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A Loewner Equation for Infinitely Many Slits

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Abstract It is well-known that the growth of a slit in the upper half-plane can be encoded via the chordal Loewner equation, which is a differential equation for schlicht functions with a certain normalisation. We prove that a multiple slit Loewner equation can be used to encode the growth of the union Γ of multiple slits in the upper half-plane if the slits have pairwise disjoint closures. Under certain assumptions on the geometry of Γ, our approach allows us to derive a Loewner equation for infinitely many slits as well.

Keywords Loewner theory · Chordal Loewner equation · Slit domain · Infinitely many slits

Mathematics Subject Classification 30C20 · 30C55

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1 Introduction and Main Result

In his celebrated paper [11], Loewner developed a fruitful approach to tackle extremal problems involving schlicht functions f, defined in the unit disk $\mathbb{D}$, with the normalisation $f(0) = 0$ and $f'(0) = 1$. This led to the so-called (radial) Loewner equation. A similar theory has been established by Kufarev et al. (cf. [9]) for schlicht functions f, defined on the upper half-plane $\mathbb{H}$, and satisfying the hydrodynamic normalisation:

$$\lim_{z\to\infty} (f(z) - z) = 0. \tag{1.1}$$

This led to the so-called chordal Loewner equation. However, the chordal Loewner equation has received much more attention since O. Schramm wrote his seminal paper [17]. Before going any further and stating our results, we will need some notation.

Definition 1.1 A bounded set $A \subseteq \mathbb{H}$ is called a (compact) $\mathbb{H}$ hull (or for short: hull) if $\operatorname{clos}(A) \cap \mathbb{H} = A$ and $\mathbb{H}\backslash A$ are simply connected. A hull A is called a slit if there is a homoemorphism $\gamma : [0, 1] \to \operatorname{clos}(A)$, such that $\gamma(0) \in \mathbb{R}$ and $\gamma(0, 1] \subseteq A$, where $\gamma\,(0, t]$ denotes the image of the half-open interval $(0, t]$ under γ. In this case, we say γ parametrises A. A multislit is a possibly finite sequence of slits Γ_j, such that $\bigcup_j \Gamma_j$ is a hull. Given $(\Gamma_j)_j$, we let $\Gamma := \bigcup_j \Gamma_j$, and also call Γ a multislit. Moreover, if for a multislit Γ, the set $\operatorname{clos}\Gamma_j$ can be separated from $\operatorname{clos}(\Gamma\backslash\Gamma_j)$ by open sets for each j, then Γ is called admissible (see Fig. 1). If we wish to emphasise that a multislit consists of only finitely many slits, then we speak of an n slit. In what follows, every multislit Γ is assumed to be admissible.

Recently, several authors, in particular mathematical physicists gazing towards conformal field theory, have studied a Loewner equation for multiple slits, to generate growing hulls or n slits; see, for example, [5,8,15,16]. However, the following geometric question has apparently received little attention: for what kind of parametrisations can any given multislit be encoded in a Loewner equation? In the radial case, there are some results for finitely many slits, see [2,3]. In the chordal case, it is, to the best of the authors' knowledge, only known that for n slits, there exists a certain (not effectively computable) parametrisation, such that a generalised Loewner equation is satisfied, see [14, Theorem 1.1]. To state our results, we recall the following well-known fact (cf. [10, p. 69]).

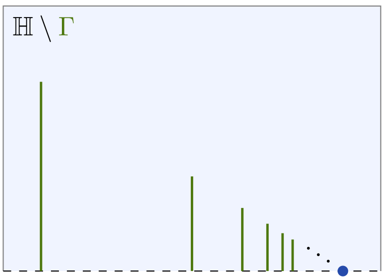

Fig. 1 Illustration of an infinite admissible multislit Γ, consisting of infinitely many (*green*) 1 slits accumulating towards a point on the boundary from the *left* (colour figure online)

Proposition 1.2 *For each hull A, there is a unique biholomorphism $g_A : \mathbb{H} \setminus A \to \mathbb{H}$ satisfying* (1.1). *Moreover,* $\lim_{z\to\infty} z(g_A(z) - z)$ *exists, and is called the half-plane capacity* $\operatorname{hcap}(A)$ *of A.*

Furthermore, we need the following notation.

Definition 1.3 Let Γ be a multislit. We call $\gamma = (\gamma_j)_j$ a parametrisation of Γ if γ_j is a parametrisation of Γ_j for every j. By a slight abuse of notation, we let $\Gamma_t := \bigcup_j \gamma_j(0, t]$.[1] We call a parametrisation γ of Γ a Loewner parametrisation of Γ if $t \mapsto \operatorname{hcap}(\Gamma_t)$ is Lipschitz continuous for $t \in [0, 1]$.

Corollary 3.3 will show that these "normalised" parametrisations can be achieved to encode a given multislit Γ in a Loewner equation. Given a multislit Γ, we write $g_t := g_{\Gamma_t}$, and denote by h_t the inverse of g_t. Consequently, we also denote by h_Γ the inverse of g_Γ. Our main result is the following:

Theorem 1.4 *Let Γ be an admissible multislit, and $(\gamma_j)_j$ a Loewner parametrisation. Then, there exist so-called driving functions $U_j : [0, 1] \to \mathbb{R}$, a constant $L > 0$, and so-called weight functions $\lambda_j : [0, 1] \to \mathbb{R}_+ \cup \{0\}$, where each U_j is continuous, and λ_j has the properties:*

(1) $0 \le \lambda_j(t) \le L$ almost everywhere,
(2) Each λ_j is measurable,
(3) $\sum_j \lambda_j(t) = \frac{\partial}{\partial t} \operatorname{hcap} \Gamma_t$ almost everywhere,

such that for almost every $t \in [0, 1]$, and all $z \in \mathbb{H} \setminus \Gamma$ it holds that

$$\frac{\partial}{\partial t} g_t(z) = \sum_j \frac{\lambda_j(t)}{g_t(z) - U_j(t)}, \quad g_0(z) = z. \tag{1.2}$$

Informally speaking, the weight function $\lambda_j(t)$ corresponds to the "speed" in which $\gamma_j(t)$ grows at the time $t \in [0, 1]$, and the driving function $U_j = g_t \circ \gamma_j$ keeps track of the position of the tip $\gamma_j(t)$ of the slit $\gamma_j(0, t]$.

Moreover, we would like to mention that the a.e. differentiability part of the theorem above is a consequence of a more general phenomenon occurring in Loewner theory, cf. [7, Theorem 3], [4, Theorem 1.1]. Furthermore, in the 1-slit case, one recovers from Theorem 1.4 the ordinary Loewner equation, which relates the existence of the derivative $\dot{g}_t$ to the existence of the weight functions λ_j.

Remark 1.5 Let us mention that the n-slit case in Theorems 1.4 and 5.2 has been proved in [1, Theorem 2.36, 2.54] (see also [2,3] for results in the radial case). Moreover, Theorem 1.4 is best possible in the sense that $t \mapsto g_t(z)$ $(z \in \mathbb{H} \setminus \Gamma)$ is in general not (everywhere) differentiable, see Remark 5.3.

The paper is structured as follows. First, we collect some basic tools in Sect. 2. These are needed to study the difference quotient of $t \mapsto g_t(z)$, where g_t is the map

[1] We do not specify the range of j, since it is irrelevant for our approach whether the multislit is an n slit or not.

from Theorem 1.4. To this end, we use classical results from (geometric) function theory, e.g., the theory of prime ends, kernel convergence, normal families, and the Nevanlinna representation. In Sect. 3, we construct the driving functions. In Sect. 4, we construct functions that will later on turn out to be the weight functions. These are the main problems when passing from the 1-slit Loewner equation to the multislit version. For overcoming this obstacle, we use tools from Lipschitz analysis. Eventually, we put the pieces together in Sect. 5 to derive the results stated above.

