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# Some sharp inequalities for integral operators with homogeneous kernel

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**Abstract**

One goal of this paper is to show that a big number of inequalities for functions in $L^p(R_+)$, $p \geq 1$, proved from time to time in journal publications are particular cases of some known general results for integral operators with homogeneous kernels including, in particular, the statements on sharp constants. Some new results are also included, *e.g.* the similar general equivalence result is proved and applied for $0 < p < 1$. Some useful new variants of these results are pointed out and a number of known and new Hardy-Hilbert type inequalities are derived. Moreover, a new Pólya-Knopp (geometric mean) inequality is derived and applied. The constants in all inequalities in this paper are sharp.

**MSC:** 26D10; 26D15

**Keywords:** inequalities; Hardy type inequalities; Hardy-Hilbert type inequalities; homogeneous kernels; sharp constants

## 1 Introduction

Let $p > 0$ and denote by $p'$ the conjugate parameter defined by $\frac{1}{p} + \frac{1}{p'} = 1$ ($p' = \infty$ when $p = 1$). We also let $f$ and $g$ denote arbitrary measurable positive functions on $(0, \infty)$. The constants in all inequalities below and in all of this paper are sharp.

*Hilbert's inequality*: The inequality

$$\int_0^\infty \int_0^\infty \frac{1}{x+y} f(x)g(y)\,dx\,dy \leq \frac{\pi}{\sin\frac{\pi}{p}} \left(\int_0^\infty f^p(x)\,dx\right)^{1/p} \left(\int_0^\infty g^{p'}(y)\,dy\right)^{1/p'} \quad \text{for } p > 1 \tag{1}$$

is called Hilbert's inequality. It can equivalently be written in the form

$$\int_0^\infty \left(\int_0^\infty \frac{1}{x+y} f(x)\,dx\right)^p dy \leq \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^p \left(\int_0^\infty f^p(x)\,dx\right). \tag{2}$$

**Remark 1** Hilbert himself considered only the case $p = 2$ and the corresponding discrete form of (1) (see his paper [1] from 1906 and also [2, 3] and the historical description in [4]). $L^p$-spaces with $p \neq 2$ appeared only later (around 1920). Concerning the equivalence of (1) and (2) see our Lemma 15 for a more general statement.

*Hardy's inequality*: The first weighted form of Hardy's inequality can be written in the following way:

$$\int_0^\infty \left( x^{\alpha-1} \int_0^x \frac{f(y)}{y\alpha}\, dy \right)^p dx \le \left( \frac{p}{p-\alpha-1} \right)^p \left( \int_0^\infty f^p(x)\, dx \right), \tag{3}$$

where $p \ge 1$, $\alpha < p-1$. The (equivalent) dual form of (3) reads

$$\int_0^\infty \left( x^{\alpha-1} \int_x^\infty \frac{f(y)}{y\alpha}\, dy \right)^p dx \le \left( \frac{p}{1+\alpha-p} \right)^p \left( \int_0^\infty f^p(x)\, dx \right), \tag{4}$$

where $p \ge 1$, $\alpha > p-1$.

**Remark 2** For the case $\alpha = 0$ (3) is the classical Hardy inequality. The almost 10 years of research until Hardy finally proved this inequality in 1925 (see [5]) is described in detail in [4]. In particular, it is completely clear that Hardy's motivation was to find an elementary proof of Hilbert's inequality for the discrete case. Also the weighted variant (3) was first proved by Hardy (see [6]). The further development of inequalities (3) and (4) to what today is called Hardy-type inequalities is very extensive and still a very active area of research (see *e.g.* the monographs [7] and [8] and [9]) and the references given there.

*Hardy-Hilbert type inequalities for homogeneous kernels*: The inequalities (2)-(4) can all be written in the unified form

$$\int_0^\infty \left( \int_0^\infty k(x,y) f(x)\, dx \right)^p dy \le C^p \int_0^\infty f^p(x)\, dx, \quad p \ge 1 \tag{5}$$

with different kernels $k(x,y)$which are homogeneous of degree $-1$. A kernel $k(x,y)$ is said to be homogeneous of degree $\lambda$, $\lambda \in R$, if

$$k(tx,ty) = t^\lambda k(x,y) \quad \text{for all } x,y \in R_+.$$

It is also well known that the inequality (5) can be equivalently rewritten in the form

$$\int_0^\infty \int_0^\infty k(x,y) f(x) g(y)\, dx\, dy \le C \left( \int_0^\infty f^p(x)\, dx \right)^{\frac{1}{p}} \left( \int_0^\infty g^{p'}(y)\, dy \right)^{\frac{1}{p'}}, \quad p \ge 1, \tag{6}$$

with the same sharp constant $C$.

**Remark 3** There are a huge number of papers devoted to the proof of (5) and (6) for concrete kernels $k(x,y)$ other than the classical Hilbert kernel $k(x,y) = 1/(x+y)$. In this connection we refer to the monograph [10] and the references there. Moreover, we announce that by using a standard dilation argument in (5)-(6) we see that such kernels must be homogeneous of degree $-1$. One weakness with many of these results is that the authors do not refer to the fact that already in 1999 (see [11] and also [12]) it was given necessary and sufficient conditions for (5) to hold and with sharp constant and general kernel of degree $-1$. See Theorem 5.

One main aim of this paper is to discuss, complete, and apply this result to get an overview of the current situation partly described in Remark 3. See Theorem 5 and the discussion in Remark 6. Moreover, the following new results are included:

(a) A general reversed version of the inequalities described in Remark 3 yielded for $0 < p < 1$. See Theorem 7.

(b) A corresponding equivalence theorem for homogeneous kernels of any order $\lambda$ but with the right-hand sides in Theorem 5 and Remark 6 replaced by some corresponding weighted $L^p$-spaces so that our main results can be used. See Theorem 10 and Remark 11.

(c) In order to be able to cover also some other results in the literature we derive a version for 'skew symmetric' kernels of order $-1$ (for the definition see (21)). See Theorem 13 and Remark 14.

(d) A completely new geometric mean (Pólya-Knopp) type inequality is derived (see Theorem 30). Moreover, we present a number of applications of this, which seems to be new too.

(e) As applications a number of new (and also well-known) sharp inequalities are presented.

