THE GEOMETRY IN GEOMETRIC ALGEBRA

By

Kristopher N. Kilpatrick

RECOMMENDED:

Dr. John A. Rhodes

Dr. Gordon I. Williams

Dr. David A. Maxwell
Advisory Committee Chair

Dr. John A. Rhodes
Chair, Department of Mathematics and Statistics

APPROVED:

Dr. Paul W. Layer
Dean, College of Natural Science and Mathematics

Dr. John C. Eichelberger
Dean of the Graduate School

12/7/14

Date

THE GEOMETRY IN GEOMETRIC ALGEBRA

A

THESIS

Presented to the Faculty

of the University of Alaska Fairbanks

in Partial Fulfillment of the Requirements

for the Degree of

MASTER OF SCIENCE

By

Kristopher N. Kilpatrick, B.S.

Fairbanks, Alaska

December 2014

Abstract

We present an axiomatic development of geometric algebra. One may think of a geometric algebra as allowing one to add and multiply subspaces of a vector space. Properties of the geometric product are proven and derived products called the wedge and contraction product are introduced. Linear algebraic and geometric concepts such as linear independence and orthogonality may be expressed through the above derived products. Some examples with geometric algebra are then given.

Table of Contents

Page

Preface

The birth of Clifford algebra can be attributed to three mathematicians: Hermann Gunther Grassmann, William Rowan Hamilton and William Kingdon Clifford. Grassmann contributed greatly to the early theory of linear algebra, and one of those contributions was the exterior, or wedge, product. Hamilton invented the quaternions, a way to extend the complex numbers into 4 dimensions. Clifford synthesized the works of the two mathematicians into an algebra that he coined geometric algebra.

The purpose of this thesis is to come to terms with geometric algebra. I originally became interested in geometric algebra from my undergraduate physics teacher. I was informed by him that geometric algebra would be the language used by physicists. I greatly admired that man, and so I picked up the book recommended by him,"Clifford Algebra to Geometric Calculus" by David Hestenes and Garret Sobczyk [HS84]. I was taken aback by the plethora of identities with no meaning what so ever behind them. My advisor David Maxwell has helped me greatly by asking me questions that I could not answer and could not find answers to in [HS84]. That book is written more for physicists than mathematicians, I believe. I realized that one way to understand this subject would be to start, as a mathematician should, from a set of axioms and build up the theory of geometric algebra.

Chapter 1 deals with establishing the product of two vectors, then the product of a vector and a blade, finally the product of a blade with a blade. The wedge, left/right contraction and scalar product are introduced and identities involving them are derived. Chapter 2 deals with establishing the correspondence between a subspace and a blade. Subspaces are then studied with the wedge and right contraction products. Chapter 3 deals with lines and planes, the Kepler problem, rotations and reflections and finding components of a vector via the wedge product.

My indebtedness goes to David Maxwell and John Rhodes for their numerous conversations about many mathematical topics and their patience with me. Finally, I should like to express my thanks to the University of Alaska Fairbanks for their financial support in my academic studies.

Chapter 1

Preliminaries

We begin by defining a Clifford algebra over the reals, or as Clifford called it, a geometric algebra. Henceforth we shall also use the name geometric algebra. After defining a geometric algebra, we establish the result of the product of two blades. We find that the product of two blades always has a highest and lowest grade resulting from multiplication. New products will be defined based on the highest and lowest grade. Useful identities between the products will be established that will facilitate quick and efficient computations.

1.1 Definitions

There are two common approaches to defining a geometric algebra. The axiomatic treatments found in [HS84] and [DL03] have the merit of being more accessible, but these works lack full rigor. On the other hand, the treatment in [Che97] is mathematically rigorous, but its abstract, formal style lacks accessibility. Our approach is intermediate between the two. We start with a set of axioms inspired by [HS84], but modified so as to allow for a rigorous subsequent development.

Definition 1.1.1. A **geometric algebra** $\mathcal{G}$ is an algebra, with identity, over $\mathbb{R}$ with the following additional structure:

1) There are distinguished subspaces $\mathcal{G}^0, \mathcal{G}^1, \ldots$ such that

$$\mathcal{G} = \mathcal{G}^0 \oplus \mathcal{G}^1 \oplus \cdots .$$

2) $1 \in \mathcal{G}^0$.

3) $\mathcal{G}^1$ is equipped with a non-degenerate, symmetric bilinear form B. Recall that a symmetric bilinear form B on a vector space V is non-degenerate if

$$\text{for all } y \in V,\ B(x,y) = 0 \text{ implies } x = 0.$$

4) For all $a \in \mathcal{G}^1$

$$a^2 = B(a,a)1 \in \mathcal{G}^0.$$

5) For each integer $r \geq 2$, $\mathcal{G}^r$ is spanned by all r-blades, where an **r-blade** is a product of r mutually anti-commuting elements from $\mathcal{G}^1$. Recall that two elements a, b anti-commute if $ab = -ba$.

Remark. The explicit multiplication by 1 will not be written. We shall write

$$a^2 = B(a,a)$$

instead of

$$a^2 = B(a,a)1;$$

that is we are identifying $\mathbb{R} \subseteq \mathcal{G}^0$.

Definition 1.1.2. Elements of $\mathcal{G}$ will be called **multivectors**. Elements of $\mathcal{G}^r$ will be called r**-vectors** and will be said to have **grade r**. Elements of $\mathcal{G}^0$ will be called scalars; elements of $\mathcal{G}^1$ will be called vectors.

A general element of $\mathcal{G}$ is then a sum of r-vectors, where each r-vector is a sum of r-blades. By our definition, a scalar is a 0-blade and a vector is a 1-blade.

Definition 1.1.3. Let A_r be an r-blade. A **representation** of A_r is a set $\{a_1, \ldots, a_r\}$ of mutually anti-commuting vectors such that

$$A_r = a_1 \cdots a_r.$$

Each a_i is called a **factor** of the representation of A_r.

Intuitively, an r-blade may be thought of as a weighted, oriented r-dimensional subspace spanned by its factors.

Example 1.1.4. Let a be a 1-blade or a vector. We view this as a arrow with an orientation specified by the direction of a. Let λ be a positive real number. Then λa is a scaling of a. We may think of λa has having a weight of λ, relative to a.

Example 1.1.5. Let $a_1 a_2$ be a 2-blade. We view this as a plane spanned by a_1, a_2 with an orientation from a_1 to a_2. Let λ be a positive real number. Then $\lambda a_1 a_2$ has a weight λ, relative to $a_1 a_2$.

These informal ideas will become more precise later.

The bilinear form B allows one to associate lengths and spatial relations between vectors. One particular spatial relation is orthogonality.

Definition 1.1.6. Let a, b be vectors. If $B(a,b) = 0$ we say that a and b are **orthogonal**. If $b^2 = B(b,b) = 0$ we say that b is a **null vector**.

A null vector is then, orthogonal to itself. We will later give an example of a geometric algebra containing null vectors.
Let us look at some examples of geometric algebras. In the following, we use $\langle A \rangle$ to denote the span of elements of A.

Example 1.1.7. Consider $\mathbb{C}$ as a vector space over $\mathbb{R}$. Let $\mathcal{G}^0 = \langle 1 \rangle$ and $\mathcal{G}^1 = \langle i \rangle$. Then $\mathbb{C} = \mathcal{G}^0 \oplus \mathcal{G}^1$. Equip $\mathcal{G}^1$ with the bilinear form B defined by $B(i,i) = -1$. Then $\mathbb{C}$ is a geometric algebra by a straightforward verification.

We now construct a geometric algebra with a subspace $\mathcal{G}^2$.

Example 1.1.8. Let $\mathcal{G}$ be the free $\mathbb{R}$-module over the set of formal symbols $\{1, e_1, e_2, e_{12}\}$. Let $\mathcal{G}^1 = \langle e_1, e_2 \rangle$ be equipped with the symmetric bilinear form B such that

$$B(e_1, e_1) = B(e_2, e_2) = 1 \text{ and } B(e_1, e_2) = 0.$$

Note that B is non-degenerate. Multiplication is defined by the following table

	1	e_1	e_2	e_{12}
1	1	e_1	e_2	e_{12}
e_1	e_1	1	e_{12}	e_2
e_2	e_2	$-e_{12}$	1	$-e_1$
e_{12}	e_{12}	$-e_2$	e_1	-1

It is a trivial, but tedious exercise, to show that this defines a group. Extending multiplication over addition by bilinearity defines a geometric algebra on $\mathcal{G}$ as can be shown by straightforward calculations.

Since

$$e_1e_2 = -e_2e_1$$

and

$$e_{12} = e_1e_2,$$

by definition e_{12} is a 2-blade. It is straightforward to show that $\mathcal{G}^2 = \langle e_{12} \rangle$. Let $\mathcal{G}^0 = \langle 1 \rangle$, then

$$\mathcal{G} = \mathcal{G}^0 \oplus \mathcal{G}^1 \oplus \mathcal{G}^2.$$

Let $a, b \in \mathcal{G}^1$. Then

$$a = \alpha_1 e_1 + \alpha_2 e_2,\ b = \beta_1 e_1 + \beta_2 e_2,\ \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$$

Observe

$$\begin{aligned} ab &= (\alpha_1 e_1 + \alpha_2 e_2)(\beta_1 e_1 + \beta_2 e_2) \\ &= (\alpha_1\beta_1 + \alpha_2\beta_2) + (\alpha_1\beta_2 - \alpha_2\beta_1)e_1e_2. \end{aligned}$$

Observe that the product ab, contains a scalar and a 2-blade. The scalar is the familiar dot product from Euclidean geometry. The coefficient of e_1e_2 can be interpreted as the signed area between a and b, signed in the sense of an orientation from e_1 to e_2. The geometric product between two vectors measures, in some sense, how parallel and how orthogonal the two vectors a and b are. For if a is a scalar multiple of b, then the coefficient of e_1e_2 is zero

and $ab = B(a,b)$. If $B(a,b) = 0$, then $ab = (\alpha_1\beta_2 - \alpha_2\beta_1)e_1e_2$.
Finally, notice that $\mathcal{G}^+ = \langle 1\rangle + \langle e_{12}\rangle$ is a subalgebra of $\mathcal{G}$. A straightforward verification shows that mapping defined by $1 \mapsto 1,\ i \mapsto e_{12}$ from $\mathbb{C}$ to $\mathcal{G}^+$ is an isomorphism. Therefore, $\mathcal{G}$ contains the complex numbers. The unit i now has a geometric interpretation as a plane. This example should be thought of as the geometric algebra, generated by e_1 and e_2, of the plane represented by e_1e_2. We denote this geometric algebra by $\mathcal{G}(\mathbb{R}^2)$.

The construction of a geometric algebra associated with a finite dimensional vector space, as presented in Example 1.1.8, can be a laborious task when the dimension of the vector space is large. In the appendix, we give a theorem about the structure of an associated geometric algebra over a finite dimensional vector space. The results of the theorem are the following.

Theorem 4.1.8 *Let V be a finite dimensional vector space over $\mathbb{R}$ equipped with a non-degenerate symmetric bilinear form B. Then there exists an orthogonal non-null basis $\{e_1, \ldots, e_n\}$ such that the geometric algebra associated of V, is the direct sum of the subspaces spanned by $e_{i_1}\cdots e_{i_r}$, $1 \le i_1 < \ldots < i_r \le n$.*

For convenience, let $e_{i_1\cdots i_k} = e_{i_1}\cdots e_{i_k}$. Given a vector space V, we use the notation $\mathcal{G}(V)$ to denote the associated geometric algebra with V. By Theorem 4.1.8, $\mathcal{G}(V)$ is a direct sum of the subspaces $\mathcal{G}^r$ spanned by $e_{i_1\cdots i_r}$, $1 \le i_1 < \cdots < i_r \le n$.

Recall the summation convention that $\sum_k \alpha^k e_k = \alpha^k e_k$. We shall use this convention unless otherwise stated. Let us look at an example of a geometric algebra of three-dimensional space.

Example 1.1.9. Let $\mathcal{G}(\mathbb{R}^3)$ be the geometric algebra generated by $\langle e_1, e_2, e_3\rangle$ with the bilinear form B defined so that the generators are orthogonal and square to 1. We have

$$\mathcal{G} = \langle 1\rangle \oplus \langle e_1, e_2, e_3\rangle \oplus \langle e_{12}, e_{23}, e_{31}\rangle \oplus \langle e_{123}\rangle.$$

Let $a, b \in \mathcal{G}^1$, with

$$a = \alpha^k e_k,\ b = \beta^k e_k.$$

Then

$$ab = (\alpha^1\beta^1 + \alpha^2\beta^2 + \alpha^3\beta^3) + (\alpha^1\beta^2 - \alpha^2\beta^1)e_{12} + (\alpha^2\beta^3 - \alpha^3\beta^2)e_{23} + (\alpha^3\beta^1 - \alpha^1\beta^3)e_{31}.$$

Observe, as in Example 1.1.8, the product contains a scalar term, $B(a, b)$ and a sum of bi-vectors. Notice also that the coefficients of the bi-vectors are the coefficients from the cross product $a \times b$.

Furthermore, notice that $\mathcal{G}^+ = \langle 1 \rangle \oplus \langle e_{12}, e_{23}, e_{31} \rangle$ forms a subalgebra of $\mathcal{G}$. Let $i = e_{12}, j = e_{31}$ and $k = e_{23}$. Then

$$i^2 = j^2 = k^2 = ijk = -1.$$

These are the rules of quaternion multiplication discovered by Hamilton. The quaternions i, j, k may be interpreted as planes.

Let us look at a geometric algebra used in special relativity.

Example 1.1.10. Let $\mathcal{G}(M)$ be the geometric algebra generated by the orthogonal vectors $\langle e_0, e_1, e_2, e_3 \rangle$ with the bilinear form B defined by

$$e_0^2 = -e_k^2 = 1 \ , k = 1, \ldots, 3.$$

We have

$$\mathcal{G} = \langle 1 \rangle \oplus \langle e_0, e_1, e_2, e_3 \rangle \oplus \langle e_{01}, e_{02}, e_{03}, e_{12}, e_{13}, e_{23} \rangle \oplus \langle e_{012}, e_{123}, e_{230}, e_{013} \rangle \oplus \langle e_{0123} \rangle.$$

Let $x \in \langle e_0, e_1, e_2, e_3 \rangle$. Then $x = x^\alpha e_\alpha, \ x^\alpha \in \mathbb{R}$. Observe

$$x^2 = B(x^\alpha e_\alpha, x^\alpha e_\alpha) = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

is the so called invariant interval of special relativity. In this geometric algebra there exist non-zero vectors that are null. Let $n = e_0 + e_1$. Then

$$n^2 = 1 - 1 = 0.$$

There are bi-vectors that square to 1 and -1. Observe

$$(e_{01})^2 = e_0 e_1 e_0 e_1 = -e_0^2 e_1^2 = 1 \ \text{ and } \ (e_{12})^2 = e_1 e_2 e_1 e_2 = -e_1^2 e_2^2 = -1.$$

With some examples in mind, let us recall a fact associated with a direct sum decomposition. Since $\mathcal{G} = \mathcal{G}^0 \oplus \mathcal{G}^1 \oplus \cdots$ we have the associated projection maps

$$\langle \cdot \rangle_r : \mathcal{G} \to \mathcal{G}^r,$$

satisfying the following properties:

$$\langle A + B \rangle_r = \langle A \rangle_r + \langle B \rangle_r$$

$$\langle \lambda A \rangle_r = \lambda \langle A \rangle_r \text{ if } \lambda \in \mathbb{R}$$

$$\langle \langle A \rangle_r \rangle_r = \langle A \rangle_r.$$

We define $\langle A \rangle_k = 0$ if $k < 0$.

There is a relationship between the bilinear form B and the geometric product between vectors.

Proposition 1.1.11. *Let a, b be vectors. Then $B(a, b) = \dfrac{ab + ba}{2}$.*

Proof. Since

$$(a + b)^2 = a^2 + b^2 + ab + ba$$

or

$$ab + ba = (a + b)^2 - a^2 - b^2$$

we have

$$\begin{aligned} ab + ba &= B(a + b, a + b) - B(a, a) - B(b, b) \\ &= B(a, a) + B(b, a) + B(a, b) + B(b, b) - B(a, a) - B(b, b) \\ &= 2B(a, b) \end{aligned}$$

as claimed. □

Corollary 1.1.12. *$ab = -ba$ if and only if $B(a, b) = 0$.*

Therefore, anti-commutativity and orthogonality are equivalent.

Proposition 1.1.13. *Let a and b be linearly dependent vectors. Then their geometric product commutes.*

Proof. If $a = \lambda b$, $\lambda \in \mathbb{R}$, then $ab = \lambda bb = \lambda b^2 = b^2\lambda = bb\lambda = ba$. □

We shall often work with inverses of elements of the geometric algebra. Let us first characterize invertibility for vectors, and then generalize to blades.

Proposition 1.1.14. *Let b be a non-zero vector. Then b is invertible if and only if b is non-null. Moreover,*

$$b^{-1} = \frac{b}{b^2}$$

when it exists.

Proof. Suppose that b is invertible. Then

$$bb^{-1} = 1$$

implies

$$b^2b^{-1} = b.$$

Since b is invertible and 0 is not, $b \neq 0$ and we find

$$b^2b^{-1} \neq 0.$$

Hence,

$$b^2 \neq 0$$

and b is non-null. We may conclude that

$$b^{-1} = \frac{b}{b^2}.$$

Conversely, suppose that b is non-null. We shall show that $b^{-1} = \dfrac{b}{b^2}$.

$$b\frac{b}{b^2} = \frac{b^2}{b^2} = 1$$

similarly $\frac{b}{b^2}b = 1$. Thus, b^{-1} is as claimed. □

We extend Proposition 1.1.14 to blades.

Proposition 1.1.15. *Let A_r be a non-zero r-blade. Then A_r is invertible if and only if each factor of each representation of A_r is non-null.*

Proof. Suppose that A_r is invertible with a representation

$$A_r = a_1 \cdots a_r$$

containing a null vector. We suppose that $a_1^2 = 0$ without loss of generality. Observe that

$$a_1 A_r = a_1^2 a_2 \cdots a_r = 0$$

implies

$$a_1 = a_1 A_r A_r^{-1} = 0$$

contradicting the fact that A_r is non-zero.
Conversely, suppose that each factor of some representation of $A_r = a_1 \cdots a_r$ is non-null. By Proposition 1.1.14, each factor of the representation is invertible. A simple verification shows that

$$A_r^{-1} = a_r^{-1} \cdots a_1^{-1}.$$

□

1.2 Grades resulting from multiplication

In this section we prove some fundamental results concerning the product of two blades. In particular, we show that the resulting grades of a product of two vectors is a scalar and a 2-vector, of a vector and an r-blade is an $r-1$ and an $(r+1)$-vector and of an r and s-blade is a sum starting at grade $|s-r|$ and incrementing by grade 2, until grade $r+s$.