2 Preliminaries and basic tools

Let us mention, for the sake of clarity, that we equip $\hat{\mathbb{C}}$ with its natural topology; in particular, the boundary of $\mathbb{H}$ is understood to contain the point ∞. Moreover, we fix an arbitrary admissible multislit Γ with Loewner parametrisation γ throughout Sects. 2–5.

The next theorem deals with biholomorphic extensions of the maps g_Γ and h_Γ, and is a direct consequence of the well-known Schwarz reflection principle combined with the classical theory of prime ends (cf. [12]).

Theorem 2.1 *Let Γ be an admissible multislit. Then, any given map g_Γ extends to a biholomorphism from $\hat{\mathbb{C}}\backslash(\mathrm{clos}(\Gamma)\cup\Gamma^*)$ onto $\hat{\mathbb{C}}\backslash\mathcal{C}$ for some $\mathcal{C}\subseteq\mathbb{R}$, where Γ^* denotes the complex conjugate of the set Γ.*

Moreover, we need the following fact about the relationship between the size of the preimage of a boundary point and the topology of the boundary of the image domain of a biholomorphism (see [12, Chapter 2, Proposition 2.5]).

Theorem 2.2 *Let $h:\mathbb{H}\to D$ be a biholomorphism, where D has locally connected boundary ∂D. Fix $w_0\in\partial D$, and consider the preimage $\mathcal{W}:=h^{-1}(\{w_0\})\subseteq\hat{\mathbb{R}}:=\mathbb{R}\cup\{\infty\}$ of w_0. Then, there is a bijection $\mathcal{C}\mapsto h(\mathcal{C})$ between the connected components $\mathcal{C}$ of $\hat{\mathbb{R}}\backslash\mathcal{W}$ and the connected components of $\partial D\backslash\{w_0\}$. In particular, $\mathcal{W}$ consists of precisely n (pairwise distinct) points if $\partial D\backslash\{w_0\}$ has n connected components.*

Let us mention that Loewner remarked that a certain group property was essential for his approach to derive his equation.[2] Therefore, similar to Loewner, we shall study $\varphi_{t,T}:=g_t\circ h_T$, where $t,T\in[0,1]$. The function $\varphi_{t,T}$ is often easier to handle than g_t, inter alia, since, as we shall see, it admits a continuous extension to $\mathbb{R}$ for $t\leq T$. To this end, it is convenient to introduce some notation which we shall use in what follows without further mention.

Definition 2.3 Let Γ be an admissible multislit with parametrisation γ. Take $0\leq t\leq T\leq 1$, and let $j\in\mathbb{N}$ be given. We define (see Fig. 2)

[2] In [11, p. 1], Loewner's exact words were: "Das charakteristische Merkmal der angewandten Untersuchungsmethode besteht in der Ausnützung des Umstandes, daß bei Zusammensetzung von schlichten konformen Abbildungen wieder eine schlichte Abbildung entsteht, daß also die schlichten Abbildungen eine Gruppe bilden." In English (translated by the authors): the characteristic property of the method applied here is the exploitation of the fact that the composition of two schlicht functions is, again, a schlicht function, that is, the schlicht mappings form a group.

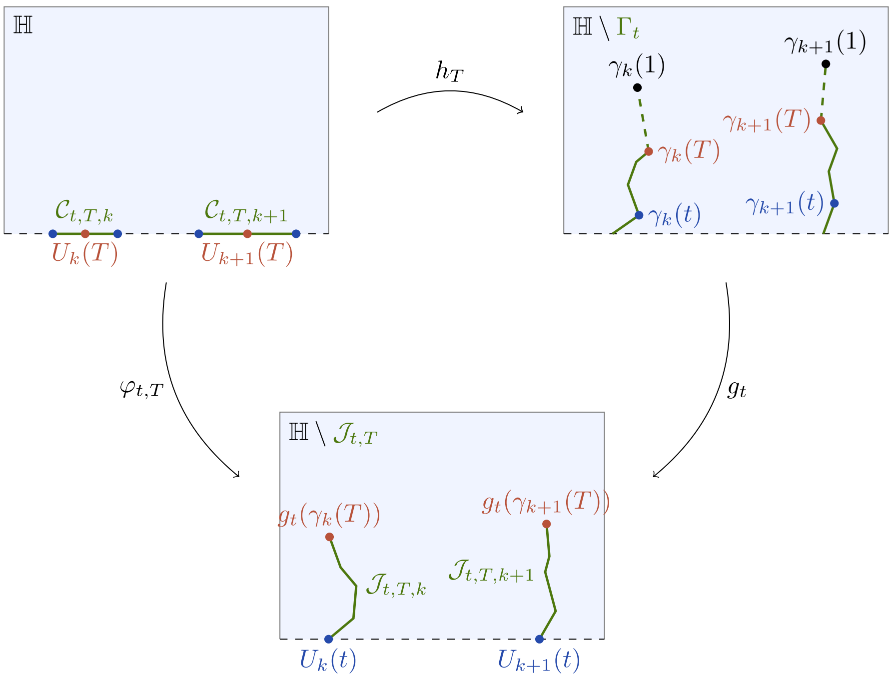

Fig. 2 Similarly *coloured symbols* are mapped onto each other. For instance, the *green-coloured* sets $\mathcal{C}_{t,T,k}$ are first formed to a slit by h_T, and then, the map g_t "bites" a piece away from the slit $\gamma_j\ (0, T]$ and manipulates the remainder slit $\gamma_j\ (t, T]$ to form a silt $\mathcal{J}_{t,T,k}$ (see also [6, Fig. 1] (colour figure online)

- $\mathcal{J}_{t,T,j} := g_t(\gamma_j\ (t, T]) \subseteq \mathbb{H}$,
- $\overline{\mathcal{J}}_{t,T,j} := g_t(\gamma_j[t, T])$ ($g_t(\gamma_j(t)) =: U_j(t)$ is well-defined by Theorem 2.2),
- $\mathcal{J}_{t,T} := \bigcup_j \mathcal{J}_{t,T,j}$,
- $\overline{\mathcal{J}}_{t,T} := \bigcup_j \overline{\mathcal{J}}_{t,T,j}$,
- $\mathcal{C}_{t,T} := \bigcup_j \mathcal{C}_{t,T,j} \subseteq \mathbb{R}$ where $\mathcal{C}_{t,T,j}$ is the preimage of $\gamma_j\ [t, T]$ under h_T (in the sense of Theorem 2.2, and observe that the normalisation in Proposition 1.2 implies that the point ∞ is not contained in $\mathcal{C}_{t,T,j}$).

We can deduce the following properties for these quantities.

Lemma 2.4 *Let $t, T \in [0, 1]$, and $t \leq T$. Then,*

(1) the function $\varphi_{t,T} : \mathbb{H} \to \mathbb{H}\backslash\mathcal{J}_{t,T}$ admits a continuous extension to the boundary,
(2) the sets $\mathcal{C}_{t,T,k}$ are pairwise disjoint intervals.

Proof We first prove (1). Recall that a biholomorphism from $\mathbb{H}$ onto a domain D admits a continuous extension if and only if ∂D is locally connected (see [12, Theorem 2.1]). To prove (1), it suffices to show that $\mathcal{J}_{t,T}$ is locally path-connected as $\partial\left(\mathbb{H}\backslash\mathcal{J}_{t,T}\right) = \hat{\mathbb{R}} \cup \mathcal{J}_{t,T}$. Therefore, we only need to show that any given $\mathcal{J}_{t,T,k}$ can be separated from $\mathrm{clos}(\mathcal{J}_{t,T}\backslash\mathcal{J}_{t,T,k})$ by some neighbourhood U. However, this is evident, since Γ was assumed to be an admissible multislit.

