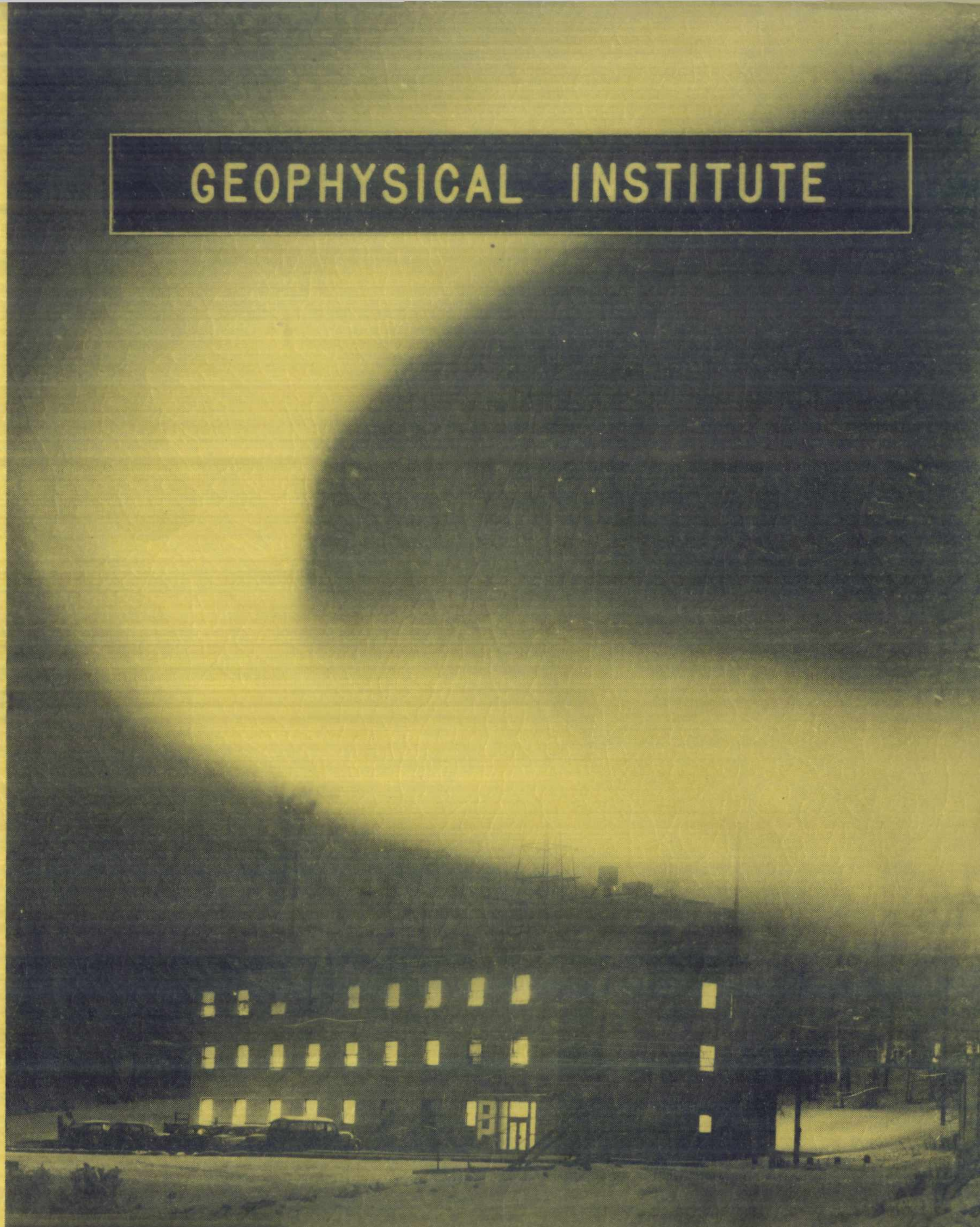

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SOME NOTES ON THE INTERPRETATION OF RAPID FLUCTUATIONS IN EARTH-CURRENTS OBSERVED IN HIGH LATITUDES

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Masahisa Sugiura

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PREFACE

This report was prepared as a part of the interdisciplinary study of the upper atmospheric disturbance in the polar regions that is conducted at the Geophysical Institute under Dr. C. T. Elvey, Director of the Institute. The report is primarily intended for the student of geophysics who is interested in this subject. A part of the mathematical procedure that was previously given by Prof. A. T. Price (reference 9 in end of paper) is included in Sections 4 to 6 with some modifications so as to enable the student to follow, without referring to Prof. Price's paper, the derivation of the formulae which are used in the present discussion, and to apply the method to similar problems.

November 15, 1958 M. S.

ABSTRACT

This paper shows that a periodically varying infinite linear current, or a periodically varying turbulent circular current of small radius (here approximated by a magnetic dipole with a changing dipole moment), in the ionosphere, which will give rise to magnetic variations of observed order of magnitude, is adequate for producing voltage differences in the ground of order 0.1 to 1 volt per kilometer that are frequently observed in high latitudes during disturbed periods. It appears difficult to interpret the earth-current record in terms of its primary origin, unless the distribution of the perturbing magnetic field and that of electric conductivity of the earth are both adequately known. However, the earth-current record is a good indicator of the upper atmospheric disturbance in the polar regions.

1. Introduction

Severe earth-current perturbations are observed in polar regions during magnetic disturbances. These earth-current variations are characterized by rapid oscillations[1,2,3,4,5,6]. According to Rooney and Sherman[2] and Rothé[4], the amplitude of such short-period variations frequently reaches one volt per kilometer or more.

Rooney[1] investigated the relation between the earth-current activity and auroral displays, using the Second International Polar Year data obtained at College, Alaska; and showed that the oscillatory disturbance in the earth-current records and moving type of aurorae are closely associated. A similar study was made by Currie[3] with the Canadian Polar Year data from Chesterfield Inlet. Rooney was aware of a possibility that earth-currents are more closely correlated to the auroral activity than the geomagnetic disturbance; he wrote:[1]

> "It is possible that the connection between earth-currents and aurorae is more direct than that between magnetic activity and aurorae, or it may be that the character of the earth-current variations is such as to bring out the relationship which exists more clearly. In support of the latter explanation is the fact that the higher frequency-components of disturbances are usually emphasized in the earth-current records as compared to those of the magnetic elements. Hence effects associated with aurorae may be less obvious when magnetic records are used in the comparison."

Earth-current records have been made in the College area since July 1955 by Dr. V. P. Hessler of the Geophysical Institute; similar but not quite continuous records have been taken at Barrow. Rapid fluctuations in earth-currents are found to be an excellent indicator of the upper atmospheric disturbance, and are used as such by the auroral and radio observers of the Institute. However, little study has been made on the physical interpretation of these rapid earth-current variations in high latitudes.

For much slower variations, earth-currents are accounted for as currents induced by the geomagnetic variations. Chapman and Whitehead[7] found a fair agreement, in type and in order of magnitude, between the diurnal variations in earth-currents observed at Ebro[8] and those computed theoretically from the diurnal magnetic variations. They found the observed variations of earth-currents to be about five times the computed ones. However, considering the inhomogeneous distribution of electrical conductivity in the earth, probably too much importance should not be attached to this discrepancy.

