Computational Complexity of Proper Equilibrium

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We study the computational complexity of proper equilibrium in finite games and prove the following results. First, for two-player games in strategic form we show that the task of simply verifying the proper equilibrium conditions of a given pure Nash equilibrium is NP-complete. Next, for n-player games in strategic form we show that the task of computing an approximation of a proper equilibrium is FIXP_a-complete. Finally, for n-player polymatrix games we show that the task of computing a symbolic proper equilibrium is PPAD-complete.

CCS Concepts: • **Theory of computation** → **Problems, reductions and completeness**; **Exact and approximate computation of equilibria**;

1 INTRODUCTION

We study the Nash equilibrium *refinement* of proper equilibrium due to Myerson [24] and obtain new results concerning the tasks of verification, approximation, and computation. Several recent works have been concerned with the complexity of such tasks for Nash equilibrium refinements in general, including proper equilibrium.

The task of verifying the conditions of a given Nash equilibrium of a game in strategic form is computationally a trivial task. On the other hand Hansen et al. [17] showed NP-hardness and SQRT-SUM-hardness for verifying the conditions of standard Nash equilibrium refinements in n-player games for $n \geq 3$, including proper equilibrium. Recently Hansen [16] showed that the problems are in fact complete for $\exists\mathbb{R}$, meaning that they are computationally equivalent to the decision problem of the existential theory of the reals.

For two-player games, similar results have so far been positive. For games in strategic form, trembling hand perfect equilibrium coincide with admissible equilibrium and can be verified in polynomial time using linear programming [28]. For games in extensive form with perfect recall, Gatti and Panozzo [15] showed that quasi-perfect equilibrium can be verified in polynomial time and Gatti et al. [14] obtained the same for the strategy part of sequential equilibrium. Determining the computational complexity of verifying a proper equilibrium was explicitly stated as an open problem by Hansen et al. [17] and Gatti and Panozzo [15]. Given the equivalence between the verification problems in n-player games and the positive results so far one may expect that proper equilibrium in two-player games can be verified in polynomial time as well. As our first result we show, perhaps surprisingly, that the problem is NP-complete. To the best of our knowledge, this is the first instance of NP-hardness for verifying whether a fully specified two player strategy profile satisfies the conditions of an equilibrium refinement. We note that Gatti and Panozzo [15] showed that it is NP-complete to verify whether a two player strategy profile given as a *realization plan* is subgame perfect or part of a sequential equilibrium, but this crucially relies on the strategy not

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being specified in the subgame. If the full strategy profile is specified, the verification can be done in polynomial time [14].

In seminal work Daskalakis et al. [8] and Chen and Deng [3] showed that computing a Nash equilibrium in a two-player game is PPAD-complete and Etessami and Yannakakis [11] showed that computing a Nash equilibrium in a n-player game is FIXP-complete, when $n \geq 3$. Thus containment in PPAD and FIXP are best possible results for computing equilibrium refinements, and such results show that computing the equilibrium refinements is polynomial time *equivalent* to computing *any* Nash equilibrium.

For two-player games in strategic form, computing a *symbolic* proper equilibrium was shown to be in PPAD by Sørensen [26]. For two-player games in extensive form with perfect recall, computing a quasi-perfect equilibrium was shown to be in PPAD by Miltersen and Sørensen [23], and computing a trembling hand perfect equilibrium was shown to be in PPAD by Farina and Gatti [12]. It was asked by Sørensen [26] whether computing a proper equilibrium by itself belongs to PPAD. The obstacle here was that it was not clear how to efficiently verify a proper equilibrium. Our NP-hardness result shows that this is not likely to be possible and underlines the importance and usefulness of computing a *symbolic* proper equilibrium rather than just a proper equilibrium by itself, since the former is straightforward to verify efficiently.

For n-player games in strategic form, computing an approximation to a trembling hand perfect equilibrium was shown to be in FIXP_a by Etessami et al. [10]. For n-player games in extensive form with perfect recall, Etessami [9] proved membership in FIXP_a for approximating a quasi-perfect equilibrium and for approximating trembling hand perfect equilibrium. It was stated as an explicit open problem by Etessami [9] whether the task of computing a proper equilibrium of a n-player game in strategic form is in FIXP_a. As our second result we give an affirmative answer to this question. Thus, in strategic form games, computing an approximating to a proper equilibrium is polynomial time equivalent to approximating any Nash equilibrium. The original proof of existence of a proper equilibrium in every finite strategic form game by Myerson [24] was based on the powerful Kakutani fixed point theorem. Our result provides an alternative proof of existence based on the Brouwer fixed point theorem.

Our final result is that in n-player *polymatrix* games the task of computing a symbolic proper equilibrium belongs to PPAD. Thus, in polymatrix games, computing an a symbolic proper equilibrium is polynomial time equivalent to computing any Nash equilibrium. This significantly strengthens the result of Sørensen [26].

2 PRELIMINARIES

2.1 Games in Strategic Form

A finite game Γ with n players in strategic form is specified as follows. Player i has a set S_i of m_i pure strategies which may be identified with the set $\{1, \ldots, m_i\}$ when needed. We denote by $\Delta(S_i) = \Delta_{m_i}$ the set of mixed strategies for Player i. That is, $\Delta_{m_i} = \{y \in \mathbb{R}^{m_i} \mid \|y\|_1 = 1; \forall j : y_j \geq 0\}$. We denote by $\Delta^o(S_i) = \Delta^o_{m_i}$ the set of fully-mixed strategies for Player i. That is, $\Delta^o m_i = \{y \in \mathbb{R}^{m_i} \mid \|y\|_1 = 1; \forall j : y_j > 0\}$. We let $D_\Gamma = \prod_{i=1}^n \Delta_{m_i}$ be the set of all mixed strategy profiles for Γ and $D^o_\Gamma = \prod_{i=1}^n \Delta^o_{m_i}$ the set of all fully mixed strategy profiles. Each combination $(a_1, \ldots, a_n)$ of pure strategies specifies a payoff $U_i(a_1, \ldots, a_n)$ to Player i, for every i. The *utility functions* U_i are extended to D_Γ in the natural way. For a *strategy profile* $x = (x_1, \ldots, x_n) \in D_\Gamma$ and a pure strategy k for Player i, we denote by $(x_{-i}; k)$ the strategy profile where Player i uses the pure strategy k and Player j uses the mixed strategy x_j, for every $j \neq i$.

2.2 Proper Equilibrium

The definition below of proper equilibrium is due to Myerson [24]. Every finite game Γ has a proper equilibrium and every proper equilibrium is in particular a Nash equilibrium.

Definition 2.1. Given $\varepsilon > 0$, a mixed strategy profile x is an ε-proper equilibrium if it is fully mixed and satisfies $x_{ik} \leq \varepsilon x_{i\ell}$ whenever $U_i(x_{-i}; k) < U_i(x_{-i}; \ell)$, for all $i \in \{1, \ldots, n\}$ and $k, \ell \in S_i$.

Definition 2.2. A mixed strategy profile x is a proper equilibrium if it is the limit point of a sequence $\{x^{\varepsilon_k}\}$ of ε_k-proper equilibria where $\varepsilon_k \to 0^+$ for $k \to \infty$.

Definition 2.3. A symbolic proper equilibrium is a collection of polynomials $P_{ij}(\varepsilon)$ in the formal variable ε, such that the strategy profile x_ε given by $(x_\varepsilon)_{ij} = P_{ij}(\varepsilon)/\sum_{k \in S_i} P_{ik}(\varepsilon)$ is an ε-proper equilibrium for every sufficiently small $\varepsilon > 0$.

2.3 Complexity Classes

Our main results are concerned with the complexity classes NP, PPAD, and FIXP$_a$. We give a brief description of the classes PPAD and FIXP and refer to Papadimitriou [25] and Etessami and Yannakakis [11] for detailed definitions of the two classes.

PPAD is a class of discrete total search problems, whose totality is guaranteed based on a parity argument on a directed graph. More formally PPAD is defined by a canonical complete problem EndOfTheLine. Here a directed graph is given implicitly by predecessor and successor circuits, and the search problem is to find a degree 1 node different from a given degree 1 node.

FIXP is the class of real-valued total search problems that can be cast as Brouwer fixed points of functions represented by $\{+, -, *, /, \max, \min\}$-circuits computing a function mapping a convex polytope described by a set of linear inequalities to itself. The class FIXP$_a$ is the class of *discrete* total search problems that reduce to *approximate* Brouwer fixed points.

3 VERIFICATION OF PROPER EQUILIBRIUM IN BIMATRIX GAMES

In this section we show that deciding whether a given pure Nash equilibrium is a proper equilibrium in a given 2-player game in strategic form is NP-hard. Our reduction is based on a modification of a game construction used by Conitzer and Sandholm [5] to show that determining existence of Nash equilibria with several natural properties in 2-player games in strategic form is NP-complete.

