AN EXISTENCE THEOREM FOR SOLUTIONS TO A MODEL PROBLEM WITH YAMABE-POSITIVE METRIC FOR CONFORMAL PARAMETERIZATIONS OF THE EINSTEIN CONSTRAINT EQUATIONS.

By

Tyler D. Knowles, A.B.

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of

Master of Science

in

Mathematics

University of Alaska Fairbanks

August 2016

APPROVED:

David Maxwell, Committee Chair
John A. Rhodes, Committee Member
Alexei Rybkin, Committee Member
Leah Wrenn Berman, Chair
 Department of Mathematics and Statistics
Paul Layer, Dean
 College of Natural Science and Mathematics
Michael Castellini, *Dean of the Graduate School*

Abstract

We use the conformal method to investigate solutions of the vacuum Einstein constraint equations on a manifold with a Yamabe-positive metric. To do so, we develop a model problem with symmetric data on $S^{n-1} \times S^1$. We specialize the model problem to a two-parameter family of conformal data, and find that no solutions exist when the transverse-traceless tensor is identically zero. When the transverse traceless tensor is nonzero, we observe an existence theorem in both the near-constant mean curvature and far-from-constant mean curvature regimes.

Table of Contents

Acknowledgements

First and foremost I would like to extend my sincerest gratitude to my advisor, Dr. David Maxwell, for suggesting this project and for his patient guidance throughout. This includes his courses in special and general relativity, topology, gauge theory, partial differential equations, and numerous discussions outside of any scheduled class time. Dr. Maxwell's dedication to and passion for both his teaching and research make him an exemplary mathematician and role model.

I would like to thank my other committee members, Dr. John A. Rhodes and Dr. Alexei Rybkin, for their helpful suggestions and feedback on the manuscript. Furthermore, I thank Dr. Rybkin for stimulating courses in real and complex analysis, a foundation in which is necessary for this work; I thank Dr. Rhodes for occasional "pep talks" and encouragement throughout the course of my graduate program.

I am also grateful to Matthew Harris for his constructive feedback on the manuscript.

Finally, I would like to gratefully acknowledge the continued support of my parents Kip and Kathryn Knowles during my graduate school experience. Their encouragement and wisdom have carried me farther than I ever thought possible. A special acknowledgement is also due Dr. Thomas C. M. Truesdell, Visiting Professor of English at Ripon College, for his support and friendship even after the completion of my undergraduate degree.

Chapter 1

Introduction and Background

1.1 Motivation

The recent detection of gravitational waves by the LIGO Scientific Collaboration has confirmed, like many previous experiments, that Einstein's Theory of General Relativity is a powerful and accurate mathematical model of gravitation. It is of mathematical and physical interest to describe, if possible, the complete set of solutions to Einstein's equation of general relativity. Einstein's equation admits an initial-value formulation, and one way to prescribe initial data is via the conformal method. The conformal method has been successfully used to show uniqueness of solutions to Einstein's equation in the cases of constant mean curvature ("CMC") and near-CMC. Work in the far-from-CMC case is ongoing, and has been investigated in several special cases. For example, [Ma11] presented a model problem with a Yamabe-null metric and showed there are regimes under which both non-existence and non-uniqueness occur. This thesis explores a model problem with a Yamabe-positive metric where a general existence theorem exists.

In this chapter, we review classical relativity and present a geometric formulation of general relativity. We then introduce the constraint equations and the conformal parameters used to discover solutions to the constraint equations. In Chapter 2, we introduce the specific manifold used to construct our model problem. In Chapter 3, we show that finding solutions to the constraint equations, in this case, reduces to finding roots of a real-valued function. Finally, we present the formal proofs of the existence of such solutions.

1.2 Overview of Relativity

The classical notion of relativity, that the laws of physics are invariant in any reference frame undergoing uniform motion, was known to Galileo. This principle can be stated in terms of Newton's First Law of Motion: there exist reference frames in which a non-accelerating particle travels in a straight line at constant speed. This law is a key hypothesis for Newton's two other laws of motion. In particular, these laws of motion hold in two spacetime reference

1

frames $R, \hat{R}$ of dimension n if and only if the coordinates $(t, \mathbf{x})$ of R and $(\hat{t}, \hat{\mathbf{x}})$ of $\hat{R}$ are related by

$$\begin{bmatrix} t \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \mathbf{v} & H \end{bmatrix} \begin{bmatrix} \hat{t} \\ \hat{\mathbf{x}} \end{bmatrix} + T, \tag{1.1}$$

where $\mathbf{v}$ is a constant column vector, H is a constant proper (that is, det $H = 1$) orthogonal matrix, and T is a constant column vector [Wo03]. A coordinate transformation in the form of Equation 1.1 is called a *Galilean transformation*, and the reference frames $R, \hat{R}$ are called *inertial frames*.

On the human scale, classical relativity and Newton's laws form accurate mathematical models for physical systems. They do not provide a complete model for physical phenomena, however, as they are incompatible with James Clerk Maxwell's equations of electromagnetism. Maxwell's theory requires that electromagnetic waves travel at a fixed speed in all inertial frames. Einstein resolved this apparent contradiction by showing that the laws of motion are invariant not under Galilean transformations, but instead under *Lorentz transformations*. In the notation of Equation 1.1, a Lorentz transformation has the form

$$\begin{bmatrix} ct \\ \mathbf{x} \end{bmatrix} = L \begin{bmatrix} c\hat{t} \\ \hat{\mathbf{x}} \end{bmatrix} + T, \tag{1.2}$$

where c is the (constant) speed of electromagnetic waves in a vacuum and L is a proper orthochronous (that is, $L_{00} > 0$) matrix satisfying

$$\eta = L^t \eta L \tag{1.3}$$

for the diagonal matrix η with entries $\eta_{00} = -1$, $\eta_{ii} = 1$, $i = 2, ..., n - 1$. It can be shown that, for systems in which massive particles move at velocities $v \ll c$, Lorentz transformations reduce to Galilean transformations. For the remainder of this thesis, we will assume geometric units where $c = 1$.

Einstein's Theory of Special Relativity does not include a description of gravity. In particular, special relativity requires that, as in classical relativity, inertial frames only undergo uniform motion. This is problematic since gravity affects all matter—even light—so there are no neutral test particles for gravity as for electromagnetism. Einstein's Theory of General Relativity takes gravity into account by assuming the *strong equivalence principle*: that there is no observable distinction between a uniform gravitational field and a uniformly

accelerating inertial frame. Thus inertial observers are those who are in free-fall. Special relativity holds in neighborhoods of these observers, and gravity is detected by observing small changes in the acceleration of nearby inertial observers.

1.3 Geometric Formulation of General Relativity

In the paradigm of General Relativity, we view spacetime as a manifold: a topological space which is looks locally like Euclidean space and which allows points to be labeled with some coordinate system. Let M be a manifold of dimension n and $p \in M$. We define a geometric object g called the *metric* (or *first fundamental form*) to be a symmetric bilinear map

$$g : T_p M \times T_p M \to \mathbb{R}$$

where $T_p M$ is the tangent space of M at p. For any $\mathbf{v}, \mathbf{u} \in T_p M$ we call $g(\mathbf{v}, \mathbf{v})$ the squared length of $\mathbf{v}$ and $g(\mathbf{v}, \mathbf{u})$ the scalar product of $\mathbf{v}$ and $\mathbf{u}$. We do not require that a metric be non-negative; indeed, we classify a vector $\mathbf{v} \in T_p M$ as *spacelike* if $g(\mathbf{v}, \mathbf{v}) > 0$, as *timelike* if $g(\mathbf{v}, \mathbf{v}) < 0$, and as *null* if $g(\mathbf{v}, \mathbf{v}) = 0$. This classification allows us to similarly classify curves and surfaces: a curve $C \subseteq M$ is spacelike if the tangent vector T^α to C is spacelike at each point of C, and a submanifold $\hat{M} \subseteq M$ is spacelike if each curve in $\hat{M}$ is spacelike. Timelike and null cuves and surfaces are defined similarly.