Rothé[4] compared earth-current records obtained at Scoresby Sund, a French Polar Year station (geomagnetic latitude 76°N.), with magnetic changes observed there, and showed that rapid earth-current changes are closely connected with abrupt changes in the magnetic field. He pointed out that the more abrupt the magnetic variations the stronger the earth-currents. This is clearly demonstrated in his reproduction of several samples of magnetic and earth-current records (Figures 12-18 in reference 4). He further compared magnetic records with earth-current traces obtained by a much less sensitive instrument, and demonstrated, with several examples, that it is the abruptness of the magnetic field change, not its amplitude, that is most closely related to the earth-current activity (Figures 19-24 in reference 4). Rothé did not give any quantitative discussion.

In this paper it is shown that a system of electric current flowing in the ionosphere that would produce magnetic changes of observed frequency and amplitude will necessarily give rise to earth-currents of amplitude of order of magnitude that is actually found in the earth-current records taken in the polar regions. As such a system of electric current two idealized models are considered: (A) an infinite linear current and, (B) a circular current of radius sufficiently small to be treated as a magnetic dipole at distances we are here concerned with. The intensity of the linear current in model (A) and the moment of the magnetic dipole in model (B) are assumed to vary periodically. The earth is represented by a semi-infinite medium of uniform conductivity with a plane boundary.

The mathematical procedure here used follows that of Price[9], which is shown to be applicable, in general, to any case when a certain inverse Laplace transform involving the functional form of the magnetic scalar potential of the inducing field exists.

2. Statement of the Problem

The system we consider consists of a semi-infinite uniform conductor with a plane boundary, and a periodic inducing (or primary) magnetic field. We compute the electric and magnetic field intensities as functions of time and space coordinates. In the conductor the electric current density can be readily obtained from the electric conductivity and the electric field.

As the inducing magnetic field, (A) a periodic infinite linear current flowing parallel to the surface of the conductor, and (B) a periodically changing magnetic dipole normal to the surface of the conductor are considered in Sections 6 and 7, respectively.

In Cartesian and cylindrical coordinates, both of which are used in this paper, the z-axis is taken to be normal to the surface of the conductor with its positive axis directed away from it.

3. The Field Equations

The electromagnetic field quantities in a continuous medium at rest satisfy the two pairs of the Maxwell equations:

$$\text{curl } \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0 \tag{1a}$$

$$\text{div } \vec{B} = 0 \tag{1b}$$

$$\text{curl } \vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \frac{4\pi}{c}\vec{j} \tag{2a}$$

$$\text{div } \vec{D} = 4\pi\rho \tag{2b}$$

where $\vec{E}$ and $\vec{H}$ are respectively the electric and magnetic field intensities, $\vec{D}$ and $\vec{B}$ the electric and magnetic inductions, $\vec{j}$ the electric current density, and ρ the charge density; c is the velocity of light. Gaussian units are used.

If the medium is isotropic, we have

$$\vec{D} = \epsilon\vec{E} \tag{3}$$

$$\vec{B} = \mu\vec{H} \tag{4}$$

$$\vec{j} = \kappa\vec{E} \tag{5}$$

where ϵ is dielectric constant, μ magnetic permeability, and κ electric conductivity; ϵ, μ and κ are, in general, functions of the space variables.

From the second pair of the Maxwell equation (2a,2b) we obtain the conservation law for charge and current densities,

$$\text{div } \vec{j} + \frac{\partial \rho}{\partial t} = 0 \tag{6}$$

Equations (3), (5) and (6) together with (2b) lead to the equation

$$\frac{\partial \rho}{\partial t} + 4\pi(\kappa/\epsilon)\rho = -(\epsilon/\kappa)\ \vec{j}\cdot\text{grad}\ (\kappa/\epsilon) \qquad (7)^*$$

As Lahiri and Price[10] have pointed out, if (a) κ/ϵ is constant or (b) $\vec{j}$ is perpendicular to grad(κ/ϵ), the charge density ρ is independent of the field vectors, and is given by $\rho_0 \exp(-4\pi\kappa t/\epsilon)$, where ρ_0 is the initial charge density. If ρ is zero initially, it remains so. If it is not zero initially, it will decay to 1/e times the initial value in $\epsilon/(4\pi\kappa)$ seconds. For $\epsilon = 1$, $\kappa = 10^{10}$ to 10^6 e.s.u. (or approximately 10^{-11} to 10^{-15} e.m.u.), this decay time is of order 10^{-11} to 10^{-7} second.

Here we consider a semi-infinite uniform conductor. Hence we can take ρ to be zero without loss of generality. Thus

$$\text{div}\ \vec{E} = 0 \qquad (8)$$

The displacement current $\frac{1}{c}\frac{\partial \vec{D}}{\partial t}$ in (2a) is small compared with $\frac{4\pi}{c}\vec{j}$. When the field is oscillatory with a period of order T seconds, the displacement current term is of order $\epsilon\vec{E}/(cT)$, and the current term of order $4\pi\ \kappa\vec{E}/c$. Hence the former is negligible in comparison with the latter, if $T \gg \epsilon/(4\pi\kappa)$. For this condition to be satisfied T has only to be greater than one second, say, even if κ is as small as 10^2 e.s.u. (or 10^{-19} e.m.u.). κ near the surface of the earth varies very widely, but it is within the range of 10^{10} to 10^4 e.s.u. (or 10^{-11} to 10^{-17} e.m.u.). Thus, in our case we can ignore the displacement current in (2a).

*For comparison with Equation (2,5) in the paper by Lahiri and Price,[10] an elementary calculation will show that

$$-(\epsilon/\kappa)\text{grad}\ (\kappa/\epsilon) = (\kappa/\epsilon)\ \text{grad}\ (\epsilon/\kappa).$$

Hence it follows, from (1a) and (2a), that the field vectors, $\vec{E}$ and $\vec{H}$, satisfy the induction equation of the form

$$\nabla^2 \vec{G} = \frac{4\pi K \mu}{c^2} \frac{\partial \vec{G}}{\partial t} \tag{9}$$

where ∇^2 is a differential operator (grad div - curl curl).

At the surface of the semi-infinite conductor the tangential components of $\vec{E}$ and $\vec{H}$, and the normal component of $\vec{B}$ are continuous.

4. Solutions of the Equations

Price(9) discussed two types of solutions of the differential equation (9). Using his terminology, "elementary solutions of the first type" represent an inducing magnetic field (external to the conductor) and the magnetic and electric fields induced by it.

"Elementary solutions of the second type" represent the free decay of certain distribution of electric current in the conductor, which has zero magnetic field outside the conductor.

Here we are only concerned with solutions of the first type, and refer only to them without further mention.

