C}_1$ be the parabola $x_2 - x_1^2 = 0$, and $\mathcal{C}_2$ the cubic $y_2 - y_1^3 = 0$ (see Fig. 4). Applying the ideas above, one gets that $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ is a curve of degree 24, whose defining polynomial has 228 nonzero terms, and factors as the line $z_1 = 0$ and a 23rd-degree curve. This whole line correspond to the single common footpoint at the origin (see Remark 2.2).

Alternatively, one may relate bisectors to offsets. For this purpose, let

$$\mathcal{B}_i = \left\{ (\mathbf{x}, \mathbf{z}, d, W) \in \mathbb{C}^6 \;\middle|\; \begin{array}{l} f_i(\mathbf{x}) = 0, \\ \mathrm{rank} \begin{pmatrix} \mathbf{z} - \mathbf{x} \\ \nabla f_i(\mathbf{x}) \end{pmatrix} = 1, \\ \|\mathbf{x} - \mathbf{z}\|^2 = d^2, \\ \|\nabla f_i(\mathbf{x})\|^2 W = 1 \end{array} \right\}, \; i = 1, 2,$$

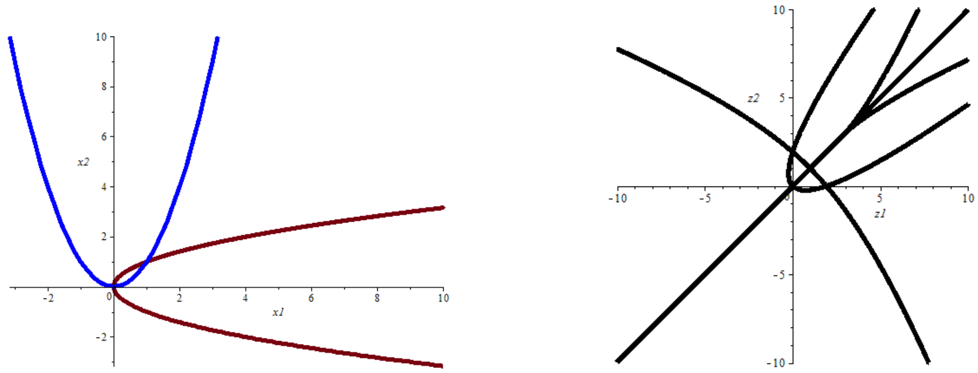


Figure 3: Left: Parabolas $x_1 = x_2^2$, $y_2 = y_1^2$. Right: Their bisector.

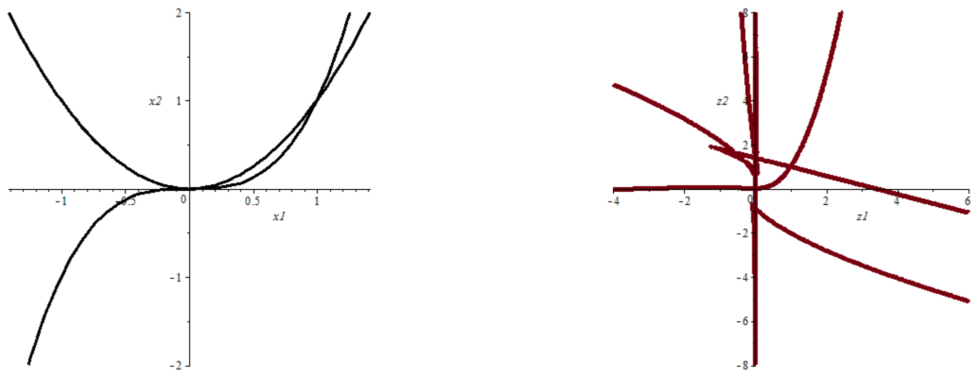


Figure 4: Left: Parabola $x_2 = x_1^2$, cubic $y_2 = y_1^3$. Right: Their bisector.

$\pi_{\mathbf{z},d} : \mathbb{C}^6 \to \mathbb{C}^2$, $\pi_{\mathbf{z},d}(\mathbf{x}, \mathbf{z}, d, W) = (\mathbf{z}, d)$, and $\pi_{\mathbf{z}} : \mathbb{C}^3 \to \mathbb{C}^2$, $\pi_{\mathbf{z}}(\mathbf{z}, d) = \mathbf{z}$. We call that $\overline{\pi_{\mathbf{z},d}(\mathcal{B}_i)}$ is the generic offset of $\mathcal{C}_i$ (see Def. 1 in [17]). In this situation, we have the following theorem.

**Theorem 2.9.** $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2) = \overline{\pi_{\mathbf{z}}(\pi_{\mathbf{z},d}(\mathcal{B}_1) \cap \pi_{\mathbf{z},d}(\mathcal{B}_2))}$.

*Proof.* Let $\mathbf{c} \in \pi_{\mathbf{z}}(\mathcal{A})$ (recall the definition of $\mathcal{A}$ in (1)). Then, there exist $\mathbf{a}, \mathbf{b} \in \mathbb{C}^2$ and $w \in \mathbb{C}$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c}, w) \in \mathcal{A}$. Because of the first, second, and last equations defining $\mathcal{A}$, we know that $\mathbf{a} \in \mathcal{C}_1$, $\mathbf{b} \in \mathcal{C}_2$ and they are not singular points, neither isotropic points on $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively. Therefore, $\|\nabla f_1(\mathbf{a})\| \neq 0$, $\|\nabla f_2(\mathbf{b})\| \neq 0$. So, by the third and fourth equations we have that

$$\mathbf{c} = \mathbf{a} + \frac{\|\mathbf{c} - \mathbf{a}\|}{\|\nabla f_1(\mathbf{a})\|} \nabla f_1(\mathbf{a}) = \mathbf{b} + \frac{\|\mathbf{c} - \mathbf{b}\|}{\|\nabla f_2(\mathbf{b})\|} \nabla f_2(\mathbf{b}) \text{ and } \|\mathbf{c} - \mathbf{a}\| = \|\mathbf{c} - \mathbf{b}\|.$$

Therefore,

$$(\mathbf{a}, \mathbf{c}, \|\mathbf{c} - \mathbf{a}\|, 1/\|\nabla f_1(\mathbf{a})\|^2) \in \mathcal{B}_1,\ (\mathbf{b}, \mathbf{c}, \|\mathbf{c} - \mathbf{a}\|, 1/\|\nabla f_2(\mathbf{b})\|^2) \in \mathcal{B}_2.$$