**Remark 4** The inequality

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln f(y)\,dy\right)dx \le e\left(\int_0^\infty f(x)\,dx\right) \tag{7}$$

is just a limit case as $p \to \infty$ of the Hardy inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\,dx. \tag{8}$$

In fact, just replace $f(x)$ by $(f(x))^{1/p}$ in (8) and use the fact that (the scale of power means) $(\frac{1}{x}\int_0^x f(y)^{1/p}\,dy)^p$ converges to the geometric mean

$$\exp\left(\frac{1}{x}\int_0^x f(y)\,dy\right) \quad \text{and} \quad \left(\frac{p}{p-1}\right)^p \to e \quad \text{as } p \to \infty.$$

Sometimes (7) is called Knopp's inequality with reference to his paper [13] from 1928 but Hardy himself in his 1925 paper [5] said that Pólya pointed out this argument to him so we prefer to call the inequality (7) the Pólya-Knopp inequality.

The paper is organized as follows: Some main results are presented and commented in Section 2. The detailed proofs are given in Section 3. Some applications concerning Hardy and Hardy-Hilbert type inequalities are presented in Section 4. Finally, Section 5 is reserved for another main result, namely the announced new Pólya-Knopp type inequality. Some applications of this result are also given. All inequalities in this paper have sharp constants.

## 2 Main results

We consider the integral operator $K$ defined by

$$Kf(x) := \int_0^\infty k(x,y)f(y)\,dy, \quad x \in R_+, \tag{9}$$

with nonnegative kernel $k(x,y)$ (a measurable function on $R_+ \times R_+$), which is homogeneous of degree $-1$, *i.e.*

$$k(tx,ty) = t^{-1}k(x,y), \quad x,y \in R^+, t > 0. \tag{10}$$

For such kernels we also define the constant

$$\kappa_p := \int_0^\infty k(1,y)y^{-1/p}\,dy = \int_0^\infty k(x,1)x^{-1/p'}\,dx, \quad p > 0. \tag{11}$$

Here and in the sequel $\frac{1}{p} + \frac{1}{p'} = 1$ when $p \neq 1$, and $p' = \infty$ when $p = 1$ (*e.g.* $x^{-1/p'} = 1$ when $p = 1$).

Our first main results reads as follows.

**Theorem 5** *Let $p \geq 1$, the kernel $k(x,y)$ satisfy* (10) *and $\kappa_p$ be the constant defined by* (11). *Then the following three statements are equivalent*:

(i) *The constant $\kappa_p < \infty$.*

(ii) *The inequality*

$$\int_0^\infty \int_0^\infty k(x,y)f(x)g(y)\,dx\,dy \leq C\|f\|_p\|g\|_{p'} \tag{12}$$

*holds for some finite constant $C$ for all $f \in L_p$ and $g \in L_{p'}$.*

(iii) *The inequality*

$$\int_0^\infty \left(\int_0^\infty k(x,y)f(x)\,dx\right)^p dy \leq C^p \int_0^\infty f^p(x)\,dx \tag{13}$$

*holds for the same finite constant $C$ as in* (12) *and all $f \in L_p$.*

*Moreover, the constant $C = \kappa_p$ is sharp in both* (12) *and* (13).

**Remark 6** The proof of (12) under the condition $\kappa_p < \infty$ was given already in the book [14], Theorem 3.19. Apart from the original proof in [14], this sufficiency part may be derived, via a change of variables, from the Young theorem for convolutions in $R$, for details see [12] and [11]. In this way the sharpness of the constant is derived from the fact that the Young inequality $\|h * f\|_p \leq \|h\|_1\|f\|_p$ holds with the sharp constants $\|h\|_1$ when $h$ is nonnegative. Hence, by using the results in [12] and [11] and the equivalence result in Lemma 15, Theorem 5 is essentially known even if it has not been formulated in this way before. However, to make our paper self-contained we include a proof which also guides us how to prove the other results in this section.

For the case $0 < p < 1$ it is expected that the inequalities (12) and (13) hold in the reversed direction but now with the natural restrictions

$$I_1 = \int_0^\infty \int_0^\infty k(x,y)f(x)g(y)\,dx\,dy < \infty \tag{14}$$

and

$$I_2 = \int_0^\infty \left(\int_0^\infty k(x,y)f(x)\,dx\right)^p dy < \infty, \tag{15}$$

so the reversed inequalities (12) and (13) make sense. We also need the following minor technical condition:

$$\lim_{\varepsilon\to 0+}\int_0^{\varepsilon_0} k(1,y)y^{-1/p}y^{(p-1)\varepsilon}\,dy=\int_0^{\varepsilon_0} k(1,y)y^{-1/p}\,dy \tag{16}$$

for some $\varepsilon_0>0$.

**Theorem 7** *Let* $0<p<1$ *and the kernel* $k(x,y)$ *satisfy* (10). *Moreover, assume that* (14)-(16) *hold. Then all the statements in Theorem* 5 *hold with inequalities* (12) *and* (13) *holding in reversed direction.*

Since $p'<0$ in this case we have $\|g\|_{p'}=(\int_0^\infty |g(y)|^{p/(p-1)}\,dy)^{\frac{p-1}{p}}$ and we assume that $0<\|g\|_{p'}<\infty$ here and in the sequel.

**Remark 8** For the proof of the fact that $\kappa_p<\infty$ implies the equivalent reversed conditions (12) and (13) we do *not* need the restriction (16).

A kernel $k(x,y)$ is said to be homogeneous of degree $\lambda_0$ if

$$k_{\lambda_0}(tx,ty)=t^{\lambda_0}k(x,y),\quad x,y\in R, t>0. \tag{17}$$

**Remark 9** By using a standard dilation argument it is seen that the inequalities considered in Theorem 5 can hold if and only if $\lambda=-1$. However, by changing the norms in the left-hand sides in (12) and (13) to power-weighted norms we can from our result obtain a similar result for homogeneous kernels of any degree $\lambda$. In order to be able to compare with a result in [15] we formulate this result as follows.

**Theorem 10** *Let* $p\geq 1$ *and* $\alpha,\beta\in R$. *Let the kernel* $k_{\lambda_0}(x,y)$ *satisfy* (17) *for* $\lambda_0=-1+\alpha+\beta$, *and define*

$$\kappa_{p,\beta}=\int_0^\infty k_{\lambda_0}(1,y)y^{-\beta-(1/p)}\,dy. \tag{18}$$

*Then the following three conditions are equivalent*:

(i*) *The constant* $\kappa_{p,\beta}<\infty$.