The game construction by Conitzer and Sandholm [5] takes any CNF formula Φ and produces a 2-player game in strategic form $G_{\mathrm{CS}}(\Phi)$ in which, except for a special Nash equilibrium, there is a one-to-one correspondence between satisfying assignments to Φ and Nash equilibria of $G_{\mathrm{CS}}(\Phi)$. We will actually describe the construction from the conference version by Conitzer and Sandholm [4], which is slightly simpler to use for our purpose. Also for simplicity, and since this is sufficient, we shall also define the game only for 3CNF formulas Φ.

Let $V = \{x_1, \ldots, x_n\}$ be a set of n Boolean variables. It is convenient to have explicit signs on literals. Thus, a variable x_i gives rise to the positive literal $+x_i$ and the negative literal $-x_i$. Thus define $L = \{+x_1, -x_1, \ldots, +x_n, -x_n\}$ to be the set of literals given by V. We say that $+x_i$ and $-x_i$ are opposite of each other and for $\ell \in L$ we denote by $-\ell$ the opposite of ℓ. Define $v : L \to V$ by $v(+x_i) = v(-x_i) = x_i$. We may now identify the clauses with subsets of L, and a CNF formula Φ is thus a set of clauses $C = \{c_1, \ldots, c_m\}$. Similarly, a truth assignment to variables V is identified with a subset $T \subseteq L$ such that for all i, exactly one of $+x_i$ and $-x_i$ belong to T. We then have that T is a satisfying assignment for Φ if and only if $c_i \cap T \neq \emptyset$, for all i.

The game $G_{\mathrm{CS}}(\Phi)$ is a symmetric game where the players have the set of strategies $S = L \cup V \cup C \cup \{f\}$. The payoffs are given as listed in table 1a given $\ell, \ell' \in L$, $x, x' \in V$, and $c, c' \in C$. Clearly (f, f) is a pure Nash equilibrium in $G_{\mathrm{CS}}(\Phi)$. Conitzer and Sandholm show that a satisfying assignment

	ℓ'	x'	c'	f
ℓ	1 if $\ell \neq -\ell'$ -2 if $\ell = -\ell'$	-2	-2	-2
x	2 if $v(\ell') \neq x$ $2-n$ if $v(\ell') = x$	-2	-2	-2
c	2 if $\ell' \notin c$ $2-n$ if $\ell' \in c$	-2	-2	-2
f	1	1	1	0

(a) $G_{\mathrm{CS}}(\Phi)$

	ℓ'	x'	c'	f
ℓ	1 if $\ell \neq -\ell'$ $-4n$ if $\ell = -\ell'$	-2	-2	-2
x	2 if $v(\ell') \neq x$ $2-n$ if $v(\ell') = x$	-2	-2	-2
c	2 if $\ell' \notin c$ $2-n$ if $\ell' \in c$	-2	-2	-2
f	$1/2$	$1/2$	$1/2$	2

(b) $\widetilde{G_{\mathrm{CS}}}(\Phi)$

Table 1. The Conitzer-Sandholm game $G_{\mathrm{CS}}(\Phi)$, with payoffs to the row player, and its robust variation $\widetilde{G_{\mathrm{CS}}}(\Phi)$

A to Φ gives a Nash equilibrium where each true literal $\ell \in A$ is played with probability $\frac{1}{n}$, and that these are the only Nash equilibria, other than (f, f). These equilibria are however not very robust, since the pure strategy f weakly dominates all $\ell \in L$. In addition, should the other player make a tremble, it would be better to play f. Thus the only trembling hand perfect equilibrium of $G_{\mathrm{CS}}(\Phi)$ is (f, f). The first step of our construction is to change $G_{\mathrm{CS}}(\Phi)$ into a more robust game $\widetilde{G_{\mathrm{CS}}}(\Phi)$ in which satisfying assignments will correspond to proper equilibria. We obtain robustness by decreasing the payoff of (f, a) to $\frac{1}{2}$ for all $a \in S \setminus \{f\}$. To make up for this in the analysis we increase the penalty to the players for playing opposite literals from -2 to $-4n$. A final change, not done for robustness but for use in the final construction is to change the payoff for (f, f) from 0 to 2. Clearly (f, f) remains a pure Nash equilibrium. The next lemma shows that every other Nash equilibrium identifies a unique satisfying assignment of Φ.

Lemma 3.1. *Let Φ be a 3CNF formula. Every Nash equilibrium (σ_1, σ_2) in $\widetilde{G_{\mathrm{CS}}}(\Phi)$ different from (f, f) corresponds to a satisfying assignment of Φ.*

Proof. For a Nash equilibrium (σ_1, σ_2) we have that $\Pr_{\sigma_2}[f] = 1$ implies that $\Pr_{\sigma_1}[f] = 1$, since f is the unique best reply to f. Likewise, $\Pr_{\sigma_1}[f] = 1$ implies that $\Pr_{\sigma_2}[f] = 1$. Since $(\sigma_1, \sigma_2) \neq (f, f)$ it follows that $\Pr_{\sigma_1}[f] < 1$ and $\Pr_{\sigma_2}[f] < 1$. We next show that significant probability mass must be placed on L. We have that $\mathrm{E}_{b\sim\sigma_2}[u_1(a, b) \mid b \neq f] \geq \frac{1}{2}$ for all $a \in \mathrm{supp}(\sigma_1) \setminus \{f\}$, since otherwise f would be a strictly better reply to σ_2 than a. Likewise $\mathrm{E}_{a\sim\sigma_1}[u_2(a, b) \mid a \neq f] \geq \frac{1}{2}$ for all $b \in \mathrm{supp}(\sigma_2) \setminus \{f\}$. Thus for $a \in \mathrm{supp}(\sigma_1) \setminus \{f\}$ we have

$$\begin{aligned}
\frac{1}{2} &\leq \mathop{\mathrm{E}}_{b\sim\sigma_2}[u_1(a,b) | b \in L] \Pr_{\sigma_2}[L \mid (S \setminus \{f\})] + \mathop{\mathrm{E}}_{b\sim\sigma_2}[u_1(a,b) | b \in V \cup C)] \Pr_{\sigma_2}[V \cup C \mid (S \setminus \{f\})] \\
&\leq 2 \Pr_{\sigma_2}[L \mid (S \setminus \{f\})] - 2 \Pr_{\sigma_2}[V \cup C \mid (S \setminus \{f\})] \\
&= 2 \Pr_{\sigma_2}[L \mid (S \setminus \{f\})] - 2 \left(1 - \Pr_{\sigma_2}[L \mid (S \setminus \{f\})]\right) \\
&= 4 \Pr_{\sigma_2}[L \mid (S \setminus \{f\})] - 2 \ ,
\end{aligned}$$

and it follows that $\Pr_{\sigma_2}[L \mid S \setminus \{f\}] \geq (\frac{1}{2} + 2)/4 = \frac{5}{8}$. Likewise $\Pr_{\sigma_1}[L \mid S \setminus \{f\}] \geq \frac{5}{8}$. We next show that the probability mass placed on L must be evenly distributed on literal pairs. Let $i \in \{1, \ldots, n\}$.

Clearly $\mathrm{E}_{b\sim\sigma_2}[u_1(a,b) \mid b \in L] \leq 1$ for all $a \in L$. Since

$$\mathop{\mathrm{E}}_{b\sim\sigma_2}[u_1(x_i,b) \mid b \in L] = 2 - n \Pr_{\sigma_2}[\{+x_i, -x_i\} \mid L] \ ,$$

it follows that $2 - n \Pr_{\sigma_2}[\{+x_i, -x_i\} \mid L] \leq 1$, since otherwise no $a \in L$ can be a best reply to σ_2. Hence $\Pr_{\sigma_2}[\{+x_i, -x_i\} \mid L] \geq \frac{1}{n}$ and we can conclude that $\Pr_{\sigma_2}[\{+x_i, -x_i\} \mid L] = \frac{1}{n}$ for all i. Likewise, $\Pr_{\sigma_1}[\{+x_i, -x_i\} \mid L] = \frac{1}{n}$ for all i.

We now show that for each literal pair, the players must both favor either the positive or the negative literal. Consider any $i \in \{1, \ldots, n\}$. Let $\ell \in \{+x_i, -x_i\}$ and suppose that $\Pr_{\sigma_2}[\ell \mid L] \geq \frac{1}{4n}$. We then have

$$\begin{aligned} \mathop{\mathrm{E}}_{b\sim\sigma_2}[u_1(-\ell,b) \mid b \neq f] &\leq 1 - 4n \Pr_{\sigma_2}[\ell \mid L] \Pr_{\sigma_2}[L \mid S \setminus \{f\}] \\ &\leq 1 - 4n \frac{1}{4n}\frac{5}{8} = \frac{3}{8} < \frac{1}{2} \ , \end{aligned}$$

and we must have $\Pr_{\sigma_1}[-\ell] = 0$. Likewise, if $\Pr_{\sigma_1}[\ell \mid L] \geq \frac{1}{4n}$ we must have $\Pr_{\sigma_2}[-\ell] = 0$.