1.2.1 Multiplication of two vectors

Definition 1.2.1. Let a and b be vectors, with b invertible. We call

$$\pi(a,b) = B(a,b)b^{-1} \text{ the } \textbf{projection} \text{ of } a \text{ onto } b$$

and

$$\rho(a,b) = a - \pi(a,b) \text{ the } \textbf{rejection} \text{ of } a \text{ from } b.$$

Lemma 1.2.2. *Let a and b be vectors, with b invertible. Then $\pi(a,b)$ commutes with b.*

Proof. By Proposition 1.1.14,

$$\pi(a,b) = B(a,b)b^{-1} = B(a,b)\frac{b}{b^2} = \frac{B(a,b)}{b^2}b.$$

Therefore, $\pi(a,b)$ and b are linearly dependent. By Proposition 1.1.13, $\pi(a,b)$ and b commute. □

Lemma 1.2.3. *Let a and b be vectors, with b invertible. Then $\rho(a,b)$ anti-commutes with b. Thus,*

$$\rho(a,b)b$$

is a 2-blade.

Proof. First note that

$$B(b,b^{-1}) = B(b,\frac{b}{b^2}) = \frac{B(b,b)}{b^2} = \frac{b^2}{b^2} = 1.$$

Observe that

$$\begin{aligned} B(b,\rho(a,b)) &= B(b, a - \pi(a,b)) \\ &= B(b,a) - B(b,\pi(a,b)) \\ &= B(b,a) - B(b, B(a,b)b^{-1}) \\ &= B(a,b) - B(a,b)B(b,b^{-1}) \\ &= B(a,b) - B(a,b) \\ &= 0. \end{aligned}$$

By Corollary 1.1.12, $\rho(a,b)$ and b anti-commute. ☐

Although Proposition 1.1.11 establishes part of the following lemma, we give a different proof using the notion invertibility.

Lemma 1.2.4. *Let a and b be vectors, with b invertible. Then*

$$\frac{ab+ba}{2} = B(a,b) \;\; \textit{and} \;\; \frac{ab-ba}{2} = \rho(a,b)b.$$

Proof. Since

$$a = \pi(a,b) + \rho(a,b)$$

we have

$$ab = \pi(a,b)b + \rho(a,b)b \text{ and } ba = b\pi(a,b) + b\rho(a,b).$$

By Lemma 1.2.2 and Lemma 1.2.3,

$$\begin{aligned} ab + ba &= \pi(a,b)b + \rho(a,b)b + b\pi(a,b) + b\rho(a,b) \\ &= 2\pi(a,b)b \\ &= 2B(a,b)b^{-1}b \\ &= 2B(a,b) \end{aligned}$$

and

$$\begin{aligned} ab - ba &= \pi(a,b)b + \rho(a,b)b - b\pi(a,b) - b\rho(a,b) \\ &= 2\rho(a,b)b. \end{aligned}$$

☐

Proposition 1.2.5. *Let a and b be vectors, with b invertible. Then*

$$ab = \langle ab\rangle_0 + \langle ab\rangle_2.$$

Moreover,

$$\langle ab\rangle_0 = B(a,b).$$

Proof. Observe

$$ab = \frac{ab + ba}{2} + \frac{ab - ba}{2}.$$

By Lemma 1.2.4,

$$ab = B(a, b) + \rho(a, b)b.$$

The result now follows since $B(a, b) \in \mathcal{G}^0$ and $\rho(a, b)b \in \mathcal{G}^2$.

□

In establishing Proposition 1.2.5 it was assumed that b was invertible. We now show the proposition holds for null vectors as well.

Lemma 1.2.6. *Let n be a null vector. Then there exists non-null vectors x, y such that $n = x + y$.*

Proof. Suppose that $n = 0$. Since B is non-degenerate, there exists a vector x such that $B(x, x) \neq 0$. Then $n = x - x$ is a sum of non-null vectors. Suppose now that $n \neq 0$. Since B is non-degenerate, there exists a vector x such that $B(n, x) \neq 0$. Let $y_\lambda = n - \lambda x$ where $\lambda \in \mathbb{R} \setminus \{0\}$. Then

$$y_\lambda^2 = \lambda(-2B(n, x) + \lambda x^2).$$

If $x^2 = 0$, then

$$y_\lambda^2 = -2\lambda B(n, x) \neq 0.$$

So,

$$n = \frac{1}{2}y_1 + \frac{1}{2}y_{-1}$$

is a sum of non-null vectors..
If $x^2 \neq 0$, then choose

$$\lambda \neq \frac{2B(n, x)}{x^2}$$

so that y_λ is non-null. Then

$$n = y_\lambda + \lambda x$$

is a sum of non-null vectors. □

Theorem 1.2.7. *Let a, b be vectors. Then*

$$ab = \langle ab \rangle_0 + \langle ab \rangle_2.$$

Proof. By Proposition 1.2.5, the result holds if b is non-null. Suppose that b is null. By Lemma 1.2.6, there exists non-null vectors x, y such that $b = x + y$. By Proposition 1.2.5,

$$\begin{aligned} ab = a(x+y) = ax + ay &= \langle ax \rangle_0 + \langle ax \rangle_2 + \langle ay \rangle_0 + \langle ay \rangle_2 \\ &= \langle a(x+y) \rangle_0 + \langle a(x+y) \rangle_2 = \langle ab \rangle_0 + \langle ab \rangle_2 \end{aligned}$$

. □

The product of two vectors gives a grade 0 and grade 2 term in the sum.

1.2.2 Multiplication of a vector and a blade

We will now generalize Theorem 1.2.7 to a vector and a blade. The resulting product will be a sum of a vector of one less and one greater in grade than the original blade.

Definition 1.2.8. Let a and A_r be a vector and an invertible r-blade with a representation $A_r = a_1 \cdots a_r$. We define the **projection** of a onto A_r by

$$\pi(a, a_1 \cdots a_r) = \sum_{k=1}^{r} B(a, a_k) a_k^{-1}$$

and the **rejection** of a from A_r by

$$\rho(a, a_1 \cdots a_r) = a - \pi(a, a_1 \cdots a_r).$$

The map π is called the projection because it measures how much of a is in the span of the factors $a_1, \ldots, a_r$ and ρ measures how much of a is not in the span of the factors $a_1, \ldots, a_r$. The formula for the projection π appears to depend on the choice of representation of the blade, but we shall see that it is independent of choice of representation and similarly for ρ.

We use the following convention. Given a product of vectors

$$a_1 \cdots \check{a}_k \cdots a_r,$$

the check indicates the the vector a_k is not present in the product.

Lemma 1.2.9. *Let a and A_r be a vector and an invertible r-blade with representation $A_r = a_1 \cdots a_r$, respectively. Then*

$$\pi(a, a_1 \cdots a_r)A_r = (-1)^{r+1} A_r \pi(a, a_1 \cdots a_r).$$

Moreover,

$$\pi(a, a_1 \cdots a_r)A_r$$

is an $(r-1)$-vector.

Proof. Since a_k^{-1} anti-commutes with a_i for $i \neq k$ and commutes with a_k we have,

$$\begin{aligned}
\pi(a, a_1 \cdots a_r)A_r &= \left(\sum_{k=1}^{r} B(a, a_k)a_k^{-1}\right) a_1 \cdots a_r \\
&= \sum_{k=1}^{r} B(a, a_k)a_k^{-1} a_1 \cdots a_r \\
&= \sum_{k=1}^{r} B(a, a_k)(-1)^{k-1} a_1 \cdots \check{a}_k \cdots a_r \qquad (1.2.2.1) \\
&= a_1 \cdots a_r \sum_{k=1}^{r} (-1)^{k-1}(-1)^{r-k} B(a, a_k)a_k^{-1} \\
&= (-1)^{r+1} A_r \pi(a, a_1 \cdots a_r).
\end{aligned}$$

Referring to (1.2.2.1), since the $r-1$ factors mutually anti-commute, we have a sum of r, $(r-1)$-blades. Hence,

$$\pi(a, a_1 \cdots a_r)A_r$$

is an $(r-1)$-vector. □

Lemma 1.2.10. *Let a and A_r be a vector and an invertible r-blade with representation $A_r = a_1 \cdots a_r$, respectively. Then*

$$\rho(a, a_1 \cdots a_r)a_k = -a_k \rho(a, a_1 \cdots a_r) \text{ for } k = 1, \ldots, r.$$

Proof. Let $1 \leq j \leq r$. Observe

$$\begin{aligned} B(a_j, \rho(a, a_1 \cdots a_r)) &= B(a_j, a - \pi(a, a_1 \cdots a_r)) = B(a_j, a) - B(a_j, \pi(a, a_1 \cdots a_r)) \\ &= B(a_j, a) - B(a_j, \sum_{k=1}^{r} B(a, a_k) a_k^{-1}) = B(a_j, a) - \sum_{k=1}^{r} B(a, a_k) B(a_j, a_k^{-1}) \\ &= B(a_j, a) - B(a, a_j) B(a_j, a_j^{-1}) - \sum_{k=1 k \neq j}^{r} B(a, a_k) B(a_j, a_k^{-1}) \\ &= B(a_j, a) - B(a_j, a) - 0 = 0. \end{aligned}$$

By Corollary 1.1.12, $\rho(a, a_1 \cdots a_r)$ and a_j anti-commute for $1 \leq j \leq r$. □

Lemma 1.2.11. *Let a and A_r be a vector and an invertible r-blade with representation $A_r = a_1 \cdots a_r$, respectively. Then*

$$\rho(a, a_1 \cdots a_r) A_r = (-1)^r A_r \rho(a, a_1 \cdots a_r).$$

Moreover,

$$\rho(a, a_1 \cdots a_r) A_r$$

is an $(r+1)$-blade.

Proof. By Lemma 1.2.10, $\rho(a, a_1 \cdots a_r)$ anti-commutes with each factor a_k. Then

$$\begin{aligned} \rho(a, a_1 \cdots a_r) A_r &= \rho(a, a_1 \cdots a_r) a_1 \cdots a_r \\ &= (-1)^r a_1 \cdots a_r \rho(a, a_1 \cdots a_r) \\ &= (-1)^r A_r \rho(a, a_1 \cdots a_r). \end{aligned}$$

Also, since each factor is anti-commuting we have that $\rho(a, a_1 \cdots a_r) A_r$ is an $(r+1)$-blade. □

Proposition 1.2.12. *Let a and A_r be a vector and an invertible r-blade, respectively. Then*

$$\frac{aA_r - (-1)^r A_r a}{2} = \langle a A_r \rangle_{r-1} \text{ and } \frac{aA_r + (-1)^r A_r a}{2} = \langle a A_r \rangle_{r+1}.$$

Consequently,

$$aA_r = \langle aA_r \rangle_{r-1} + \langle aA_r \rangle_{r+1} \textit{ and } A_r a = \langle A_r a \rangle_{r-1} + \langle A_r a \rangle_{r+1}.$$

Proof. Since

$$a = \pi(a, a_1 \cdots a_r) + \rho(a, a_1 \cdots a_r),$$

by Lemmas 1.2.9 and 1.2.11,

$$\begin{aligned} aA_r + (-1)^r A_r a &= \pi(a, a_1 \cdots a_r)A_r + \rho(a, a_1 \cdots a_r)A_r + (-1)^r A_r \pi(a, a_1 \cdots a_r) + \\ &(-1)^r A_r \rho(a, a_1 \cdots a_r) \\ &= \pi(a, a_1 \cdots a_r)A_r + \rho(a, a_1 \cdots a_r)A_r + (-1)^{2r+1} \pi(a, a_1 \cdots a_r)A_r + \\ &(-1)^{2r} \rho(a, a_1 \cdots a_r)A_r \\ &= 2\rho(a, a_1 \cdots a_r)A_r \end{aligned}$$

or

$$\frac{aA_r + (-1)^r A_r a}{2} = \rho(a, a_1 \cdots a_r)A_r.$$

A similar calculation shows that

$$\frac{aA_r - (-1)^r A_r a}{2} = \pi(a, a_1 \cdots a_r)A_r.$$

Then

$$\begin{aligned} aA_r &= \frac{aA_r - (-1)^r A_r a}{2} + \frac{aA_r + (-1)^r A_r a}{2} \\ &= \pi(a, a_1 \cdots a_r)A_r + \rho(a, a_1 \cdots a_r)A_r \\ &= \langle aA_r \rangle_{r-1} + \langle aA_r \rangle_{r+1}. \end{aligned}$$

Similarly,

$$A_r a = \langle A_r a \rangle_{r-1} + \langle A_r a \rangle_{r+1}.$$

□

Remark. This shows that π, ρ are independent of choice of representation.

Corollary 1.2.13. *Let a and A_r be a vector and an invertible r-blade, respectively. Then*

$$\langle aA_r\rangle_{r+1} = (-1)^r\langle A_r a\rangle_{r+1}$$

and

$$\langle aA_r\rangle_{r-1} = (-1)^{r+1}\langle A_r a\rangle_{r-1}.$$

Proof. By Proposition 1.2.12 and Lemmas 1.2.10 and 1.2.9,

$$\langle aA_r\rangle_{r+1} = \rho(a, a_1, \dots, a_r)A_r = (-1)^r A_r\rho(a, a_1, \dots, a_r) = (-1)^r\langle A_r a\rangle_{r+1}.$$

The other equality is established similarly. □

We assumed throughout the invertibility of the blade to establish Proposition 1.2.12. In the case when the vector space is finite dimensional, the result still holds for a vector and any blade. Let A_r be an r-blade with a representation $A_r = a_1 \cdots a_r$. Each factor of the representation is a sum of orthogonal non-null vectors $e_1, \dots, e_n$

$$a_k = \sum_{i=1}^{n} \alpha_{ik}e_i,\ k = 1, \dots, r.$$

Then

$$A_r = a_1 \cdots a_r = \sum_{i_1=1}^{n} \alpha_{i_1 1}e_{i_1} \cdots \sum_{i_r=1}^{n} \alpha_{i_r r}e_{i_r}.$$

After expanding we find that A_r is a sum of invertible r-blades. Then as in the case of vectors, by linearity of the projection operators we have

Theorem 1.2.14. *Suppose that $\mathcal{G}^1$ is finite dimensional. Let a and A_r be a vector and r-blade, respectively. Then*

$$aA_r = \langle aA_r\rangle_{r-1} + \langle aA_r\rangle_{r+1}.$$

Remark. The author is working on a way to generalize Theorem 1.2.14 when the vector space is not necessarily finite dimensional. As it currently stands, the following results involving the product of blades only are known to hold in the finite dimensional case.

Note the general structure of the geometric product of a vector with an r-blade has a sum of an $r-1$ and $(r+1)$-vector.

Now that we have established the grades resulting from the product of a vector with a blade we shall generalize to the product of a blade with a blade. To help with this generalization we introduce a new map.

1.2.3 Reversion

We shall introduce a very useful map called reversion. Reversion allows one to reverse the order of a product of multivectors and this is useful for algebra manipulations.

Definition 1.2.15. Let $A \in \mathcal{G}$ given by the unique sum $A = \sum_{r=1}^{n} A_r$, where $A_r = \langle A \rangle_r$. Then the **reverse** of A is

$$A^{\dagger} = \sum_{r=1}^{n} (-1)^{\frac{r(r-1)}{2}} A_r.$$

Remark. Note that for an r-blade A_r, $A_r^{\dagger} = (-1)^{\frac{r(r-1)}{2}} A_r$. So, $A_r^{\dagger}$ is an r-blade as well. Also, $(A^{\dagger})^{\dagger} = A$; reversion is an involution.

If λ is a scalar

$$\lambda^{\dagger} = (-1)^{\frac{0(0-1)}{2}} \lambda = \lambda,$$

and if a is a vector

$$a^{\dagger} = (-1)^{\frac{1(1-1)}{2}} a = a.$$

The general pattern is as follows

$$+ \quad + \quad - \quad - \quad + \quad + \quad \cdots,$$

which we see has a period of 4.

Lemma 1.2.16. *Let a and A be a vector and multivector, respectively. Then*

$$(aA)^{\dagger} = A^{\dagger} a.$$

Proof. We show the result holds for an r-blade A_r. By Proposition 1.2.12 and Corollary

1.2.13,

$$\begin{aligned}
(aA_r)^\dagger &= (\langle aA_r\rangle_{r-1} + \langle aA_r\rangle_{r+1})^\dagger \\
&= \langle aA_r\rangle_{r-1}^\dagger + \langle aA_r\rangle_{r+1}^\dagger \\
&= (-1)^{\frac{(r-1)((r-1)-1)}{2}}\langle aA_r\rangle_{r-1} + (-1)^{\frac{(r+1)((r+1)-1)}{2}}\langle aA_r\rangle_{r+1} \\
&= (-1)^{\frac{(r-1)((r-1)-1)}{2}}(-1)^{r+1}\langle A_r a\rangle_{r-1} + (-1)^{\frac{(r+1)((r+1)-1)}{2}}(-1)^{r}\langle A_r a\rangle_{r+1} \\
&= (-1)^{\frac{r^2-r+4}{2}}\langle A_r a\rangle_{r-1} + (-1)^{\frac{r^2-r}{2}}\langle A_r a\rangle_{r+1} \\
&= (-1)^{\frac{r(r-1)}{2}}(\langle A_r a\rangle_{r-1} + \langle A_r a\rangle_{r+1}) \\
&= (-1)^{\frac{r(r-1)}{2}}A_r a \\
&= A_r^\dagger a^\dagger.
\end{aligned}$$

If $A \in \mathcal{G}$, then $A = \sum_k \langle A\rangle_k$. Hence,

$$(aA)^\dagger = (a\sum_k \langle A\rangle_k)^\dagger = (\sum_k a\langle A\rangle_k)^\dagger = \sum_k (a\langle A\rangle_k)^\dagger = \sum_k \langle A\rangle_k^\dagger a^\dagger = (\sum_k \langle A\rangle_k)^\dagger a = A^\dagger a.$$

□

Proposition 1.2.17. *Reversion satisfies the following properties:*

1) $(AB)^\dagger = B^\dagger A^\dagger$

2) $(A+B)^\dagger = A^\dagger + B^\dagger$

3) $\langle A^\dagger\rangle_r = \langle A\rangle_r^\dagger$

Proof. Properties 2) and 3) follow straightforwardly from the definition of reversion. To establish 1) we shall show that $(B_s A_r)^\dagger = A_r^\dagger B_s^\dagger$ for any r-blade and s-blade by induction on s. By Lemma 1.2.16, the results holds for $s = 1$. Now suppose that the statement holds for some fixed s. Let B_{s+1} be an $(s+1)$-blade. We can factor $B_{s+1} = B_s b$ for some s-blade B_s

and some vector b. By our induction hypothesis and Lemma 1.2.16,

$$\begin{aligned}
(B_{s+1}A_r)^\dagger &= (B_s b A_r)^\dagger \\
&= (B_s\langle bA_r\rangle_{r-1} + B_s\langle bA_r\rangle_{r+1})^\dagger \\
&= (B_s\langle bA_r\rangle_{r-1})^\dagger + (B_s\langle bA_r\rangle_{r+1})^\dagger \\
&= \langle bA_r\rangle_{r-1}^\dagger B_s^\dagger + \langle bA_r\rangle_{r+1}^\dagger B_s^\dagger \\
&= (\langle bA_r\rangle_{r-1}^\dagger + \langle bA_r\rangle_{r+1}^\dagger)B_s^\dagger \\
&= (\langle bA_r\rangle_{r-1} + \langle bA_r\rangle_{r+1})^\dagger B_s^\dagger \\
&= (bA_r)^\dagger B_s^\dagger \\
&= (A_r^\dagger b^\dagger)B_s^\dagger \\
&= A_r^\dagger(b^\dagger B_s^\dagger) \\
&= A_r^\dagger(B_s b)^\dagger \\
&= A_r^\dagger B_{s+1}^\dagger.
\end{aligned}$$

By the principle of mathematical induction the result holds for all $s \in \mathbb{N}$. That $(AB)^\dagger = B^\dagger A^\dagger$ for all $A, B \in \mathcal{G}$ now follows from expanding A and B as their unique sums of r-vectors.

