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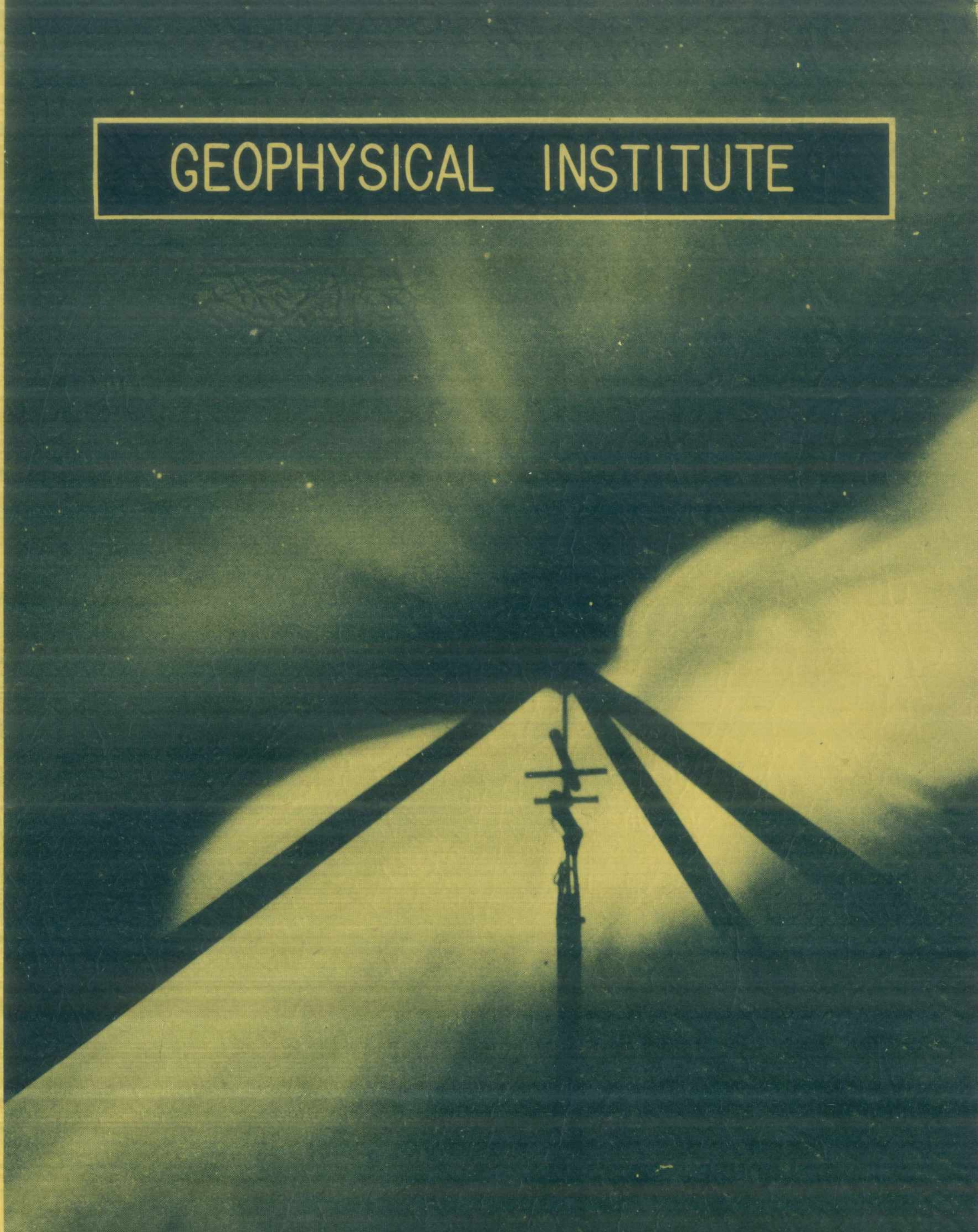

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A Note on Harmonic Analysis of Geophysical Data with Special Reference to the Analysis of Geomagnetic Storms

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Masahisa Sugiura

Scientific Report No. 1
Contract No. AF 19(604)-2163
April 18, 1960

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Geophysics Research Directorate
Air Force Cambridge Research Center
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Bedford, Massachusetts

