VERTEX ARBORICITY OF TRIANGLE-FREE GRAPHS

By

Samantha Warren

RECOMMENDED: Elizabeth S. Allman

Dr. Elizabeth Allman

Jill Faudree

Dr. Jill Faudree

Dr. John Gimbel
Advisory Committee Chair

John G. Rhodes

Dr. John Rhodes
Chair, Department of Mathematics and Statistics

April 20, 2016
Date

VERTEX ARBORICITY OF TRIANGLE-FREE GRAPHS

A

PROJECT

Presented to the Faculty

of the University of Alaska Fairbanks

in Partial Fulfillment of the Requirements

for the Degree of

MASTER OF SCIENCE

By

Samantha Warren, B.S.

Fairbanks, Alaska

May 2016

Abstract

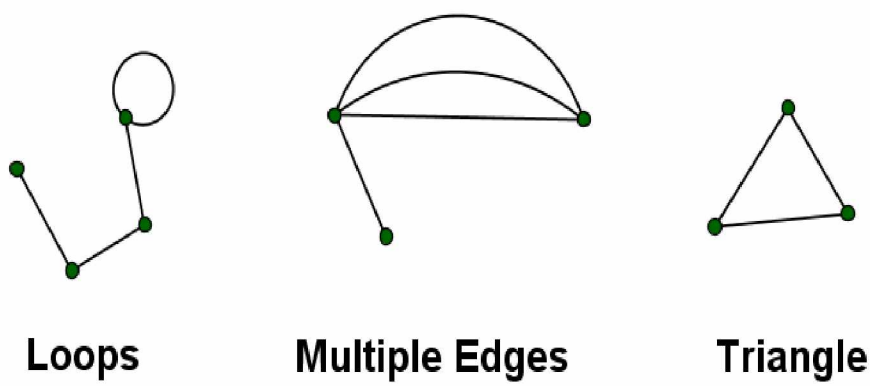
The vertex arboricity of a graph is the minimum number of colors needed to color the vertices so that the subgraph induced by each color class is a forest. In other words, the vertex arboricity of a graph is the fewest number of colors required in order to color a graph such that every cycle has at least two colors. Although not standard, we will refer to vertex arboricity simply as arboricity. In this paper, we discuss properties of chromatic number and k-defective chromatic number and how those properties relate to the arboricity of triangle-free graphs. In particular, we find bounds on the minimum order of a graph having arboricity three. Equivalently, we consider the largest possible vertex arboricity of triangle-free graphs of fixed order.

Table of Contents

Chapter 1

Introduction

A fundamental part of graph theory is the idea of partitioning graphs to satisfy given constraints. There are many ways in which a person can partition a graph. For example, when finding the arboricity of a graph, we partition the vertices of a graph into partite sets in which the vertices of the sets induce acyclic subgraphs. We can also put restrictions on the graph such as considering only graphs that are triangle-free. In this paper, we will do exactly that. In addition, we consider only simple finite graphs. In other words, we are interested in only those graphs that have a finite set of vertices and edges, contain no loops, and contain no multiple edges.

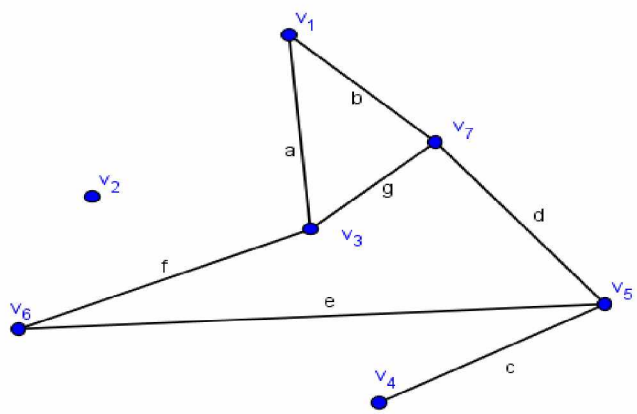


We will begin by covering definitions required in order to discuss arboricity. Some of these include chromatic number, k-defective chromatic number, and a particular function $f(m, k)$. After defining and discussing arboricity, we will summarize results and theorems of

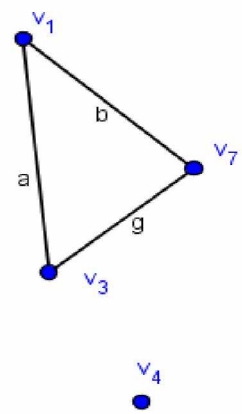
each of these parameters that will help provide a broader understanding of arboricity. From these, we pose a question that has motivated this work. Namely, what is the largest possible vertex arboricity of triangle-free graphs of fixed order? In Chapter 2, we present some results to this question. And in Chapter 3, we discuss possible further inquiry.

1.1 Definitions

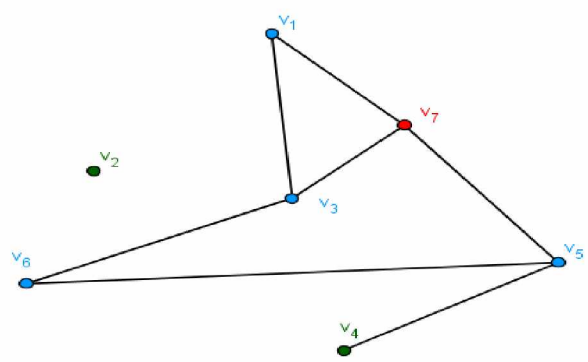
We first present some basic definitions. For undefined terms, refer to [CLZ11]. To help illustrate the first of our definitions, consider the graph below.



For a graph G, let $V(G)$ be the set of vertices of G and $E(G)$ be the set of edges of G. Thus, if G is our graph above, $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $E(G) = \{a, b, c, d, e, f, g\}$. We say the *order* of a graph G is $|V(G)|$. For vertices v and w, we say w is a *neighbor* of v if w is adjacent to v. For example, v_3 is a neighbor of v_7, v_1, and v_6 while v_2 has no neighbors. We define the *degree* of v to be the number of neighbors of v. The set including v and all neighbors of v is called the *closed neighborhood* of v. The *maximum degree*, which we denote $\Delta(G)$, is the largest degree of all vertices of G. A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a subset A of $V(G)$, the subgraph *induced by* A is the subgraph, H, such that $V(H) = A$ and an edge e is an element of $E(H)$ if e is an element of $E(G)$ such that both endpoints of e are contained in A. To illustrate this, the subgraph of the graph above induced by the set $\{v_1, v_7, v_3, v_4\}$ is the following.

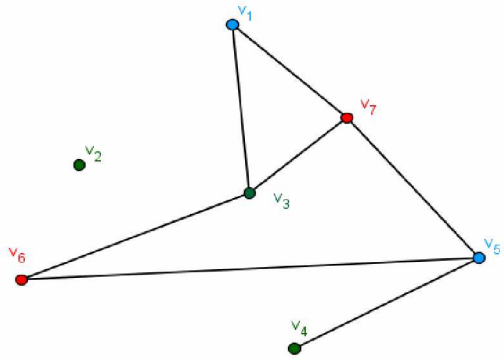


A *coloring* of a graph G is a labeling $f : V(G) \to S$, where S is some arbitrary finite set. We refer to the elements of S as colors. Typically, when S is small, we let elements be actual colors. So if we have a set of three colors, say red, blue, and green, one possible coloring of the graph above is the following.