If we assume that the source of the inducing field lies in the region $z \geq h$, the (total) magnetic field in the region $0 < z < h$ is conservative, and Equation (9) becomes trivial for $\vec{H}$, because of (1b). In this region, $\vec{H}$ is derived from a scalar potential, say, Ω, which satisfies Laplace's equation. Hence Ω can be expressed as a summation, or an integral, of functions of the form

$$\Omega = -\left\{ A(t)\, e^{\lambda z} + B(t)\, e^{-\lambda z} \right\} P(x,y) \tag{10}$$

where λ is real and positive (or zero) and P(x,y) satisfies the equation

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \lambda^2 P = 0 \tag{11}$$

In (10) we can identify $-A(t)e^{\lambda z}P(x,y)$ as the potential of the inducing field and $-B(t)e^{-\lambda z}P(x,y)$ as the induced field.

$\vec{H}$ is given by

$$\begin{aligned}\vec{H} &= -\ \mathrm{grad}\ \Omega \\ &= \mathrm{grad}\left[\left\{A(t)e^{\lambda z} + B(t)e^{-\lambda z}\right\}P(x,y)\right]\end{aligned} \tag{12}$$

The solution of (9) for $\vec{E}$ is readily obtained in the region $0 < z < h$ as

$$\vec{E} = -\frac{1}{c\lambda}\left\{\dot{A}(t)\ e^{\lambda z} - \dot{B}(t)e^{-\lambda z}\right\}\left(\frac{\partial P}{\partial y}, -\frac{\partial P}{\partial x}, 0\right) \tag{13}$$

where the factor in front of the brackets is introduced so that $\vec{H}$, derived from Ω, and $\vec{E}$ satisfy Equation (1a); dots signify time derivatives. Physically, the part corresponding to the first term in the brackets in the expression (13) refers to the electric field associated with the inducing magnetic field, and that corresponding to the second term to the electric field associated with the induced field.

In the conductor, i.e. in the region $z < 0$, $\vec{E}$ can be written as

$$\vec{E} = Z(z,t)\left(\frac{\partial P}{\partial y}, -\frac{\partial P}{\partial x}, 0\right) \tag{14}$$

where Z(z,t) satisfies a differential equation

$$\frac{\partial^2 Z}{\partial z^2} - \lambda^2 Z - \frac{4\pi K\mu}{c^2}\frac{\partial Z}{\partial t} = 0 \tag{15}$$

The boundary condition that the tangential components of $\vec{E}$ are continuous gives a relation

$$Z(-0,t) = -\frac{1}{c\lambda}\left\{\dot{A}(t) - \dot{B}(t)\right\} \tag{16}$$

The corresponding magnetic field in this region is readily obtained from (13) and (1a):

$$\mu\dot{\vec{H}} = -c\left(\frac{\partial Z}{\partial z}\frac{\partial P}{\partial x}, \frac{\partial Z}{\partial z}\frac{\partial P}{\partial y}, \lambda^2 ZP\right) \tag{17}$$

This expression and $\vec{H}$ given by (12) satisfy the boundary condition that the normal component of $\vec{B}$ (or $\dot{\vec{B}}$) is continuous at $z = 0$ by virtue of (16). The remaining boundary condition, that the tangential components of $\vec{H}$ are continuous at $z = 0$, leads to

$$Z'(-0,t) = -\frac{\mu}{c}\left\{\dot{A}(t) + \dot{B}(t)\right\} \tag{18}$$

where Z' signifies $\partial Z/\partial z$.

The electric current density in the conductor is given by $\vec{j} = \kappa\vec{E}$ with the expression (14) for E. The immediate inference is that <u>the induced current flows everywhere parallel to the surface of the conductor</u>.

5. Applications of the Present Method

The solutions discussed in Section 4 can be applied to any case when the scalar potential of the inducing field can be written in the region $0 < z < h$, as a summation over discrete values of λ, or an integral over λ, of functions of the form

$$\Omega_o = -A(t)\, e^{\lambda z}\, P(x,y)$$

or, by introducing a constant factor

$$\Omega_o = -A(t)\, e^{\lambda(z-h)}\, P(x,y).$$

Putting $z' = h - z$, Ω_o is

$$\Omega_o = -A(t)\, e^{-\lambda z'} P(x,y)$$

Obviously the case when Ω_o takes an integral form has wide applications. Suppose that the potential Ω_o is given by $-A(t)\,F(x,y,z')$. Then the problem is to express $F(x,y,z')$ in the form

$$F(x,y,z') = \int_0^{\infty} e^{-\lambda z'} P_\lambda (x,y)\, d\lambda \tag{19}$$

Regarding $F(x,y,z')$ as a function of z' and denoting it by $h(z')$, Equation (19) may be looked upon as the inverse Laplace transform of the given function $h(z')$, i.e.

$$h(z') = \int_0^{\infty} e^{-\lambda z'} f(\lambda)\, d\lambda \tag{20}$$

where $z' > 0$.

When the inducing field is periodic in time, the problem is much simplified because of the linearity of the equations. Taking a single harmonic of period $2\pi/p$, the expression (10) may be written as

$$\Omega = -\left(A e^{\lambda z} + B e^{-\lambda z}\right) e^{ipt} P(x,y) \tag{21}$$

where A and B are complex constants and the real part of the right-hand side is to be taken.

The equation (15) for $Z(z,t)$ now becomes

$$\frac{\partial^2 Z}{\partial z^2} - \left(\frac{i4\pi K\mu p}{c^2} + \lambda^2\right) Z = 0 \tag{22}$$

The solution of (22) is

$$Z = (a\, e^{\theta z} + b\, e^{-\theta z})\, e^{ipt} \tag{23}$$

where

$$\theta^2 = i4\pi \kappa \mu\, p/c^2 + \lambda^2 \tag{24}$$

or

$$\theta = \frac{1}{\sqrt{2}}\left[\sqrt{\sqrt{\alpha^4 + \lambda^4} + \lambda^2} + i\sqrt{\sqrt{\alpha^4 + \lambda^4} - \lambda^2}\right] \tag{25}$$

where

$$\alpha^2 = 4\pi \kappa \mu \;\; p/c^2 \tag{26}$$

Because of the form of (23) the real part of θ can be taken to be positive without loss of generality.