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C. T. Elvey, Director

ABSTRACT

Some geophysical characteristics tend to have a fixed distribution relative to the sun. An example is the distribution of air temperature on an ideal earth that is perfectly symmetrical (e.g., in its pattern of land and water) about its axis of rotation. In such a case the geophysical characteristic at any fixed station on the earth undergoes a daily variation that depends only on local time (and latitude and season). This simple pattern of daily change may be modified by intrinsic changes in the solar influences on the earth. The harmonic components of the daily variation at any station may in this case undergo phase changes, in some respects corresponding to Doppler shifts of frequency in optical or sonic phenomena. Care is then needed if the results of harmonic analysis are to be properly interpreted. Such interpretation is discussed with reference to the parts Dst and DS of the magnetic storm variations.

Like caution must be observed in cases where the amplitude of a harmonic variation changes, with fixed phase.

1. Introduction

In geophysics one frequently deals with phenomena periodic both in space and time. Because of the rotation of the earth about its axis, temporal coordinates - local, or universal, time, or time defined in a specific manner with reference to some event - and the spatial coordinates that specify the positions of observatories relative to the sun are, in general, uniquely related to each other. In analyzing observational records the distinction between the temporal and geometrical coordinates must be clearly borne in mind, though they are often interchangeable.

In particular, progressive change of a phenomenon with local time as observed at a fixed observatory, and the spatial distribution, at some instant of time, of the same phenomenon at different longitudes, relative to the sun, should not be confused. These two variations may be regarded as being equivalent only when the phenomenon concerned stays stationary to the sun.

When one analyzes a set of data harmonically, care must be taken regarding the question: with respect to what coordinate is the variation analyzed?

2. Harmonic analysis of a variation with constant amplitude and changing phase

Let us suppose that a wave is stationary in the reference frame S that is at rest relative to the sun, and that in S the wave is described by $c \sin m\lambda$, where c is constant and m an integer; the variable λ is reckoned in angular measure (See Fig.1a).

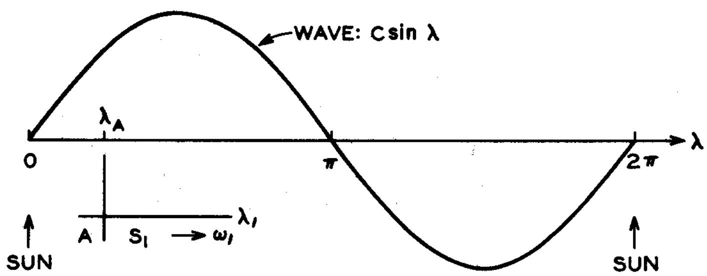

Fig. 1a. The wave is stationary relative to S. The reference frame S_1 travels with a constant velocity ω_1. The observer A, at rest in S_1, sees the wave to be c sin λ_A, where λ_A is the coordinate of A, or local time of A.

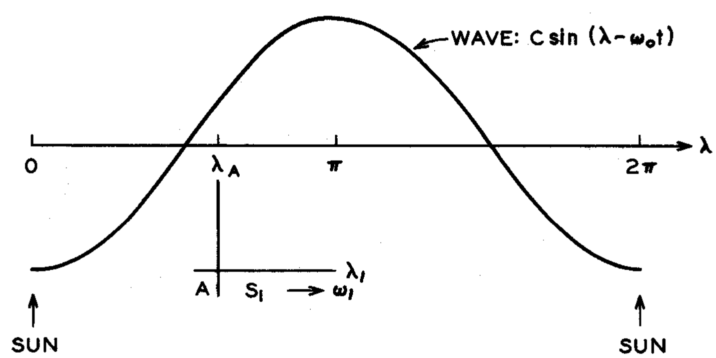

Fig. 1b. The wave travels with a constant velocity ω_o relative to S. The wave appears in S as c sin $(\lambda-\omega_o t)$. In S_1, traveling with a constant velocity ω_1, the wave is c sin $\{\lambda_1+(\omega_1-\omega_o)t\}$. The observer A at the origin of S_1 sees c sin $(1-\omega_o/\omega_1)\omega_1 t$ or c sin $(1-\omega_o/\omega_1)\lambda_A$.

The reference frame S_1 is traveling with a constant (angular) velocity ω_1 in the direction of increasing λ. Motions considered in this paper are such that the Galilean transformation of coordinates is valid.