□

1.2.4 Multiplication of two blades

We will now generalize Theorem 1.2.14 to a blade and a blade. The result will be a generalization of our previous results and will be called the grade expansion identity.

Proposition 1.2.18 (Grade Expansion Identity). *Let A_r and B_s be an r- and s-blade, respectively. Then*

$$A_r B_s = \sum_{k=0}^{m} \langle A_r B_s\rangle_{|s-r|+2k} \text{ where } m = \min\{r, s\}.$$

We begin with two lemmas.

Lemma 1.2.19. *Let A_r and B_s be an r- and s-blade, respectively. If $s \geq r$, then*

$$A_r B_s = \sum_{k=0}^{r} \langle A_r B_s \rangle_{s-r+2k}.$$

Proof. We proceed by induction on r. When $r = 0$ the result is evident. Suppose that for some r the result holds for all s blades such that $s \geq r$. Let A_{r+1} be an $(r+1)$-blade and B_s be an s-blade such that $s \geq r+1$. Since A_{r+1} is an $(r+1)$-blade we can write $A_{r+1} = aA_r$, where A_r is an r-blade and a is a vector. By the induction hypothesis and Theorem 1.2.14,

$$\begin{aligned} A_{r+1}B_s &= aA_rB_s \\ &= a\sum_{k=0}^{r} \langle A_r B_s \rangle_{s-r+2k} \\ &= \sum_{k=0}^{r} a\langle A_r B_s \rangle_{s-r+2k} \\ &= \sum_{k=0}^{r} (\langle a\langle A_r B_s \rangle_{s-r+2k} \rangle_{s-r+2k-1} + \langle a\langle A_r B_s \rangle_{s-r+2k} \rangle_{s-r+2k+1}) \\ &= \sum_{k=0}^{r} (\langle a\langle A_r B_s \rangle_{s-r+2k} \rangle_{s-(r+1)+2k} + \langle a\langle A_r B_s \rangle_{s-r+2k} \rangle_{s-(r+1)+2k+2}). \end{aligned}$$

Also since $A_{r+1}B_s \in \mathcal{G}$, we have the unique direct sum decomposition

$$A_{r+1}B_s = \sum_{k} \langle A_{r+1}B_s \rangle_k.$$

We must have

$$\langle A_{r+1}B_s \rangle_{s-(r+1)+2k+1} = 0$$

for $k = 0, \ldots, r$ and

$$\langle A_{r+1}B_s \rangle_k = 0$$

for $k \geq s + (r+1) + 1$. Hence,

$$A_{r+1}B_s = \sum_{k=0}^{r+1} \langle A_{r+1}B_s \rangle_{s-(r+1)+2k}.$$

By the principle of mathematical induction, the result holds. □

Lemma 1.2.20. *Let A_r and B_s be an r- and s-blade, respectively. If $r \geq s$, then*

$$A_rB_s = \sum_{k=0}^{s} \langle A_rB_s \rangle_{r-s+2k}.$$

Proof. Reversion is an involution and hence

$$A_rB_s = ((A_rB_s)^\dagger)^\dagger = (B_s^\dagger A_r^\dagger)^\dagger.$$

Since reversion doesn't alter grade, by Lemma 1.2.19,

$$B_s^\dagger A_r^\dagger = \sum_{k=0}^{s} \langle B_s^\dagger A_r^\dagger \rangle_{r-s+2k}.$$

Then

$$A_rB_s = \left(\sum_{k=0}^{s} \langle B_s^\dagger A_r^\dagger \rangle_{r-s+2k} \right)^\dagger = \sum_{k=0}^{s} \langle B_s^\dagger A_r^\dagger \rangle_{r-s+2k}^\dagger = \sum_{k=0}^{s} \langle A_rB_s \rangle_{r-s+2k}.$$

□

Proposition 1.2.18 follows from Lemmas 1.2.19 and 1.2.20.
We find from the grade expansion identity that the product of two blades is not necessarily a blade but a sum of blades where the grade increases by two.

1.3 Products derived from the geometric product

In this section we introduce four new products: the wedge product, left/right contraction and the scalar product. The right contraction and wedge product shall be used extensively in Chapter 2.

1.3.1 Outer or wedge product

The wedge product is the product of the so-called exterior algebra. We will show that the wedge product is alternating and associative, an r-blade is the wedge product of r vectors, and the wedge of r vectors is an r-blade. Finally, in the case of a finite dimensional vector space, it is shown that the wedge of linearly independent vectors is not zero.

Definition 1.3.1. Let A_r and B_s be an r- and s-blade, respectively.

Then the **outer** or **wedge product** is

$$A_r \wedge B_s = \langle A_r B_s \rangle_{r+s}.$$

We extend the definition of the wedge product to multivectors by bilinearity.

Example 1.3.2. Consider $\mathcal{G}(\mathbb{R}^3)$, the geometric algebra from Example 1.1.9. Let

$$A = e_{12} \text{ and } a = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3.$$

Consider

$$\begin{aligned} a \wedge A &= \langle aA \rangle_3 \\ &= \langle (\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3) e_{12} \rangle_3 \\ &= \langle \alpha_1 e_{112} + \alpha_2 e_{212} + \alpha_3 e_{312} \rangle_3 \\ &= \langle \alpha_1 e_2 - \alpha_2 e_1 + \alpha_3 e_{123} \rangle_3 \\ &= \alpha_3 e_{123}. \end{aligned}$$

Then, since $e_{123} \neq 0$,

$$a \wedge A = 0 \text{ iff } \alpha_3 = 0 \text{ iff } a = \alpha_1 e_1 + \alpha_2 e_{32} \text{ iff } a \in \langle e_1, e_2 \rangle$$

Intuitively, if $a \wedge A = 0$, then the line a is contained in the plane A. If $a \wedge A \neq 0$ then the line a and plane A form a volume, which is a grade 3 object, formed from grade 1 and 2 objects.

Proposition 1.3.3. *Let A_r, B_s and C_t be r, s and t-blades, respectively. Then*

$$A_r \wedge (B_s \wedge C_t) = (A_r \wedge B_s) \wedge C_t \tag{1.3.1.1}$$

$$A_r \wedge B_s = (-1)^{rs} B_s \wedge A_r \tag{1.3.1.2}$$

Proof. We follow [HS84]. By associativity and expanding $(A_r B_s)C_t = A_r(B_s C_t)$ each side with the grade expansion identity (Proposition 1.2.18) we have

$$\begin{aligned}
A_r \wedge (B_s \wedge C_t) &= A_r \wedge \langle B_s C_t \rangle_{s+t} \\
&= \langle A_r \langle B_s C_t \rangle_{s+t} \rangle_{r+(s+t)} \\
&= \langle A_r (B_s C_t) \rangle_{r+s+t} \\
&= \langle (A_r B_s) C_t \rangle_{(r+s)+t} \\
&= (A_r \wedge B_s) \wedge C_t.
\end{aligned}$$

Noting that reversion is an involution and grade preserving, we further have

$$\begin{aligned}
A_r \wedge B_s &= \langle A_r B_s \rangle_{r+s} \\
&= (\langle (A_r B_s)^\dagger \rangle_{r+s})^\dagger \\
&= (-1)^{\frac{(r+s)((r+s)-1)}{2}} (-1)^{\frac{s(s-1)}{2}} (-1)^{\frac{r(r-1)}{2}} \langle B_s A_r \rangle_{r+s} \\
&= (-1)^{rs} B_s \wedge A_r
\end{aligned}$$

□

1.3.2 Linear independence and the wedge product

We establish the connection between linear independence and the wedge of a product of vectors when the vector space is finite dimensional.

Corollary 1.3.4. *Let a, b be vectors. Then*

$$a \wedge b = -b \wedge a.$$

Consequently,

$$a \wedge a = 0.$$

Proof. This follows immediately from Proposition 1.3.3. □

We now show that the wedge of linearly dependent vectors is zero.

Proposition 1.3.5. *If the collection of vectors $\{a_1, \ldots, a_r\}$ is linearly dependent, then $a_1 \wedge \cdots \wedge a_r = 0$.*

Proof. Since $a_1, \ldots, a_r$ are linearly dependent there exist scalars $\lambda_1, \ldots, \lambda_r$ not all zero for which

$$\lambda_1 a_1 + \cdots + \lambda_r a_r = 0.$$

Without loss of generality suppose that $\lambda_1 \neq 0$. By Corollary 1.3.4,

$$a_1 \wedge a_2 \wedge \cdots \wedge a_r = \lambda_1^{-1}(\lambda_1 a_1) \wedge a_2 \wedge \cdots \wedge a_r = -\lambda_1^{-1}(\lambda_2 a_2 + \cdots + \lambda_r a_r) \wedge a_2 \wedge \cdots \wedge a_r = 0.$$

□

The converse of Proposition 1.3.5 will be given next, but first a lemma.

Lemma 1.3.6. *Let $a_1 \cdots a_r$ be a representation of an r-blade. Then*

$$a_1 \cdots a_r = a_1 \wedge \cdots \wedge a_r.$$

Proof. We proceed by induction on r. When $r = 1$ the result holds trivially. Suppose the result holds for any r-blade. By the induction hypothesis,

$$\begin{aligned} a_1 a_2 \cdots a_{r+1} = a_1(a_2 \wedge \cdots \wedge a_{r+1}) &= \langle a_1(a_2 \wedge \cdots \wedge a_{r+1})\rangle_{r-1} + \langle a_1(a_2 \wedge \cdots \wedge a_{r+1})\rangle_{r+1} \\ &= \langle a_1 a_2 \cdots a_{r+1}\rangle_{r-1} + \langle a_1(a_2 \wedge \cdots \wedge a_{r+1})\rangle_{r+1} = a_1 \wedge (a_2 \wedge \cdots \wedge a_{r+1}) \\ &= a_1 \wedge a_2 \wedge \cdots \wedge a_{r+1}. \end{aligned}$$

□

Corollary 1.3.7. *Let $\dim \mathcal{G}^1 = n$. Let $\mathcal{A} = \{a_1, \ldots, a_r\}$ be a set of $r \leq n$ linearly independent vectors. Then $a_1 \wedge \cdots \wedge a_r \neq 0$.*

Proof. Since $\mathcal{A}$ is linearly independent we may extend it to a basis of $\mathcal{G}^1$

$$\mathcal{A} \cup \{a_{r+1}, \ldots, a_n\}.$$

Furthermore, by Lemma 4.1.2 proved in the appendix, there exists an orthogonal non-null basis $\{e_1, \ldots, e_n\}$ for $\mathcal{G}^1$ where each basis vector squares to ± 1. Let $L : \mathcal{G}^1 \to \mathcal{G}^1$ be the linear map defined by

$$a_k = L(e_k),\ k = 1, \ldots, n.$$

Since L maps a basis to a basis, L is non-singular. Suppose that

$$a_k = L(e_k) = \alpha_k^s e_s,\ k = 1, \ldots, n.$$

By Lemma 1.3.6,

$$\begin{aligned} a_1 \wedge \cdots \wedge a_r \wedge \cdots \wedge a_n &= \alpha_1^{j_1} e_{j_1} \wedge \cdots \wedge \alpha_n^{j_n} e_{j_n} \\ &= \sum_{\sigma \in S_n} (-1)^\sigma \alpha_1^{\sigma(1)} \cdots \alpha_n^{\sigma(n)} e_1 \wedge \cdots \wedge e_n \\ &= \det(L) e_1 \cdots e_n \\ &= \det(L) I, \end{aligned}$$

where $I = e_1 \cdots e_n$. Since each basis vector e_i squares to ± 1, we have $II^\dagger = \pm 1$. Therefore, $I \neq 0$. Then since $\det(L) \neq 0$ also, we cannot have $a_1 \wedge \cdots \wedge a_r = 0$; else

$$\det(L) I = a_1 \wedge \cdots \wedge a_r \wedge \cdots \wedge a_n = (a_1 \wedge \cdots \wedge a_r) \wedge \cdots \wedge a_n = 0,$$

a contradiction. □

If the wedge of r vectors is zero, then the vectors must be linearly dependent. We have the following result.

Proposition 1.3.8. *An r-blade $a_1 \wedge \cdots \wedge a_r = 0$ if and only if $\{a_1, \ldots, a_r\}$ is linearly dependent.*

In Chapter 2, Proposition 1.3.8 will allow the correspondence between blades and subspaces to be made.

1.3.3 The wedge of r vectors is an r-blade

We now show that the wedge of r vectors is an r-blade. Recall some basic facts from linear algebra.

Lemma 1.3.9. *Let V be a finite dimensional inner product space. Let T be a self-adjoint linear operator on V. Then there is an orthonormal basis for V each of which is a characteristic vector for T.*

Translating Lemma 1.3.9 into the language of matrices, we have the following.

Corollary 1.3.10. *Let A be an $n \times n$ real symmetric matrix. Then there is an orthogonal matrix P such that $P^T AP$ is diagonal.*

Proposition 1.3.11. *Let $\{a_1, \ldots, a_r\}$ be a set of vectors of $\mathcal{G}^1$. Then*

$$a_1 \wedge \cdots \wedge a_r$$

is an r-blade.

Proof. We follow the strategy of [DL03]. If the set of vectors is linearly dependent then $a_1 \wedge \cdots \wedge a_r = 0$, hence an r-blade. Suppose now that the vectors are linearly independent. Let M be the matrix with entries $B(a_i, a_j)$. Then M is real symmetric. So, there exists an orthogonal matrix $P = (p^i_j)$ such that $P^T MP = D$ is diagonal. Let

$$e_k = p^j_k a_j, \ k = 1, \ldots, r.$$

Then

$$\begin{aligned} B(e_k, e_s) &= B(p^j_k a_j, p^i_s a_i) = p^j_k B(a_j, a_i) p^i_s \\ &= (P^T)_{kj} (M)_{ji} (P)_{is} \\ &= (D)_{ks}. \end{aligned}$$

Hence, with respect to our bilinear form B, the vectors $e_1, \ldots, e_r$ are orthogonal. This means that

$$\begin{aligned} e_1 \cdots e_r &= e_1 \wedge \cdots \wedge e_r \\ &= p_{j_1 1} a_{j_1} \wedge \cdots \wedge p_{j_r r} a_{j_r} \\ &= \sum_{\sigma \in S_r} (-1)^\sigma p_{\sigma(1)1} \cdots p_{\sigma(r)r} a_1 \wedge \cdots \wedge a_r \\ &= \det(P) a_1 \wedge \cdots \wedge a_r. \end{aligned}$$

We used here the anti-symmetry of the wedge product. Since P is an orthogonal matrix, we have $\det(P) \neq 0$. Then

$$a_1 \wedge \cdots \wedge a_r = \det(P)^{-1} e_1 \cdots e_r$$

is an r-blade. □

1.3.4 Left/right contraction

The left/right contraction products might be less familiar operations. The contractions are not associative like the wedge product but an identity will be established connecting the two products.

Definition 1.3.12. Let A_r and B_s be an r and s-blade, respectively. Then the

right contraction is

$$A_r \rfloor B_s = \langle A_r B_s \rangle_{s-r} \text{ (read } A_r \text{ contracted onto } B_s\text{)}$$

and the

left contraction is

$$A_r \lfloor B_s = \langle A_r B_s \rangle_{r-s}.$$

Let a, b be vectors. Then

$$a \rfloor b = \langle ab \rangle_0 = B(a, b).$$

The right contraction between the vector a and b is just the bilinear form B evaluated on them. Furthermore, by definition if $s < r$ then $A_r \rfloor B_s = 0$. In general the right contraction is not a commutative product. We extend the definition of the left/right contraction to multivectors by bilinearity.

Example 1.3.13. Consider $\mathcal{G}(\mathbb{R}^3)$, the geometric algebra from Example 1.1.9. Let $A = e_{12}$ and $a = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$. Observe

$$\begin{aligned} a \rfloor A &= \langle aA \rangle_1 \\ &= \langle (\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3) e_{12} \rangle_1 \\ &= \langle \alpha_1 e_{112} + \alpha_2 e_{212} + \alpha_3 e_{312} \rangle_1 \\ &= \langle \alpha_1 e_2 - \alpha_2 e_1 + \alpha_3 e_{123} \rangle_1 \\ &= -\alpha_2 e_1 + \alpha_1 e_2. \end{aligned}$$

Intuitively, this is a line in the plane A. Furthermore, observe

$$\begin{aligned} a \rfloor (a \rfloor A) &= \langle (\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3)(\alpha_2 e_1 - \alpha_1 e_2) \rangle_0 \\ &= \langle \alpha_1 \alpha_2 e_{11} - \alpha_1^2 e_{12} + \alpha_2^2 e_{21} - \alpha_2 \alpha_1 e_{22} + \alpha_3 \alpha_2 e_{31} - \alpha_3 \alpha_1 e_{32} \rangle_0 \\ &= \langle \alpha_1 \alpha_2 - \alpha_2 \alpha_1 - \alpha_1^2 e_{12} + \alpha_2^2 e_{21} + \alpha_3 \alpha_2 e_{31} - \alpha_3 \alpha_1 e_{32} \rangle_0 \\ &= 0. \end{aligned}$$

We interpret $a \rfloor A$ as the line contained in the plane, orthogonal to the line a.
Consider

$$A \rfloor e_{123} = \langle e_{12123} \rangle_1 = -e_3.$$

A more general interpretation for the intuition of this result is as follows. In the volume represented by e_{123}, the line represented by $-e_3$ is most unlike the plane represented by A.

Proposition 1.3.14. *Let A_r, B_s and C_t be r-,s- and t-blades, respectively. Then*

$$A_r \rfloor (B_s \rfloor C_t) = (A_r \wedge B_s) \rfloor C_t. \tag{1.3.4.1}$$

Proof. We follow [HS84]. By associativity and expanding $(A_r B_s) C_t = A_r (B_s C_t)$ each side with the grade expansion identity (Proposition 1.2.18) we have if $t \geq s + r$

$$\begin{aligned} A_r \rfloor (B_s \rfloor C_t) &= A_r \rfloor \langle B_s C_t \rangle_{t-s} \\ &= \langle A_r \langle B_s C_t \rangle_{t-s} \rangle_{t-s-r} \\ &= \langle \langle A_r B_s \rangle_{s+r} C_t \rangle_{t-s-r} \\ &= \langle (A_r \wedge B_s) C_t \rangle_{t-(r+s)} \\ &= (A_r \wedge B_s) \rfloor C_t \end{aligned}$$

and if $t < s + r$

$$A_r \rfloor (B_s \rfloor C_t) = 0 = (A_r \wedge B_s) \rfloor C_t,$$

since negative grades are zero. □

Identity (1.3.4.1) will be very useful in Chapter 2.

