A STUDY OF SATURATION NUMBER

By

Erika Burr, B.A.

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of

MASTER OF SCIENCE
In
Mathematics

University of Alaska Fairbanks
August 2017

APPROVED:
Dr. Jill Faudree, Committee Chair
Dr. Gordon Williams, Committee Member
Dr. Leah Berman-Williams, Chair
Department of Mathematics and Statistics
Dr. Paul Layer, Dean
College of Natural Science and Mathematics
Dr. Michael Castellini, Dean of the Graduate School

Abstract

This paper seeks to provide complete proofs in modern notation of (early) key *saturation number* results and give some new results concerning the *semi-saturation number.* We highlight relevant results from extremal theory and present the saturation number for the complete graph K_k, and the star $K_{1,t}$, elaborating on the proofs provided in the 1964 paper *A Problem in Graph Theory* by Erdős, Hajnal and Moon and the 1986 paper *Saturated Graphs with Minimal Number of Edges* by Kászonyi and Tuza. We discuss the proof of a general bound on the saturation number for a family of target graphs provided by Kászonyi and Tuza. A discussion of related results showing that the complete graph has the maximum saturation number among target graphs of the same order and that the star has the maximum saturation number among target trees of the same order is included. Before presenting our result concerning the *semi-saturation number* for the path P_k, we discuss the structure of some P_k-saturated trees of large order as well as the saturation number of P_k with respect to host graphs of large order.

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Acknowledgments

I would like to express the deepest gratitude to my advisor Dr. Jill Faudree for her clear, focused and frequent guidance in the research and writing process, for her patience and willingness to understand my questions, and for her dedication to finding answers to satisfy us both.

I would also like to express appreciation to my committee for their thoughful questions and for the time and energy they spent carefully reviewing and providing suggestions for this document.

In addition, I would like to thank Dr. John Gimbel. It was my experience (albeit struggle!) in his introductory Graph Theory course that inspired me to continue my study of simple graphs. I wish to express general thanks to the University of Alaska Fairbanks (UAF) Department of Mathematics and Statistics including the capable Kellie Dolan, my classmates in the graduate program, and my professors for all that I have learned and been able to accomplish as a mathematics student at UAF!

Chapter 1

Introduction

1.1 Basic Definitions

A *finite simple graph* $G = \big(V(G), E(G)\big)$ is a set of vertices $V(G)$ together with a set of *edges* $E(G)$ where each edge $e \in E(G)$ is an unordered pair of distinct vertices $u, v \in V(G)$. When $e = \{u, v\}$ belongs to $E(G)$, we write $e = uv$ and we say that the vertices u and v are *adjacent* in G. If v is adjactent to exactly j vertices in G, we say that v *has degree* j, and we write $\deg_G(v) = j$.

The complement $\overline{G}$ of a graph G is the set of vertices $V(G)$ together with the set of exactly those edges *not* in $E(G)$. That is, for any vertices $u, v \in V(G)$ with $u \neq v$, if the edge $uv \notin E(G)$ then $uv \in E(\overline{G})$.

When $|V(G)| = n$ we say the G is a graph of *order* n. When $|E(G)| = m$ we will say that G has *size* m.

The understanding of definitions and proofs is aided by diagrams. When we use a diagram to represent a graph, the edges are represented by line segments or arcs and the vertices are respresented by small circles.

Observe that the graph G represented in Figure 1.1a has vertex set $V(G) = \{u, v, w, x, y, z\}$ and edge set $E(G) = \{yz, zu, wz, zx, zv, yv, wv, vu, vx\}$. Notice that $yu \notin E(G)$ and that $yu \in E(\overline{G})$. (See Figure 1.1b.)

The graph represented in Figure 1.2 belongs to a family of graphs known as complete graphs. In the complete graph K_n of order n, each pair of distinct vertices is joined by an edge. Thus K_n has size $m = \binom{n}{2}$.

A graph G is called *bipartite* if $V(G)$ can be partitioned into two sets U and W such that every edge of G joins a vertex of U and a vertex of W. That is, for all $e \in E(G)$, we have $e = uw$ where $u \in U$ and $w \in W$. The bipartite graph G is said to be *complete bipartite* if $E(G) = \{uw \ : \ u \in U, \ w \in W\}$. That is, if G is biparitie with all possible edges. If G is complete bipartite with $|U| = s$ and $|W| = t$, we write $G = K_{s,t}$. A graph H is called *k-partite* if $V(H)$ can be partitioned into k sets $V_1, V_2, \ldots, V_k$ with $k \geq 2$ such that for all $e \in E(H)$ we have $e = v_i v_j$ where $v_i \in V_i$ and $v_j \in V_j$ with $1 \leq i, j \leq k$ and $i \neq j$. When $|V_i|, |V_j| \in \{\lfloor \frac{n}{k} \rfloor, \lceil \frac{n}{k} \rceil\}$ for all $i, j \in \{1, 2, \ldots, k\}$ we will say that H is *balanced*. That is, H is balanced if all of the partite sets have as near the same order as possible.

In a graph H, a set of vertices $S \subseteq V(H)$ is an *independent set* if for all pairs $u, v \in S$ the vertices u and v are not adjacent. Notice that since each edge of a $k-$partite graph joins vertices from distinct partite sets, the partite set V_i forms an independent set for all $i \in \{1, 2, \ldots, k\}$.

A graph is $r-$*regular* if every vertex of the graph has degree r. Notice that K_4 is $3-$regular and for any n, the graph K_n is $(n-1)-$regular. For any integers n and r with $0 \leq r \leq n-1$, there exists an $r-$regular graph of order n whenever at least one of r and n is even. An elementary degree-sum argument shows that when both r and n are odd, no $r-$regular graph exists.

Let G and G_1 be graphs with $k = |V(G_1)| \leq |V(G)| = n$. If $V(G_1) \subseteq V(G)$ and $E(G_1) \subseteq E(G)$ then we say the G_1 is a *subgraph* of G. We say that two graphs H and H' are *isomorphic* if there exists an adjacency-preserving bijection between their vertex sets and we write $H \cong H'$. If there exists a subset $V' \subseteq V(G)$ of the vertices of G and a subset $E' \subseteq E(G)$ of the edges of G such that H is isomorphic to the subgraph $H' = (V', E')$, we say that G *contains a copy* of H.

If I is a subgraph of G such that $E(I) = \{uv \ : \ u, v \in V(I) \text{ and } uv \in E(G)\}$, we say that I is an *induced subgraph* of G and we may write $I = G[V(I)]$. Note that by our definition above, when we say that G contains a copy of H, the subgraph H' of G such that $H' \cong H$ need not be an induced subgraph of G.

If we obtain the graph G' from G by adding the edge $e \in E(\overline{G})$, we have $E(G') = E(G) \cup \{e\}$ and we write $G' = G + e$. If $e = uv$, we may write $G' = G + uv$.

(a) A graph G. **(b)** The complement $\overline{G}$ of G.

Figure 1.1: A graph G and its complement. By inspection, G is K_4-saturated.

(a) (b)

Figure 1.2: Two representations of the complete graph K_4 of order 4.

1.2 Saturation Number

We say that G is $H-$***saturated*** if G contains no copy of H and for all $e \in E(\overline{G})$ the graph $G+e$ does contain a copy of H.

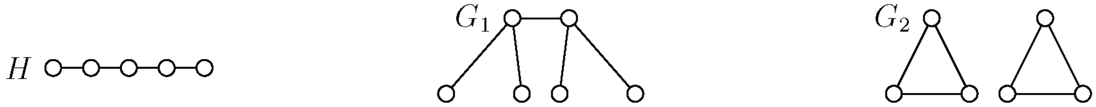
By inspection, we see that G in Figure 1.1a contains no copy of K_4. By checking all six edges $e \in E(\overline{G})$, we can see that G is K_4-saturated. That is, for any edge in $e \in E(\overline{G})$, the graph $G+e$ contains a copy of K_4. In this context, it is common to refer to G as the *host* graph and to refer to K_4 as the *target* graph.

Let H be a target graph of order k. Observe that any host graph G of order $n < k$ must necessarily be complete. In this case, we say that G is *trivially* $H-$saturated. We will generally assume that the host G has order $n \geq k$ unless otherwise stated.

Figure 1.3 shows a target graph H along with two distinct $H-$saturated graphs. Notice that $|E(G_1)| < |E(G_2)|$.

The ***saturation number*** of a target graph H with respect to host graphs of order n, denoted $sat(n,H)$, is the minimum number of edges in an $H-$saturated graph on n vertices. That is,

$$sat(n,H) := \min\{|E(G)| : G \text{ is } H\text{-saturated}, \ |V(G)| = n\}.$$



(a) A target graph H.

(b) A host G_1 of order 6 and size 5.

(c) A host G_2 of order 6 and size 6.

Figure 1.3: Both G_1 and G_2 are saturated with respect to H.

If G is an $H-$saturated graph of order n and $|E(G)| = sat(n, H)$, we say that G is a *minimal $H-$saturated graph.* In Chapter 4 we will see that G_1 in Figure 1.3 is a minimal $H-$saturated graph of order 6.

In general, to satisfactorily prove that $sat(n, H) = m$, we must both show that there exists an $H-$saturated graph G on n vertices with m edges, and we must show that we cannot do better. That is, for any $H-$saturated graph F of order n, we must show that $|E(F)| \geq m$. The first task is "easy" because we typically have a particular graph G in mind. We can describe this graph G and prove it is $H-$saturated, by cases (kinds of pairs of vertices) as needed. The second task *might* be "easy." Either it can be directly shown by making some straight-forward observations, or we may suppose that an $H-$saturated graph on n vertices with fewer than m edges exists and reach a contradiction.

This paper seeks to clarify proofs of key historical results related to saturation number by offering complete, detailed proofs in modern notation. The proofs we provide for the results discussed in Chapters 2, 3, and 4 are adapted from those originally published with their corresponding results. Each result includes a citation idicating both the original paper and the result's label in that paper.

This paper concludes with a consideration of the notion of *saturation number* as used in [EHM64]. We explore a weakening of the saturation number called the *semi-saturation number*, and we find the semi-saturation number of the path P_k on k vertices.

1.3 Chapter Overview

In Chapter 2 we give a brief history of Extremal Graph Theory as it relates to saturation number. We consider the family of extremal K_k-saturated graphs and we present the family of minimal K_k-saturated graphs. The paper [EHM64], in which minimal K_k-saturated graphs were first characterized, is widely regarded as the founding result in saturation theory.

In Chapter 3, we give a few general results regarding bounds on the saturation number and elaborate (by supplying details and illustrations) upon proofs presented in [KT86]. In particular, we discuss results showing that the complete graph has the maximum saturation number among target graphs of the same order and that the star has the maximum saturation number among target trees of the same order.

In Chapter 4, we summarize what is known about the saturation number for families of trees. In particular, we discuss a few results concerning the saturation number for paths and subdivided stars. For a given target tree T, we also discuss conditions that guarantee the existence of a minimal $T-$saturated forest.

Finally in Chapter 5, we consider the semi-saturation number for the P_k, motivated by the notion of saturation introduced in [EHM64]. We show that for n and k large enough, the semi-saturation number for P_k is strictly smaller than the saturation number for P_k.

In Chapter 6 we list a few questions for further research.

Chapter 2

A Brief History of Saturation Number

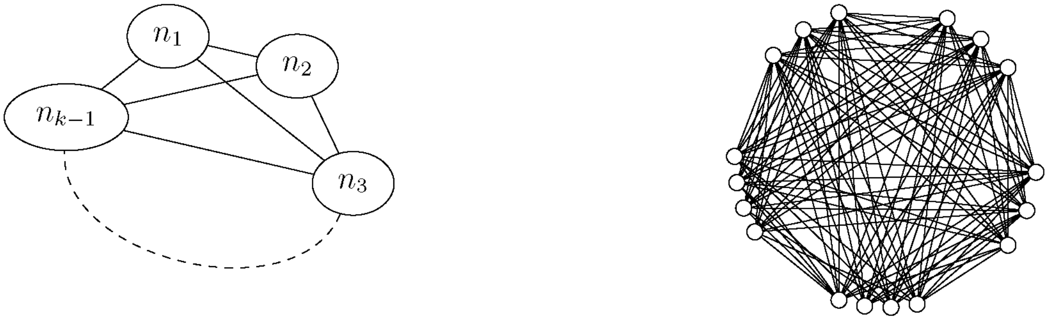
2.1 Extremal Theory

The study of saturation number falls into the area of reasearch known as extremal graph theory which explores questions regarding upper and lower bounds on the size of a class of graphs (or other graph parameters such as order, minimum degree, or girth) that guarantee certain properties. We use the notation $ex(n, H) = \max\{|E(G)| \ : \ G \text{ is } H-\text{saturated}\}$ to indicate the extremal number for the target graph H with respect to $H-$saturated graphs of order n. Thus for all $H-$saturated graphs G of order n we have,

$$sat(n, H) \leq |E(G)| \leq ex(n, H).$$

One of the earliest results in extremal theory is due to Mantel (1907), who showed that any graph of order n with at least $\lfloor n^2/4 \rfloor + 1$ edges contains a triangle (or a copy of K_3) [CLZ11]. The complete balanced bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ of order n motivates this bound. That is, $ex(n, K_3) = \lfloor n^2/4 \rfloor = |E(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil})|$. We say that a host graph G of order n is *extremal* if G is $H-$saturated and has $ex(n, H)$ edges. Thus $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is extremal with respect to K_3.

In 1941, Turán extended Mantel's result to characterize the class of K_k-saturated graphs, of maximum size [BB98]. The extremal graph for K_k, known as the Turán graph $T_{n,k-1}$ is the $(k-1)-$partite graph on n vertices with partite sets of order n_i where $n_i = \lfloor \frac{n}{k-1} \rfloor$ or $n_i = \lceil \frac{n}{k-1} \rceil$ so that $\sum_i n_i = n$. (See Figures 2.1a and 2.1b.)



(a) The Turán Graph $T_{n,k-1}$, the complete balanced $(k-1)$-partite graph, is the unique K_k-saturated graph of maximum size.

(b) $T_{17,5}$ is the graph of maximum size on 17 vertices that is K_6-saturated. The addition of any edge produces a copy of K_6.

Figure 2.1: The Turán graph $T_{n,k-1}$ is the unique K_k-saturated graph on n vertices with the maximum number of edges. That is, $T_{n,k-1}$ is the extremal K_k-saturated graph.

The graph $T_{n,k-1}$ is the unique K_k-saturated graph on n vertices with the maximum number of edges [CLZ11]. Thus,

$$ex(n, K_k) = |E(T_{n,k-1})|.$$

Considering this construction, calculations [BB98] reveal that

$$ex(n, K_k) \geq \left(1 - \frac{1}{k-1}\right)\binom{n}{2}.$$

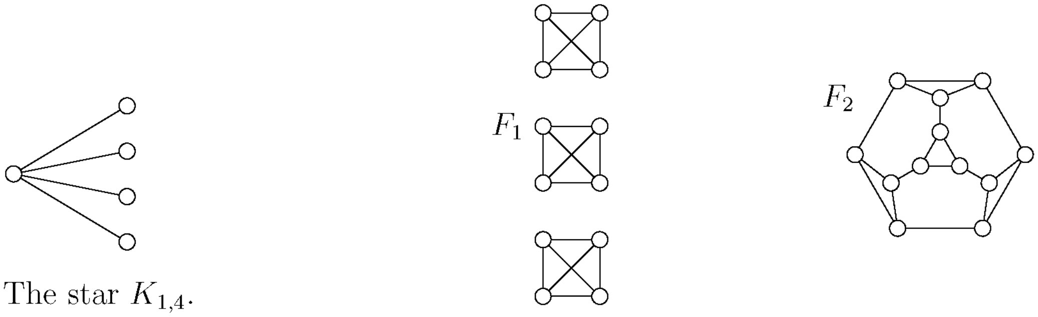
Thus we see that for fixed k, the rate of growth of the extremal number of K_k is quadratic with respect to n. In the following section we will see how this compares to the rate of growth of the *saturation* number with respect to n. (See Corollary 2.)