If the origins of S and S_1 coincide at time t=o, λ and λ_1 are connected by $\lambda=\lambda_1 + \omega_1 t$, where t is 'absolute' time. To the observer A, who is fixed at the origin of S_1, the wave appears as $c \sin m\omega_1 t$. If he reckons time in angular measure, according to $t'=\omega_1 t$, his description of the wave is $c \sin mt'$ in terms of his (local) time t', or $c \sin m\lambda_A$ in terms of <u>his</u> longitude λ_A relative to the sun.

An idealized quiet daily variation (S_q) that changes neither its amplitude nor phase may be regarded as an example of such a wave.

Next we consider a wave, traveling relative to S with a constant (angular) velocity ω_o in the direction of increasing λ. In S the wave is $c \sin m(\lambda-\omega_o t)$. In S_1, which is moving with velocity ω_1, the wave is $c \sin m\,[\lambda_1 + (\omega_1 - \omega_o)t]$. The observer A, who is fixed at the origin of S_1, therefore, sees the wave to be $c \sin m\,(1-\omega_o/\omega_1)\omega_1 t$; in terms of t', or λ_A, this latter is

$$c \sin \underset{-}{m}^* t', \text{ or } c \sin \underset{-}{m}^* \lambda_A,$$

where

$$\underset{-}{m}^* = m\,(1-\omega_o/\omega_1) \tag{1}$$

The frequency $\underset{-}{m}^*$ is the familiar (classical) Doppler-shifted frequency.

The DS variation of magnetic storms is an example of this kind of wave except that its phase changes with a varying rate.

Without knowing the changing phase (with respect to S), which the motion of the wave amounts to, the observer A analyzes his observational data of the variation $c \sin m^{*}\lambda_A$ in the interval $\lambda_A = -\pi$ to π, as if the wave were stationary to S; this interval is obviously not equal to the wave-length of the fundamental harmonic (m=1). He will, therefore, obtain not the m-th harmonic alone, but, in general, all frequencies from 0 to ∞, the spectrum being dependent on ω_o/ω_1.

Expanding the function $c \sin m^{*}\lambda$ in a Fourier sine series in the interval $\lambda=-\pi$ to π, we obtain

$$c \sin m^{*}\lambda = c \sum_{k=1}^{\infty} b_k \sin k\lambda$$

where

$$b_k = \frac{(-1)^k}{\pi} \frac{2k \sin m^{*}\pi}{m^{*2} - k^2} \qquad (2)$$

if m^{*} is not an integer; and

$$b_k = \delta_{km^{*}} \qquad (2')$$

if m^{*} is an integer, where $\delta_{k\ell} = o$ when $k \neq \ell$ and $\delta_{k\ell} = 1$ when $k=\ell$.

When m^{*} deviates only by a small amount, say Δn, from an integer, say n, so that $m^{*} = n+\Delta n$ ($\Delta n \ll n$), then from (2) we see that b_n tends to 1 as Δn tends to 0:

$$b_n = \frac{(-1)^n}{\pi} \quad \frac{2n \sin (n+\Delta n)\pi}{(n+\Delta n)^2 - n^2}$$

$$= \frac{(-1)^n}{\pi} \quad \frac{2n \cos (n\pi) \sin (\Delta n\pi)}{n^2 \left[(1+\Delta n/n)^2 - 1\right]}$$

$$\rightarrow \quad \frac{2n \; \Delta n\pi}{\pi \; n^2 \; 2\frac{\Delta n}{n} \; (1+1/2\frac{\Delta n}{n})}$$

$$\rightarrow \quad 1-1/2\frac{\Delta n}{n} \quad \rightarrow \quad 1 \quad \text{as} \quad \Delta n \quad \rightarrow 0.$$

This corresponds to a complete shift of frequency from m to n; in this case other harmonics have zero amplitude.

Fig. 2 shows the coefficients b_k as functions of m^*, for $k=1,2,3,4$. For each value of k, b_k is 1 at $m^*=k$, and is maximum at m^* a little less than k; b_k is maximum at m^* satisfying $\cot m^*\pi = \frac{2\,m^*}{\pi\,(m^{*2}-k^2)}$, and the maximum value is $\frac{2k}{\sqrt{\pi^2(m^{*2}-k^2)^2+4m^{*2}}}$.