We now established that the strategy profile identifies a truth assignment A to the variables. Namely for each $i \in \{1, \ldots, n\}$ there is $\ell \in \{+x_i, -x_i\}$ with $\Pr_{\sigma_1}[\ell \mid L] = \Pr_{\sigma_2}[\ell \mid L] = \frac{1}{n}$ and $\Pr_{\sigma_1}[-\ell \mid L] = \Pr_{\sigma_2}[-\ell \mid L] = 0$, and we define A to be the set of these ℓ. We will finally show that A satisfies all clauses. Let $c = \{\ell_1, \ell_2, \ell_3\}$ be a clause. Note that

$$\mathop{\mathrm{E}}_{b\sim\sigma_2}[u_1(c,b) \mid b \in L] = 2 - n \Pr_{\sigma_2}[\{\ell_1, \ell_2, \ell_3\} \mid L] \ ,$$

We must have $2 - n \Pr_{\sigma_2}[\{\ell_1, \ell_2, \ell_3\} \mid L] \leq 1$, since otherwise no $a \in L$ can be a best reply to σ_2. It follows that $\Pr_{\sigma_2}[\{\ell_1, \ell_2, \ell_3\} \mid L] \geq \frac{1}{n}$, and there must then be $j \in \{1, 2, 3\}$ for which $\Pr_{\sigma_2}[\ell_j \mid L] \geq \frac{1}{3n}$, which means that $\ell_j \in A$. □

We are now ready to define final game $G_{\mathrm{Prop}}(\Phi)$ for any 3CNF formula Φ. The game $G_{\mathrm{Prop}}(\Phi)$ is simply obtained from $\widetilde{G_{\mathrm{CS}}}(\Phi)$ by giving each player an additional pure strategy g. By playing the new strategy g both players are able to fix the payoff to 1, and otherwise the payoffs are unchanged. The resulting game is shown in table 2.

	ℓ'	x'	c'	f	g
ℓ	1 if $\ell \neq -\ell'$ $-4n$ if $\ell = -\ell'$	-2	-2	-2	1
x	2 if $v(\ell') \neq x$ $2-n$ if $v(\ell') = x$	-2	-2	-2	1
c	2 if $\ell' \notin c$ $2-n$ if $\ell' \in c$	-2	-2	-2	1
f	$1/2$	$1/2$	$1/2$	2	1
g	1	1	1	1	1

Table 2. $G_{\mathrm{Prop}}(\Phi)$.

Lemma 3.2. *Let Φ be a 3CNF formula. If Φ has a satisfying assignment then (g, g) is a proper Nash equilibrium of $G_{\mathrm{Prop}}(\Phi)$.*

Proof. Let $\varepsilon > 0$ be arbitrary and suppose $t_i \in \{+\ell_i, -\ell_i\}$ such that $A = \{(t_1, \ldots, t_n\}$ is a satisfying assignment of Φ. Write $C = C_1 \cup C_2 \cup C_3$, where C_k is the set of clauses containing k literals from A.

Define σ^ε according to table 3, where $x \in V$, $c_1 \in C_1$, $c_2 \in C_2$, $c_3 \in C_3$, and we let $N = 1 + n\varepsilon + n\varepsilon^2 + |C_1|\varepsilon^3 + \varepsilon^4 + |C_2|\varepsilon^5 + |C_3|\varepsilon^6 + n\varepsilon^7$ be a normalizing factor.

i	g	t_i	x	c_1	f	c_2	c_3	$-t_i$
$\sigma^\varepsilon(i)N$	1	ε	ε^2	ε^3	ε^4	ε^5	ε^6	ε^7

Table 3. An ε-proper equilibrium of $G_{\mathrm{Prop}}(\Phi)$.

By construction we have that σ^ε is a fully mixed strategy. We claim that $(\sigma^\varepsilon, \sigma^\varepsilon)$ is an ε-proper equilibrium. In order to verify this we compute the payoffs for all pure strategies played against σ^ε. Clearly $u_1(g, \sigma^\varepsilon) = 1$. Next we have that $u_1(a, \sigma^\varepsilon)$ for $a \in A \cup V \cup C_1$ are all exactly equal and have value of the form[1] $1 + O(\varepsilon^2)$. For instance we have

$$\begin{aligned} u_1(x, \sigma^\varepsilon)N &= 1 + (n-1)2\varepsilon + (2-n)\varepsilon \\ &\quad - 2(n\varepsilon^2 + |C_1|\varepsilon^3 + \varepsilon^4 + |C_2|\varepsilon^5 + |C_3|\varepsilon^6) - n\varepsilon^7 \\ &= 1 + n\varepsilon - 2(n\varepsilon^2 + |C_1|\varepsilon^3 + \varepsilon^4 + |C_2|\varepsilon^5 + |C_3|\varepsilon^6) - n\varepsilon^7 \\ &= 1 + n\varepsilon + O(\varepsilon^2)\ , \end{aligned}$$

from which the stated bound follows using $1/N = 1 - n\varepsilon + O(\varepsilon^2)$.

Similarly we obtain $u_1(f, \sigma^\varepsilon) = 1 - n\varepsilon/2 + O(\varepsilon^2)$, $u_1(c_2, \sigma^\varepsilon) = 1 - n\varepsilon + O(\varepsilon^2)$, $u_1(c_3, \sigma^\varepsilon) = 1 - 2n\varepsilon + O(\varepsilon^2)$, and $u_1(-t_i, \sigma^\varepsilon) = 1 - n(4n+1)\varepsilon + O(\varepsilon^2)$. With these computed it is now straightforward to observe that $(\sigma^\varepsilon, \sigma^\varepsilon)$ is an ε-proper equilibrium for all sufficiently small $\varepsilon > 0$. Since $(\sigma^\varepsilon, \sigma^\varepsilon) \to (g, g)$ for $\varepsilon \to 0^+$, it follows that (g, g) is a proper equilibrium. □

Lemma 3.3. *Let Φ be a 3CNF formula. If (g, g) is a proper Nash equilibrium of $G_{\mathrm{Prop}}(\Phi)$ then Φ has a satisfying assignment.*

Proof. By definition of (g, g) being a proper equilibrium there exist a sequence $(\sigma_1^\varepsilon, \sigma_2^\varepsilon)$ of ε-proper equilibrium with $(\sigma_1^\varepsilon, \sigma_2^\varepsilon) \to (g, g)$ for $\varepsilon \to 0^+$. Assume $\varepsilon \leq \frac{1}{2}$, $\Pr_{\sigma_1^\varepsilon}[g] \geq \frac{1}{2}$, and $\Pr_{\sigma_2^\varepsilon}[g] \geq \frac{1}{2}$. Then g is a best reply to σ_1^ε and σ_2^ε. This means in particular that $\mathrm{E}_{b\sim\sigma_2^\varepsilon}[u_1(a, b) \mid b \neq g] \leq 1$ for all $a \in S$. Likewise $\mathrm{E}_{a\sim\sigma_2^\varepsilon}[u_2(a, b) \mid a \neq g] \leq 1$ for all $b \in S$.

Define the conditional strategies $\tau_1^\varepsilon = \sigma_1^\varepsilon \mid (S \setminus \{g\})$ and $\tau_2^\varepsilon = \sigma_2^\varepsilon \mid (S \setminus \{g\})$. By our construction of the game $G_{\mathrm{Prop}}(\Phi)$ we have that $\mathrm{E}_{b\sim\tau_2^\varepsilon}[u_1(a, b)] < \mathrm{E}_{b\sim\tau_2^\varepsilon}[u_1(a', b)]$ if and only if $\mathrm{E}_{b\sim\sigma_2^\varepsilon}[u_1(a, b)] < \mathrm{E}_{b\sim\sigma_2^\varepsilon}[u_1(a', b)]$ for all $a, a' \in S$. Likewise $\mathrm{E}_{a\sim\tau_1^\varepsilon}[u_2(a, b)] < \mathrm{E}_{a\sim\tau_1^\varepsilon}[u_2(a, b')]$ if and only if $\mathrm{E}_{a\sim\sigma_1^\varepsilon}[u_2(a, b)] < \mathrm{E}_{a\sim\sigma_1^\varepsilon}[u_2(a, b')]$ for all $b, b' \in S$.

Also $\Pr_{\tau_1^\varepsilon}[a] \leq \varepsilon \Pr_{\tau_1^\varepsilon}[a']$ if and only if $\Pr_{\sigma_1^\varepsilon}[a] \leq \varepsilon \Pr_{\sigma_1^\varepsilon}[a']$ for all $a, a' \in S \setminus \{g\}$, and likewise $\Pr_{\tau_2^\varepsilon}[b] \leq \varepsilon \Pr_{\tau_2^\varepsilon}[b']$ if and only if $\Pr_{\sigma_2^\varepsilon}[b] \leq \varepsilon \Pr_{\sigma_2^\varepsilon}[b']$ for all $b, b' \in S \setminus \{g\}$. It follows that $(\tau_1^\varepsilon, \tau_2^\varepsilon)$ is an ε-proper equilibrium in $\widetilde{G_{\mathrm{CS}}}(\Phi)$.