So $\mathbf{c} \in \pi_{\mathbf{z}}(\pi_{\mathbf{z},d}(\mathcal{B}_1) \cap \pi_{\mathbf{z},d}(\mathcal{B}_2))$, and taking closures, $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2) \subset \overline{\pi_{\mathbf{z}}(\pi_{\mathbf{z},d}(\mathcal{B}_1) \cap \pi_{\mathbf{z},d}(\mathcal{B}_2))}$. Conversely, let $\mathbf{c} \in \pi_{\mathbf{z}}(\pi_{\mathbf{z},d}(\mathcal{B}_1) \cap \pi_{\mathbf{z},d}(\mathcal{B}_2))$. Then, there exist $d_0 \in \mathbb{C}$ such that $(\mathbf{c}, d_0) \in \pi_{\mathbf{z},d}(\mathcal{B}_1) \cap \pi_{\mathbf{z},d}(\mathcal{B}_2)$. So, there exist $\mathbf{a}, \mathbf{b} \in \mathbb{C}^2$ and $W_1, W_2 \in \mathbb{C}$ such that $(\mathbf{a}, \mathbf{c}, d_0, W_1) \in \mathcal{B}_1, (\mathbf{b}, \mathbf{c}, d_0, W_2) \in \mathcal{B}_2$. Then, $(\mathbf{a}, \mathbf{b}, \mathbf{c}, W_1 W_2) \in \mathcal{A}$ and hence, $\mathbf{c} \in \pi_{\mathbf{z}}(\mathcal{A})$. Taking closures one gets the other inclusion. □

**Remark 2.10.** Let $\mathcal{O}_i$ denote the generic offset of $\mathcal{C}_i$ then

$$\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2) = \overline{\pi_{\mathbf{z}}(\pi_{\mathbf{z},d}(\mathcal{B}_1) \cap \pi_{\mathbf{z},d}(\mathcal{B}_2))} \subset \overline{\pi_{\mathbf{z}}(\overline{\pi_{\mathbf{z},d}(\mathcal{B}_1)} \cap \overline{\pi_{\mathbf{z},d}(\mathcal{B}_2)})} = \overline{\pi_{\mathbf{z}}(\mathcal{O}_1 \cap \mathcal{O}_2)}.$$

**Theorem 2.11.** *If* $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ *is not empty, then all its components have dimension 1.*

*Proof.* From Lemma 3 in [17], the generic offset $O_i$ of $\mathcal{C}_i$ is a surface in $\mathbb{C}^3$. So, each irreducible component of $\pi_{\mathbf{z},d}(\mathcal{B}_i)$ is a quasiprojective variety of dimension 2. Furthermore, since $\mathcal{C}_1$ and $\mathcal{C}_2$ are irreducible and different, $\pi_{\mathbf{z},d}(\mathcal{B}_1)$ and $\pi_{\mathbf{z},d}(\mathcal{B}_2)$ have none common component. So, applying Corollary 1, page 75 in [21] to each component of $\pi_{\mathbf{z},d}(\mathcal{B}_1)$ and each component of $\pi_{\mathbf{z},d}(\mathcal{B}_2)$, we get that either $\pi_{\mathbf{z},d}(\mathcal{B}_1) \cap \pi_{\mathbf{z},d}(\mathcal{B}_2) = \emptyset$ or all its components have dimension 1. However, by Theorem 2.9, if $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2) \neq \emptyset$ then $\pi_{\mathbf{z},d}(\mathcal{B}_1) \cap \pi_{\mathbf{z},d}(\mathcal{B}_2) \neq \emptyset$. Now, let us prove that for every $\mathbf{c} \in \pi_z(\pi_{\mathbf{z},d}(\mathcal{B}_1) \cap \pi_{\mathbf{z},d}(\mathcal{B}_2))$, $\pi_{\mathbf{z}}^{-1}(\mathbf{c})$ is finite. Indeed, if $\mathrm{card}(\pi^{-1}(\mathbf{c})) = \infty$ then there exist infinitely many $d_i \in \mathbb{C}$ such that $O_1(\mathbf{c}, d_i) = O_2(\mathbf{c}, d_i) = 0$. But this implies that $\mathbf{c}$ belongs to the offset of $\mathcal{C}_i$ for almost all distances, which is impossible (see Lemma 4 in [16]). Therefore, by Theorem 11.12 in [11], the dimension of the components of $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ and of $\pi_{\mathbf{z},d}(\mathcal{B}_1) \cap \pi_{\mathbf{z},d}(\mathcal{B}_2)$ is the same. □

**Remark 2.12.** If we allow the curves not to be real, and we exclude the isotropic condition in the incidence variety $\mathcal{A}$, the dimension of the bisector may drop to 0. For instance, let $\mathcal{C}_1$ be the parabola defined by $f_1 = x_2^2 - ix_1$ and $\mathcal{C}_2$ be the line defined by $f_2 = y_2 + iy_1 + 1$. Applying Def. 2.1, we get that $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ is the algebraic set defined by $\{(3 + 8z_2)(z_2^2 + z_2 + 1)^2 = 0, -iz_2 - i + z_1 = 0\}$, namely

$$\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2) = \{(5/8i, -3/8), (i(-1/2 \pm (1/2)i\sqrt{3}) + i, -1/2 \pm (1/2)i\sqrt{3})\}.$$

Therefore, $\dim(\pi_{\mathbf{z}}(\mathcal{A})) = 0$.

# 3 Untrimmed Bisectors: Parametric Case

In Def. 2.1, we have introduced the notion of bisector of two plane curves, independently of the representation. Nevertheless, in many situations, the used curves are rational, and hence admit a rational parametric representation. In the following we see how to adapt the incidence variety $\mathcal{A}$ to the parametric case, such that the bisector computation is simplified.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be rational, and $\mathcal{P}_1(t_1)$ and $\mathcal{P}_2(t_2)$ be (non necessarily proper) rational parametrizations of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. We assume that $\mathcal{P}_1(t_1)$ and $\mathcal{P}_2(t_2)$ are expressed as

$$\mathcal{P}_1(t_1) = \left(\frac{a_1(t_1)}{c(t_1)}, \frac{a_2(t_1)}{c(t_1)}\right), \ \mathcal{P}_2(t_2) = \left(\frac{b_1(t_2)}{d(t_2)}, \frac{b_2(t_2)}{d(t_2)}\right)$$

where $\gcd(a_1, a_2, c) = 1$ and $\gcd(b_1, b_2, d) = 1$. Besides $\nabla f_i$, we consider

$$\mathcal{T}_1 = \left(-\frac{\partial f_1}{\partial x_2}, \frac{\partial f_1}{\partial x_1}\right), \quad \mathcal{T}_2 = \left(-\frac{\partial f_2}{\partial y_2}, \frac{\partial f_2}{\partial y_1}\right).$$

Associated with $\mathcal{P}_i$, we introduce the incidence variety $\mathcal{A}_{\mathcal{P}}$ defined as (we write $\mathbf{t} = (t_1, t_2)$)

$$\mathcal{A}_{\mathcal{P}} = \left\{ (\mathbf{t}, \mathbf{z}, W) \in \mathbb{C}^5 \;\middle|\; \begin{array}{l} (z_1\, c(t_1) - a_1(t_1), z_2\, c(t_1) - a_2(t_1)) \cdot \mathcal{T}_1(\mathcal{P}_1(t_1)) = 0, \\ (z_1\, d(t_2) - b_1(t_2), z_2\, d(t_2) - b_2(t_2)) \cdot \mathcal{T}_2(\mathcal{P}_2(t_2)) = 0, \\ \mathrm{num}(\|\mathbf{z} - \mathcal{P}_1(t_1)\|^2) = \mathrm{num}(\|\mathbf{z} - \mathcal{P}_2(t_2)\|^2), \\ \Delta(\mathbf{t})W = 1 \end{array} \right\}, \quad (2)$$

where $\mathrm{num}(R)$ denotes the numerator of the rational function $R$ and where

$$\Delta(\mathbf{t}) = c(t_1)d(t_2)\mathrm{num}(\|\nabla f_1(\mathcal{P}_1(t_1))\|^2)\mathrm{num}(\|\nabla f_2(\mathcal{P}_2(t_2))\|^2).$$

In the following theorem we assume that the parametrizations are normal (see Section 6.3 in [20] for details on normal parametrizations).