(ii*) *The inequality*

$$\int_0^\infty\int_0^\infty k_{\lambda_0}(x,y)f(x)g(y)\,dx\,dy\leq C\|f\|_{p,x^\alpha}\|g\|_{p',x^\beta} \tag{19}$$

*holds for some finite constant* $C$ *for all* $f\in L_{p,x^\alpha}$ *and* $g\in L_{p',x^\beta}$.

(iii*) *The inequality*

$$\int_0^\infty\left(y^{-\beta}\int_0^\infty k(x,y)f(x)\,dx\right)^p dy\leq C^p\int_0^\infty f^p(x)x^{\alpha p}\,dx \tag{20}$$

*holds for the same finite constant* $C$ *as in* (19) *and all* $f\in L_{p,x^\alpha}$.

(iv*) *The constant* $C = \kappa_{p,\beta}$ (*defined by* (18)) *is sharp in both* (19) *and* (20).

**Remark 11** By choosing $\lambda = -\lambda_0$, $\alpha = 1 - \frac{\lambda}{r} - \frac{1}{p}$, $\beta = 1 - \frac{\lambda}{s} - \frac{1}{p'}$ $(= -\frac{\lambda}{s} + \frac{1}{p})$ with $s > 1$, $\frac{1}{r} + \frac{1}{s} = 1$ we can compare with Theorem 2.1 in [15]. For the case $p > 1$, $\lambda_0 > 0$ the equivalence in (ii*) and (iii*) were established already in this Theorem and also the sharpness in (iv*) for these cases. However, the necessity pointed out in (i*) was not explicitly pointed out in this paper.

**Remark 12** By using our Theorem 7 and making similar calculations as in the proof of Theorem 10 we can obtain a similar complement and strengthening of Theorem 2.2 in [15] yielding for $0 < p < 1$ and kernels of any homogeneity $\lambda_0 \in R$.

In order to cover even more direct applications we finally also state another consequence (but also formal generalization) of Theorem 5. We consider here (skew-symmetric) kernels with the following generalized homogeneity of order $-1$:

$$k\bigl(t^a x, t^b y\bigr) = t^{-1} k(x,y), \quad a, b \neq 0. \tag{21}$$

**Theorem 13** *Let* $p \geq 1$ *and let the kernel* $k(x,y)$ *satisfy* (21) *with* (*generalized duality*) *condition* $\frac{a}{p'} + \frac{b}{p} = 1$ *and define*

$$\kappa_{p,\beta}(a,b) := \left(\frac{a}{b}\right)^{\frac{1}{p'}} \int_0^\infty k(1,t) t^{\frac{1}{b}[(\frac{b-2}{p}+1)]-1}\, dt.$$

*Then the following conditions are equivalent*:

(i) *The constant* $\kappa_p(a,b) < \infty$.

(ii) *The inequality*

$$\int_0^\infty \int_0^\infty k(x,y) f(x) g(y)\, dx\, dy \leq C \|f\|_p \|g\|_{p'} \tag{22}$$

*holds for some finite constant* $C$ *for all* $f \in L_p$ *and* $g \in L_{p'}$.

(iii) *The inequality*

$$\int_0^\infty \left(\int_0^\infty k(x,y) f(x)\, dy\right)^p dx \leq C^p \int_0^\infty f^p(x)\, dx \tag{23}$$

*holds for the same finite constant* $C$ *as in* (22) *and all* $f \in L_p$.

(iv) *The sharp constant in both* (22) *and* (23) *is* $C = \kappa_p(a,b)$.

**Remark 14** By using a similar proof to that of Theorem 13 we can obtain a similar consequence (and formal extension) also of our Theorem 7.

For the proof of these Theorems we need a lemma of independent interest, which we state and prove in a little more general form. Let $k(x,y)$ denote a positive kernel on $R_+ \times R_+$.

**Lemma 15**

(a) *Let* $p \geq 1$. *The following statements are equivalent*:

(i) *The inequality*

$$\int_0^\infty \int_0^\infty k(x,y)f(x)g(y)\,dx\,dy \le C\|f\|_p\|g\|_{p'} \tag{24}$$

*holds for some finite constant* $C$ *and all* $f \in L_p$ *and* $g \in L_{p'}$.

(ii) *The inequality*

$$\int_0^\infty \left(\int_0^\infty k(x,y)f(x)\,dx\right)^p dy \le C^p \int_0^\infty f^p(x)\,dx \tag{25}$$

*holds for the same finite constant* $C$ *as in* (24) *and all* $f \in L_p$.

(b) *Let* $0 < p < 1$. *A similar equivalence to that in* (a) *holds also in this case but with the inequalities in* (24) *and* (25) *reversed* (*here we use the same convention concerning* $\|g\|_{p'}$ *as before*, *see the sentence after Theorem* 7).

**Remark 16** The statement in (a) is well known and follows from a more general statement in functional analysis. However, we give here another simple direct proof which works also to prove that part (b) holds, which seems not to have been explicitly stated before.

## 3 Proofs

*Proof of Lemma* 15 (a) Let $p > 1$. Assume that (25) holds. Then, by using Hölder's inequality, we find that

$$\begin{aligned} I_1 &= \int_0^\infty \int_0^\infty k(x,y)f(x)g(y)\,dx\,dy \\ &\le \left(\int_0^\infty \left(\int_0^\infty k(x,y)f(x)\,dx\right)^p dy\right)^{1/p} \left(\int_0^\infty g^{p'}(y)\,dy\right)^{\frac{1}{p'}} \\ &\le C\|f\|_p\|g\|_{p'}, \end{aligned}$$

so (24) holds. Now assume that (24) holds and choose

$$g(y) = \left(\int_0^\infty k(x,y)f(x)\,dx\right)^{p-1} \in L_{p'}.$$

With this choice

$$I_1 = \int_0^\infty \left(\int_0^\infty k(x,y)f(x)\,dx\right)^p dy := I_2.$$

Thus, by (24),

$$\begin{aligned} I_2 &\le C\|f\|_p \left(\int_0^\infty \left(\int_0^\infty k(x,y)f(x)\,dx\right)^p dy\right)^{\frac{1}{p'}} \\ &= C\|f\|_p I_2^{\frac{1}{p'}}. \end{aligned}$$

Hence,

$$I_2 \le C\|f\|_p$$

so (25) holds.

Let $p = 1$ so $p' = \infty$. By applying (24) with $g(y) \equiv 1$ we see that (24) implies (25). Moreover, by using that $g(y) \leq \|g\|_\infty$, $y \in (0,\infty)$, we find that (25) implies (24).