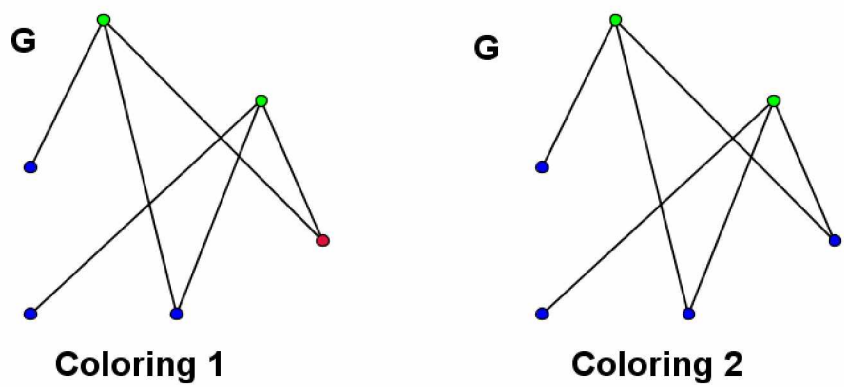


The set of all vertices assigned to any one color is called a *color class*. If a vertex v is assigned to an element a of S, we say v has color a. In the graph above, the red color class consists of only one element, v_7. We say that v_7 has color red. A set of vertices, A, of a graph is *independent* if no two vertices in A are adjacent. In a traditional coloring, which we call a proper coloring, each color class is independent. The coloring of the graph above

is not a proper coloring, however, the following is a proper coloring.



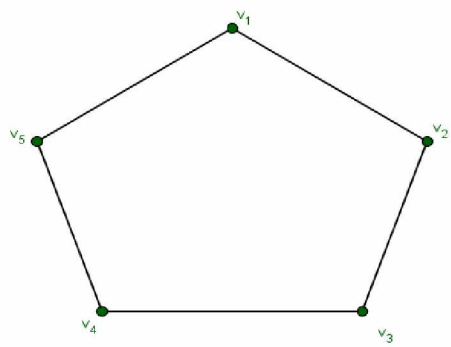
Not all colorings that we discuss will be proper. A *k-coloring* of a graph is a coloring where $|S| = k$. We say a graph is *k-colorable* if it has a proper k-coloring. The *chromatic number*, $\chi(G)$, is the least k such that G is k-colorable. A graph G, is *k-chromatic* if $\chi(G) = k$. Consider the following two colorings of the same graph.



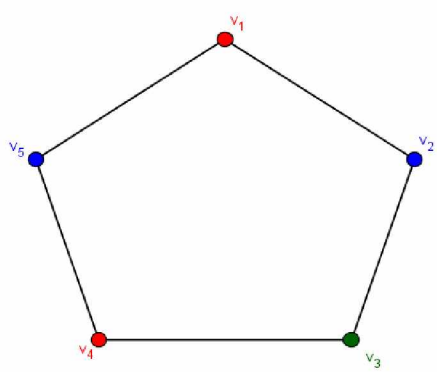
Notice that since we can color G with three colors as seen in coloring 1, G is 3-colorable. However, we can also properly color G with two colors as seen in coloring 2. Thus G is 2-colorable as well. Since there are two vertices that are adjacent, we cannot properly color

G with one color. Thus G is not 1-colorable. We can therefore conclude that two is the least number of colors needed to properly color G. In other words, G has chromatic number two. Thus G is 2-chromatic.

Consider the following 5-cycle.



Let us try to properly color the 5-cycle, C_5 with as few colors as possible. Without loss of generality, we can color v_1 red. This would force v_5 and v_2 to be another color, say blue. Which would, in turn, force v_4 and v_3 to be red. However, this would result in an improper coloring. So we will need to use a different color, say green, to color either v_3 or v_4. And such a coloring would be proper.

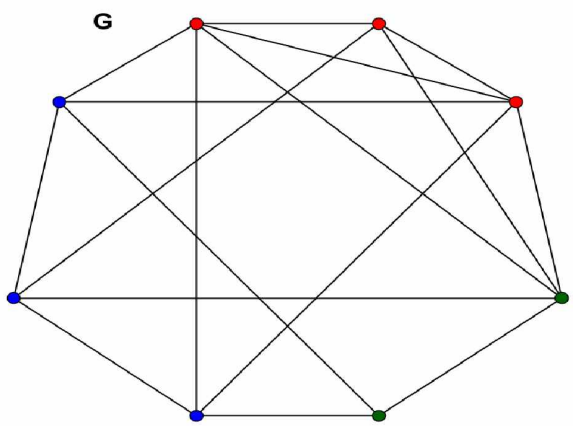


Therefore, $\chi(C_5) = 3$. We can generalize this and show that every odd cycle will have chromatic number three. Furthermore, any proper coloring of a graph containing an odd cycle requires three colors to properly color the odd cycle. Thus we have the following.

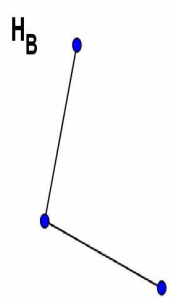
Remark 1. *If G contains a cycle of odd length, then $\chi(G) \geq 3$.*

We will soon prove that arboricity is bounded above by chromatic number. By considering properties and bounds on chromatic number, we can further understand arboricity.

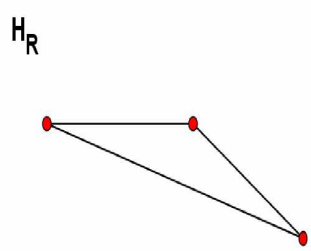
As a general case of colorability, we define the following. We say a graph is *(m,k)-colorable* if its vertices can be colored with m colors such that each vertex, v, is adjacent to at most k vertices having the same color as v. To illustrate this, consider the following graph.



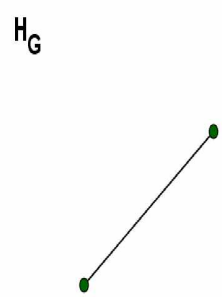
Notice that we have colored the graph, G, above using three colors. Consider the subgraphs induced by each color.