**Theorem 3.1.** *Let $\mathcal{P}_1(t_1)$, $\mathcal{P}_2(t_2)$ be normal. Then* $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2) = \overline{\pi_z(\mathcal{A}_{\mathcal{P}})}$.

*Proof.* We consider the rational map

$$\begin{array}{rccc} \varphi: & \mathcal{A}_{\mathcal{P}} & \to & \mathbb{C}^7 \\ & (\mathbf{t}, \mathbf{z}, W) & \mapsto & \left(\mathcal{P}_1(t_1), \mathcal{P}_2(t_2), \mathbf{z}, \dfrac{W}{c(t_1)d(t_2)}\right). \end{array}$$

We observe that because of the last equation of $\mathcal{A}_{\mathcal{P}}$, $\varphi$ is well-defined on all points of $\mathcal{A}_{\mathcal{P}}$. Moreover, $\varphi(\mathcal{A}_{\mathcal{P}}) \subset \mathcal{A}$ (see the definition of $\mathcal{A}$ (1)). So, since the $\mathbf{z}$ component is invariant under $\varphi$, $\pi_{\mathbf{z}}(\mathcal{A}_{\mathcal{P}}) = \pi_{\mathbf{z}}(\varphi(\mathcal{A}_{\mathcal{P}})) \subset \pi_{\mathbf{z}}(\mathcal{A})$. Conversely, let $\mathbf{z}_0 \in \pi_{\mathbf{z}}(\mathcal{A})$. Then, there exists $\mathbf{x}_0, \mathbf{y}_0, W_0$ such that $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0, W_0) \in \mathcal{A}$. Moreover, since $\mathcal{P}_i$ are normal, there exist $t_0, h_0$ such that $\mathbf{x}_0 = \mathcal{P}_1(t_0), \mathbf{y}_0 = \mathcal{P}_2(h_0)$. Furthermore, since $\mathbf{x}_0, \mathbf{y}_0$ are isotropic and their first component is nonzero, $\Delta(t_0, h_0)$ is not identically zero. Therefore, $(t_0, h_0, \mathbf{z}_0, 1/\Delta(t_0, h_0)) \in \mathcal{A}_{\mathcal{P}}$. So, $\mathbf{z}_0 \in \pi_{\mathbf{z}}(\mathcal{A}_{\mathcal{P}})$. Thus, $\pi_{\mathbf{z}}(\mathcal{A}_{\mathcal{P}}) = \pi_{\mathbf{z}}(\mathcal{A})$. □

**Remark 3.2.** We observe the following

1. Every rational curve can be parametrized proper and normally (see Theorem 6.26 in [20]).

2. If we use non-normal parametrizations, it may happen that $\pi_{\mathbf{z}}(\mathcal{A}_{\mathcal{P}}) \subsetneq \pi_{\mathbf{z}}(\mathcal{A})$, and hence the parametrization may not compute the whole bisector (see Example 3.3). If this happens, the missing subset is included in the union of the normal lines to $\mathcal{C}_1$ and $\mathcal{C}_2$ at the critical points of the parametrizations (see Corollary 6.23 in [20] for the notion of critical point).

   *Proof.* If $\mathcal{P}_1(t_1)$ is not normal (similarly for $\mathcal{P}_2(t_2)$) then exactly one point on $\mathcal{C}_1$ is not reachable by $\mathcal{P}_1$, namely, the critical point. So, the argument of the inclusion $\pi_{\mathbf{z}}(\mathcal{A}) \subset \pi_{\mathbf{z}}(\mathcal{A}_{\mathcal{P}})$, in the proof of Theorem 3.1, may fail, and one can only ensure that $\pi_{\mathbf{z}}(\mathcal{A}_{\mathcal{P}}) \subset \pi_{\mathbf{z}}(\mathcal{A})$. Indeed, the argument fails if for $\mathbf{z}_0 \in \mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ it holds that $\pi_{\mathbf{x}}(\pi_{\mathbf{z}}^{-1}(\mathbf{z})) = \emptyset$ for almost all $\mathbf{z}$ in the components of $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ that contains $\mathbf{z}_0$. In this situation, if $\mathbf{a}$ is the critical point of $\mathcal{P}_1$, by the definition of $\mathcal{A}$, almost all points, in those components of $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$, belong to the normal line to $\mathcal{C}_1$ at $\mathbf{a}$. Now, the result follows using that $\dim(\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)) = 1$ (see Theorem 2.11) and that the normal line is unique. □

3. If one does not want to use normal parametrizations, because of the previous remark, one can directly check whether the normal lines to the critical points are included in $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$.

4. In the definition of $\mathcal{A}_{\mathcal{P}}$ one can replace $\nabla f_i(\mathcal{P}_i)$ by $(\mathcal{P}_i)'$, where $(\mathcal{P}_i)'$ denotes the velocity vector. However, in this case a similar phenomenon to the normality can happen if the parametrization has singular points not being singular points of the curve. Nevertheless, this case may be avoided by checking whether the corresponding normal lines are in the bisector.

**Example 3.3.** Let $\mathcal{C}_1$ be the circle $x_1^2 + x_2^2 = 1$ and $\mathcal{C}_2$ the line $y_1 + 1 = 0$. Using directly the definition of bisector, we get that $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ is the line and the parabola defined as $z_2(-z_2^2 + 4z_1 + 4) = 0$. We consider now the parametrizations

$$\mathcal{P}_1(t_1) = \left( \frac{-t_1^2 + 1}{t_1^2 + 1}, \frac{2t_1}{t_1^2 + 1} \right), \mathcal{P}_2(t_2) = (-1, t_2).$$

Note that $\mathcal{P}_1(t_1)$ is not normal and its critical point is $(-1, 0)$. $\mathcal{P}_2(t_2)$ is normal. The computation returns the correct answer. However, if we take $\mathcal{P}_2$ as the non-normal parametrization $\mathcal{P}_2(t_2) = (-1, 1/t_2)$, the computation returns the parabola $-z_2^2 + 4z_1 + 4 = 0$ and the line $z_2 = 0$ is missed. Note that the missing line is the normal line to both, $\mathcal{C}_1, \mathcal{C}_2$, at $(-1, 0)$.