(b) Hölder's inequality holds in the reversed direction in this case. Hence, the proof of (b) only consists of obvious modifications of the proof of (a). □

*Proof of Theorem* 5 Let $p > 1$ and assume that i) holds. Then, by Hölder's inequality and $K$ defined by (9), we have

$$\begin{aligned} Kf(x) &= \int_0^\infty k(x,y)f(y)\,dy \\ &= \int_0^\infty y^{-\frac{1}{p'p}}\big(k(x,y)\big)^{\frac{1}{p'}} y^{\frac{1}{p'p}}\big(k(x,y)\big)^{\frac{1}{p}}\,dy \\ &\leq \left(\int_0^\infty y^{-\frac{1}{p}}k(x,y)\,dy\right)^{\frac{1}{p'}}\left(\int_0^\infty y^{-\frac{1}{p'}}k(x,y)f^p(y)\,dy\right)^{\frac{1}{p}} := I_1^{\frac{1}{p'}} I_2^{\frac{1}{p}}. \end{aligned}$$

In $I_1$ we change the variable $y$ to $yx$ and use (10) and (11) to obtain

$$\begin{aligned} I_1 &= \int_0^\infty (yx)^{-\frac{1}{p}}k(x,xy)x\,dy \\ &= x^{-\frac{1}{p}}\int_0^\infty k(1,y)y^{-\frac{1}{p}}\,dy = x^{-\frac{1}{p}}\kappa_p. \end{aligned}$$

We conclude that

$$\|Kf\|_{L_p} \leq \kappa_p^{\frac{1}{p'}}\left(\int_0^\infty x^{-\frac{1}{p}}\int_0^\infty y^{\frac{1}{p'}}k(x,y)f^p(y)\,dy\right)^{\frac{1}{p}}. \tag{26}$$

We now change the variable $x$ to $xy$ using (10) and (11) to find that

$$\begin{aligned} I_3 &:= \int_0^\infty x^{-\frac{1}{p'}}y^{\frac{1}{p'}}k(x,y)f^p(y)\,dy \\ &= \int_0^\infty (xy)^{-\frac{1}{p'}}y^{\frac{1}{p'}}k(xy,y)f^p(y)x\,dy = \int_0^\infty x^{-\frac{1}{p}}k(x,1)f^p(y)\,dy. \end{aligned}$$

Hence, by (11), (26), and the Fubini theorem,

$$\|Kf\|_{L_p} \leq \kappa_p^{\frac{1}{p'}}\kappa_p^{\frac{1}{p}}\|f\|_p = \kappa_p\|f\|_p,$$

which means that (13) holds with $C = \kappa_p^p$ for any $f \in L_p$.

Next we assume that (13) holds for some $C < \infty$ and all $f \in L_p$. By using the sharpness in Hölder's inequality we have the following representation formula:

$$\|Kf\|_{L_p} = \sup_{\|\Psi\|_{p'}=1}\int_0^\infty\int_0^\infty k(x,y)f(y)\,dy\,\Psi(x)\,dx. \tag{27}$$

Let $\varepsilon > 0$ and consider the following test function:

$$f_\varepsilon(y) = \begin{cases} y^{-\frac{1}{p}-\varepsilon}, & y \geq 1, \\ 0, & 0 \leq y < 1. \end{cases}$$

Moreover, let

$$\Psi = \Psi_\varepsilon(y) = (\varepsilon p)^{\frac{1}{p'}} \left(f_\varepsilon(y)\right)^{p-1},$$

which has the property $\|\Psi\|_{p'} = 1$.

We note that

$$\|f_\varepsilon\|_p = \left(\frac{1}{\varepsilon p}\right)^{\frac{1}{p}}. \tag{28}$$

Moreover,

$$\begin{aligned} I_4 &:= \int_0^\infty \int_0^\infty k(x,y) f_\varepsilon^p(y)\,dy\,\Psi_\varepsilon(x)\,dx \\ &= (\varepsilon p)^{\frac{1}{p'}} \int_1^\infty x^{-\frac{1}{p'}-\varepsilon(p-1)} \left(\int_1^\infty k(x,y) y^{-\frac{1}{p}-\varepsilon}\,dy\right) dx. \end{aligned} \tag{29}$$

Furthermore, by changing the variable $y$ to $yx$ and using (10) we find that

$$\begin{aligned} \int_1^\infty k(x,y) y^{-\frac{1}{p}-\varepsilon}\,dy &= \int_{\frac{1}{x}}^\infty k(x,yx) x y^{-\frac{1}{p}-\varepsilon}\,dy \\ &= x^{-\frac{1}{p}-\varepsilon} \int_{\frac{1}{x}}^\infty k(1,y) y^{-\frac{1}{p}-\varepsilon}\,dy. \end{aligned}$$

We insert this into (29) and use Fubini's theorem to obtain

$$\begin{aligned} I_4 &= (\varepsilon p)^{\frac{1}{p'}} \int_1^\infty x^{-1-\varepsilon p} \int_{\frac{1}{x}}^\infty k(1,y) y^{-\frac{1}{p}-\varepsilon}\,dy\,dx \\ &= (\varepsilon p)^{\frac{1}{p'}} \int_0^\infty k(1,y) y^{-\frac{1}{p}-\varepsilon} \left(\int_{\max(1,1/y)}^\infty x^{-1-\varepsilon p}\,dx\right) dy \\ &= (\varepsilon p)^{-\frac{1}{p}} \int_0^\infty k(1,y) y^{-\frac{1}{p}-\varepsilon} \left(\max(1,1/y)\right)^{\varepsilon p}\,dy. \end{aligned}$$

Hence, by using (12), (27), (28), together with this inequality, we conclude that

$$C \geq \int_0^\infty k(1,y) y^{-\frac{1}{p}-\varepsilon} \left(\max(1,1/y)\right)^{\varepsilon p}\,dy. \tag{30}$$

Thus, by letting $\varepsilon \to 0^+$ in (30) and using the Fatou lemma, we see that (i) holds and

$$\kappa_p \leq C < \infty.$$

The proof of the equivalence of (i) and (iii) is complete including the fact that $C = \kappa_p^p$ is the sharp constant in (13).

Moreover, by using Lemma 15, we see that statements (i) and (ii) are equivalent including the fact that the constant $C = \kappa_p$ is sharp also in (12). We have thus also proved that statement (iv) is correct.