Proposition 1.3.15. *Let A_r and B_s be an r- and s-blade, respectively. Then*

$$A_r \lfloor B_s = (-1)^{r(s-1)} B_s \rfloor A_r \tag{1.3.4.2}$$

Proof. Recall that the reversion map is an involution and grade preserving. Observe

$$\begin{aligned} A_r \lfloor B_s &= \langle A_r B_s \rangle_{r-s} \\ &= (\langle (A_r B_s)^\dagger \rangle_{r-s})^\dagger \\ &= (-1)^{\frac{(r-s)((r-s)-1)}{2}} (-1)^{\frac{r(r-1)}{2}} (-1)^{\frac{s(s-1)}{2}} \langle B_s A_r \rangle_{r-s} \\ &= (-1)^{s(r-1)} B_s \rfloor A_r. \end{aligned}$$

□

1.3.5 Scalar product

We introduce the scalar product to define the magnitude of a blade, which will be used in Chapter 3. The product of two multivectors will be shown to commute under the scalar product. The scalar product is used heavily in [DL03].

Definition 1.3.16. Let A_r and B_s be an r- and s-blade, respectively. Then the **scalar product** is

$$A_r * B_s = \langle A_r B_s^\dagger \rangle_0.$$

We extend the definition of the scalar product to multivectors by bilinearity.

1.3.6 Cyclic permutation of the scalar product

Lemma 1.3.17. *Let A_r and B_r be r-blades, respectively. Then*

$$A_r * B_r = B_r * A_r.$$

Proof. Recall that the reversion map is an involution. Observe

$$\begin{aligned} A_r * B_s &= \langle A_r B_s^\dagger \rangle_0 \\ &= \langle (A_r B_s^\dagger)^\dagger \rangle_0 \\ &= \langle B_s A_r^\dagger \rangle_0 \\ &= B_s * A_r. \end{aligned}$$

□

Proposition 1.3.18. *Let $A, B \in \mathcal{G}$. Then*

$$A * B = B * A.$$

Proof. Let A, B be given by their unique sum

$$A = \sum_r A_r, B = \sum_s B_s,$$

respectively. Note first that $\langle A_r B_s \rangle_0 = 0$ unless $r = s$, by the grade expansion identity (Proposition 1.2.18). By Lemma 1.3.17,

$$\langle AB \rangle_0 = \langle \sum_{r,s} A_r B_s \rangle_0 = \sum_r \langle A_r B_r \rangle_0 = \sum_r \langle B_r A_r \rangle_0 = \sum_{s,r} \langle B_s A_r \rangle_0 = \langle BA \rangle_0.$$

□

Corollary 1.3.19. *Let* $A_1, \ldots, A_n \in \mathcal{G}$. *Then*

$$\langle A_1 A_2 A_3 \cdots A_n \rangle_0 = \langle A_2 A_3 \cdots A_n A_1 \rangle_0. \tag{1.3.6.1}$$

Products of multivectors can be cyclically permuted under the scalar product.

1.3.7 Signed magnitude

For an r-blade A_r with a representation $A_r = a_1 \cdots a_r$,

$$A_r * A_r = \langle A_r A_r^\dagger \rangle_0 = \langle a_1 \cdots a_r a_r \cdots a_1 \rangle_0 = a_1^2 \cdots a_r^2.$$

The scalar product defines in some sense a "magnitude" of a blade. But one must be careful for the square of a factor may be non-positive.

Definition 1.3.20. Let A_r be an r-blade. Then the **(squared) magnitude** of A_r is

$$|A_r|^2 = A_r * A_r.$$

If λ is a scalar

$$|\lambda|^2 = \lambda * \lambda = \lambda^2,$$

if a is a vector

$$|a|^2 = a * a = a^2.$$

In general, for an r-blade,

$$|A_r|^2 = A_r * A_r = \langle A_r A_r^\dagger \rangle_0 = (-1)^{\frac{r(r-1)}{2}} A_r^2. \tag{1.3.7.1}$$

In the case that an r-blade is invertible, we have $|A_r|^2 \neq 0$. So,

$$|A_r|^2 = A_r A_r^\dagger$$

which means

$$A_r^{-1} = \frac{A_r^\dagger}{|A_r|^2} = \frac{(-1)^{\frac{r(r-1)}{2}}}{|A_r|^2} A_r.$$

The inverse of a blade is just a scalar multiple of the blade.

Example 1.3.21. Consider $\mathcal{G}(\mathbb{R}^2)$. Let $a = \alpha_1 e_1 + \alpha_2 e_2,\ b = \beta_1 e_1 + \beta_2 e_2$. Then $a \wedge b = (\alpha_1\beta_2 - \alpha_2\beta_1)e_{12}$. The magnitude of $a \wedge b$ is

$$|a \wedge b|^2 = (a \wedge b)(a \wedge b)^\dagger = (\alpha_1\beta_2 - \alpha_2\beta_1)^2.$$

The squared magnitude of $a \wedge b$, may be interpreted as the square of the Euclidean area of the parallelogram with sides a and b.

With the contraction and wedge products, we may write in our new symbols

$$aA_r = a\rfloor A_r + a \wedge A_r, \tag{1.3.7.2a}$$

$$a\rfloor A_r = \frac{aA_r - (-1)^r A_r a}{2}, \tag{1.3.7.2b}$$

$$a \wedge A_r = \frac{aA_r + (-1)^r A_r a}{2}. \tag{1.3.7.2c}$$

1.4 Additional identities

Proposition 1.4.1. *Let A_r, B_s and a be an r-,s-blade and a vector, respectively. Then*

$$a\rfloor(A_r B_s) = (a\rfloor A_r)B_s + (-1)^r A_r(a\rfloor B_s) \tag{1.4.0.3a}$$

$$= (a \wedge A_r)B_s - (-1)^r A_r(a \wedge B_s), \tag{1.4.0.3b}$$

$$a \wedge (A_r B_s) = (a \wedge A_r)B_s - (-1)^r A_r(a\rfloor B_s) \tag{1.4.0.4a}$$

$$= (a\rfloor A_r)B_s + (-1)^r A_r(a \wedge B_s) \tag{1.4.0.4b}$$

Proof. Identity (1.4.0.3a) will be proven, the others follow similarly. Let $m = \min\{r, s\}$. We follow the strategy of [HS84]. By the grade expansion identity (Proposition 1.2.18), and equation (1.3.7.2b),

$$\begin{aligned}
2a\rfloor(A_rB_s) &= 2a\rfloor\sum_{k=0}^{m}\langle A_rB_s\rangle_{|r-s|+2k} \\
&= 2\sum_{k=0}^{m} a\rfloor\langle A_rB_s\rangle_{|r-s|+2k} \\
&= \sum_{k=0}^{m} a\langle A_rB_s\rangle_{|r-s|+2k} - (-1)^{|r-s|+2k}\langle A_rB_s\rangle_{|r-s|+2k}a \\
&= a\sum_{k=0}^{m}\langle A_rB_s\rangle_{|r-s|+2k} - (-1)^{|r-s|}\sum_{k=0}^{m}\langle A_rB_s\rangle_{|r-s|+2k}a \\
&= aA_rB_s - (-1)^{r+s}A_rB_sa \\
&= (aA_r - (-1)^rA_ra + (-1)^rA_ra)B_s + (-1)^rA_r(-aB_s + aB_s - (-1)^sB_sa) \\
&= 2(a\rfloor A_r)B_s + (-1)^rA_r2(a\rfloor B_s)
\end{aligned}$$

□

Corollary 1.4.2. *Let A_r, B_s and a be an r-,s-blade and a vector, respectively. Then*

$$a\rfloor(A_r \wedge B_s) = (a\rfloor A_r) \wedge B_s + (-1)^r A_r \wedge (a\rfloor B_s) \tag{1.4.0.5}$$

$$a \wedge (A_r\rfloor B_s) = (a\rfloor A_r)\rfloor B_s + (-1)^r A_r\rfloor(a \wedge B_s) \tag{1.4.0.6}$$

Proof. The identities follow from Proposition 1.4.1 by projecting out and collecting the highest and lowest grades. □

We introduce the notion of containment only briefly. Much more will be said in Chapter 2.

Definition 1.4.3. Let A_r, B_s be r, s-blades, respectively. If $a \wedge A_r = 0$ implies $a \wedge B_s = 0$ we write $A_r \subset B_s$ and say that A_r is **contained** in B_s.

Proposition 1.4.4. *Let A_r, B_s and C_t be r-,s- and t-blades, respectively. If $A_r \subset C_t$, then*

$$(A_r\rfloor B_s)\rfloor C_t = A_r \wedge (B_s\rfloor C_t). \tag{1.4.0.7}$$

Proof. We proceed by induction on r. Let $r = 1$. By identity (1.4.0.6),

$$A_1 \wedge (B_s \rfloor C_t) = (A_1 \rfloor B_s) \rfloor C_t + (-1)^s B_s \rfloor (A_1 \wedge C_t) = (A_1 \rfloor B_s) \rfloor C_t.$$

Suppose the statement holds for some fixed r. Let A_{r+1} be an $(r+1)$-blade, with $A_{r+1} \subset C_t$. We may write $A_{r+1} = a \wedge A_r$ where a and A_r is a vector and an r-blade, respectively. Note that $a, A_r \subset C_t$. By the induction hypothesis, base case, and identity (1.3.4.1),

$$\begin{aligned} A_{r+1} \wedge (B_s \rfloor C_t) &= a \wedge A_r \wedge (B_s \rfloor C_t) \\ &= a \wedge ((A_r \rfloor B_s) \rfloor C_t) \\ &= (a \rfloor (A_r \rfloor B_s)) \rfloor C_t \\ &= ((a \wedge A_r) \rfloor B_s) \rfloor C_t \\ &= (A_{r+1} \rfloor B_s) \rfloor C_t. \end{aligned}$$

By the Principle of Mathematical induction the result holds for all $r \in \mathbb{N}$. □

Lemma 1.4.5. *Let $a, a_1, \ldots, a_r \in \mathcal{G}^1$. Then*

$$\frac{1}{2}(aa_1 \cdots a_r - (-1)^r a_1 \cdots a_r a) = \sum_{k=1}^{r} (-1)^{k-1} B_k a_1 \cdots \check{a}_k \cdots a_r$$

where $B_k = B(a, a_k)$.

Proof. We induct on r and use Proposition 1.1.11. Let $r = 1$. By Proposition 1.1.11, the result follows trivially. Suppose the statement holds for some r. By the induction hypothesis and Proposition 1.1.11,

$$
\begin{aligned}
aa_1a_2\cdots a_ra_{r+1} &= (\sum_{k=1}^{r}(-1)^{k-1}2B_ka_1\cdots\check{a}_k\cdots a_r + (-1)^r a_1\cdots a_r a)a_{r+1} \\
&= \sum_{k=1}^{r}(-1)^{k-1}2B_ka_1\cdots\check{a}_k\cdots a_ra_{r+1} + (-1)^r a_1\cdots a_r a a_{r+1} \\
&= \sum_{k=1}^{r}(-1)^{k-1}2B_ka_1\cdots\check{a}_k\cdots a_ra_{r+1} + (-1)^r a_1\cdots a_r(2B_{r+1} - a_{r+1}a) \\
&= \sum_{k=1}^{r+1}(-1)^{k-1}2B_ka_1\cdots\check{a}_k\cdots a_ra_{r+1} + (-1)^{r+1} a_1\cdots a_r a_{r+1} a.
\end{aligned}
$$

By the Principle of Mathematical Induction the result holds for all $r \in \mathbb{N}$. □

Note that the result holds for any product of vectors regardless of commuting or anti-commuting and does not depend on the vectors being invertible.

Proposition 1.4.6 (Reduction Identity)**.** *Let a and A_r be a vector and an r-blade with representation $A_r = a_1\cdots a_r$, respectively. Then*

$$
a\rfloor A_r = \sum_{k=1}^{r}(-1)^{k-1}B(a, a_k)a_1\cdots\check{a}_k\cdots a_r.
$$

Proof. By equation (1.3.7.2b) and Lemma 1.4.5,

$$
a\rfloor A_r = \frac{aA_r - (-1)^r A_r a}{2} = \sum_{k=1}^{r}(-1)^{k-1}B(a, a_k)a_1\cdots\check{a}_k\cdots a_r.
$$

□

Chapter 2

The geometry of blades

This chapter shall discuss properties of the lowest and highest grade of a product of blades, the contraction and wedge. With the wedge product, the notion of containment will be introduced, which will allow the correspondence between blade and subspace. The right contraction, which we shall call just the contraction, will allow us to generalize orthogonality between blades. Much use of our algebraic machinery established in Chapter 1 will take place.

2.1 Wedge product and containment

In this section a correspondence between subspaces and blades is developed.

Definition 2.1.1. Let A_r and B_s be r- and s-blades, respectively. Then A_r is **contained** in B_s, written $A_r \subset B_s$, if $a \wedge A_r = 0$ implies $a \wedge B_s = 0$.

Containment is a transitive relation. That is, if $A_r \subset B_s$ and $B_s \subset C_t$, then $A_r \subset C_t$; for if $a \wedge A_r = 0$ then $a \wedge B_s = 0$ so $a \wedge C_t = 0$.

Example 2.1.2. Consider the geometric algebra $\mathcal{G}(\mathbb{R}^3)$. Let $A = e_{12}$ and suppose that $a \wedge A = 0$. We showed in Example 1.3.2 that this is equivalent to $a \in \langle e_1, e_2 \rangle$. Then $a = \alpha_1 e_1 + \alpha_2 e_2$ for some scalars α_1, α_2. Observe

$$a \wedge e_{123} = \langle (\alpha_1 e_1 + \alpha_2 e_2) e_{123} \rangle_4 = \langle \alpha_1 e_{23} + \alpha_2 e_{31} \rangle_4 = 0.$$

Therefore, $A \subset e_{123}$. Intuitively, e_{123} represents a volume, containing the plane A.

We show that containment in the sense of the wedge product means containment in the sense of the span of a collection of vectors.

Proposition 2.1.3. *Let A_r be a non-zero r-blade with a representation $A_r = a_1 \cdots a_r$. Then $b \wedge A_r = 0$ if and only if $b \in \langle a_1, \ldots, a_r \rangle$.*

Proof. If $b \in \langle a_1, \ldots, a_r \rangle$, then the set $\{b, a_1, \ldots, a_r\}$ is linearly dependent. By Proposition 1.3.8,

$$b \wedge A_r = 0.$$

Conversely, suppose that $b \wedge A_r = 0$. By Proposition 1.3.8, the set $\{b, a_1, \ldots, a_r\}$ is linearly dependent and since $\{a_1, \ldots, a_r\}$ is linearly independent we have

$$b \in \langle a_1, \ldots, a_r \rangle.$$

□

Definition 2.1.4. Let A_r be an r-blade. Then $\mathcal{G}^1(A_r) = \{a \in \mathcal{G}^1 : a \wedge A_r = 0\}$. We shall call $\mathcal{G}^1(A_r)$ the **subspace representation** of A_r.

Corollary 2.1.5. *Let A_r be a non-zero r-blade with a representation $A_r = a_1 \wedge \cdots \wedge a_r$. Then $\mathcal{G}^1(A_r) = \langle a_1, \ldots, a_r \rangle$.*

Proof. This follows immediately from Proposition 2.1.3. □

Given a blade, by Corollary 2.1.5, there is a corresponding subspace. In the next section we show the converse.

2.1.1 Blades represent oriented weighted subspaces

With the notion of containment we now establish a correspondence between a subspace and a blade.

Proposition 2.1.6. *Let H be a subspace of vectors, $\dim H = m$. Then there exists a non-zero m-blade H_m for which $\mathcal{G}^1(H_m) = H$.*

Proof. There exists a basis for H, $\{h_1, \ldots, h_m\}$. By Proposition 1.3.11,

$$H_m = h_1 \wedge \cdots \wedge h_m$$

is an m-blade and is non-zero by Corollary 1.3.8. By Proposition 2.1.3,

$$\mathcal{G}^1(H_m) = H.$$

□

This is a key result, for it says that given a subspace there is a corresponding blade, and that blade is the wedge between the basis elements. This allows one to do algebra with subspaces.

Example 2.1.7. Consider the geometric algebra $\mathcal{G}(\mathbb{R}^3)$ and the subspace $H = \langle e_1, e_2 \rangle$. A blade representing the subspace is $e_1 \wedge e_2 = e_{12}$. With this association, we may interpret the subspace as having an orientation, from e_1 to e_2; and weight given by the area of the parallelogram determined by e_1 and e_2 which is 1.
Consider the subspace $L = \langle e_1 \rangle$. A blade representing the subspace is e_1. With this association, we may interpret the line as having the orientation specified by e_1; and the weight, which is 1. We could also use the blade $-2e_1$. In this instance, the orientation is that of $-e_1$; and the weight is twice of e_1, or 2.

Note the correspondence between the grade of a blade and the dimension of a subspace. It will now be shown that any blade representing a subspace, will have its grade equal to the dimension of the subspace.

Proposition 2.1.8. *Let H be a subspace of vectors,* $\dim H = m$*, and let A_k be a non-zero k-blade. If $\mathcal{G}^1(A_k) = H$, then $k = m$.*

Proof. Suppose that A_k has a representation $A_k = a_1 \wedge \cdots \wedge a_k$ and suppose that $\{h_1, \ldots, h_m\}$ is a basis of H. By our hypothesis,

$$\langle h_1, \ldots, h_m \rangle = \langle a_1, \ldots, a_k \rangle.$$

Since the dimension of a vector space is unique and the $a_k's$ are linearly independent, we conclude that $k = m$. □

We now show that two blades representing the same subspace, are scalar multiples of each other.

Proposition 2.1.9. *If A_m, B_m are non-zero m-blades that represent the same subspace of dimension m, then $B_m = \lambda A_m,\ \lambda \in \mathbb{R}$. Conversely, if $B_m = \lambda A_m$, then $\mathcal{G}^1(B_m) = \mathcal{G}^1(A_m)$.*

Proof. By Proposition 2.1.6, the blades A_m and B_m have the form

$$A_m = h_1 \wedge \cdots \wedge h_m,\ B_m = h'_1 \wedge \cdots \wedge h'_m$$

where $\{h_1, \ldots, h_m\}, \{h'_1, \ldots, h'_m\}$ are bases for the subspace. Then

$$h'_k \in \langle h_1, \ldots, h_m \rangle,\ k = 1, \ldots, m.$$

Therefore, there exist scalars η for which

$$h'_k = \eta_k^s h_s \text{ (sum on } s),\ k = 1, \ldots, m.$$

Hence,

$$\begin{aligned} B_m &= h'_1 \wedge \cdots \wedge h'_m \\ &= \eta_1^{j_1} h_{j_1} \wedge \cdots \wedge \eta_m^{j_m} h_{j_m} \\ &= \sum_{\sigma \in S_n} (-1)^\sigma \eta_1^{\sigma(1)} \cdots \eta_n^{\sigma(n)} h_1 \wedge \cdots \wedge h_m \\ &= \lambda A_m \end{aligned}$$

where

$$\lambda = \sum_{\sigma \in S_n} (-1)^\sigma \eta_1^{\sigma(1)} \cdots \eta_n^{\sigma(n)}.$$

The converse of the statement is trivial to prove. □

If we take A_m in the proposition above to have unit weight, then B_m will have a weight of λ. The wedge product allows the correspondence between blades and subspaces. Even though the correspondence is not unique it is essentially unique up to a scalar multiple. A blade can be thought of as an oriented weighted r dimensional subspace.