The extremal number for various families of graphs has been the subject of extensive study. We discuss here the extremal number for two families of graphs whose saturation numbers will be treated in Chapters 3 and 4.

(a) The path P_5 of order 5 has size 4.

(b) The extremal P_5-saturated graph of order 20.

Figure 2.2: The extremal P_5-saturated graph of order 20 has size $ex(20, P_5) = 30$ and consists of five copies of K_4, denoted $5K_4$.



(a) The star $K_{1,4}$.

(b) The graphs F_1 and F_2 are both extremal $K_{1,4}-$saturated graphs of order 12.

Figure 2.3: Any 3−regular graph of order n is $K_{1,4}-$saturated with size $ex(n, K_{1,4}) = \frac{3}{2}n$. For example, for $n = 12$ both $F_1 = 3K_4$ and the connected graph F_2 are extremal $K_{1,4}-$saturated graphs. These graphs do not exhaust the possibilities.

Given an integer $k \geq 1$, a graph of order k and size $k-1$ is a $k-$*path*, denoted P_k, if its vertices can be labelled $v_1, v_2, \ldots, v_k$ and its edge set consists exactly of the pairs $\{v_i, v_{i+1}\}$ with $1 \leq i \leq k-1$. If $v_1 = u$ and $v_k = w$ we say that P_k is a $uw-$*path* and u and w are the *end-vertices* of P. (See Figure 2.2a.)

The P_k-saturated graph of order n and maximum size has at most $\binom{k-2}{2} n$ edges. When $k-1$ divides n, the bound is sharp: the extremal graph with exactly $ex(n, P_k) = \binom{k-2}{2} n$ edges is a union of complete graphs K_{k-1} of order $k-1$ [BB98]. (See Figure 2.2b.) When $k-1$ *does not* divide n, the extremal graph consists of $\lfloor \frac{n}{k-1} \rfloor$ disjoint copies of K_{k-1} and one copy of K_r, a complete graph on the remaining $r = n - \lfloor \frac{n}{k-1} \rfloor$ vertices.

A *star* of order k and size $k-1$, denoted $K_{1,k-1}$, is the graph with one vertex of degree $k-1$ and $k-1$ vertices of degree one. (See Figure 2.3a.) The average degree of the vertices in a graph G of order n and size m is equal to $\frac{2m}{n}$. By average degree considerations, observe that any graph G of order $n \geq k$ with $|E(G)| > \binom{k-2}{2} n$ must have at least one vertex of degree at least $k-1$ and hence contains a copy of $K_{1,k-1}$. That is, the $K_{1,k-1}-$saturated graph of maximum size has at most $\binom{k-2}{2} n$ edges. For n large enough and appropriate parity, any $(k-2)-$regular graph of order n is an extremal $K_{1,k-1}-$saturated graph. (See Figure 2.3.) We note that this is consistent with the bound offered in the Erdős-Sós Conjecture. The Conjecture states that for n large enough, every graph of order n and size at least $\binom{k-2}{2} n+1$ contains as a subgraph every tree of order k [E63].

2.2 The Minimal K_k-saturated Graph: $A_k(n)$

If we desire the K_k-saturated graph on n vertices with the *fewest* possible edges, that is with $sat(n, K_k)$ edges, we redistribute the vertices of the Turán graph in Figure 2.1a. Let all but the $(k-1)^{\text{st}}$ partite set contain exactly one vertex. That is, $n_i = 1$ for all $i \in \{1, 2, \ldots, k-2\}$. Let the $(k-1)^{\text{st}}$ partite set contain (the remaining) $n-k+2$ vertices. That is, $n_{k-1} = n-k+2$. (See Figure 2.4a.)

Notice that the first $k-2$ parts form a copy of K_{k-2}. If we redraw the graph to group these vertices together, we obtain the representation shown in Figure 2.4b. We will call this graph $A_k(n)$.

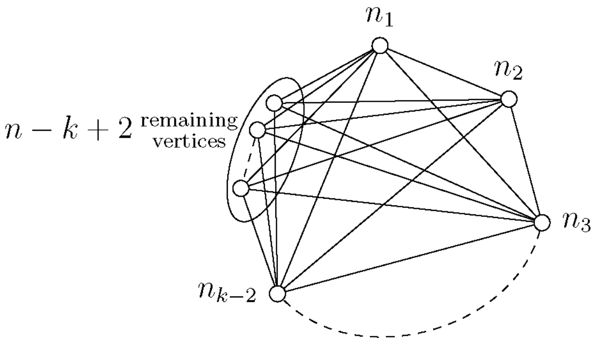
Below we present a result and its proof using the proof-technique and similar notation as in the 1964 paper *A Problem in Graph Theory* by Erdős, Hajnal and Moon [EHM64]. This proof is of particular interest because of its use of induction, an approach which is rather unusual in saturation theory. Saturation number proofs typically employ exhaustive case analysis.

Notation 1. *Define the* neighborhood *of v in G, denoted $N_G(v)$ to be the set of vertices adjacent to v in the graph G. That is,*

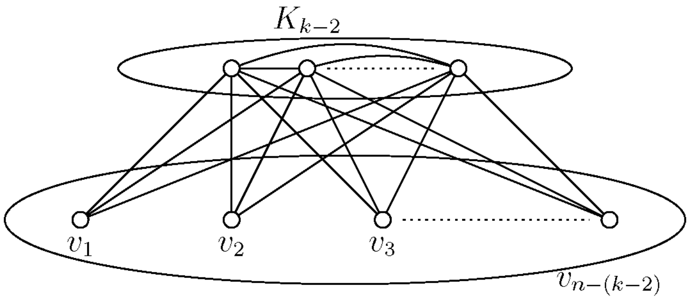
$$N_G(v) = \{u \in V(G) : uv \in E(G)\}.$$

We say the the edge uv joins *the vertices u and v. The* join *of a vertex w to a set of vertices U, denoted $w \vee U$, is a graph with vertex set $\{w\} \cup U$ and edge set $\{wu_i \ : \ u_i \in U\}$.*

We say that a graph G has *property* (n, k) if G is a graph of order n, and for any edge $e \in E(\overline{G})$, the graph $G + e$ contains a *new* copy of K_k. Note that we do *not* in general say



(a) The unique K_k-saturated graph of order n and minimum size.



(b) The graph $A_k(n)$.

Figure 2.4: The unique minimal K_k-saturated graph of order n has $k-2$ vertices of degree $n-1$ and $n-(k-2)$ vertices of degree $k-2$ [EHM64]. Two different represention are shown.

that a graph G with property (n,k) is K_k−saturated because G may contain a copy of K_k. Note also that if $n < k$, any graph G with property (n,k) is necessarily complete. Thus we will assume that $n \geq k$.

Consider the graph $A_k(n)$ on n vertices (see Figure 2.4b) which can be described as a complete graph on $k-2$ vertices each of which is joined to each of the remaining $n-(k-2)$ vertices. Note that $A_k(n)$ has size

$$\big|E\big(A_k(n)\big)\big| = \binom{k-2}{2} + (k-2)\big(n-(k-2)\big) = n(k-2) - \binom{k-1}{2},$$

and notice that $A_k(n)$ has property (n,k).

We say that G is a *minimal* (n,k) graph if G has property (n,k) and $|E(G)| = \min\{|E(H)| : H$ has property $(n,k)\}$. Notice that for $n \geq k$, the complete graph K_n is a *not* minimal (n,k) graph since the graph $K_n - e$, obtained from K_n by deleting any single edge, has property (n,k) and fewer edges.

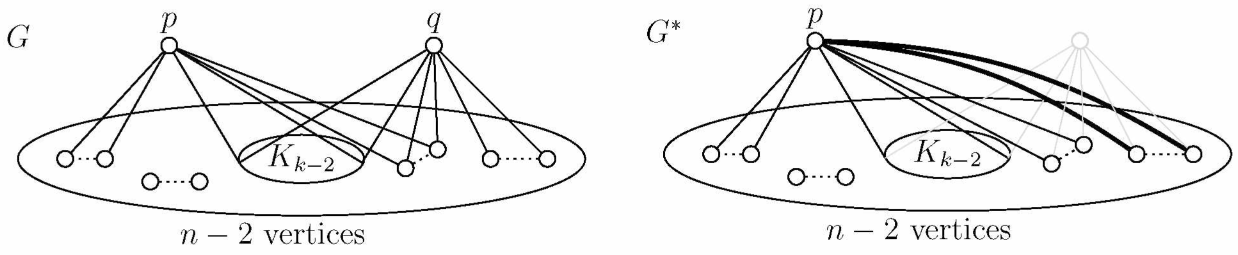
Theorem 1 ([EHM64] Theorem 1). *For every pair of integers n and k, satisfying $2 \leq k \leq n$, the only minimal (n,k) graph is $A_k(n)$.*

Proof. First we show that $A_k(n)$ is a minimal (n,k) graph; second we show that $A_k(n)$ is unique. We proceed by induction on n.

We will show that $A_k(n)$ is minimal by establishing and solving a recurrence relation for the minimum number of edges in a graph with property (n,k). Observe that for $n = k$, the unique minimal (k,k) graph is $K_k - e = A_k(k)$ with $\binom{k}{2} - 1$ edges. Let G be a minimal (n,k) graph with $n \geq k+1$. Then there exist vertices $p, q \in V(G)$ such $pq \notin E(G)$. Further, since $G + pq$ contains a new copy of K_k, it must be that p and q share at least $k-2$ neighbors and that there exist $k-2$ of these common neighbors that form a copy of K_{k-2}. (See Figure 2.5a.)

Let G^* be the graph formed by removing q (and its incident edges) from G and adding edges joining p to any neighbors of q not adjacent to p in G. (See Figure 2.5.)

Let $e \in E(\overline{G^*})$. If $e = ab$ for some $a, b \neq p$, then $e \in E(\overline{G})$. We know that $G + e$ contains a new copy of K_k, and we know that this copy of K_k cannot contain both p and q since p and



(a) G, a minimal (n,k) graph. **(b)** $G^* = G[V(G)-q] \cup \Big(p \vee (N_G(q) \setminus N_G(p))\Big)$

Figure 2.5: The inductive step: Choose a pair nonadjacent vertices p and q. Note that $N_G(p) \cap N_G(q)$ contains a copy of K_{k-2}. We obtain G^* from G by deleting q and adding edges joining p to any neighbors of q not already adjacent to p in G. To illustrate the formation of G^* from G, in the representation of G^* the deleted vertex q and its incident edges are shown faintly while the edges added between p and neighbors of q are shown thicker.

q are nonadjacent in G. Then in $G^* + e$, we obtain the same new K_k subgraph (up to the role of p which serves as q when needed). On the other hand, if $e = pb$ for some $b \in V(G^* - p)$, then $pb \notin E(G)$, and since G has property (n,k), we know that $G + e$ contains a new copy of K_k. Again, the new copy of K_k we obtain in $G + e$ is also contained in $G^* + e$ (since, as above, this copy of K_k cannot contain both p and q). Thus G^* has property $(n-1,k)$.

Let $f_k(n)$ denote the number of edges in a minimal (n,k) graph. We claim that

$$f_k(n) \geq f_k(n-1) + (k-2). \tag{2.1}$$

To see this, note that since G^* has property $(n-1,k)$, we know that $|E(G^*)| \geq f_k(n-1)$. Further, since $N_G(p) \cap N_G(q)$ contains at least $k-2$ vertices, at least $k-2$ edges were removed from G to obtain G^*. Hence,

$$|E(G^*)| \leq |E(G)| - (k-2), \qquad \text{or equivalently,} \qquad |E(G)| \geq |E(G^*)| + (k-2).$$

We have seen that for $k = n$, $K_k - e$ is the unique minimal (k, k) graph on k vertices. Hence

$$f_k(k) = \binom{k}{2} - 1. \tag{2.2}$$

Now, solving the recurrence relation in (2.1) with initial condition (2.2) we have

$$f_k(n) \geq \binom{k}{2} - 1 + (n-k)(k-2) = n(k-2) - \binom{k-1}{2} = |E\big(A_k(n)\big)|. \tag{2.3}$$

Thus (2.3) confirms that $A_k(n)$ is a minimal (n, k) graph.

The proof that $A_k(n)$ is unique proceeds by induction on the order of G. We have seen that $A_k(k) = K_k - e$ is the unique minimal (k, k) graph. Now assume that $A_k(n)$ is the unique minimal (n, k) graph for all n satisfying $k \leq n < m$ for some integer m. Let G be a minimal (m, k) graph. Then $|E(G)| = |E\big(A_k(m)\big)|$. As above, construct G^* on $m-1$ vertices from G by choosing two nonadjacent vertices $p, q \in V(G)$, removing q (and its incident edges), and adding edges as needed to join p to any neighbors of q in G that are not also neighbors of p in G. By our previous result, G^* has property $(m-1, k)$. We claim that $G^* = A_k(m-1)$. We must show that G^* is a minimal $(m-1, k)$ graph. By construction we have

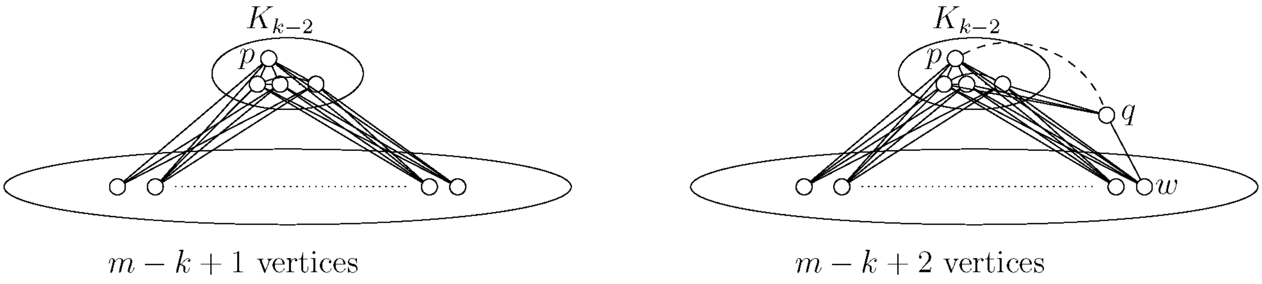
$$|E(G^*)| \leq |E(G)| - (k-2) = |E\big(A_k(m)\big)| - (k-2). \tag{2.4}$$

By (2.3) we know that $|E\big(A_k(m-1)\big)| = |E\big(A_k(m)\big)| - (k-2)$. But since $|E(G)| = |E\big(A_k(m)\big)|$ and G^* has property $(m-1, k)$ we must have

$$|E(G^*)| \geq |E\big(A_k(m-1)\big)| = |E\big(A_k(m)\big)| - (k-2). \tag{2.5}$$

Hence $|E(G^*)| = |E\big(A_k(m-1)\big)|$ and G^* is a minimal $(m-1, k)$ graph. Then by the induction hypothesis, $G^* = A_k(m-1)$.