We now show (2). By (1), the map $\varphi_{t,T}$ extends continuously to $\hat{\mathbb{R}}$. Note that by the path-connectedness of $\mathbb{H}\backslash\mathcal{J}_{t,T}$, we can consider a simple curve J_k^- which connects the tip of a given $\mathcal{J}_{t,T,k}$, i.e., the point $g_t(\gamma_k(T))$, with its starting point, i.e., the point $g_t(\gamma_k(t))$, from the left.[3] The preimage $\tilde{J}_k^- := \varphi_{t,T}^{-1}(J_k^-)$ is a simple curve in $\mathbb{H}$ that connects two distinct boundary points $\alpha, \beta \in \mathbb{R}$. Denote by ω_k^-, Ω_k^- the interior of $[\alpha, \beta] \cup \tilde{J}_k^-$, and of $\mathcal{J}_{t,T,k} \cup J_k^-$, respectively. We can extend the homoemorphism $\varphi_{t,T}|_{\omega_k^-} : \omega_k^- \to \Omega_k^-$ to a homoemorphism from $\operatorname{clos}\omega_k^-$ onto $\operatorname{clos}\Omega_k^-$. Then, the preimage of $\mathcal{J}_{t,T,k}$ under $\varphi_{t,T}|_{\operatorname{clos}\omega_k^-}$ has to be the interval $[\alpha, \beta]$. By applying the same reasoning to a curve J_k^+ that connects the tip of $\mathcal{J}_{t,T,k}$ with its starting point from the right, we get that the preimage of $\mathcal{J}_{t,T,k}$ under $\varphi_{t,T}|_{\operatorname{clos}\omega_k^+}$ is an interval of the form $\left[\beta, \alpha'\right]$. In view of Theorem 2.2, we find that $\mathcal{C}_{t,T,j} = [\alpha, \beta] \cup \left[\beta, \alpha'\right]$, so $\mathcal{C}_{t,T,j}$ is an interval. Theorem 2.2 yields that these intervals are disjoint. □

Furthermore, there is a simple but crucial integral representation for $\varphi_{t,T}$:

Lemma 2.5 *Let $s \in [0, 1]$, and choose $t, T \in [0, s]$, such that $t \leq T$. Then,*

(1) it holds that $\varphi_{t,T} \circ \varphi_{T,s} = \varphi_{t,s}$,
(2) for all $z \in \mathbb{H}$:

$$\varphi_{t,T}(z) = z + \frac{1}{\pi}\int_{\mathcal{C}_{t,T}} \frac{\operatorname{Im}(\varphi_{t,T}(\xi))}{\xi - z}\,\mathrm{d}\xi = z + \frac{1}{\pi}\sum_k \int_{\mathcal{C}_{t,T,k}} \frac{\operatorname{Im}(\varphi_{t,T}(\xi))}{\xi - z}\,\mathrm{d}\xi,$$

(3) we have that

$$\operatorname{hcap}(\Gamma_T) - \operatorname{hcap}(\Gamma_t) = \frac{1}{\pi}\int_{\mathcal{C}_{t,T}} \operatorname{Im}(\varphi_{t,T}(\xi))\,\mathrm{d}\xi.$$

Proof (1) is a simple calculation.

The first equality in (2) follows from the well-known Nevanlinna representation, cf. [13, Theorem 5.3] or [7], via the Stieltjes inversion formula. The second equality is due to the decomposition

$$\mathcal{C}_{t,T} = \bigcup_j \mathcal{C}_{t,T,j},$$

and the fact that $\mathcal{C}_{t,T,j}$ are pairwise disjoint, by Lemma 2.4 (2).

To show (3), we exploit [10, Eq. (3.8)] to get $\operatorname{hcap}(\Gamma_T) - \operatorname{hcap}(\Gamma_t) = \operatorname{hcap}(g_t(\Gamma_T\backslash\Gamma_t))$. Furthermore, since

$$g_t(\Gamma_T\backslash\Gamma_t) = \bigcup_j g_t(\gamma_j\,(t, T]) = \bigcup_j \mathcal{J}_{t,T,j} = \mathcal{J}_{t,T},$$

[3] The distinction between "left" and "right" is obtained in the following manner: Extend the simple curve $\mathcal{J}_{t,T,k}$ to ∞, thereby cutting $\hat{\mathbb{H}}\backslash\mathcal{J}_{t,T}$ into two disjoint path-connected components, the "left" component being the one, whose boundary contains an interval $(-\infty, b]$, where $b > 0$ is some real number.

expanding the left-hand side and the right-hand side of Lemma 2.5 (2) into Laurent series, and comparing coefficients yields the claim. □

Now, we are in the position to conclude a simple, but very useful lemma.

Lemma 2.6 *Let* $z \in \mathbb{H}$*, and* $0 \le t \le T \le s \le 1$ *be fixed. Then, the increment* $\varphi_{T,s}(z) - \varphi_{t,s}(z)$ *equals*

$$\sum_j \left(\operatorname{Re} \frac{1}{\varphi_{T,s}(z) - \xi_j} + i \operatorname{Im} \frac{1}{\varphi_{T,s}(z) - \xi_j'} \right) \frac{1}{\pi} \int_{\mathcal{C}_{t,T,j}} \operatorname{Im} \varphi_{t,T}(\xi)\, \mathrm{d}\xi$$

for some suitably chosen $\xi_j, \xi_j' \in \mathcal{C}_{t,T,j}$.[4]

Proof Using Lemma 2.5 (1) and (2), we can write

$$\varphi_{T,s}(z) - \varphi_{t,s}(z) = \frac{1}{\pi} \int_{\mathcal{C}_{t,T}} \frac{\operatorname{Im}(\varphi_{t,T}(\xi))}{\varphi_{T,s}(z) - \xi}\, \mathrm{d}\xi = \frac{1}{\pi} \sum_j \int_{\mathcal{C}_{t,T,j}} \frac{\operatorname{Im}(\varphi_{t,T}(\xi))}{\varphi_{T,s}(z) - \xi}\, \mathrm{d}\xi.$$

Considering the integral

$$\int_{\mathcal{C}_{t,T,j}} \frac{\operatorname{Im}(\varphi_{t,T}(\xi))}{\varphi_{T,s}(z) - \xi}\, \mathrm{d}\xi,$$

the claim follows after splitting the integrand into real and imaginary parts, and applying the mean value theorem. □

Using the previous lemma, we can derive a crucial fact about the differentiability of $\tau \mapsto g_\tau(z)$. Namely, we have:

Corollary 2.7 *Let* $0 \le t \le T \le s \le 1$*, and* γ *be a parametrisation of an admissible multislit* Γ*. Then*

$$|\varphi_{T,s}(z) - \varphi_{t,s}(z)| \le 2 \frac{\operatorname{hcap}(\Gamma_T) - \operatorname{hcap}(\Gamma_t)}{\operatorname{Im} z}.$$

In particular, if $t \mapsto \operatorname{hcap}(\Gamma_t)$ *is Lipschitz continuous, then, for any fixed* $z \in \mathbb{H}$*,* $[0, s] \ni \tau \mapsto \varphi_{\tau,s}(z)$ *has the same property. In this case,* $\tau \mapsto \varphi_{\tau,s}(z)$ *(*$\tau \in [0, s]$*) is differentiable almost everywhere.*

Proof Lemma 2.6 yields

$$\varphi_{T,s}(z) - \varphi_{t,s}(z) = \sum_j \left(\operatorname{Re} \frac{1}{\varphi_{T,s}(z) - \xi_j} + i \operatorname{Im} \frac{1}{\varphi_{T,s}(z) - \xi_j'} \right) \frac{1}{\pi} \int_{\mathcal{C}_{t,T,j}} \operatorname{Im} \varphi_{t,T}(\xi)\, \mathrm{d}\xi.$$

[4] Of course, ξ_j, ξ_j' depend on z, t, T. However, for the ease of notation, we drop these dependencies from the notation. The only exception to this will be the proof of Theorem 1.4.