The matrix η in Equation 1.3 defines the *Minkowski metric*. The Minkowski metric is an important example of a *Lorentzian metric*: one which has either $n-1$ positive eigenvalues and one negative eigenvalue, or vice-versa. In general relativity, we assume spacetime is a four-dimensional manifold V with a Lorentzian metric λ. The pair (V, λ) is called a *Lorentzian manifold*.

It will be convenient to use tensor index notation in the remainder of this chapter. Recall that a type (i, j) tensor is a multilinear map

$$Y : \underbrace{Q^* \times \ldots \times Q^*}_{i \text{ times}} \times \underbrace{Q \times \ldots \times Q}_{j \text{ times}} \to \mathbb{R}$$

where Q is a finite-dimensional vector space with dual space Q^* of covectors. "Up" indices (Y^α) indicate vector components while "down" indices (Y_α) indicate covector components. We also implement the Einstein summation notation, where an index appearing in both the up and down positions in an expression indicates an implied summation.

Before we exhibit Einstein's Equation of General Relativity, we need to state several definitions. Choose a manifold M and a metric $g_{\alpha\beta}$. We define the *Christoffel symbol* $\Gamma^\alpha_{\beta\gamma}$ to

3

be
$$\Gamma^\alpha_\beta \gamma := \frac{1}{2} g^{\alpha\delta} (\partial_\gamma g_{\delta\beta} + \partial_\beta g_{\delta\gamma} - \partial_\delta g_{\beta\gamma}).$$

Given a vector field X^α, we define the *covariant derivative* ∇_β of X^α to be

$$\nabla_\beta X^\alpha := \frac{\partial}{\partial x^\alpha} X^\alpha + \Gamma^\alpha_{\beta\gamma} X^\gamma.$$

We measure the commutability of covariant derivatives with the *Riemann tensor*

$$R_{\alpha\beta\gamma}{}^\delta X^\gamma := \nabla_\alpha \nabla_\beta X^\delta - \nabla_\beta \nabla_\alpha X^\delta.$$

Although in general a tensor on a four-dimensional manifold having four indices has 256 independent components, the Riemann tensor has symmetries which reduce the number of independent components to 20. We use the Riemann tensor to define the *Ricci tensor*

$$R_{\alpha\beta} := R_{\alpha\gamma\beta}{}^\gamma$$

and the *scalar curvature*

$$R := R_\alpha{}^\alpha,$$

both of which measure the intrinsic curvature of M. Finally, we may define the *Einstein tensor*

$$G_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}.$$

Einstein's Theory of General Relativity says that gravity is not, as Newton described, an attractive force between masses; instead, gravity is defined by the curvature of spacetime. In particular, the theory relates the Einstein tensor $G_{\alpha\beta}$ (and thus the curvature of spacetime) to the distribution of matter in the spacetime via the *energy-momentum tensor* $T^{\alpha\beta}$. For example, consider a cloud of particles with smoothly-varying velocity described by the vector field U. Let the cloud have a uniform rest density per unit volume ρ. In this case, we define

$$T^{\alpha\beta} := \rho U^\alpha U^\beta$$

where $U^\alpha U^\beta$ is the tensor product of U^α and U^β. This definition may be extended to other distributions of matter, as well as to fluids (see [Wo07]). The relationship between $G_{\alpha\beta}$ and $T^{\alpha\beta}$ is given by Einstein's famous equation of general relativity,

$$G_{\alpha\beta} + \lambda_{\alpha\beta}\Lambda = 8\pi G T_{\alpha\beta}, \tag{1.4}$$

where G is Newton's gravitational constant and Λ is the cosmological constant. We view $T_{\alpha\beta}$ as a source term, similar to the source term in Poisson's equation describing Newtonian gravity. Remarkably, Equation 1.4 relates the physically-meaningful quantity $T_{\alpha\beta}$ to the geometrically-meaningful quantity $G_{\alpha\beta}$. Colloquially, matter tells space how to curve and space tells matter how to move [MTW73].

Given a spacetime (V, λ) of dimension n, let $\Sigma \subseteq V$ be a spacelike submanifold of dimension $n - 1$. As a subspace of V, Σ inherits the metric

$$h_{\alpha\beta} := \lambda_{\alpha\beta}\big|_{\Sigma}$$

from V. We may also assign Σ a *second fundamental form*: a symmetric bilinear map

$$K : T_p\Sigma \times T_p\Sigma \to \mathbb{R}$$

defined by

$$K_{\alpha\beta} = h_\alpha{}^\gamma \nabla_\gamma n_\beta \tag{1.5}$$

where n^α is a unit-length timelike vector field normal to Σ; that is, $n^\alpha n_\alpha = -1$. Geometrically, $K_{\alpha\beta}$ measures the extrinsic curvature of Σ in V by measuring the change in normal vectors as we change position in Σ.

1.4 The Constraint Equations

In this thesis we consider Equation 1.4 in a vacuum ($T_{\alpha\beta} \equiv 0$) with $\Lambda = 0$:

$$G_{\alpha\beta} = 0. \tag{1.6}$$

The initial-value formulation of Equation 1.6 can be derived in the following manner. Let Σ, h, and $K_{\alpha\beta}$ be as described above. Let n^α be a unit timelike vector normal to Σ. Applying n^α to Equation 1.4, we note that $n^\alpha T_{\alpha\beta}$ is the momentum density observed by n^α and $n^\alpha n^\beta T_{\alpha\beta}$ is the energy density observed by n^α. It is a geometric fact that $n^\alpha G_{\alpha\beta}$ can be computed in terms of $h_{\alpha\beta}$ and $K_{\alpha\beta}$ [Wa84]. Furthermore, it can be shown that these considerations lead to a system of constraint equations for Equation 1.6 [Wa84]. In particular, $n^\alpha n^\beta G_{\alpha\beta} = 0$ and $h^\beta{}_\alpha G_{\beta\gamma} n^\alpha = 0$ reduce to the vacuum Einstein constraint equations

$$R - K^{\alpha\beta} K_{\alpha\beta} + \tau^2 = 0, \tag{1.7a}$$

$$\nabla^\alpha K_{\alpha\beta} - \nabla_\beta \tau = 0, \tag{1.7b}$$

respectively. Here $\tau = h^{\alpha\beta}K_{\alpha\beta}$ is the *mean curvature* of Σ in V. Thus we think of Equation 1.6 as an initial-value problem where the initial data are a Riemannian manifold Σ with metric $h_{\alpha\beta}$ and a second fundamental form $K_{\alpha\beta}$ satisfying the constraint equations 1.7 [CB52]. In particular, satisfying the constraint equations is both necessary and sufficient to ensure that an ambient spacetime (V, λ) containing (Σ, h) exists and satisfies Equation 1.6 [CB52].

Note that Equation 1.7a is a single equation, while Equation 1.7b is a system of three equations. Thus System 1.7 is composed of four equations in twelve variables (six in $g_{\alpha\beta}$ and six in $K_{\alpha\beta}$ due to symmetry in each) and is therefore underdetermined. This is not unexpected since we are free to specify an initial slice Σ with $h_{\alpha\beta}$ and $K_{\alpha\beta}$ satisfying the Constraint Equations 1.7. The space of solutions is not well understood. Furthermore, we note that solutions to Equation 1.6, though modeling a vacuum universe, are of interest because gravitational waves can propagate through spacetime even in the absence of matter.

1.5   Conformal Parameterizations

Ideally, we would like to describe all solutions of Equation 1.6 on a given spacelike hypersurface of spacetime. This would provide a procedure for finding all possible sets of initial data. In general this is an open problem; only in the case of constant mean curvature has this problem been completely solved. This solution uses the conformal method of Lichnerowicz, Choquet-Bruhat, and York and extends to non-CMC cases where much less is known.