2.1.2 Direct sum

We show, under a simple condition, that the wedge of two blades may be interpreted as the direct sum of their representative subspaces. We begin with a vector and blade then generalize to a blade with a blade.

Example 2.1.10. Consider $\mathcal{G}(\mathbb{R}^3)$. Informally, if we interpret e_1 and e_2 as lines and $e_1 \wedge e_2$ representing a plane containing e_1, e_2. We may view $e_1 \wedge e_2$ as the direct sum of the lines e_1, e_2.

Often we do not need a representation for a blade. When this is the case we will use the symbol A_r to denote the collection of its factors; for example we may say $\{a, A_r\}$ is a linearly dependent set, which means that if A_r has a representation $A_r = a_1 \wedge \cdots \wedge a_r$, then $\{a, a_1, \ldots, a_r\}$ is a linearly dependent set.

Lemma 2.1.11. *Let a and A_r be a vector and an r-blade, respectively. Then there exists a non-zero vector $b \subset a, A_r$ if and only if $a \wedge A_r = 0$.*

Proof. Suppose there exists a non-zero vector $b \subset a, A_r$. By Proposition 2.1.3, $b \in \langle a\rangle, \langle A_r\rangle$ Then the set $\{a, A_r\}$ is linearly dependent, so that by Corollary 1.3.8, $a \wedge A_r = 0$.
Conversely, suppose that $a \wedge A_r = 0$. Then $\{a, A_r\}$ is linearly dependent. Suppose that A_r has a representation $A_r = a_1 \wedge \cdots \wedge a_r$. Then there exists scalars $\lambda, \lambda_1, \ldots, \lambda_r$ not all zero for which

$$\lambda a + \lambda_1 a_1 + \cdots + \lambda_r a_r = 0.$$

The set $\{a_1, \ldots, a_r\}$ is linearly independent so $\lambda \neq 0$ and there is a $\lambda_k \neq 0$. Then the vector

$$b = -\lambda a = \lambda_1 a_1 + \cdots + \lambda_r a_r$$

is contained in a and A_r and is non-zero. □

Proposition 2.1.12. *Let a and A_r be a vector and an r-blade, respectively. If $a \wedge A_r \neq 0$, then $\langle a\rangle \oplus \mathcal{G}^1(A_r) = \mathcal{G}^1(a \wedge A_r)$.*

Proof. Since $a \wedge A_r \neq 0$, by Lemma 2.1.11,

$$\langle a\rangle \cap \mathcal{G}^1(A_r) = \{0\}.$$

Let $b \in \mathcal{G}^1(a \wedge A_r)$. Then $b \wedge (a \wedge A_r) = 0$ or $b \in \langle a, A_r \rangle$. If A_r has a representation $A_r = a_1 \cdots a_r$, then $b = \lambda a + \mu_1 a_1 + \cdots \mu_r a_r$. Thus, $b \in \langle a \rangle \oplus \mathcal{G}^1(A_r)$. So,

$$\mathcal{G}^1(a \wedge A_r) \subseteq \langle a \rangle \oplus \mathcal{G}^1(A_r).$$

Conversely, let $v \in \langle a \rangle \oplus \mathcal{G}^1(A_r)$. Then $v = \lambda a + \mu_1 a_1 + \cdots \mu_r a_r$. Therefore, $v \wedge a \wedge A_r = 0$. So, $v \in \mathcal{G}^1(a \wedge A_r)$. Hence,

$$\langle a \rangle \oplus \mathcal{G}^1(A_r) \subseteq \mathcal{G}^1(a \wedge A_r).$$

Thus,

$$\langle a \rangle \oplus \mathcal{G}^1(A_r) = \mathcal{G}^1(a \wedge A_r).$$

□

Let's generalize Proposition 2.1.12. First we give a lemma similar in flavor to Lemma 2.1.11.

Lemma 2.1.13. *Let A_r and B_s be non-zero r- and s-blades, respectively. Then $A_r \wedge B_s = 0$ if and only if there exists a non-zero vector $b \in \mathcal{G}^1(A_r) \cap \mathcal{G}^1(B_s)$.*

Proof. Suppose there exists a non-zero vector $b \in \mathcal{G}^1(A_r) \cap \mathcal{G}^1(B_s)$. Then

$$b \wedge A_r = b \wedge B_s = 0.$$

Suppose that we have a representation

$$A_r = a_1 \wedge \cdots \wedge a_r, \; B_s = b_1 \wedge \cdots \wedge b_s.$$

Then

$$\{b, a_1, \ldots, a_r\}, \; \{b, b_1, \ldots, b_s\}$$

are linearly dependent sets. Hence, there exists scalars $\lambda, \lambda_1, \ldots, \lambda_r, \mu, \mu_1, \ldots, \mu_s$ not all zero for which

$$\lambda b + \lambda_1 a_1 + \cdots + \lambda_r a_r = 0, \tag{2.1.2.1}$$

$$\mu b + \mu_1 b_1 + \cdots + \mu_s b_s = 0. \tag{2.1.2.2}$$

Since

$$\{a_1, \ldots, a_r\},\ \{b_1, \ldots, b_s\}$$

are linearly independent sets, $\mu, \lambda \neq 0$ and there is non-zero λ_k and μ_k. Multiplying (2.1.2.1) and (2.1.2.2) by μ and λ, respectively, then subtracting we obtain

$$\mu\lambda_1 a_1 + \cdots + \mu\lambda_r a_r - \lambda\mu_1 b_1 - \cdots - \lambda\mu_s b_s = 0.$$

Hence, the set

$$\{a_1, \ldots, a_r, b_1, \ldots, b_s\}$$

is linearly dependent. Thus,

$$A_r \wedge B_s = 0.$$

Conversely, suppose that $A_r \wedge B_s = 0$. Then $\{A_r, B_s\}$ is a linearly dependent set. Suppose that A_r and B_s have a representation $A_r = a_1 \wedge \cdots \wedge a_r$ and $B_s = b_1 \wedge \cdots \wedge b_s$, respectively. There exist scalars $\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_s$ not all zero such that

$$\lambda_1 a_1 + \cdots + \lambda_r a_r + \mu_1 b_1 + \cdots + \mu_s b_s = 0.$$

All the λ cannot be simultaneously zero; for B_s is non-zero, hence the factors of the representation are linearly independent. Similarly, all the μ cannot be simultaneously zero. Then the vector

$$b = \lambda_1 a_1 + \cdots + \lambda_r a_r = -\mu_1 b_1 + \cdots + -\mu_s b_s$$

is contained in A_r and B_s and is non-zero. □

Proposition 2.1.14. *Let A_r and B_s be an r- and s-blade, respectively. If $A_r \wedge B_s \neq 0$, then $\mathcal{G}^1(A_r \wedge B_s) = \mathcal{G}^1(A_r) \oplus \mathcal{G}^1(B_s)$.*

Proof. Since $A_r \wedge B_s \neq 0$ by Lemma 2.1.13,

$$\mathcal{G}^1(A_r) \cap \mathcal{G}^1(B_s) = \{0\}.$$

Suppose that A_r and B_s have a representation $A_r = a_1 \wedge \cdots \wedge a_r$ and $B_s = b_1 \wedge \cdots \wedge b_s$. Let $v \in \mathcal{G}^1(A_r \wedge B_r)$. Then $v \wedge (A_r \wedge B_s) = 0$ or $v \in \langle a_1, \ldots, a_r, b_1, \ldots, b_s \rangle$. Therefore,

$$v = \lambda_1 a_1 + \cdots + \lambda_r a_r + \mu_1 b_1 + \cdots + \mu_s b_s \in \mathcal{G}^1(A_r) \oplus \mathcal{G}^1(B_s).$$

Hence,

$$\mathcal{G}^1(A_r \wedge B_s) \subseteq \mathcal{G}^1(A_r) \oplus \mathcal{G}^1(B_s).$$

Let $v \in \mathcal{G}^1(A_r) \oplus \mathcal{G}^1(B_s)$. Then

$$v = \lambda_1 a_1 + \cdots + \lambda_r a_r + \mu_1 b_1 + \cdots + \mu_s b_s,$$

Ergo,

$$v \wedge A_r \wedge B_s = 0.$$

Hence,

$$\mathcal{G}^1(A_r) \oplus \mathcal{G}^1(B_s) \subseteq \mathcal{G}^1(A_r \wedge B_s).$$

Thus,

$$\mathcal{G}^1(A_r) \oplus \mathcal{G}^1(B_s) = \mathcal{G}^1(A_r \wedge B_s).$$

□

Example 2.1.15. Consider the geometric algebra $\mathcal{G}(\mathbb{R}^3)$. Let $a = e_1 + e_3$ and $b = e_2 + e_3$. Let $H_1 = \langle a, b \rangle$ and $H_2 = \langle e_1 \rangle$. Then a blade representation of H_1 and H_2 is $A_2 = a \wedge b$ and $A_1 = e_1$, respectively. Since $A_2 \wedge A_1 = a \wedge b \wedge e_1 = e_{321} \neq 0$. We have $\mathcal{G}^1(A_2) \oplus \mathcal{G}^1(A_1) = \mathcal{G}^1(e_{321}) = \mathcal{G}^1$, the whole vector space.

2.2 Contraction and orthogonality

In this section we show that the contraction of a blade onto another blade is a blade, the notion of orthogonality is generalized to blades. We will further introduce the notion of orthogonal complement.

Definition 2.2.1. Let A_r and B_s be an r- and s-blade, respectively, with $r \leq s$. Then A_r is **orthogonal** to B_s if $A_r \rfloor B_s = 0$.

Note that the blade being contracted onto must have the greater grade.

Example 2.2.2. Consider $\mathcal{G}(\mathbb{R}^3)$. Observe

$$e_1 \rfloor (e_{12}) = \langle e_1 e_{12} \rangle_1 = e_2.$$

Since the line e_1 is contained in the plane e_{12}, the line and plane are not orthogonal. Note that the line obtained from e_1 contracted onto the plane e_{12}, $e_1 \rfloor e_{12}$, is orthogonal to e_1.
Observe

$$e_3 \rfloor e_{12} = \langle e_3 e_{12} \rangle_1 = 0.$$

The line e_3 is orthogonal to the plane e_{12}.
Observe

$$e_{12} \rfloor e_{23} = \langle e_{1223} \rangle_0 = 0.$$

Therefore, the planes e_{12} and e_{23} are orthogonal. Note that the line e_2 is contained in both planes.

We have shown that a vector contained in a blade is in the span of the blade. We should then expect a vector to be orthogonal to a blade precisely when it is orthogonal to each vector in its span.

Proposition 2.2.3. *Let a and A_r be a vector and an r-blade, respectively. Then $a \rfloor b = 0$ for all $b \subset A_r$ if and only if $a \rfloor A_r = 0$.*

Proof. Let A_r have a representation $A_r = a_1 \wedge \cdots \wedge a_r$. Suppose that $b \subset A_r$ implies that $a \rfloor b = 0$. In particular, $a \rfloor a_k = 0,\ k = 1, \ldots, r$. By the reduction identity (Proposition 1.4.6),

$$a \rfloor A_r = \sum_{k=1}^{r} (-1)^{k-1} (a \rfloor a_k) a_1 \wedge \cdots \check{a}_k \cdots \wedge a_r = 0.$$

Conversely, suppose that $a\rfloor A_r = 0$. By the reduction identity (Proposition 1.4.6),

$$0 = a\rfloor A_r = \sum_{k=1}^{r}(-1)^{k-1}(a\rfloor a_k)a_1 \wedge \cdots \check{a}_k \cdots \wedge a_r = 0. \tag{2.2.0.3}$$

Wedge equation (2.2.0.3) with a_1 to obtain

$$0 = (a\rfloor a_1)a_2 \wedge \cdots \wedge a_r \wedge a_1$$

Since $A_r \neq 0$, $a\rfloor a_1 = 0$. Similarly, $a\rfloor a_k = 0, k = 1, \cdots, r$. Now, if $b \subset A_r$, then $b \in \langle a_1, \cdots, a_r\rangle$. Hence, $a\rfloor b = 0$ since a is orthogonal to each of the spanning vectors. □

If a vector is not orthogonal to a blade then there exists a vector in the span of the blade not orthogonal to the vector.

2.2.1 The contraction of blades is a blade

We have shown that $a\rfloor A_r$ is an $(r-1)$-vector, but a stronger statement can be made.

Lemma 2.2.4. *Let a be a vector, A_r an r-blade. Then there exists vectors $b_1, \ldots, b_r$ such that $a\rfloor b_k = 0,\ k \geq 2$ and*

$$A_r = b_1 \wedge \cdots \wedge b_r.$$

Proof. Using an argument from [Chi12], suppose that A_r has a representation $A_r = a_1 \wedge \cdots \wedge a_r$. By the reduction identity (Proposition 1.4.6),

$$a\rfloor A_r = \sum_{k=1}^{r}(a\rfloor a_k)a_1 \wedge \cdots \check{a}_k \cdots \wedge a_r. \tag{2.2.1.1}$$

If a is orthogonal to all but one of the a_k, say, a_1 we have

$$a\rfloor A_r = (a\rfloor a_1)a_2 \wedge \cdots \wedge a_r,$$

an $(r-1)$-blade. Suppose after a suitable rearrangement of the given factors of A_r, that a is orthogonal to $a_{k+1}, \cdots, a_r$ and not orthogonal to $a_1, \cdots, a_k$, so that $a\rfloor a_1 \neq 0$. Let

$$a'_j = a\rfloor(a_1 \wedge a_j)(a\rfloor a_1)^{-1}, j = 2, \cdots, k \tag{2.2.1.2}$$

Then

$$
\begin{aligned}
a \rfloor a_j' &= a \rfloor (a \rfloor (a_1 \wedge a_j)(a \rfloor a_1)^{-1}) \\
&= (a \wedge a) \rfloor ((a_1 \wedge a_j)(a \rfloor a_1)^{-1}) \\
&= 0,
\end{aligned}
$$

so a is orthogonal to a_j' for $j = 2, \cdots, k$. By the reduction identity (Proposition 1.4.6) equation (2.2.1.2) may be rewritten to obtain,

$$a_j = a_j' + a_1 a \rfloor a_j (a \rfloor a_1)^{-1}, j = 2, \cdots, k.$$

Then

$$
\begin{aligned}
A_r &= a_1 a_2 \cdots a_k a_{k+1} \cdots a_r \\
&= a_1 \wedge a_2 \wedge \cdots \wedge a_k \wedge a_{k+1} \wedge \cdots \wedge a_r \\
&= a_1 \wedge (a_2' + a_1 a \rfloor a_2 (a \rfloor a_1)^{-1}) \wedge \cdots (a_k' + a_1 a \rfloor a_k (a \rfloor a_1)^{-1}) \wedge a_{k+1} \wedge \cdots \wedge a_r \\
&= a_1 \wedge \cdots \wedge a_2' \wedge \cdots \wedge a_k' \wedge a_{k+1} \wedge \cdots \wedge a_r.
\end{aligned}
$$

By replacing this factorization for A_r into equation (2.2.1.1), we have returned to the initial case. □

Corollary 2.2.5. *Let A_r and B_s be r- and s-blades, respectively, $r \leq s$. Then there exist vectors $b_1, \ldots, b_{s-r}$ such that*

$$b_k \wedge B_s = 0, \ k = 1, \ldots, s - r,$$

and

$$A_r \rfloor B_s = \lambda b_1 \wedge \cdots \wedge b_{s-r}, \ \textit{for some } \lambda \in \mathbb{R}.$$

Proof. We proceed by induction on r. By Lemma 2.2.4, the statement holds for $r = 1$. Suppose the statement holds for all $(r-1)$-blades. Let A_r be an r-blade. We may write $A_r = a \wedge A_{r-1}$ where a is a vector and A_{r-1} is an $(r-1)$-blade. By the induction hypothesis there exist vectors $b_1, \ldots, b_{s-(r-1)}$ such that

$$b_k \wedge B_s = 0,\ k = 1, \ldots, s-(r-1)$$

and

$$A_{r-1} \rfloor B_s = \lambda b_1 \wedge \cdots \wedge b_{s-(r-1)}.$$

Then

$$A_r \rfloor B_s = (a \wedge A_{r-1}) \rfloor B_s = a \rfloor (A_{r-1} \rfloor B_s) = \lambda a \rfloor (b_1 \wedge \cdots \wedge b_{s-(r-1)}).$$

By Lemma 2.2.4, there exist vectors $b'_1, \ldots, b'_{s-(r-1)}$ such that

$$b'_k \wedge (A_{r-1} \rfloor B_s) = 0,\ k = 1, \ldots, s-(r-1)$$

and

$$\lambda a \rfloor (b_1 \wedge \cdots \wedge b_{s-(r-1)}) = \lambda (a \rfloor b'_1) b'_2 \wedge \cdots \wedge b'_{s-(r-1)}$$

that is

$$A_r \rfloor B_s = \lambda (a \rfloor b'_1) b'_2 \wedge \cdots \wedge b'_{s-(r-1)}.$$

Finally, since

$$b'_1, \ldots, b'_{s-(r-1)} \in \langle b_1, \ldots, b_{s-(r-1)} \rangle$$

and

$$b_1, \ldots, b_{s-(r-1)} \in \langle B_s \rangle$$

we may conclude that

$$b'_1, \ldots, b'_{s-(r-1)} \in \langle B_s \rangle$$

or

$$b'_k \wedge B_s = 0,\ k = 1, \ldots, s-(r-1).$$

□

Corollary 2.2.6. *Let a and A_r be a vector and an r-blade, respectively. Then $a \rfloor A_r$ is an $(r-1)$-blade.*

Proof. This follows directly from Lemma 2.2.4 and the reduction identity (Proposition 1.4.6).

□

2.2.2 Orthogonal complement

Recall that $A_r \rfloor B_s$ can be thought of as the geometric object contained in B_s, that is most unlike A_r, "most unlike" in this context meaning, orthogonal. This is a generalized notion of the orthogonal complement.

Proposition 2.2.7. *Let a be a vector and A_r a non-zero r-blade, $r \neq 1$. Then*

$$a \rfloor A_r \subset A_r$$

and

$$a \rfloor (a \rfloor A_r) = 0.$$

Proof. By Lemma 2.2.4 there exists vectors $a_1, \ldots, a_r$ such that

$$A_r = a_1 \wedge \cdots \wedge a_r$$

and

$$a \rfloor A_r = (a \rfloor a_1) a_2 \wedge \cdots \wedge a_r.$$

Suppose that $b \wedge (a \rfloor A_r) = 0$. Then

$$b \in \langle a_2, \ldots, a_r \rangle \subset \langle a_1, \ldots, a_r \rangle.$$

Hence,

$$b \wedge A_r = 0.$$

Thus,

$$a \rfloor A_r \subset A_r.$$

Finally, by identity (1.3.4.1),

$$a \rfloor (a \rfloor A_r) = (a \wedge a) \rfloor A_r = 0.$$

□

Note that the restriction on $r \neq 1$ in Proposition 2.2.7 is essential. Suppose that $a \rfloor A_1 \neq 0$.