Subgraph induced by blue color class

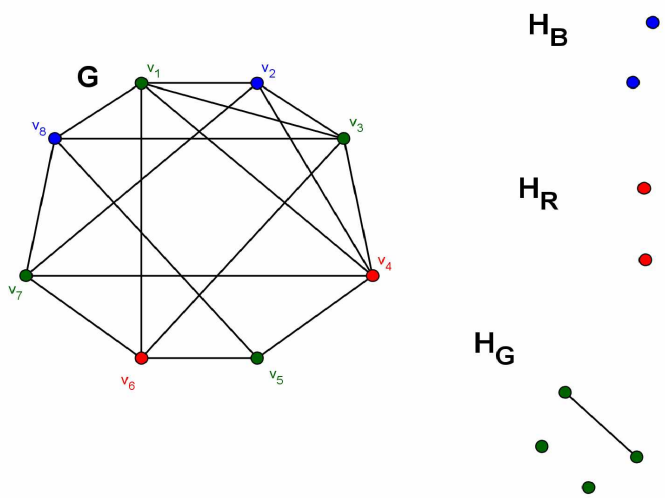


Subgraph induced by red color class



Subgraph induced by green color class

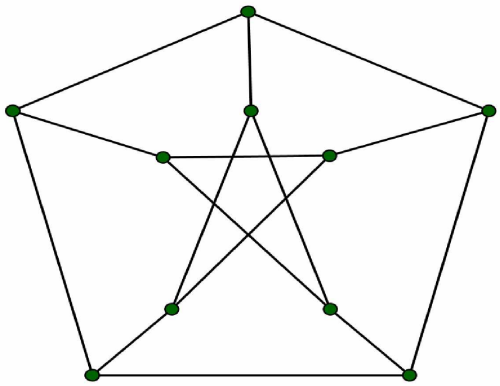
Notice that $\Delta(H_B) = 2$, $\Delta(H_R) = 2$, and $\Delta(H_G) = 1$. Since the maximum degree of the subgraph induced by each color is at most 2, we reason that G is $(3, 2)$-colorable. From this coloring, one may wish to claim that G is not $(3, 1)$-colorable. However, the following coloring of G along with the corresponding subgraphs induced by each color class will prove otherwise.



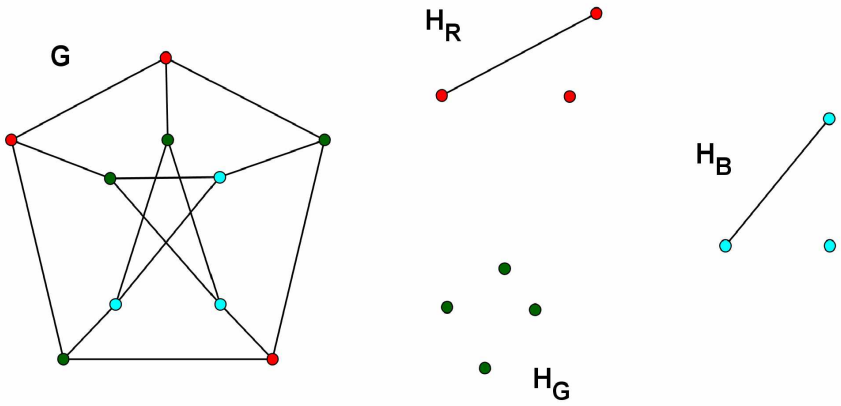
Note here that $\Delta(H_B) = 0$, $\Delta(H_R) = 0$, and $\Delta(H_G) = 1$. Thus G is $(3, 1)$-colorable. We can reason that G is not $(3, 0)$-colorable. We do this by observing that v_1, v_2, v_3, and v_4 are all pairwise adjacent, in other words, they form a clique. For any coloring of G with three colors, at least two of v_1, v_2, v_3, v_4 will have the same color. Therefore, G is not $(3, 0)$-colorable.

In our example of (m, k)-colorability, we kept the number of colors we used constant and found the minimum k. Doing the opposite gives us our next parameter. For a graph G, and a fixed integer k, the *k-defective chromatic number*, which we will denote $\chi_k(G)$, is the least positive integer m such that G is (m, k)-colorable.

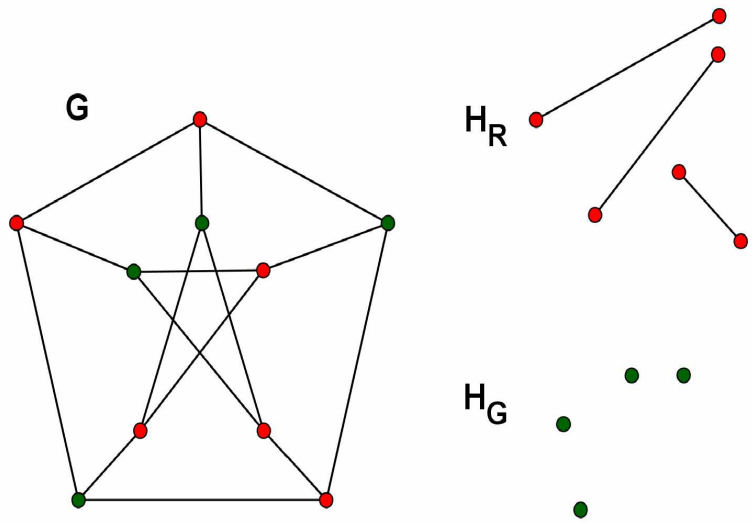
As an example, let us fix $k = 1$ and find the 1-defective chromatic number of the following graph, G, also called the Petersen Graph.



First consider the following coloring of G using three colors along with the subgraphs induced by each color class.



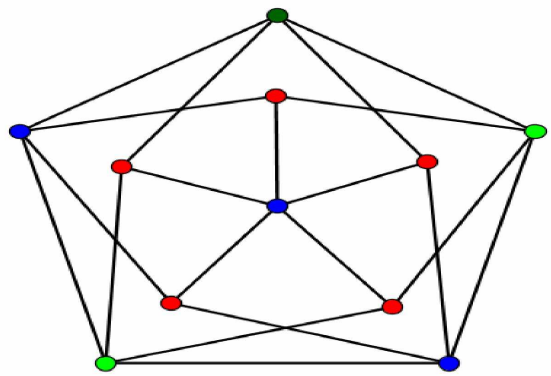
As the maximum degree of each color class is at most 1, this coloring shows that the 1-defective chromatic number of G is at most three. Now consider the next coloring of G using only two colors and the subgraphs induced by each color class.



This shows us that $\chi_1(G) \leq 2$. Since $\Delta(G) = 3$, we know G is not $(1,1)$-colorable. Therefore, we conclude that $\chi_1(G) = 2$. Note k-defective chromatic number generalizes chromatic number. In particular, $\chi_0(G) = \chi(G)$.

For the parameters we have looked at so far, we begin with a graph and then determine the value of the parameter. If, instead, we began with a given property of the graph, such as $\chi_k(G) = m$ for some k and m, then the next parameter, would give us the fewest number of vertices such that there exists a graph with that property. In particular, let $f(m,k)$ be the smallest order of a triangle-free graph such that $\chi_k(G) = m$. Notice that $f(m,0)$ is the smallest number of vertices required for a triangle-free graph to have chromatic number m.