The initial data for the conformal method are:

1. *A conformal class of metrics* $[g_{\alpha\beta}]$. Recall that two metrics $g_{\alpha\beta}, h_{\alpha\beta}$ are *conformally related* if there exists a positive function ℓ such that $h_{\alpha\beta} = \ell^2 g_{\alpha\beta}$. We will write $\ell^2 = \phi^{q-2}$ where ϕ is a positive function and $q = \frac{2n}{n-2}$. Note that the relation "is conformally related to" forms equivalence classes, so we may say $h_{\alpha\beta}$ is in the conformal class of $g_{\alpha\beta}$.

2. *A transverse-traceless tensor* σ. A transverse-traceless tensor is one that is symmetric and both divergence-free and trace-free.

3. *A mean curvature* τ. (Recall the definition of τ from the discussion after System 1.7.)

4. *A lapse function* N. The lapse is a smooth function assuring that changing the representative of the conformal class does not affect the solution found.

We use these initial data to seek solutions $(h_{\alpha\beta}, K_{\alpha\beta})$ of the Constraint Equations 1.7 such that $h_{\alpha\beta}$ is in the conformal class of $g_{\alpha\beta}$,

$$h_{\alpha\beta} = \phi^{q-2}g_{\alpha\beta},$$

and with $K_{\alpha\beta}$ decomposed as in [Yo74]:

$$K_{\alpha\beta} = \phi^{-2}\left(\sigma_{\alpha\beta} + \frac{1}{2N}\mathbb{L}W_{\alpha\beta}\right) + \frac{\tau}{n}\bar{g}_{\alpha\beta}.$$

Here n is the dimension of the hypersurface, ϕ is an unknown positive function, W^α is an unknown vector field, and $\mathbb{L}$ is the conformal Killing operator

$$(\mathbb{L}W)_{\alpha\beta} = \nabla_\alpha W_\beta + \nabla_\beta W_\alpha - \frac{2}{n}\nabla^\gamma W_\gamma g_{\alpha\beta}.$$

Letting R_h denote the scalar curvature of Σ as computed with $h_{\alpha\beta}$, a standard computation gives

$$R_h = \phi^{1-q}(-2\kappa q\Delta\phi + R_g\phi).$$

It then follows from straightforward computations that we may write the Constraint Equations 1.7 as

$$-2\kappa q\Delta_g\phi + R_g\phi - \left|\sigma_{\alpha\beta} + \frac{1}{2N}\mathbb{L}W_{\alpha\beta}\right|_g^2 \phi^{-q-1} + \kappa\tau^2\phi^{-q-1} = 0, \tag{1.8a}$$

$$\nabla^\beta \frac{1}{2N}\mathbb{L}W_{\alpha\beta} - \kappa\phi^q\nabla_\alpha\tau = 0. \tag{1.8b}$$

We wish to solve these equations for (ϕ, W).

It was expected that selection of generic conformal initial data would lead to a unique solution of the Constraint Equations 1.7; in the case of constant mean curvature this is in fact the case. Special cases in the near- and far-from-CMC regime have been investigated, including in [HNT09] and [Ma09], in hopes of discovering a general theorem. Later, [Ma11] showed that, for symmetric data on conformally flat tori, there existed regimes of both non-existence and non-uniqueness of solutions to System 1.7.

In light of the suppositions for previously-proven existence theorems, we would like to consider model problems on three cases of manifolds: those with Yamabe-null, -positive, and -negative metrics. Given a manifold M and a metric g on M, we define the *Yamabe invariant* of g to be

$$\mathcal{Y}_g = \inf_{\substack{f\in C^\infty(M)\\ f\not\equiv 0}} \frac{\int_M 2\kappa q|\nabla f|_g^2 + R_g f^2 dV_g}{\|f\|_{L^q}^2}.$$

It can be shown that $\mathcal{Y}_g > 0$ (that is, g is Yamabe-positive) if and only if there exists a positive function ϕ and a metric h such that h is in the conformal class of g and $R_h > 0$ everywhere on M (for details, refer to the summary paper [LP87]). A metric being Yamabe-

null or Yamabe-negative is defined similarly. The paper [Ma11] considered the case of a Yamabe-null metric, and in the remainder of this thesis we consider the case of a Yamabe-positive metric.

Chapter 2

Symmetric Data on $S^{n-1} \times S^1$

Constructing and finding solutions of model problems for the Constraint Equations 1.8 provides insight for discovering a general theorem about the existence or nonexistence of solutions in the non-CMC case. To construct our model problem, let $n \geq 3$ and denote by S_r^{n-1} the $(n-1)$-sphere of radius r. We will take S^1 to be $[-\pi, \pi]$ with endpoints identified. The manifold we consider is $S_r^{n-1} \times S^1$, with the metric $\bar{g}_r + ds^2$ where $\bar{g}_r$ is the usual round metric on S_r^{n-1}. Note in particular that $S_r^{n-1} \times S^1$ has positive scalar curvature, which we denote by R, and thus $\bar{g}_r + ds^2$ is a Yamabe-positive metric. We seek solutions (h, K) of the Constraint Equations 1.8 such that h and K are periodic functions of $x \in S^1$ alone. We denote derivatives with respect to x by primes.

As in [Ma11], we will prescribe the mean curvature τ using the formula

$$\tau_t := t + \nu \tag{2.1}$$

where $t \in \mathbb{R}$ is constant and ν is the jump function

$$\nu(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}.$$

For our purposes, "near-CMC" refers to taking t sufficiently large so that τ does not change sign. We also define

$$\gamma_N := \frac{\int_{S^1} \nu N}{\int_{S^1} N}, \tag{2.2}$$

whose significance will be explained in the next chapter.

In the near-CMC regime, [Ma11] showed that for a particular Yamabe-null metric on $S_{r_1}^1 \times \ldots \times S_{r_n}^1$ there exists a solution to the Constraint Equations 1.8 if and only if $\sigma \not\equiv 0$. We will show a similar result on $S_r^{n-1} \times S^1$. When the transverse-traceless tensor is small and not identically 0, [Ma11] showed that there exists at least one solution if $|t|$ is sufficiently

9

large and $|t| \neq 1$. Again, we will find a similar result on $S_r^{n-1} \times S^1$. Our results will differ from [Ma11] when $|t| < 1$; in his case, there were regions of existence and nonexistence. In our case, a solution exists.

Working concretely with $n = 3$, choose a mean curvature τ_t, a transverse traceless tensor σ, and a lapse function N. Then System 1.8 becomes

$$-8\Delta_g \phi + R_g \phi - \left| \sigma_{\alpha\beta} + \frac{1}{2N} \mathbb{L} W_{\alpha\beta} \right|_g^2 \phi^{-7} + \frac{2}{3} \tau_t^2 \phi^6 = 0, \tag{2.3a}$$

$$\nabla^\beta \frac{1}{2N} \mathbb{L} W_{\alpha\beta} - \frac{2}{3} \phi^6 \nabla_a \tau_t = 0. \tag{2.3b}$$

Assuming W is a function of x alone, we take $W = (0, 0, w(x))$ and find

$$\nabla \left(\frac{1}{2N} \mathbb{L} W_{\alpha\beta} \right) = \begin{pmatrix} 0 & 0 & \frac{2}{3} \left(\frac{1}{2N} w' \right)' \end{pmatrix}.$$

Furthermore,

$$\nabla \tau = \begin{pmatrix} 0 & 0 & \tau_t'(x) \end{pmatrix}$$

so Equation 2.3b becomes

$$\frac{2}{3} \left(\frac{1}{2N} w(x)' \right)' = \frac{2}{3} \phi^6 \tau'.$$

It follows that

$$\left(\frac{1}{2N} w'(x) \right)' = \phi^6 \tau_t'.$$

We now turn to Equation 2.3a. We define

$$\hat{\sigma}_{\alpha\beta} := \frac{1}{3} ds^2 - \frac{2}{3} \bar{g}_r$$

and note that $\hat{\sigma}$ is transverse traceless on S^2 and that $|\sigma_1|^2 = \frac{2}{3}$. We write $\sigma_{\alpha\beta} = \mu \hat{\sigma}_{\alpha\beta}$ where μ is constant, and then find

$$\left| \sigma_{\alpha\beta} + \frac{1}{2N} \mathbb{L} W_{\alpha\beta} \right|_g^2 = \left| \mu \hat{\sigma}_{\alpha\beta} + \frac{1}{2N} w'(x) \right|^2 = \frac{2}{3} \left(\mu + \frac{1}{2N} w'(x) \right)^2. \tag{2.4}$$