For the case $p = 1$ we again change the variable $x$ to $yx$ and use (10) to obtain

$$\begin{aligned}\int_0^\infty \int_0^\infty k(x,y)f(y)\,dy\,dx &= \int_0^\infty \int_0^\infty k(yx,y)f(y)x\,dy\,dx \\ &= \int_0^\infty \int_0^\infty k(x,1)\,dx f(y)\,dy = \kappa_1 \int_0^\infty f(y)\,dy\end{aligned}$$

*i.e.* (13) holds even with equality with constant $\kappa_1$ and all $f \in L_1$. In particular, the equivalence of (i) and (iii) is proved. The equivalence of (ii) and (iii) follows from Lemma 15 and the statement (iv) is obvious. The proof is complete. □

*Proof of Theorem* 7 First we note that Hölder's inequality holds in the reversed direction so the proof of the necessity part follows exactly as in the proof of Theorem 5. For the proof of the sufficiency part instead of the representation formula (27) in the case $0 < p < 1$ we use the corresponding representation formula,

$$\|Kf\|_{L_p} = \inf_{\|\Psi\|_{p'}=1} \int_0^\infty \int_0^\infty k(x,y)f(y)\,dy\Psi(x)\,dx,$$

with the same interpretation of $\|\Psi\|_{p'}$ as mentioned just after Theorem 7. By using the same test function $f_\varepsilon$ and the corresponding $\Psi_\varepsilon$ we now come to that (30) holds in the reversed direction but the problem is now that we cannot use the Fatou lemma. However, according to (16) we have

$$\begin{aligned}C^{\frac{1}{p}} \le & \int_0^{\varepsilon_0} k(1,y)y^{-\frac{1}{p}}y^{(p-1)\varepsilon}\,dy \\ & + \varepsilon_0^{(p-1)\varepsilon}\int_{\varepsilon_0}^1 k(1,y)y^{-\frac{1}{p}}\,dy \\ & + \int_1^\infty k(1,y)y^{-\frac{1}{p}}\,dy \\ \to \kappa_p & \quad \text{as } \varepsilon \to 0.\end{aligned}$$

This shows that the constant $C = \kappa_p^p$ is sharp in the reversed form of (13). The remaining part of the proof follows by applying Lemma 15(b). □

*Proof of Theorem* 10 Consider Theorem 5 with $f$ replaced by $fx^\alpha$, $g$ replaced by $gx^\beta$, and the kernel

$$k(x,y) := \frac{k_{\lambda_0}(x,y)}{x^\alpha y^\beta}$$

which is homogeneous of degree –1. Hence, Theorem 10 follows from Theorem 5. (Note that $k(1,y) = k_{\lambda_0}(1,y)y^{-\beta}$.) □

*Proof of Theorem* 13 Introduce the auxiliary kernel $k_1(x,y) := k(a^a, y^b)$ which obviously is homogeneous of order –1 in usual sense. Moreover, in (22) we make the changes of variables $x = u^a$ and $y = v^b$ and define

$$F(u) := f\left(u^a\right)u^{\frac{a-1}{p}}$$

and

$$G(v) := g^{p'}\left(v^b\right)v^{\frac{b-1}{p}}.$$

This leads us to consider the kernel

$$k_2(u,v) := k_1(u,v)u^{\frac{a-1}{p'}}v^{\frac{b-1}{p}}.$$

In order that also this kernel shall have homogeneity –1 we must assume that

$$\frac{a-1}{p'} + \frac{b-1}{p} = 0 \quad \textit{i.e.} \text{ that} \quad \frac{a}{p'} + \frac{b}{p} = 1.$$

We now apply Theorem 5 with $f$ and $g$ replaced by $F$ and $G$ and with the kernel $k_2(u,v)$ and the proof follows. □

## 4 Examples of inequalities covered by the results in Section 3

First we present two simple standard examples.

**Example 17** Let $f(x,y) = \frac{1}{x+y}$ and $p > 1$. Then Theorem 5 guarantees that the following equivalent inequalities hold:

$$\int_0^\infty \left(\int_0^\infty \frac{f(y)}{x+y}\,dy\right)^p dx \le \kappa_p^p \int_0^\infty f^p(x)\,dx$$

and

$$\int_0^\infty \int_0^\infty \frac{f(x)f(y)}{x+y}\,dx\,dy \le \kappa_p \left(\int_0^\infty f^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^{p'}(y)\,dy\right)^{\frac{1}{p'}}$$

with the sharp constant

$$\kappa_p = \int_0^\infty \frac{1y^{-\frac{1}{p}}}{1+y}\,dy = \frac{\pi}{\sin\frac{\pi}{p}}.$$

In a similar way we can get a great number of so called Hardy-Hilbert type inequalities by using other related kernels of homogeneous type –1. For example, if $\lambda p' > 1$ we have the following equivalent inequalities:

$$\int_0^\infty \left(x^{\lambda-1}\int_0^\infty \frac{f(y)}{x^\lambda + y^\lambda}\,dy\right)^p dx \le \kappa_p^p \int_0^\infty f^p(x)\,dx \tag{31}$$

and

$$\int_0^\infty \int_0^\infty \frac{x^{\lambda-1}f(y)g(x)}{x^\lambda + y^\lambda}\,dx\,dy \le \kappa_p \left(\int_0^\infty f^p(y)\,dy\right)^{\frac{1}{p}} \left(\int_0^\infty g^{p'}(x)\,dx\right)^{\frac{1}{p'}} \tag{32}$$

with sharp constant

$$\kappa_p = \int_0^\infty \frac{y^{-\frac{1}{p}}}{1+y^\lambda}\,dy = \frac{1}{\lambda}\int_0^\infty \frac{y^{\frac{1}{\lambda p'}-1}}{1+y}\,dy = \frac{\pi}{\lambda\sin\frac{\pi}{\lambda p'}}.$$

**Remark 18** Inequalities of the type (31) and (32) are in several papers called Hardy-Hilbert or Hilbert type inequalities. As we have pointed out they are a consequence of Theorem 5 and can be obtained if and only if the kernel $k(x,y)$ is homogeneous of type $-1$. A great number of examples have been presented in the literature but most such results can also be derived from Theorem 5 for $1 \leq p < \infty$ and from the reversed forms from Theorem 7 for $0 < p < 1$.