Combining the formula above with the estimate

$$\left|\varphi_{T,s}(z)-\xi\right| \geq \left|\operatorname{Im}\bigl(\varphi_{T,s}(z)-\xi\bigr)\right| = \operatorname{Im}\varphi_{T,s}(z), \quad \xi \in \mathcal{C}_{t,T},$$

gives

$$\left|\varphi_{T,s}(z)-\varphi_{t,s}(z)\right| \leq \frac{2}{\pi}\frac{1}{\operatorname{Im}\varphi_{T,s}(z)}\int_{\mathcal{C}_{t,T}} \operatorname{Im}\varphi_{t,T}(\xi)\,\mathrm{d}\xi.$$

Lemma 2.5 yields $\operatorname{Im}\varphi_{T,s}(z) \geq \operatorname{Im} z$, and hence

$$\left|\varphi_{T,s}(z)-\varphi_{t,s}(z)\right| \leq \frac{2}{\operatorname{Im} z}\frac{1}{\pi}\int_{\mathcal{C}_{t,T}} \operatorname{Im}\varphi_{t,T}(\xi)\,\mathrm{d}\xi = 2\frac{\operatorname{hcap}(\Gamma_T)-\operatorname{hcap}(\Gamma_t)}{\operatorname{Im} z}.$$

The additional assertion follows from Rademacher's theorem. □

3 Driving Functions

By the previous lemma, there exists

$$\Xi_j := \Xi_j(z,t,T) = \operatorname{Re}\frac{1}{\varphi_{T,s}(z)-\xi_j} + i\operatorname{Im}\frac{1}{\varphi_{T,s}(z)-\xi_j'}, \quad \xi_j, \xi_j' \in \mathcal{C}_{t,T,j}, \tag{3.1}$$

such that we can write

$$\frac{\varphi_{T,s}(z)-\varphi_{t,s}(z)}{T-t} = \sum_j \Xi_j \frac{1}{T-t}\frac{1}{\pi}\int_{\mathcal{C}_{t,T,j}} \operatorname{Im}\varphi_{t,T}(\xi)\,\mathrm{d}\xi.$$

Now, it is natural to proceed by showing that both factors in the product

$$\Xi_j \times \frac{1}{\pi(T-t)}\int_{\mathcal{C}_{t,T,j}} \operatorname{Im}\varphi_{t,T}(\xi)\,\mathrm{d}\xi$$

converge as $|t-T|$ tends to 0. In light of this thought, we prove, in the first part of the next section, that $\mathcal{C}_{t,T,j}$ tends[5] to some point $U_j(T)$ as $t \nearrow T$, or to $U_j(t)$ for $T \searrow t$. After that, we turn our attention to the more delicate problem of deciding whether

$$\frac{1}{T-t}\frac{1}{\pi}\int_{\mathcal{C}_{t,T,j}} \operatorname{Im}\varphi_{t,T}(\xi)\,\mathrm{d}\xi$$

exists for $t \nearrow T$, or $T \searrow t$. Moreover, we need the following simple estimate which controls how much g_A, for a given hull A, can differ from the identity map $z \mapsto z$.

[5] We say that a sequence (M_k) of sets tends to the point p if $\sup_{m\in M_k}|m-p|$ converges to 0.

Lemma 3.1 *Let A be a hull. Then, it holds that*

$$\sup_{z\in\mathbb{H}\setminus A} |g_A(z) - z| = \sup_{w\in\mathbb{H}} |w - h_A(w)| \leq 3\,\mathrm{diam}\,(A)\,.$$

In particular, for every 1-slit Γ and every fixed $t_0 \in [0, 1]$, the maps g_{Γ_t} converge locally uniformly to the identity mapping on $\mathbb{H}$ as $t \to 0$.

Proof If $0 \in \mathrm{clos}(A)$, then the claim follows from [10, Corollary 3.44]. By taking $c \in \mathbb{R}$, such that $B := A - c$ satisfies $0 \in \mathrm{clos}(B)$, we can deduce the general case from $g_B(z) = g_A(z + c) - c$ and $g_A(z + c) - (z + c) = g_B(z) - z$. $\square$

Theorem 3.2 *If T respectively t is fixed, then, for fixed k, there is a $\delta > 0$, such that for all $t \in [T - \delta, T]$, respectively, $T \in [t, t + \delta]$, we can separate $\mathcal{C}_{t,T,k}$ from $\mathcal{C}_{t,T}\setminus\mathcal{C}_{t,T,k}$ by a (fixed) open set for any k.*

Proof We consider the case $t \nearrow T$. Since $\mathcal{C}_{t,T,k} = h_T^{-1}(\gamma_k[t, T])$ is becoming smaller as $t \nearrow T$, it suffices to separate $\mathcal{C}_{t,T,k}$ from $\mathcal{C}_{t,T}\setminus\mathcal{C}_{t,T,k}$ for some t. Note that we can separate $\mathcal{J}_{t,T,k}$ from $\mathcal{J}_{t,T}\setminus\mathcal{J}_{t,T,k}$. Hence, by continuity of $\varphi_{t,T}$, the assertion is clear in the case of $t \nearrow T$. In the remaining case, we can separate $\mathcal{J}_{t,T,k} = g_t(\gamma_k(t, T])$, which are getting smaller from $\mathcal{J}_{t,T}\setminus\mathcal{J}_{t,T,k}$ by a simple curve J. Using Carathéodory's Kernel theorem,[6] we get that the simple curves $\tilde{J}_{t,T,k} := \varphi_{t,T}^{-1} \circ J$ converge to J, which separates $\mathcal{C}_{t,T,k}$ from $\mathcal{C}_{t,T}\setminus\mathcal{C}_{t,T,k}$. $\square$

The next corollary demonstrates how one can normalise a given parametrisation of a multislit.

Corollary 3.3 *Let Γ be an admissible multislit with parametrisation $\gamma = (\gamma_j)$, such that $t \mapsto \mathrm{hcap}\,\Gamma_t$ is strictly increasing. Then, there exists a Loewner parametrisation of Γ.*

Proof Let $L := \mathrm{hcap}\,\Gamma$. Note that, for $R > 0$ sufficiently large, one has the representation

$$f : [0, 1] \to [0, L]\,, \quad t \mapsto \mathrm{hcap}\,\Gamma_t = \frac{1}{2\pi i}\int_{\partial B_R(0)} g_{\Gamma_t}(\xi)\,\mathrm{d}\xi\,.$$

By Carathéodory's kernel theorem, f is continuous. By assumption, f is strictly increasing, and hence a homoemorphism. Therefore, $\tilde{\gamma}_j(t) := (\gamma_j \circ f^{-1})\,(Lt)\,,\ t \in [0, 1]$ satisfies the requirements of Definition 1.3, because of

$$\mathrm{hcap}\left(\bigcup_j \tilde{\gamma}_j\,(0, t]\right) = \mathrm{hcap}\left(\bigcup_j \gamma_j\,\left(0, f^{-1}\,(Lt)\right]\right) = f(f^{-1}(Lt)) = Lt.$$

$\square$

[6] By the kernel theorem, we refer to the following statement. Let $\Omega_n \subset \mathbb{C}$ denote a sequence of domains. Let X and Ω denote domains, where X is unbounded. If Ω_n converges in the kernel sense to a domain Ω, and the sequence $f_n : X \to \Omega_n$ of biholomorphisms satisfying (1.1) is locally bounded, then it converges locally uniformly to the unique biholomorphism $f : X \to \Omega$ with (1.1). This can be proved in the same manner as the ordinary kernel convergence theorem.