Analogously if only one of the curves, say $\mathcal{C}_1$, is expressed parametrically, we can consider the incidence variety

$$\mathcal{A}_{\mathcal{P}}^I = \left\{ (t, \mathbf{y}, \mathbf{z}, W) \in \mathbb{C}^6 \;\middle|\; \begin{array}{l} (z_1\, c(t_1) - a_1(t_1), z_2\, c(t_1) - a_2(t_1)) \cdot \mathcal{T}_1(\mathcal{P}_1(t)) = 0 \\ (\mathbf{z} - \mathbf{y}) \cdot \mathcal{T}_2(\mathbf{y}) = 0 \\ \mathrm{num}(\|\mathbf{z} - \mathcal{P}_1(t)\|^2 - \|\mathbf{z} - \mathbf{y}\|^2) = 0 \\ \Delta(t) W = 1 \end{array} \right\} \subset \mathbb{C}^5,$$

where $\Delta(t) = c(t)\mathrm{num}(\|\nabla f_1(\mathcal{P}_1(t))\|^2))\|\nabla f_2(\mathbf{y})\|^2$. Reasoning similarly, we have the following theorem.

**Theorem 3.4.** *Let $\mathcal{P}_1(t)$ be normal. Then* $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2) = \overline{\pi_z(\mathcal{A}^I_{\mathcal{P}})}$.

# 4 Parametric Representation of the Untrimmed Bisector

Throughout this section, we consider that $\mathcal{C}_1, \mathcal{C}_2$ are rational, and we keep the notation used in Section 2. It is clear that, using the fact that $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ is algebraic, the untrimmed bisector can be represented by means of its implicit equations. Nevertheless, these equations can be huge, and hence hard to manage (see e.g. Examples 2.7, 2.8). An alternative would be to use rational parametric representations of the bisector. However, in general, the bisector can be reducible (see e.g. Examples 2.3, 2.4, 2.5, 2.7, 2.8). Furthermore, some of the bisector components have positive genus (see e.g. Example 2.7), or the bisector is irreducible with positive genus (see e.g. Examples 2.6). In [7], an alternative representation for irreducible bisectors, based on the parameters space, is introduced. They assume the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ to be rational, regular and $C^1$-continuous. Here, in this section, we formally extend this representation to the general case.

In this situation, let us consider the following diagram (see (2) in Section 3 for the definition of $\mathcal{A}_{\mathcal{P}}$)

$$\begin{array}{ccc} & \mathcal{A}_{\mathcal{P}} & \\ & \swarrow \pi_{\mathbf{z}} \qquad \searrow \pi_{\mathbf{t}} & \\ \mathbb{C}^2 \supset \mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2) \supset \pi_{\mathbf{z}}(\mathcal{A}_{\mathcal{P}}) & & \pi_{\mathbf{t}}(\mathcal{A}_{\mathcal{P}}) \subset \mathbb{C}^2 \end{array}$$

where $\pi_{\mathbf{z}}(\mathbf{t}, \mathbf{z}, W) = \mathbf{z}$, and $\pi_{\mathbf{t}}(\mathbf{t}, \mathbf{z}, W) = \mathbf{t}$. We denote by $\mathcal{M}_{\mathcal{P}}$ the Zariski closure of $\pi_{\mathbf{t}}(\mathcal{A}_{\mathcal{P}})$, that is

$$\mathcal{M}_{\mathcal{P}} := \overline{\pi_{\mathbf{t}}(\mathcal{A}_{\mathcal{P}})}.$$

**Theorem 4.1.** *Let $\Gamma \subset \mathcal{M}_{\mathcal{P}}$ be an irreducible component. Then,*

1. *If* $\dim(\Gamma) > 0$ *then* $\pi_{\mathbf{t}} : \overline{\pi_{\mathbf{t}}^{-1}(\Gamma)} \to \Gamma$ *is a birational map.*
2. *If* $\dim(\Gamma) = 0$, *say* $\Gamma = \{(\alpha, \beta)\}$ *then*
   - *(a) If $\mathcal{P}_1(\alpha) \neq \mathcal{P}_2(\beta)$, $\pi_{\mathbf{t}}^{-1}((\alpha, \beta))$ has cardinality 1.*
   - *(b) If $\mathcal{P}_1(\alpha) = \mathcal{P}_2(\beta)$ and $\nabla f_1(\mathcal{P}_1(\alpha)), \nabla f_2(\mathcal{P}_2(\beta))$ are not parallel, $\pi_{\mathbf{t}}^{-1}((\alpha, \beta))$ has cardinality 1.*
   - *(c) If $\mathcal{P}_1(\alpha) = \mathcal{P}_2(\beta)$ and $\nabla f_1(\mathcal{P}_1(\alpha)), \nabla f_2(\mathcal{P}_2(\beta))$ are parallel, then $\pi_{\mathbf{t}}^{-1}((\alpha, \beta))$ has dimension 1.*

*Proof.* Let $\dim(\Gamma) > 0$. Since $\pi_{\mathbf{t}}$ is rational, we only need to prove that the generic fiber of $\pi_{\mathbf{t}}$ on $\Gamma$ has cardinality 1. For this purpose, we consider the set $\Sigma$ of all $\mathbf{t}$ such that $\mathcal{P}_1(t_1) = \mathcal{P}_2(t_2)$. Since $\mathcal{C}_1 \neq \mathcal{C}_2$, $\Sigma$ is finite, and since $\dim(\Gamma) > 0$ and irreducible, then $\Gamma^* := (\pi_{\mathbf{t}}(\mathcal{A}_{\mathcal{P}}) \cap \Gamma) \setminus \Sigma$ is non-empty and dense in $\Gamma$. Let us study $\pi_{\mathbf{t}}^{-1}(\mathbf{t}_0)$, for $\mathbf{t}_0 = (\alpha, \beta) \in \Gamma^*$. Since $\Gamma^* \subset \pi_{\mathbf{t}}(\mathcal{A}_{\mathcal{P}})$, the fiber is non-empty. Moreover, since $W$ only appears in one equation, and with degree 1, the cardinality $\pi_{\mathbf{t}}^{-1}(\mathbf{t}_0)$ is equal to the number of $\mathbf{z}$ solutions of the first three equations. Because of the last equation of $\mathcal{A}_{\mathcal{P}}$, $\|\nabla f_1(\mathcal{P}_1(\alpha))\|^2 \neq 0, \|\nabla f_2(\mathcal{P}_2(\beta))\|^2 \neq 0$. So, the first and second equations of $\mathcal{A}_{\mathcal{P}}$ imply that the corresponding $\mathbf{z}$ in the fiber are in the intersection of the normal lines to $\mathcal{C}_1$ at $\mathcal{P}_1(\alpha)$ and to $\mathcal{C}_2$ at $\mathcal{P}_2(\beta)$. If $\nabla f_1(\mathcal{P}_1(\alpha)), \nabla f_2(\mathcal{P}_2(\beta))$ are not parallel, the two lines intersect at a point, and hence the fiber has cardinality 1. Let us assume that $\nabla f_1(\mathcal{P}_1(\alpha)), \nabla f_2(\mathcal{P}_2(\beta))$ are parallel. We know that the fiber is not empty, hence both normal lines must be equal. So, since $\mathcal{P}_1(\alpha) \neq \mathcal{P}_2(\beta)$, the only $\mathbf{z} = \mathcal{P}_1(\alpha) + \lambda\mathbf{n} = \mathcal{P}_2(\beta) + \mu\mathbf{n}$, such that $\|\mathbf{z} - \mathcal{P}_1(\alpha)\|^2 = \|\mathbf{z} - \mathcal{P}_2(\beta)\|^2$, where $\mathbf{n} = \nabla f_1(\mathcal{P}_1(\alpha))/\|\nabla f_1(\mathcal{P}_1(\alpha))\|$, is the point $\mathbf{z} = \mathcal{P}_1(\alpha) + \frac{1}{2}(\mathcal{P}_2(\beta) - \mathcal{P}_1(\alpha))$.