Then we may write Equation 2.3a as

$$-8\Delta_g \phi + R_g \phi - \frac{2}{3} \left(\mu + \frac{1}{2N} w'(x) \right)^2 \phi^{-7} + \frac{2}{3} \tau_t^2 \phi^6 = 0.$$

10

A similar derivation works in higher dimensions, so for any $n \geq 3$ we may write the Constraint Equations 1.8 as

$$-2\kappa q\phi'' + R_g\phi - \kappa\left(\mu + \frac{1}{2N}w'(x)\right)^2 \phi^{-q-1} + \kappa\tau_t^2\phi^{q-1} = 0, \tag{2.5a}$$

$$\left(\frac{1}{2N}w'\right)' - \phi^q\tau_t' = 0. \tag{2.5b}$$

Finding solutions (h, K) of the Einstein Constraint Equations 1.8 on $S_r^{n-1} \times S^1$ requires that we parameterize solutions (ϕ, μ, w) of the Constraint Equations 2.5.

We remark that Equation 2.3a involves the Laplacian on S^1; this Laplacian admits a Maximum Principle whose properties we exploit to find solutions in the next chapter. There we show that we can use Equation 2.3b to rewrite Equation 2.3a in such a way as to reduce finding solutions of System 1.8 to finding roots of a real-valued function. Furthermore, System 2.5 only differs from the system given in [Ma11] for conformally flat tori in that we have an additional term: $R_g\phi$. Our model problem is constructed specifically to explore the effect of this term; in particular, we find that when $R_g > 0$ the nonexistence regime in [Ma11] vanishes. We conjecture, however, that the non-uniqueness regime of [Ma11] extends to a non-uniqueness regime in the $R_g > 0$ case.

Chapter 3

Solutions of the Constraint Equations

3.1 Summary of Results

The prescribed data for the Constraint Equations 2.5 are a constant μ, a lapse function N, and a mean curvature function τ. Recall the definition of γ_N from Equation 2.2. In the remainder of this chapter, we establish the following results:

Theorem 1. *(Near-CMC Results) If $|t + \gamma_N| > 2$, then there exists a solution (ϕ, w) of the Constraint Equations 2.5 if and only if $\mu \neq 0$.*

Theorem 2. *(Existence) Suppose $|t| \neq 1$ and $\mu \neq 0$. Then there exists at least one solution of the Constraint Equations 2.5.*

3.2 Reduction to Root Finding

In order to determine the existence of solutions to the Constraint Equations 2.5, we first show that the solution of Equation 2.5b can be determined up to the value of $\phi(0)$. Substituting this value into Equation 2.5a will allow us to define a differential equation whose solutions form the basis for the remainder of our study.

Let $\Omega \subseteq \mathbb{R}$ be an arbitrary interval, $1 \leq p \leq \infty$, m a positive integer, and define

$$W^{m,p}(\Omega) := \{ u \in L^p(\Omega) \ : \ D^k u \in L^p(\Omega) \text{ for } 0 \leq k \leq m \}$$

to be the Sobolev space over Ω [Ad03]. Here D^k denotes the kth weak partial derivative. We will take $W_+^{m,p}$ to be the set of positive functions in $W^{m,p}$. This definition may be extended to subsets of $L_{\text{loc}}^p(S^1)$, where $u \in W^{m,p}(S^1)$ if u is measurable on $\mathbb{R}$, u has m distributional derivatives in $L_{\text{loc}}^p(\mathbb{R})$, and u is 2π periodic. The solution space for our model problem is $W_+^{m,p}(S^1)$.

We now show that, given τ_t, the existence of solutions of the Constraint Equations 2.5 reduces to finding roots of a real-valued function. First, we show that the solutions of Equation 2.5b can be determined up to the value of $\phi(0)$.

Proposition 3. *Suppose $(\phi, w) \in W^{2,\infty}_+(S^1) \times W^{1,\infty}(S^1)$ is a solution of the Constraint Equations 2.5. Define*

$$\gamma_N = -\frac{\int_{S^1} \nu N}{\int_{S^1} N}. \tag{3.1}$$

Then

$$\frac{w'}{2N} = \phi(0)^q [\nu + \gamma_N]. \tag{3.2}$$

Proof. First, note that $\tau' = 2[\delta_0 - \delta_\pi]$ where δ_y is the Dirac delta distribution with singularity at y. Suppose (ϕ, w) is a solution of the Constraint Equations 2.5. Then

$$\left(\frac{w'}{2N}\right)' = 2\phi^q[\delta_0 - \delta_\pi].$$

Letting $\langle \cdot, \cdot \rangle$ denote the pairing of distributions on test functions, it is clear that

$$\left\langle \left(\frac{w'}{2N}\right)', 1 \right\rangle = 0.$$

Thus

$$0 = \langle \phi^q(\delta_0 - \delta_\pi), 1 \rangle = \phi(0)^q - \phi(\pi)^q,$$

and we conclude $\phi(0) = \phi(\pi)$. We now write Equation 2.5b as

$$\left(\frac{w'}{2N}\right)' = \phi(0)^q \nu',$$

and thus

$$\frac{w'}{2N} = \phi(0)^q[\nu + C]$$

where C is some constant of integration. Then

$$\frac{w'}{2} = \phi(0)^q N[\nu + C],$$

and since $\int_{S^1} w' = 0$ we find

$$0 = \phi(0)^q \int_{S^1} N[\nu + C].$$

14

Since ϕ is a positive function, we conclude

$$\int_{S_1} N[\nu + C] = 0 \tag{3.3}$$

and thus can determine C. In particular, Equation 3.3 gives

$$\int_{S_1} N\nu = -C \int_{S_1} N \implies C = \gamma_N.$$

□

If we substitute Equations 2.1 and 3.2 into Equation 2.5a, we find

$$-2\kappa q\phi'' + R\phi - \kappa \left[\mu + \phi(0)^q(\nu + \gamma_N) \right]^2 \phi^{-q-1} + \kappa(t + \nu)^2\phi^{q-1} = 0 \tag{3.4}$$

which is a nonlocal equation for ϕ since it relies on the value of $\phi(0)$ for all points of S^1. The following proposition proves that finding solutions to Equation 3.4 is both necessary and sufficient for finding solutions of the Constraint Equations 2.5.