Now, we can characterise the limit behaviour of $\mathcal{C}_{t,T,k}$ and $\mathcal{J}_{t,T,k}$ as $t \nearrow T$, or $T \searrow t$ as follows:

Lemma 3.4 *The following statements hold:*

(1) $\mathcal{C}_{t,T,k}$ *shrinks*[7] *to* $U_k(T)$ *as* $t \nearrow T$.
(2) $\mathcal{C}_{t,T,k}$ *tends to* $U_k(t)$ *as* $T \searrow t$.
(3) $\overline{\mathcal{J}}_{t,T,k}$ *shrinks to* $U_k(t)$ *as* $T \searrow t$.
(4) $\mathcal{J}_{t,T,k}$ *tends to* $U_k(T)$ *as* $t \nearrow T$.

To prove this, we recall the following lemma (cf. [6, Lemma 2.1]):

Lemma 3.5 *Let* $K_1, K_2 \subset \mathbb{C}$ *be compact sets, and let* $\varphi : \hat{\mathbb{C}}\backslash K_1 \to \hat{\mathbb{C}}\backslash K_2$ *be biholomorphic, moreover, denote by* $B_r(c)$ *the closed ball of radius* $r > 0$ *and centre* $c \in \mathbb{C}$. *If* φ *satisfies* (1.1)*, then*

(1) $K_1 \subseteq B_{2\,\mathrm{diam}(K_2)}(w_0)$ *for every* $w_0 \in K_2$,
(2) $K_2 \subseteq B_{2\,\mathrm{diam}(K_1)}(z_0)$ *for every* $z_0 \in K_1$.

Proof of Lemma 3.4 (1) and (3): By definition, we have $\mathcal{C}_{t,T,k} = h_T^{-1}(\gamma_k[t,T])$ and $\overline{\mathcal{J}}_{t,T,k} = g_t(\gamma_k[t,T])$, so the assertions are clear.

(2): Let $\mathfrak{J}_{k,t,T}$ denote the union of all $\mathcal{J}_{t,T,j}$ with $j \neq k$. We consider the function $g_{\mathfrak{J}_{k,t,T}}$. Then, we have

$$h_{\tilde{\mathcal{J}}_{t,T,k}} = g_{\mathfrak{J}_{k,t,T}} \circ \varphi_{t,T} \quad \text{where} \quad \tilde{\mathcal{J}}_{t,T,k} := g_{\mathfrak{J}_{k,t,T}}(\mathcal{J}_{t,T,k}).$$

By Theorem 2.1, the map $h_{\tilde{\mathcal{J}}_{t,T,k}}$ extends to a biholomorphism from $\mathbb{C}\backslash\tilde{\mathcal{C}}_{t,T,k}$ onto $\mathbb{C}\backslash(\mathrm{clos}\,\tilde{\mathcal{J}}_{t,T,k} \cup \tilde{\mathcal{J}}^*_{t,T,k})$, where

$$h^{-1}_{\tilde{\mathcal{J}}_{t,T,k}}(\mathrm{clos}\,\tilde{\mathcal{J}}_{t,T,k}) =: \tilde{\mathcal{C}}_{t,T,k} \subseteq \mathbb{R}.$$

Because of $h^{-1}_{\tilde{\mathcal{J}}_{t,T,k}} \circ g_{\mathfrak{J}_{k,t,T}} = \varphi^{-1}_{t,T}$ and $\mathcal{C}_{t,T,k} = \varphi^{-1}_{t,T}(\overline{\mathcal{J}}_{t,T,k})$, it turns out that

$$\mathcal{C}_{t,T,k} = h^{-1}_{\tilde{\mathcal{J}}_{t,T,k}}(g_{\mathfrak{J}_{k,t,T}}(\overline{\mathcal{J}}_{t,T,k})) = h^{-1}_{\tilde{\mathcal{J}}_{t,T,k}}(\mathrm{clos}\,\tilde{\mathcal{J}}_{t,T,k}) = \tilde{\mathcal{C}}_{t,T,k}. \tag{3.2}$$

We claim that $\tilde{\mathcal{J}}_{t,T,k}$ tends to $U_k(t)$ as $T \searrow t$. By Lemma 3.5 (1) with $\varphi := h_{\tilde{\mathcal{J}}_{t,T,k}}$

$$\mathcal{C}_{t,T,k} = \tilde{\mathcal{C}}_{t,T,k} \subseteq B_{2\,\mathrm{diam}(\tilde{\mathcal{J}}_{t,T,k})}(g_{\mathfrak{J}_{k,t,T}}(U_k(t))). \tag{3.3}$$

Carathéodory's kernel theorem yields that $g_{\mathfrak{J}_{k,t,T}}$ converges locally uniformly on $\mathbb{C}\backslash(\overline{\mathfrak{J}}_{k,t,T} \cup \mathfrak{J}^*_{k,t,T})$ to the identity mapping on $\mathbb{C}\backslash\bigcup_{j\neq k} U_j(t)$ as $T \searrow t$, i.e., we have[8]

[7] A sequence of sets (M_k) shrinks to a point p if $\bigcap_{k=1}^{\infty} M_k = \{p\}$, and $M_{k+1} \subseteq M_k$.

[8] Note that $U_j(t) \neq U_k(t)$ for $j \neq k$ and $U_k(t) \in \mathbb{R}$ which implies $U_j(t) \in \mathbb{C}\backslash(\overline{\mathfrak{J}}_{k,t,T} \cup \overline{\mathfrak{J}}^*_{k,t,T})$.

$$g_{\mathfrak{J}_{k,t,T}}\big(U_k(t)\big) \to U_k(t) \quad \text{as} \quad T \searrow t.$$

Therefore, (2) is proved if we show that $\operatorname{diam}(\tilde{\mathcal{J}}_{t,T,k})$ converges to zero as $T \searrow t$. Using Lemma 3.1

$$\begin{aligned}\operatorname{diam}(\tilde{\mathcal{J}}_{t,T,k}) &= \sup_{z,w\in\mathcal{J}_{t,T,k}} |g_{\mathfrak{J}_{k,t,T}}(z) - g_{\mathfrak{J}_{k,t,T}}(w)| \\ &\leq 2 \sup_{z\in\mathcal{J}_{t,T,k}} |g_{\mathfrak{J}_{k,t,T}}(z) - z| + \operatorname{diam} \mathcal{J}_{t,T,k}.\end{aligned}$$

Due to $\overline{\mathcal{J}}_{t,T,k} \subseteq \mathbb{C}\backslash(\overline{\mathfrak{J}}_{k,t,T} \cup \mathfrak{J}^*_{k,t,T})$ and Carathéodory's kernel theorem, $g_{\mathfrak{J}_{k,t,T}}$ converges uniformly on $\overline{\mathcal{J}}_{t,T',k}$, for some fixed $T' \in (t, 1]$, to $\mathrm{id}_{\overline{\mathcal{J}}_{t,T',k}}$. Hence

$$\sup_{z\in\mathcal{J}_{t,T,k}} |g_{\mathfrak{J}_{k,t,T}}(z) - z| \to 0 \quad \text{as} \quad T \searrow t.$$

By (3), $\overline{\mathcal{J}}_{t,T,k}$ shrinks to $U_k(t)$ as $T \searrow t$, which implies $\operatorname{diam}(\mathcal{J}_{t,T,k}) \to 0$ as $T \searrow t$. Therefore, we get $\operatorname{diam}(\tilde{\mathcal{J}}_{t,T,k}) \to 0$ as $T \searrow t$. By (3.3), we infer that $\mathcal{C}_{t,T,k}$ tends to $U_k(t)$.