Now, let $\Gamma = \{(\alpha, \beta)\}$. If $\mathcal{P}_1(\alpha) \neq \mathcal{P}_2(\beta)$, then $\Gamma^* = \Gamma$ and the above reasoning is valid and hence the fiber has cardinality 1. If $\mathcal{P}_1(\alpha) = \mathcal{P}_2(\beta)$ and $\nabla f_1(\mathcal{P}_1(\alpha)), \nabla f_2(\mathcal{P}_2(\beta))$ are not parallel, again, the reasoning is also valid. However, if $\mathcal{P}_1(\alpha) = \mathcal{P}_2(\beta)$ and $\nabla f_1(\mathcal{P}_1(\alpha)), \nabla f_2(\mathcal{P}_2(\beta))$ are parallel then $\pi_1^{-1}((\alpha, \beta))$ have dimension 1; namely the lifting to $\mathcal{A}_{\mathcal{P}}$ of the normal line at $\mathcal{P}_1(\alpha)$. $\square$

**Remark 4.2.** In Theorem 4.1 we have considered zero-dimension components of $\mathcal{M}_{\mathcal{P}}$. However, in all checked examples $\mathcal{M}_{\mathcal{P}}$ has none zero-dimensional component.

Based on the previous theorem, we introduce the following definition.

**Definition 4.3.** Let $\Gamma \subset \mathcal{M}_{\mathcal{P}}$ be irreducible of positive dimension. We associate with $\Gamma$ the rational map $\Upsilon_\Gamma : \Gamma \to \mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$, where

$$\Upsilon_\Gamma = \pi_{\mathbf{z}} \circ (\pi_{\mathbf{t}}|_\Gamma)^{-1}.$$

**Remark 4.4.** Observe that if $\Gamma$ is a rational curve and $\mathcal{Q}$ is a parametrization, then $\Gamma \circ \mathcal{Q}$ is a parametrization of the component $\pi_{\mathbf{z}}(\pi_{\mathbf{t}}^{-1}(\Gamma))$ of the bisector.

Taking into account the proof of Theorem 4.1, we get the following theorem.

**Theorem 4.5.** *Let $\Gamma$ be an irreducible component of positive dimension of $\mathcal{M}_{\mathcal{P}}$ and let $\mathbb{C}(\Gamma)$ denote the field of rational functions of $\Gamma$.*

1. *If $\mathcal{T}_1(\mathcal{P}_1(t_1)), \mathcal{T}_2(\mathcal{P}_2(t_2))$ are parallel as vectors in $\mathbb{C}(\Gamma)^2$, then*

$$\Upsilon_\Gamma : \Gamma \to \mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2); \Upsilon_\Gamma(\mathbf{t}) = \mathcal{P}_1(t_1) + \frac{1}{2}(\mathcal{P}_2(t_2) - \mathcal{P}_1(t_1)).$$

2. *If $\mathcal{T}_1(\mathcal{P}_1(t_1)), \mathcal{T}_2(\mathcal{P}_2(t_2))$ are not parallel as vectors in $\mathbb{C}(\Gamma)^2$, then*

$$\Upsilon_\Gamma : \Gamma \to \mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2); \Upsilon_\Gamma(\mathbf{t}) = \begin{pmatrix} \mathcal{T}_1(\mathcal{P}_1(t_1)) \\ \mathcal{T}_2(\mathcal{P}_2(t_2)) \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{P}_1(t_1) \cdot \mathcal{T}_1(\mathcal{P}_1(t_1)) \\ \mathcal{P}_2(t_2) \cdot \mathcal{T}_2(\mathcal{P}_2(t_2)) \end{pmatrix}$$

**Remark 4.6.** From the computational point of view, if $\dim(\Gamma) = 1$ and its defining polynomial is $\gamma(\mathbf{t})$, then $\mathcal{T}_1(\mathcal{P}_1(t_1)), \mathcal{T}_2(\mathcal{P}_2(t_2))$ are parallel in $\mathbb{C}(\Gamma)^2$ iff $\gamma$ divides the numerator of the determinant of the matrix whose rows are $\mathcal{T}_i(\mathcal{P}_i(t_i))$.

**Remark 4.7.** If $\Gamma$ is an irreducible component of $\mathcal{M}_{\mathcal{P}}$ with $\dim(\Gamma) = 0$, say $\Gamma = \{(\alpha, \beta)\}$, then

1. if $\nabla f_1(\mathcal{P}_1(\alpha)), \nabla f_2(\mathcal{P}_2(\beta))$ are not parallel,

$$\pi_{\mathbf{z}}(\pi_{\mathbf{t}}^{-1}(\Gamma)) = \left\{ \begin{pmatrix} \mathcal{T}_1(\mathcal{P}_1(\alpha)) \\ \mathcal{T}_2(\mathcal{P}_2(\beta)) \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{P}_1(\alpha) \cdot \mathcal{T}_1(\mathcal{P}_1(\alpha)) \\ \mathcal{P}_2(\beta) \cdot \mathcal{T}_h(\mathcal{P}_2(\beta)) \end{pmatrix} \right\}.$$

2. If $\nabla f_1(\mathcal{P}_1(\alpha)), \nabla f_2(\mathcal{P}_2(\beta))$ are parallel, and $\mathcal{P}_1(\alpha) \neq \mathcal{P}_2(\beta)$ then

$$\pi_{\mathbf{z}}(\pi_{\mathbf{t}}^{-1}(\Gamma)) = \left\{ \mathcal{P}_1(\alpha) + \frac{1}{2}(\mathcal{P}_2(\beta) - \mathcal{P}_1(\alpha)) \right\}.$$

3. If $\nabla f_1(\mathcal{P}_1(\alpha)), \nabla f_2(\mathcal{P}_2(\beta))$ are parallel, and $\mathcal{P}_1(\alpha) = \mathcal{P}_2(\beta)$ then

$$\pi_{\mathbf{z}}(\pi_{\mathbf{t}}^{-1}(\Gamma)) = \{\mathcal{P}_1(\alpha) + \lambda \nabla f_1(\mathcal{P}_1(\alpha)) \,|\, \lambda \in \mathbb{C}\}.$$

See Remark 2.2.