This result has an important implication in the interpretation of harmonic analysis. Take, for instance, the case when $m^* = 1.4$. For this value of m^*, b_1 and b_2 are nearly equal (Fig. 2). Thus the observer A obtains the first and second harmonics that are nearly of equal magnitude, although the wave is a single harmonic in S. If $m=1$, $\omega_o/\omega_1= -0.4$ gives $m^*=1.4$; or if $m=2$, $\omega_o/\omega_1= 0.3$ gives $m^*=1.4$, etc. From the harmonic analysis alone, there is no way of discriminating between these possibilities. If the wave is not a single harmonic in S, or if the velocity ω_o is not constant in time, the situation is even more complex.

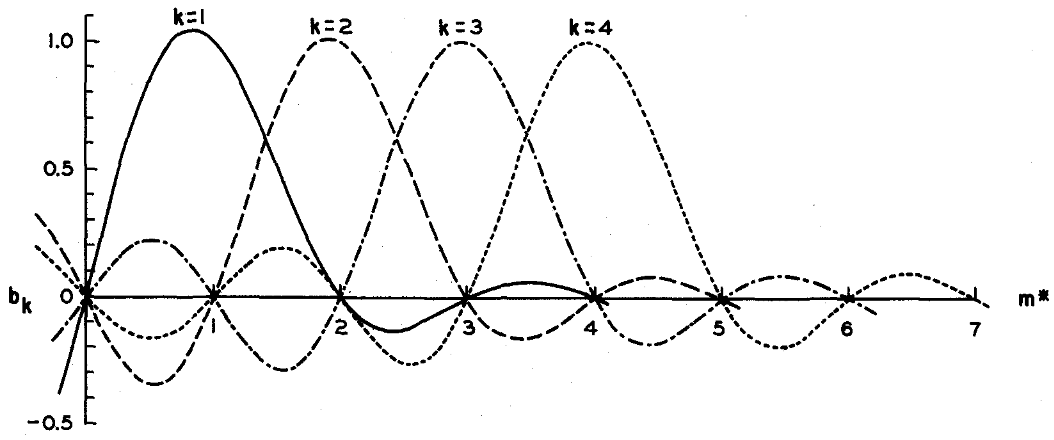

Fig. 2. The coefficients b_k as functions of m^*, for k=1,2,3,4.

This example illustrates the importance of careful examination of the treatment of data, and the interpretation of the result of harmonic analysis or of the synoptic representation of data.

It is further pointed out here that in general the variation to be expanded in the interval $-\pi$ to π will be of the form $\sum_{m=0}^{\infty} a_m \cos m\,(1-\omega_o/\omega_1)\lambda + b_m \sin m\,(1-\omega_o/\omega_1)\lambda$; in this case, because of the cosine terms, Dst will not be simply a_o, but $a_o + \sum_{m=1}^{\infty} a_m \frac{\sin m(1-\omega_o/\omega_1)\pi}{m(1-\omega_o/\omega_1)\pi}$. Therefore, in determining Dst from the observed data some precaution must be taken to eliminate any influence of the Doppler shift involved in DS.

3. <u>Harmonic analysis of a variation with changing amplitude and constant phase</u>

Another example of somewhat different nature is the harmonic analysis of a variation, which is a simple harmonic in λ, in S_1 at any instant of time, but which changes its amplitude with time.

Suppose that such a wave is expressed by $c(t) \sin m\lambda$, where $c(t)$ is a function of t. With respect to S_1 this wave is $c(t) \sin m(\lambda_1+\omega_1 t)$, since the coordinate transformation is $\lambda=\lambda_1+\omega_1 t$. If the observer A uses this coordinate λ_A as his time coordinate, he could describe the wave as $c'(\lambda_A) \sin m\lambda_A$, where $c'(\lambda_A)\equiv c(\lambda_A/\omega_1)$.

If he analyzes this variation harmonically in the interval $\lambda_A = -\pi$ to π, he will, in general, obtain harmonics of harmonic numbers from o to ∞.