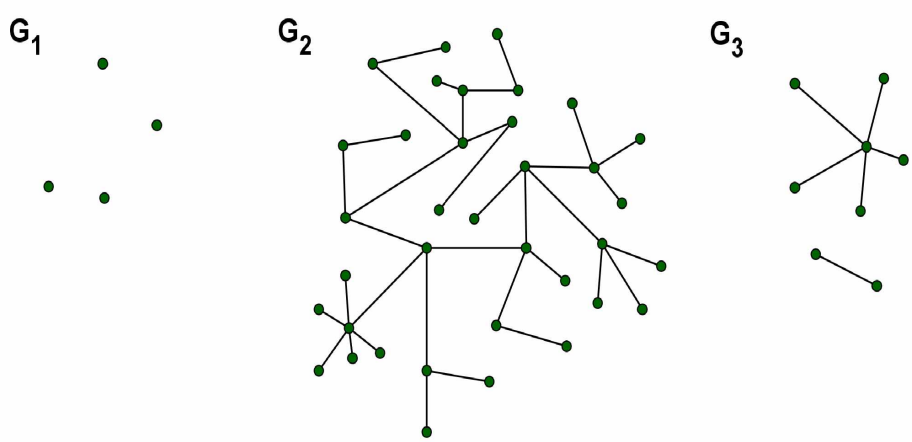
The Grötzsch graph, seen below, was proven by Chvátal in [Chv74] to be the unique smallest graph that is both 4-chromatic and triangle-free.



Since, as shown above, we can properly color the Grötzsch graph with four colors, we know the chromatic number of the Grötzsch graph is at most four. We will see in section 1.3 that the Grötzsch graph indeed has chromatic number four.

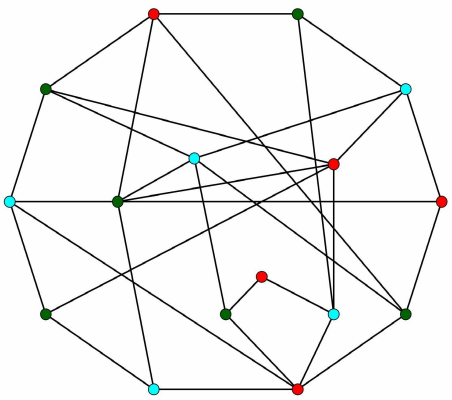
Since the order of the Grötzsch graph is 11, we have that $f(4,0) = 11$. In this document, we look at properties and theorems regarding $f(m,1)$ because, as we will soon see, it is closely related to arboricity.

Before discussing arboricity further, we define a graph to be *acyclic* if it contains no cycle. For example, consider the graphs below.

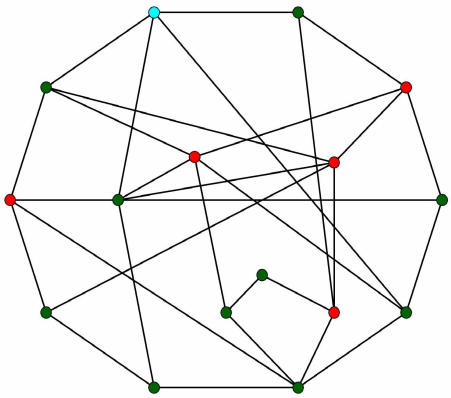


We sometimes refer to acyclic graphs as forests. A connected forest is a tree. An *acyclic*

coloring of G is a coloring where each color class induces a forest. For example, consider the following colorings of the same graph on 16 vertices.

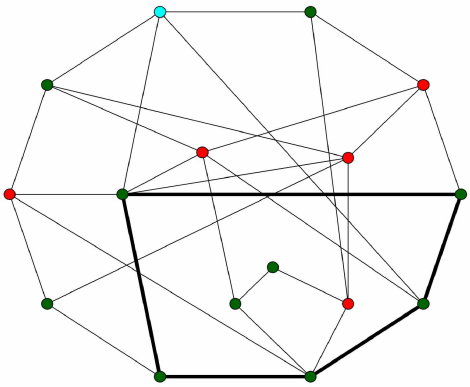


Coloring 1



Coloring 2

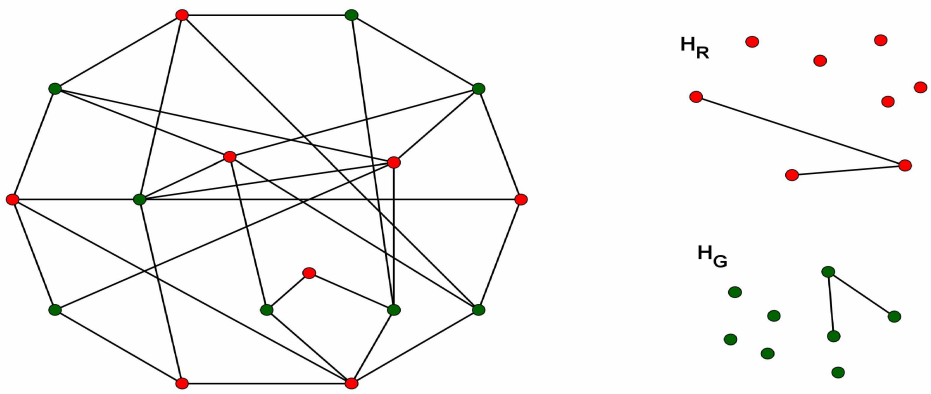
It may be difficult to see, but the graph above is triangle-free. Notice that both colorings use three colors. Coloring 1, as it is a proper coloring, has the property that the set of vertices of each color class is independent. Since independent sets are forests, Coloring 1 is an acyclic coloring. On the other hand, Coloring 2 contains a cycle in which each vertex has the same color, as seen below. This tells us that Coloring 2 is not an acyclic coloring.



Coloring 2

The *vertex arboricity*, $a(G)$, of a graph G, is the minimum number of colors needed in an

acyclic coloring of G. As stated in the abstract, although not standard, we will refer to vertex arboricity simply as arboricity. In the previous graph, we found an acyclic coloring using three colors. This implies the arboricity of the graph is at most three. Note that any graph containing a cycle will need at least two colors for an acyclic coloring. Since there exists a cycle in the graph above, we know the arboricity is at least two. The following coloring of the graph along with the subgraphs induced by each of the two color classes proves that indeed, the arboricity is two.



Coloring 3

The topic of arboricity was introduced and discussed in [CKW68] by Chartrand, Kronk, and Hudson. The terminology "arboricity" is often reserved for what we call line or edge arboricity. The *line arboricity* of the graph is the minimum number of colors needed to color the edges of a graph such that every cycle has at least two edges of different colors. A natural progression from line arboricity was to vertex arboricity. Since 1968, there have been others who have studied arboricity. One of these include s [CL95] which discusses arboricity and how it relates to maximum degree. Others are [CK69] and [RW08] both of which cover arboricity of planar graphs. A couple others are [AAC99], which focuses on arboricity of graphs with prescribed size, and [CP11] which explains an intermediate value theorem for arboricities.

