Inverted Sugar: Generalized Ray Tracing Algorithm

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Abstract

Though particular algorithms utilizing Snell's Law effectively describe most ray tracing, issues arise with infrasound applications, particularly in understanding waveguides. A more generalized differential equation derived from Fermat's principle of least time and general functions may provide a computational solution to coupling ray tracing with wavelength related issues. However, due to its complexity, this differential equation deserves its own experimentation to confirm or deny its effectiveness. In this experiment, a gradient of index of refraction established by the diffusion of sugar emulated an atmospheric temperature inversion (hence "Inverted Sugar") and comparison of actual rays to computer generated rays in the same gradient confirmed the accuracy of this algorithm.

Background/Theory

Most individuals encounter Snell's Law in an introductory physics course. The underlying principle from which it arises, Fermat's Theorem of Least Time, is avoided due to it's mathematical opaqueness. Beyond the math, though, the principle is quite elegant: rays travel along the path that minimizes time of travel. Starting in one medium where speed is constant and ending in another where the speed is another constant, the light travels in two straight lines with different slopes, but in a continuous medium, the light constantly bends to match this minimal time path. Snell's Law, even applied with tiny layers of different medium, loses this information, possibly losing the more nuanced behaviors that nature loves. Infrasound waveguides manifest these, showing peculiar preference for particular frequencies and even splitting them and demonstrating a faux Doppler effect. When current analysis tools fail, return to first principles. From Fermat's theorem, assuming sole travel in the xz plane, and a general function of speed with height v(z), the following differential equation arises:

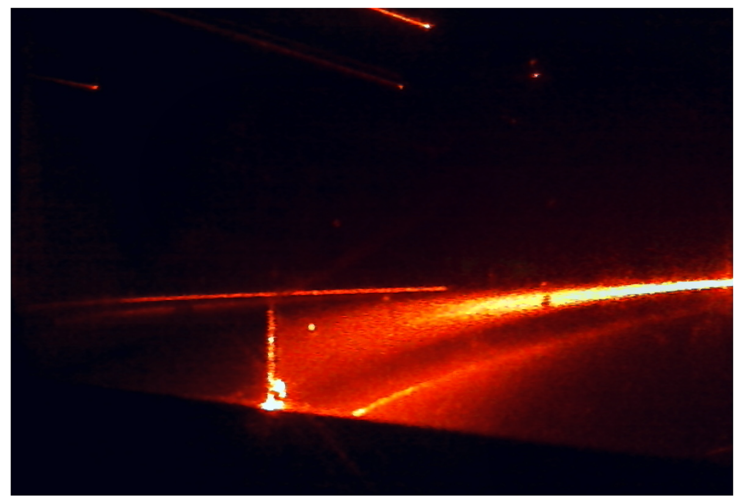


Figure 1: Detail of light's refraction through an index of refraction gradient

$$\frac{d^2z}{dx^2} = -\frac{1}{v(z)}\frac{dv(z)}{dz}(1+\frac{dz}{dx}^2)^2$$

This equation serves as the computational backbone of this ray tracing algorithm. It is observed that this algorithm is computationally stable in the backwards direction when the velocity gradient is negative, therefore the ray solutions were developed from the final conditions backwards.

The speed v(z) = c/n(z), where n(z) is the index of refraction. To model n(z) for this experiment, it is known that the index of refraction varies linearly with density (Heidcamp) and that the solution to the diffusion equation in one dimension comprises exponentials. The following model was thus chosen:

$$n(z) = Ae^{-kz} + n_w$$

where n_w is the index of refraction of sugarless water and *A* and *k* are real positive coefficients determined by the fit of the data. With this information one can completely determine the value of the second derivative of z computationally and therefore predict the path of a ray through this medium.

Procedure

The basic measurements occurred in two main parts:
1.Index of refraction measurements at constant heights (henceforth referred to as z) to determine index of refraction profiles.
2.Ray traces of constant y (the rays travel in an xz plane perpendicular to the smallest faces of the tank) that exhibit the path of the beam described by the least time formulation.