Proposition 4. *Suppose* $(\phi, w) \in W_+^{2,\infty}(S^1) \times W^{1,\infty}(S^1)$ *solves the Constraint Equations 2.5. Then* ϕ *satisfies Equation 3.4. Conversely, if* $\phi \in W_+^{2,\infty}(S^1)$ *is a solution of Equation 3.4 then there exists a solution* $w \in W^{1,\infty}(S^1)$*, determined uniquely up to a constant, of Equation 3.2 and* (ϕ, w) *is a solution of the Constraint Equations 2.5.*

Proof. First, if (ϕ, w) solves the Constraint Equations 2.5 then by Proposition 3 we know w solves Equation 3.2. Substituting this into Equation 2.5a yields Equation 3.4. Conversely, suppose ϕ solves Equation 3.4. Since Equation 3.2 is integrable, the solution $w \in W^{1,\infty}(S^1)$ determined up to a constant. Suppose w is of this form. By construction, w solves Equation 2.5b for ϕ and ϕ solves Equation 2.5a for w. We conclude (w, ϕ) is a solution of the Constraint Equations 2.5. □

Our goal, therefore, is to study Equation 3.4. To do so, we introduce a family of Lichnerowicz equations depending on a positive parameter b:

$$-2\kappa q\phi_b'' + R\phi_b - \kappa[\mu + b^q(\gamma_N + \nu)]^2\phi_b^{-q-1} + \kappa(t + \nu)^2\phi_b^{q-1} = 0. \tag{3.5}$$

Note that solutions of Equation 3.5 satisfying $\phi_b(0) = b$ are in one-to-one correspondence with solutions of Equation 3.4. The functions ϕ_b grow as b increases, so it will be convenient to to work with a rescaled function that is bounded as $b \to \infty$. Define $\psi_b = b^{-1}\phi_b$. Then Proposition 5 follows from Proposition 4, and thus finding solutions of the Constraint Equations 2.5 reduces to finding solutions of Equation 3.6:

Proposition 5. *The solutions of the Constraint Equations 2.5 are in one-to-one correspondence with the functions $\psi_b \in W_+^{2,\infty}(S^1)$ satisfying*

$$b^{-q+2}(-2q\kappa\psi_b'' + R\psi_b) - \kappa(\mu b^{-q} + \gamma_N + \nu)^2\psi_b^{-q-1} + \kappa(t+\nu)^2\psi_b^{q-1} = 0 \qquad (3.6)$$

and

$$\psi_b(0) = 1$$

for some $b > 0$. Given a solution ψ_b solving Equation 3.6 and satisfying $\psi_b(0) = 1$, the corresponding solution ϕ of Equation 3.4 is $b\psi_b$.

Establishing Theorems 1 and 2 relies on properties of solutions to Equation 3.6. In particular, write Equation 3.6 as a Lichnerowicz equation of the form

$$-u'' + Ru - \alpha^2 u^{-q-1} + \beta^2 u^{q-1} = 0. \qquad (3.7)$$

If $p \in W_+^{2,\infty}(S^1)$ and

$$-p'' + Rp - \alpha^2 p^{-q-1} + \beta^2 p^{q-1} \leq 0$$

we say p is a *subsolution* of Equation 3.7. Similarly, if $v \in W_+^{2,\infty}(S^1)$ and

$$-v'' + Rv - \alpha^2 v^{-q-1} + \beta^2 v^{q-1} \geq 0$$

we say v is a *supersolution* of Equation 3.7. Subsolutions and supersolutions of Equation 3.6 are our main tools for proving the results leading to Theorems 1 and 2. Arguing as in the proof of Proposition 3.10 of [Ma11], we may establish the following properties of solutions of Equation 3.7 which are the other necessary tools.

Proposition 6. *Suppose α and β as in Equation 3.7 belong to $L^\infty(S^1)$ and that $\alpha \not\equiv 0$ and $\beta \not\equiv 0$. Let $p > 1$.*

1. *There exists a unique solution $u \in W_+^{2,p}(S^1)$, and moreover $u \in W^{2,\infty}(S^1)$.*

2. *If $p \in W_+^{2,\infty}(S^1)$ is a subsolution of Equation 3.7 then $p \leq u$.*

3. *If $v \in W_+^{2,\infty}(S^1)$ is a supersolution of Equation 3.7 then $v \geq u$.*

4. *The solution $u \in W_+^{2,p}$ depends continuously on $(\alpha, \beta) \in L^\infty \times L^\infty$.*

Note that Proposition 6 Parts 2 and 3 are related to the Maximum Principle for the Laplacian on S^1.

We now define a map $\mathcal{F} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ which forms the basis of the remainder of our study.

16

Definition 7. *Let $t \in \mathbb{R}$ be constant and let τ_t be defined as in Equation 2.1. Let N be a smooth lapse function, γ_N be defined by Equation 3.1, and $R > 0, \mu$ be constants. For $b > 0$, Proposition 6 Part 1 implies that there exists a corresponding solution $\psi_b \in W_+^{2,\infty}(S^1)$ of Equation 3.6. We define $\mathcal{F} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ by*

$$\mathcal{F}(b) = \psi_b(0).$$

There is one special case that interests us and warrants special notation. If $\mu = 0$, we denote the solution of Equation 3.6 by $\psi_{0,b}$ and denote $\mathcal{F}_0(b) := \psi_{0,b}(0)$.

Proposition 5 implies that the existence theory of the conformal method for the family of data (ν, N, R) reduces to the study of the algebraic solutions of $\mathcal{F}(b) = 1$:

Proposition 8. *The solutions $(\phi, w) \in W_+^{2,\infty} \times W^{1,\infty}(S^1)$ of Equations 2.5 are in one-to-one correspondence with the positive solutions of $\mathcal{F}(b) = 1$.*

3.3 Solutions of $\mathcal{F}(b) = 1$

3.3.1 Elementary Estimates for $\mathcal{F}$

Proving Theorems 1 and 2 require that we establish the following:

1. If $\mu \neq 0$ (that is, if the transverse-traceless tensor is not identically zero) then $\mathcal{F}(b)$ is $O(b^{-1})$ for b sufficiently small.

2. If $\mu = 0$ then $\mathcal{F}$ is uniformly bounded on $(0, \infty)$.

3. For all values μ, $\mathcal{F}(b)$ is bounded above for b sufficiently large.

4. If $\mu \neq 0$, then a solution of $\mathcal{F}(b) = 1$ exists if and only if $\mathcal{F}(b) \leq 1$ for some $b > 0$.

We first find bounds for ψ_b.

Lemma 9. *Fix $b > 0$ and suppose $|t| \neq 1$. Define the constants*

$$M_b = \max\{M_{b,+}, M_{b,-}\}$$
$$m_b = \min\{\tilde{m}_{b,+}, \tilde{m}_{b,-}, \hat{m}_{b,+}, \hat{m}_{b,-}\}$$

where

$$M_{b,\pm} = \left[\frac{(\mu b^{-q} + \gamma_N \pm 1)^2}{(t \pm 1)^2}\right]^{\frac{1}{2q}},$$

$$\tilde{m}_{b,\pm} = \left[\frac{(\mu b^{-q} + \gamma_N \pm 1)^2}{2(t \pm 1)^2}\right]^{\frac{1}{2q}},$$

$$\hat{m}_{b,\pm} = \left[\frac{\kappa(\mu b^{-q} + \gamma_N \pm 1)^2}{2Rb^{-q+2}}\right]^{\frac{1}{q+2}}.$$

Then $m_b \leq \psi_b \leq M_b$. In particular,

$$m_b \leq \mathcal{F}(b) \leq M_b.$$

Proof. A constant M is a supersolution of Equation 3.6 if

$$Rb^{-q+2}M - \kappa(\mu b^{-q} + \gamma_N + \nu)^2 M^{-q-1} + \kappa(t + \nu)^2 M^{q-1} \geq 0.$$

In particular, note that this can be restated as

$$Rb^{-q+2}M^{q+2} + \kappa(t + \nu)^2 M^{2q} \geq \kappa(\mu b^{-q} + \gamma_N + \nu)^2$$

which since $R > 0$ holds if

$$\kappa(t + \nu)^2 M^{2q} \geq \kappa(\mu b^{-q} + \gamma_N \pm 1)^2.$$

Thus M_b is a supersolution. Proposition 6 Part 3 now implies that $\psi_b \leq M_b$ on S^1.