Similar to chromatic number, there is no algorithm to determine the arboricity of a graph. To aid in finding the arboricity of a graph, we look to known bounds and related parameters. We have several theorems and properties that will help us determine the arboricity of triangle-

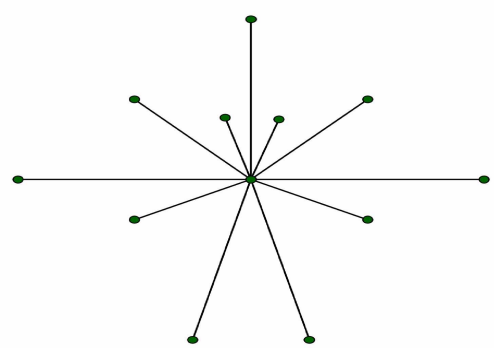
free graphs. Thus, the broad question we are interested in is "what is the arboricity of a triangle-free graph?" To begin, we first look at properties of chromatic number and k-defective chromatic number and apply these properties to arboricity. While some properties of these parameters may not explicitly answer the question we posed, we find bounds and relate them to triangle-free graphs.

1.2 Previous Results

To provide some context to our results, we present some theorems and properties of chromatic number, k-defective chromatic number, and arboricity. The following theorem by Brooks provides us with a bound on chromatic number related to maximum degree.

Theorem 1. *(Brooks) Let G be a simple connected graph. If G is neither a complete graph nor an odd cycle, then $\chi(G) \leq \Delta(G)$.*

Notice that for K_n, the complete graph on n vertices $\chi(K_n) = n$ while $\Delta(K_n) = n - 1$. By Remark 1, we know that an odd cycle has chromatic number three and maximum degree two. Therefore the hypothesis for Brooks's theorem are both necessary. Many proofs for this theorem are known and most are lengthy. One known proof is by induction on the number of vertices and the use of a greedy algorithm to color the vertices. Brooks's theorem oftentimes is far from a good bound. For example, the following graph has maximum degree 11, but chromatic number two.



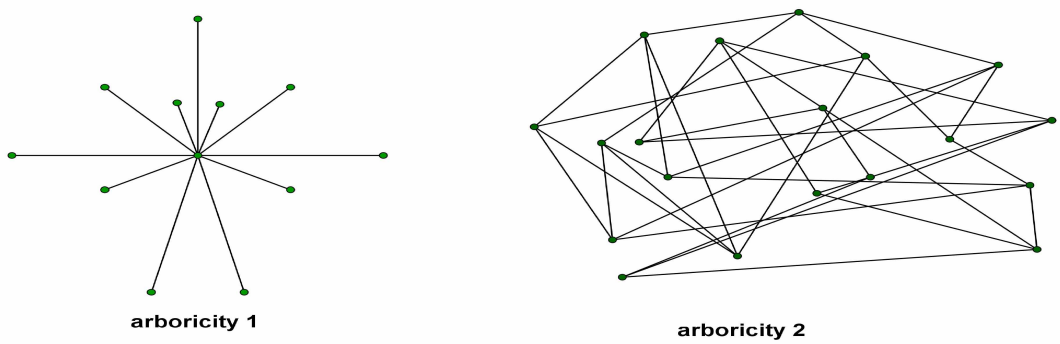
Other times, Brook's theorem theorem gives the correct chromatic number. For example, consider the Petersen graph. Since the Petersen graph contains an odd cycle, the chromatic

number is at least three. By Brooks's theorem, the chromatic number is at most three. Therefore, we know without explicitly coloring the Petersen graph, that its chromatic number is three.

Relating the previous theorem to arboricity, Kronk and Mitchem [KM75] proved the following theorem which bounds arboricity by relating it to maximum degree.

Theorem 2. *(Kronk and Mitchem) Let G be a simple connected graph. If G is neither a cycle nor a clique of odd order, then $a(G) \leq \lceil \Delta(G)/2 \rceil$.*

Similar to Theorem 1, the hypotheses are necessary. In a similar way that Brooks' Theorem can be useful, Theorem 2 can be useful. For example, consider the graphs below.



Theorem 2 tells us that the arboricity of the graph on the left is at most six. However, as a tree, the graph on the left has arboricity one. Since the graph on the right contains a cycle, we know the arboricity is at least two, and using Theorem 2 tells us that the arboricity is at most two, we can conclude the graph on the right has arboricity two.

The following is a well known result of graph theory proven in [Kon36].

Remark 2. *For G, a graph, $\chi(G) \leq 2$ if and only if G has no odd cycles.*

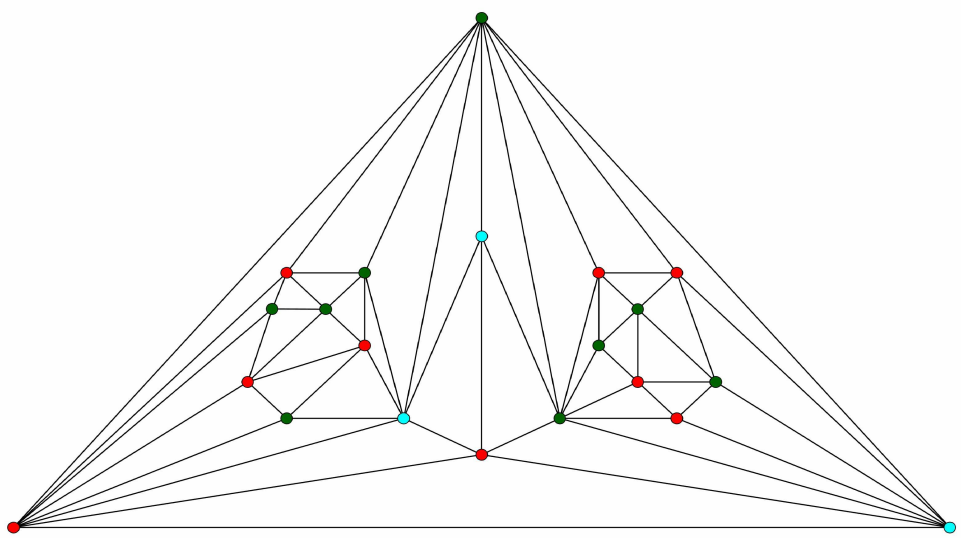
The forward statement is the contrapositive of Remark 1 which we proved earlier. To prove the reverse direction, suppose G has no odd cycles. Choose vertex $v \in V(G)$ and it blue. Color every vertex whose shortest distance to v is even red. Color every vertex whose shortest distance to v is odd blue. Note that if two vertices of the same color are adjacent,

then there is an odd cycle. Thus G is bipartite with the vertices of one partite set colored red and the other vertices colored blue. Therefore $\chi(G) \leq 2$.

From the previous remark, we know that acyclic graphs have chromatic number at most two. From Theorem 2, we know that graphs with maximum degree at most four also have arboricity at most two. The following theorem proved by Raspaud and Wang in [RW08] tells us of another condition where arboricity is at most two.

Theorem 3. *If G is a plane graph with $|V(G)| \leq 20$, then $a(G) \leq 2$.*

The proof of this uses properties we have not discussed such as notions of dual graphs, hamiltonian graphs, and Euler's formula. We refer the reader to [RW08] to see the proof. Raspaud and Wang proved this result to be sharp as they presented a planar graph on 21 vertices with arboricity three, as seen below.



If H is a subgraph of G, then any acyclic coloring of G can be restricted to an acyclic coloring of H, hence, we have the following.