(4): Lemma 3.5 (2) yields, similarly as above, the inclusion

$$\tilde{\mathcal{J}}_{t,T,k} \subseteq B_{2\operatorname{diam}(\mathcal{C}_{t,T,k})}(U_k(T)),$$

and hence $\tilde{\mathcal{J}}_{t,T,k}$ tends to $U_k(T)$ as $t \nearrow T$.

To conclude that $\mathcal{J}_{t,T,k}$ tends to $U_k(T)$ as $t \nearrow T$, it is enough to show that for any given sequence $t_n \nearrow T$, there is a subsequence for which this claim holds. After choosing an appropriate subsequence which we denote again by $(t_n)_n$, the maps $g_{\mathfrak{J}_{k,t_n,T}}$ tend to some schlicht g on a compact set K containing $\mathcal{J}_{t,T,k}$. Therefore, $\tilde{\mathcal{J}}_{t_n,T,k} = g_{\mathfrak{J}_{k,t_n,T}}(\mathcal{J}_{t_n,T,k})$ implies by taking $n \to \infty$ that $\mathcal{J}_{t_n,T,k}$ has to converge to some point. By arguing similarly, for $k' \neq k$, we get that for some appropriate subsequence of $(t_n)_n$, all $\mathcal{J}_{t_n,T,k'}$ converge to points. Hence, $g_{\mathfrak{J}_{k,t_n,T}}$ converges to $\mathrm{id}_{\mathbb{C}}$. This implies that $\mathcal{J}_{t_n,T,k}$ tends to $U_k(T)$. □

The result above immediately implies the following important corollary.

Corollary 3.6 *Let Γ be an admissible multislit with Loewner parametrisation γ. Then, the driving functions $U_j : [0, 1] \to \mathbb{R}$, given by Definition 2.3, are continuous.*

Proof For $0 \leq t \leq T \leq 1$, we get $U_j(t) \in \mathcal{C}_{t,T,j}$ and $U_j(T) \in \overline{\mathcal{J}}_{t,T,j}$. Therefore, Lemma 3.4 implies the right continuity and left continuity of $t \mapsto U_j(t)$, and guarantees, furthermore, that $\lim_{t\nearrow\tau} U_j(t) = U_j(\tau) = \lim_{T\searrow\tau} U_j(T)$. □

4 Weight Functions

Now, we are in the position to define the weight functions from Theorem 1.4. In view of (1.2), we define these functions, roughly speaking, as the "residues" of the derivative $z \mapsto \dot{\varphi}_{t,1}(z)$ in the "dynamical boundary points" $U_k(t)$. However, doing so requires some involved analysis.

Theorem 4.1 *Let γ be a Loewner parametrisation of an admissible multislit Γ. If $\tau \mapsto \varphi_{\tau,1}(u_k)$ is differentiable at t on a sequence $(u_k)_k$ converging to a point in the set $\varphi_{t,1}^{-1}(U_j(t))$, then the limits*

$$\lim_{\tau \nearrow t} \frac{1}{t-\tau} \frac{1}{\pi} \int_{\mathcal{C}_{\tau,t,j}} \operatorname{Im} \varphi_{\tau,t}(\xi)\, \mathrm{d}\xi, \quad \lim_{\tau \searrow t} \frac{1}{\tau - t} \frac{1}{\pi} \int_{\mathcal{C}_{t,\tau,j}} \operatorname{Im} \varphi_{t,\tau}(\xi)\, \mathrm{d}\xi \tag{4.1}$$

exist, and are both equal to

$$\lim_{k\to\infty} \dot{\varphi}_{t,1}(u_k) \left[\varphi_{t,1}(u_k) - U_j(t)\right] =: \lambda_j(t). \tag{4.2}$$

In particular, λ_j is defined for almost every $t \in [0, 1]$.

Proof By Corollary 2.7, the second assertion is an immediate consequence of the first one, which we prove in several steps. To simplify the notation, we assume $j = 1$. The case $j \neq 1$ can be treated similarly.

(i) Let $0 \le \tau \le t$. Using Lemma 2.6 with the abbreviation

$$\xi_{j,k}(\tau) := \operatorname{Re} \frac{1}{\varphi_{t,1}(u_k) - \xi_j(\tau)} + i \operatorname{Im} \frac{1}{\varphi_{t,1}(u_k) - \xi_j'(\tau)}, \quad \xi_j(\tau), \xi_j'(\tau) \in \mathcal{C}_{\tau,t,j},$$

we find that

$$D_k(t,\tau) := \frac{\varphi_{t,1}(u_k) - \varphi_{\tau,1}(u_k)}{t-\tau} = \sum_j \xi_{j,k}(\tau) \frac{1}{t-\tau} \frac{1}{\pi} \int_{\mathcal{C}_{\tau,t,j}} \operatorname{Im}(\varphi_{\tau,t}(\xi))\, \mathrm{d}\xi. \tag{4.3}$$

Since $\operatorname{Im} \xi_{j,k}(\tau) \neq 0$ is strictly negative, we can define the quantity

$$\Delta_k(\tau) := \frac{\varphi_{t,1}(u_k) - \varphi_{\tau,1}(u_k)}{(t-\tau)\xi_{1,k}(\tau)}, \quad \tau \in [0, t). \tag{4.4}$$

Now let $w_{k,j}(\tau) := \xi_{j,k}(\tau)\xi_{1,k}(\tau)^{-1}$, and note that

$$\Delta_k(\tau) = \frac{1}{t-\tau} \frac{1}{\pi} \int_{\mathcal{C}_{\tau,t,1}} \operatorname{Im} \varphi_{\tau,t}(\xi)\, \mathrm{d}\xi + \sum_{j\neq 1} w_{k,j}(\tau) \frac{1}{t-\tau} \frac{1}{\pi} \int_{\mathcal{C}_{\tau,t,j}} \operatorname{Im} \varphi_{\tau,t}(\xi)\, \mathrm{d}\xi. \tag{4.5}$$

Since, by Lemma 3.4, $\xi_j(\tau)$ and $\xi_j'(\tau)$ both converge to $U_j(t)$ as $\tau \nearrow t$, Corollary 2.7 provides the existence of

$$\lim_{\tau \nearrow t} w_{k,j}(\tau) = \frac{\varphi_{t,1}(u_k) - U_1(t)}{\varphi_{t,1}(u_k) - U_j(t)}. \tag{4.6}$$