Then

$$(a\rfloor A_1) \wedge A_1 = \langle (a\rfloor A_1)A_1 \rangle_{1+0} = (a\rfloor A_1)A_1 \neq 0$$

and we do not have containment.

Let us generalize to the contraction of two blades. We shall see that Proposition 2.2.7 generalizes for blades. We show first that $A_r\rfloor B_s$ is an $(s-r)$-blade, for an r- and s-blade A_r and B_s, respectively.

Proposition 2.2.8. *Let A_r and B_s be an r- and s-blade, respectively. Then $A_r\rfloor B_s$ is an $(s-r)$-blade.*

Proof. This follows directly from Corollary 2.2.5.

□

Proposition 2.2.9. *Let A_r and B_s be r- and s-blades, respectively, $r \leq s$. Then*

$$A_r\rfloor B_s \subset B_s$$

and

$$A_r\rfloor (A_r\rfloor B_s) = 0.$$

Proof. By Corollary 2.2.5,

$$A_r\rfloor B_s = \lambda b_1 \wedge \cdots \wedge b_{s-r}$$

where

$$b_k \wedge B_s = 0,\ k = 1, \ldots, s-r.$$

We shall show containment. Suppose that

$$b \wedge (A_r\rfloor B_s) = 0.$$

Then

$$b \in \langle b_1, \ldots, b_{s-r} \rangle \subset \langle B_s \rangle$$

or

$$b \wedge B_s = 0.$$

Thus,

$$A_r \rfloor B_s \subset B_s.$$

By equation 1.3.4.1,

$$A_r \rfloor (A_r \rfloor B_s) = (A_r \wedge A_r) \rfloor B_s = 0.$$

□

2.3 Geometric, wedge and contraction relations

In this section we find conditions for when the geometric product of two blades contains only the highest or lowest possible grade in the resulting multiplication.

Proposition 2.3.1. *Let A_r and B_s be r- and s-blades, respectively.*

i) If for all $a \subset A_r$, $a \rfloor B_s = 0$ then $A_r B_s = A_r \wedge B_s$.

ii) If A_r is invertible, then the converse of i) holds.

Proof. To show (i), we proceed by induction on r. When $r = 1$ the result follows from equation (1.3.7.2a). Suppose the statement holds for all r-blades. Let A_{r+1} be an $(r+1)$-blade for which $a \subset A_{r+1}$ implies that $a \rfloor B_s = 0$. We may write $A_{r+1} = bA_r$ where b is a vector and A_r is an r-blade. Since $b, A_r \subset A_{r+1}$, $b \rfloor B_s = 0$ and if $a \subset A_r$ then $a \rfloor B_s = 0$ since $A_r \subset A_{r+1}$. By the induction hypothesis, the base case and identity (1.4.0.5)

$$\begin{aligned} A_{r+1} B_s &= a A_r B_s \\ &= a(A_r \wedge B_s) \\ &= a \rfloor (A_r \wedge B_s) + a \wedge A_r \wedge B_s \\ &= (a \rfloor A_r) \wedge B_s + (-1)^r A_r \wedge (a \rfloor B_s) + A_{r+1} \wedge B_s \\ &= A_{r+1} \wedge B_s. \end{aligned}$$

Conversely, to show (ii), suppose that A_r is invertible and $A_r B_s = A_r \wedge B_s$. Suppose that $a \subset A_r$. We will show that $a \rfloor B_s = 0$. We have

$$a A_r = a \rfloor A_r.$$

Consider

$$\begin{aligned} a\rfloor B_s &= \langle aB_s\rangle_{s-1} \\ &= \langle (a\rfloor A_r)A_r^{-1}B_s\rangle_{s-1} \\ &= \langle (a\rfloor A_r)A_r^{-1}\wedge B_s\rangle_{s-1} \\ &= 0 \end{aligned}$$

for the lowest grade is $r+s-(r-1)=s+1$ by the grade expansion identity. □

Remark. If A_r is not invertible we are not guaranteed property ii); Consider e_1, e_2, e_3 orthogonal vectors with $e_1^2 = e_2^2 = 1$ and $e_3^2 = 0$. Let $A_2 = e_1e_3, B_3 = e_3e_2e_1$. Since A_2 has a null vector it is not invertible. Observe

$$A_2B_3 = 0 = A_2 \wedge B_3$$

and $e_1 \subset A_2$ while

$$e_1\rfloor B_3 = \langle e_1e_3e_2e_1\rangle_2 = e_3e_2 \neq 0.$$

Proposition 2.3.2. *Let A_r and B_s be r- and s-blades, respectively.*

i) If $A_r \subset B_s$, then $A_rB_s = A_r\rfloor B_s$.

ii) If A_r or B_s is invertible, then the converse of i) holds.

Proof. To show (i), we proceed by induction on r. When $r = 1$ the result is trivial. Suppose the statement holds for all r-blades. Consider A_{r+1} an $(r+1)$-blade for which $A_{r+1} \subset B_s$. We may write $A_{r+1} = aA_r$ where A_r is an r-blade and a is a vector. Note that since $A_{r+1} \subset B_s$, then $a, A_r \subset B_s$ and we may use identity (1.4.0.7). Observe

$$\begin{aligned} A_{r+1}B_s &= a(A_rB_s) \\ &= a(A_r\rfloor B_s) \\ &= a\rfloor(A_r\rfloor B_s) + a\wedge(A_r\rfloor B_s) \\ &= (a\wedge A_r)\rfloor B_s + (a\rfloor A_r)\rfloor B_s \\ &= (aA_r)\rfloor B_s \\ &= A_{r+1}\rfloor B_s. \end{aligned}$$

Conversely, to show (ii), suppose that A_r is invertible and $a \wedge A_r = 0$. We shall show that $a \wedge B_s = 0$. By our hypothesis, we may write

$$a = (a \rfloor A_r) A_r^{-1}.$$

Then

$$\begin{aligned} a \wedge B_s &= \langle a B_s \rangle_{s+1} \\ &= \langle a A_r^{-1} A_r B_s \rangle_{s+1} \\ &= \langle (a \rfloor A_r^{-1})(A_r \rfloor B_s) \rangle_{s+1} \\ &= 0 \end{aligned}$$

since the highest grade will be, by the grade expansion identity, $(s - r) + (r - 1) = s - 1$. Suppose now that B_s is invertible. Then

$$A_r = (A_r \rfloor B_s) B_s^{-1}$$

since B_s^{-1} is a scalar multiple of B_s we have

$$A_r = (A_r \rfloor B_s) \rfloor B_s^{-1} \subset B_s.$$

□

2.4 Duality

We introduce the notion of dual when $\mathcal{G}^1$ has dimension n. It is shown that the subspace $\mathcal{G}^r$ is isomorphic to $\mathcal{G}^{n-r}$.
In Theorem 4.1.8 of the appendix, we show that

$$\mathcal{G}^n = \langle e_1 \wedge \cdots \wedge e_n \rangle$$

where

$$\{e_1, \ldots, e_n\}$$

is an orthogonal basis of non-null vectors squaring to ± 1. Note that $\mathcal{G}^n$ has the same dimension as $\mathcal{G}^0$.

Definition 2.4.1. Let $I = e_1 \wedge \cdots \wedge e_n$. Let $A \in \mathcal{G}$. Then the **dual** of A with respect to I is given by

$$A^* = AI.$$

When the context is clear we just say A^* is the dual of A.

We will now show that $\mathcal{G}^r$ and $\mathcal{G}^{n-r}$ are dual to each other.

Proposition 2.4.2. *The subspaces $\mathcal{G}^r$ and $\mathcal{G}^{n-r}$ are canonically isomorphic as vector spaces.*

Proof. It is straightforward to show $*$ is a linear map on $\mathcal{G}$. Since each factor of I is invertible, I is invertible. Therefore, $*$ is invertible and defines a linear isomorphism of $\mathcal{G}$. We will show that $(\mathcal{G}^r)^* = \mathcal{G}^{n-r}$. By Corollary 2.1.5,

$$\mathcal{G}^1(I) = \langle e_1, \ldots, e_n \rangle = \mathcal{G}^1.$$

Let A_r be an r-blade. If $a \wedge A_r = 0$, then $a \wedge I = 0$ since $a \in \langle e_1, \ldots, e_n \rangle$. Therefore, $A_r \subset I$. By Proposition 2.3.2,

$$A_r^* = A_r I = A_r \rfloor I,$$

so A_r^* is an $(n-r)$-blade.

Let A_{n-r} be an $(n-r)$-blade, then $A_{n-r} I^{-1} = A_{n-r} \rfloor I^{-1}$ is an r-blade and

$$(A_{n-r} I^{-1})^* = A_{n-r}.$$

Thus,

$$\mathcal{G}^r \cong \mathcal{G}^{n-r}.$$

□

Let us now characterize relationships between a blade and its dual. Let A_r be an r-blade such that $A_r^2 \neq 0$. Then

$$A_r \wedge A_r^* = \langle A_r A_r I \rangle_n = A_r^2 I \neq 0.$$

By Proposition 2.1.14,

$$\mathcal{G}^1(A_r) \oplus \mathcal{G}^1(A_r^*) = \mathcal{G}^1(A_r \wedge A_r^*) = \mathcal{G}^1(A_r^2 I) = \mathcal{G}^1.$$

The direct sum of the subspaces corresponding to A_r and its dual is the whole vector space, when $A_r^2 \neq 0$. Furthermore,

$$A_r \rfloor A_r^* = \langle A_r A_r I \rangle_{n-2r} = A_r^2 \langle I \rangle_{n-2r} = 0.$$

A blade and its dual are orthogonal.

Proposition 2.4.3. *(Duality relationships) Let A_r and B_s be an r- and s-blade, respectively. Then*

$$(A_r \wedge B_s)^* = A_r \rfloor B_s^* \qquad (2.4.0.1)$$

$$(A_r \rfloor B_s)^* = A_r \wedge B_s^*. \qquad (2.4.0.2)$$

Proof. Observe

$$(A_r \wedge B_s)^* = (A_r \wedge B_s) \rfloor I = A_r \rfloor (B_s \rfloor I) = A_r \rfloor B_s^*$$

and since $A_r \subset I$, we have by identity (1.4.0.7)

$$(A_r \rfloor B_s)^* = (A_r \rfloor B_s) \rfloor I = A_r \wedge (B_s \rfloor I) = A_r \wedge B_s^*.$$

□

With the use of the dual we see how the contraction and the wedge are dual to each other. Let us show how a statement about the wedge product can be used to establish a statement about the contraction product with the use of the duality relations.

Proposition 2.4.4. *Let A_r and B_s be an r- and s-blade, respectively, $r \leq s$. Then $A_r \rfloor B_s = 0$ if and only if there exists a non-zero vector $a \subset A_r$ such that $a \rfloor B_s = 0$.*

Proof. By Lemma 2.1.13 and identity (2.4.0.2),

$$\begin{aligned} A_r \rfloor B_s = 0 &\Leftrightarrow\ A_r \wedge B_s^* = 0 \\ &\Leftrightarrow\ a \in \mathcal{G}^1(A_r) \cap \mathcal{G}^1(B_s^*) \setminus \{0\} \\ &\Leftrightarrow\ a \wedge A_r = 0 \text{ and } a \wedge B_s^* = 0 \\ &\Leftrightarrow\ a \subset A_r \text{ and } a \rfloor B_s = 0. \end{aligned}$$

□

Note though that it need not be the case that all vectors contained in A_r are orthogonal to B_s; consider the geometric algebra $\mathcal{G}(\mathbb{R}^3)$. Let $A = e_1 e_2$ and $B = e_2 e_3$. Then

$$A \rfloor B = \langle e_1 e_2 e_2 e_3 \rangle_0 = 0$$

while $e_2 \subset A$ but

$$e_2 \rfloor B = \langle e_2 e_2 e_3 \rangle_1 = e_3 \neq 0.$$

We now show that a vector orthogonal to a blade is orthogonal to any blade contained in that blade.

Proposition 2.4.5. *If $A_r \subset B_s$ and $b \rfloor B_s = 0$, then $b \rfloor A_r = 0$.*

Proof. Suppose A_r has a representation $A_r = a_1 \wedge \cdots \wedge a_r$. Since $A_r \subset B_s$, $a_1, \ldots, a_r \in \langle B_s \rangle$. By Proposition 2.2.3, b is orthogonal to the $a_k's$. Again by Proposition 2.2.3, b is orthogonal to A_r. □

We now show that the dual reverses inclusion.

Proposition 2.4.6. *If $A_r \subset B_s$, then $B_s^* \subset A_r^*$.*

Proof. Let $b \subset B_s^*$. By identity (2.4.0.1),

$$\begin{aligned} b \wedge B_s^* = 0 &\Rightarrow\ b \rfloor B_s = 0 \\ &\Rightarrow\ b \rfloor A_r = 0 \\ &\Rightarrow\ b \wedge A_r^* = 0. \end{aligned}$$

So, $b \subset A_r^*$. Thus,

$$B_s^* \subset A_r^*.$$

□

2.5 Projection operator

In this section we show how to project blades onto other blades.

Definition 2.5.1. Let B_s be an invertible s-blade. Given an r-blade A_r the **projection** of A_r onto B_s is

$$P_{B_s}(A_r) = (A_r \rfloor B_s) B_s^{-1}.$$

Since $A_r \rfloor B_s \subset B_s$, by Proposition 2.3.2,

$$P_{B_s}(A_r) = (A_r \rfloor B_s) \rfloor B_s^{-1}.$$

We may interpret the projection as the composition of two contractions onto the same space. The projection of A_r onto B_s, is then the object most unlike $A_r \rfloor B_s$ in B_s or the object most like A_r in B_s. When discussing projections, the blade being projected onto is assumed to be invertible.

Example 2.5.2. Consider $\mathcal{G}(\mathbb{R}^3)$. Let $a = \alpha^k e_k,\ A = e_{12}$. Observe,

$$P_A(a) = (a \rfloor A) A^{-1} = (\alpha^k e_k) \rfloor e_{12} = (\alpha^1 e_2 - \alpha^2 e_1) \rfloor e_{21} = \alpha^1 e_1 + \alpha^2 e_2.$$

The projection thus projected the line a onto a line in the plane A. If $a = \alpha^3 e_3$, then $P_A(a) = 0$. If $\alpha^3 = 0$, then $P_A(a) = a$. We may interpret this as geometric objects in the plane are left fixed by P_A, while geometric objects orthogonal to the plane are mapped to zero.

The projection operator is now shown to have the properties of a projection.

Proposition 2.5.3. *Let A_r and B_s be r- and s-blades, respectively. Then*

(i) If $A_r \subset B_s$, then $P_{B_s}(A_r) = A_r$.

(ii) *If* $A_r \rfloor B_s = 0$*, then* $P_{B_s}(A_r) = 0$.

(iii) $P^2_{B_s}(A_r) = P_{B_s}(A_r)$, *(*$P_{B_s}$ *is idempotent)*

(iv) $\langle P_{B_s}(A) \rangle_r = P_{B_s}(\langle A \rangle_r)$, $A \in \mathcal{G}$.

Proof. (i) If $A_r \subset B_s$, then by Proposition 2.3.2, $A_r B_s = A_r \rfloor B_s$, since B_s is invertible. Therefore,

$$P_{B_s}(A_r) = (A_r \rfloor B_s)B_s^{-1} = A_r B_s B_S^{-1} = A_r.$$

(ii) This is immediate from the definition of P_{B_s}.

(iii) Since $P_{B_s}(A_r) \subset B_s$, by Proposition 2.3.2,

$$P_{B_s}(A_r) \rfloor B_s = P_{B_s}(A_r)B_s.$$

Observe,

$$P^2_{B_s}(A_r) = P_{B_s}(P_{B_s}(A_r)) = (P_{B_s}(A_r) \rfloor B_s)B_s^{-1} = P_{B_s}(A_r)B_s B_s^{-1} = P_{B_s}(A_r).$$

(iv) This follows from linearity of the projection operator and the direct sum decomposition.

□

Chapter 3

Examples with geometric algebra

The language of geometric algebra is used to discuss analytic geometry, namely lines and planes; reflections and rotations; the Kepler problem; and finding components of vectors.

3.1 Lines and planes

In this section the wedge product is used to describe lines and planes. The geometric algebra $\mathcal{G}(\mathbb{R}^n)$, will be used to model n-dimensional Euclidean space.

3.1.1 Lines

We follow [Hes99]. In this subsection it will be shown that a line is uniquely specified by a vector u and a bivector M such that $u \wedge M = 0$. The vector specifies the direction of the line and the bivector specifies the angular momentum of the line.
Let u be a fixed, non-zero, vector and let x be a vector. By Proposition 1.3.8,

$$x \wedge u = 0$$

is equivalent to

$$x = \lambda u, \ \lambda \in \mathbb{R}.$$

The equation

$$x \wedge u = 0 \tag{3.1.1.1}$$

therefore describes a line passing through the origin. A line passing through $a \neq 0$, is given by

$$(x-a)\wedge u = 0 \tag{3.1.1.2}$$

or

$$x = a + \lambda u. \tag{3.1.1.3}$$

Equation (3.1.1.3) gives the familiar parameterized form of a line passing through some point offset from the origin, while equation (3.1.1.2) describes the line by an equation (i.e. implicitly).

Let $M = a \wedge u$. We may write (3.1.1.2) as

$$x \wedge u = M. \tag{3.1.1.4}$$

Since equation (3.1.1.2) is homogeneous, we may suppose that u is of unit magnitude so that $u^{-1} = u$. Since

$$xu = x\rfloor u + x \wedge u = x\rfloor u + M,$$

we obtain

$$x = (\alpha + M)u,\ \ \alpha = x\rfloor u.$$

Let $d = Mu = M\lfloor u$. Since $du = Muu = M$, we obtain

$$d \wedge u = M \text{ and } d\rfloor u = 0.$$

This means that d is contained on the line and is orthogonal to u. We may write

$$x = \alpha u + d$$

and observe that

$$x^2 = \alpha^2 + d^2$$

is minimal when $\alpha = 0$. Hence, d is on the line and of minimal distance from the origin. In the literature, d is called the *directance* for 'directed distance' and the bivector M, is called the *moment* of the line. The name *moment* may be made clearer with the following example. Suppose a particle of unit mass has position x given by equation (3.1.1.3) with parameter λ

representing time. Then the velocity of the particle is $v = u$. The moment of momentum, or angular momentum, is then $x \wedge v = (a + \lambda u) \wedge u = a \wedge u$, which is precisely the bivector M as given in equation (3.1.1.4).