By the Bolzano-Weierstrass Theorem the sequence $(\tau_1^\varepsilon, \tau_2^\varepsilon)$ has a limit point (τ_1, τ_2) which then by definition is a proper equilibrium of $\widetilde{G_{\mathrm{CS}}}(\Phi)$. In particular is (τ_1, τ_2) a Nash equilibrium. As observed $\mathrm{E}_{b\sim\tau_2^\varepsilon}[u_1(a, b)] \leq 1$ for all $a \in S \setminus \{g\}$. It follows that also $\mathrm{E}_{b\sim\tau_2}[u_1(a, b)] \leq 1$ for all $a \in S \setminus \{g\}$, and in particular $\mathrm{E}_{a\sim\tau_1, b\sim\tau_2}[u_1(a, b)] \leq 1$. Likewise we obtain that $\mathrm{E}_{a\sim\tau_1, b\sim\tau_2}[u_2(a, b)] \leq 1$. This rules out

[1] Here and below we use $O(g(\varepsilon))$ to denote a function f satisfying $|f(\varepsilon)| \leq M|g(\varepsilon)|$ for every $0 < \varepsilon < \varepsilon_0$ given some $M, \varepsilon_0 > 0$. The actual value of $u_1(a, \sigma^\varepsilon)$ is strictly less than 1.

the possibility that $(\tau_1, \tau_2) = (f, f)$, and it follows from Lemma 3.1 that (τ_1, τ_2) identifies a satisfying assignment of Φ. □

Theorem 3.4. *It is* NP*-complete to decide if a given pure Nash equilibrium of a two-player game in strategic form is a proper equilibrium.*

Proof. Combining Lemma 3.2 and Lemma 3.3 we have a reduction from 3SAT to the problem of verifying that a given pure Nash equilibrium is a proper equilibrium, which establishes NP-hardness. We next sketch the proof of containment in NP. For a given pure Nash equilibrium that is a proper equilibrium also a symbolic proper Nash equilibrium with coefficients of polynomial bitsize. Given such a proposed symbolic proper Nash equilibrium it may be checked in polynomial time that it does define a sequence of ε-proper equilibria for all sufficiently small ε. □

The result directly translates to extensive form as well, so the following corollary follows from the standard reduction from strategic form to extensive form.

Corollary 3.5. *It is* NP*-hard to decide if a given pure Nash equilibrium of a two-player game in extensive form is an extensive form proper equilibrium.*

4 APPROXIMATING A PROPER EQUILIBRIUM IN STRATEGIC FORM GAMES

In this section we show that approximating a proper equilibrium for a finite game in strategic form with $n \geq 3$ players is FIXP_a-complete. On a high level, our proof strategy is similar to previous work of Etessami et al. [10] who proved the analogous result for approximating (trembling hand) perfect equilibrium as well as the follow-up work of Etessami [9] who proved several analogous results for approximating refinements of Nash equilibrium in *extensive form* games of perfect recall. As shown by Etessami and Yannakakis [11] computing a Nash equilibrium for a finite game in strategic form with $n \geq 3$ is FIXP-complete and as an immediate consequence of that it is FIXP_a-complete to approximate a Nash equilibrium. This directly implies FIXP_a-hardness for approximating refinements of Nash equilibrium the nontrivial task is to prove containment in FIXP_a. Like proper equilibrium, a perfect equilibrium is defined as a limit point of so-called ε-perfect equilibrium. Etessami et al. [10] showed that in the perturbed game, where each strategy must be played with probability at least ε, computing an ε-perfect equilibrium is in FIXP. It is then shown that for ε sufficiently small relative to a given $\delta > 0$, any ε-perfect equilibrium is δ-close to an actual perfect equilibrium, and furthermore that such an ε may be computed from δ by means of repeated squaring, thereby leading to the result.

Given this, it would be natural to try to analogously prove FIXP_a-completeness for approximating a proper equilibrium by showing that computing an ε-proper equilibrium (for a perturbed game) is in FIXP. Whether this is possible remains an open problem; on the other hand we show how to *approximate* an ε-proper equilibrium, and by doing so also approximate an actual proper equilibrium. This result is obtained by showing that computing a so-called δ-almost ε-proper equilibrium (in a perturbed game) is in FIXP. The definition of a δ-almost ε-proper equilibrium was suggested by Etessami [9] as a possible way to define a relaxation of proper equilibrium computable in PPAD, analogously to other relaxations defined by Etessami [9]. Whether this turns out to be the case remains to be seen. But the notion does turns out to be the right technical definition for our purposes.

Let in the following Γ be a fixed finite game in strategic form with n players and with Player i having m_i pure strategies.

Definition 4.1. Given $\varepsilon > 0$ and $\delta > 0$, a mixed strategy profile x is a δ-almost ε-proper equilibrium if it is fully mixed and satisfies $x_{ik} \leq \varepsilon x_{i\ell}$ whenever $U_i(x_{-i}; k) + \delta \leq U_i(x_{-i}; \ell)$, for all $i \in \{1, \ldots, n\}$ and $k, \ell \in S_i$.

4.1 Uniform Approximation

Note that an ε-proper equilibrium is a δ-almost ε-proper equilibrium for all $\delta > 0$. We will show that a proper equilibrium can be approximated by a ε-proper equilibrium, for sufficiently small ε, which in turn can be approximated by a δ-almost ε-proper equilibrium, for sufficiently small δ. This is done by invoking the powerful "almost implies near" paradigm of Anderson [1] twice. First we show that any proper equilibrium may be uniformly approximated by an ε-proper equilibrium.

Lemma 4.2. *For any fixed strategic form game Γ, and any $\gamma > 0$, there is an $\varepsilon > 0$, so that any ε-proper equilibrium of Γ has ℓ_∞-distance at most γ to some proper equilibrium of Γ.*

Proof. Suppose to the contrary there is a game Γ and $\gamma > 0$ so that for all $\varepsilon > 0$ there is an ε-proper equilibrium x_ε of Γ so that there is no proper equilibrium in a γ-neighborhood (with respect to the ℓ_∞ norm) of x_ε. Consider the sequence $(x_{1/n})_{n\in\mathbb{N}}$. Since this is a sequence in a compact space, by the Bolzano-Weierstrass Theorem it has a limit point, x^*, which is a proper equilibrium of Γ by definition. But this contradicts the statement that there is no proper equilibrium in a γ-neighborhood of any of the strategy profiles $x_{1/n}$. □

We shall next consider a *perturbed* version of Γ. Let $\varepsilon > 0$ be given and define $\eta_i(\varepsilon) = \varepsilon^{m_i}/m_i$ for $i \in \{1, \dots, n\}$. When ε is understood from the context we shall simply denote $\eta_i(\varepsilon)$ by η_i. We denote by $\Delta_{m_i}^{\eta_i}$ the set of η_i-perturbed mixed strategies for Player i, meaning that each pure strategy is played with probability at least η_i. Thus $\Delta_{m_i}^{\eta_i} = \{y \in \mathbb{R}^m \mid \|y\|_1 = 1; \forall j : y_j \geq \eta_i\}$. Let $\eta = (\eta_1, \dots, \eta_n)$, and define $D_\Gamma^\eta = \prod_{i=1}^n \Delta_{m_i}^{\eta_i}$ to be the set of all η-perturbed mixed strategy profiles for Γ. Clearly $D_\Gamma^\eta \subset D_\Gamma^o \subset D_\Gamma$. The η-perturbed game Γ_η restricts the set of mixed strategy profiles to the set of η-perturbed mixed strategy profiles. The game Γ_η was used also by Myerson [24] in his existence proof of proper equilibrium. One of the important properties of Γ_η is that any limit point of a sequence of strategy profiles is fully mixed, which is used in the following proof that any ε-proper equilibrium may be uniformly approximated by a δ-almost ε-proper equilibrium.

Lemma 4.3. *For any fixed strategic form game Γ, any $\varepsilon > 0$ and any $\gamma > 0$, there is an $\delta > 0$, so that any δ-almost ε-proper equilibrium of Γ in D_Γ^η has ℓ_∞-distance at most γ to some ε-proper equilibrium of Γ in D_Γ^η.*

Proof. Suppose to the contrary there is a game Γ, $\varepsilon > 0$, and $\gamma > 0$ so that for all $\delta > 0$ there is a δ-almost ε-proper equilibrium x_δ of Γ in D_Γ^η so that there is no ε-proper equilibrium in D_Γ^η and in a γ-neighborhood (with respect to the ℓ_∞ norm) of x_δ. Consider the sequence $(x_{1/n})_{n\in\mathbb{N}}$. Since this is a sequence in a compact space, by the Bolzano-Weierstrass Theorem is has a convergent subsequence $(x_{1/n_r})_{r\in\mathbb{N}}$. Let $x^* = \lim_{r\to\infty} x_{1/n_r}$. We now claim that x^* is an ε-proper equilibrium, which will contradict the statement that there is no ε-proper equilibrium in a γ-neighborhood of any of the strategy profiles $x_{1/n}$.