Now consider p and q as above. We will show that the vertex q in G is a vertex of degree exactly $k-2$ with the same neighborhood as p. We proceed by contradiction. Recall that $A_k(n)$ has $k-2$ vertices of degree $n-1$ (which form a copy of K_{k-2}) and $n-(k-2)$ vertices of degree $k-2$. Suppose that in $G^* = A_k(m-1)$ the vertex p is one of the $k-2$ vertices



(a) G^* if we suppose $p \in V(K_{k-2})$

(b) G if we suppose $p \in V(K_{k-2})$.

Figure 2.6: If we suppose for the sake of contradiction that p belongs to the copy of K_{k-2} in $A_k m - 1 = G^*$, then since $pq \notin E(G)$ and G has property (n, k), the vertex q must have at least one neighbor w that does not belong to the copy of K_{k-2} in G^*. The dashed arc indicates a non-edge.

of degree $m-2$, (that is, $p \in V(K_{k-2})$ and p is adjacent to all other vertices in G^*). Then $V\big(K_{k-2} - p\big) \subseteq N_G(q)$, and since $G + pq$ contains at least one new copy of K_k, the vertex q must be adjacent to at least one more vertex, w in G. (See Figures 2.6a and 2.6b.)

Now, for each $x \in V(G^* - p)$, we have $px \in E(G^*)$, and, by our method of obtaining G^* from G, we know that in G the vertex x falls into one of three categories.

Category 1: $px, qx \in E(G)$. Note that for each x in this category, we lose exactly one edge (namely qx) when we form G^* from G. (Equivalently, for each such x the graph G has one more edge than does G^*.)

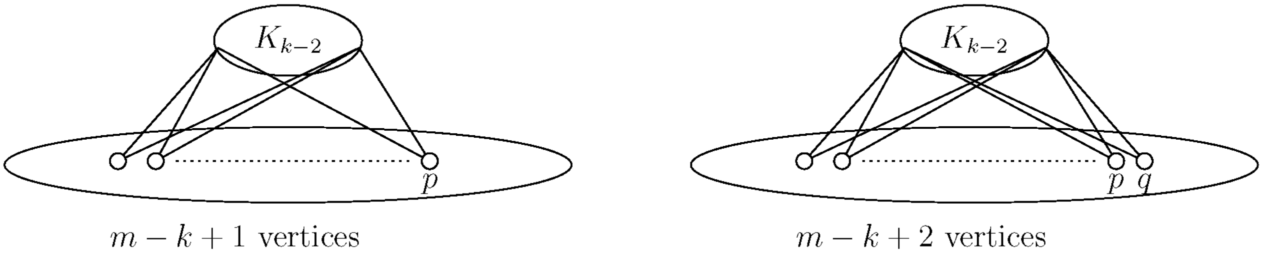
Category 2: $px \in E(G)$ and $qx \notin E(G)$. Note that for each x is this catergory, there is no change in the number of edges when we form G^* from G.

Category 3: $qx \in E(G)$ and $px \notin E(G)$. Note that for each x in this category, in constructing G^* from G, the edge qx is deleted and the edge px is added, so there is no change in the *number* of edges when we form G^* from G.

By (2.4) and (2.5) we know that $|E(G^*)| = |E(G)| - (k-2)$. That is, G^* has exactly $k-2$ fewer edges than does G. Since G has property (m,k), we know that in G, the common neighborhood of p and q contains a copy of K_{k-2}. Then in G we have exactly $k-2$ vertices adjacent to both p and q. Thus there are exactly $k-2$ vertices of Category 1 (and up to isomorphism we know which ones they are: here $x \in V(K_{k-2} - p)$ or $x = w$ as labeled in Figure 2.6b).

Now in G^* for any vertex $x \notin V(K_{k-2})$ and $x \neq w$, we know that either $px \in E(G)$ and $qx \notin E(G)$ (Category 2) or $qx \in E(G)$ and $px \notin E(G)$ (Category 3). If $px \in E(G)$ then in $G + qx$ we do not obtain a new copy of K_k since $|N_G(x) \cap N_G(q)| = k-3$. By symmetry, if $qx \in E(G)$ then in $G + px$ we do not obtain a new copy of K_k for the same reason. In either case, then G does not have property (m,k), a contradiction. Therefore, the vertex p must be one of the $m-k+1$ vertices of degree $k-2$ in G^* as shown in Figure 2.7a.

Since p and q are non-adjacent in G, and p and q must have at least $k-2$ common neighbors, q must be adjacent to all $k-2$ neighbors of p in G. Observe that by the construction of G^*, and since $|E(G^*)| = |E(G)| - (k-2)$, if $\deg_{G^*}(p) = k-2$ then necessarily $\deg_G(q) = k-2$.



(a) $G^* = A_k(m-1)$ and $G^* = G - \{q\}$

(b) $G = A_k(m)$

Figure 2.7: Since $G^* = A_k(m-1)$ and $G^* = G - \{q\}$, and since $A_k(m)$ differs from $A_k(m-1)$ only by a single vertex joined to all vertices of the K_{k-2} subgraph in G^*, it follows that $G = A_k(m)$.

But now we see that the only difference between G and G^* is the vertex $q \in V(G)$ and $k-2$ edges in $E(G)$ joining q to each of the vertices of the K_{k-2} subgraph. Thus

$$G = A_k(m).$$

(See Figure 2.7b.)

We conclude that $A_k(n)$ is the unique minimal (n, k) graph for any $n, k \in \mathbb{N}$ satisfying $2 \leq k \leq n$. $\square$

(a) A graph G. **(b)** The complement $\overline{G}$ of G.

Figure 2.8: The graph G of order 6 has property $(6, 3)$. The addition of any non-edge to G produces a new copy of K_3.

Remark 1. *We say that a graph is K_k−free if it contains no copy of K_k. Note that the minimal (n, k) graph $A_k(n)$ is K_k−free, though we did not require that a graph with property (n, k) be K_k−free. (For example, the graph G shown in Figure 2.8 has property $(6, 3)$ since for any $e \in E(\overline{G})$ the graph $G + e$ contains a new copy of K_3.)*

In the case of $H = K_k$, the unique minimal (n, k) graph is also the unique minimal H−saturated graph. As we will see in Chapter 5, in general, the minimal H−saturated graph need not be the minimal graph G such that $G + e$ contains a new copy of H for all $e \in E(\overline{G})$.

Corollary 2 (Corollary to Theorem 1)**.** *For any integers n and k satisfying $2 \leq k \leq n$,*

$$sat(n, K_k) = (k-2)n - \binom{k-1}{2}.$$

Proof of Corollary 2. Since $A_k(n)$ is the unique minimal K_k-saturated graph of order n and since

$$|E(A_k(n)| = \binom{k-2}{2} + (k-2)(n-k+2) = (k-2)n - \binom{k-1}{2},$$

the result follows. □

Recall from Section 2.1 that the extremal number $ex(n, K_k)$ is quadratic in n. Observe that the saturation number $sat(n, K_k)$ is linear in n.

Chapter 3

General Saturation Number Results

We present here an upper bound on the saturation number for general families of target graphs. The results and proof technique are taken from the 1986 paper *Saturated Graphs with Minimal Number of Edges* by Kászonyi and Tuza [KT86]. We expand on the proofs presentated in [KT86] by offering further details and some illustration. In the first section we show that for n sufficiently large, the saturation number for any non-complete graph of order k is strictly smaller than $sat(n, K_k)$, and in the second section we show that for n sufficiently large, the saturation number for any non-star tree of order $t+1$ is strictly less than $sat(n, K_{1,t})$.

3.1 General Bounds for $sat(n, \mathcal{F})$

We begin with some definitions and notation, primarily taken from [CLZ11]. We say that a graph G is *connected* if for any pair of distinct vertices $u, v \in V(G)$, there exists a $uv-$path in G. In general, a *component* of a graph is a maximal connected subgraph. The graph in Figure 3.2 is connected while parts a, b, and d of Figure 3.4 show *disconnected* graphs with two (connected) components.

We often describe a graph by referencing its components. Let H be the graph consisting of three components: exactly two copies of K_4 and exactly one copy of K_3. Then we write $H = 2K_4 \cup K_3$. Note that in this example H has order 11 and size 15.

Given an integer $k \geq 3$, a graph of order k is a $k-$*cycle*, denoted C_k, if its vertices can be labelled $v_1, v_2, \ldots, v_k$ and its edge set consists exactly of the pairs $\{v_1, v_k\}$ and $\{v_i, v_{i+1}\}$ with $1 \leq i \leq k-1$. Observe that a $k-$cycle has size k. A graph is said to be *acyclic* if it does not contain a copy of C_k for any $k \geq 3$. A connected acyclic graph is called a *tree*. A


(a) $K_{1,2} = P_3$ (b) $K_{1,3}$ (c) $K_{1,4}$ (d) $K_{1,5}$

Figure 3.1: Some representations of stars of small order.

tree of order t necessarily has size $t-1$. A *forest* is a graph all of whose components are trees. The graph P_5 in Figure 2.2a is an example of a tree on 5 vertices. Every tree has at least two vertices of degree one. Vertices of degree one are called *pendant vertices* or *leaves*. Note that K_1 and K_2 are both trees. We refer to K_1 and K_2 as *trivial* trees.

A tree of particular interest to us is $K_{1,t}$, the complete bipartite graph of order $t+1$ with partite sets of order 1 and t. As noted in Chapter 2, $K_{1,t}$ is often referred to as a *star*. Recall that $K_{1,t}$ has one vertex of degree t and t pendant vertices. See Figure 3.1.

We define the *maximum degree* of a graph H, by $\Delta(H) = \max_{v \in V(H)}\{\deg(v)\}$. Observe that if H has order k, then $\Delta(H) \leq k-1$. In determining whether a host graph G contains a copy of a given target graph H, it can be useful to consider maximum degree. For example, since $\Delta(K_{1,t}) = t$, we immediately see that any graph containing a vertex of degree t contains a copy of $K_{1,t}$.

Recall that in a graph H, a set of vertices $S \subseteq V(H)$ is an *independent set* if for all pairs $u, v \in S$, the vertices u and v are not adjacent. The *independence number* of H, denoted

$$\alpha(H) = \max_{S \subseteq V(H)} \{|S| \ : \ uv \notin E(H) \text{ for all } u, v \in S\},$$

is the cardinality of the largest independent set of vertices in H. We may write α in place of $\alpha(H)$ when H is clear from the context. The graph in Figure 3.2 has independence number $\alpha = 3$ and $\{c_1, c_3, c_6\}$ is a largest independent set.

Let $U \subseteq V(H)$. We define $H[U]$ to be the subgraph of H with vertex set U and edge set $\{uw \in E(H) \mid u, w \in U\}$. We say that $H[U]$ is the subgraph of H *induced by U*. For example, in Figure 3.2, the subgraph induced by $U = \{c_1, c_3, c_4, c_6\}$ is a copy of $K_{1,3}$.

We next present and prove an upper bound on the saturation number for a general family $\mathcal{F}$ of target graphs. We say that a graph G is $\mathcal{F}-$*saturated* if G contains no copy of any target graph $F \in \mathcal{F}$ and if for all $e \in E(\overline{G})$ the graph $G + e$ contains a copy of some $F \in \mathcal{F}$.

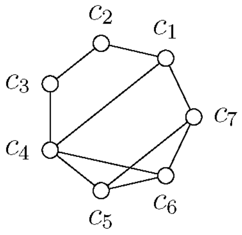
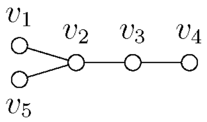


Figure 3.2: A connected graph with $\alpha = 3$.

We begin with two useful lemmas.

Lemma 3 ([KT86] Lemma 8)**.** *If F is connected for all $F \in \mathcal{F}$ and G is $\mathcal{F}-$saturated, then every (connected) component of G is $\mathcal{F}-$saturated.*

Proof. Let G_1 be a component of G. If G_1 is complete, then G_1 is trivially $\mathcal{F}-$saturated, so we may assume that G_1 is not complete. Then there exist vertices $u, v \in V(G_1)$ such that $uv \notin E(\overline{G})$. Since G is $\mathcal{F}-$saturated, $G + uv$ contains a copy of F for some $F \in \mathcal{F}$. Since F is connected, this copy of F is entirely contained in $G_1 + uv$. Hence G_1 is $\mathcal{F}-$saturated. □



(a) F

(b) $F - v_1$ **(c)** $F - v_2$ **(d)** $F - v_3$ **(e)** $F - v_4$

Figure 3.3: If $F \in \mathcal{F}$ then $F - v_i \in \mathcal{F}'$ for $i \in \{1, 2, \ldots, 5\}$. Notice that $F - v_1 \cong F - v_5$.

We call $\mathcal{F}' = \{F - v : F \in \mathcal{F}, v \in V(F)\}$ the *deleted family* of $\mathcal{F}$. That is, the members of $\mathcal{F}'$ are the graphs obtained by deleting a single vertex from some member of $\mathcal{F}$. For example, in Figure 3.3, given $F \in \mathcal{F}$ we have $F - v_i \in \mathcal{F}'$ for $i \in \{1, 2, \ldots, 5\}$. Observe that in Figure 3.3, $F - v_2$ consists of a pair of isolated vertices (vertices of degree zero) and a copy of the star $K_{1,1}$, while $F - v_3$ consists of one isolated vertex and a copy of the star $K_{1,2}$, and $F - v_4$ is a copy of $K_{1,3}$.

From $\mathcal{F}'$ we obtain the deleted family $\mathcal{F}'' = \{F - v : F \in \mathcal{F}', v \in V(F)\}$. If we continue in this manner for l iterations, we obtain $\mathcal{F}^{(l)} = \{F - v : F \in \mathcal{F}^{(l-1)}, v \in V(F_i)\}$, the deleted family obtained by deleting every possible subset of l vertices from each member of $\mathcal{F}$. We will make use of this construction and the next lemma to prove Theorem 5 below.

Lemma 4 ([KT86] Lemma 9)**.** *Let $\mathcal{F}$ be a family of graphs and let G_n be a graph of order n. Let $\mathcal{F}' = \{F - v : v \in V(F), F \in \mathcal{F}\}$ and suppose that there exists some vertex $x \in V(G_n)$ where $deg_{G_n}(x) = n - 1$. Then G_n is $\mathcal{F}-$saturated if and only if $G_n - \{x\}$ is $\mathcal{F}'-$saturated.*

Proof. Let G_n be a graph on n vertices with some $x \in V(G_n)$ of degree $n - 1$. Denote $G_{n-1} = G_n - \{x\}$.