If we are given a bivector M and a vector u, with the condition that $u \wedge M = 0$, we may uniquely form a line by defining the set of points satisfying

$$x = (\alpha + M)u, \ \alpha \in \mathbb{R}.$$

We see

$$(x - Mu) \wedge u = \alpha u \wedge u = 0$$

is a line with direction u, passing through the point Mu.

3.1.2 Planes

Let U be a fixed, non-zero, bivector and x a vector. The equation of a plane passing through the point a is given by

$$(x - a) \wedge U = 0 \tag{3.1.2.1}$$

Since equation (3.1.2.1) is homogeneous we may suppose that U has unit magnitude, so that $U^2 = -1$. We may express this in a more familiar form. Suppose that U has a representation $U = u_1 u_2$. Then equation (3.1.2.1) is equivalent to

$$(x - a) \in \langle u_1, u_2 \rangle$$

or

$$x = a + \alpha_1 u_1 + \alpha_2 u_2, \ \alpha_1, \alpha_2 \in \mathbb{R}.$$

Let $T = a \wedge U$. Suppose that a is not contained in U, so that T is a nonzero trivector. We may write equation (3.1.2.1) as

$$x \wedge U = T.$$

Let $d = TU^{-1} = U^{-1} \rfloor T$.

Since

$$dU = TU^{-1}U = T,$$

we obtain

$$d \wedge U = T \text{ and } d \rfloor U = 0.$$

This means that d is contained on the plane and is orthogonal to U. We solve for x,

$$xU = x \rfloor U + x \wedge U = x \rfloor U + T.$$

Hence,

$$x = (x \rfloor U)U^{-1} + TU^{-1} = (x \rfloor U)U^{-1} + d.$$

The quantity

$$(x \rfloor U)U^{-1}$$

is the projection of x into the bivector U. We have

$$x^2 = ((x \rfloor U)U^{-1})^2 + d^2$$

which is a minimum when

$$(x \rfloor U)U^{-1} = 0.$$

Hence, d is on the plane and of minimal distance from the origin. The vector d is called the directance.

3.1.3 The point of intersection of a line and a plane

Let $n = 3$. Then given a 3-blade I and a vector z, by Theorem 4.1.8 in the appendix,

$$z \wedge I = \langle zI \rangle_4 = 0. \tag{3.1.3.1}$$

Suppose that we have a line and plane given by

$$(x - a) \wedge u = 0, \ (y - b) \wedge U = 0$$

for fixed vectors a, b, u and bivector U and $u, U \neq 0$. Suppose there exists a point of intersection p, i.e.

$$(p - a) \wedge u = 0 \ , \ (p - b) \wedge U = 0$$

Let us determine the point. Define

$$z = p - a.$$

Then we have from our relations

$$z \wedge u = 0, \ (z + a - b) \wedge U = 0$$

or

$$z \wedge u = 0, \ z \wedge U = (b - a) \wedge U.$$

By equations (3.1.3.1) and (1.4.0.5),

$$\begin{aligned}
z(u \wedge U) &= z\rfloor(u \wedge U) \\
&= (z\rfloor u) \wedge U - u \wedge (z\rfloor U) \\
&= (z\rfloor u)U - u \wedge (z\rfloor U) \\
&= zuU - u \wedge (z\rfloor U) \\
&= uzU - u \wedge (z\rfloor U) \\
&= u(z\rfloor U + z \wedge U) - u \wedge (z\rfloor U) \\
&= u\rfloor(z\rfloor U) + u \wedge (z\rfloor U) + u(z \wedge U) - u \wedge (z\rfloor U) \\
&= (u \wedge z)\rfloor U + u(z \wedge U) \\
&= ((b - a) \wedge U)u
\end{aligned}$$

Hence,

$$(p - a)(u \wedge U) = z(u \wedge U) = ((b - a) \wedge U)u \tag{3.1.3.2}$$

From analytical geometry we would expect either 0, 1 or infinitely many solutions. Let us examine the possible cases:
Case 1: suppose that $a = b$.
If $u \wedge U = 0$, then the line, u, is contained in the plane, U, and we should expect infinitely many solutions. Indeed, by equation 3.1.3.2,

$$(p - a)0 = 0.$$

This suggests that there are infinitely many solutions.
If $u \wedge U \neq 0$, then the line is passing through the plane at $a = b$ and we should expect one point of intersection, namely $a = b$. Indeed, since $u \wedge U$ is invertible, by equation 3.1.3.2,

$$p = a.$$

Case 2: suppose that $a \neq b$.
If $u \wedge U = 0$, then when translated to the origin the line is contained in the plane. Then either $(b - a) \wedge U = 0$ or $(b - a) \wedge U \neq 0$. If $(b - a) \wedge U = 0$, then a and b are on the plane and there should be infinitely many solutions. Indeed, by equation 3.1.3.2,

$$(p - a)0 = 0.$$

This suggests that there are infinitely many solutions. If $(b - a) \wedge U \neq 0$, then a is not on the plane and there should be no solution. Indeed, since u and $(b - a) \wedge U$ are invertible, if such a solution existed, by equation 3.1.3.2,

$$0 = 1,$$

a contradiction.
If $u \wedge U \neq 0$, then the line is not in the plane and we should expect one solution. Indeed, by equation 3.1.3.2,

$$p = \frac{((b - a) \wedge U)u}{u \wedge U} + a$$

which is a unique point.

3.2 The Kepler Problem

We follow [Hes99] and [DL03]. The Kepler Problem is to determine the orbit given the inverse square law of gravitation. Before we begin to solve for the orbit, let us recall a definition of a conic section.
Let $d = \delta \hat{e}$, $\hat{e}^2 = 1$, be the directance of some plane offset from the origin. A point

$r = |r|\hat{r},\ \hat{r}^2 = 1$, is on the conic section if its magnitude satisfies the relation

$$\frac{|r|}{\delta - r \rfloor \hat{e}} = \varepsilon,$$

this is known as the polar form of a conic section and the quantity $\hat{r} \rfloor e = e \cos\theta$. The number ε is called the *eccentricity*. Let $e = \varepsilon\hat{e}$. We may solve for $|r|$,

$$|r| = \frac{\varepsilon\delta}{1 + \hat{r} \rfloor e}. \tag{3.2.0.3}$$

The number $\varepsilon\delta$, is the *semi-latus rectum*. We shall show that the orbit satisfies (3.2.0.3). Suppose that we are given a force

$$F = -\frac{k}{|r|^3} r,$$

k is a proportionality constant. By Newton's second law,

$$m\ddot{r} = m\dot{v} = -\frac{k}{|r|^3} r.$$

Define the angular momentum by

$$L = mr \wedge v.$$

Note that $v \wedge L = r \wedge L = 0$, i.e., v, r are contained in the plane L. (Therefore, all the dynamics takes place on the plane L.) Observe

$$\begin{aligned} \dot{L} &= m\dot{r} \wedge v + mr \wedge \dot{v} \\ &= mr \wedge -\frac{k}{m|r|^3} r \\ &= 0, \end{aligned}$$

and so L is conserved. Since L is a 2-blade, this means that neither the orientation, weight nor attitude in space changes. Let

$$\hat{r} = \frac{r}{|r|}.$$

Then

$$\hat{r} \cdot \hat{r} = 1.$$

After differentiation we obtain

$$2\hat{r} \cdot \dot{\hat{r}} = 0.$$

Hence,

$$\hat{r}\dot{\hat{r}} = -\dot{\hat{r}}\hat{r}.$$

Since

$$v = \dot{r} \text{ and } r = |r|\hat{r}$$

we have

$$v = \dot{|r|}\hat{r} + |r|\dot{\hat{r}}.$$

The angular momentum is

$$L = mr \wedge v = m(|r|\hat{r}) \wedge (\dot{|r|}\hat{r} + |r|\dot{\hat{r}}) = m|r|^2\hat{r} \wedge \dot{\hat{r}} = m|r|^2\hat{r}\dot{\hat{r}}.$$

Then

$$L\dot{v} = -L\frac{k}{m|r|^3}r = -m|r|^2\hat{r}\dot{\hat{r}}\frac{k}{m|r|^3}r = k\dot{\hat{r}}.$$

Since L, k are constants,

$$\frac{d}{dt}(Lv - k\hat{r}) = 0.$$

Then

$$Lv - k\hat{r} = ke$$

or

$$Lv = k(\hat{r} + e)$$

where e is a constant vector. In the literature, e is called the *eccentricity vector*. The scalar part of the equation

$$L(vr) = k(\hat{r} + e)r$$

is

$$L(v \wedge r) = k(\hat{r} + e)\lfloor r.$$

Then

$$k(|r| + r\rfloor e) = \frac{LL^\dagger}{m}$$

or

$$|r| = \frac{\frac{|L|^2}{mk}}{1 + \hat{r} \rfloor e}.$$

This is the equation of a conic with eccentricity $\epsilon = |e|$ and semi-latus rectum $\frac{|L|^2}{mk}$.
As is known the energy, E, is conserved and is given by

$$E = \frac{1}{2}mv^2 - \frac{k}{r}.$$

We show that if $L \neq 0$, then E is determined by the angular momentum L and the eccentricity vector e. Recall that

$$Lv = k(\hat{r} + e).$$

After solving for ke and squaring we obtain

$$\begin{aligned}
k^2e^2 &= (ke)(ke)^\dagger \\
&= (Lv - k\hat{r})(Lv - k\hat{r})^\dagger \\
&= (Lv - k\hat{r})(vL^\dagger - k\hat{r}) \\
&= LvvL^\dagger - Lvk\hat{r} - k\hat{r}vL^\dagger + k\hat{r}k\hat{r} \\
&= |L|^2v^2 + k^2 - k(Lv\hat{r} + \hat{r}vL^\dagger) \\
&= |L|^2v^2 + k^2 + k(vL\hat{r} + \hat{r}vL) \\
&= |L|^2v^2 + k^2 + 2k(\hat{r} \rfloor (v \rfloor L) \\
&= |L|^2v^2 + k^2 + \frac{2k}{m|r|}(mr \wedge v) \rfloor L \\
&= |L|^2v^2 + k^2 - \frac{2k}{m|r|}|L|^2
\end{aligned}$$

so that

$$k^2(e^2 - 1) = |L|^2(v^2 - \frac{2k}{m|r|}).$$

The quantity

$$v^2 - \frac{2k}{m|r|}$$

must be constant for every other quantity is a constant. Observe that the quantity is

$$\frac{2E}{m}.$$

Hence, E is a constant that is already built into the other constants of motion L and e.

3.3 Reflections and rotations

In this section we will express reflections and rotations in the language of geometric algebra. The effect of two reflections is a rotation, which takes place in a plane. The result holds for any dimension. We shall suppose the standard Euclidean inner product for the bilinear form B.

3.3.1 Reflections

Definition 3.3.1. Let v be a vector of unit magnitude. The reflection of u about the vector v is given by

$$U_v(u) = vuv.$$

Let $u_{\|} = (u \rfloor v)v$ and $u_{\perp} = u - u_{\|}$ denote the component of u parallel to v and perpendicular to v, respectively. To compare this with the standard formula of a reflection about a line, we compute

$$\begin{aligned} U_v(u) &= vuv \\ &= v(uv) \\ &= v(-vu + 2u \rfloor v) \\ &= -v^2 u + 2(u \rfloor v)v \\ &= -u + 2(u \rfloor v)v \\ &= u_{\|} - u_{\perp}. \end{aligned}$$

We see explicitly, that the reflection of u about the vector v reflects the component perpendicular to v.

3.3.2 Rotations

Let us reflect u about v and then w. Observe,

$$\begin{aligned} U_w U_v(u) &= U_w(vuv) \\ &= wvuvw \\ &= (vw)^\dagger u(vw). \end{aligned} \tag{3.3.2.1}$$

Let $\psi = vw$. The multivector ψ is called a *rotor*. Note that

$$\psi^\dagger \psi = \psi \psi^\dagger = 1.$$

In the literature, see [DS07] for example, a rotor R is defined as a geometric product of an even number of unit vectors such that $RR^\dagger = 1$.

Let us examine ψ when $B(v, w) = 0$. Since $B(v, w) = 0$,

$$\psi^\dagger = wv = -vw = -\psi.$$

Notice that

$$v\psi = v(vw) = w \text{ and } w\psi = w(vw) = -v.$$

We may interpret this as v has been rotated by $\frac{\pi}{2}$ into w. Similarly, w has been rotated by $\frac{\pi}{2}$ into $-v$. Intuitively, the bivector $\psi = v \wedge w$ specifies a plane, and when acting on a vector on the right produces a rotation by $\frac{\pi}{2}$. Let us examine this action on a general vector in the plane $v \wedge w$. If $u \in \langle v, w \rangle$, then $u = \alpha v + \beta w$. We have

$$u\psi = \alpha w - \beta v.$$

The vector u has been rotated in the plane $v \wedge w$ by $\pi/2$ radians. We may compound this rotation further by an additional $\pi/2$. Observe,

$$\psi^\dagger u \psi = -u\psi^\dagger \psi = -u.$$

We observe that u has been rotated π in the plane $v \wedge w$.

Let s be a vector such that $s\rfloor(v \wedge w) = 0$, i.e., s is perpendicular to the plane $v \wedge w$. By identity (1.3.1.2),

$$s\psi = s(v \wedge w) = (v \wedge w)s = \psi s.$$

Observe,

$$\psi^\dagger s\psi = \psi^\dagger \psi s = s.$$

We see that vectors perpendicular to the plane $v \wedge w$ are not effected by the two sided action of $v \wedge w$. Let $x \in V$. We may write

$$x = u + s$$

where u and s are contained in and perpendicular to the plane $v \wedge w$, respectively. Then

$$\psi^\dagger x\psi = \psi^\dagger (u+s)\psi = \psi^\dagger u\psi + \psi^\dagger s\psi = -u + s.$$

Intuitively, given a vector, we may rotate, in this particular case by π, only the component of the vector contained in the plane. This result holds in any dimension.

Let us consider the general case when $B(v, w)$ is not necessarily zero. Then

$$\psi = vw = v\rfloor w + v \wedge w.$$

Let $\frac{\theta}{2}$, $0 < \frac{\theta}{2} < \pi$, be the angle between v and w. Then

$$v\rfloor w = \cos\ \frac{\theta}{2}.$$

This means that

$$\begin{aligned}
|v \wedge w|^2 &= \langle (v \wedge w)(w \wedge v)\rangle_0 \\
&= \langle (vw - v\rfloor w)(wv - w\rfloor v)\rangle_0 \\
&= \langle vwwv + (v\rfloor w)^2 - 2(v\rfloor w)^2\rangle_0 \\
&= 1 - (v\rfloor w)^2 \\
&= 1 - \cos^2\ \frac{\theta}{2} \\
&= \sin^2\ \frac{\theta}{2}.
\end{aligned}$$

So,

$$|v \wedge w| = \sin\ \frac{\theta}{2}.$$

Let

$$\hat{R} = \frac{v \wedge w}{\sin\ \frac{\theta}{2}}.$$

Observe, by equation 1.3.7.1,

$$\hat{R}^2 = \frac{(v \wedge w)^2}{\sin^2\ \frac{\theta}{2}} = \frac{-|v \wedge w|^2}{\sin^2\ \frac{\theta}{2}} = -1.$$

Then

$$\psi = \cos\ \frac{\theta}{2} + \hat{R} \sin\ \frac{\theta}{2}.$$

Observe that,

$$\psi^\dagger \psi = \psi \psi^\dagger = 1.$$

Let us examine equation 3.3.2.1 when $u \in \langle v, w \rangle$. In this case we have

$$u\hat{R} = -\hat{R}u$$

which means that

$$\psi^\dagger u = u\psi.$$

Hence,

$$\psi^\dagger u \psi = u\psi^2 = u(\cos\ \theta + \hat{R} \sin \theta).$$

We may interpret this result as u has been rotated in the plane $v \wedge w$ by an angle θ. To see this, recall that $\hat{R}$ be interpreted as rotating u by $\pi/2$ in the $v \wedge w$ plane. Then $u\psi^2$ is a sum of orthogonal vectors u and $u\hat{R}$ that have been scaled appropriately by $\cos \theta$ and $\sin \theta$ such that the sum of each length, gives the length of u.
Let us now examine equation 3.3.2.1 when $s \rfloor (v \wedge w) = 0$. In this case we have

$$s\hat{R} = \hat{R}s$$

which means that

$$\psi^\dagger s \psi = \psi^\dagger \psi s = s.$$

We see that vectors perpendicular to the plane $v \wedge w$ are not rotated. Let $x \in V$. We may write

$$x = u + s$$

where u and s are contained in and perpendicular to the plane $v \wedge w$, respectively. Then

$$\psi^\dagger x \psi = \psi^\dagger (u + s)\psi = \psi^\dagger u \psi + \psi^\dagger s \psi = u\psi^2 + s.$$

Intuitively, given a vector, we may rotate the component of the vector contained in the plane by twice the angle between the vectors representing the blade. This result holds in any dimension.

3.4 Finding the components of a vector

We follow the strategy of [Lou04]. One is familiar with finding the components of a vector with respect to a basis by projecting the length of the vector in question onto the basis. Another method to calculate the components, may be viewed by using ratios of Euclidean areas.

Let $a, b \in \mathbb{R}^n$ linearly independent, so $a \wedge b$ is invertible. Suppose that $v \in \mathbb{R}^n$ is such that $v \wedge a \wedge b = 0$, so $v \in \langle a, b\rangle$. Then

$$v = \alpha a + \beta b, \ \alpha, \beta \in \mathbb{R}.$$

We proceed to determine α and β. Wedging by a we find that

$$a \wedge v = \beta a \wedge b$$

or

$$\beta = \frac{a \wedge v}{a \wedge b}$$

and wedging by b we find

$$\alpha = \frac{v \wedge b}{a \wedge b}.$$

So

$$v = \frac{v \wedge b}{a \wedge b} a + \frac{a \wedge v}{a \wedge b} b.$$

Example 3.4.1. Let e_1, e_2 be an orthonormal basis. Let

$$a = e_1 - 2e_2, b = e_1 + e_2, \text{ and } v = 5e_1 - e_2.$$

We calculate α and β in the linear combination

$$v = \alpha a + \beta b.$$

We have

$$a \wedge b = \langle (e_1 - 2e_2)(e_1 + e_2) \rangle_2 = 3e_1e_2,$$

$$v \wedge b = \langle (5e_1 - e_2)(e_1 + e_2) \rangle_2 = 6e_1e_2$$

and

$$a \wedge v = \langle (e_1 - 2e_2)(5e_1 - e_2) \rangle_2 = 9e_1e_2.$$

Then

$$\alpha = \frac{6e_1e_2}{3e_1e_2} = 2$$

and

$$\beta = \frac{9e_1e_2}{3e_1e_2} = 3$$

so that

$$v = 2a + 3b.$$

We may generalize to n vectors as follows. Let $\{b_i\}$ be a linearly independent set of vectors. Then if $v \wedge b_1 \wedge \cdots \wedge b_r = 0$ (i.e., $v \in \langle b_1, \ldots, b_r \rangle$) we have

$$v = \sum_{k=1}^{r} (-1)^{r-k} \frac{b_1 \wedge \cdots \wedge \check{b}_k \wedge \cdots \wedge b_r \wedge v}{b_1 \wedge \cdots \wedge b_r} b_k.$$

We see that the components of v are ratios of "hypervolumes".