Based on the previous result we introduce the following representation of the bisector.

**Definition 4.8.** We define the parametric representation of $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ as the set

$$\mathrm{PBis}(\mathcal{P}_1, \mathcal{P}_2) = \bigcup_{\Gamma \in \mathcal{M}_{\mathcal{P}}^{+}} \{(\Gamma, \Upsilon_\Gamma(\mathbf{t}))\} \bigcup_{\Gamma \in \mathcal{M}_{\mathcal{P}}^{0}} \{(\Gamma, \pi_{\mathbf{z}}(\pi_{\mathbf{t}}^{-1}(\Gamma)))\}$$

where $\mathcal{M}_{\mathcal{P}}^{+}$ denotes the set of all irreducible components of positive dimension of $\mathcal{M}_{\mathcal{P}}$, and $\mathcal{M}_{\mathcal{P}}^{0}$ denotes the set of all irreducible 0-dimensional components of $\mathcal{M}_{\mathcal{P}}$.

If a component $\Gamma$ is rational, and $\mathcal{Q}(h)$ a parametrization of $\Gamma$ (see Remark 4.4), we will write $(\mathbb{C}, \Upsilon_\Gamma(\mathcal{Q}(h)))$, instead of $(\Gamma, \Upsilon_\Gamma(\mathbf{t}))$.

**Remark 4.9.** If a normal to the original curves appears in the bisector, this line may be lost in the parametric representation; see Example 4.12. The reason is that $\pi_{\mathbf{t}} \circ \pi_{\mathbf{z}}^{-1}$ may send the whole line onto a point on a 1-dimensional component of $\mathcal{M}_{\mathcal{P}}$. Nevertheless, analyzing the points of $\mathcal{M}_{\mathcal{P}}$ where the tangent vectors $\mathcal{T}_i(\mathcal{P}_i)$ are parallel, and applying Remark 4.7, one can reach these lines; see again Examples 4.12 and 4.15.

We illustrate these ideas with some examples.

**Example 4.10.** We consider Example 2.3. Taking $\mathcal{P}_1(t_1) = (t_1, 0)$ and $\mathcal{P}_2(t_2) = (0, t_2)$, we get that $\mathcal{M}_{\mathcal{P}} = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \equiv t_1 - t_2 = 0$ and $\Gamma_2 \equiv t_1 + t_2 = 0$. On the other hand, $\mathcal{T}_1(\mathcal{P}_1(t_1)) = (-1, 0), \mathcal{T}_2(\mathcal{P}_2(t_2)) = (0, 1)$. So, they are not parallel over $\mathbb{C}(\Gamma_i)$. Therefore,

$$\Upsilon_{\Gamma_i} : \Gamma_i \to \text{Bis}(\mathcal{C}_1, \mathcal{C}_2); \mathbf{t} \mapsto (t_1, t_2)$$

according to Theorem 4.5. Furthermore, since $\Gamma_i$ is rational, composing $\Upsilon_{\Gamma_i}$ with the parametrization $\mathcal{Q}_1(h) = (h, h)$ of $\Gamma_1$ and $\mathcal{Q}_2(h) = (h, -h)$, we get

$$\text{PBis}(\mathcal{P}_1, \mathcal{P}_2) = \{(\mathbb{C}, (h, h)), (\mathbb{C}, (h, -h))\}.$$

Recall that $\text{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ is defined by $z_1^2 - z_2^2 = 0$.

**Example 4.11.** We consider Example 2.4. We take

$$\mathcal{P}_1(t_1) = \left(\frac{4t_1}{t_1^2+1}, \frac{2(t_1^2-1)}{t_1^2+1}\right), \quad \mathcal{P}_2(t_2) = \left(\frac{2t_2}{t_2^2+1}, \frac{t_2^2-1}{t_2^2+1}\right).$$

Then, $\mathcal{M}_{\mathcal{P}} = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \equiv t_1 t_2 + 1 = 0$ and $\Gamma_2 \equiv t_1 - t_2 = 0$. On the other hand,

$$\det \begin{pmatrix} \mathcal{T}_1(\mathcal{P}_1(t_1)) \\ \mathcal{T}_2(\mathcal{P}_2(t_2)) \end{pmatrix} = \frac{-16(t_1 t_2 + 1)(-t_2 + t_1)}{(t_1^2+1)(t_2^2+1)}$$

that is zero in $\mathbb{C}(\Gamma_1)$ and in $\mathbb{C}(\Gamma_2)$. So, $\mathcal{T}_1(\mathcal{P}_1(t_1)), \mathcal{T}_2(\mathcal{P}_2(t_2))$ are parallel on both components. Therefore,

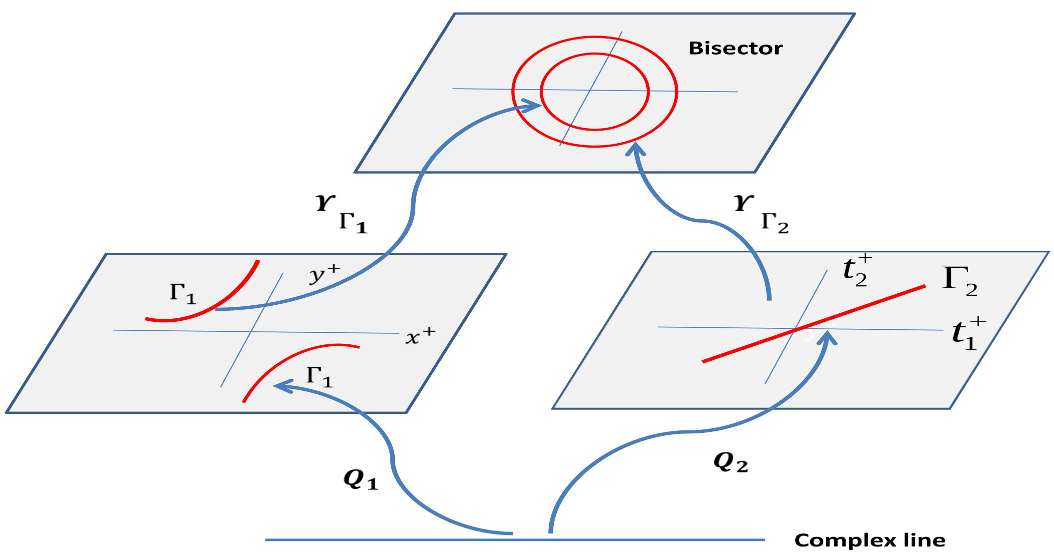


Figure 5: $\text{PBis}(\mathcal{P}_1, \mathcal{P}_2)$ in Example 4.11

$$\Upsilon_{\Gamma_i}:\Gamma_i\rightarrow \mathrm{Bis}(\mathcal{C}_1,\mathcal{C}_2);\mathbf{t}\mapsto \mathcal{P}_1(t_1)+\frac{1}{2}(\mathcal{P}_2(t_2)-\mathcal{P}_1(t_1)),$$