Z must tend to zero as z tends to $-\infty$. Therefore, the coefficient b is zero. The two conditions (16) and (18) determine a and B in terms of A:

$$a = -\frac{2\mu\, ip}{c(\theta + \lambda\mu)} A \tag{27}$$

$$B = \frac{\theta - \lambda\mu}{\theta + \lambda\mu} A \tag{28}$$

Thus the scalar potential of the induced magnetic field outside the conductor is

$$\Omega_i = -\frac{\theta - \lambda\mu}{\theta + \lambda\mu} A\, e^{-\lambda z + ipt} P(x,y) \tag{29}$$

The electric field outside the conductor is

$$\vec{E} = -\frac{ip}{c\lambda} A \left[e^{\lambda z} - \frac{\theta - \lambda\mu}{\theta + \lambda\mu} e^{-\lambda z}\right] e^{ipt}\left(\frac{\partial P}{\partial y}, -\frac{\partial P}{\partial x}, 0\right) \tag{30}$$

The electric field inside the conductor is given by

$$E = -\frac{2\mu ip}{c(\theta+\lambda\mu)} A e^{ipt+\theta z} \left(\frac{\partial P}{\partial y}, -\frac{\partial P}{\partial x}, 0\right) \tag{31}$$

The current density inside the conductor is

$$\vec{j} = K\vec{E}$$
$$= -\frac{2K\mu ip}{c(\theta+\lambda\mu)} A e^{ipt+\theta z}\left(\frac{\partial P}{\partial y}, -\frac{\partial P}{\partial x}, 0\right) \tag{32}$$

6. Periodic Infinite Linear Current

In this Section we consider, as the source of the inducing field, a periodic infinite linear current of intensity $cj_o e^{ipt}$ e.s.u. flowing in the direction of the (positive) x-axis parallel to the surface of the conductor. The magnetic scalar potential of this inducing field can be written as

$$\Omega_o = -2\, j_o e^{ipt} \tan^{-1}(y/(h-z)) \tag{33}$$

Using the Laplace transform[11]

$$\tan^{-1}(ap^{-1}) = \int_0^\infty e^{-pt}\, t^{-1} \sin(at)\, dt, \tag{34}$$

Ω_o can be expressed as

$$\Omega_o = -2\, j_o\, e^{ipt} \int_0^\infty e^{-\lambda(h-z)} \sin\lambda y \frac{d\lambda}{\lambda} \tag{35}$$

From (29) Ω_i becomes

$$\Omega_i = -2\, j_o\, e^{ipt} \int_0^\infty \frac{\theta-\lambda\mu}{\theta+\lambda\mu} e^{-\lambda(h+z)} \sin\lambda y \frac{d\lambda}{\lambda} \tag{36}$$

From (30) $\vec{E}$ inside the conductor is given by

$$\vec{E} = \left(-2\, j_o\, e^{ipt} \int_0^\infty \frac{2\mu ip}{c(\theta+\lambda\mu)} e^{-\lambda h+\theta z} \cos\lambda y\; d\lambda\, 0,\ 0\right) \tag{37}$$

When $\mu = 1$, Equation (37) is much simplified; the x-component of $\vec{E}$ becomes

$$E_x = - \frac{c\, j_0}{\pi K} e^{ipt} \int_0^\infty (\theta - \lambda)\, e^{-\lambda h + \theta z} \cos \lambda y \, d\lambda \qquad (37a)$$

where θ is a complex number containing α and λ ; it is given by (25).

At z = 0, Equation (37a) is further simplified; the real part of (37a) becomes

$$E_x = \frac{2\, j_0 p}{c} \left[I_1\,(\xi, \eta) \cos pt + I_2\,(\xi, \eta) \sin pt \right] \qquad (38)$$

where

$$I_1\,(\xi, \eta) = \int_0^\infty \left[2u - \sqrt{2} \sqrt{\sqrt{1 + u^4} + u^2} \right] e^{-\xi u} \cos \eta u \, du \qquad (39)$$

$$I_2\,(\xi, \eta) = \int_0^\infty \sqrt{2} \sqrt{\sqrt{1 + u^4} - u^2}\; e^{-\xi u} \cos \eta u \, du \qquad (40)$$

and $\xi = \alpha h$, $\eta = \alpha y$. Here the variable of integration was changed from λ to u by a transformation $u = \lambda / \alpha$.

The integrals I_1 and I_2 can be written as

$$I_1(\zeta) = \frac{1}{2} \int_0^\infty \left[2u - \sqrt{2} \sqrt{\sqrt{1 + u^4} + u^2} \right] \left(e^{-\zeta u} + e^{-\bar{\zeta} u} \right) du \qquad (41)$$

$$I_2(\zeta) = \frac{1}{2} \int_0^\infty \sqrt{2} \sqrt{\sqrt{1 + u^4} - u^2}\; \left(e^{-\zeta u} + e^{-\bar{\zeta} u} \right) du \qquad (42)$$

where

$$\zeta = \xi + i\eta, \quad \bar{\zeta} = \xi - i\eta$$

For large values of $|\zeta|$, I_1 and I_2 can be evaluated by expanding the coefficients of the exponential functions in series. The results are

$$I_1(\zeta) = \frac{1}{2} \Big[2\zeta^{-2} - \sqrt{2}\,(\zeta^{-1} + \zeta^{-3} + 3\zeta^{-5} - 45\zeta^{-7} \pm \ldots.)$$
$$+ 2\bar{\zeta}^{-2} - \sqrt{2}\,\,(\bar{\zeta}^{-1} + \bar{\zeta}^{-3} + 3\bar{\zeta}^{-5} - 45\bar{\zeta}^{-7} \pm \ldots) \Big]$$

$$I_2(\zeta) = \frac{1}{2}\left[\sqrt{2}\,(\zeta^{-1} - \zeta^{-3} + 3\zeta^{-5} + 45\zeta^{-7} \pm \ldots\ldots) + \sqrt{2}\,(\bar{\zeta}^{-1} - \bar{\zeta}^{-3} + 3\bar{\zeta}^{-5} + 45\bar{\zeta}^{-7} \pm \ldots\ldots)\right]$$

For $y = 0$, $\bar{\zeta} = \zeta$; and I_1 and I_2 become

$$I_1(\xi, 0) = 2\xi^{-2} - \sqrt{2}\,(\xi^{-1} + \xi^{-3} + 3\xi^{-5} - 45\xi^{-7} \pm \ldots\ldots)$$

$$I_2(\xi, 0) = \sqrt{2}\,(\xi^{-1} - \xi^{-3} + 3\xi^{-5} + 45\xi^{-7} \pm \ldots\ldots)$$

Values of ξ which we are interested in are not always large compared with unity. Hence the integrals $I_1(\xi,0)$ and $I_2(\xi,0)$ are computed numerically for ξ = 0.1, 0.2, 0.25, 0.5 and 1. For ξ = 5, 8, 10, 100 and 1000, the integrals are estimated by the expansion method shown above. The details of the numerical integration, and the estimate of the remainders of the integrals are given in Appendix.

Values of $I_1(\xi,0)$, $I_2(\xi,0)$ and $\sqrt{I_1^2 + I_2^2}$ are given in Table 1. Fig. 1 graphically shows $I_1(\xi,0)$ and $I_2(\xi,0)$ as functions of ξ in the range 10^{-1} to 10^3; logarithmic scales are used for I_1, I_2 and ξ.