A constant m is a subsolution of Equation 3.6 if

$$Rb^{-q+2}m - \kappa(\mu b^{-q} + \gamma_N + \nu)^2 m^{-q-1} + \kappa(t + \nu)^2 m^{q-1} \leq 0.$$

In particular, note that m is a subsolution if both inequalities

$$2Rb^{-q+2}m^{q+2} \leq \kappa(\mu b^{-q} + \gamma_N + \nu)^2, \tag{3.8a}$$

$$2\kappa(t + \nu)^2 m^{2q} \leq \kappa(\mu b^{-q} + \gamma_N + \nu)^2 \tag{3.8b}$$

18

hold. Now since $\nu = \pm 1$ on S^1, Inequalities 3.8 hold if both

$$m^{q+2} \leq \frac{\kappa(\mu b^{-q} + \gamma_N \pm 1)^2}{2Rb^{-q+2}},$$
$$m^{2q} \leq \frac{\kappa(\mu b^{-q} + \gamma_N + \pm)^2}{2\kappa(t \pm 1)^2}$$

hold. Thus m_b is a subsolution. By Proposition 6 Part 2, we have $\psi_b \geq m_b$ on S^1. We conclude $m_b \leq \psi_b \leq M_b$ and it follows that $m_b \leq \mathcal{F}(b) \leq M_b$. $\qquad\square$

We may now estimate ψ_b and $\mathcal{F}(b)$ for large values of b using the limiting behavior of M_b and m_b as $b \to \infty$.

Lemma 10. *Suppose $|t| \neq 1$. Define*

$$M_\infty = \max \left[\left| \frac{\gamma_N - 1}{t - 1} \right|^{\frac{1}{q}}, \left| \frac{\gamma_N + 1}{t + 1} \right|^{\frac{1}{q}} \right],$$
$$m_\infty = \min \left[\frac{1}{2} \left| \frac{\gamma_N - 1}{t - 1} \right|^{\frac{1}{q}}, \frac{1}{2} \left| \frac{\gamma_N + 1}{t + 1} \right|^{\frac{1}{q}} \right]$$

Given $\epsilon > 0$,

$$m_\infty - \epsilon \leq \psi_b \leq M_\infty + \epsilon$$

holds for b sufficiently large. If $\mu = 0$ then

$$m_\infty \leq \psi_b \leq M_\infty$$

for b sufficiently large.

Proof. It is clear that
$$\lim_{b \to \infty} M_b = M_\infty.$$
As $b \to \infty$, note that $\tilde{m}_{b,\pm}$ remains finite but $\hat{m}_{b,\pm} \to \infty$. Thus

$$\lim_{b \to \infty} m_b = \tilde{m}_\infty,$$

so the bounds $\tilde{m}_\infty - \epsilon \leq \psi_b \leq M_\infty + \epsilon$ hold for b sufficiently large.

If $\mu = 0$, then $M_\infty = M_b$ and $m_b = m_\infty$ for b sufficiently large, so $m_\infty \leq \psi_b \leq M_\infty$ for all $b > 0$. $\qquad\square$

The behavior of $\mathcal{F}$ near zero is determined by the behavior of the associated sub- and supersolutions near zero, which we now investigate.

Lemma 11. *If $\mu = 0$, then*

$$\mathcal{F}_0(b) \leq M_\infty \tag{3.9}$$

for all $b > 0$. If $\mu \neq 0$ there is a positive constant k such that

$$\mathcal{F}(b) \geq kb^{-1} \tag{3.10}$$

for b sufficiently small.

Proof. The uniform upper bound in Equation 3.9 with $\mu = 0$ follows from Lemma 10.

If $\mu \neq 0$ then $m_b = \min\{\tilde{m}_{b,+}, \tilde{m}_{b,-}\}$ for b sufficiently small. Then $m_b \geq kb^{-1}$ with k constant, so Inequality 3.10 follows from Lemma 9. $\square$

The singularity of $\mathcal{F}$ at $b = 0$ gives a simple test for determining if there is at least one solution of $\mathcal{F}(b) = 1$. The following lemma concludes the proofs of the facts listed at the beginning of this section.

Lemma 12. *Suppose $\mu \neq 0$. Then there exists a solution $\mathcal{F}(b) = 1$ if and only if for some $b > 0$, $\mathcal{F}(b) \leq 1$.*

Proof. It follows from Lemma 11 that $\mathcal{F} > 1$ for b sufficiently small. Fixing $p > 1$, by Proposition 6 Part 4 the map $b \to \psi_b$ from $(0, \infty)$ to $W^{2,p}(S^1)$ is continuous. Furthemore, $\mathcal{F}$ is continuous since $W^{2,p} \hookrightarrow C(S^1)$ is a continuous embedding (see the Sobolev embedding theorem of [Ad03]). By our suppositions, there exists a b such that $\mathcal{F}(b) \leq 1$ so the desired result follows from the Intermediate Value Theorem. $\square$

We are now ready to prove Theorem 1.

3.3.2 Proof of Theorem 1 (Near-CMC Results)

To prove Theorem 1, it remains to establish the following facts:

1. If $|t + \gamma_N| > 2$ then $\limsup_{b \to \infty} \mathcal{F}(b) < 1$.

2. $\mathcal{F}_0(b) < 1$ for all b.

20

The existence of a solution of $\mathcal{F}(b) = 1$ follows from Fact 1 and Lemma 12. The non-existence of solutions of $\mathcal{F}_0(b) = 1$ follows from Fact 2. The upper bounds of Facts 1 and 2 follow from the constant supersolutions of Lemma 9.

We first establish Fact 1.

Lemma 13. *Suppose $|t + \gamma_N| > 2$. Then*

$$M_\infty < 1.$$

Proof. First note that by the triangle inequality for integrals, $|\gamma_N| \leq 1$. Thus $|t| > 1$.

Suppose $t > 1$. Then

$$M_\infty^q = \max \left[\left| \frac{\gamma_N - 1}{t - 1} \right|, \left| \frac{\gamma_N + 1}{t + 1} \right| \right] = \max \left[\frac{1 - \gamma_N}{t - 1}, \frac{\gamma_N + 1}{t + 1} \right].$$

By our suppositions, $2 < t + \gamma_N$ so $1 - \gamma_N < t - 1$. Also, $\gamma_N \leq 1 < t$ so $\gamma_N + 1 < t + 1$. We conclude $M_\infty < 1$.

Now suppose $t < -1$. Then

$$M_\infty^q = \max \left[\left| \frac{\gamma_N - 1}{t - 1} \right|, \left| \frac{\gamma_N + 1}{t + 1} \right| \right] = \max \left[\frac{1 - \gamma_N}{1 - t}, \frac{\gamma_N + 1}{-t - 1} \right].$$

By our suppositions, $2 < -t - \gamma_N$ so $1 + \gamma_N < -t - 1$. Also, $t < -1 \leq \gamma_N$ so $1 - \gamma_N < 1 - t$. Again $M_\infty < 1$.

We conclude $M_\infty < 1$ for all $|t| > 1$. $\qquad\square$

If $\mu = 0$, we have the following non-existence result. This corollary proves Fact 2.

Corollary 14. *If $|t + \gamma_n| > 2$ then $\mathcal{F}_0(b) < 1$ for all $b > 0$. In particular, there are no solutions of $\mathcal{F}_0(b) = 1$.*

Proof. Recall from Lemma 10 that $\mu = 0$ implies $M_b = M_\infty$ for all $b > 0$. By Lemma 13, $M_\infty < 1$. Thus $\mathcal{F}_0(b) < 1$ for all $b > 0$. $\qquad\square$

The proof of Theorem 1 follows from Corollary 14 and Proposition 8: there exists a solution of the Constraint Equations 2.5 if and only if $\mu \neq 0$.

We now turn to the proof of Theorem 2.

3.3.3   Proof of Theorem 2 (Existence)

To establish Theorem 2, it remains to prove the following facts:

21

1. If $|t| \neq 1$ then $\lim_{b \to \infty} \psi_{0,b}(0) = \begin{cases} 1, & |t| < 1 \\ |t|^{-1/q}, & |t| > 1. \end{cases}$

2. There exists a constant c such that $\psi_{0,b} \leq 1 - cb^{-q+2}$ for b sufficiently large.

3. There exists a constant k such that $\psi_b \leq \psi_{0,b} + kb^{-q}$ for b sufficiently large.

When $|t| < 1$, it follows from Fact 1 and the Intermediate Value Theorem that there exists a b for which $\mathcal{F}(b) = 1$. Assuming $|t| > 1$, the same conclusion follows from Facts 2 and 3.