according to Theorem 4.5. Furthermore, since $\Gamma_i$ is rational, composing $\Upsilon_{\Gamma_i}$ with the parametrization $\mathcal{Q}_1(h)=(h,1/h)$ of $\Gamma_1$ and $\mathcal{Q}_2(h)=(h,h)$, we get (see Fig. 5)

$$\mathrm{PBis}(\mathcal{P}_1,\mathcal{P}_2)=\left\{\left(\mathbb{C},\left(\frac{h}{h^2+1},\frac{1}{2}\frac{h^2-1}{h^2+1}\right)\right),\left(\mathbb{C},\left(\frac{3h}{h^2+1},\frac{3}{2}\frac{h^2-1}{h^2+1}\right)\right)\right\}.$$

Recall that $\mathrm{Bis}(\mathcal{C}_1,\mathcal{C}_2)$ is defined by $(4z_1^2+4z_2^2-1)(4z_1^2+4z_2^2-9)=0$.

**Example 4.12.** We consider Example 2.5. Taking $\mathcal{P}_1(t_1)=(t_1,t_1^2)$ and $\mathcal{P}_2(t_2)=(t_2,0)$, we get that $\mathcal{M}_{\mathcal{P}}$ is the rational curve $\Gamma$ defined as $t_1^4+t_1t_2-t_2^2=0$. On the other hand, the determinant of the tangent vectors is $2t_1$ that is not zero over $\mathbb{C}(\Gamma)$. So, they are not parallel. Therefore,

$$\Upsilon_{\Gamma}:\Gamma\rightarrow \mathrm{Bis}(\mathcal{C}_1,\mathcal{C}_2);\mathbf{t}\mapsto\left(t_2,\frac{2t_1^3+t_1-t_2)}{2t_1}\right);$$

compare to Theorem 4.5. Furthermore, since $\Gamma$ is rational, composing $\Upsilon_{\Gamma}$ with the parametrization

$$\mathcal{Q}(h)=\left(\frac{h}{h^2-1},\frac{-h}{h^4-2h^2+1}\right)$$

of $\Gamma$ we get

$$\mathrm{PBis}(\mathcal{P}_1,\mathcal{P}_2)=\left\{\left(\mathbb{C},\left(\frac{-h}{h^4-2h^2+1},\frac{(h^2+1)h^2}{2(h^4-2h^2+1)}\right)\right)\right\}.$$

See the implicit equation of $\mathrm{Bis}(\mathcal{C}_1,\mathcal{C}_2)$ in Example 2.5. One may observe that the parametric representation is missing the line $z_1=0$ that is the normal line at $(0,0)$ of both initial curves $\mathcal{C}_1$ and $\mathcal{C}_2$. What happens is that $\pi_{\mathbf{t}}^{-1}(\pi_{\mathbf{z}}(0,\lambda))=\{(0,0)\}\subset\Gamma$; compare to Remark 4.9. Now, we consider the intersection of $\Gamma$ with the determinant of the tangent vectors $\mathcal{T}_i(\mathcal{P}_i)$, namely $2t_1$. This gives, precisely the point $(\alpha,\beta)=(0,0)$, and applying Remark 4.7, we get

$$\pi_{\mathbf{z}}(\pi_{\mathbf{t}}^{-1}((0,0)))=\{\mathcal{P}_1(0)+\lambda\nabla f_1(\mathcal{P}_1(0))\,|\,\lambda\in\mathbb{C}\}=\{(0,\lambda)\,|\,\lambda\in\mathbb{C}\}.$$

So, we have

$$\mathrm{PBis}(\mathcal{P}_1,\mathcal{P}_2)=\left\{\left(\mathbb{C},\left(\frac{-h}{h^4-2h^2+1},\frac{(h^2+1)h^2}{2(h^4-2h^2+1)}\right)\right),(\mathbb{C},(0,h))\right\}.$$

**Example 4.13.** We consider Example 2.6. Taking $\mathcal{P}_1(t_1)=(0,t_1)$ and $\mathcal{P}_2(t_2)=(t_2,t_2^3)$, we get that $\mathcal{M}_{\mathcal{P}}=\Gamma$ where $\Gamma$ is the elliptic curve $5t_2^6-4t_1t_2^3-t_1^2+t_2^2=0$ (observe that $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ is irreducible of genus 1). On the other hand, the determinant of the tangent vectors is 1 that is not zero over $\mathbb{C}(\Gamma)$. So, they are not parallel. Therefore,

$$\Upsilon_\Gamma : \Gamma \to \mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2); \mathbf{t} \mapsto \left(3t_2^5 - 3t_1t_2^2 + t_2, t_1\right);$$

compare to Theorem 4.5. Thus

$$\mathrm{PBis}(\mathcal{P}_1, \mathcal{P}_2) = \left\{\left(\mathbb{C}, \left(3t_2^5 - 3t_1t_2^2 + t_2, t_1\right)\right)\right\}.$$

**Example 4.14.** We consider Example 2.7. Taking $\mathcal{P}_1(t_1) = (t_1^2, t_1)$ and $\mathcal{P}_2(t_2) = (t_2, t_2^2)$, we get that $\mathcal{M}_{\mathcal{P}} = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1$ is the line $t_1 = t_2$ and $\Gamma_2$ is the genus 4, 5-degree, curve defined as (observe that one component of $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ has genus 4)

$$4t_1^4t_2 + 4t_1^3t_2^2 + 4t_1^2t_2^3 + 4t_1t_2^4 - 4t_1^2t_2^2 - 3t_1^3 - 3t_1^2t_2 - 3t_1t_2^2 - 3t_2^3 + 2t_1t_2 - t_1 - t_2 = 0$$

On the other hand, the determinant of the tangent vectors is $4t_1t_2 - 1$ that is not zero over $\mathbb{C}(\Gamma_i)$. So, they are not parallel. Therefore,