Remark 3. *If H is a subgraph of G, then $a(H) \leq a(G)$.*

In [SAA97], M. Simanihuruk, et al. discuss some results on k-defective chromatic number relevant to this work. If G is a graph of order n and k is a natural number, then dividing

the vertices into sets of order at most $k+1$, we know the subgraph induced by each set has maximum degree at most k. Thus we have the following.

Remark 4. *Let G be a graph of order n and k be a natural number, then $\chi_k(G) \leq \lceil \frac{n}{k+1} \rceil$.*

The following was proven by Lovasz [Lov66] and independently by Hopkins and Staton [HS86].

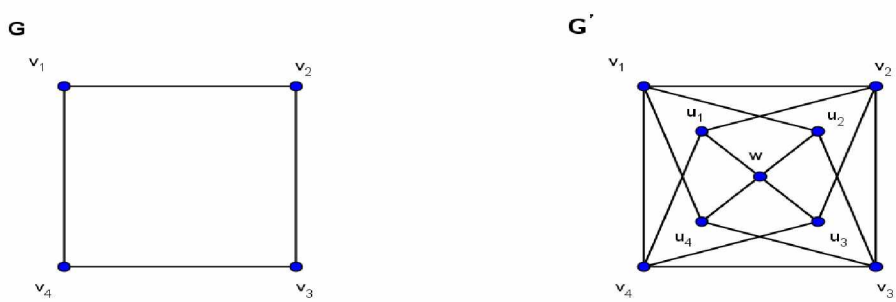
Theorem 4. *Let G be a graph with maximum degree Δ and k, a natural number. Then $\chi_k(G) \leq \lceil \frac{\Delta+1}{k+1} \rceil$.*

As arboricity of a graph G is bounded above by $\chi_1(G)$, it is useful to note the previous theorem demonstrates that $\chi_1(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$.

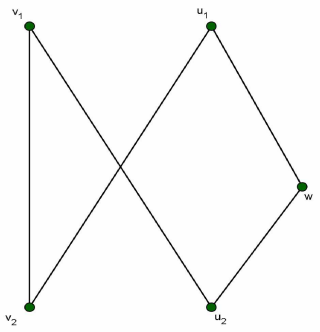
1.3 Mycielski's Construction

While none of the theorems or properties we have discussed so far are restricted to triangle-free graphs, we now consider Mycielski's construction, which is.

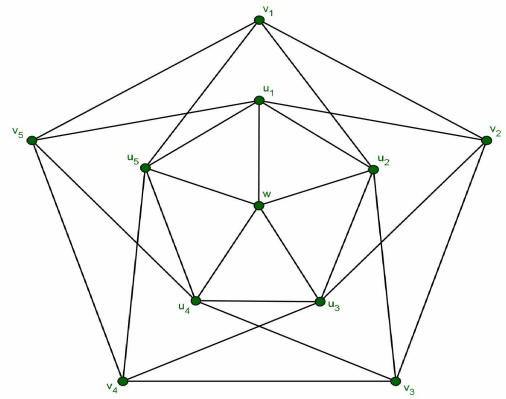
Given a graph G on vertices $\{v_1, v_2, \ldots, v_n\}$, we construct a graph G' by adding vertices $\{u_1, u_2, \ldots, u_n, w\}$ to G such that for each $i \in \{1, 2, \ldots, n\}$, we have u_i is adjacent to w and all neighbors in G of v_i. For example:



Given any triangle-free k-chromatic graph G of order n, Mycielski's construction produces a triangle-free $(k+1)$-chromatic graph G' of order $2n+1$. If we perform Mycielski's construction on K_2, we produce C_5. Performing Mycielski's construction again on C_5, we construct the Grötzsch graph.



Mycielski's Construction on C_2



Mycielski's Construction on C_5

We proved earlier that $\chi(C_5) = 3$. Since performing Mycielski's construction on C_5 produces the Grötzsch graph, we know that the Grötzsch graph is 4-chromatic. By an induction proof using Mycielski's construction with K_2 as a base case, it is clear that for every positive integer k, there exists a triangle-free k-chromatic graph.

Chapter 2

Some Findings

With all of the bounds we have discussed so far, we would like to somehow relate them to arboricity. The next theorems establish relationships between arboricity, chromatic number and k-defective chromatic number. Some of these relationships, like Theorems 8 and 9 and Corollary 10, are new. Others, like Theorems 5 and 6 are typically left to the reader to prove; we provide proofs below.

The following theorem gives us a relationship between chromatic number and arboricity.

Theorem 5. *Let G be a graph, then $\lceil \chi(G)/2 \rceil \leq a(G) \leq \chi(G)$.*

Proof. Suppose we have a proper coloring with $\chi(G)$ colors, then every color class induces a set of isolated vertices. So every color class induces a forest. Thus $a(G) \leq \chi(G)$.

Now suppose we have a coloring on $a(G)$ colors such that every color class induces a forest. Then by properly coloring each of the $a(G)$ acyclic color classes with at most two colors, we can properly color G with at most $2a(G)$ colors. Thus $\chi(G) \leq 2a(G)$ or equivalently, $\chi(G)/2 \leq a(G)$. Since $a(G)$ is an integer, we conclude $\lceil \chi(G)/2 \rceil \leq a(G)$. □

We similarly want to find a relationship between arboricity and 1-defective chromatic number.

Theorem 6. *Let G be a graph, then $\lceil \chi_1(G)/2 \rceil \leq a(G) \leq \chi_1(G)$.*

Proof. Let G be a graph with $\chi_1(G) = m$. Suppose we have a $(m, 1)$-coloring. Then every color class induces a graph with maximum degree one. So every color class induces a forest. Thus $a(G) \leq \chi_1(G)$. As $\chi_1(G) \leq \chi(G)$, the lower bound follows from Theorem 5. □

Using this result and Theorem 4 with $k = 1$, we have the following.

Corollary 7. *If G is a graph, then $a(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$.*

Mycielski's construction allows us to find triangle-free graphs with arbitrarily large chromatic number. Combining Mycielski's Construction and Theorem 5, we can produce triangle-free graphs with arbitrarily large arboricity. A question we ask ourselves next is "what is the largest arboricity of a triangle-free graph when we restrict the number of vertices?" To put it another way, "what is the fewest number of vertices needed for a triangle-free graph to have large given arboricity?" Note that any graph with a cycle will have arboricity greater than or equal to two.

We ask the more specific question "what is the fewest number of vertices needed for a triangle-free graph to have arboricity three?" This work presents some partial results for these questions.

2.1 Particular Bounds

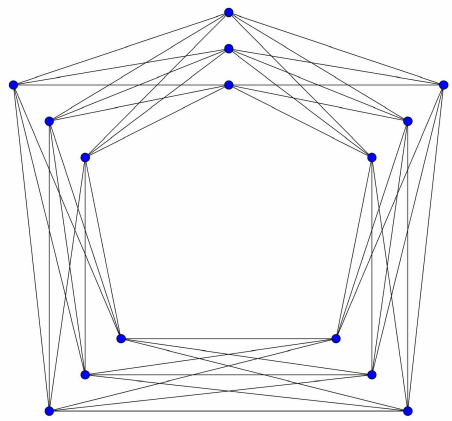
We begin by considering a result of [SAA97].