Chapter 4

Appendix

4.1 Construction of a geometric algebra

We will show in Theorem 4.1.8 below, how, given a finite dimensional vector space and a non-degenerate symmetric bilinear form, one can construct an associated finite dimensional geometric algebra.

Proposition 4.1.1. *(Polarization Identity) Let B be a symmetric bilinear from on a vector space V over $\mathbb{R}$. Then for all $v, w \in V$*

$$B(v,w) = \frac{1}{2}(B(v+w, v+w) - B(v,v) - B(w,w)).$$

Proof. Let $v, w \in V$. Then

$$B(v+w, v+w) = B(v,v) + B(v,w) + B(w,v) + B(w,w) = 2B(v,w) + B(v,v) + B(w,w)$$

or

$$B(v,w) = \frac{1}{2}(B(v+w, v+w) - B(v,v) - B(w,w)).$$

□

Intuitively, the polarization identity says that to know a symmetric bilinear form along the diagonal, is to know the bilinear form for all pairs of vectors.

Lemma 4.1.2. *Let V be a finite dimensional vector space over $\mathbb{R}$ with a non-degenerate symmetric bilinear form B. Then there exists a basis of orthogonal non-null vectors that square to ± 1.*

Proof. Suppose for all $x \in V, B(x,x) = 0$, i.e., all vectors of V are null. By the polarization identity

$$2B(v,w) = B(v+w, v+w) - B(v,v) - B(w,w) = 0$$

for all $v, w \in V$. Since B is non-degenerate, this cannot happen. Hence, there exists $x \in V$ such that $B(x,x) \neq 0$. Define the function

$$\hat{B} : V \to \mathbb{R}$$

by

$$v \mapsto B(x,v).$$

It is a straightforward verification that $\hat{B}$ belongs to the dual space of V. Let $K = \ker \hat{B}$. We will show that V is the direct sum of $\langle x \rangle$ and K. Let $v \in V$. Then we may write, for $b \in K$,

$$v = \frac{\hat{B}(v)}{\hat{B}(x)} x + b.$$

This is well-defined since

$$\hat{B}(x) = B(x,x) \neq 0.$$

Hence,

$$V = \langle x \rangle + K.$$

Furthermore, if $y \in \langle x \rangle \cap K$, then $y = \lambda x,\ \lambda \in \mathbb{R}$ and

$$0 = \hat{B}(y) = B(x, \lambda x) = \lambda B(x,x).$$

Since $B(x,x) \neq 0$, $\lambda = 0$. Hence,

$$V = \langle x \rangle \oplus K.$$

We now show that $B|_K$ is non-degenerate. Let $k \in K$. Since B is non-degenerate there exists $v \in V$ such that $B(k,v) \neq 0$. By our direct sum decomposition, we may write

$$v = \lambda x + b,\ \lambda \in \mathbb{R}, b \in K.$$

By bilinearity, we have

$$B(k,b) = \lambda B(k,x) + B(k,b) = B(k,\lambda x + b) = B(k,v) \neq 0.$$

Hence, $B|_K$ is non-degenerate. By induction it follows that V is the direct sum of one dimensional orthogonal spaces whose generator is non-null. So, V has an orthogonal non-null basis. Since the vectors forming the basis are non-null, they can be normalized to square to ± 1. $\square$

Definition 4.1.3. Let R be an algebra over $\mathbb{R}$ and let A be a subspace of R. We define the map

$$D : A \times \cdots \times A \to R$$

by

$$D(a_1, \ldots, a_r) = \frac{1}{k!} \sum_{\sigma \in S_r} (-1)^\sigma a_{\sigma(1)} \cdots a_{\sigma(r)},$$

where S_r is the set of all permutations of the numbers 1 to r and $(-1)^\sigma$ is the sign of the permutation σ, $+1$ or -1 if σ is even or odd, respectively.

Proposition 4.1.4. *The map D is an alternating map, i.e., D is r-multilinear and*

$$D(a_1, \ldots, a_j, \ldots, a_i, \ldots, a_r) = -D(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_r).$$

Proof. By distributivity of the ring structure on R, D is r-multilinear. We follow [Spi65] to show that D is alternating. Let $(i\ j)$ be the permutation that interchanges i and j. If

$\sigma \in S_k$, let $\tau = \sigma \circ (i\ j)$. Then

$$\begin{aligned} D(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_r) &= \frac{1}{k!} \sum_{\sigma \in S_r} (-1)^{\sigma} a_{\sigma(1)} \cdots a_{\sigma(i)} \cdots a_{\sigma(j)} \cdots a_{\sigma(r)} \\ &= \frac{1}{k!} \sum_{\sigma \in S_r} (-1)^{\sigma} a_{\sigma(1)} \cdots a_{\sigma(i\ j)(j)} \cdots a_{\sigma(i\ j)(i)} \cdots a_{\sigma(r)} \\ &= \frac{1}{k!} \sum_{\sigma \in S_r} (-1)^{\sigma} a_{\tau(1)} \cdots a_{\tau(j)} \cdots a_{\tau(i)} \cdots a_{\tau(r)} \\ &= (-1)^{(i\ j)} \frac{1}{k!} \sum_{\tau \in S_r} (-1)^{\tau} a_{\tau(1)} \cdots a_{\tau(j)} \cdots a_{\tau(i)} \cdots a_{\tau(r)} \\ &= -D(a_1, \ldots, a_j, \ldots, a_i, \ldots, a_r). \end{aligned}$$

□

Corollary 4.1.5. *Let R be an algebra over $\mathbb{R}$ and let A be a subspace of R. Then*

$$D(\ldots, a, \ldots, a, \ldots) = 0. \tag{4.1.0.1}$$

Proof. Since D is alternating,

$$D(\ldots, a, \ldots, a, \ldots) = -D(\ldots, a, \ldots, a, \ldots),$$

and the result follows.

□

Corollary 4.1.6. *If $a_1, \ldots, a_r$ mutually anti-commute, then*

$$D(a_1, \ldots, a_r) = a_1 \cdots a_r. \tag{4.1.0.2}$$

Proof. Since $a_1, \ldots, a_r$ mutually anti-commute,

$$a_{\sigma(1)} \cdots a_{\sigma(r)} = (-1)^{\sigma} a_1 \cdots a_r.$$

Hence,

$$D(a_1, \ldots, a_r) = \frac{1}{k!} \sum_{\sigma \in S_r} (-1)^\sigma a_{\sigma(1)} \cdots a_{\sigma(r)} = \frac{1}{k!} \sum_{\sigma \in S_r} a_1 \cdots a_r = a_1 \cdots a_r.$$

□

Proposition 4.1.7. *Let T be a ring; let I be an ideal of T; let $\pi : T \to T/I$ be the canonical projection; let $\phi : T \to T$ be a homomorphism. If $\pi(\phi(I)) = 0$, then there is an induced homomorphism $\tilde{\phi} : T/I \to T/I$ defined by $\tilde{\phi}(\bar{y}) = \pi(\phi(x))$ for any $x \in \pi^{-1}(\bar{y})$. That is, the following diagram commutes:*

$$\begin{array}{ccc} T & \xrightarrow{\phi} & T \\ \pi \downarrow & & \downarrow \pi \\ T/I & \xrightarrow{\tilde{\phi}} & T/I \end{array}$$

Proof. We show that the function $\tilde{\phi}$ is well-defined. Suppose that $\bar{y} = \bar{y'}$ in T/I. Then $y = y' + i$ for some $i \in I$, and

$$\pi\phi(y) = \pi\phi(y' + i) = \pi\phi(y') + \pi\phi(i) = \pi\phi(y').$$

Hence, $\tilde{\phi}$ is well-defined. It is straightforward to show $\tilde{\phi}$ is a homomorphism. □

The construction of a geometric algebra from a finite dimensional vector space and a non-degenerate symmetric bilinear form is now given.

Theorem 4.1.8. *Let V be a n-dimensional vector space over $\mathbb{R}$ equipped with a non-degenerate symmetric bilinear form B. Let $T(V) = \sum_{k=1}^{\infty} T_k(V)$ be the tensor algebra over V. Let I be the two-sided ideal generated by elements of the form*

$$v \otimes v - B(v, v)1, \quad v \in V \tag{4.1.0.3}$$

and let $\mathcal{G}(V, B) = T(V)/I$. Then $\mathcal{G}(V, B)$ is a geometric algebra such that

$$\mathcal{G}(V, B) = \mathcal{G}^0 \oplus \mathcal{G}^1 \oplus \cdots \oplus \mathcal{G}^n$$

where each subspace $\mathcal{G}^r$ has dimension $\binom{n}{r}$. Moreover, the subspaces $\mathcal{G}^r$ are spanned by

$e_{i_1}\cdots e_{i_r}$ $1 \le i_1 < \cdots < i_r \le n$, where $\{e_1, \ldots, e_n\}$ is the basis of V guaranteed by Lemma 4.1.2.

Proof. Since $\mathcal{G}(V,B) = T(V)/I$, it is an algebra, with identity, over $\mathbb{R}$. When working in the quotient space $\mathcal{G}(V,B)$, we shall drop the tensor product symbol. By Lemma 4.1.2, V has an orthogonal non-null basis $\{e_1, \ldots, e_n\}$. We introduce some convenient notation. Let $\sigma = (i_1 \cdots i_r)$, $1 \le i_1 < \cdots < i_r \le n$. Then

$$e_\sigma = e_{i_1} \cdots e_{i_r}.$$

We say that σ has length r. For an empty sequence $\sigma_0 = ()$, we define $e_{\sigma_0} = 1$. Note that since the $e_i's$ are non-null, in $\mathcal{G}(V,B)$, they are invertible with $e_i^{-1} = \frac{1}{B(e_i,e_i)} e_i$. Therefore, the $e_\sigma's$ are invertible.
Our strategy will be to first establish Property 1; then identify $\mathbb{R}$ with $\mathcal{G}^0$ and V with $\mathcal{G}^1$, respectively, Properties 2, 3 and 4 will follow; and finally Property 5.
The elements $e_{i_1} \otimes \cdots \otimes e_{i_r}$ form a basis of $T(V)$, therefore their cosets span $\mathcal{G}(V,B)$. Since, in $\mathcal{G}(V,B)$,

$$(e_i + e_j)(e_i + e_j) = B(e_i + e_j, e_i + e_j)1,$$

we see

$$e_i e_j = -e_j e_i, \ i \neq j. \tag{4.1.0.4}$$

By equation (4.1.0.4), the factors of each coset of $e_{i_1} \otimes \cdots \otimes e_{i_k}$ may be shuffled so to have increasing indices and by 4.1.0.3, any repeated factors may be identified with a scalar. Hence, the $e_\sigma's$ span $\mathcal{G}(V,B)$. We now show that the $e_\sigma's$ are linearly independent, to establish property 1.

We follow the strategy of [Rie58] to show that the $e_\sigma's$ are linearly independent. Observe,

$$e_j e_\sigma = e_j e_{i_1} \cdots e_{i_r} = \begin{cases} (-1)^r e_\sigma e_j & \text{if } j \neq i_s, \ s = 1, \ldots, r \\ (-1)^{r-1} e_\sigma e_j & \text{if } j = i_s, \text{ for some } s, \ s = 1, \ldots, r \end{cases}$$

We see if the length of $\sigma \neq \sigma_0$ is less than n, then there exists an e_j that anti-commutes with e_σ. If the length of σ is n, then any basis element will anti-commute with e_σ when n is even, but not when n is odd.

Let I be a finite indexing set. Consider,

$$\sum_{\sigma \in I} A_\sigma e_\sigma = 0 \tag{4.1.0.5}$$

and suppose to produce a contradiction that there exists a $\beta \in I,\ \beta \neq \sigma_0$, such that $A_\beta \neq 0$. If β has length less than n, or length n with n even, multiply equation (4.1.0.5) by $A_\beta^{-1} e_\beta^{-1}$. Then equation (4.1.0.5) becomes

$$1 + \sum_{\sigma \in J} A'_\sigma e'_\sigma = 0 \tag{4.1.0.6}$$

where $J = I \setminus \{\beta\},\ A'_\sigma = A_\sigma A_\beta^{-1},\ e'_\sigma = e_\sigma e_\beta^{-1}$. If the coefficients A'_σ are zero, we obtain $1 = 0$ and since the ideal I is a proper ideal, we reach a contradiction. Suppose that $A_\alpha \neq 0,\ \alpha \in J$. There exists an e_j that anti-commutes with e'_α. Multiply equation (4.1.0.6) on the right by e_j and on the left by e_j^{-1} and add to equation (4.1.0.6). After dividing the result by 2, we obtain

$$1 + \sum_{\sigma \in J} \frac{1}{2} A'_\sigma (e'_\sigma + e_j e'_\sigma e_j^{-1}) = 0.$$

We have eliminated the coefficient A_α and will have

$$1 = 0$$

if all other coefficients are zero. Assuming we never have the $A_\beta \neq 0$ in (4.1.0.5) with $\beta \neq \sigma_0$ of length n with n odd, iterating a finite number of times we will obtain $1 = 0$. Thus, the $e'_\sigma s$ are linearly independent.

Suppose now that the only $A_\beta \neq 0$ in equation (4.1.0.5) with $\beta \neq \sigma_0$ such that β has length n with n odd. Since there may not exist a basis vector of V to anti-commute with $e_1 \cdots e_n$, we proceed by a different route. We will introduce a homomorphism on $\mathcal{G}(V, B)$, that will allow for a proof of the linear independence of the $e'_\sigma s$. Let

$$\hat{\phi} : V \to V$$

defined by

$$e_i \mapsto -e_i.$$

This map extends to a homomorphism

$$\phi : T(V) \to T(V)$$

defined by

$$e_{i_1} \otimes \cdots \otimes e_{i_k} \mapsto (-1)^k e_{i_1} \otimes \cdots \otimes e_{i_k}.$$

Let

$$\pi : T(V) \to G$$

be the canonical projection map. Observe, for the generators of I,

$$\pi(\phi((x \otimes x - B(x,x)1)) = \pi(x \otimes x - B(x,x)1) = 0.$$

Hence, $\pi\phi(I) = 0$. Thus, by Proposition 4.1.7, we obtain an induced homomorphism

$$\tilde{\phi} : \mathcal{G}(V,B) \to \mathcal{G}(V,B)$$

defined by

$$\tilde{\phi}(y) = \pi(\phi(x))$$

where $x \in \pi^{-1}(\bar{y})$. Proceeding with the same argument as the earlier case, suppose that equation (4.1.0.5) states

$$1 + A_\beta e_1 \cdots e_n = 0. \tag{4.1.0.7}$$

Apply $\tilde{\phi}$ to equation (4.1.0.7). We obtain

$$1 - \beta e_1 \cdots e_n = 0. \tag{4.1.0.8}$$

Add equations (4.1.0.7) and (4.1.0.8) and divide by 2, we obtain

$$1 = 0,$$

a contradiction. Hence, in this case the $e'_\sigma s$ are linearly independent.

Thus, in all cases, the $e'_\sigma s$ are linearly independent. Let $\mathcal{G}^r$ denote the subspaces spanned by e_σ, $\sigma = (i_1 \cdots i_r)$. By what we have just shown, $\mathcal{G}$ is a direct sum of the subspaces $\mathcal{G}^r$.

We now establish Properties 2, 3 and 4. Since V and $\mathcal{G}^1$ have the same dimension, they are isomorphic. We then identify $\mathcal{G}^1$ with V. Then $\mathcal{G}^1$ is equipped with the bilinear form B. Furthermore, $\mathcal{G}^0$ contains 1 and is isomorphic to $\mathbb{R}$.

We now establish Property 5. Let

$$D : \mathcal{G}^1 \times \cdots \times \mathcal{G}^1 \to \mathcal{G}$$

be defined as in Definition 4.1.3. We now show that all r-blades are contained in $\mathcal{G}^r$. Let $a_1 \cdots a_r$ be an r-blade. Since $a_k \in \mathcal{G}^1$, $a_k = \sum_s \alpha_{ks} e_s,\ k = 1, \ldots, r$. By Corollary 4.1.6,

$$\begin{aligned} a_1 \cdots a_r &= D(a_1, \ldots, a_r) \\ &= D(\sum_{s_1} \alpha_{1s_1} e_{s_1}, \ldots, \sum_{s_r} \alpha_{rs_r} e_{s_r}) \\ &= \sum_{s_1,\ldots,s_r} \alpha_{1s_1} \cdots \alpha_{rs_r} D(e_{s_1}, \ldots, e_{s_r}) \end{aligned} \tag{4.1.0.9}$$

By (4.1.0.1) and (4.1.0.2), the sum on the right of (4.1.0.9) may be written as a linear combination of the e_σ, where $\sigma = (i_1 \cdots i_r)$. Hence,

$$a_1 \cdots a_r \in \mathcal{G}^r.$$

Furthermore, if $r > n$ there will be at least one repeated basis vector e_i in e_σ, which means that by Corollaries 4.1.5 and 4.1.6,

$$e_\sigma = D(e_{i_1}, \ldots, e_{i_r}) = 0.$$

Thus, $\mathcal{G}^k = \{0\}$ for $k > n$. Thus, $\mathcal{G}(V, B)$ is the direct sum of $\mathcal{G}^0, \ldots \mathcal{G}^n$ where $\mathcal{G}^r$ contains all r-vectors.

Since each $\mathcal{G}^r$ is generated by e_σ, $\sigma = (i_1 \ldots i_r)$, $i_1 < \cdots < i_r$, there are $\binom{n}{r}$ basis vectors. □

References

[Che97] Claude Chevalley. *The algebraic theory of spinors and Clifford algebras.* Springer-Verlag, Berlin, 1997. Collected works. Vol. 2, Edited and with a foreword by Pierre Cartier and Catherine Chevalley, With a postface by J.-P. Bourguignon.

[Chi12] E. Chisolm. Geometric algebra. *Cornell University Library*, 2012.

[DL03] C. Doran and A. Lasenby. *Geometric Algebra for Physicists.* Cambridge University Press, 2003.

[DS07] Fontijne D. Dorst, L. and M. Stephen. *Geometric Algebra for Computer Science.* Morgan Kaufmann, 2007.

[Hes99] David Hestenes. *New foundations for classical mechanics*, volume 99 of *Fundamental Theories of Physics.* Kluwer Academic Publishers Group, Dordrecht, second edition, 1999.

[HS84] D. Hestenes and G. Sobczyk. *Clifford Algebra to Geometric Calculus: A Unified Language for Mathematical Physics.* D. Reidel Publishing Company, 1984.

[Lou04] Pertti Lounesto. Introduction to Clifford algebras. In *Lectures on Clifford (geometric) algebras and applications*, pages 1–29. Birkhäuser Boston, Boston, MA, 2004.

[Rie58] Marcel Riesz. *Clifford numbers and spinors (Chapters I–IV).* Lectures delivered October 1957-January 1958. Lecture Series, No. 38. The Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Md., 1958.

[Spi65] Michael Spivak. *Calculus on manifolds. A modern approach to classical theorems of advanced calculus.* W. A. Benjamin, Inc., New York-Amsterdam, 1965.