Suppose that G_n is $\mathcal{F}-$saturated. First we show that G_{n-1} contains no member of $\mathcal{F}'$. Suppose to the contrary that there exists $F' \in \mathcal{F}'$ such that G_{n-1} contains a copy of F'. Since $F' \in \mathcal{F}'$, there exists $F \in \mathcal{F}$ such that $F - v = F'$. Then we can let $x \in V(G_n)$ play the role of $v \in V(F)$, and we see that G_n contains a copy of F. But this contradicts the assumption that G_n is $\mathcal{F}-$free. Thus there can be no member F' of $\mathcal{F}'$ such that G_{n-1} contains a copy of F'.

Next we show that for any $e \in E(\overline{G})$, there exists $F' \in \mathcal{F}'$ such that $G_{n-1} + e$ contains a copy of F'. Choose $e \in E(\overline{G})$. Since G_n is $\mathcal{F}-$saturated, we know that $G_n + e$ contains a copy of some $F \in \mathcal{F}$. If x belongs to this copy of F in $G_n + e$, then there exists a copy of $F' \in \mathcal{F}'$ in $G_{n-1} + e$. If x does not belong to this copy of F in $G_n + e$, then there exists a copy of F in $G_{n-1} + e$. Then there is also necessarily a copy of $F' \in \mathcal{F}'$ in $G_{n-1} + e$. Thus, in either case, G_{n-1} is $\mathcal{F}'-$saturated.

Conversely, suppose that G_{n-1} is $\mathcal{F}'-$saturated. Then for all $e \in E(\overline{G_{n-1}})$, the graph $G_{n-1} + e$ contains a copy of some $F' \in \mathcal{F}'$. Since $F' \in \mathcal{F}'$ we know that there exists $F \in \mathcal{F}$ such that $F - v = F'$ for some $v \in V(F)$. Now if $G_{n-1} + e$ contains a copy of F, then $G_n + e$ must also contain a copy of F. Notice that if $G_{n-1} + e$ does not contain a copy of F, since $G_n = G_{n-1} \vee \{x\}$, the vertex x can play the role of $v \in V(F)$ and $G_n + e$ contains a copy of $F \in \mathcal{F}'$. Thus $G_n + e$ contains a copy of some $F \in \mathcal{F}$ for all $e \in E(\overline{G_{n-1}})$.

Notice that G_n cannot contain a copy of any $F \in \mathcal{F}$ since then G_{n-1} would contain a copy of $F' = F - v$ for some $v \in V(F)$, contradicting the assumption that G_{n-1} is $\mathcal{F}'-$free. Thus G_n is $\mathcal{F}-$saturated. □

With the help of Lemma 4 and the notion of the deleted family, we can prove a general bound on the saturation number for an arbitrary family $\mathcal{F}$ of target graphs. We first introduce and illustrate two key parameters.

Notation 2. *Given a family $\mathcal{F}$ of graphs, denote*

$$l = l(\mathcal{F}) = \min_{F \in \mathcal{F}} \{|V(F)| - \alpha(F) - 1\}$$

and

$$d = d(\mathcal{F}) = \min_{\hat{F} \subset F \in \mathcal{F}} \Big\{ |E(\hat{F})| : \hat{F} \textit{ is a subgraph induced by an independent set } S \subset V(F) \\ \textit{and some other vertex } x \in V(F) \setminus S \quad \textit{where } |S| = |V(F)| - l - 1 \Big\}.$$

Notice that the induced subgraph $\hat{F} = F[S \cup \{x\}]$ always consists of a star and a (possibly empty) set of isolated vertices.

To find l, we choose the smallest value of $|V(F)| - \alpha(F) - 1$ among all $F \in \mathcal{F}$. To find d, we restrict the search to members F of $\mathcal{F}$ such that $|V(F)| = \alpha(F) + l + 1$. For each such F, we search over all independent sets $S \subseteq V(F)$ with $|S| = \alpha(F)$ and all vertices $x \in V(F) \setminus S$, and we choose the smallest value of $|E(F[S \cup \{x\}])|$. Note that for any particular $F \in \mathcal{F}$ such that $|V(F)| > \alpha(F) + l + 1$, there is no independent set S of the required order. Thus, effectively, the search for d is restricted to members $F \in \mathcal{F}$ whose maximum independent sets are "large" relative to $|V(F)|$.

One way to think about the parameters l and d is that l is the smallest integer such that at least one member of $\mathcal{F}^{(l)}$ consists of a star together with a possibly empty set of isolated vertices, and among these star-like members of $\mathcal{F}^{(l)}$, the minimum size is d.

We also note that the search over all graphs F satisfying $|V(F)| - \alpha(F) - 1 = l$ must truly be over all possible independent sets S of order $|V(F)| - l - 1$. This issue is illustrated by the graph in Figure 3.2. Notice that for $S = \{c_2, c_4, c_7\}$, we cannot obtain $d = 1$. However, there are choices of S (such as $S = \{c_1, c_3, c_5\}$ with $x = c_6$) that do give $d = 1$.

Using the above parameter notation,

Theorem 5 ([KT86] Theorem 1). *For n sufficiently large,*

$$sat(n,\mathcal{F}) \leq ln + \frac{1}{2}(d-1)(n-l) - \binom{l+1}{2}. \tag{3.1}$$

Proof. First suppose that $l = 0$. Then $\mathcal{F}$ contains the union of a star and a (possibly empty) set of isolated vertices. Furthermore, the smallest such star is $K_{1,d}$, by our definition of d. More precisely, there exists $F \in \mathcal{F}$ such that $F = K_{1,d} \cup \overline{K_r}$ where r is a nonnegative integer. Now, if G is $\mathcal{F}-$saturated, we know that $\deg_G(v) \leq d-1$ for all $v \in V(G)$. Then $|E(G)| \leq \frac{1}{2}(d-1)n$, and (3.1) holds when $l = 0$.

Now suppose that $l \geq 1$. Then no member of $\mathcal{F}$ can be a star. Construct the deleted families $\mathcal{F}', \mathcal{F}'', \ldots, \mathcal{F}^{(l)}$. As noted above, by our definitions of l and d, we must now have $K_{1,d} \cup \overline{K_r} \in \mathcal{F}^{(l)}$ for some integer $r \geq 0$. Furthermore, $K_{1,p} \notin \mathcal{F}^{(l)}$ for any $p < d$, again by our definition of d. That is, $\mathcal{F}^{(l)}$ contains a smallest star (together with $\overline{K_r}$), namely $K_{1,d}$.

Since $K_{1,d} \cup \overline{K_r} \in \mathcal{F}^{(l)}$, for an arbitrary $\mathcal{F}^{(l)}-$saturated graph G_{n-l}, we have

$$|E(G_{n-l})| \leq \frac{1}{2}(d-1)(n-l). \tag{3.2}$$

By repeated application of Lemma 4, we see that the graph G_n, obtained from G_{n-l} by adding l vertices of degree $n-1$, is $\mathcal{F}-$saturated if and only if G_{n-l} is $\mathcal{F}^{(l)}-$saturated.

We have

$$\begin{aligned} |E(G_n)| &= |E(G_{n-l})| + l(n-1) - \binom{l}{2} \\ &\leq \frac{1}{2}(d-1)(n-l) + l(n-1) - \binom{l}{2} \\ &= \frac{1}{2}(d-1)(n-l) + ln - \binom{l+1}{2}, \end{aligned}$$

which is the bound in (3.1). □

The above proof is constructive. With the help of Lemma 4, we have shown that given an $\mathcal{F}^{(l)}-$saturated graph G_{n-1}, we can obtain an $\mathcal{F}-$saturated graph whose size satisfies

(3.1). Hence we know that the saturation number for the family $\mathcal{F}$ with respect to a host graph of order n respects this bound.

Theorem 5, together with Theorem 1 in Chapter 2, allows us to prove that the complete graph has the largest saturation number among all target graphs of the same order.

Theorem 6 ([KT86] Theorem 3(a))**.** *For fixed k, let H_k be a graph on k vertices, then*

$$sat(n, H_k) < sat(n, K_k)$$

for $H_k \not\cong K_k$ and $n \geq k$.

Proof. From Corollary 2, we know

$$sat(n, K_k) = |E(A_k(n)| = (k-2)n - \binom{k-1}{2}.$$

Observe that the coefficient of n in (3.1), which gives an upper bound on $sat(n, \mathcal{F})$, is $l + \frac{1}{2}(d-1)$. Thus for n sufficiently large and fixed k, $sat(n, H_k) < sat(n, K_k)$ provided that $l + \frac{1}{2}(d-1) < k-2$.

Let $\mathcal{F} = \{H_k\}$. Then using the parameters of Notation 2, we know that $l = k - \alpha(H_k) - 1$, or equivalently

$$k = l + \alpha(H_k) + 1. \tag{3.3}$$

Furthermore,

$$l \leq d \leq \alpha(H_k). \tag{3.4}$$

Case 1: $d > 1$. Note that if $d > 1$, then $d - 1 > \frac{1}{2}(d-1)$. Thus

$$\begin{aligned} k - 2 &= l + \alpha(H_k) - 1 && \text{by (3.3)} \\ &\geq l + d - 1 && \text{by (3.4)} \\ &> l + \frac{1}{2}(d-1), \end{aligned}$$

which is what we needed to show.

Case 2: $d = 1$. Note that K_k is the unique graph of order k with independence number $\alpha = 1$. Since $H_k \not\cong K_k$, if $d = 1$, by (3.4) we know $d < \alpha(H_k)$. Then we have

$$\begin{aligned} k-2 &= l + \alpha(H_k) - 1 \\ &> l + d - 1 \\ &= l + \frac{1}{2}(d-1), \end{aligned}$$

as needed. $\square$

3.2 Stars

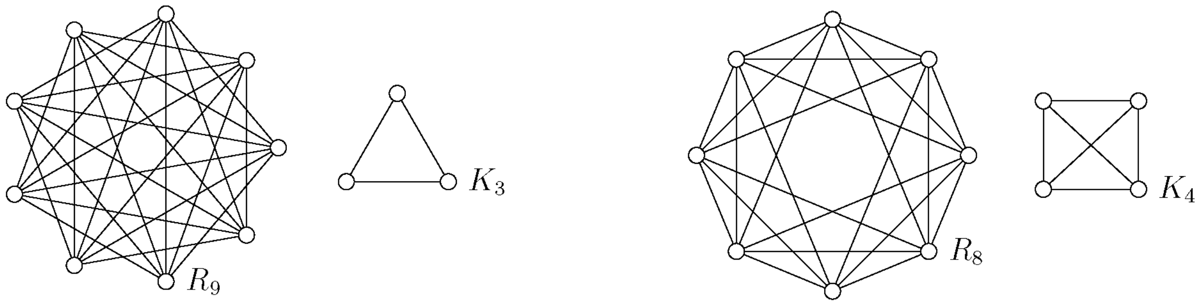
Theorem 7 ([KT86] Theorem 4). *The saturation number for the star $K_{1,t}$ is given by*

$$sat(n, K_{1,t}) = \begin{cases} \frac{t-1}{2}n - \frac{1}{2}\lfloor \frac{t^2}{4} \rfloor & n \geq t + \lfloor \frac{t}{2} \rfloor \\ \\ \binom{t}{2} + \binom{n-t}{2} & t+1 \leq n \leq t + \lfloor \frac{t}{2} \rfloor \end{cases}. \tag{3.5}$$

Remark 2. *We can describe the minimal $K_{1,t}-$saturated graphs as follows.*

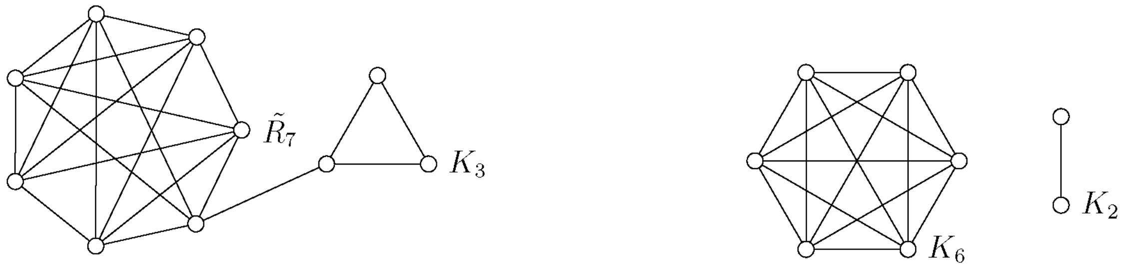
1. For n "large" relative to t (that is, $n \geq \lfloor \frac{3t}{2} \rfloor$), if either of the quantities $t-1$ or $n-t/2$ is even, then a minimal $K_{1,t}-$saturated graph G consists of two components: a complete graph $K_{\lfloor \frac{t}{2} \rfloor}$ or $K_{\lceil \frac{t}{2} \rceil}$ (both will work) and any $(t-1)-$regular graph on the remaining $n - \lfloor \frac{t}{2} \rfloor$ or $n - \lceil \frac{t}{2} \rceil$ vertices. (See Figures 3.4a and 3.4b.)

2. If $n \geq \lfloor \frac{3t}{2} \rfloor$ and both of the quantities $t-1$ and $n-t/2$ are odd, then G consists of one component: a complete graph on $\frac{t}{2}$ vertices and a *nearly* $(t-1)-$regular graph on $n - \frac{t}{2}$ vertices joined by exactly one edge. Here by *nearly* $(t-1)-$regular we mean that all vertices but one have degree $t-1$ and exactly one vertex x has degree $t-2$. In G this vertex x is adjacent to exactly one vertex in the complete graph on $\lfloor \frac{t}{2} \rfloor$ or $\lceil \frac{t}{2} \rceil$ vertices. (See Figure 3.4c.)



(a) For $n = 12$, the graph $K_3 \cup R_9$ is minimal $K_{1,7}-$saturated.

(b) For $n = 12$, the graph $K_4 \cup R_8$ is also minimal $K_{1,7}-$saturated.



(c) For $n = 10$, the graph obtained by joining K_3 to $\tilde{R}_7$ by one edge incident to the vertex of degree 4 in $\tilde{R}_7$ is minimal $K_{1,6}-$saturated.

(d) For $n = 8$, the graph $K_6 \cup K_2$ is the minimal $K_{1,6}-$saturated graph.

Figure 3.4: The minimal $K_{1,7}-$saturated graphs for $n = 12 > \lfloor \frac{3(7)}{2} \rfloor$ and the minimal $K_{1,6}-$saturated graphs for $n = 10 > \lfloor \frac{3(6)}{2} \rfloor$ and $n = 8 < \lfloor \frac{3(6)}{2} \rfloor$.

3. If n is “small” relative to t (specifically, $n < \lfloor \frac{3t}{2} \rfloor$), then $G = K_t \cup K_{n-t}$. That is, G is the union of the smallest possible $(t-1)-$regular graph, namely K_t, and a complete graph on the remaining vertices. (See Figure 3.4d.)

Proof of Theorem 7. Suppose G is a $K_{1,t}-$saturated graph on n vertices with the minimum number of edges. Then for all $v \in V(G)$, $\deg(v) \leq t-1$.

Given $v_1, v_2 \in V(G)$ with $\deg(v_1) \leq \deg(v_2) \leq t-2$, the vertices v_1 and v_2 must be adjacent. For if G does not contain the edge $e = \{v_1, v_2\}$, then in $G + e$ all vertices are of degree at most $t-1$ and $G + e$ does not contain a copy of $K_{1,t}$.