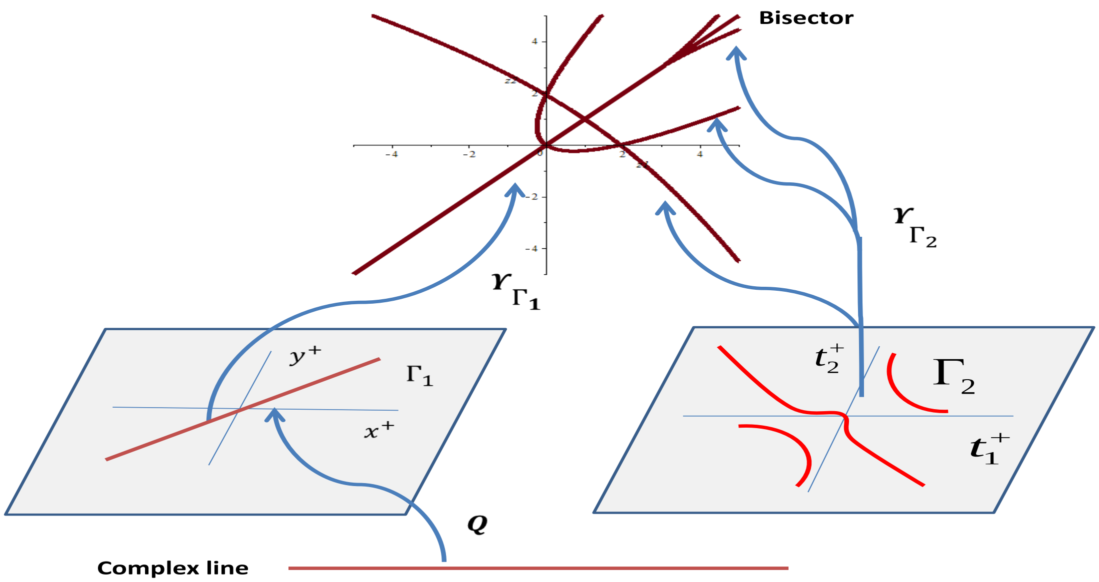


Figure 6: $\mathrm{PBis}(\mathcal{P}_1, \mathcal{P}_2)$ in Example 4.14

$$\Upsilon_{\Gamma_i} : \Gamma_i \to \mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2); \mathbf{t} \mapsto \left(\frac{t_2\,(4\,{t_1}^3 - 2\,{t_2}^2 + 2\,t_1 - 1)}{4\,t_2\,t_1 - 1}, -\frac{t_1\,(-4\,{t_2}^3 + 2\,{t_1}^2 - 2\,t_2 + 1)}{4\,t_2\,t_1 - 1}\right);$$

compare to Theorem 4.5. Furthermore, since $\Gamma_1$ can be parametrized by $(h, h)$, we can take $\Upsilon_{\Gamma_1}(h, h)$ that is

$$\left(\frac{(2\,h+1)\,h^2}{2\,h^2+1}, \frac{(2\,h+1)\,h^2}{2\,h^2+1}\right) \sim (h, h)$$

$$\mathrm{PBis}(\mathcal{P}_1,\mathcal{P}_2) =$$
$$\left\{((\mathbb{C},(h,h)),\left(\Gamma_2,\left(\frac{{t_2}^2\,(4\,{t_1}^3+2\,{t_1}^2-2\,t_2-1)}{4\,{t_2}^2{t_1}^2-1},-\frac{{t_1}^2\,(-4\,{t_2}^3-2\,{t_2}^2+2\,t_1+1)}{4\,{t_2}^2{t_1}^2-1}\right)\right)\right\}.$$

Observe that this representation is much simpler that the implicit equation of $\mathrm{Bis}(\mathcal{C}_1,\mathcal{C}_2)$ (see details in Example 2.7).

**Example 4.15.** We consider Example 2.8. Taking $\mathcal{P}_1(t_1) = (t_1, t_1^2)$ and $\mathcal{P}_2(t_2) = (t_2, t_2^3)$, we get that $\mathcal{M}_{\mathcal{P}} = \Gamma$ where $\Gamma$ is the genus 10, 8-degree, curve defined as

$$3{t_2}^8+6{t_1}^2{t_2}^5-10t_1{t_2}^6-9{t_1}^4{t_2}^2+8{t_1}^3{t_2}^3+2{t_1}^5-3{t_1}^2{t_2}^2+4t_1{t_2}^3-{t_2}^4+2{t_1}^2t_2-2t_1{t_2}^2=0$$

On the other hand, the determinant of the tangent vectors is $3t_2^2-2t_1$ that is not zero over $\mathbb{C}(\Gamma)$. So, they are not parallel. Therefore,

$$\Upsilon_\Gamma : \Gamma \to \mathrm{Bis}(\mathcal{C}_1,\mathcal{C}_2); \mathbf{t} \mapsto \left(-\frac{t_1t_2\,(-6\,{t_2}^4+6\,{t_1}^2t_2+3\,t_2-2)}{-3\,{t_2}^2+2\,t_1},\frac{-3\,{t_2}^5+2\,{t_1}^3+t_1-t_2}{-3\,{t_2}^2+2\,t_1}\right);$$

compare to Theorem 4.5. One may observe that the parametric representation is missing the line $z_1 = 0$ that is the normal line at $(0,0)$ of both initial curves $\mathcal{C}_1$ and $\mathcal{C}_2$. What happens is that

$$\pi_{\mathbf{t}}^{-1}(\pi_{\mathbf{z}}(0,\lambda)) = \{(0,0),(0,(1/3)3^{3/4},(0,(1/3i)3^{3/4}),(0,-(1/3)3^{3/4}),(0,-(1/3i)3^{3/4})\} \subset \Gamma;$$

compare to Remark 4.9. Now, we consider the intersection of $\Gamma$ with the determinant of the tangent vectors $\mathcal{T}_i(\mathcal{P}_i)$, namely $3t_2^2-2t_1$. This gives, 7 points in $\Gamma$ of the form $(0,\beta)$, and applying Remark 4.7, we get

$$\pi_{\mathbf{z}}(\pi_{\mathbf{t}}^{-1}((0,\beta))) = \{\mathcal{P}_1(0)+\lambda\nabla f_1(\mathcal{P}_1(0))\,|\,\lambda\in\mathbb{C}\} = \{(0,\lambda)\,|\,\lambda\in\mathbb{C}\}.$$

So, we have

$$\mathrm{PBis}(\mathcal{P}_1,\mathcal{P}_2) =$$
$$\left\{\left(\Gamma,\left(-\frac{t_1t_2\,(-6\,{t_2}^4+6\,{t_1}^2t_2+3\,t_2-2)}{-3\,{t_2}^2+2\,t_1},\frac{-3\,{t_2}^5+2\,{t_1}^3+t_1-t_2}{-3\,{t_2}^2+2\,t_1}\right)\right),(\mathbb{C},(0,h))\right\}.$$

Observe that this representation is much simpler than the implicit equation of $\mathrm{Bis}(\mathcal{C}_1,\mathcal{C}_2)$

# 5 Conclusions and future work

While all previous work about bisectors is mainly motivated by applications, in this article a general theoretical study of the untrimmed bisector of two real algebraic plane curves has been presented. It remains as an open question to prove that the parametric representation of the untrimmed bisector presented in Section 4 does not produce isolated points ever.

In the near future, we aim to devise a trimming method within the framework of the present paper. In addition, we plan to extend this study to the bisector surface of a pair of algebraic surfaces.

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