To establish the most consistent measurements possible, the aquarium was fixed to an optics bench. A laser was set up with a constant angle of incidence (in this case, 40 degrees was arbitrarily chosen for all index of refraction measurements.) With the tank unfilled, the laser shone through two panes of glass to the opposite face of glass (parallel to the xz plane); the image is the reference d = 0. Any media placed within the tank with an index of refraction n differing from the index of air n_{air} will cause displacement of the image by d in a predictable fashion found through basic geometry and Snell's law, yielding the relationship:

$$n = n_{air}\sqrt{sin^2(\theta_i) + [\frac{d}{y}sin(\theta_i) + cos(\theta_i)]^2}$$

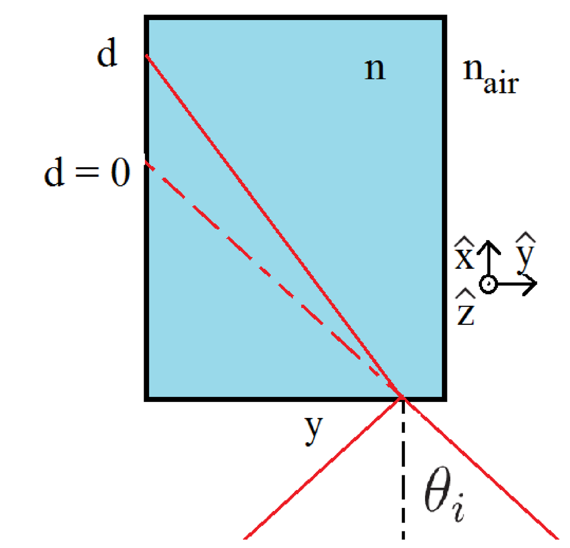


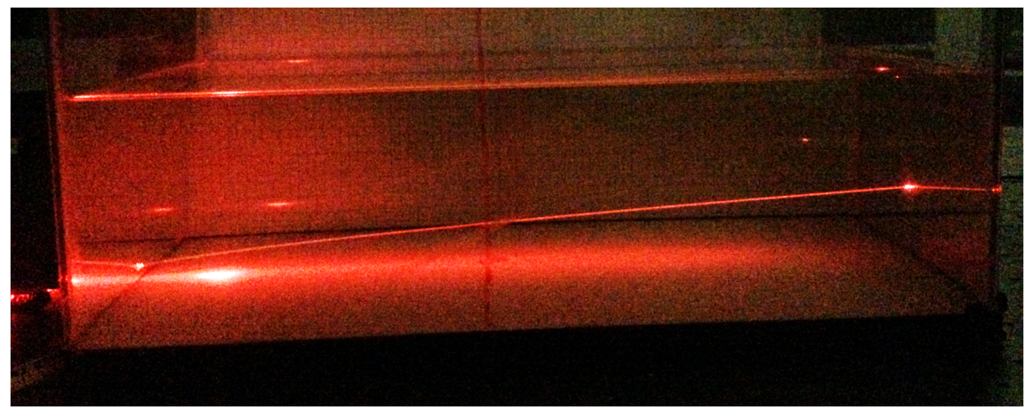
Figure 2: Schematic of tank and index of refraction measurements. The dotted line represents the laser's path when no media diffracts it.

The experimental procedure is as follows:
1. Four lbs of sugar were distributed at the bottom of a 10 gallon aquarium.
2. A magazine cover shielded the sugar from turbulent mixing while 4 gallons of water were added slowly, after which the cover was removed.
3. After 9 hours of diffusion, the first measurements were taken: at least 7 in the vertical direction and a photograph containing the trace in the xz plane.
4. Measurements were repeated about every 8 hours, 3 sets of measurements total.
5. Steps 1-4 were repeated once.

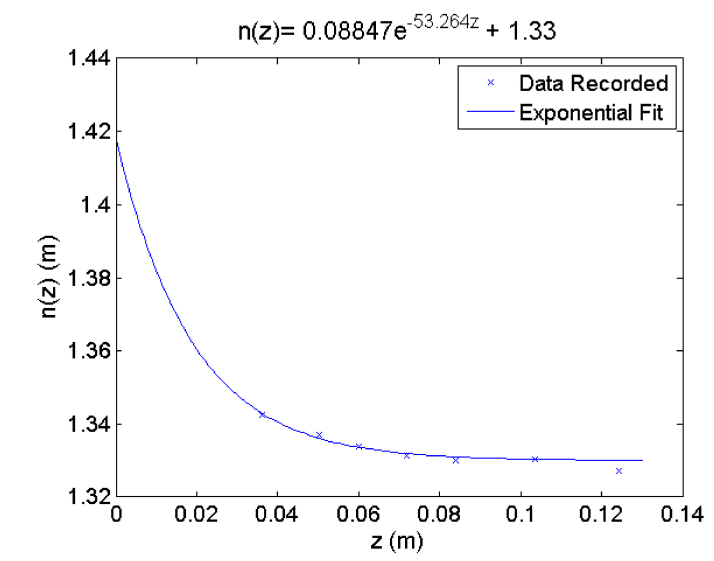
After the data was digitized, vertical index of refraction measurements were analyzed using Matlab to fit to an exponential curve by minimizing the least squares. This gives a continuous vertical index of refraction profile. Next, the position and slope at one end of the ray was used to initialize a RK2 differential equation solver for the generalized ray tracing equation. After solving for the path of the ray, each data point on the ray was compared to the predicted point at the corresponding horizontal distance and an average sum of squared deviations was computed. Finally, these errors were compared to the measurement error in recording the actual ray trace.

