Amalgam Decomposition and Cohomology of the Group $SL_2(\mathbb{Z})$ and the Bianchi Groups

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Abstract

This thesis contains a couple of examples of how the cohomology groups of some groups of matrices can be computed using their decomposition as an amalgamated product and, the relation between those groups and their corresponding classifying spaces.

The document is divided in two parts. The first part describes a geometric method to prove that the special linear group $SL_2(\mathbb{Z})$ is an amalgamated product of cyclic groups, using the action of the group on the hyperbolic plane. Then, we use this decomposition and a Mayer–Vietoris long exact sequence to compute the cohomology groups of this group.

The second part of the thesis deals with Bianchi groups, which are defined as $PSL_2(\mathbb{O}_d)$, where $\mathbb{O}_d$ is the ring of integers of an imaginary quadratic extension of the field of rational numbers. The amalgam decomposition of a particular groups, the Euclidean Bianchi groups, is given, and we conclude with the computation of the cohomology groups of the group $\Gamma_1 = PSL_2(\mathbb{O}_1)$.

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Contents

1 Introduction

This work started with the interest of study the action of the special linear group $SL_2(\mathbb{Z})$ on the hyperbolic plane, this is, by Möbuis transformations. This action leads to describe the group as the amalgamated product

$$\mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z},$$

using some results from Serre [16], which explain how a group can be decomposed as an amalgamated product when it acts on a tree in a certain way.

As a topological matter, there is a well-known long exact sequence, the Mayer–Vietoris long exact sequence, that relates the (co)homology of two spaces, a common subspace, and the union of the spaces, but we can use this long exact sequence identifying that setting of spaces as the classifying spaces of the factors in an amalgamated product of groups. This identification let us find the cohomology groups of $SL_2(\mathbb{Z})$.

The above is the content of the first part of this thesis, which leaves a bunch of different ways to take of similar topics. The second part proceeds with an interesting study of the Bianchi groups

$$\Gamma_d = PSL_2(\mathbb{O}_d),$$

with $\mathbb{O}_d$ denoting the ring of integers of the field extension $\mathbb{Q}(\sqrt{-d})$. These groups turn out to be amalgamated products, all but Γ_3. We describe this decomposition completely for the Euclidean Bianchi groups, which are those where the ring $\mathbb{O}_d$ is an Euclidean domain, this is, for $d = 1, 2, 3, 7, 11$.

The final result of the document is the computation of the cohomology groups of the group Γ_1, the only Euclidean Bianchi group that does not have an HNN group as a factor in the amalgam decomposition. For this, we use the theory of spectral sequences, in particular, a Mayer–Vietoris spectral sequence, which gets its name for being, in a way, a generalization of the long exact sequence used in the first part.

This work could continue with the description of the ring structure of the cohomology already computed, the computation of the cohomology of the other Euclidean Bianchi groups, the study of the amalgam decomposition of the non-Euclidean Bianchi groups, or, for sure, with many other topics.

Part I
$SL_2(\mathbb{Z})$ as an amalgam and its cohomology

We start with some basic results with graphs and trees with a group action. Using this, it is shown that the group $SL_2(\mathbb{Z})$ is an amalgamated product of cyclic groups, making it to act (by Möbius transformations) on a segment of the unit circle in the hyperbolic plane.

Then, we compute the cohomology of $SL_2(\mathbb{Z})$ with integer coefficients using a Mayer–Vietoris long exact sequence obtained from considering a diagram of the union of the classifying spaces associated to the amalgamated product.

2 Trees and amalgams

The results in this section are taken from [16], where the Bass–Serre theory is developed. We will use this theory later to proof the decomposition as an amalgamated product of the group $SL_2(\mathbb{Z})$.

2.1 Graphs and trees

Definition 2.1. *A* ***graph*** Γ *consists of two sets,* $X = \mathrm{vert}\ \Gamma$ *and* $Y = \mathrm{edge}\ \Gamma$, *and two maps,*

$$Y \to X \times X, \qquad y \mapsto (o(y), t(y)),$$

and

$$Y \to Y, \qquad y \mapsto \overline{y},$$

such that for each $y \in Y$, *we have* $\overline{y} \neq y$, $\overline{\overline{y}} = y$, *and* $o(y) = t(\overline{y})$.

An element $P \in X$ is called a **vertex** of Γ. An element $y \in Y$ is called an **edge** and $\overline{y}$ is called the **inverse** edge. The vertex $o(y) = t(\overline{y})$ is called the **origin** of y and the vertex $t(y) = o(\overline{y})$ is called the **terminus** of y.

An **oriented graph** is defined by giving two sets X and Y_+ and a map $Y_+ \to X \times X$, where X is the set of vertices of the graph and the set of edges will be $Y = Y_+ \coprod \overline{Y}_+$, where $\overline{Y}_+$ is a copy of Y_+.

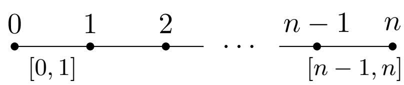
A graph will be described sometimes with a diagram and the obvious interpretation, that is, the points are the vertices and the lines are the edges, whose orientation may be given by an arrow.