Table 1. $I_1(\xi,0)$, $I_2(\xi,0)$ and $\sqrt{I_1^2 + I_2^2}$

ξ	$I_1(\xi,0)$	$I_2(\xi,0)$	$\sqrt{I_1^2 + I_2^2}$
0.1	-0.743	2.918	3.011
0.2	-0.706	2.313	2.419
0.25	-0.672	2.114	2.218
0.5	-0.586	1.524	1.633
1	-0.460	1.011	1.110
5	-0.217	0.274	0.350
8	-0.148	0.174	0.229
10	-0.123	0.140	0.186
10^2	-0.0139	0.0141	0.0198
10^3	-0.0014	0.0014	0.0020

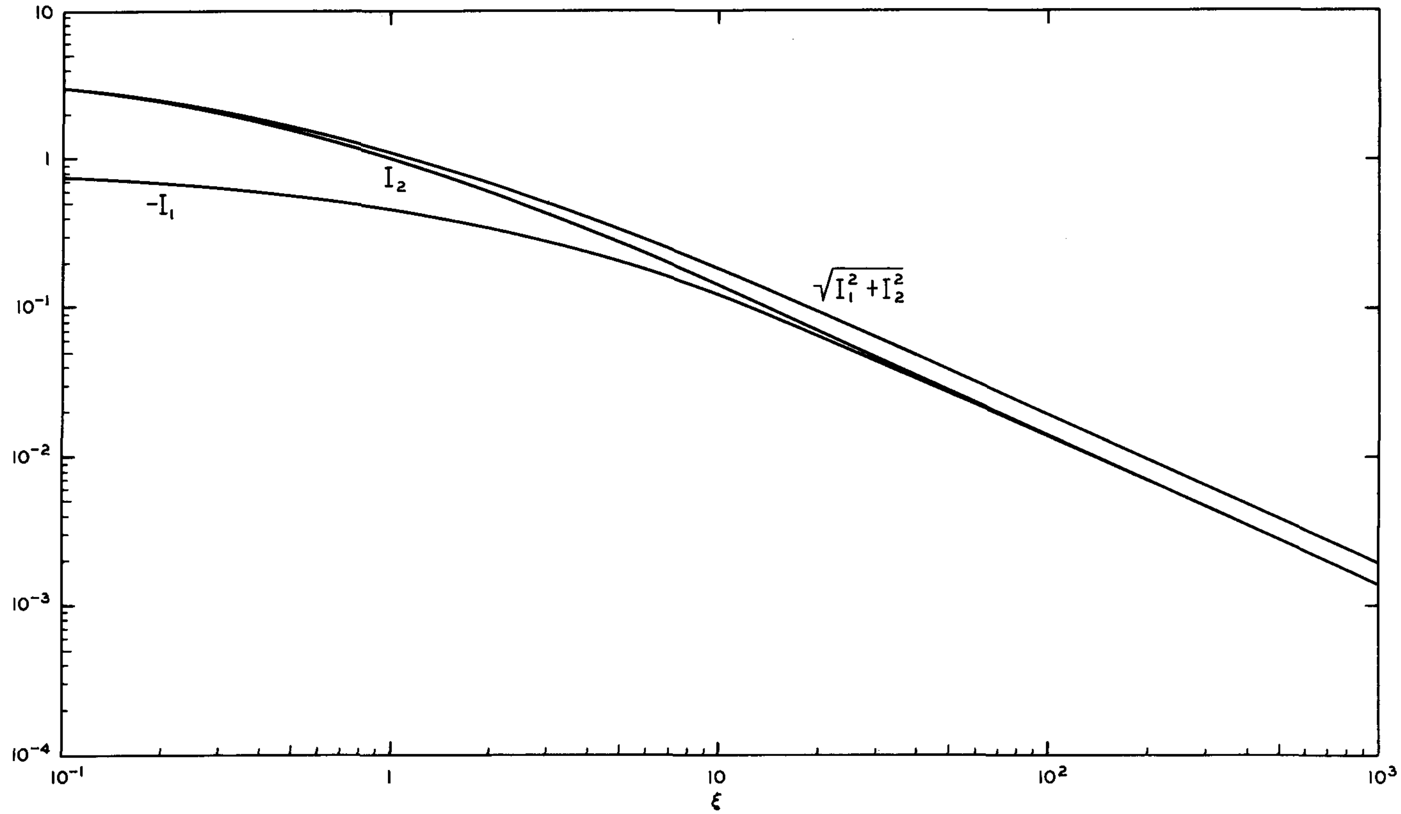

Fig. 1. $I_1(\xi,0)$, $I_2(\xi,0)$, and $\sqrt{I_1^2(\xi,0) + I_2^2(\xi,0)}$ as functions of ξ. Logarithmic scale is used both for ordinates and abscissae.

7. Magnetic Dipole

In this Section we consider a magnetic dipole of moment $\vec{M}$ at R = 0, z = h, with respect to cylindrical coordinates (R, θ, z). We assume that $\vec{M} = (0, 0, m_o e^{ipt})$. The magnetic potential of the dipole is

$$\Omega_o = - m_o e^{ipt} \frac{h - z}{\left\{R^2 + (z-h)^2\right\}^{3/2}}$$

Using the Laplace transform* [p. 182 in reference 11]

$$\frac{p}{(a^2 + p^2)^{3/2}} = \int_0^\infty t\, J_o(at)\, e^{-pt}\, dt,$$

we can express Ω_o as

$$\Omega_o = - m_o e^{ipt} \int_0^\infty \lambda J_o(\lambda R)\, e^{-\lambda(h-z)}\, d\lambda \quad (h-z > 0) \qquad (39)$$

where $J_o(\lambda R)$ is the Bessel function of order zero.

By means of Equation (29) we obtain the expression for the scalar potential of the induced field:

$$\Omega_i = - m_o e^{ipt} \int_0^\infty \frac{\theta - \lambda\mu}{\theta + \lambda\mu} \lambda\, J_o(\lambda R)\, e^{-\lambda(h-z)}\, d\lambda \qquad (40)$$

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* This expression can also be derived from the integral of Lipschitz[12]

$$\int_0^\infty e^{-at} J_o(bt)dt = (a^2 + b^2)^{-1/2}$$

with certain conditions imposed on a and b to secure the convergence of this integral. By differentiating both sides with respect to a, one obtains

$$\int_0^\infty t\, e^{-at} J_o(bt)dt = a\,(a^2 + b^2)^{-3/2}$$

The electric field $\vec{E}$ has only one non-vanishing component E_θ in both regions $z > 0$ and $z < 0$. The lines of electric force are everywhere circular and parallel to the surface of the conductor.

In the region $z > 0$,

$$\vec{E} = -\frac{ip}{c} m_0 e^{ipt} \int_0^\infty e^{-\lambda h} \left(e^{\lambda z} - \frac{\theta - \lambda\mu}{\theta + \lambda\mu} e^{-\lambda z} \right) (0, \lambda J_1(\lambda R), 0)\, d\lambda \quad (41)$$

where $J_1(\lambda R)$ is the Bessel Function of order 1.