**Example 19** Let $k(x,y) = x^{\alpha-1}y^{-\alpha}$, $0 < y \leq x$, $k(x,y) = 0$, $y > x$. Then Theorem 5 implies the following equivalent inequalities:

$$\int_0^\infty \left( x^{\alpha-1} \int_0^x \frac{f(y)}{y^\alpha} \, dy \right)^p dx \leq \kappa_p^p \int_0^\infty f^p(x) \, dx \tag{33}$$

and

$$\int_0^\infty \int_0^x \frac{x^{\alpha-1} f(y) g(x)}{y^\alpha} \, dy \, dx \leq \kappa_p \left( \int_0^\infty f^p(y) \, dy \right)^{\frac{1}{p}} \left( \int_0^\infty g^{p'}(x) \, dx \right)^{\frac{1}{p'}}$$

with the sharp constant

$$\kappa_p = \int_0^1 y^{-\alpha} y^{-\frac{1}{p}} \, dy = \frac{p}{p - 1 - \alpha p}, \quad \alpha < \frac{1}{p'}, 1 \leq p \leq \infty.$$

By instead using the kernel $k(x,y) = x^{\alpha-1}y^{-\alpha}$, $y \geq x$, $k(x,y) = 0$, $0 < y < x$, Theorem 5 implies the equivalent inequalities

$$\int_0^\infty \left( x^{\alpha-1} \int_x^\infty \frac{f(y)}{y^\alpha} \, dy \right)^p dx \leq \kappa_p^p \int_0^\infty f^p(x) \, dx \tag{34}$$

and

$$\int_0^\infty \int_x^\infty \frac{x^{\alpha-1} f(y) g(x)}{y^\alpha} \, dy \, dx \leq \kappa_p \left( \int_0^\infty f^p(y) \, dy \right)^{\frac{1}{p}} \left( \int_0^\infty g^{p'}(x) \, dx \right)^{\frac{1}{p'}}$$

with the sharp constant

$$\kappa_p = \frac{p}{\alpha p - p + 1}, \quad \alpha > \frac{1}{p'}, 1 \leq p \leq \infty.$$

**Remark 20** The inequality (33) is the first weighted form of Hardy's original inequality proved by Hardy himself in 1928 (see [6]). Equation (34) is sometimes called the dual form of (33), in fact these inequalities are in a sense equivalent.

In our next example we unify and generalize the inequalities in Examples 17 and 19 by presenting a scale of inequalities between these inequalities (a genuine Hardy-Hilbert inequality).

**Example 21** Apply Theorem 5 with the kernel

$$k(x,y) = \frac{x^{\alpha+\beta-1}}{y^\alpha (x+y)^\beta}, \quad 0 < y \leq x \quad \text{and} \quad k(x,y) = 0, \quad y > x.$$

We find that the (Hardy-Hilbert type) inequality

$$\int_0^\infty \left( x^{\alpha+\beta-1} \int_0^{ax} \frac{f(y)}{y^\alpha (x+y)^\beta}\, dy \right)^p dx \le \kappa_p^p \int_0^\infty f^p(x)\, dx, \tag{35}$$

where $0 < a \le \infty$, holds with the sharp constant

$$\begin{aligned} \kappa_p &= \int_0^a \frac{y^{-\frac{1}{p}}}{y^\alpha (x+y)^\beta}\, dy = \int_0^{\frac{a}{1+a}} t^{-\alpha-\frac{1}{p}} (1-t)^{\alpha+\frac{1}{p}+\beta-2}\, dx \\ &= B_{\frac{a}{1+a}} \left( \frac{1}{p'} - \alpha, \alpha + \beta - \frac{1}{p'} \right), \end{aligned}$$

where $0 < a \le \infty$, $1 \le p \le \infty$,

$$\begin{cases} \alpha < \frac{1}{p'}, & \beta \in R, \quad \text{if } a < \infty, \\ \alpha < \frac{1}{p'}, & \alpha + \beta > \frac{1}{p'}, \quad \text{if } a = \infty. \end{cases}$$

and $B_z(u,v)$ denotes the incomplete beta-function

$$B_z(u,v) = \int_0^z t^{1-u} (1-t)^{v-1}\, dt, \quad 0 < z \le 1.$$

**Remark 22** Concerning (35) note especially that

(∗) if $a = 1$, $\beta = 0$, we obtain the Hardy inequality (33) in Example 19,
(∗∗) if $a = \infty$, $\beta = 1$, $\alpha = 0$ we get the Hilbert inequality in Example 17,
(∗∗∗) in all (Hardy like) cases $\beta = 0$ we have the sharp constant

$$\frac{a^{\frac{1}{p'}-\alpha}}{\frac{1}{p'} - \alpha}, \quad \alpha < \frac{1}{p'}.$$

**Remark 23** Recall also that the incomplete beta-function is a particular case of the Gauss hypergeometric function: $B_z(u,v) = (\frac{z^u}{u}) {}_2F_1(u, 1-v; u+1; z)$, which gives an alternative expression for the sharp constant

$$\varkappa_p = \frac{p'}{1-\alpha p'} \left( \frac{a}{1+a} \right)^{\frac{1}{p'}-\alpha} {}_2F_1\left( \frac{1}{p'} - \alpha, \frac{1}{p'} + 1 - \alpha - \beta; \frac{1}{p'} + 1 - \alpha; \frac{a}{1+a} \right).$$

Making use of the various known properties of the Gauss function, one can produce further particular cases of the above Hardy-Hilbert inequality with 'nice' sharp constants. For instance, it is known that

$${}_2F_1(1,1;2;z) = \frac{1}{z} \ln \frac{1}{1-z},$$

see [16], formula 9.12.6. Then, under the choice $\beta = 1$ and $\alpha = -\frac{1}{p}$ in (35), this yields the following particular case of (35):

$$\int_0^\infty \left| \int_0^{ax} \left( \frac{y}{x} \right)^{\frac{1}{p}} \frac{f(y)\, dy}{x+y} \right|^p dx \le \varkappa_p^p \int_0^\infty |f(x)|^p\, dx$$

with the sharp constant $\varkappa_p = \ln(1+a)$, $0 < a < \infty$.

The following example is a dual counterpart to Example 21.