Let $X = \{v \in V(G) : \deg(v) \leq t-2\}$. Then by the previous observation, all vertices of X are adjacent to each other, so G contains a $K_{|X|}$ subgraph, and the adjacencies of X contribute at least $\binom{|X|}{2}$ edges. Let $Y = \{v \in V(G) : \deg(v) = t-1\}$. Then the adjacencies of Y contribute at least $\frac{1}{2}(t-1)(|Y|)$ edges. Since $V(G) = X \cup Y$, we have

$$|E(G)| \geq \binom{|X|}{2} + \frac{1}{2}(t-1)(n-|X|). \tag{3.6}$$

Define

$$f(x) := \binom{x}{2} + \frac{1}{2}(t-1)(n-x). \tag{3.7}$$

Then $f'(x) = x - \frac{t}{2}$, and $x = \frac{t}{2}$ is a critical point of f. Observe that $f''(\frac{t}{2}) = 1 > 0$, and thus f has a minimum at $x = \frac{t}{2}$. Then in (3.6) when $|X| = x = \frac{t}{2}$ we have

$$|E(G)| \geq \binom{t/2}{2} + \frac{1}{2}(t-1)\left(n - \frac{t}{2}\right). \tag{3.8}$$

Let $R_{n-\lfloor\frac{t}{2}\rfloor}$ denote any $(t-1)-$regular graph on $n - \lfloor\frac{t}{2}\rfloor$ vertices. Observe that if $n \geq t + \lfloor\frac{t}{2}\rfloor$, and if t is odd or $n - \lfloor\frac{t}{2}\rfloor$ is even, then the graph $K_{\lfloor\frac{t}{2}\rfloor} \cup R_{n-\lfloor\frac{t}{2}\rfloor}$ is $K_{1,t}-$saturated and achieves the bound in (3.8). That is,

$$sat(n, K_{1,t}) = \left\lceil \binom{\lfloor t/2 \rfloor}{2} + \frac{1}{2}(t-1)\left(n - \left\lfloor\frac{t}{2}\right\rfloor\right)\right\rceil = \frac{1}{2}\left[(t-1)n - \left\lfloor\frac{t^2}{4}\right\rfloor\right]. \tag{3.9}$$

There are two remaining cases:

Case 1: *Suppose that* $n \geq t + \lfloor\frac{t}{2}\rfloor$, *and that* t *is even and* $n - \frac{t}{2}$ *is odd.*

Then there exists no $(t-1)-$regular graph on $n - \frac{t}{2}$ vertices. (Since no graph can have an odd number of vertices of odd degree.) Let $\tilde{R}_{n-\frac{t}{2}}$ denote a *nearly* $(t-1)-$regular graph on $n - \frac{t}{2}$ vertices: that is, all vertices in $\tilde{R}_{n-\frac{t}{2}}$ have degree $t-1$ except one vertex x that has degree exactly $t-2$. (See Figure 3.4c for an example.) Now observe that the graph obtained by joining $\tilde{R}_{n-\frac{t}{2}}$ to $K_{\frac{t}{2}}$ by one edge e incident to x and some vertex $v \in V(K_{\frac{t}{2}})$ is a minimal

$K_{1,t}-$saturated graph on n vertices. We have

$$\begin{aligned} |E(\tilde{R}_{n-\frac{t}{2}} \cup K_{\frac{t}{2}} + e)| &= \binom{t/2}{2} + \frac{1}{2}[(n - \frac{t}{2} - 1)(t-1) + (t-2)] + 1 \\ &= \binom{t/2}{2} + \frac{1}{2}\left[\left(n - \frac{t}{2}\right)(t-1) + 1\right] \\ &= \left\lceil \binom{t/2}{2} + \frac{1}{2}(t-1)\left(n - \frac{t}{2}\right) \right\rceil \qquad \text{as in (3.9).} \end{aligned}$$

Case 2: *Suppose* $n < t + \lfloor \frac{t}{2} \rfloor$.

Case 2.1: *Suppose* $n - |X| \geq t$. Then the graph $G = K_t \cup K_{n-t}$ with

$$|E(G)| = \frac{t(t-1)}{2} + \frac{(n-t)(n-t-1)}{2} \tag{3.10}$$

is $K_{1,t}-$saturated and achieves the bound in (3.6).

Case 2.2 *Suppose that* $n - |X| < t$. Observe that since $G[Y]$ is not a $(t-1)-$regular graph and we overcount by at most $\binom{n-|X|}{2}$ edges,

$$|E(G)| \geq \binom{|X|}{2} + (n - |X|)(t-1) - \binom{n-|X|}{2}. \tag{3.11}$$

Define

$$g(x) := \binom{x}{2} + (n-x)(t-1) - \binom{n-x}{2} = (n-t)x - \frac{n}{2}(n - 2t + 1).$$

Observe that g is linear in x and that $g'(x) = n - t > 0$. Since, in this case, $|X| = x > n - t$, we know that $g(x) > g(n-t) = \binom{t}{2} + \binom{n-t}{2}$. Thus the bound in (3.10) cannot be improved. Hence for $n < t + \lfloor \frac{t}{2} \rfloor$, we have

$$sat(n, K_{1,t}) = \binom{t}{2} + \binom{n-t}{2}.$$

□

Theorem 8 ([KT86] Theorem 3(b)). *For fixed $t \geq 3$, let T_{t+1} be a tree on $t+1$ vertices, then*

$$sat(n, T_{t+1}) < sat(n, K_{1,t})$$

for $T_{t+1} \ncong K_{1,t}$ and n sufficiently large.

Proof. Let T be any tree of order $t+1$ such that $T \ncong K_{1,t}$. We exhibit a $T-$saturated graph G of order n such that $|E(G)| < sat(n, K_{1,t})$ for n large enough. Let r and q be non-negative integers such that $n = (t-1)q + r$ and $r \leq t-2$. We claim that the graph $G = \left(\lfloor \frac{n}{t-1} \rfloor - r\right) K_{t-1} \cup rK_t$ is $T-$saturated. Note first that G contains no copy of T since all components of G have order at most t. Trivially, K_{t-1} contains a copy of any tree of order $t-1$. Since T is not a star, there exists an edge $e' = uv \in E(T)$ such that neither u nor v is a pendant vertex. Then each component of $T - e'$ has order at least two. Hence for all $e \in E(\overline{G})$, there is a copy of T in $G + e$ where e plays the role of e'. Notice that since $r \leq t-2$ we have $|E(G)| \leq \binom{t-2}{2} n + \binom{t-2}{2} t$. Then for $n > \lfloor \frac{9t^2}{4} \rfloor$,

$$\begin{aligned} sat(n, T) &\leq \binom{t-2}{2} n + (t-1)(t-2) \\ &= \binom{t-1}{2} n - \frac{n}{2} + t^2 - 3t + 2 \\ &< \binom{t-1}{2} n - \frac{1}{2} \left\lfloor \frac{t^2}{4} \right\rfloor \\ &= sat(n, K_{1,t}). \end{aligned}$$

□

Chapter 4

Saturation Numbers for Paths and other Families of Trees

Here we collect some saturation results for a few families of trees including paths and subdivided stars. We also briefly discuss a few subtree properties that are known to guarantee relatively large or relatively small saturation numbers.

4.1 Isolated Edges

Before directly addressing trees, we begin with a few results concerning target graphs with isolated edges offered in [M73] and [KT86]. We say that a vertex w in a graph G is *isolated* if $\deg_G(w) = 0$. We say that an edge $uv \in E(G)$ is isolated if $\deg_G(u) = 1 = \deg_G(v)$. The complete graph $K_2 = P_2$ is an isolated edge, and a collection of m copies of K_2, denoted mK_2, is called a collection of *isolated* or *vertex-disjoint edges.*

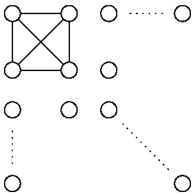


Figure 4.1: For any graph F of order 5 with no isolated vertices and at least one isolated edge, the graph $K_4 \cup \overline{K_{n-4}}$ is $F-$saturated.

First, we consider a family $\mathcal{F}$ of graphs none of whose members has an isolated vertex. If the member of smallest order F_1 of the family $\mathcal{F}$ has an isolated edge, then for n large enough, we will see that the saturation number for $\mathcal{F}$ is bounded above by some constant c based solely on $|V(F_1)|$. Moreover, the saturation number of $\mathcal{F}$ is bounded above by such a constant if and only if some member of $\mathcal{F}$ has an isolated edge.

Let F be a graph of order k with no isolated vertices and at least one isolated edge. Then for $n \geq 5$, the graph $G = K_{k-1} \cup \overline{K_{n-(k-1)}}$ is $F-$saturated. Note that $|E(G)| = \binom{k-1}{2}$. (See Figure 4.1 for an example.)

Theorem 9. ([KT86] Theorem 2)

1. *Let $\mathcal{F}$ be a family of graphs. Suppose that no member of $\mathcal{F}$ contains isolated vertices. Let $F \in \mathcal{F}$ be an element of smallest order and suppose that F has an isolated edge. Then*
$$\lim_{n \to \infty} sat(n, \mathcal{F}) = c$$
for some constant c.
2. *If $\lim_{n\to\infty} sat(n, \mathcal{F}) = c$ then there exists $F_i \in \mathcal{F}$ containing an isolated edge.*
3. *If no $F_i \in \mathcal{F}$ contains an isolated edge, then $sat(n, \mathcal{F}) \geq \lfloor \frac{n}{2} \rfloor$.*

Proof of (1). Observe that if $|V(F)| = k$, the graph $H = K_{k-1} \cup \overline{K_{n-k-1}}$ is $\mathcal{F}-$saturated. Thus for $n \geq k$, we have $sat(n, \mathcal{F}) \leq \binom{k-1}{2}$. □

Proof of (2). Suppose there exists $N \in \mathbb{N}$ such that for $n \geq N$, we have $sat(n, \mathcal{F}) = c$ for some constant $c \in \mathbb{N}$. Then for all $n \geq N$ there exists an $\mathcal{F}-$saturated graph G on n vertices with $|E(G)| = c$. If $n \geq \max\{N, 2c+2\}$ then any G of size c must have at least two isolated vertices x and y. Since G is $\mathcal{F}-$saturated with the fewest possible edges, namely c, G contains no copy of $F_i \in \mathcal{F}$ for any i, but $G + xy$ must contain a copy of $F_i \in \mathcal{F}$ for some i. Then for some i, F_i must have at least one isolated edge. □

Proof of (3). Suppose that for all i, the graph $F_i \in \mathcal{F}$ contains no isolated edge. Let G be an arbitrary minimal $\mathcal{F}-$saturated graph. By part (2) above, G can have at most one isolated vertex. Thus $|E(G)| \geq \left\lfloor \frac{n}{2} \right\rfloor$. □

Next, we consider a collection of isolated edges mK_2 and discuss the saturation number for mK_2. With the help of Theorem 10, a result that originally appeared in different but equivalent terminology in [M73], we can completely characterize the minimal mK_2-saturated graphs [KT86].

Theorem 10 (Mader's Characterization as stated in [KT86]). *Let mK_2 denote the graph consisting of m vertex-disjoint edges. Let G be an mK_2−saturated graph of order n. The structure of G can be characterized as follows:*

1. *If G is* not *connected, then every component of G is a complete graph with an odd number of vertces.*
2. *If G is connected, and $G \not\cong K_n$, with $n \geq 2m$, then G has a vertex x of degree $n-1$ and $G-x$ is $(m-1)K_2$−saturated.*

A set of vertex-disjoint edges in a graph is called a *matching.* Observe that a matching of maximum cardinality, or *size*, contained in K_{2t+1} consists of exactly t edges. By Mader's Characterization, if G is an mK_2−saturated graph and G is not connected, then $G = \bigcup_i K_{2t_i+1}$ where $t_i \in \{0, 1, 2, \ldots\}$. Since the size of the largest matching contained in each component $C_i = K_{2t_i+1}$ of G is t_i, we see that the size of the largest matching contained in G is $\sum_i t_i$. Let $e \in E(\overline{G}.)$ Since all components of G are complete graphs of odd order, $e = uv$ where u and v belong to distinct components of G. Furthermore, the largest matching contained in $G + e$ has size $1 + \sum_i t_i = m$. Hence G contains a (largest) matching of size $m - 1 = \sum_i t_i$.

Corollary 11 ([KT86] Corollary to Theorem 10). *If $n \geq 3(m-1)$ then*

$$sat(n, mK_2) = 3(m-1)$$

and if G is a minimal mK_2−saturated graph, $G = (m-1)K_3 \cup \overline{K_{n-3(m-1)}}$ or $m = 2$, $n = 4$ and $G = K_{1,3}$.

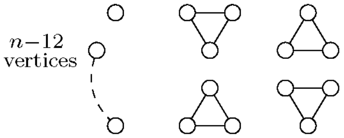


Figure 4.2: If $m = 5$ and $n \geq 12$, then the minimal $5K_2$−saturated graph is $G = 4K_3 \cup \overline{K_{n-12}}$. Notice that for any edge $e \in E(\overline{G})$, the graph $G + e$ contains a copy of $5K_2$.

Proof of Corollary 11. We proceed by induction on m. The corollary follows by simple case analysis for $m = 1$ and $m = 2$. Assume that for some $m_0 \geq 2$, the corollary holds. Let $m = m_0 + 1$ and let G be a minimal mK_2-saturated graph of order $n \geq \max\{5, 3(m-1)\}$. First we show that G is not connected. Suppose to the contrary that there exists $x \in V(G)$ with $\deg_G(x) = n - 1$. Then by Mader's Characterization, $G - x$ is $(m_0)K_2-$saturated. By the induction hypothesis, then $|E(G-x)| \geq 3(m_0-1) = 3(m-2)$. Now since $G = (G-x) \vee x$, we have

$$|E(G)| \geq n - 1 + 3(m-2) \geq 4 + 3(m-2) > 3m - 3.$$

But there exists a minimal mK_2-saturated graph of size $3(m-1)$, namely $(m-1)K_3 \cup \overline{K_{n-3(m-1)}}$, a contradiction to our assumption that G is mK_2-minimal. Hence G has no vertex of degree $n-1$. Then, since G is mK_2-saturated, by part (2) of Mader's Characterization, G is not connected. Now by part (1) of Mader's Characterization, G is a (disjoint) union of two or more complete graphs of odd order.

Let C be a component of G. We show that $C \in \{K_3, K_1\}$. Suppose to the contrary that $C = K_{2t+1}$ with $t \geq 2$.

Case 1: G contains at most one isolated vertex. Then at most one component of G is a copy of K_1. Let n_0 denote that number of vertices of degree zero in G. Then $n_0 \in \{0, 1\}$. Let n_1 denote the number of vertices (of degree 2) in G belonging to the components of G that are triangles. Then $n_1 \geq 0$. Let n_2 denote the number of vertices of G belonging to components of order 5 or more. Then $n_2 \geq 5$. By assumption G has order $n \geq 3m - 3$ and thus $n_2 + n_1 + n_0 \geq 3m - 3$. Since the vertices of G belonging to the necessarily complete components of order 5 or more have degree at least 4, and since the number of edges in a graph is equal to half its degree sum, we have

$$\begin{aligned}
|E(G)| &= \frac{1}{2} \sum_{v \in V(G)} \deg_G(v) \\
&\geq \frac{1}{2}(4n_2 + 2n_1 + 0n_0) \\
&= 2n_2 + n_1 \\
&> n_2 + n_1 + n_0 \\
&\geq 3m - 3.
\end{aligned}$$

But then

$$|E(G)| > \left|E\Big((m-1)K_3 \cup \overline{K_{n-3(m-1)}}\Big)\right| = 3m-3,$$

contradicting the minimaity of G.