First, since $x_{1/n_r} \in D_\Gamma^\eta$ for all n we also have $x^* \in D_\Gamma^\eta$, and in particular is x^* fully mixed. Define $\nu > 0$ by

$$\nu = \min_{i,k,\ell} \left\{ U_i(x_{-i}^*; \ell) - U_i(x_{-i}^*; k) \mid U_i(x_{-i}^*; k) < U_i(x_{-i}^*; \ell) \right\} \quad .$$

By continuity of the functions U_i we have

$$\lim_{r\to\infty} U_i\left((x_{1/n_r})_{-i}; k\right) = U_i(x_{-i}^*; k) \quad ,$$

for all i and k. Thus let N be an integer such that

$$|U_i\left((x_{1/n_r})_{-i}; k\right) - U_i(x_{-i}^*; k)| \leq \nu/3 \quad ,$$

and such that $1/n_r \leq \nu/3$, for all i,k, and $r \geq N$.

Consider now k and ℓ such that $U_i(x^*_{-i};k) < U_i(x^*_{-i};\ell)$. By construction, for any $r \geq N$ we also have $U_i((x_{1/n_r})_{-i};k) + 1/n_r \leq U_i((x_{1/n_r})_{-i};\ell)$. Since x_{1/n_r} is a $(1/n_r)$-almost ε-proper equilibrium it follows that $(x_{1/n_r})_{ik} \leq \varepsilon(x_{1/n_r})_{i\ell}$. Taking the limit $r \to \infty$ we also have $x^*_{ik} \leq \varepsilon x^*_{i\ell}$, which shows that x^* is a ε-proper equilibrium as claimed. □

4.2 Approximate Selection and Fixed Points

We shall now work toward a fixed point characterization of δ-almost ε-proper equilibrium in Γ_η. We shall first generalize the property defining a δ-almost ε-proper equilibrium in two ways. First, the definition applies to all scalar multiples of mixed strategies. Secondly, the definition incorporates a notion of *action valuation* which take the place of the utilities $U_i(x_{-i};k)$.

Definition 4.4. An action valuation for Player i is an assignment $v_i \in \mathbb{R}^{m_i}$ of valuations to actions. We say that $x_i \in \mathbb{R}^{m_i}_+$ satisfies the δ-almost ε-proper property with respect to action valuation v_i if and only if $x_{ik} \leq \varepsilon x_{i\ell}$ whenever $v_{ik} + \delta \leq v_{i\ell}$, for all $k, \ell \in S_i$.

Note that while we do not require that $x_i \in \Delta^o_{m_i}$ we have that x_i satisfies the δ-almost ε-proper property with respect to action valuation v_i if and only if $x_i/\|x_i\|_1$ does as well.

There is a discontinuity in the δ-almost ε-proper property on the upper bounds imposed on the individual coordinates of x_i as a function of the action valuation v_i. For $\delta = 0$ this discontinuity is inherent, but for $\delta > 0$ we can resolve it by defining a notion of δ-approximate selection that linearly interpolates between two values x and y in an interval of length δ on the right side of a selection point 0.

Definition 4.5. For given $\delta > 0$, the δ-approximate selection function Sel_δ is defined by

$$\mathrm{Sel}_\delta(x,y,z) = \begin{cases} x & \text{if } z \leq 0 \\ (1 - z/\delta)x + (z/\delta)y & \text{if } 0 \leq z \leq \delta \\ y & \text{if } \delta \leq z \end{cases}$$

We next incorporate the δ-approximate selection function into a function $P_{i,\varepsilon}$ that captures the upper bounds imposed on the individual coordinates of x_i.

Definition 4.6. Define the function $P_{i,\varepsilon} : \mathbb{R}^{m_i}_+ \times \mathbb{R}^{m_i} \to \mathbb{R}^{m_i}_+$ by

$$(P_{i,\varepsilon}(x_i, v_i))_k = \min_\ell \{\mathrm{Sel}_\delta(x_{ik}, \varepsilon x_{i\ell}, v_{i\ell} - v_{ik})\}$$

For a fixed action valuation v_i we obtain an operator $P^{v_i}_{i,\varepsilon} : \mathbb{R}^{m_i}_+ \to \mathbb{R}^{m_i}_+$ in the natural way.

Definition 4.7. Let v_i be an action valuation. Then the operator $P^{v_i}_{i,\varepsilon} : \mathbb{R}^{m_i}_+ \to \mathbb{R}^{m_i}_+$ is defined by

$$P^{v_i}_{i,\varepsilon}(x_i) = P_{i,\varepsilon}(x_i, v_i) \enspace .$$

A simple but important property of the operator $P^{v_i}_{i,\varepsilon}$ is that it is *monotone*.

Lemma 4.8. *For any $x_i \in R^{m_i}_+$ and $v \in \mathbb{R}^{m_i}$ we have $P_{i,\varepsilon}(x_i, v_i) \leq x_i$.*

Proof. This follows from the definition of $P_{i,\varepsilon}$ since $\mathrm{Sel}_\delta(x_{ik}, \varepsilon x_{i\ell}, v_{i\ell} - v_{ik}) = x_{ik}$ for $k = \ell$. □

An immediate consequence is that fixed points of the operator are exactly the points whose L^1-norm is preserved by the operator.

Corollary 4.9. *Let $x_i \in \Delta^o_{m_i}$. Then x_i is a fixed point of $P^{v_i}_{i,\varepsilon}$ if and only if $\|P_{i,\varepsilon}(x_i, v_i)\|_1 = 1$.*

The operator was defined precisely for the purpose that fixed points will satisfy the δ-almost ε-proper property. This is easy to verify.

Lemma 4.10. *Suppose $x_i \in \mathbb{R}_+^{m_i}$ is a fixed point of $P_{i,\varepsilon}^{v_i}$. Then x_i satisfies the δ-almost ε-proper property with respect to v_i.*

Proof. Suppose that $v_{ik} + \delta \leq v_{i\ell}$. Then $v_{i\ell} - v_{ik} \geq \delta$, and since x_i is a fixed point of $P_{i,\varepsilon}^{v_i}$ we must have $x_{ik} = (P_{i,\varepsilon}(x_i, v_i))_k \leq \mathrm{Sel}_\delta(x_{ik}, \varepsilon x_{i\ell}, v_{i\ell} - v_{ik}) = \varepsilon x_{i\ell}$. □

The operator $P_{i,\varepsilon}^{v_i}$ is by itself not useful for expressing δ-almost ε-proper equilibrium as fixed points precisely for the reason that it is monotone and hence only preserves that L^1 norm at fixed points. We will instead be able to use it for actually computing a mixed strategy that satisfy the δ-almost ε-proper property with respect to v_i, for a given action valuation v_i. We will see that iterating the $P_{i,\varepsilon}^{v_i}$ operator on the uniform distribution will work for this purpose.

4.3 Computing a δ-almost ε-proper Mixed Strategy

Let $\tau_{m_i} \in \Delta_{m_i}$ be the uniform distribution over the pure strategies of Player i, i.e. $\tau_{m_i k} = 1/m_i$ for $k = 1, \dots, m_i$. For a fixed action valuation v_i, let $(P_{i,\varepsilon}^{v_i})^{\circ j}$ denote the j-th iteration of the operator $P_{i,\varepsilon}^{v_i}$. That is we let $(P_{i,\varepsilon}^{v_i})^{\circ 0}$ simply denote the identity function on $\mathbb{R}_+^{m_i}$ and define $(P_{i,\varepsilon}^{v_i})^{\circ(j+1)} := P_{i,\varepsilon}^{v_i} \circ (P_{i,\varepsilon}^{v_i})^{\circ j}$. Define for all $j \geq 0$ the vectors $\tau_{m_i}^{(j)} = (P_{i,\varepsilon}^{v_i})^{(j)}(\tau_{m_i})$. Starting the iteration of $P_{i,\varepsilon}^{v_i}$ on the uniform distribution means that individual coordinates can not become very small.