**Example 24** Applying Theorem 5 with the kernel

$$k(x,y) = \frac{x^{\alpha+\beta}}{y^{\alpha}(x+y)^{\beta}}, \quad y \geq x \quad \text{and} \quad k(x,y) = 0, \quad 0 < y \leq x,$$

and we find that

$$\int_0^\infty \left| x^{\alpha+\beta} \int_{ax}^\infty \frac{f(y)\,dy}{y^{1+\alpha}(x+y)^{\beta}} \right|^p dx \leq \varkappa_p^p \int_0^\infty \left|f(x)\right|^p dx,$$

where $0 \leq a < \infty$, with the sharp constant

$$\begin{aligned} \varkappa_p &= \int_a^\infty \frac{dy}{y^{1+\alpha+\frac{1}{p}}(1+y)^{\beta}} = \int_0^{\frac{1}{1+a}} t^{\alpha+\beta+\frac{1}{p}-1}(1-t)^{-\alpha-\frac{1}{p}-1}\,dt \\ &= B_{\frac{1}{1+a}}\left(\alpha+\beta+\frac{1}{p}, -\alpha-\frac{1}{p}\right), \end{aligned}$$

where $0 \leq a < \infty$, $1 \leq p \leq \infty$, and

$$\begin{cases} -\beta < \alpha + \frac{1}{p}, & \beta \in R, \quad \text{if } a > 0, \\ -\beta < \alpha + \frac{1}{p} < 0, & \beta > 0, \quad \text{if } a = 0. \end{cases}$$

**Example 25** (Hardy-Littlewood inequality [17]) We have

$$\int_0^\infty \frac{1}{x^{\alpha}} \left| \int_0^x \frac{f(y)\,dy}{(x-y)^{1-\alpha}} \right|^p dx \leq \varkappa_p^p \int_0^\infty \left|f(x)\right|^p dx$$

with the sharp constant

$$\varkappa_p = \int_0^1 \frac{dy}{y^{\frac{1}{p}}(1-y)^{1-\alpha}} = B\left(\alpha, \frac{1}{p'}\right), \quad \alpha > 0, 1 < p < \infty.$$

The following example is also a particular case of Theorem 5.

**Example 26** (Unifying Examples 19 and 25) We have

$$\int_0^\infty x^{\alpha+\beta-1} \left| \int_0^x \frac{f(y)\,dy}{y^{\alpha}(x-y)^{\beta}} \right|^p dx \leq \varkappa_p^p \int_0^\infty \left|f(x)\right|^p dx$$

with the sharp constant

$$\varkappa_p = \int_0^1 \frac{dy}{y^{\alpha+\frac{1}{p}}(1-y)^{\beta}} = B\left(1-\beta, \frac{1}{p'} - \alpha\right), \quad \alpha < \frac{1}{p'}, \beta < 1, 1 < p < \infty.$$

As a simple generalization of Example 17, the next example also easily follows from Theorem 5.

**Example 27** (Hilbert type inequality) We have

$$\int_0^\infty \left| \int_0^\infty \frac{a(\frac{y}{x})}{x+y} f(y)\,dy \right|^p dx \le \varkappa_p^p \int_0^\infty |f(x)|^p \,dx$$

under the assumption that

$$\varkappa_p = \int_0^\infty \frac{|a(y)|\,dy}{y^{\frac{1}{p}}(1+y)} < \infty, \quad 1 \le p < \infty.$$

This constant $\varkappa_p$ is sharp when $a(y) \ge 0$. In particular,

$$\int_0^\infty \left| \int_0^\infty \left(\frac{x}{y}\right)^\gamma \frac{f(y)}{x+y}\,dy \right|^p dx \le \varkappa_p^p \int_0^\infty |f(x)|^p \,dx$$

with the sharp constant

$$\varkappa_p = \int_0^\infty \frac{dy}{y^{\gamma+\frac{1}{p}}(1+y)} = \frac{\pi}{\sin\pi(\gamma+\frac{1}{p})}, \quad -\frac{1}{p} < \gamma < \frac{1}{p'}, 1 \le p < \infty$$

and

$$\int_0^\infty \left| \int_0^\infty \left(\frac{x}{y}\right)^\gamma \frac{\ln(1+\frac{y}{x})}{x+y} f(y)\,dy \right|^p dx \le \varkappa_p^p \int_0^\infty |f(x)|^p \,dx$$

with the sharp constant

$$\varkappa_p = \int_0^\infty \frac{\ln(1+y)\,dy}{y^{\frac{1}{p}}(1+y)} = p^2, \quad 1 < p < \infty.$$

We finish this section by also giving the following application of our Theorem 13.

**Example 28** Let $\alpha > 0$, $p > 1$, and $\lambda$, $\mu$ satisfy that

$$\frac{1}{\lambda p'} + \frac{1}{\mu p} = \alpha, \qquad 2 - p < \frac{1}{\alpha\mu} < 2.$$

Then the following inequalities hold and are equivalent:

(i) $\int_0^\infty \int_0^\infty (\frac{1}{x^\lambda + y^\mu})^\alpha f(x)g(y)\,dx\,dy \le C\|f\|_p\|g\|_{p'}$ for all $f \in L_p$ and $g \in L_{p'}$.

(ii) $\int_0^\infty (\int_0^\infty (\frac{1}{x^\lambda + y^\mu})^\alpha f(x)\,dy)^p\,dx \le C^p \int_0^\infty f^p(x)\,dx$ for all $f \in L_p$.

The sharp constant $C$ in both (i) and (ii) is

$$C = \frac{1}{|\lambda|^{\frac{1}{p'}}|\mu|^{\frac{1}{p}}} B(a_0, a_1),$$

with $a_0 = \frac{1}{p}(2\alpha - \frac{1}{\mu})$ and $a_1 = \alpha - a_0$.

In fact, the proof follows by just using Theorem 13 with $a = \frac{1}{\lambda\alpha}$, $b = \frac{1}{\mu\alpha}$ and making some straightforward calculations.

**Remark 29** In the classical Hilbert case $\alpha = \lambda = \mu = 1$ we obtain

$$C = B\left(\frac{1}{p}, \frac{1}{p'}\right) = \frac{\pi}{\sin(\frac{\pi}{p})}$$

so that (i) coincides with the classical form (1) of Hilbert's inequality.

## 5 A new general geometric mean type inequality

In addition to the constant $\varkappa_p$ defined in (11), we also introduce the constants

$$\varkappa_\infty := \int_0^\infty k(1,y)\,dy$$

and

$$\varkappa^* := \frac{\int_0^\infty k(1,y)\ln\frac{1}{y}\,dy}{\int_0^\infty k(1,y)\,dy}$$

assuming that $k(x,y) \geq 0$ and maybe zero only on a set of measure zero.

Our new general geometric mean inequality reads as follows.