In the region $z < 0$,

$$\vec{E} = -\frac{2\mu ip\, m_0}{c} e^{ipt} \int_0^\infty \frac{\lambda}{\theta + \lambda\mu} e^{-\lambda h + \theta z} (0, \lambda J_1(\lambda R), 0)\, d\lambda \quad (42)$$

Taking $\mu = 1$, we simplify the expression of E_θ in the region $z < 0$:

$$\frac{1}{\theta + \lambda} = \frac{\theta - \lambda}{i\alpha^2} = \frac{(\theta - \lambda)\, c^2}{i 4\pi K p}$$

so that we have

$$E_\theta = -\frac{m_0 c}{2\pi K} \int_0^\infty \lambda^2 (\theta - \lambda) J_1(\lambda R)\, e^{-\lambda h + \theta z + ipt} d\lambda \quad (z > 0)$$

$$j_\theta = -\frac{m_0 c}{2\pi} \int_0^\infty \lambda^2 (\theta - \lambda) J_1(\lambda R)\, e^{-\lambda h + \theta z + ipt} d\lambda \quad (z > 0)$$

where θ, as before, is

$$\theta = \frac{1}{\sqrt{2}} \left\{ \sqrt{\sqrt{\alpha^4 + \lambda^4} + \lambda^2} + i \sqrt{\sqrt{\alpha^4 + \lambda^4} - \lambda^2} \right\}$$

Taking the real part of the right-hand side of the expression for E_θ, we have

$$E_\theta = \frac{m_0 p \alpha^2}{c} \left[K_1(\alpha h, \alpha R) \cos pt + K_2(\alpha h, \alpha R) \sin pt \right] \quad (43)$$

where

$$K_1(\xi,\eta) = \int_0^{\infty}\left[2u-\sqrt{2}\sqrt{\sqrt{1+u^4}+u^2}\right]u^2 J_1(\eta u)\, e^{-\xi u}\, du \qquad (44)$$

$$K_2(\xi,\eta) = \int_0^{\infty}\sqrt{2}\, u^2 \sqrt{\sqrt{1+u^4}-u^2}\; J_1(\eta u)\, e^{-\xi u}\, du \qquad (45)$$

Noting that for real x $|J_0(x)| \leqslant 1$, $|J_r(x)| \leqslant 1/\sqrt{2}$ (r=1,2,...)(p.31 of reference 12), and expanding the coefficients of $J_1(\eta u)e^{-\xi u}$ in powers of u^{-1}, one can easily show that the two integrals are convergent. For the purpose of order estimation K_1 and K_2 are numerically integrated for a particular set of values of ξ and η, $\xi = 1$ and $\eta = 0.1$. The numerical integration is made for u = 0 to 4 with intervals of 0.2 and for u = 4 to 10 with intervals of 0.25.

The valuse of K_1 and K_2 for $\xi = 1$ and $\eta = 0.1$ are

$$K_1(1.0,\ 0.1) = -\ 0.006$$

$$K_2(1.0,0.1) = 0.095$$

K_1 is rapidly convergent; the remainder of the integral is of order of 0.001 or likely to be less. K_2 converges less rapidly than K_1, but the remainder of the integral is of order 0.01 or less.

8. Discussions

8.1 Periodic linear current

From Equation (38) the amplitude of the electric field, $|E|$, is

$\frac{2j_0 p}{c}\sqrt{I_1^2 + I_2^2}$, or $\frac{4\pi j_0}{cT}\sqrt{I_1^2 + I_2^2}$, where T is the period of the variation ($T = 2\pi/p$).

At the ground directly beneath the current, i.e., $z = 0$ and $y = 0$, the amplitude of the total magnetic field, $|H|$, is $2j_o/h$ Gauss; its period is T seconds.

Hence we can express $|E|$ in terms of $|H|$ as follows.

$$|E| = \frac{2\pi h}{cT} |H| \sqrt{I_1^2 + I_2^2}$$

For $T = 60$ seconds and $K = 10^{-14}$ e.m.u., α is 1.1×10^{-7}. We assume that the height of the current is of order 100 km, or 10^7 cm. Then ξ ($= \alpha h$) is of order 1. For $\xi = 1$ and $\eta = 0$, $\sqrt{I_1^2 + I_2^2}$ is very nearly 1 (Table 1). Thus

$$\begin{aligned} |E| \approx \frac{2\pi \times 10^7}{3 \times 10^{10} \times 60} |H| &\approx 3 \times 10^{-5} \; |H| \quad \text{e.s.u.} \\ &= 300 \times 3 \times 10^{-5} \; |H| \quad \text{volts/cm} \\ &\approx 10^{-2} \; |H| \quad \text{volts/cm} \\ &= 10^3 \; |H| \quad \text{volts/km} \end{aligned}$$

Thus, for $|H|$ of order 100 gamma ($= 10^{-3}$ Gauss) $|E|$ is of order 1 volt/km, or 1000 milli-volts/km.

8.2 Periodic magnetic dipole

From Equation (43) the amplitude of the electric field $|E|$ is $\frac{m_o p \alpha^2}{c} \sqrt{K_1^2 + K_2^2}$, or $\frac{2\pi m_o \alpha^2}{c\,T} \sqrt{K_1^2 + K_2^2}$, where T is the period of the variation of the dipole moment. The magnetic field of a dipole with a moment m_o is of order m_o/r^3 at distance $\underline{r}$. The electric field is zero directly beneath the dipole ($R = 0$), because $J_1(0) = 0$. We estimate the electric field at a distance 10 km ($= 10^6$ cm) from the point on the ground directly below the dipole. We again assume that $T = 60$ seconds, $K = 10^{-14}$

e.m.u., h = 100 km (= 10^7 cm). Then $\alpha = 10^{-7}$; $\xi = \alpha h = 1$, and $\eta = \alpha R = 0.1$. For these values of ξ and η, $\sqrt{K_1^2 + K_2^2} \approx 0.1$ (Section 6). The amplitude of the magnetic field $|H|$ is of order m_o/h^3. Hence we have

$$
\begin{aligned}
|E| &= \frac{2\pi m_o \alpha^2}{cT} \sqrt{K_1^2 + K_2^2} \\
&\approx \frac{2\pi h^3 \alpha^2}{cT} \sqrt{K_1^2 + K_2^2} \; |H| \\
&\approx \frac{2\pi \times 10^{21} \times 10^{-14}}{3 \times 10^{10} \times 60} \times 10^{-1} \; |H| \\
&\approx 3 \times 10^{-6} \; |H| \quad \text{e.s.u.} \\
&\approx 10^{-3} \; |H| \quad \text{volts/cm} \\
&= 10^{2} \; |H| \quad \text{volts/km}
\end{aligned}
$$

For $|H|$ of order 100 gamma (= 10^{-3} Gauss), $|E|$ is of order 0.1 volt/km, or 100 milli-volts/km.

8.3 General discussion

The infinite linear current may be considered as being a typical example of a system of electric current of fairly large scale, and the circular current as representing an example of a regional current system of limited size. The above results indicate that in both cases the calculated electric field is of the same order of magnitude as that actually observed in high latitudes.

In the above discussions the electric conductivity of the earth is assumed to be uniform and isotropic. This is, of course, an idealized model of the earth. In reality the conductivity varies horizontally as well as vertically, and is, in general, not isotropic near the surface of the earth. However, the non-isotropy may be ignored in the order of magnitude estimate we are concerned with.