Figure 3 (above): The faces of my physics peers prepare to minimize turbulent mixing by impacting incident streams of water

Figure 4 (right): Ray traveling through xz plane bent through index of refraction gradient. Though subtle, a similar magnitude of bending was predicted by the generalized ray tracing algorithm



Results



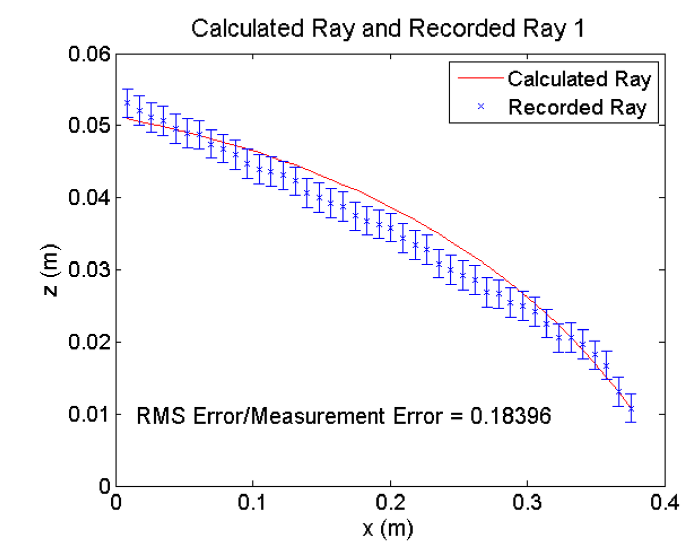
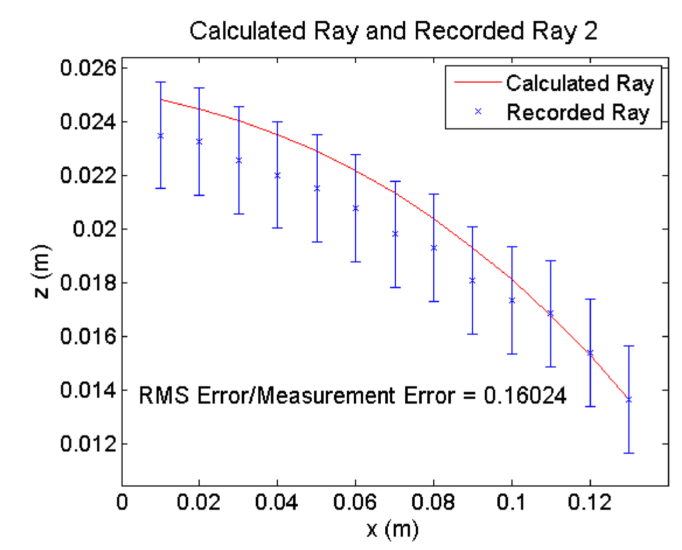


Figure 5 (above): Index of refraction measures fit to an exponential at 22 hours of diffusion. This function was used to generate Figure 6 (top right) and Figure 7 (bottom right). Both rays were captured less than 5 minutes after the index of refraction gradient was measured. The fit for these and all other data remained under a fifth of the measurement error. Ray 1's curve is more gradual as it was started from an initial position further from the more intense index of refraction gradient.



Conclusions

The backwards ray tracing algorithm matched the path of the laser consistently to within 0.2 times the uncertainty in the measured position. Computational stability occurred in the backward direction as expected. Though further computational and experimental testing is necessary to validate this as an analytic tool for prediction *e.g.* of faux Doppler shifts and other peculiar effects, this algorithm is at least able to predict the position of a single beam of light traveling through a one dimensional gradient. Additionally, this algorithm holds promise for inferring traits of a gradient from known ray tracing information, potentially allowing for sonic measurements of temperature inversions or other atmospheric or seismic profiles.

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