To establish Fact 1, we reduce to the case considered in Theorem 4.1 of [Ma11]. To this end, consider the singularly perturbed Lichnerowicz equation

$$-\epsilon^2(u'' - Ru) - \alpha^2 u^{-q-1} + \beta^2 u^{q-1} = 0 \tag{3.11}$$

where α, β are constant on $(-\pi, 0)$ and $(0, \pi)$ with values $\alpha_\pm$ and $\beta_\pm$, respectively, and $u \in W^{2,\infty}_+(S^1)$. We seek to construct supersolutions of Equation 3.11 in the form $u_\epsilon - k_\epsilon$ where k_ϵ is a constants such that $k_\epsilon \to 0$ as $\epsilon \to 0$ and u_ϵ satisfies

$$-\epsilon^2 u_\epsilon'' - \alpha^2 u_\epsilon^{-q-1} + \beta^2 u_\epsilon^{q-1} = 0. \tag{3.12}$$

Equation 3.12 was analyzed in [Ma11], where the following Theorem 15 was proven as Theorem 4.1. Recall that a function $f(x)$ converges *rapidly* to L at zero if

$$\lim_{x \to 0} |f(x) - L| x^{-n} = 0$$

for all $n \in \mathbb{N}$.

Theorem 15 (Proven in [Ma11]). *Suppose that $\beta_- \neq 0$ and $\beta_+ \neq 0$. Then*

$$\lim_{\epsilon \to 0} u_\epsilon(0) = \left[\frac{|\alpha_+| + |\alpha_-|}{|\beta_+| + |\beta_-|} \right]^{\frac{1}{q}}$$

and this convergence is rapid.

We remark that Equation 3.12 becomes algebraic in u_ϵ as $\epsilon \to 0$ and thus we expect $u_\epsilon \to |\alpha_\pm/\beta_\pm|^{1/q}$ away from discontinuities of α and β [Ma11]. In particular, u_ϵ is finite and

$$u_\epsilon = \max\left(\left|\frac{\alpha_+}{\beta_-}\right|^{1/q}, \left|\frac{\alpha_-}{\beta_-}\right|^{1/q} \right) \tag{3.13}$$

is a constant supersolution of Equation 3.12 for all ϵ so long as $\beta_\pm \neq 0$.

We now investigate supersolutions of Equation 3.11. These supersolutions must satisfy

$$-\epsilon^2(u_\epsilon - k_\epsilon)'' + \epsilon^2 R(u_\epsilon - k_\epsilon) - \alpha^2(u_\epsilon - k_\epsilon)^{-q-1} + \beta^2(u_\epsilon - k_\epsilon)^{q-1} \geq 0. \tag{3.14}$$

Since k_ϵ is constant, we have $-\epsilon^2(u_\epsilon - k_\epsilon)'' = -\epsilon^2 u_\epsilon''$ and substituting Equation 3.12 into Equation 3.14 we find

$$\alpha^2 u_\epsilon^{-q-1} - \beta^2 u_\epsilon^{q-1} + \epsilon^2 R(u_\epsilon - k_\epsilon) - \alpha^2(u_\epsilon - k_\epsilon)^{-q-1} + \beta^2(u_\epsilon - k_\epsilon)^{q-1} \geq 0.$$

(Note that letting $k_\epsilon = 0$ gives an immediate supersolution of Equation 3.14. This, however, is insufficient for our purposes.) We consider two conditions which, when simultaneously true, guarantee a supersolution:

$$\epsilon^2 R(u_\epsilon - k_\epsilon) \geq 2\alpha^2((u_\epsilon - k_\epsilon)^{-q-1} - u_\epsilon^{-q-1}), \tag{3.15a}$$

$$\epsilon^2 R(u_\epsilon - k_\epsilon) \geq 2\beta^2(u_\epsilon^{q-1} - (u_\epsilon - k_\epsilon))^{q-1}. \tag{3.15b}$$

We first address Equation 3.15a. We have

$$2\epsilon^2 R u_\epsilon \left(1 - \frac{k_\epsilon}{u_\epsilon}\right) \geq \alpha^2 u_\epsilon^{-q-1}\left(\left(1 - \frac{k_\epsilon}{u_\epsilon}\right)^{-q-1} - 1\right),$$

which holds if and only if

$$\frac{2\epsilon^2 R u_\epsilon^{q+2}}{\alpha^2} \geq \frac{\left(1 - \frac{k_\epsilon}{u_\epsilon}\right)^{-q-1} - 1}{1 - \frac{k_\epsilon}{u_\epsilon}}.$$

Let $y = \frac{k_\epsilon}{u_\epsilon}$ and define

$$f(y) = \frac{(1-y)^{-q-1} - 1}{1-y}.$$

Recall u_ϵ is a positive function, so we may assume $u_\epsilon - k_\epsilon > 0$. Then $0 < y < 1$ and we seek a constant c such that

$$cy \leq f(y). \tag{3.16}$$

Note that $f > 0$ and

$$f'(y) = \frac{(q+2)(1-y)^{-q-1} - 1}{(1-y)^2},$$

23

and $(q + 2) (1 - y)^{-q-1} > 1$ so $f' > 0$. In particular, f is increasing on $0 < y < 1$ and therefore a c as in Equation 3.16 exists and we take

$$k_\epsilon = \frac{2\epsilon^2 R \zeta^{q+3}}{c\alpha}$$

where $\zeta = \sup_\epsilon u_\epsilon$. That ζ is finite follows from the fact that Equation 3.13 defines a constant supersolution of Equation 3.12. The argument for Equation 3.15b is similar, so we let

$$k_\epsilon = \max \left\{ \frac{2\epsilon^2 R \zeta^{q+3}}{c\alpha^2}, \frac{2\epsilon^2 R \zeta^{3-q}}{\hat{c}\beta^2} \right\} \tag{3.17}$$

where $\hat{c}$ is a constant taking the place of c. In particular, we now have $\psi_{0,b} \leq u_\epsilon - k_\epsilon$.

We may now apply Theorem 15 to prove Fact 1.

Proposition 16. *Assuming* $|t| \neq 1$,

$$\lim_{b \to \infty} \psi_{0,b}(0) = \begin{cases} 1, & |t| < 1 \\ |t|^{-1/q}, & |t| > 1. \end{cases}$$

Proof. Let $\epsilon^2 = 2\kappa q b^{-q+2}$, $\alpha_\pm = \gamma_N \pm 1$, and $\beta_\pm = t \pm 1$. Applying Theorem 15 to Equation 3.12, we find

$$\lim_{b \to \infty} \psi_b(0) = \left[\frac{|\gamma_N + 1| + |\gamma_N - 1|}{|t + 1| + |t - 1|} | \right]^{\frac{1}{q}}.$$

From $|\gamma_N| < 1$, we have

$$|1 + \gamma_N| + |\gamma_N - 1| = 2,$$

while $|t| < 1$ implies

$$|1 + t| + |1 - t| = 2.$$

Also, $|t| > 1$ gives

$$|1 + t| + |1 - t| = 2|t|$$

and the desired result follows. $\qquad\square$

Recalling that $\psi_{0,b} \leq u_\epsilon - k_\epsilon$, Fact 2 follows from Fact 1, the definition of k_ϵ in Equation 3.17, and the rapid convergence of u_ϵ to 1. Thus it remains to prove Fact 3.

When b is large, note that the contribution of μb^{-q} to Equation 3.6 is small so we expect $\psi_b \to \psi_{0,b}$ as $b \to \infty$. To establish this, we show that perturbations of $\psi_{0,b}$ are sub- and supersolutions of Equation 3.6 for ψ_b.