Theorem 8. *The smallest order of a triangle-free graph G with $\chi_1(G) = 4$ is at least 17, that is, $f(4,1) \geq 17$.*

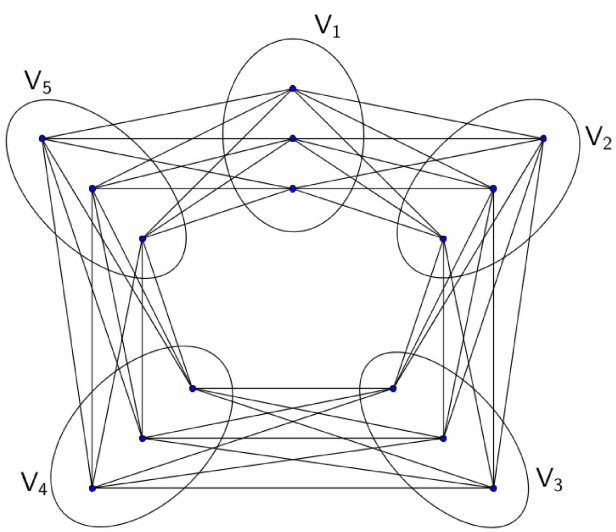
The proof of this theorem deals with multiple cases and is quite long. This result, together with Theorem 5 implies every triangle-free graph of order 16 has an arboricity of at most three. With this information, we search for a tighter bound.

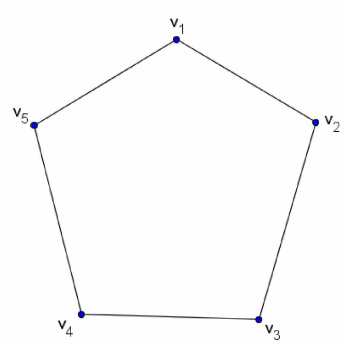
The next construction will show us the existence of a triangle-free graph with 15 vertices that has arboricity three. We do not know if 15 is best possible.

Construction 9. *The graph below has arboricity three.*



Proof. Let G be the graph above. Suppose for the sake of contradiction, G has arboricity at most two. Then there is a 2-coloring f of G, where each color class, say red and blue, induces a forest. Build graph G', a pentagon with vertices v_1, v_2, v_3, v_4, v_5 where the vertices of G' correspond respectively to the five three-vertex sets V_1, V_2, V_3, V_4, V_5 of G as seen below.





For $i \in \{1, 2, 3, 4, 5\}$, color $v_i \in V(G')$ red (or blue) if the three vertices of G that correspond to v_i have, according to the f-coloring, a majority color red (or blue). If G' is 2-colored, then two adjacent vertices in G' must have the same color. Corresponding to

those two vertices in G' are four vertices in G of the same color which induce a cycle, a contradiction. $\square$

In addition to an upper bound, [SAA97] offers us a lower bound.

Theorem 10. *The smallest order of a triangle-free graph G such that $\chi_1(G) = 3$ is 9, that is, $f(3,1) = 9$.*

The proof approach of this theorem has similarities to the one we take in proving the next theorem.

Using Theorem 6 and the previous result, we see that at least nine vertices will be needed in order for a triangle-free graph to have arboricity three.

With the next theorem, we prove that at least 10 vertices are required for a triangle-free graph to have arboricity at least three.

Theorem 11. *If G is a triangle-free graph on 9 vertices, then $a(G) \leq 2$.*

Proof. Let G be a triangle-free graph on 9 vertices and suppose for the sake of contradiction that $a(G) \geq 3$. If G has at least two components, then we can add edges to G to connect G without changing the arboricity. Thus we can assume G is connected. Then for any partition of the vertices of G into two parts, A and B, there must be a cycle in the subgraph induced by A or a cycle in the subgraph induced by B.

Suppose G has a vertex, v of degree five or more. Let A be the closed neighborhood of v. Let B be the other three or fewer vertices of G. Notice that A and B partition the vertices of G. Since G is triangle-free and B has three or fewer vertices, we know the subgraph induced by B is acyclic. Note that if any neighbors of v are adjacent, then G would contain a triangle, a contradiction. Thus the induced subgraph of A is acyclic as well. Thus $a(G) \leq 2$. Hence, we can assume G cannot have a vertex of degree five or higher.

Now suppose G has a vertex, v of degree four. Let A be the closed neighborhood of v. Let B be the other four vertices of G. Notice that A and B partition $V(G)$. As above, the subgraph induced by A must be acyclic. If the subgraph induced by B is acyclic, we are done. Suppose the subgraph induced by B is not acyclic. Note that the only way the subgraph induced by B is triangle-free and not acyclic is if the subgraph induced by B is a 4-cycle. Since the four cycle is 2-colorable, give the vertices of B a traditional coloring with the colors red and blue. Note that a vertex in A cannot be adjacent to both a red and a blue

vertex or G would contain a triangle. If a vertex in A is not adjacent to any vertex of B, color it blue. Color the vertices of A that are adjacent to any red vertex of B, blue. Color the remaining vertices red. Notice that the blue and red vertices partition $V(G)$. Notice also that the subgraph induced by the blue color class is acyclic as is the subgraph induced by the red color class. Therefore, we can assume G cannot have a vertex of degree four or higher. Hence, assume the maximum degree of G is at most three.

From Corollary 7, we conclude $a(G) \leq \lceil 4/2 \rceil = 2$.

□

In order to make sure the lower bound is nine, we need to verify that no graph on fewer than nine vertices can have arboricity three. We can use the results of Theorem 11 to give bounds on graphs of order less than ten.

Corollary 12. *If G is a triangle-free graph on 9 vertices or fewer, then $a(G) \leq 2$.*

Proof. Let G be a triangle-free graph on fewer than 9 vertices. Then consider the graph G' on 9 vertices where G' is G together with isolated vertices. By Theorem 11, $a(G') \leq 2$. Since G is a subgraph of G', we know that $a(G) \leq a(G')$. Thus $a(G) \leq 2$. □

With Construction 9 and Corollary 12, we see that the fewest number of vertices in a triangle-free graph with arboricity three is between 10 and 15. Chvatal proved [Chv74], the smallest 4-chromatic triangle-free graph has 11 vertices and is unique. Using this fact, we know that for a triangle-free graph G, on 10 vertices, $\chi(G) \leq 3$. Hence, from Theorem 5, for such graphs $a(G) \leq 3$. We do not know if three can be replaced with two.

2.2 General Bounds

So far, we have considered specific values for the order of a graph. We now consider the general case. The following was proved by J. Gimbel and C. Thomassen [GT00] and independently by A. Nilli [Nil00].