Case 2: G contains at least two isolated vertices. Then we can replace $K_{2t+1} \cup 2K_1$ with t vertex-disjoint triangles tK_3 to obtain G', an mK_2-saturated graph of the same order. To see that G' is indeed mK_2-saturated, recall that by our observations above, the size of the largest matching in $K_{2t+1} \cup 2K_1$ is t, as is the size of the largest matching in tK_3. Hence the size of the largest matching contained in G' is again $m-1$. Now since G' is also a union of complete graphs of odd order, G' is mK_2-saturated. Since $t \geq 2$, we have

$$|E(K_{2t-1})| = \binom{2t+1}{2} = \frac{(2t+1)2t}{2} = 2t^2 + t > 3t$$

and then $|E(G')| < |E(G)|$. But this cannot be since G is mK_2-minimal. Thus G contains no K_{2t+1} for $t \geq 2$. Hence for $n \geq \max\{5, 3(m-1)\}$, we must have $G = (m-1)K_3 \cup \overline{K_{n-3(m-1)}}$. □

4.2 Paths

We have seen that for a collection of m copies of P_2, that is mK_2, the minimal saturated graphs consist of a union of triangles and isolated vertices. We now consider small paths P_k for $k \in \{3, 4\}$. We will see that for $k = 3$ and $k = 4$, unions of copies of P_2 together with (possibly) one isolated vertex or one triangle, respectively, are minimal P_k-saturated graphs.

Theorem 12 (KT86 Proposition 6)**.** *Let n be a positive integer. Then*

$$sat(n, P_3) = \begin{cases} \frac{n-1}{2} & n \text{ odd} \\ \\ \frac{n}{2} & n \text{ even} \end{cases}$$

and

$$sat(n, P_4) = \begin{cases} \frac{n+3}{2} & n \ odd \\ \\ \frac{n}{2} & n \ even \end{cases}.$$

Proof of $sat(n, P_3)$. Let G be a P_3-saturated graph on n vertices. Observe that every connected component G_i of G contains at most two vertices. Notice that G can have at most one isolated vertex, for if there are two isolated vertices $x, y \in V(G)$, then $G + xy$ does not contain a copy of P_3 and G is not P_3-saturated. It follows that if n is odd, $G = \frac{n-1}{2}K_2 \cup K_1$, and if n is even, $G = \frac{n}{2}K_2$. □

Proof of $sat(n, P_4)$. For $j \geq 2$ let G be a graph of order $n = 2j - 1$ consisting of $j - 1$ components: $G = (j-2)K_2 \cup K_3$. Notice that G has $j - 2 + 3 = j + 1$ edges. Observe that for any two non-adjacent vertices $u, w \in V\big((j-2)K_2\big) \subseteq V(G)$, the graph $G + uw$ contains a copy of P_4. Notice also that for any vertex $v \in V(K_3) \subseteq V(G)$, the graph $G + uv$ contains a copy of P_4. Thus G is P_4-saturated. We claim that G is P_4-minimal.

Let G' be a minimal P_4-saturated graph. We will show that all components of G' belong to the set $\{K_2, K_3, K_{1,4}\}$. Note that any (connected) component of G' of order $k \geq 4$ must be a star since G' contains no copy of P_4. For even values of k, we can replace a star on k vertices with the graph $\frac{k}{2}K_2$ which requires only $\frac{k}{2} < k - 1$ edges. Thus no connected component of G' can have order $k \in \{4, 6, 8, \ldots\}$. For odd values of k, we can replace a star on k vertices with $\frac{k-3}{2}K_2 \cup K_3$ which requires only $\frac{k-3}{2} + 3 \leq k - 1$ edges. Here equality occurs only in the case of $k = 5$. Since $|E(K_{1,4}) = 4 = E(K_2 \cup K_3)|$, the star $K_{1,4}$ can be a component of G'.

Any component on fewer than 4 vertices must be complete. If we suppose that K_1 is a component of G', then observe that the remaining components of G' must be copies of K_3. But note that $K_1 \cup K_3$ contains three edges and can be replaced by $2K_2$ which requires only two edges. Thus G' contains no isolated vertices.

Since G' contains only components $G_i \in \{K_2, K_3, K_{1,4}\}$ and $|E(K_{1,4}) = 4 = E(K_2 \cup K_3)|$, we have shown that G is a minimal P_4-saturated graph and $sat(n, P_4) = j + 1 = \frac{n+3}{2}$ for odd n.

If G' is of order $n = 2j$, by our above observations, it must be that $G' = jK_2$ which has exactly $j = \frac{n}{2}$ edges. Thus $sat(n, P_4) = \frac{n}{2}$ for even n. □

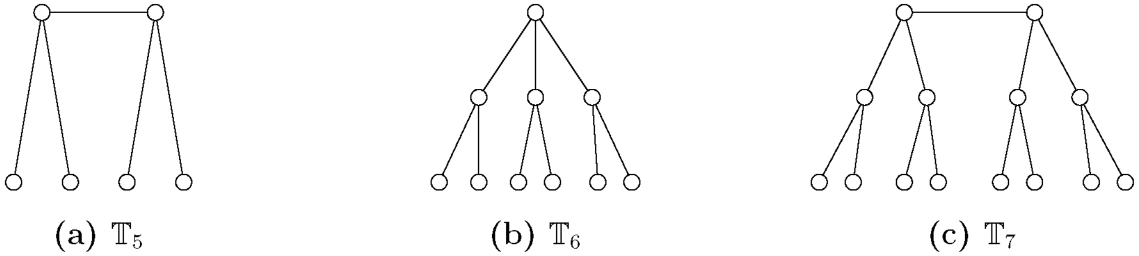

(a) $\mathbb{T}_5$ (b) $\mathbb{T}_6$ (c) $\mathbb{T}_7$

Figure 4.3: Representations of $\mathbb{T}_k$ for $k \in \{5, 6, 7\}$.

Now we turn our attention to paths P_k with $k \geq 5$. We will describe a minimal P_k-saturated forest and discuss the saturation number for P_k for large n.

Recall that given distinct vertices u and v in a connected graph G, there exists a $uv-$path in G. The number of edges in a minimal $uv-$path is the *distance* between u and v in G. The *eccentricity* of $u \in V(G)$ is the distance from u to a vertex farthest from u in G. The *central vertices* of G are the vertices of G with minimum eccentricity. Every tree has either one or two central vertices [CLZ11].

Notation 3. *In [KT86], we are given a description of* $\mathbb{T}_k$ *for any* $k \geq 5$ *where* $\mathbb{T}_k$ *is a (non-trivial)* P_k-*saturated tree of smallest order. This tree* $\mathbb{T}_k$ *has* $\lfloor \frac{k}{2} \rfloor$ *levels, and the highest level contains the central vertices (known as the* root *or* roots *depending on the parity of* k*) of the tree. We say* $\mathbb{T}_k$ *is "almost binary" since all vertices (including the root(s)) have degree exactly* 3 *except for the lowest level which contains only pendant vertices. We say that the vertices adjacent to the pendant vertices of* $\mathbb{T}_k$ *form the* second level *of* $\mathbb{T}_k$*. (See Figures 4.3 and 4.4.)*

Let

$$a_k = \begin{cases} 3 \cdot 2^{j-1} - 2 & k = 2j \\ \\ 4 \cdot 2^{j-1} - 2 & k = 2j + 1 \end{cases}.$$

Observe that a_k *is the order of* $\mathbb{T}_k$*. (See Figure 4.4.) Notice that* a_k *grows exponentially with* k*.*

Theorem 13 (KT86 Theorem 7). *Let* T *be a* P_k-*saturated tree with* $k \geq 5$*. Then* T *contains a copy of* $\mathbb{T}_k$*.*

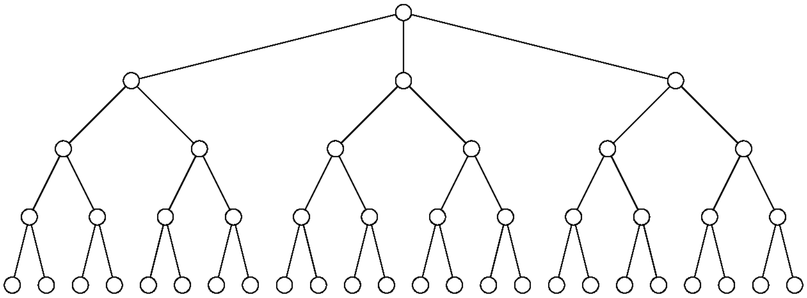


Figure 4.4: The tree $\mathbb{T}_{10}$ is the minimal $P_{10}-$saturated graph of order $a_{10} = 3{\cdot}2^{5-1} - 2 = 46$. Notice that $\mathbb{T}_{10}$ has $\lfloor \frac{10}{2} \rfloor = 5$ levels with 24 pendant vertices and 12 vertices in the second level. Notice that starting an any internal (non-pendant) vertex x, there are at least two paths of equal length sharing only x and terminating in (nearest) pendant vertices.

Proof Theorem 13. Observe that if G is P_k-saturated, any vertex v of degree two in G must belong to a triangle. This is because the addition of the edge joining the neighbors of v cannot increase the length of the longest path in G. Hence any P_k-saturated tree T contains no vertices of degree two. Choose $x \in V(T)$ such that $\deg(x) > 1$, and let $x_1, x_2, \ldots, x_p$ with $p \geq 3$ be the neighbors of x. Let ℓ_i with $1 \leq i \leq p$ denote the maximum number of vertices in a path starting at x and containing x_i. Let the index labels i be assigned to neighbors of x such that $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_p$. Since T is P_k-free, by following a longest path through x_1 and x_2, we see that $\ell_1 + \ell_2 - 1 \leq k - 1$. (Note that we subtract 1 here since we have double counted x.) Since there exists a copy of P_k in $T + x_2x_3$, we know that $\ell_1 + \ell_2 \geq k$. Hence $\ell_1 + \ell_2 = k$.

Further, $\ell_2 = \ell_3$. To see this, suppose to the contrary that $\ell_2 > \ell_3$. Then the addition of the edge $e = x_1x_2$ does not produce a copy of P_k in $T + e$ because the longest path in $T + e$ containing e has $\ell_1 + \ell_3 < \ell_1 + \ell_2 = k$ vertices. That is, the longest path containing $e = x_1x_2$ uses the path from x_1 on ℓ_1 vertices, the path from x_3 on ℓ_3 vertices, and connects them via the path (x_1, x_2, x, x_3). Since x is not a pendant vertex, we now see that T contains at least two paths of equal length (namely of length $\ell_2 = \ell_3$) that start at x, share only the vertex x, and terminate in pendant vertices. Hence T contains a copy of $\mathbb{T}_k$. $\square$

We note that the above proof implies that, for $k \geq 5$, except in the trivial case of $T \in \{K_1, K_2\}$, any P_k-saturated tree T has order at least a_k. Thus for $2 < n < a_k$, if G

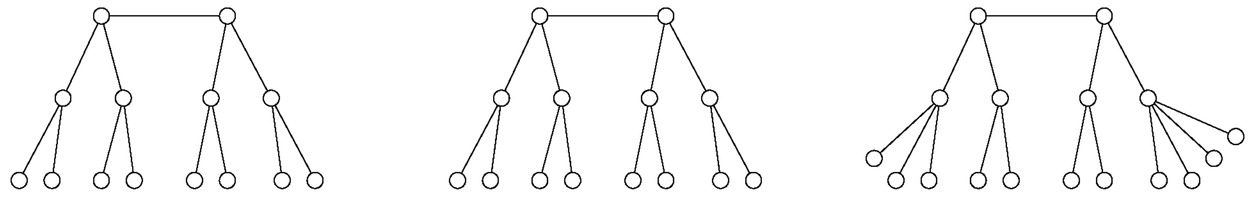


Figure 4.5: The forest composed of copies of two copies of $\mathbb{T}_7$ and one copy of $\mathbb{T}_7^*$ is a minimal P_7-saturated graph for $n = 45$.

is a P_k-saturated graph of order n, we know that G is *not* a tree. A characterization of the structure of such G, as well as $sat(n, P_k)$ for small n, remain open questions. However, Theorem 13 allows us to establish the saturation number of P_k for large n.

Remark 3. *Theorem 13 gives a method for constructing a P_k-saturated forest F. We construct F of order n where $n = a_kq + r$ with $0 \leq r < a_k$ by taking q copies of $\mathbb{T}_k$ and adding a total of r pendant vertices to the copies of $\mathbb{T}_k$. In this construction, each of these r pendant vertices must be joined to some vertex in the second level of some copy of $\mathbb{T}_k$ in F. We call this* multiplying *the pendant vertices of $\mathbb{T}_k$.*

Recall that since a tree of order t has $t - 1$ edges, the size of a forest of order n with q components is $n - q$. Thus a forest of order n with $\lfloor \frac{n}{a_k} \rfloor$ trees has size $n - \lfloor \frac{n}{a_k} \rfloor$.

Example 1. *If we take $k = 7$, then $j = 3$ and $a_7 = 4 \cdot 2^{3-1} - 2 = 14$. Let $n = 45$. Then $q = \lfloor \frac{45}{14} \rfloor = 3$ and our forest can contain up to 3 copies of $\mathbb{T}_7$ using 42 vertices. We also have $r = 45 - 3(14) = 3$ remaining vertices. We can join these vertices as leaves (pendant vertices) to the second-lowest level of one copy of $\mathbb{T}_7$ to obtain $\mathbb{T}_7^*$, a P_k-saturated graph on 17 vertices and 16 edges. We now have the forest $2\mathbb{T}_7 \cup \mathbb{T}_k^*$ with 42 edges. See Figure 4.5*

By Theorem 13, any P_k-saturated tree T contains $\mathbb{T}_k$. By Lemma 3, all connected components of G must be complete and P_k-free or must contain $\mathbb{T}_k$. Hence for large n, a P_k-saturated graph has at least $n - \lfloor \frac{n}{a_k} \rfloor$ edges. The details follow.

Corollary 14. *If $n \geq a_k$, and $k \geq 6$, then*

$$sat(n, P_k) = n - \left\lfloor \frac{n}{a_k} \right\rfloor . \tag{4.1}$$

Proof of Corollary 14. Let G be a minimal P_k-saturated graph of order $n \geq a_k$. Let $r = n - qa_k$ with $0 \leq r < a_k$. Let $\mathbb{T}_k^*$ denote a copy of $\mathbb{T}_k$ with r additional pendant vertices joined to the second level. Observe that $\mathbb{T}_k^*$ is P_k-saturated and that $(q-1)\mathbb{T}_k \cup \mathbb{T}_k^*$ is a P_k-saturated forest on $n - q$ edges.