Lemma 4.11. *For all $j \geq 0$ and k we have $\tau_{m_i k}^{(j)} \geq \varepsilon^{m_i - 1}/m_i$.*

Proof. By possibly reordering coordinates, we may without loss of generality assume that $v_1 \geq \cdots \geq v_{m_i}$. Note then that $\mathrm{Sel}_\delta(x, y, v_{i\ell} - v_{ik}) = x$ for all $\ell \geq k$. We shall then show by induction in k that $\tau_{m_i k}^{(j)} \geq \varepsilon^{k-1}/m_i$ for all j. For the base case of $k = 1$, by the above we have $\mathrm{Sel}_\delta(\tau_{m_i k}^{(j)}, \varepsilon\tau_{m_i \ell}^{(j)}, v_{i\ell} - v_{ik}) = \tau_{m_i k}^{(j)}$ for all j and ℓ, and hence $\tau_{m_i k}^{(j)} = \tau_{m_i k} = 1/m_i$ for all j. Consider now a general $k \geq 2$. We get $\mathrm{Sel}_\delta(\tau_{m_i k}^{(j)}, \varepsilon\tau_{m_i \ell}^{(j)}, v_{i\ell} - v_{ik}) \geq \min\{\tau_{m_i k}^{(j)}, \varepsilon^{\ell-1}/m_i\}$ for all j and $\ell < k$ and $\mathrm{Sel}_\delta(\tau_{m_i k}^{(j)}, \varepsilon\tau_{m_i \ell}^{(j)}, v_{i\ell} - v_{ik}) = \tau_{m_i k}^{(j)}$ for all j and $\ell \geq k$, from which the result follows. □

Note that by the above lemmas as j tends to infinity the sequence $\tau_{m_i}^{(j)}$ converges to a limit point τ^* that satisfies the δ-almost ϵ-proper property and furthermore satisfies $\tau_k^* \geq \varepsilon^{m_i-1}/m_i$ for all k. It follows that $\tau^*/\|\tau^*\|_1 \in \Delta_{m_i}^{\eta_i}$ and satisfies the δ-almost ϵ-proper property as well. Of course we will only be able to iterate $P_{i,\varepsilon}^{v_i}$ polynomially many times. We will next show that this is sufficient for our purposes.

Lemma 4.12. *For a given k suppose that for every ℓ with $v_{i\ell} \geq v_{ik}$ we have that $\tau_{m_i \ell}^{(j+1)} \geq \sqrt{\varepsilon}\tau_{m_i \ell}^{(j)}$. Then $\tau_{m_i k}^{(j+2)} \geq \sqrt{\varepsilon}\tau_{m_i k}^{(j+1)}$.*

Proof. By assumption, if $v_{i\ell} \geq v_{ik}$ we have

$$\begin{aligned}\mathrm{Sel}_\delta(\tau_{m_i k}^{(j+1)}, \varepsilon\tau_{m_i \ell}^{(j+1)}, v_{i\ell} - v_{ik}) &\geq \mathrm{Sel}_\delta(\sqrt{\varepsilon}\tau_{m_i k}^{(j)}, \varepsilon\sqrt{\varepsilon}\tau_{m_i \ell}^{(j)}, v_{i\ell} - v_{ik}) \\ &= \sqrt{\varepsilon}\,\mathrm{Sel}_\delta(\tau_{m_i k}^{(j)}, \varepsilon\tau_{m_i \ell}^{(j)}, v_{i\ell} - v_{ik})\end{aligned},$$

and by since by definition of Sel_δ, if $v_{i\ell} \leq v_{ik}$ we have $\mathrm{Sel}_\delta(x, y, v_{i\ell} - v_{ik}) = x$ it follows that

$$\begin{aligned}\tau_{m_i k}^{(j+2)} &= \min_\ell \left\{\mathrm{Sel}_\delta(\tau_{m_i k}^{(j+1)}, \varepsilon\tau_{m_i \ell}^{(j+1)}, v_{i\ell} - v_{ik})\right\} \\ &= \sqrt{\varepsilon}\min_\ell \left\{\mathrm{Sel}_\delta(\tau_{m_i k}^{(j)}, \varepsilon\tau_{m_i \ell}^{(j)}, v_{i\ell} - v_{ik})\right\} \\ &= \sqrt{\varepsilon}\tau_{m_i k}^{(j+1)} .\end{aligned}$$

□

Thus, as soon $\tau^{(j)}_{m_i\ell}$ decreases by a factor at no less than $\sqrt{\varepsilon}$ with each application of $P^{v_i}_{i,\varepsilon}$, for all ℓ such that $v_{i\ell} \geq v_{ik}$, the same will be true for $\tau_{m_i k}$. Since each $\tau^{(j)}_{m_i k}$ is bounded from below by ε^{m_i-1}/m_i they must each decrease by a factor no less than $\sqrt{\varepsilon}$ after polynomially many iterations. More precisely we have the following precise bound.

Corollary 4.13. *If $\varepsilon \leq 1/m_i$ and $j \geq 2m_i^2$ then $\tau^{(j+1)}_{m_i k} \geq \sqrt{\varepsilon}\tau^{(j)}_{m_i k}$ for all k.*

Proof. Since $\varepsilon \leq 1/m_i$ we have $\tau^{(j)}_{m_i k} \geq \varepsilon^{m_i}$ for all k and $j \geq 0$. Thus every $\tau^{(j)}_{m_i k}$ can decrease by a factor less than $\sqrt{\varepsilon}$ at most $2m_i$ times. By possibly reordering coordinates, we may without loss of generality assume that $v_1 \geq \cdots \geq v_{m_i}$. By induction in k follows that for every $\ell \leq k$ we have that $\tau^{(j+1)}_{m_i\ell} \geq \sqrt{\varepsilon}\tau^{(j)}_{m_i\ell}$ whenever $j \geq 2m_i k$ by the previous lemma. □

We next show that $\tau^{(j)}_{m_i}$ for our purposes will be essentially as useful as the limit point τ^* for $j \geq 2m_i^2$. First we need another simple observation about the operator $P^{v_i}_{i,\varepsilon}$.

Lemma 4.14. *Suppose k and ℓ are such that $v_{ik} + \delta \leq v_{i\ell}$. For $x_i \in \mathbb{R}^{m_i}_+$ and $x'_i = P^{v_i}_{i,\varepsilon}(x_i)$. If $x'_{ik} > \sqrt{\epsilon}x'_{i\ell}$ it follows that $x'_{i\ell} < \sqrt{\varepsilon}x_{i\ell}$.*

Proof. By the assumptions and the definition of $P^{v_i}_{i,\varepsilon}$ we have $x'_{ik} \leq \epsilon x_{i\ell}$. Thus we have $x'_{i\ell} < 1/\sqrt{\epsilon}x'_{ik} \leq \sqrt{\epsilon}x_{i\ell}$ as stated. □

We can now finally state our conclusion about $\tau^{(j)}_{m_i}$, which follows directly from the above statements.

Proposition 4.15. *Let $\varepsilon \leq 1/m_i$. Then $\tau^{(j)}_{m_i}$ satisfy the δ-almost $\sqrt{\varepsilon}$-proper property for all $j \geq 2m_i^2$.*

4.4 Fixed Point Characterization

We are now finally in position to give a fixed point characterization of δ-almost ε-proper equilibrium. For this we define a function $F^{\varepsilon,\delta}_\Gamma : D^{\eta(\varepsilon^2)}_\Gamma \to D^{\eta(\varepsilon^2)}_\Gamma$ such that every fixed point of $F^{\varepsilon,\delta}_\Gamma$ is a δ-almost ε-proper equilibrium of Γ. Let $x \in D^{\eta(\varepsilon^2)}_\Gamma$ be given and define the following for all i and all $k \in S_i$:

(1) $v_{ik} = U_i(x_{-1}; k)$.
(2) $y_i = (P^{v_i}_{i,\varepsilon^2})^{\circ(2m_i^2)}(\tau_{m_i})$.

We now simply define $(F^{\varepsilon,\delta}_\Gamma(x))_i = y_i/\|y_i\|_1$. Note that by Lemma 4.11 we have that $y_i \in \Delta^{\eta_i(\varepsilon^2)}_{m_i}$ for all i. Hence $(F^{\varepsilon,\delta}_\Gamma(x)) \in D^{\eta(\varepsilon^2)}_\Gamma$ thereby making $F^{\varepsilon,\delta}_\Gamma$ well defined. We next consider the fixed points of $F^{\varepsilon,\delta}_\Gamma$.

Proposition 4.16. *Let $\delta > 0$ and $0 < \varepsilon < 1$. Then every fixed point $x \in D^{\eta(\varepsilon^2)}_\Gamma$ is a δ-almost ε-proper equilibrium of Γ.*

Proof. Suppose that x is a fixed point of $F^{\varepsilon,\delta}_\Gamma(x)$. For each i we then have that $x_i = y_i/\|y_i\|_1$. By Proposition 4.15 we have that y_i satisfies the δ-almost ε-proper property with respect to action valuation v_i by construction. This implies that x_i satisfies the δ-almost ε-proper property with respect to action valuation v_i as well. Since this holds for each i, we conclude that x is a δ-almost ε-proper equilibrium. □

From the above description it is easy to construct in polynomial time a $\{+, -, *, /, \max, \min\}$-circuit computing $F_\Gamma^{\varepsilon,\delta}(x)$, where x, ε, and δ are inputs of the circuit. The function Sel_δ is computable by the formula $\mathrm{Sel}_\delta(x, y, z) = (1 - \max(\min(z, \delta), 0)/\delta)x + (\max(\min(z, \delta), 0)/\delta)y$. The operator $P_{i,\varepsilon}^{v_i}$ was already defined in the required form, and the description of $F_\Gamma^{\varepsilon,\delta}(x)$ above consists mainly of polynomially many such functions composed with each other.