(ii) We claim that for any $\varepsilon > 0$, there is some $\delta > 0$, and $n_0 \in \mathbb{N}$, such that for all $j \neq 1$ and $\tau \in [t-\delta, t]$, we have $|w_{k,j}(\tau)| < \varepsilon$ for $k \geq n_0$. To see this, let $K := \min\{\mathrm{dist}(\mathcal{C}_{\tau,t,j}, \mathcal{C}_{\tau,t,1}) : j \neq 1\}$, and note that we have $K > 0$, by Theorem 3.2. Therefore, we use the estimate

$$|\xi_{j,k}(\tau)| \leq 2\Big(\min_{\xi\in\mathcal{C}_{\tau,t,j}} |\xi - \varphi_{\tau,1}(u_k)|\Big)^{-1},$$

and

$$\min_{\xi\in\mathcal{C}_{\tau,t,j}} |\xi - \varphi_{\tau,1}(u_k)| \geq \min_{\xi'\in\mathcal{C}_{\tau,t,j},\xi\in\mathcal{C}_{\tau,t,1}} |\xi - \xi'| - \max_{\xi'\in\mathcal{C}_{\tau,t,1}} |\xi' - \varphi_{\tau,1}(u_k)|.$$

Hence, for k large enough, τ sufficiently close to t and $j \neq 1$, we conclude that

$$\min_{\xi\in\mathcal{C}_{\tau,t,j}} |\xi - \varphi_{\tau,1}(u_k)| \geq \frac{K}{2},$$

implying that $|\xi_{j,k}(\tau)| \leq \frac{4}{K}$ and consequently

$$|w_{k,j}(\tau)| = \frac{|\xi_{j,k}(\tau)|}{|\xi_{1,k}(\tau)|} \leq \frac{4}{K|\xi_{1,k}(\tau)|},$$

which shows that $|w_{k,j}(\tau)| < \varepsilon$.

(iii) Next, we claim that the sequence defined by

$$a_k := \lim_{\tau\nearrow t} \Delta_k(\tau) = \dot{\varphi}_{\tau,1}(u_k)(\varphi_{t,1}(u_k) - U_j(t)) \tag{4.7}$$

is a Cauchy sequence. Let L denote the Lipschitz constant of $t \mapsto \mathrm{hcap}(\Gamma_t)$. By employing (4.5), Lemma 2.5 (3), and Step (ii), we deduce that

$$\begin{aligned}
|a_M - a_m| &= \left|\lim_{\tau\nearrow t} \sum_{j\neq 1} (w_{M,j}(\tau) - w_{m,j}(\tau)) \frac{1}{t-\tau}\frac{1}{\pi}\int_{\mathcal{C}_{\tau,t,j}} \mathrm{Im}\,\varphi_{\tau,t}(\xi)\,\mathrm{d}\xi\right| \\
&\leq \limsup_{\tau\nearrow t} \sum_{j\neq 1} |w_{M,j}(\tau) - w_{m,j}(\tau)| \frac{1}{t-\tau}\frac{1}{\pi}\int_{\mathcal{C}_{\tau,t,j}} \mathrm{Im}\,\varphi_{\tau,t}(\xi)\,\mathrm{d}\xi \\
&\leq \frac{4}{K}\left(\limsup_{\tau\nearrow t} |\xi_{1,M}(\tau)|^{-1} + \limsup_{\tau\nearrow t} |\xi_{1,m}(\tau)|^{-1}\right) \\
&\quad \times \limsup_{\tau\nearrow t} \sum_{j\neq 1} \frac{1}{t-\tau}\frac{1}{\pi}\int_{\mathcal{C}_{\tau,t,j}} \mathrm{Im}\,\varphi_{\tau,t}(\xi)\,\mathrm{d}\xi \\
&\leq \frac{4L}{K}\big(|\varphi_{t,1}(u_M) - U_1(t)| + |\varphi_{t,1}(u_m) - U_1(t)|\big),
\end{aligned}$$

thereby proving the claim.

(iv) We let $a := \lim_{k\to\infty} a_k \in \mathbb{C}$, and show that

$$\lim_{\tau\nearrow t} \frac{1}{\pi}\frac{1}{t-\tau}\int_{\mathcal{C}_{\tau,t,1}} \operatorname{Im}\varphi_{\tau,t}(\xi)\,\mathrm{d}\xi$$

exists, and is equal to a.[9] To this end, let $(\tau_m)_m$ be a sequence converging to t from below. We let

$$b_m := \frac{1}{\pi}\frac{1}{t-\tau_m}\int_{\mathcal{C}_{\tau_m,t,1}} \operatorname{Im}\varphi_{\tau_m,t}(\xi)\,\mathrm{d}\xi \in [0,1]\,.$$

With the help of Lemma 2.5 (3), we find that for all $m, k \in \mathbb{N}$

$$\begin{aligned}
|b_m - a| &\le \Bigg|\overbrace{b_m + \sum_{j\neq 1} w_{k,j}(\tau_m)\frac{1}{\pi}\frac{1}{t-\tau_m}\int_{\mathcal{C}_{\tau_m,t,j}} \operatorname{Im}\varphi_{\tau_m,t}(\xi)\,\mathrm{d}\xi}^{\Delta_k(\tau_m)} - a\Bigg| \\
&\quad + \Bigg|\sum_{j\neq 1} w_{k,j}(\tau_m)\frac{1}{t-\tau_m}\frac{1}{\pi}\int_{\mathcal{C}_{\tau_m,t,j}} \operatorname{Im}\varphi_{\tau_m,t}(\xi)\,\mathrm{d}\xi\Bigg| \\
&\le |\Delta_k(\tau_m) - a| + \frac{4}{K|\xi_{1,k}(\tau_m)|}\sum_{j\neq 1}\frac{1}{t-\tau_m}\frac{1}{\pi}\int_{\mathcal{C}_{\tau_m,t,j}} \operatorname{Im}\varphi_{\tau_m,t}(\xi)\,\mathrm{d}\xi \\
&\le |\Delta_k(\tau_m) - a_k| + |a_k - a| + \frac{4L}{K}\frac{1}{|\xi_{1,k}(\tau_m)|}.
\end{aligned}$$

Since $(b_m)_m$ is a bounded sequence of real numbers, its limes superior exists. The estimate above yields, that for any $k \in \mathbb{N}$

$$\begin{aligned}
\Big|\limsup_{m\to\infty} b_m - a\Big| &\le \limsup_{m\to\infty} |\Delta_k(\tau_m) - a_k| + |a_k - a| + \frac{4L}{K}\limsup_{m\to\infty} |\xi_{1,k}(\tau_m)|^{-1} \\
&= |a_k - a_k| + |a_k - a| + \frac{4L}{K}|\varphi_{t,1}(u_k) - U_1(t)|.
\end{aligned}$$

Taking $k \to \infty$, we get $\limsup_{m\to\infty} b_m = a$. By arguing in the same way as above, we conclude that $\liminf_{m\to\infty} b_m = a$, which entails $\lim_{m\to\infty} b_m = a$. Hence, indeed

$$\lim_{\tau\nearrow t} \frac{1}{t-\tau}\frac{1}{\pi}\int_{\mathcal{C}_{\tau,t,j}} \operatorname{Im}\varphi_{\tau,t}(\xi)\,\mathrm{d}\xi = a.$$

Lemma 3.4, and Corollary 2.7 yield that

$$a_k = \lim_{\tau\nearrow t} \Delta_k(\tau) = \lim_{\tau\nearrow t} D_k(t,\tau) = \dot{\varphi}_{t,1}(u_k) = \lim_{T\searrow t} D_k(T,t).$$

[9] Note that thus the limit a is in fact independent of the choice of the sequence $(u_k)_k$.

By arguing as we did in the Steps (ii) to (iv), simply replacing τ by T, and $\tau \nearrow t$ by $T \searrow t$, we find that

$$\lim_{T \searrow t} \frac{1}{T-t} \frac{1}{\pi} \int_{\mathcal{C}_{t,T,j}} \operatorname{Im} \varphi_{t,T}(\xi)\, \mathrm{d}\xi = a = \lim_{\tau \nearrow t} \frac{1}{t-\tau} \frac{1}{\pi} \int_{\mathcal{C}_{\tau,t,j}} \operatorname{Im} \varphi_{\tau,t}(\xi)\, \mathrm{d}\xi.$$

□

5 The Multislit Equation

Now, we combine the results from the previous sections, and deduce a generalised Loewner equation.