**Theorem 30** *Let $f(x) \geq 0$, let $\varkappa_\infty < \infty$ for some $p > 1$. If $\varkappa^* < \infty$, then*

$$\int_0^\infty \exp\left(\frac{1}{\varkappa_\infty}\int_0^\infty k(x,y)\ln f(y)\,dy\right)dx \leq e^{\varkappa^*}\int_0^\infty f(x)\,dx \tag{36}$$

*and the constant $e^{\varkappa^*}$ is sharp.*

*Proof* First we observe that

$$\varkappa_\infty < \infty \quad \text{and} \quad \varkappa_p < \infty \quad \text{for some } p \quad \Longrightarrow \quad \varkappa_q < \infty \quad \text{for all } q > p,$$

because

$$\varkappa_q \leq \int_0^1 k(1,t)t^{-\frac{1}{p}}\,dt + \int_1^\infty k(1,t)\,dt \leq \varkappa_p + \varkappa_\infty.$$

Therefore, we can apply the inequality (13) for all sufficiently large $p$.

We rewrite this inequality as

$$\left\|\frac{1}{\varkappa_\infty}\int_0^\infty k(x,y)f(y)\,dy\right\|_{L^p(R_+)} \leq \frac{\varkappa_p}{\varkappa_\infty}\|f\|_{L^p(R_+)}.$$

Here, we replace $f(x)$ by $f(x)^\lambda$, and also $p$ by $\frac{1}{\lambda}$, where $\lambda$ is an arbitrarily small positive number, and we make use of the relation

$$\|f^\lambda\|_p = \|f\|_{\lambda p}^\lambda.$$

We get

$$\left\|\left(\frac{1}{\varkappa_\infty}\int_0^\infty k(x,y)f(y)^\lambda\,dy\right)^{\frac{1}{\lambda}}\right\|_{L^1(R_+)} \leq \left(\frac{\varkappa_{\frac{1}{\lambda}}}{\varkappa_\infty}\right)^{\frac{1}{\lambda}}\|f\|_{L^1(R_+)}. \tag{37}$$

Denote

$$g_\lambda(x) = \frac{1}{\varkappa_\infty}\int_0^\infty k(x,y)f(y)^\lambda\,dy.$$

Since $\lim_{\lambda\to 0}(g_\lambda(x)) = 1$ for almost all $x$ we have

$$\lim_{\lambda\to 0}\bigl(g_\lambda(x)\bigr)^{\frac{1}{\lambda}} = \lim_{\lambda\to 0} e^{\frac{\ln g_\lambda(x)}{x}} = e^{\lim_{\lambda\to 0}\frac{d}{d\lambda}\ln g_\lambda(x)} = \exp\biggl(\frac{1}{\varkappa_\infty}\int_0^\infty k(x,y)\ln f(y)\,dy\biggr).$$

Similarly

$$\lim_{\lambda\to 0}\biggl(\frac{\varkappa_{\frac{1}{\lambda}}}{\varkappa_\infty}\biggr)^{\frac{1}{\lambda}} = e^{\frac{1}{\varkappa_\infty}\int_0^\infty k(1,t)\ln\frac{1}{t}\,dt}$$

and from (37) we arrive at (36). □

**Example 31** (Generated by a weighted Hardy inequality) Take $k(x,y) = \frac{x^{a-1}}{y^\alpha}$ when $y \le x$ and $k(x,y) = 0$ otherwise, where $\alpha < 1$. Then $\varkappa_\infty = \frac{1}{1-\alpha}$ and

$$\varkappa^* = (1-\alpha)\int_0^1 y^{-\alpha}\ln\frac{1}{y}\,dy = (1-\alpha)\int_0^\infty te^{-(1-\alpha)t}\,dt = \frac{1}{1-\alpha}$$

and (36) turns into

$$\int_0^\infty \exp\biggl((1-\alpha)x^{\alpha-1}\int_0^x \frac{\ln f(y)\,dy}{y^\alpha}\biggr)\,dx \le e^{\frac{1}{1-\alpha}}\int_0^\infty f(x)\,dx$$

with $e^{\frac{1}{1-\alpha}}$ as the sharp constant. For $\alpha = 0$ this is the classical Pólya-Knopp inequality (see (7)).

**Example 32** (Generated by weighted Hilbert inequality) Take $k(x,y) = (\frac{x}{y})^\alpha\frac{1}{x+y}$ where $0 < \alpha < 1$. Then

$$\varkappa_\infty = \int_0^\infty \frac{dy}{y^\alpha(1+y)} = \frac{\pi}{\sin\alpha\pi}.$$

To calculate $\varkappa^*$ we differentiate the last equality in $\alpha$ and get

$$\int_0^\infty \frac{\ln\frac{1}{y}\,dy}{y^\alpha(1+y)} = -\frac{\pi^2\cos\alpha\pi}{\sin^2\alpha\pi}$$

so that $\varkappa^* = -\pi\cot\alpha\pi$ and (36) turns into the sharp inequality

$$\int_0^\infty \exp\biggl(\frac{\pi x^\alpha}{\sin\alpha\pi}\int_0^\infty \frac{\ln f(y)}{y^\alpha(x+y)}\,dy\biggr)\,dx \le e^{-\pi\cot\alpha\pi}\int_0^\infty f(x)\,dx.$$

**Example 33** (Generated by the Hardy-Littlewood inequality) Take $k(x,y) = \frac{1}{x^\alpha(x-y)^{1-\alpha}}$ when $y < x$ and $k(x,y) = 0$ otherwise, where $\alpha > 0$. Then $\varkappa_\infty = \frac{1}{\alpha}$. Via integration by parts and some additional tricks it may be shown that

$$\int_0^\infty k(1,y)\ln\frac{1}{y}\,dy = \int_0^\infty \frac{\ln\frac{1}{y}}{(1-y)^{1-\alpha}}\,dy = \frac{\psi(1+\alpha)-\psi(1)}{\alpha},$$

where $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the Euler psi function and we find that (36) turns into the Pólya-Knopp type inequality

$$\int_0^\infty \exp\left(\frac{\alpha}{x^\alpha}\int_0^\infty \frac{\ln f(y)}{(x-y)^{1-\alpha}}\,dy\right)dx \le e^{\frac{\psi(1+\alpha)-\psi(1)}{\alpha}}\int_0^\infty f(x)\,dx. \tag{38}$$

Note that in the case $\alpha = 1$ the inequality (38) turns into the classical Pólya-Knopp inequality (see (7)) with the sharp constant $e$ in view of the property $\psi(2) = \psi(1) + 1$ of the psi function.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors have on an equal level discussed and contributed to the proofs of the theorems in this paper. All authors have read and approved the final manuscript.

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