Theorem 13. *There exists an absolute positive constant c so that for G, a triangle-free graph with at most m edges,*

$$\chi(G) \leq c\frac{m^{1/3}}{(\log m)^{2/3}}.$$

In both [GT00] and [Nil00], the above result was proven to be tight, up to a constant factor, as seen in the following.

Theorem 14. *For every m sufficiently large, there exists a triangle-free graph G with size m and a positive constant c_1 such that*

$$\chi(G) \geq c_1 \frac{m^{1/3}}{(\log m)^{2/3}}.$$

These results give asymptotically tight bounds on chromatic number. We would like to find similar bounds for arboricity. Using these results and Theorem 5, we have the following.

Theorem 15. *For a triangle-free graph G with at most m edges, there is an absolute positive constant c such that*

$$a(G) \leq c \frac{m^{1/3}}{(\log m)^{2/3}}.$$

Similarly, we know the following.

Theorem 16. *For m sufficiently large, and c_1 the constant from Theorem 14, there exists a triangle-free graph G with size m and*

$$a(G) \geq c_1 \frac{m^{1/3}}{2(\log m)^{2/3}}.$$

While the general theorems we have seen so far have given us a bound for arbocicity in terms of the number of edges, we would like to have bounds for arboricity in terms of the order of a graph. In [Kim95], Kim proved tight bounds on chromatic number in terms of order of a graph.

Theorem 17. *For a triangle-free graph G on n vertices, there exists a positive constant c_2 such that*

$$\chi(G) \leq c_2 \frac{\sqrt{n}}{\sqrt{\log n}}.$$

To relate this to arboricity, we again use Theorem 5 and have the following.

Theorem 18. *For a triangle-free graph, G, on n vertices, there exists a positive constant c_2 such that*

$$a(G) \leq c_2 \frac{\sqrt{n}}{\sqrt{\log n}}.$$

Kim proved that the result in the theorem above is sharp up to a constant factor.

Theorem 19. *There exists a constant $c_1 > 0$ such that for sufficiently large n, there exists a triangle-free graph of order n such that*

$$\chi(G) \geq c_1 \frac{\sqrt{n}}{\sqrt{\log n}}.$$

This result and Theorem 5 tells us the following.

Theorem 20. *For sufficiently large n, there exists a triangle-free graph of order n and a positive constant c_3 such that*

$$a(G) \geq c_3 \frac{\sqrt{n}}{\sqrt{\log n}}.$$

Chapter 3

Further Inquiry

To conclude, we consider some questions not answered in the paper.

We looked at bounds on the number of vertices in triangle-free graphs with arboricity at most three. We further ask, is there a natural number $n \leq 14$ such that there exists a triangle-free graph of order n with arboricity three? If there were such a number it would be greater than nine. After many months of searching through many randomly chosen graphs and throughout the process of attempting to find counterexamples, we have the following conjecture.

Conjecture 21. *If G is a triangle-free graph of order $n \leq 14$, then $a(G) \leq 2$.*

Furthermore, what can we say about the exact number of vertices in triangle-free graphs with arboricity at most four? Five? etc. Using the results from [SAA97] such as Theorem 8 or the following theorem might provide some ideas about where to begin the search.

Theorem 22. *For any integer $m \geq 4, f(m, 1) \geq m^2$.*

We were able to relate 1-defective chromatic number and $f(m, 1)$ with arboricity. Can we find more relationships between arboricity and 1-defective chromatic number? For example, how far apart can we push arboricity and 1-defective chromatic number? How far apart can they be in triangle-free graphs? Similarly, what relationships can be found between arboricity and 2-defective chromatic number? k-defective chromatic number?

It is not difficult to show the arboricity of planar graphs is at most three [CKW68]. Also note that the chromatic number of a triangle-free planar graph is at most three [Grö59]. We do not know if there exists a triangle-free graph that has arboricity equal to three. What is the arboricity of planar triangle-free graphs? Recall from Theorem 3 that planar graphs on fewer than 21 vertices have arboricity at most two. How many vertices would be needed?

References

[AAC99] Nirmala Achuthan, N. R. Achuthan, and L. Caccetta. On the vertex arboricity of graphs with prescribed size. *The Australasian Journal of Combinatorics*, 19:17–36, 1999.

[Chv74] V. Chvátal. The minimality of the Mycielski graph. In *Graphs and combinatorics (Proc. Capital Conf., George Washington Univ., Washington, D.C., 1973)*, pages 243–246. Lecture Notes in Math., Vol. 406. Springer, Berlin, 1974.

[CK69] Gary Chartrand and Hudson V. Kronk. The point-arboricity of planar graphs. *Journal of the London Mathematical Society*, s1-44(1):612–616, 1969.

[CKW68] Gary Chartrand, Hudson V. Kronk, and Curtiss E. Wall. The point-arboricity of a graph. *Israel Journal of Mathematics*, 6(2):169–175, 1968.

[CL95] Paul A. Catlin and Hong-Jian Lai. Vertex arboricity and maximum degree. *Discrete Mathematics*, 141(13):37 – 46, 1995.

[CLZ11] G. Chartrand, Linda Lesniak, and P. Zhang. *Graphs & Digraphs*, volume 5. Chapman & Hall/CRC, 2011.

[CP11] Avapa Chantasartrassmee and Narong Punnim. An intermediate value theorem for the arboricities. *International Journal of Mathematics and Mathematical Sciences*, 2011:947151:1–947151:7, 2011.

[Grö59] Herbert Grötzsch. Zur Theorie der diskreten Gebilde. VII. Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel. *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe*, 8:109–120, 1958/1959.

[GT00] John Gimbel and Carsten Thomassen. Coloring triangle-free graphs with fixed size. *Discrete Mathematics*, 219(13):275 – 277, 2000.

[HS86] Glenn Hopkins and William Staton. Vertex partitions and k-small subsets of graphs. *Ars Combinatoria*, 22:19–24, 1986.

[Kim95] Jeong Han Kim. The ramsey number r(3, t) has order of magnitude t2/log t. *Random Struct. Algorithms*, 7(3):173–207, October 1995.

[KM75] Hudson V. Kronk and John Mitchem. Critical point-arboritic graphs. *J. London Math. Soc. (2)*, 9:459–466, 1974/75.

[Kon36] Denes Konig. *Theorie der endlichen und unendlichen Graphen : kombinatorische Topologie der Streckenkomplexe*. Chelsea Publ. Co. [c1936], New York, 1936.

[Lov66] L. Lovász. On decomposition of graphs. *Studia Scientiarum Mathematicarum Hungarica. A Quarterly of the Hungarian Academy of Sciences*, 1:237–238, 1966.

[Nil00] A. Nilli. Triangle-free graphs with large chromatic numbers. *Discrete Mathematics*, 211(13):261 – 262, 2000.

[RW08] André Raspaud and Weifan Wang. On the vertex-arboricity of planar graphs. *Eur. J. Comb.*, 29(4):1064–1075, May 2008.

[SAA97] M. Simanihuruk, Nirmala Achuthan, and N. R. Achuthan. On minimal triangle-free graphs with prescribed 1-defective chromatic number. *The Australasian Journal of Combinatorics*, 16:203–227, 1997.