Define

$$\mathcal{G}_b : \left[\frac{-m_\infty}{2}, M_\infty\right] \to L^\infty$$

via

$$\mathcal{G}_b(K) = \mathcal{N}_b(\psi_{0,b} + K) \tag{3.18}$$

where, we recall, $\psi_{0,b}$ solves

$$b^{-q+2}(-2q\kappa\psi_{0,b}'' + R\psi_{0,b}) - \kappa(\gamma_N + \nu)^2\psi_{0,b}^{-q-1} + \kappa(t+\nu)^2\psi_{0,b}^{q-1} = 0, \tag{3.19}$$

and $\mathcal{N}_b : W_+^{2,p}(S^1) \to L^p(S^1)$ is the nonlinear Lichnerowicz operator

$$\mathcal{N}_b(w) = (-2q\kappa w'' + Rw)b^{-q+2} - \kappa(\mu b^{-q} + \gamma_N + \nu)^2 w^{-q-1} + \kappa(t+\nu)^2 w^{q-1}.$$

Note that $\psi_{0,b} + K$ is a subsolution of Equation 3.6 with ψ_b if $\mathcal{G}_b(K) \le 0$ a.e. It follows from Equation 3.19 that

$$\mathcal{G}_b(K) = \mathcal{D}(K) + \mathcal{E}(K), \tag{3.20}$$

where

$$\mathcal{D}(K) = \kappa(t+\nu)^2[(\psi_{0,b} + K)^{q-1} - \psi_{0,b}^{q-1}] + Rb^{-q+2}[(\psi_{0,b} + K) - \psi_{0,b}],$$
$$\mathcal{E}(K) = \kappa(\gamma_N + \nu)^2\psi_{0,b}^{-q-1} - [\kappa(\mu b^{-q} + \gamma_N + \nu)^2](\psi_{0,b} + K)^{-q-1}.$$

The following lemma establishes constants which we will use to bound $\mathcal{E}$, which in turn allows us to bound ψ_b.

Lemma 17. *There exist positive constants* E_-, E_+ *such that*

$$E_-K \le \kappa(\gamma_N + \nu)^2[\psi_{0,b}^{-q-1} - (\psi_{0,b} + K)^{-q-1}] \le E_+K, \quad K \ge 0,$$
$$E_+K \le \kappa(\gamma_N + \nu)^2[\psi_{0,b}^{-q-1} - (\psi_{0,b} + K)^{-q-1}] \le E_-K, \quad K \le 0 \tag{3.21}$$

for all $b > 1$ *and all* $K \in [-m/2, M_\infty]$.

Proof. Let $A \in [m, M_\infty]$ and $K \in [-m/2, M_\infty]$ and define

$$f_A(K) = A^{-q-1} - (A + K)^{-q-1}.$$

25

Then

$$f_A(K) = K \int_0^1 (q+1)(A+tK)^{-q-2} dt,$$

so by the triangle inequality for integrals we find

$$(q+1)(2M_\infty)^{-q-2} K \le f_A(K) \le (q+1)(m/2)^{-q-2} K$$

when $K \ge 0$ and

$$(q+1)(m/2)^{-q-2} K \le f_A(K) \le (q+1)(2M_\infty)^{-q-2} K$$

when $K \le 0$. Letting

$$E_+ = \max[(\gamma_N - 1)^2, (\gamma_N + 1)^2](q+1)(m/2)^{-q-2},$$
$$E_- = \min[(\gamma_N - 1)^2, (\gamma_N + 1)^2](q+1)(m/2)^{-q-2}$$

it is clear that Equation 3.21 holds. $\qquad\square$

We now have the tools to prove Fact 3.

Proposition 18. *There exists a constant $k > 0$ such that*

$$\psi_b < \psi_{0,b} + kb^{-q} \qquad (3.22)$$

for all b sufficiently large.

Proof. First, observe that $\mathcal{D}(K)$ has the same sign as K and thus $\mathcal{G}_b(K) > 0$ when $K > 0$ and $\mathcal{E}(K) > 0$. Suppose $0 < K \le M_\infty$. Then from Lemma 17 we find

$$
\begin{aligned}
\mathcal{E}(K) &= \kappa(\gamma_N + \nu)^2 (\psi_{0,b}^{-q-1} - (\psi_{0,b} + K)^{-q-1} - [\kappa(\mu b^{-q} + 2\mu b^{-q}(\gamma_N + \nu))](\psi_{0,b} + K)^{-q-1} \\
&\ge E_- K - [\kappa\mu^2 b^{-2q} + 4\kappa|\mu|b^{-q}](m/2)^{-q-1}.
\end{aligned}
$$

Define

$$K_+(b) = \frac{[\kappa\mu^2 b^{-q} + 4\kappa|\mu|](m/2)^{-q-1}}{E_-} b^{-q}.$$

In particular, $0 < K_+(b) < M_\infty$ when b is sufficiently large. Then $\mathcal{E}(K_+(b)) \ge 0$ and it follows that $\mathcal{G}_b(K_+(b)) \ge 0$.

In this construction, $K_+(b) = O(b^{-q})$. Since $\mathcal{G}_b(K_+(b)) \geq 0$, it follows that $\psi_{0,b} + K_+(b)$ is a supersolution of Equation 3.6. That is,

$$\psi_b \leq \psi_{0,b} + K_+(b)$$

when b is sufficiently large. Taking $b \to \infty$ therefore results in Equation 3.22. $\qquad\square$

Having established Facts 1 through 3 provides the proof for Theorem 2. In particular, taking Propositions 16 and 18 together we find

$$
\begin{aligned}
\psi_b \quad &< \quad \psi_{0,b} + kb^{-q} \\
&< \quad 1 - cb^{-q+2} + kb^{-2} \\
&< \quad 1
\end{aligned}
$$

for b sufficiently large. Theorem 2 then follows from the Intermediate Value Theorem and Lemma 12.

We remark that the existence of solutions to our model problem, in contrast to the nonexistence in [Ma11], shows the fragility of the nonexistence regime to the condition $R = 0$.

Chapter 4

Conclusion and Future Work

We have shown that in both the near- and far-from-CMC regimes of our model problem on $S_r^{n-1} \times S^1$, a solution to the Constraint Equations 2.5 exists so long as the transverse-traceless tensor is nonzero. We believe that by arguing as in [Ma11], we can establish uniqueness in the near-CMC regime. In the far-from CMC regime, we conjecture that the non-uniqueness of solutions in [Ma11] suggests the possibility of non-uniqueness in our model problem. Future work will address these questions. Furthermore, another natural case to consider is that of a model problem on a manifold with a Yamabe-negative metric.

References

[Ad03] Adams, R., Fournier J. J. F.: *Sobolev Spaces*, 2$^{\text{nd}}$ Edition. Oxford: Elsevier Science Ltd, 2003

[CB52] YChoquet-Bruhat, Y.: Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires. Acta Mathematica **88**(1), 141-225 (1952)

[HNT09] Holst, M., Nagy, G., Tsogtgerel, G.: Rough solutions of the Einstein constraint equations on closed manifolds without near-CMC conditions. Comm. Math. Phys. **288**, 547-613 (2009)

[LP87] Lee, J., Parker, T.: The Yamabe problem. Bull. Amer. Math. Soc. **17**(1), 37-91 (1987)

[Ma09] Maxwell, D.: A class of solutions of the vacuum Einstein constraint equations with freely specified mean curvature. Math. Res. Lett. **16**(4), 627-645 (2009)

[Ma11] Maxwell, D.: A model problem for conformal parameterizations of the Einstein constraint equations. Commun. Math. Phys. **302**, 697-736 (2011)

[MTW73] Misner, C., Thorne, K., Wheeler, J. A.: *Gravitation*. San Fancisco: W. H. Freeman and Company, 1973

[Wa84] Wald, R. M.: *General Relativity*. Chicago: The University of Chicago Press, 1984

[Wo03] Woodhouse, N. M. J.: *Special Relatvity*. London: Springer Verlag, 2003

[Wo07] Woodhouse, N. M. J.: *General Relatvity*. London: Springer Verlag, 2007

[Yo74] York, J.: Covariant decomposition of symmetric tensors in the theory of gravitation. Ann. Inst. H. Poincaré **21**, 319-32 (1974)