Theorem 4.17. *There exist a function $F_\Gamma^{\varepsilon,\delta} : D_\Gamma^{\eta(\varepsilon^2)} \to D_\Gamma^{\eta(\varepsilon^2)}$ given by a $\{+, -, *, /, \max, \min\}$-circuit computable in polynomial time from Γ, with the circuit having inputs x,$\varepsilon > 0$, and $\delta > 0$ as inputs, such that for all fixed $0 < \varepsilon < 1$ and $\delta > 0$, every fixed point of $F_\Gamma^{\varepsilon,\delta}$ is a δ-almost ε-proper equilibrium of Γ. In particular is the problem of computing a δ-almost ε-proper equilibrium of a finite game with n players in strategic form in* FIXP.

4.5 Approximating Proper Equilibrium

We have now established that proper equilibrium may be approximated by a ε-proper equilibrium which in turn may be approximated by a δ-almost ε-proper equilibrium which in turn can be computed in FIXP. What remains for proving that approximating a proper equilibrium can be done in FIXP_a are effective bounds on how small ε and δ need to be in order to guarantee a good approximation.

We can obtain such bounds in a generic way using the general machinery of real algebraic geometry, cf. Basu et al. [2]. We shall here just outline how this is done and refer to Etessami et al. [10] for full details of an analogous derivation given for trembling hand perfect equilibrium. The main idea is to formalize the "almost implies near" statements of Lemma 4.2 and Lemma 4.3 as formulas in the first order theory of the reals. For Lemma 4.2 a first-order formula is constructed from Γ and γ with a single free variable ε, expressing that ε satisfies the conclusion of the lemma. Applying quantifier elimination to that formula and employing known bounds on the result of this we obtain the following statement.

Lemma 4.18. *There is a constant c_1, such that for all integers n,m, and B and every $\gamma > 0$, the following holds for every $\varepsilon \leq min(\gamma, 1/B)^{n^{c_1 m^3}}$. For any finite game Γ with n players having a total of at most m pure strategies, and with integer payoffs of absolute value at most B, any ε-proper equilibrium of Γ has L^∞-distance at most γ to some proper equilibrium of Γ.*

Similarly, for Lemma 4.3 a first-order formula is constructed from Γ, γ *and* ε with a single free variable δ, expressing that δ satisfies the conclusion of the lemma. Again applying quantifier elimination to that formula and employing known bounds on the result of this we obtain the following statement.

Lemma 4.19. *There is a constant c_2, such that for all integers n,m, and B, every $\gamma > 0$ and every $\varepsilon > 0$, the following holds for every $\delta \leq min(\gamma, \varepsilon, 1/B)^{n^{c_2 m^2}}$. For any finite game Γ with n players having a total of at most m pure strategies, and with integer payoffs of absolute value at most B, any δ-almost ε-proper equilibrium of Γ has L^∞-distance at most γ to some ε-proper equilibrium of Γ.*

We can now finally state the main theorem of this section. The proof is again analogous to a corresponding proof for trembling hand perfect equilibrium by Etessami et al. [10] except that we here construct *two* virtual infinitesimals $\delta \ll \varepsilon$ by means of repeated squaring instead of a single such virtual infinitesimal.

Theorem 4.20. *Given as input a finite game Γ with n players in strategic form having integer payoffs and a rational $\gamma > 0$, the problem of computing a strategy profile x' such that there is a proper equilibrium x of Γ with $\|x' - x\|_\infty < \gamma$ is* FIXP_a*-complete.*

Proof. Let $m = \sum_{i=1}^{n} m_i$ be the total number of pure strategies for the players. Let B be the largest magnitude of a payoff in Γ. By definition of FIXP_a our task is to given $\gamma > 0$ construct a polytope P, a circuit C computing a function $F : P \to P$, and a number γ', such that γ'-approximations to fixed points of F can be efficiently transformed into γ-approximations of proper equilibrium of Γ. We simply let $\gamma' = \gamma/2$ and ensure that fixed points of F are themselves γ-approximations of proper equilibrium of Γ. The polytope P is just the polytope D_Γ of all strategy profiles of Γ, whose defining inequalities can clearly be computed in polynomial time. We next describe the circuit C. First a suitable $\varepsilon > 0$ satisfying the conditions of Lemma 4.18 is computed by repeated squaring of the number $\min(\gamma/4, 1/B)$ exactly $\lceil c_1 m^3 \lg n \rceil$ times. Next a suitable $\delta > 0$ satisfying the conditions of Lemma 4.19 is computed by repeated squaring of the number $\min(\gamma/4, \varepsilon, 1/B)$ exactly $\lceil c_2 m^2 \lg n \rceil$ times. Next the input $x \in D_\Gamma$ is mapped to $D_\Gamma^{\eta(\varepsilon^2)}$ by any mapping that is the identity function on $D_\Gamma^{\eta(\varepsilon^2)}$. This may be done by computing for each i a number t_i such that $\sum_{j=1}^{m_i} \max(x_{ij} - t_i, \eta_i(\varepsilon^2)) = 1$ using a sorting network like Etessami and Yannakakis [11] and then mapping each x_{ij} to $\max(x_{ij} - t_i, \eta_i(\varepsilon^2))$. Next we compute the function $F_\Gamma^{\varepsilon,\delta}$ on the transformed input with the constructed ε and δ, and let this be the output of C. By Theorem 4.17, any fixed point of F is a δ-almost ε-proper equilibrium of Γ. By Lemma 4.19 this is a $\gamma/4$-approximation to an ε-proper equilibrium, which in turn by Lemma 4.18 is a $\gamma/4$-approximation to a proper equilibrium of Γ. Finally, by the triangle inequality, any $\gamma' = \gamma/2$-approximation to a fixed point of F is a $\gamma/2 + \gamma/4 + \gamma/4 = \gamma$-approximation of a proper equilibrium of Γ. This completes the proof. □

5 COMPUTING PROPER EQUILIBRIUM IN POLYMATRIX GAMES

In this section, we show that computing a proper equilibrium can be done in PPAD for some multiplayer games. Specifically, we provide a PPAD algorithm for computing a proper equilibrium of a polymatrix game. We do this by observing that the strategy constraints from Sørensen [26], which gives a symbolic ε-proper equilibrium, can be incorporated into the LCP formulation of Howson [18] for equilibria of polymatrix games. Lemke's algorithm is then shown to always find a solution to the LCP, which thus yields a PPAD algorithm for computing a symbolic ε-proper equilibrium of the given polymatrix game. The symbolic ε-proper equilibrium serves both as a representation of the proper equilibrium (by letting $\varepsilon \to 0^+$), but also as a witness that the equilibrium is indeed proper, thereby circumventing the hardness of recognizing a solution that is given only as standard probabilities.

A polymatrix game G is an n-player game, where the utility of each player can be separated into independent terms for each opponent. Each player plays their chosen strategy pairwise against all other players in independent bimatrix games, and receives the sum as the combined utility. As with strategic form games, player i has m_i pure strategies. The game is specified by a collection of matrices, one matrix $A_{ij} \in \mathbb{R}^{m_i \times m_j}$ for each ordered pair of players (i, j). If each player i plays mixed strategies $x_i \in \Delta_{m_i}$, the expected utility for player i is $U_i(x_1, ..., x_n) = \sum_{j \neq i} x_i^\top A_{ij} x_j$.

These games can be formulated as symmetric bilinear games [13, 22]. While bilinear games is a useful abstraction that would allow us to add the needed strategy constraints, that formulation is not suitable for our purpose for two reasons. First, it needlessly doubles the dimension of the LCP, which is a bad idea, since the running time for solving LCP is in general exponential in the dimension. This would not be a problem in itself, if one only cares about the complexity class. However, secondly, it adds the requirement on the solution to the LCP that it must correspond to a symmetric equilibrium, and not just any equilibrium. This either adds even more constraints to the LCP, or requires using a non-standard LCP solver. We therefore stick closer to the original formulation of Howson [18].

The approach of Howson [18] can be understood as reformulating the n-player polymatrix game into a form where a single meta-player specifies the strategy for all players. For conciseness, we pack all the payoff matrices A_{ij} into one matrix A and the strategy vectors x_i into one vector x in the following way:

$$A = \begin{bmatrix} 0 & A_{12} & \dots & A_{1n} \\ A_{21} & 0 & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & 0 \end{bmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

With this representation, the sum of utilities for all players, when they combined play strategy x, is given by $x^\top Ax$. A strategy profile x is then an equilibrium exactly when $x^\top Ax = \max_y y^\top Ax$.