If $c'(\lambda)$ is expressible by a power series of λ, i.e. $c'(\lambda) = \sum_{\ell=0}^{\infty} c_\ell \lambda^\ell$, the integrals giving the coefficients of the Fourier expansion of $c'(\lambda)\sin m\lambda$ can readily be integrated by repeated application of integration by parts.

For an example, let us suppose that $c'(\lambda)$ is expressed as

$$c'(\lambda) = c_o + c_1\lambda + c_2\lambda^2 \qquad (3)$$

Then the coefficients of the expansion

$$c'(\lambda)\sin m\lambda = a_o + \sum_{k=1}^{\infty}(a_k \cos k\lambda + b_k \sin k\lambda) \qquad (4)$$

are given by

$$a_o = (-1)^{m+1} \frac{c_1}{m} \qquad (5)$$

and for $k>o$, $k \neq m$

$$a_k = (-1)^{k+m+1} \frac{2m}{m^2-k^2} c_1 \qquad (6)$$

$$b_k = (-1)^{k+m} \frac{8\,km}{(m^2-k^2)^2} c_2 \qquad (7)$$

and for $k=m$

$$a_m = -\frac{c_1}{2m} \qquad (8)$$

$$b_m = c_o + \left(\frac{\pi^2}{3} - \frac{1}{2m^2}\right) c_2 \qquad (9)$$

As is obvious from considerations of symmetry, the contribution from the linear term, $c_1\lambda$, only appears in the constant term a_o, and the coefficients a_k of cosine terms; and the contribution from the second power, $c_2\lambda^2$, appears only in the coefficients b_k of sine terms.

Table 1 gives the coefficients a_k and b_k for m=1,2,3,4 and k=o,1,2,3----,6.

Table 1. a_k and b_k for m = 1, 2, 3, 4.

a_o

m	1	2	3	4
a_o	c_1	$-\frac{1}{2}c_1$	$\frac{1}{3}c_1$	$-\frac{1}{4}c_1$

a_k

k \ m	1	2	3	4
1	$-\frac{1}{2}c_1$	$\frac{2}{3}c_1$	$-\frac{3}{4}c_1$	$\frac{8}{15}c_1$
2	$-\frac{2}{3}c_1$	$-\frac{1}{4}c_1$	$\frac{6}{5}c_1$	$-\frac{2}{3}c_1$
3	$\frac{1}{4}c_1$	$\frac{2}{5}c_1$	$-\frac{1}{6}c_1$	$\frac{8}{7}c_1$
4	$-\frac{2}{15}c_1$	$\frac{1}{6}c_1$	$-\frac{6}{7}c_1$	$-\frac{1}{8}c_1$
5	$\frac{1}{12}c_1$	$-\frac{2}{21}c_1$	$\frac{3}{8}c_1$	$-\frac{8}{9}c_1$
6	$-\frac{2}{35}c_1$	$\frac{1}{16}c_1$	$\frac{2}{9}c_1$	$\frac{2}{5}c_1$

b_k

k \ m	1	2	3	4
1	*	$-\frac{16}{9}c_2$	$\frac{3}{8}c_2$	$-\frac{32}{225}c_2$
2	$-\frac{16}{9}c_2$	*	$-\frac{48}{25}c_2$	$\frac{4}{9}c_2$
3	$\frac{3}{8}c_2$	$-\frac{48}{25}c_2$	*	$-\frac{96}{49}c_2$
4	$-\frac{32}{225}c_2$	$\frac{4}{9}c_2$	$-\frac{96}{49}c_2$	*
5	$\frac{5}{72}c_2$	$-\frac{80}{881}c_2$	$\frac{15}{32}c_2$	$-\frac{160}{81}c_2$
6	$\frac{48}{1225}c_2$	$\frac{3}{32}c_2$	$-\frac{16}{81}c_2$	$\frac{12}{25}c_2$

* m=1, $b_1=c_o+(\frac{\pi^2}{3}-\frac{1}{2})\,c_2$

m=2, $b_2=c_o+(\frac{\pi^2}{3}-\frac{1}{8})\,c_2$

m=3, $b_3=c_o+(\frac{\pi^2}{3}-\frac{1}{18})\,c_2$

m=4, $b_4=c_o+(\frac{\pi^2}{3}-\frac{1}{32})\,c_2$

This effect is unlikely to be important in the analysis of the solar, or luni-solar, daily variations in the geomagnetic field or in the ionosphere, in which the day-to-day change is not so great. Even at the geomagnetic equator where the day-to-day change in S_q is appreciable, the effect is only slight. However, when variations with rapidly changing amplitude are analyzed, this problem may be worth considering. For a linear change of amplitude, the effect must be taken into consideration if the change $\Delta c' (\equiv c'(\pi)-c'(-\pi))$ is of the order of $2\pi c_0$ or greater.