Lemma 5.1 *The map $z \mapsto \partial_t \varphi_{t,s}(z)$ is defined for almost every $t \in [0, s]$, and is holomorphic in $\mathbb{H}$.*

Proof Let $(z_k)_k$ be a sequence of pairwise distinct elements of $\mathbb{H}$, such that $M := \{z_k : k \in \mathbb{N}\}$ is dense in $\mathbb{H}$, i.e., $\operatorname{clos} M = \operatorname{clos} \mathbb{H}$. Then, $t \mapsto \partial_t \varphi_{t,s}(w)$ exists for every $w \in M$, and every $t \in D$ for some set $D \subseteq [0, 1]$ of full measure. Suppose that $t_n \in [0, 1]$ tends to $t \in D$ as $n \to \infty$, but $t_n \neq t$ for all n. Then

$$\Delta_n : \mathbb{H} \longrightarrow \mathbb{C}, \quad z \longmapsto \frac{\varphi_{t_n,s}(z) - \varphi_{t,s}(z)}{t_n - t}$$

is holomorphic. Since $(\Delta_n(z_k))_n$ is convergent, and Δ_n is locally bounded due to Corollary 2.7, Δ_n converges locally uniformly to some holomorphic $\Delta : \mathbb{H} \to \mathbb{C}$. Using the identity principle, we see that Δ is independent of the choice of $(t_n)_n$, which yields the claim. □

We are now able to prove our main result.

Proof of Theorem 1.4. We work in several steps. We fix $s \in [0, 1]$. By the relation $\varphi_{t,s} = g_t \circ h_s$, it suffices to deduce the Loewner equation for $\varphi_{t,s}$ instead of g_t.

(i) Let $0 \leq \tau \leq t \leq s \leq 1$. By Corollary 2.7, we already know that the function $t \mapsto \varphi_{t,s}(z)$ is differentiable for almost every t. Therefore, it suffices to calculate its left derivative. We let $\Delta_j(z, \tau, t) := \Xi_j \lambda_j(\tau, t)$ where

$$\lambda_j(\tau, t) := \frac{1}{t-\tau} \frac{1}{\pi} \int_{\mathcal{C}_{\tau,t,j}} \operatorname{Im} \varphi_{\tau,t}(\xi)\, \mathrm{d}\xi,$$

and $\Xi_j = \Xi_j(z, \tau, t)$ is from (3.1) (see also Lemma 2.6). By Lemma 3.4,

$$\Xi_j \to \frac{1}{\varphi_{t,s}(z) - U_j(t)} \quad \text{as} \quad \tau \nearrow t$$

and consequently

$$\frac{\varphi_{t,s}(z) - \varphi_{\tau,s}(z)}{t - \tau} = \sum_j \Delta_j(z, \tau, t). \tag{5.1}$$

Taking the limit $\tau \nearrow t$ in (5.1), and interchanging limit with summation, by the dominated convergence theorem, yields that for almost every $t \in [0, s]$, and $z \in \mathbb{H}$ the equations

$$\partial_t \varphi_{t,s}(z) = \sum_j \frac{\lambda_j(t)}{\varphi_{t,s}(z) - U_j(t)}, \quad \text{and} \quad \varphi_{0,s}(z) = h_s(z)$$

hold true. This implies (1.2).

(ii) We now prove that λ_j actually has the properties it is claimed to have in Theorem 1.4. Using Lemma 5.1, we note that $t \mapsto \partial_t \varphi_{t,1}(\varphi_{t,1}^{-1}(U_j(t) + k^{-1}))$ is measurable for k sufficiently large. We use Eq. (4.2) with the sequence $(k^{-1})_k$, and write

$$\lambda_j(t) = \lim_{k \to \infty} \partial_t \varphi_{t,1}\big(\varphi_{t,1}^{-1}(U_j(t) + k^{-1})\big) \cdot k^{-1}.$$

Hence, λ_j is measurable. Equation (4.1), in combination with Lemma 2.5 (3), yields that $0 \leq \lambda_j(t) \leq L$ holds almost everywhere, where L denotes a Lipschitz constant of $t \mapsto \operatorname{hcap}(\Gamma_t)$. Comparing coefficients in the expansion of

$$\begin{aligned}\sum_k \frac{\lambda_k(t)}{\varphi_{t,s}(z) - U_k(t)} &= \frac{1}{\varphi_{t,s}(z)} \sum_k \sum_{m \geq 0} \lambda_k(t) \left(\frac{U_k(t)}{\varphi_{t,s}(z)}\right)^m \\ &= \frac{1}{\varphi_{t,s}(z)} \sum_{m \geq 0} \sum_k \lambda_k(t) \left(\frac{U_k(t)}{\varphi_{t,s}(z)}\right)^m, \quad |z| \to \infty,\end{aligned}$$

and the expansion of

$$\begin{aligned}\frac{\partial}{\partial t}\varphi_{t,s}(z) &= \frac{\partial}{\partial t}\left(z + \frac{\operatorname{hcap}\Gamma_t - \operatorname{hcap}\Gamma_s}{z} + O\big(|z|^2\big)\right) \\ &= \frac{1}{z}\frac{\partial}{\partial t}\operatorname{hcap}\Gamma_t + \frac{\partial}{\partial t} O\big(|z|^2\big), \quad |z| \to \infty,\end{aligned}$$

yields (3). This concludes the proof.

□

Moreover, we can now deduce a relation between the existence of the limits defining the weight functions in (4.1) at time $t \in [0, 1]$ and the validity of the Loewener equation (1.2) at t.

Theorem 5.2 *Using the notation of Theorem 1.4, the following statements are equivalent for an admissible multislit Γ with Loewner parametrisation γ, and $t \in [0, 1]$:*

(1) $\tau \mapsto g_\tau(z)$ *is differentiable at* t *for all* $z \in \mathbb{H}\backslash\Gamma$.
(2) The following two limits exist, and are equal for any j

$$\lim_{\tau \nearrow t} \frac{1}{t-\tau}\frac{1}{\pi}\int_{\mathcal{C}_{\tau,t,j}} \operatorname{Im}\varphi_{\tau,t}(\xi)\,\mathrm{d}\xi, \quad \lim_{\tau \searrow t} \frac{1}{\tau-t}\frac{1}{\pi}\int_{\mathcal{C}_{t,\tau,j}} \operatorname{Im}\varphi_{t,\tau}(\xi)\,\mathrm{d}\xi.$$

Proof Theorem 4.1 yields that (1) implies (2). By carefully reviewing the proof of Theorem 1.4, we see that we could have written $\tau \searrow t$ instead of $\tau \nearrow t$. Hence, we conclude that (2) implies (1). □

Remark 5.3 One can show, by replacing hcap by the logarithmic mapping radius lmr, and arguing along the lines of [3], that the weight function λ_k of an admissible multislit exists in t if and only if the map

$$X_{k,t} : (-t, 1-t) \to [0,\infty), \quad h \mapsto \operatorname{hcap}\left(\gamma_k\,(0, t+h] \cup \bigcup_{j\neq k} \gamma_j\,(0,t]\right)$$

is differentiable at 0, and in this case, $X'_{k,t}(0) = \lambda_k(t)$. In view of this and since one can vary γ_k suitably, it is clear that Theorem 1.4 is best possible in the sense that one cannot get a Loewner equation for all t for an arbitrary Loewner parametrisation.

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