We now need a brief aside to introduce the strategy constraints from Sørensen [26] that will ensure the ε-proper equilibria. The constraints are built on a proof of existence of proper equilibria from Kohlberg and Mertens [19, Prop. 5]. While Sørensen only applies the construction to two-player games, the proof of Kohlberg and Mertens works for any number of players. In their proof, an auxiliary game G' is constructed where the m_i strategies of player i are replaced by $m_i!$ pure strategies, one for each of the ways that m_i first powers of ε can be assigned to the m_i pure strategies of the original game G. Equilibria of G' are proved to be ε-proper of the original game. The construction by Sørensen allows for representing the mixed strategies of G' directly as a restriction of the strategy space of G, thus avoiding an exponential blowup in size. Specifically, for any number of pure strategies m_i, it gives strategy constraints $E_{m_i}x = e_{m_x}$, with all entries of E_{m_i} being integers and all entries of e_{m_x} being either 0 or a power of ε. In the full construction, the constraints also include inequalities, but standard LP tricks lets us express everything as equalities for brevity, e.g., by using auxiliary variables. In our construction, we will pack these constraints into one constraint matrix and vector as follows:

$$E = \begin{bmatrix} E_{m_1} & 0 & \dots & 0 \\ 0 & E_{m_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & E_{m_n} \end{bmatrix} \qquad e = \begin{pmatrix} e_{m_1} \\ e_{m_2} \\ \vdots \\ e_{m_n} \end{pmatrix}$$

With this, $Ex = e$ will restrict the strategy profile x such that each player of the polymatrix game to play within the ε-permutahedron as defined by Sørensen.

The following derivation will stay fairly closely to that of Koller et al. [20] for the sequence form, since their proofs of correctness almost directly works for our case as well. In fact, the following derivation is almost identical, except that it is specialized to finding an x that is a best response to itself instead of an (x, y) pair that are mutual best responses.

Given a strategy profile y, the task of computing a best response within the strategy constraints is captured by the following LP in the left and its dual on the right

$$\begin{array}{lll} \underset{x}{\text{maximize}} & x^\top(Ay) & \\ \text{subject to} & x^\top E^\top & = e^\top \\ & x & \geq 0 \end{array} \qquad \begin{array}{ll} \underset{p}{\text{minimize}} & e^\top p \\ \text{subject to} & E^\top p \geq Ay \end{array} \tag{1}$$

By complementary slackness, x is a best response to y whenever

$$x^\top(-Ay + E^\top p) = 0 \tag{2}$$

This also holds true for $y = x$, so replacing y with x and constraining x to the desired strategy space, we get

$$\begin{aligned} x^\top(-Ax + E^\top p) &= 0 \\ Ex &= e \\ x &\geq 0 \end{aligned} \tag{3}$$

An LCP in standard form is specified by a pair (b, M) with $b \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$. The problem is to find $z \in \mathbb{R}^n$ such that

$$\begin{aligned} b + Mz &\geq 0 \\ z^\top(b + Mz) &= 0 \\ z &\geq 0 \end{aligned} \tag{4}$$

The constraints specified in this section can be packed into an LCP in standard form in the following way. If we let $z = (x, p', p'')^\top$, where $p = p' - p''$, and

$$M = \begin{pmatrix} -A & E^\top & -E^\top \\ -E & 0 & 0 \\ E & 0 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ e \\ -e \end{pmatrix}. \tag{5}$$

then the LCP standard form of (4) gives us exactly the desired constraints. This means that in a solution to the LCP, the strategy profile x is a best response to itself within the restricted strategy space, and therefore an equilibrium of the auxiliary game, and therefore an ε-proper equilibrium of the given polymatrix game.

It is worth making two observations here. First, the above construction generalizes that of Sørensen, since a bimatrix game is the special case of polymatrix games with only two players. This generalization is direct in that if a bimatrix game is seen as a polymatrix game, the above construction will yield exactly the LCP of Sørensen. Secondly, the derivation above is the same for many other strategy constraints. In fact, if we just restrict the strategies to be the normal strategies of polymatrix games, we get exactly the original LCP of Howson.

5.1 Solving the LCP with Lemke's algorithm

The standard algorithm for solving LCPs in standard form is the one provided by Lemke [21]. The algorithm performs a sequence of *complementary pivots* on a relaxation of the LCP, maintaining the complementarity condition $z^\top(b+Mz) = 0$ while searching for a basic solution where the relaxation disappears. The pivots are similar to those of the simplex method [7] for linear programming, but the complementarity condition specifies the entering and leaving variables (for non-degenerate LCPs). Pivoting continues until the variable introduced by the relaxation can leave the basis and become 0. If this happens, the current solution to the relaxed problem is also a solution to the original problem. For a thorough exposition of Lemke's algorithm, see the monograph by Cottle et al. [6].

When applying Lemke's algorithm to find a solution to an LCP, there are two pitfalls that must be avoided. The first was hinted at in the above; an LCP can be degenerate, in which case the complementarity condition is not enough to specify the next pivot. This is typically handled by slightly perturbing b with different powers of an indeterminate infinitesimal ε, which is ignored when reading off the solution at the end. Just like for Sørensen [26], this perturbation of the b vector can also give an way of keeping the powers of ε needed for the ε-permutahedra in the strategy constraints. Those ε end up in the b vector of the LCP, so they can serve a dual purpose. Any implementation of Lemke's algorithm using the perturbation trick can easily output the solution with these perturbations intact.

The second, and more troublesome pitfall, is an alternative undesirable way to terminate: the current entering variable might not be restricted by a basic variable. This is known as ray termination. To prove that the algorithm works for our case, we must prove that ray termination cannot happen. To do this, we use the following theorem by Koller et al. [20]

Theorem 5.1 (Koller et.al.'96). *If (i) $z^\top Mz \geq 0$ for all $z \geq 0$, and (ii) $z \geq 0$, $Mz \geq 0$ and $z^\top Mz = 0$ imply $z^\top b \geq 0$, then Lemke's algorithm computes a solution of the LCP (2.9) and does not terminate with a secondary ray.*

Our application of the theorem is similar to that of Koller et al., and we need a similar set of lemmas.

Lemma 5.2. *The only non-negative solution x to $Ex = 0$ is $x = 0$.*

Proof. Observe that the constraint $Ex = 0$ unfolds into independent constraints $E_{m_1}x_1 = e_{m_1}$, $E_{m_2}x_2 = e_{m_2}, \ldots, E_{m_n}x_n = e_{m_n}$. By Lemma 5.2 of Sørensen [26], the statement holds true for each of those individually, and therefore also for the combined constraint. □

Lemma 5.3. *If $E^\top p \geq 0$, then $e^\top p \geq 0$.*

Proof. Consider the LP for finding any feasible strategy profile, (and it's dual on the right):

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & 0 \\ \text{subject to} \quad & Ex = e \\ & x \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \underset{p}{\text{minimize}} \quad & e^\top p \\ \text{subject to} \quad & E^\top p \geq 0 \end{aligned} \tag{6}$$

Since the primal is feasible and has value 0, by weak duality the objective function of the dual is lower bounded by 0, i.e., $e^\top p \geq 0$. □

Theorem 5.4. *If $A \leq 0$, then M and b in (5) satisfy all assumptions of Theorem 5.1.*

Proof. Let $z = (x, p', p'')^\top \geq 0$ and $p = p' - p''$. Then we have that $z^\top Mz = -x^\top Ax \geq 0$, satisfying condition (i) of Theorem 5.1. Furthermore, $Mz \geq 0$ implies $-Ax + E^\top p \geq 0$ and $Ex = 0$. Combining the latter with non-negativity of x, Lemma 5.2 implies that $x = 0$. Combining this with the first, we get $E^\top p \geq 0$. By Lemma 5.3, this implies that $e^\top p \geq 0$. Finally, $z^\top b = e^\top p \geq 0$, showing that we satisfy condition (ii) of Theorem 5.1. □

The condition $A \leq 0$ can be ensured by subtracting a suitably large constant from the payoff of all players. This does not change the set of proper equilibria, as the value of best replies are shifted by the same amount. Thus, ray termination is not a possibility, and Lemke's algorithm will terminate with an equilibrium of G'. The solution can be read of with the values being formal polynomials in ε, with ε itself being an indeterminate infinitesimal. For all sufficiently small values of ε, this solution is an ε-proper equilibrium of the original polymatrix game. The limit point for $\varepsilon \to 0^+$ is simply the 0-th order terms of the polynomials. Thus the solution provides both a proper equilibrium, and a witness of this in the form of a symbolic sequence of ε-proper equilibria.

All of this combined allows us to strengthen the main result of Sørensen [26] from bimatrix games to polymatrix games:

THEOREM 5.5. *A symbolic ε-proper equilibrium for a given polymatrix game can be computed by applying Lemke's algorithm to an LCP of polynomial size.*

The last step of the way to PPAD is by using an orientation [27] of Lemke's algorithm, thus again strengthening the matching corollary of Sørensen [26]

COROLLARY 5.6. *The refinement of proper equilibria, corresponding to Kohlberg and Mertens' proof of existence, is* PPAD*-complete to compute for a given polymatrix game.*

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