Suppose, to produce a contradiction, that $|E(G)| < n - q$. Then G must have at least $q+1$ tree-components, at least one of which must have fewer than a_k vertices. By the proof of Theorem 13, the only P_k-saturated trees on fewer than a_k vertices are K_1 and K_2. Notice that neither $\mathbb{T}_k \cup K_1$ nor $\mathbb{T}_k \cup K_2$ is P_k-saturated for $k \geq 6$. To see this consider $e = uv$ where $u \in V(K_p)$ for $p \in \{1, 2\}$ and $v \in V(\mathbb{T}_k)$ is a central vertex, or *root*, of $\mathbb{T}_k$. Then $\mathbb{T}_k \cup K_p + e$ does not contain a copy of P_k. Thus for $k \geq 6$, G cannot have tree components on fewer than a_k vertices.

Then G can have at most q tree components. Since there exists a P_k-saturated forest composed of a total of $q = \lfloor \frac{n}{a_k} \rfloor$ of trees: $q-1$ copies of $\mathbb{T}_k$ and one copy of $\mathbb{T}_k^*$,

$$sat(n, P_k) = |E(G)| = n - \left\lfloor \frac{n}{a_k} \right\rfloor.$$

We note that in the case where $k = 5$, $a_k = 6$, the previous argument applies, but for $r \in \{2, 3, 4, 5\}$, the minimal P_k-saturated graph must have K_2 as a component and

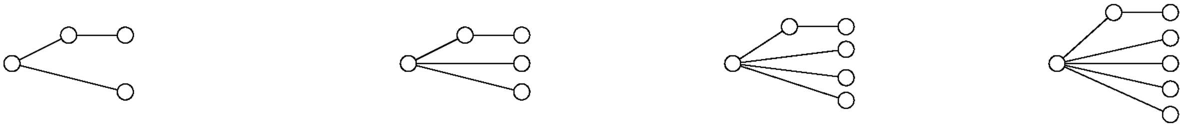
$$sat(n, P_5) \geq n - \left(\left\lfloor \frac{n-2}{6} \right\rfloor + 1 \right).$$

□

4.3 Trees of Minimum Saturation Number

We now adopt an alternative notation for the star $K_{1,t}$. We write $S_{t+1} = K_{1,t}$. This notation will aid in our discussion of the "star-like" trees called *subdivided stars*. To *subdivide the edge* $e = uv$ is to replace the edge e with a copy of P_3 such that u and v are its end vertices. Note that for each subdivided edge, both the order and the size of the graph increase by one.

Notation 4. *Let S_{k-1}^1 denote the graph obtained by subdividing exactly one edge of a star on $k-1$ vertices ($S_{k-1} = K_{1,k-2}$). See Figure 4.6.*



(a) S_3^1 (b) S_4^1 (c) S_5^1 (d) S_6^1

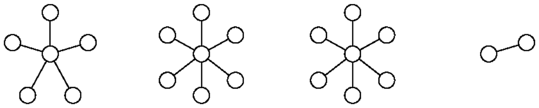
Figure 4.6: Some subdivided stars of small order.

Thus S_{k-1}^1 has k vertices: one vertex of degree $k-2$, one vertex of degree two, and $k-2$ pendant vertices.

For an arbitrary tree T_k of order k, we have seen that $sat(n, T_k) \leq sat(n, S_k)$ [KT86]. It has also been shown that $sat(n, T_k) \geq sat(n, S_{k-1}^1)$ [FFGJ09]. We summarize some of the details below. The [FFGJ09] proof of Theorem 16, which we present here uses Lemma 15 and leads to a characterization, Corollary 17, of minimal S_{k-1}^1−saturated graphs. We give examples in Figure 4.7.

(a) For $n = 19$, the minimal S_5^1−saturated graph is a forest of stars: $3S_6 \cup K_1$.

(b) For $n = 20$, the minimal S_5^1−saturated graph is a forest of stars: $3S_6 \cup K_2$.



(c) For $n = 22$, the minimal S_5^1−saturated graph is a forest of stars, three of which have order 6 or more, one of which is $K_2 = S_2$. The forest $S_6 \cup 2S_7 \cup K_s$ is shown here. The forest $2S_6 \cup S_8 \cup S_2$ is also a minimal S_5^1−saturated graph.

Figure 4.7: The minimal S_{k-1}^1−saturated graphs are star forests. Here $k = 6$.

When $n = k$, the only tree that has a connected host graph of order k and size $k-1$ is the subdivided star S_{k-1}^1, and S_k is the S_{k-1}^1−saturated host:

Lemma 15 ([FFGJ09] Lemma 1)**.** *Let T_k be a tree of order k. If there exists a tree T_k' of order k such that T_k' is T_k−saturated, then $T_k = S_{k-1}^1$ and $T_k' = S_k$.*

Theorem 16 (FFGJ09 Theroem 5)**.** *Let T_k be a tree of order $k \geq 5$ and choose $n \geq k+2$. Then*

$$sat(n, T_k) \geq n - \left\lfloor \frac{n+k-2}{k} \right\rfloor .$$

Moreover, S^1_{k-1} is the only tree T_k that attains this minimum for all such n.

Corollary 17 ([FFGJ09] Corollary 1)**.** *For $k \geq 5$, if G is a minimal $S^1_{k-1}-$saturated graph of order n, then G is a forest of $\lfloor (n+k-2)/k \rfloor$ stars. If $n - k\lfloor n/k \rfloor \geq 2$, then exactly one star is $S_2 = K_2$.*

Proof of Theorem 16. Let G be a minimal T_k-saturated graph of order n for a fixed tree T_k with $k \geq 5$. Note that

1. Any component G_{p_i} of G with order $p_i \leq k-1$ must be complete,
2. For any two components G_{p_i} and G_{p_j}, of orders p_i and p_j respectively, we must have $p_i + p_j \geq k$, and as a consequence
3. G can have at most one component $G_{p_i} \in \{K_1 = S_1, K_2 = S_2\}$.

Suppose $T_k \not\cong S^1_{k-1}$. Then by Lemma 15, any non-complete tree component of G has order at least $k+1$. Since a forest with t components has $n-t$ edges, the forest composed of the union of trees on $k+1$ vertices and one copy of K_1 or K_2 has at least $n - \left(\lfloor \frac{n-1}{k+1} \rfloor + 1\right)$ edges. Then we obtain the lower bound

$$sat(n, T_k) = |E(G_n)| \geq n - \left(\left\lfloor \frac{n-1}{k+1} \right\rfloor + 1 \right).$$

From this bound we can see that for $n \geq k+2$ we have

$$sat(n, T_k) \geq n - \left(\left\lfloor \frac{n-2}{k} \right\rfloor + 1 \right) = n - \left\lfloor \frac{n+k-2}{k} \right\rfloor .$$

Suppose instead that $T_k = S^1_{k-1}$. If $|E(G)| < n - \lfloor \frac{n+k-2}{k} \rfloor$, then G must have more than $\lfloor \frac{n+k-2}{k} \rfloor$ components and at least two components G_{p_i} and G_{p_j} are of order $k-1$ or less and thus must be complete. But we can replace these components with a star on $p_i + p_j$ vertices and obtain a T_k-saturated graph G' with fewer edges, a contradiction. Let F be the forest of order n composed of $\lfloor\frac{n+k-2}{k}\rfloor$ stars, all of order at least k, except at most one ($S \in \{K_1, K_2\}$,). Notice that F has size $|E(F)| = n - \lfloor\frac{n+k-2}{k}\rfloor$. Then in this case

$$sat(n, T_k) = n - \left\lfloor \frac{n+k-2}{k} \right\rfloor .$$

$\square$

4.4 Other Tree Saturation Number Results

4.4.1 Properties of subtrees and saturation number bounds

In addition to the lower bound on saturation number for trees, some interesting subtree properties guaranteeing relatively high or relatively low saturation numbers have been identified [FFGJ09]. We briefly discuss a few of these results below.

Since all trees have at least two pendant vertices, the minimum degree of every nontrivial tree is one. However, for a given tree T, the *second smallest* degree $\delta_2(T)$ is of interest. Indeed, given a non-star tree T, it is known that $sat(n,T) \geq \left(\frac{\delta_2(T)-1}{2}\right) n$, for n sufficiently large. Relatively speaking, trees with high second smallest degree have high saturation numbers. In particular, if $\delta_2(T) \geq 3$, the bound in Theorem 18 below shows that there exists no (non-trivial) $T-$saturated tree.

Theorem 18 ([FFGJ09] Theorem 6). *If $T_k \not\cong K_{1,k-1}$ is a tree, $k \geq 5$, with $\delta_2 = \delta_2(T_k)$, then for $n \geq (\delta_2 - 1)^3$,*

$$sat(n, T_k) \geq \left(\frac{\delta_2 - 1}{2}\right) n.$$

On the other hand, for n large enough, a non-path tree T with a relatively long induced path has saturation number at most $n-1$. Suppose T is a tree with maximum degree $\Delta(T) \geq 3$ and whose longest path contains ℓ vertices. If the vertices of a path $P = (v_1, v_2, \ldots, v_\ell)$ of order ℓ in T can be labeled such that deg$(v_2), \ldots,$deg$(v_{\lceil \ell/2 \rceil}) = 2$, then we can find a $T-$saturated tree of similar structure to the [KT86] tree described in Notation 3. Specifically, for target trees with long induced paths, we can modify the $P_\ell-$saturated tree $\mathbb{T}_\ell$ (see Figure 4.4 and Notation 3), by "multiplying the branches" so that all internal (non-pendant) vertices have degree $\Delta + 1$, and we obtain a target-saturated host tree whose longest path contains $\ell - 1$ vertices. Such host trees are called $\mathbb{T}_{\ell-1,\Delta+1}$ in the theorem below. (Notice that in this notation, for the P_k−saturated tree of Notation 3 we have $\mathbb{T}_k = \mathbb{T}_{k-1,3}$.)

Theorem 19 ([FFGJ09] Theorem 7). *Let T be a tree with maximum degree $\Delta \geq 3$ and whose longest path contains exactly ℓ vertices, the first $\lceil \ell/2 \rceil$ of which have degree at most 2. Then the tree $\mathbb{T}_{\ell-1,\Delta+1}$ is T−saturated and $sat(n, T) \leq n - 1$ for $n \geq |V(\mathbb{T}_{\ell-1,\Delta+1})|$.*

4.4.2 Trees T for which there exists a minimal T−saturated forest

In the case that we can find a smallest non-trivial T_k−saturated tree T_s, the following technical lemma (Lemma 20) can simplify the proof of the saturation number [FFGJ09]. If there exist T_k−saturated trees of all orders $p \in \{s, s+1, \ldots, 2s-1\}$ and the (disjoint) union of any pair of such trees is again T_k−saturated, then there exists a minimal T_k−saturated forest. In the case that the (disjoint) union of K_1 or K_2 and a T_k−saturated tree of order p is T_k−saturated, this leads to the lower bound $sat(n, T_k) \geq n - \lfloor \frac{n-1}{s} \rfloor - 1$. We also obtain the upper bound $sat(n, T_k) \leq n - \lfloor \frac{n}{s} \rfloor$ corresponding to a forest of $\lfloor \frac{n}{s} \rfloor - 1$ trees of order s and one tree of order p. When it applies, Lemma 20 provides an outline for the saturation number proof. We will see such a proof in Chapter 5, and this technique was already seen in the proof of Corollary 14.

Lemma 20 ([FFGJ09] Lemma 3). *Suppose that T_k is a tree of order $k \geq 5$ and that T_s is a T_k−saturated tree of order $s \geq k$ such that*

1. *$s \leq |V(T)|$ for all T_k−saturated trees T,*
2. *for all j with $1 \leq j \leq s-1$, there exists a T_k−saturated tree T_{s+j} of order $s+j$, and*
3. *the union of any pair of T_k−saturated trees $T_{s+j_1}, T_{s+j_2} \in \mathcal{T} = \{T_s, T_{s+1}, \ldots, T_{2s-1}\}$ is T_k−saturated,*

then for $n \geq s$, there exists a minimal T_k−saturated forest, and

$$n - \left\lfloor \frac{n-1}{s} \right\rfloor - 1 \leq sat(n, T_k) \leq n - \left\lfloor \frac{n}{s} \right\rfloor .$$

The above theorems and lemma provide useful tools for establishing the saturation numbers for a variety of trees. The saturation number bounds for specific trees including particular brooms, twice-or-more subdivided stars, and double stars are also established in [FFGJ09].

Chapter 5

Semi-Saturation Number

In this chapter we state and prove a new theorem concerning the semi-saturation number of paths.

5.1 Motivation

In order for a graph G to be $H-$saturated, G must satisfy two requirements:

1. G must be $H-$free, and
2. for all $e \in E(\overline{G})$ the graph $G + e$ must contain a copy of H, necessarily containing e.

The notion of an $H-$semi-saturated graph arises from eliminating the first of these two requirements. We say that a graph G is $H-$*semi-saturated* if for all $e \in E(\overline{G})$, the graph $G + e$ contains a *new* copy of H. That is, $G + e$ contains a copy of H that contains e. Thus any $H-$saturated graph is also an $H-$semi-saturated graph.

The *semi-saturation number* for a target graph H with respect to host graphs of order n is the minimum number of edges in an $H-$semi-saturated graph of order n. We adopt the notation of [FK12] and write

$$ssat(n, H) := \min\{|E(G)| : G \text{ is } H - \text{semi-saturated},\ |V(G)| = n\}.$$

Since any $H-$saturated graph is also $H-$semi-saturated, we have

$$ssat(n, H) \leq sat(n, H).$$

By the proof of Theorem 1, the saturation number and the semi-saturation number for K_k coincide. On the other hand, as we shall see below, there exist families of graphs for which the semi-saturation number is strictly smaller than the saturation number.

In fact, both the saturation number and the semi-saturation number for a $k-$cycle are known. For $n \geq k \geq 3$, a minimal C_k-saturated graph on n vertices has $sat(n, C_k) = n + \frac{n}{k} + \mathcal{O}(\frac{n}{k^2} + k^2)$ edges [FK12]. In [FK12], it is proved constructively that the number of edges in a minimal C_k-semi-saturated graph on n vertices is on the order of $n + \frac{n}{2k}$. That is, for $n \geq k \geq 6$, it is known that

$$ssat(n, C_k) = n + \frac{n}{2k} + \mathcal{O}\Big(\frac{n}{k^2} + k\Big).$$

5.2 The Semi-Saturation Number for P_k

We will establish the semi-saturation number of P_k, and we will prove that for $k \geq 6$ and $n \geq 2\lfloor\frac{3(k-1)}{2}\rfloor$, the semi-saturation number of P_k is strictly less than the saturation number of P_k with respect to host graphs of the same order. That is, $ssat(n, P_k) < sat(n, P_k)$. We first prove a semi-saturated version of Lemma 20.