We now give some consideration to the non-uniformity of the electric conductivity. The penetration of the inducing magnetic field into the earth, and hence the spatial distribution of the induced electric current, depends on the period of the inducing field, if it is periodic, or on the rapidity of its change, if aperiodic. Therefore, when a model with a uniform conductivity is adopted to represent the earth, the 'effective' conductivity is dependent on the period, or rapidity, of the change in the inducing magnetic field.

With this in mind, let us re-examine the result for the infinite linear current. Suppose that the amplitude of the electric field, $|E|$, is in the range $10 \leqslant E \leqslant 10^3$ milli-volts/km, or $3 \times 10^{-10} \leqslant |E| \leqslant 3 \times 10^{-8}$ e.s.u. in c.g.s. units, and that the amplitude of the magnetic field, $|H|$, is not greater than 100 gamma, or 10^{-3} Gauss. We examine if the 'effective' conductivity that is compatible with the relation between $|E|$ and $|H|$ with $|E|$ and $|H|$ within the prescribed ranges is of a reasonable order of magnitude. Here we consider changes with period (i) 1 second, (ii) 1 minute, and (iii) 5 minutes.

(i) $T = 1$ second.

For $|H| = 10^{-3}$ Gauss (100 gamma), the inequalities

$$3 \times 10^{10} \leqslant |E| = 2 \times 10^{-3} \frac{\sqrt{I_1^2 + I_2^2}}{T} |H| \leqslant 3 \times 10^{-8} \text{ e.s.u.}$$

give us a range for $\sqrt{I_1^2 + I_2^2}$ as

$$1.5 \times 10^{-4} \leqslant \sqrt{I_1^2 + I_2^2} \leqslant 1.5 \times 10^{-2}$$

For $|H| = 10^{-4}$ Gauss (10 gamma), the inequalities give us

$$1.5 \times 10^{-3} \leqslant \sqrt{I_1^2 + I_2^2} \leqslant 1.5 \times 10^{-1}$$

For large values of ξ, I_1 tends to $-\sqrt{2}/\xi$, and I_2 to $\sqrt{2}/\xi$; thus $\sqrt{I_1^2 + I_2^2}$ is very nearly $2/\xi$, when ξ is large. Using this asymptotic expression and the curves in Fig. 1, the above inequalities for $\sqrt{I_1^2 + I_2^2}$ can readily be transformed to those for ξ.

For $|H| = 100$ gamma: $1.3 \times 10^4 \geqslant \xi \geqslant 1.3 \times 10^2$

for $|H| = 10$ gamma: $1.3 \times 10^3 \geqslant \xi \geqslant 15$

In terms of K these relations give

for $|H| = 100$ gamma: $2 \times 10^{-8} \geqslant K \geqslant 3 \times 10^{-12}$ e.m.u.

for $|H| = 10$ gamma: $2 \times 10^{-10} \geqslant K \geqslant 3 \times 10^{-14}$ e.m.u.

Thus $|H|$ of order of 10 gamma and less will account for the variation of $\vec{E}$ with this period and its amplitude in the range considered here.

(ii) T = 60 seconds.

By a similar procedure as in (i), the inequalities

$$3 \times 10^{-10} \leqslant |E| \leqslant 3 \times 10^{-8} \quad \text{e.s.u.}$$

give

for $|H| = 100$ gamma: $0.01 \leqslant \sqrt{I_1^2 + I_2^2} \leqslant 1$

for $|H| = 10$ gamma: $0.1 \leqslant \sqrt{I_1^2 + I_2^2} \leqslant 10$

In terms of ξ these relations become

for $|H| = 100$ gamma: $200 \geqslant \xi \geqslant 1.5$

for $|H| = 10$ gamma: $20 \geqslant \xi$ (lower limit for $\xi < 0.1$)

Or, in terms of K we have

for $|H| = 100$ gamma: $3 \times 10^{-10} \geqslant K \geqslant 1.7 \times 10^{-14}$

for $|H| = 10$ gamma: $3 \times 10^{-12} \geqslant K$ (lower limit for $K < 10^{-16}$)

Thus $|H|$ of order 100 to 10 gamma and less can produce the variations of $\vec{E}$ of the period and amplitude considered here.

It is now apparent that we can interpret without any inconsistency the electric field of order 10 to 100 milli-volts with period of order 1 second to several minutes as being the electric field induced by some inducing magnetic field with amplitude of order 100 to 10 gamma or less and with the same period as the electric field.

An exact interpretation of the observed earth-current data is extremely difficult and is practically impossible because of the following two reasons. First, the complete knowledge of the inducing field is required to compute the induced magnetic or electric field, and conversely the distribution of the induced magnetic or electric field must be known in order to deduce the inducing field. Secondly, in addition to this, the distribution of the electric conductivity in the earth must be known through the depth in which the inducing magnetic field penetrates.

Another approach to the problem of earth-currents is to assume the inducing magnetic field, which may, to a certain extent, if not completely, be inferred from the magnetic observations made over areas comparative with the extension of the inducing field, and to determine the distribution of the electric conductivity of the earth so as to give as good an agreement as possible with the actually observed induced field. This method does not, in general, determine the distribution of conductivity uniquely. But with suitable assumptions, we can obtain useful informations on the distribution of the electric conductivity of the earth. This method has been widely employed to determine the conductivity in the interior of the earth, e.g. by Chapman(13), Chapman and Whitehead(14), Chapman and Price(15), Price(16), Lahiri and Price[10], and others. A similar method may be used for a regional conductivity survey. Such a study will also be valuable in evaluating the observational results of the geomagnetic field on a regional basis. The utilization of the (natural) electromagnetic field for geological surveying is now studied in Russia under the sponsorship of the Geophysical Institute of the Academy of Sciences of the U.S.S.R.[17]

8.4 Some remarks on a simple method of estimating the electric field which is frequently misused

By integrating Equation (1a) over a surface Σ with a closed boundary Γ, we obtain

$$\int_{\Sigma} (\mathrm{curl}\,\vec{E}) \cdot d\vec{\Sigma} = -\frac{1}{c} \int_{\Sigma} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{\Sigma} \qquad (46)$$

where $d\vec{\Sigma} = \vec{n}\, d\Sigma$, $\vec{n}$ being a unit vector normal to the surface element $d\Sigma$. By applying Stokes's theorem to the member on the left-hand side of (46), we have

$$\oint_{\Gamma} \vec{E} \cdot d\vec{\ell} = -\frac{1}{c} \int_{\Sigma} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{\Sigma} \qquad (47)$$

where $d\vec{\ell} = \vec{\tau}\, d\ell$, τ being a unit vector tangent to the line element $d\ell$. The direction of $\vec{n}$ and $\vec{\tau}$ are to be taken so that looking from the side of the surface to which $\vec{n}$ is pointed, $\vec{\tau}$ is directed counterclockwise.