4. DS variation in magnetic storms

Chapman (1952) and Sugiura and Chapman (1956,1957,1958, 1960) analyzed the magnetic storm field into two parts, Dst and DS. In our conception the separation of the two parts was made for each instant of storm time, regarding the storm field as being a function of longitude relative to the sun, and of geomagnetic latitude. Dst was defined as the mean over all longitudes, and DS the deviation from it.

It was assumed that in each latitude belt, variations, observed at different stations and for different storms, are all equivalent.

The method, used by us for the analysis of DS for each of the first six hours (§ 14 of our paper), expresses the above 'geometrical' concept of DS. For the subsequent phases of the storm, DS was determined for each of six, or eight, hour intervals.

It was found that both the phase and amplitude of DS vary very rapidly during the first several hours, and that these changes become slower as the storm progresses.

Though the concept of Dst and DS was clear in our mind, the question of the 'Doppler effect' was not realized, when the analysis was made.

The fact of the appearance and non-appearance of the Doppler effect in DS according to the method of analysis used may be illustrated in the following way.

Suppose that a wave is $c \sin m(\lambda-\omega_0 t)$ in the reference frame S that is stationary with regard to the sun. With reference to the frame S_1 that is moving with velocity ω_1, the wave is $c \sin m[\lambda_1+(\omega_1-\omega_0)t]$. If this wave is observed at an instant t, for all λ_1, it varies as $c \sin m\lambda_1$ in <u>space</u>, and hence there is no Doppler effect. If the wave is seen by an observer at $\lambda_1=\lambda_A$, he observes the wave to vary as $c \sin m(\omega_1-\omega_0)t$ in <u>time</u>. Hence, the frequency is Doppler-shifted.

If he observes the (time) rate of change, instead of the change itself, he is always observing the rate of change at the Doppler-shifted frequency.

With these remarks in mind one can draw the following conclusions regarding the actual treatment of the data.

(i) If DS is determined for any instant of storm time, the Doppler effect does not appear. In this type of analysis each observatory contributes only one (instantaneous or mean) value per storm.

(ii) If DS is derived from data for any interval of time, regardless of the length of interval, the question of Doppler effect arises. In this case each observatory contributes two or more values per storm, according to the length of the interval.

(iii) If DS is determined from the rate of change, such as hour-to-hour differences, the Doppler effect inevitably appears, even if the rate of change refers to some instant of time; in fact this is the limiting case of (ii) when the interval is made infinitely short.

In our analysis the rate of change was used to determine DS for six, or eight, hour intervals. For the first six hours, DS was also determined for individual hours, likewise using the rate of change. Though the phase (with respect to the reference frame fixed to the sun) changes rapidly during these hours, the rate of change may be considered approximately constant in each one-hour interval for which the rate of change (hour-to-hour difference) was computed. For the later intervals the rate of change in the phase may be regarded nearly constant within each interval. Therefore, the harmonic coefficients of DS, and Dst, for the first six individual hours and for the subsequent intervals, can, in principle, be interpreted in a definite, if not simple, manner with a set of parameters.

Yokouchi (1958) determined DS (and Dst) for overlapping intervals of twenty-four hours and for individual storm hours, using instantaneous hourly values, measured from pre-storm level, for many storms for the Kakioka Magnetic Observatory. He showed some important features that had not been found by Chapman and Sugiura because of the difference in the method of analysis. Professor Chapman brought to my attention this discrepancy between Yokouchi's and our results. In the process of studying the difference in the treatment of data, the present paper was written as a by-product. A preliminary study has already been made on the subject that was brought up by Yokouchi; his result has been confirmed. A further detailed study is made, and it is planned that a separate paper will be published in the near future.

In conclusion it is emphasized that due caution should be taken in data analysis or graphical representation, when the variation dealt with involves simultaneously both time and space coordinates as variables.

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