Lemma 21. *Suppose that T_k is a tree of order $k \geq 5$ and that T_s is a T_k-semi-saturated tree of order $s \geq k$ such that*

1. *$s \leq |V(T)|$ for all T_k-semi-saturated trees T,*
2. *for all j with $1 \leq j \leq s-1$, there exists a T_k-semi -saturated tree T_{s+j} of order $s+j$, and*
3. *the union of any pair of T_k-semi-saturated trees $T_{s+j_1}, T_{s+j_2} \in \mathcal{T} = \{T_s, T_{s+1}, \ldots, T_{2s-1}\}$ is T_k-semi-saturated,*

then for $n \geq s$, there exists a minimal T_k-semi-saturated forest, and

$$n - \left\lfloor\frac{n-1}{s}\right\rfloor - 1 \leq ssat(n, T_k) \leq n - \left\lfloor\frac{n}{s}\right\rfloor. \qquad (5.1)$$

Proof of Lemma 21. We show that the forest $F = (\lfloor \frac{n}{s} \rfloor - 1)T_s \cup T_{s+j}$, where $j = n - \lfloor \frac{n}{s} \rfloor$, is T_k-semi-saturated. Let $e = uv \in E(\overline{F})$. Then either $u, v \in V(T)$ for some T_k-semi-saturated tree T in F, or $u \in T_1$ and $v \in T_2$ where T_1 and T_2 are distinct T_k-semi-saturated trees in F. In either case, $F + e$ contains a new copy of T_k since the union of any pair of trees in F is T_k-semi-saturated. Notice that since F is a forest of $\lfloor \frac{n}{s} \rfloor$ trees, we have

$$|E(F)| = n - \left\lfloor \frac{n}{s} \right\rfloor.$$

This establishes the upper bound in (5.1).

The lower bound holds since any T_k-semi-saturated graph G can essentially be replaced by a forest of trees selected from $\mathcal{T}$ without adding any edges. Observe that G will have at most one component of order 2 or less. Let $G = H_1 \cup H_2$, where H_1 consists of all the components of G that are trees and have at least 3 vertices. Since G is T_k-semi-saturated, this means that all components of H_1 have order at least s. Let $H_2 = G - H_1$. Thus, any component that is a K_1 or K_2 lies in H_2.

Case 1: Assume $H_2 = \emptyset$.

Then G is a forest and each component of G is a tree of order at least s. Thus, since $\lfloor (n-1)/s \rfloor + 1 \geq \lfloor n/s \rfloor$,

$$|E(G)| \geq n - \left\lfloor \frac{n}{s} \right\rfloor \geq n - \left\lfloor \frac{n-1}{s} \right\rfloor - 1.$$

Case 2: Assume K_1 or K_2 is a component of H_2.

Then $|E(H_2)| \geq |V(H_2)| - 1$ and $|V(H_1)| \leq |V(G)| - 1$. Thus,

$$\begin{aligned} |E(G)| &= |E(H_1)| + |E(H_2)| \\ &\geq |V(H_1)| - \left\lfloor \frac{|V(H_1)|}{s} \right\rfloor + |V(H_2)| - 1 \\ &\geq |V(G)| - \left\lfloor \frac{|V(G)| - 1}{s} \right\rfloor - 1, \end{aligned}$$

which is what we needed to show.

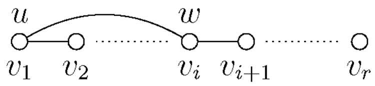


Figure 5.1: If either of u or w is an end vertex of P_r, then $P_r + uw$ contains the path $P'_r = (v_{i-1}, v_{i-2}, \ldots, v_1 = u, v_i = w, v_{i+1}, \ldots v_r)$, a new copy of P_r.

Case 3: Assume that no component of H_2 is a K_1 or K_2 but $H_2 \neq \emptyset$.

The the argument from Case 2 is repeated. That is, $|E(H_2)| \geq |V(H_2)|$ and $|V(H_1)| \leq |V(G)| - 1$. Thus,

$$\begin{aligned}
|E(G)| &= |E(H_1)| + |E(H_2)| \\
&\geq |V(H_1)| - \left\lfloor \frac{|V(H_1)|}{s} \right\rfloor + |V(H_2)| \\
&\geq |V(G)| - \left\lfloor \frac{|V(G)| - 1}{s} \right\rfloor \\
&\geq |V(G)| - \left\lfloor \frac{|V(G)| - 1}{s} \right\rfloor - 1,
\end{aligned}$$

which is what we needed to show. □

Lemma 22. *Let P_k be a path on $k \geq 2$ vertices and let*

$$r = \left\lfloor \frac{3(k-1)}{2} \right\rfloor .$$

Then the path P_r is P_k−semi-saturated.

Proof of Lemma 22. Let $P_r = (v_1, v_2, \ldots, v_r)$. Let u and w be any pair of distinct nonadjacent vertices in $V(P_r)$. We must show that $P_r + uw$ contains a copy of P_k using the edge uw. First we consider the case where u or w, say u, is an end vertex of of P_r. Suppose that $u = v_1$ and let $w = v_i$ for some $i \in \{3, 4, \ldots, r\}$. Then $P_r + uw$ contains a path on r vertices: $(v_{i-1}, v_{i-2}, \ldots, v_1 = u, v_i = w, v_{i+1}, \ldots v_r)$. (See Figure 5.1.) Thus $P_r + uw$ also contains a path on k vertices containing the edge uw.

Now suppose that neither u nor w is an end vertex of P_r. Then the choice of u and w induces a tripartition on the vertices of $P_r - \{u, w\}$. That is, the choice of u and w partitions $V(P_r) - \{u, w\}$ into three parts: the vertices between u and w on the path from u to w, the vertices on the path not containing w from u to one end-vertex of P_r, and the vertices on the path from w to the other end-vertex of P_r. We claim that the induced subgraph on the two larger parts together with u and w contains a new copy of P_k. To see this, observe that the smallest part has order at most

$$\frac{1}{3}\Big(\Big\lfloor\frac{3(k-1)}{2}\Big\rfloor - 2\Big) \leq \Big\lfloor\frac{r-2}{3}\Big\rfloor.$$

Note that for even values of k, we have $r = \frac{3k-4}{2}$ and for odd values of k, we have $r = \frac{3k-3}{2}$. Then, if k is even, the longest path containing uw in $P_r + uw$ has order at least

$$\begin{aligned} r - \Big\lfloor\frac{r-2}{3}\Big\rfloor &= \frac{3k-4}{2} - \Big\lfloor\frac{1}{3}\Big(\frac{3k-4}{2} - 2\Big)\Big\rfloor \\ &= \frac{3k}{2} - 2 - \Big\lfloor\frac{k}{2} - \frac{4}{3}\Big\rfloor \\ &= \frac{3k}{2} - 2 - \frac{k}{2} + 2 \\ &= k. \end{aligned}$$

If k is odd, the longest path containing uw in $P_r + uw$ has order at least

$$\begin{aligned} r - \Big\lfloor\frac{r-2}{3}\Big\rfloor &= \frac{3k-3}{2} - \Big\lfloor\frac{1}{3}\Big(\frac{3k-3}{2} - 2\Big)\Big\rfloor \\ &= \frac{3k}{2} - \frac{3}{2} - \Big\lfloor\frac{k-1}{2} - \frac{2}{3}\Big\rfloor \\ &= \frac{3k}{2} - \frac{3}{2} - \frac{k}{2} + \frac{1}{2} + 1 \\ &= k. \end{aligned}$$

□

Lemma 23. *Let T_p be any tree of order p with $3 \leq p \leq r - 1$ where $r = \lfloor\frac{3(k-1)}{2}\rfloor$. Then T_p is not P_k−semi-saturated.*

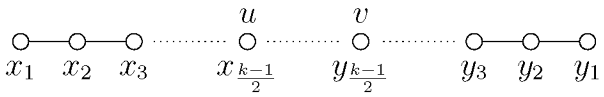


Figure 5.2: A longest path P_ℓ of order $\ell \geq k$ in T_p, a tree of order $p \leq r-1$. When k is odd, the graph $T_p + uv$ does not contain a new copy of P_k.

Proof of Lemma 23. First, suppose that k is odd. Then T_p has order

$$p \leq r - 1 = \frac{3k-5}{2}.$$

Recall from Notation 3 that for $k \geq 5$ where $k = 2j+1$, the order of the minimal P_k−saturated tree is

$$a_k = 4 \cdot 2^{j-1} - 2 = 4 \cdot 2^{\frac{k-3}{2}} - 2.$$

Observe that for $k \geq 5$, we have $a_k > \frac{3k-5}{2}$. Suppose that the longest path in T_p has order $\ell \leq k-1$. Then if T_p is P_k−semisaturated, T_p is also P_k−saturated, and then by Theorem 13 we know that T_p has order at least a_k, a contradiction since $p \leq \frac{3k-5}{2}$. So we may assume that the longest path P_ℓ in T_p has order $\ell \geq k$.

Let the end vertices of P_ℓ be labeled x_1 and y_1 and let their neighbors be labeled x_2 and y_2, respectively. Continue labeling along the path as shown in Figure 5.2 up to $x_{\frac{k-1}{2}}$ and $y_{\frac{k-1}{2}}$. Choose $u = x_{\frac{k-1}{2}}$ and $v = y_{\frac{k-1}{2}}$. Note that $uv \in E(\overline{T_p})$ since $\ell \geq k$. Observe that this choice of u and v partitions the vertices of $T_p \setminus \{u, v\}$ into at least three components, exactly three of which correspond to the connected components of $T_p \setminus \{u, v\}$ containing vertices of P_ℓ. The two parts of this partition containing $X = \{x_1, x_2, \ldots, x_{\frac{k-3}{2}}\}$ and $Y = \{y_1, y_2, \ldots, y_{\frac{k-3}{2}}\}$, respectively, have order at least $\frac{k-3}{2}$. The third part contains at least one vertex w since u and v are not adjacent.

Now consider $T_p + uv$. Since

$$\begin{aligned} p - \left(2\left(\frac{k-3}{2}\right) + 2\right) &\leq \frac{3k-5}{2} - (k-1) \\ &= \frac{k-3}{2}, \end{aligned}$$

we know that the third part has order at most $\frac{k-3}{2}$. Hence the longest path in $T_p + uv$ containing uv has order at most $2\left(\frac{k-3}{2}\right) + 2 = k - 1$ and thus T_p is not T_k- semi-saturated for odd values of k.

When k is even, we have $p \leq \frac{3k-6}{2}$. Observe that for $k \geq 4$ with $k = 2j$ we have

$$a_k = 3 \cdot 2^{j-1} - 2 = 3 \cdot 2^{\frac{k-2}{2}} - 2 > \frac{3k-6}{2}.$$

Thus, as above, we may assume that the longest path P_ℓ in T_p has order at least k. Labeling the vertices of P_ℓ in a similar manner as above, we now choose $u = x_{\frac{k-2}{2}}$ and $v = y_{\frac{k-2}{2}}$. The resulting partition of $V(T_p \setminus \{u, v\})$ contains two parts of order at least $\frac{k-4}{2}$ (the parts containing x_1 and y_1). By a similar computation as above, we see that there are at least two vertices w_1 and w_2 on the path from u to v in T_p, and the part containing w_1 and w_2 has order at most $\frac{k-2}{2}$. Then in $T_p + uv$ the longest path containing uv has order at most

$$\frac{k-2}{2} + \frac{k-4}{2} + 2 = k - 1.$$

Hence T_p is not P_k-semi-saturated for even values of k. $\square$

Theorem 24. *Let $n \geq 2r$, with $r = \lfloor \frac{3(k-1)}{2} \rfloor$. Then*

$$n - \left\lfloor \frac{n-1}{r} \right\rfloor - 1 \leq ssat(n, P_k) \leq n - \left\lfloor \frac{n}{r} \right\rfloor.$$

Proof of Theorem 24. By Lemmas 22 and 23, we know that P_r with $r = \lfloor \frac{3(k-1)}{2} \rfloor$ is a P_k-semi-saturated tree of smallest order. It follows from the proof of Lemma 22 that for j with $r \leq j \leq 2r - 1$, the path P_j is P_k-semi-saturated. Further, the (disjoint) union of any two paths P_{j_1} and P_{j_2} with $j_1, j_2 \in \{r, r+1, \ldots, 2r-1\}$ is P_k-semi-saturated. To see this, let $e = uv$ where $u \in V(P_{j_1})$ and $v \in V(P_{j_2})$. Then the longest path P in $\left(P_{j_1} \cup P_{j_2}\right) + e$ such that $e \in E(P)$ has order at least $\lceil \frac{j_1}{2} \rceil + \lceil \frac{j_2}{2} \rceil \geq r$. ($P$ has smallest order when u, v are central vertices of P_{j_1} and P_{j_2}, respectively.) Now by Lemma 21, for $n \geq r$, there exists a

Figure 5.3: For $k = 7$, we have $r = \lfloor \frac{3(7-1)}{2} \rfloor = 9$. For $n = 45$ the forests F_1 and F_2 are both minimal P_7-semi-saturated graphs with $45 - \lfloor \frac{45}{9} \rfloor = 40$ edges.

P_k-semi-saturated forest and

$$n - \left\lfloor \frac{n-1}{r} \right\rfloor - 1 \leq ssat(n, P_k) \leq n - \left\lfloor \frac{n}{r} \right\rfloor.$$

□

See Figure 5.3 for some examples of minimal P_7-semi-saturated forests for $n = 42$. When compared to a minimal P_7-saturated forest of the same order, such as that shown in Figure 4.5, we see that $ssat(42, P_7) < sat(42, P_7)$. This result is generalized in Theorem 25.

Theorem 25. *For $k \geq 6$, and $n \geq 2r = 2\lfloor \frac{3(k-1)}{2} \rfloor$,*

$$ssat(n, P_k) < sat(n, P_k).$$

Proof of Theorem 25. By Corollary 14, for $n \geq a_k$ we have $sat(n, P_k) = n - \lfloor \frac{n}{a_k} \rfloor$. Since for $k \geq 6$, we have $r < a_k$, and $ssat(n, P_k) \leq n - \lfloor \frac{n}{r} \rfloor$, then $ssat(n, P_k) < sat(n, P_k)$. For n such that $2r \leq n < a_k$, any minimal P_k-saturated graph G can have at most one tree component, and if G has a tree component T, then $T \in \{K_1, K_2\}$. All non-tree components of G have at least as many edges as vertices and thus

$$|E(G)| \geq n - 1.$$

Since for such n, there exists a P_k-semi-saturated forest of order n composed of at least two trees and thus having at most $n - 2$ edges, we have $|E(G)| > ssat(n, P_k)$. □

Chapter 6

Further Questions

In [EHM64] it is established that $ssat(n, K_k) = sat(n, K_k)$. We have seen that for P_k, the saturation number is larger than the semi-saturation number. This raises the questions: for which families of graphs is the semi-saturation number the same as the saturation number? Further, what properties do these families of graphs have that guarantees $ssat(n, H_k) = sat(n, H_k)$? Can a bound analogous to that given in Theorem 5 be established for the semi-saturation number of an arbitrary family $\mathcal{F}$ of graphs?

In the notation of Theorem 19, we have seen that for $n \geq a_k$, a forest of the trees $T_{k-1,3}$ (with pendant vertices multiplied as needed) is a minimal P_k-saturated graph, and we know that any P_k-saturated tree of order a_k or more contains $T_{k-1,3}$. (See the proof of Corollary 14 and Theorem 13.) But what do non-tree path-saturated graphs look like? In particular, what is the structure of a small (order $n < a_k$) P_k-saturated graph?

[EHM64] completely characterizes the minimal K_k-saturated graphs, and [KT86] characterizes the minimal $K_{1,t}-$saturated graphs. For the graphs whose saturation number is known, what is the structure of the minimal saturated graphs that correspond to this saturation number?

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