The equation of lines of force is given by $\vec{E} \times d\vec{\ell} = 0$; with respect to Cartesian coordinates the equations are $dx/E_x = dy/E_y = dz/E_z$. If the contour Γ coincides with a line of electric force, $\vec{E} \cdot d\vec{\ell} = |\vec{E}|\, d\ell$, and if further $|\vec{E}|$ is constant along the contour, one can write Equation (47) in the form

$$|\vec{E}| = -\frac{1}{c}\int_{\Sigma}\frac{\partial\vec{B}}{\partial t}\cdot d\vec{\Sigma}/L \qquad (48)$$

where $L = \oint_{\Gamma} d\ell$, i.e. the length of the contour.

Equation (48) is frequently used for the purpose of estimating the intensity of the induced electric field. This procedure is valid only (i) when the contour Γ that forms the boundary of the surface Σ is coincident with a line of electric force*, and further (ii) when $|\vec{E}|$ is constant along it. In general, lines of electric force are unknown; even if a line of force is known, $|\vec{E}|$ is not necessarily constant along it. It should be borne in mind that the equation of lines of force, i.e. $\vec{E} \times d\vec{\ell} = 0$, does not contain the magnitude of the field vector, but only its direction.

A simple case for which Equation (48) can be used is the one in which the field quantities are axially symmetrical. In this case, if the axis of symmetry is known in some way, Equation (48) may be used.

Summarizing, while Equation (47) is always valid for any contour Γ and any surface that is bounded by Γ, Equation (48) is valid only under limited conditions.

9. Conclusion

There appears to be no internal inconsistency in the interpretation that the rapid fluctuations in earth-currents observed in high latitudes

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* Since div $\vec{E} = 0$, the vector field $\vec{E}$ is solenoidal, and lines of force are always closed.

are due to the electromagnetic induction by some inducing magnetic field.

Exact physical interpretations of the observed earth-current records are practically impossible. However, assuming a suitable distribution of the inducing magnetic field, the earth-current records may be used to determine the distribution of the electric conductivity of the earth.

Though the amplitude analysis is probably not helpful in constructing a physical picture of the upper atmospheric disturbance, the period analysis may contribute greatly to our understanding of the physical properties of the upper atmosphere.

As a disturbance indicator the measurement of earth-currents appears to be superior to that of the geomagnetic field with the instruments currently used.

10. Acknowledgements

The author is indebted to Mrs. Sandra J. Fuller for carrying out the numerical integration involved in this work. He wishes to express his thanks to Mr. Wallace B. Murcray for the discussions that motivated the author to examine the problems of earth-currents. The author's thanks are also due to Drs. Manfred H. Rees, George C. Reid, and Gian-Carlo Rumi for reading the manuscript and for valuable suggestions.

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APPENDIX

The integrals $I_1(\xi,0)$ and $I_2(\xi,0)$ are

$$I_1(\xi,0) = \int_0^\infty \left[2u - \sqrt{2}\sqrt{\sqrt{1+u^4}+u^2} \right] e^{-\xi u}\, du$$

$$I_2(\xi,0) = \int_0^\infty \sqrt{2}\sqrt{\sqrt{1+u^4}-u^2}\; e^{-\xi u}\, du$$

where $\xi > 0$.

$I_1(\xi,0)$ and $I_2(\xi,0)$ were numerically integrated from 0 to 4 with intervals of 0.2, from 4 to 10 with intervals of 0.25 from 10 to 15 with intervals of 0.5, and from 15 to 20 with intervals of 1. The remainders of the integrals were estimated in the following way.

For large values of u, $2u - \sqrt{2}\sqrt{\sqrt{1+u^4}+u^2}$ can be expanded in a convergent series

$$-\frac{1}{4}\left(\frac{1}{u}\right)^3 + \frac{5}{64}\left(\frac{1}{u}\right)^7 \mp \;\ldots\ldots$$

Hence, when u_1 is large, we have

$$\int_{u_1}^\infty \left[2u - \sqrt{2}\sqrt{\sqrt{1+u^4}+u^2}\right] e^{-\xi u} du = \int_{u_1}^\infty \left[-\frac{1}{4}\left(\frac{1}{u}\right)^3 + \frac{5}{64}\left(\frac{1}{u}\right)^7 \mp \ldots\right] e^{-\xi u}\, du$$

Thus this integral tends to $-\frac{1}{4}\int_{u_1}^\infty u^{-3} e^{-\xi u} du$, when u_1 is large. Since

$\xi > 0$, $\frac{1}{4}\int_{u_1}^\infty u^{-3} e^{-\xi u}\, du < \frac{1}{4}\int_{u_1}^\infty u^{-3}\, du = \frac{1}{8} u_1^{-2}$, which is 0.0003125 for $u_1 = 20$. The values of $I_1(\xi,0)$ evaluated by the numerical integration from $u = 0$ to 20 range from - 0.743 to - 0.460 for $\xi = 0.1$ to 1. Hence the remainder of the integral can be considered as being negligible in all cases.

For the integral $I_2(\xi,0)$, $\sqrt{2}\sqrt{\sqrt{1+u^4}-u^2}$ can be expanded in a convergent power series

$$\frac{1}{u} - \frac{1}{8}\left(\frac{1}{u}\right)^5 \pm \;\ldots\ldots$$

Hence, when u_1 is large, we have

$$\int_{u_1}^{\infty} \sqrt{2}\sqrt{\sqrt{1+u^4}-u^2}\; e^{-\xi u} du = \int_{u_1}^{\infty} \left[\frac{1}{u} - \frac{1}{8}\left(\frac{1}{u}\right)^5 \pm \ldots\right] e^{-\xi u} du$$

Thus the integral tends to $\int_{u_1}^{\infty} u^{-1} e^{-\xi u} du$. Now

$$0 < \int_{u_1}^{\infty} u^{-1} e^{-\xi u} du = \frac{e^{-\xi u_1}}{\xi u_1} - \frac{1}{\xi}\int_{u_1}^{\infty} u^{-2} e^{-\xi u} du.$$

Since ξ and u_1 are positive,

$$\frac{e^{-\xi u_1}}{\xi u_1} > 0, \quad \frac{1}{\xi}\int_{u_1}^{\infty} u^{-2} e^{-\xi u} du > 0.$$

Hence

$$\int_{u_1}^{\infty} u^{-1} e^{-\xi u} du < \frac{e^{-\xi u_1}}{\xi u_1}$$

For $u_1 = 20$, the member on the right-hand side of this inequality is 0.068, 0.005, 0.001, for $\xi = 0.1$, 0.2, 0.25, respectively; it is less than 0.0001 for $\xi = 0.5$ and 1. The values of I_2 evaluated by a numerical integration over $u = 0$ to 20 for $\xi = 0.1$, 0.2, 0.25, 0.5 and 1 are 2.918, 2.314, 2.115, 1.526 and 1.015, respectively. Therefore, the remainder is not more than a few percent of the integral in each of the cases.

For $\xi = 5$, I_1 and I_2 were computed both by the numerical integration and the series expansion. The value of I_1 obtained by the former method is - 0.2147, and that obtained by the latter - 0.2174; the two values agree within 1.3%. The value of I_2 evaluated by the numerical integration is 0.2742 and that by the series expansion 0.2737; these agree within 0.2 %.