Definition 2.2. *A* ***morphism*** *between two graphs* Γ *and* Γ' *is a pair of maps*

$$\alpha : \text{vert } \Gamma \to \text{vert } \Gamma' \qquad \textit{and} \qquad \beta : \text{edge } \Gamma \to \text{edge } \Gamma'$$

such that $\alpha(o(y)) = o(\beta(y))$ *and* $\overline{\beta(y)} = \beta(\overline{y})$. *The morphism is said to be* ***injective*** *if* α *and* β *are injective. The morphism is an* ***isomorphism*** *if* α *and* β *are bijective. The set of isomorphisms of a graph* Γ *is a group with composition, which is denoted* $\mathrm{Aut}(\Gamma)$.

Let n be a nonnegative integer. Consider the oriented graph given by the following diagram.

Path_n:


It has $n+1$ vertices and an orientation given by the n edges $[i, i+1]$, $0 \leq i \leq n-1$, with $o([i, i+1]) = i$ and $t([i, i+1]) = i+1$.

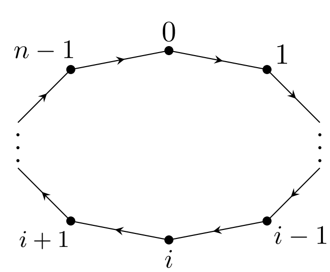
Definition 2.3. *A* ***path*** *of length* n *in a graph* Γ *is a morphism* c *of* Path_n *into* Γ.

For $n \geq 1$, c is determined by the sequence $(y_1, \ldots, y_n)$, where $y_i = c([i-1, i])$ and $t(y_i) = o(y_{i+1})$, for $1 \leq i \leq n$. The path will be denoted simply by c; we say that c is a path from $o(y_1)$ to $t(y_n)$ and that these vertices are the extremities of the path.

If $y_{i-1} = \overline{y_i}$ for some i, the pair (y_{i-1}, y_i) is called a **backtracing**. A new path of length $n-2$ with the same extremities can be obtained with the sequence $(y_1, \ldots, y_{i-2}, y_{i+1}, \ldots, y_n)$, so if there is a path between two vertices then there is one without backtracing.

Definition 2.4. *A graph is said to be* ***connected*** *if any two vertices are the extremities of some path. The maximal connected subgraphs, under the relation of inclusion, are called the* ***connected components*** *of the graph.*

Let n be a positive integer. Consider the oriented graph given by the following diagram.

Circ_n:


The set of vertices is $\mathbb{Z}/n\,\mathbb{Z}$, and the orientation is given by the n edges $[i, i+1]$ ($i \in \mathbb{Z}/n\,\mathbb{Z}$) with $o([i, i+1]) = i$ and $t([i, i+1]) = i+1$.

Definition 2.5. *A **circuit** of length n in a graph is any subgraph isomorphic to* Circ_n.

Now, we define a tree.

Definition 2.6. *A **tree** is a connected non-empty graph without circuits.*

There is something called the realization of a graph, which is the construction of a topological space associated to a graph. This is simple and intuitive; we will not use this notion again.

As before, let Γ be a graph with set of vertices X and set of edges Y. Let T be the topological space obtained from the disjoint union of X and $Y \times [0,1]$ with the discrete topology, and let $\sim$ be the *minimal* equivalence relation on T such that

$$(y,t) \sim (\overline{y}, 1-t), \qquad o(y) \sim (y,0), \qquad \text{and} \qquad t(y) \sim (y,1)$$

for all $y \in Y$ and $t \in [0,1]$. The **realization of** Γ is defined as $\mathrm{real}(\Gamma) := T/{\sim}$. It can be shown that this is a CW-complex (see Section 3.2 for the definition). We can see that the realization is a connected (and path connected) space if and only if the graph is connected.

Also, if Γ is a tree, it can be proved that its realization is contractible. This is because any path inside Γ can be contracted to a point, so this can be done simultaneously, or successively, to contract all the tree. See [16] for more information.

2.2 Groups acting on trees

Let G be a group and X be a graph, and let G **act on** X. This action means that there is a group homorphism $G \to \mathrm{Aut}(X)$.

An **inversion** is a pair consisting of an element $g \in G$ and an edge y of X such that $gy = \overline{y}$. The group is said to act **without inversion** if there is no such pair.

If G acts without inversion we can define the **quotient graph** $G\backslash X$, whose vertex set and edge set are the quotient of vert X and edge X, respectively, under the action of G. This graph is also called X **mod** G.

We say that a group acts **freely** on a graph X if it acts without inversion and no element $g \neq 1$ of G leaves a vertex of X fixed.

A **fundamental domain** of X mod G is a subgraph T of X such that $T \to G\backslash X$ is an isomorphism. It can be shown that, when X is a tree, a fundamental domain exists if and only if $G\backslash X$ is a tree.

2.3 Amalgamated free product of groups

Consider a collection of groups $\{G_i\}_{i \in I}$ together with a set $F_{i,j}$ of homomorphisms $G_i \to G_j$, for each pair $i, j \in I$.

Proposition 2.7. *There exists a group G and a collection of homomorphisms $\{f_i\}_{i\in I}$, with $f_i : G_i \to G$ and $f_j \circ f = f_i$, $\forall f \in F_{i,j}$, such that the following property is satisfied:*

- *If there is a group H and a collection of homomorphisms $\{h_i\}_{i\in I}$, $h_i : G_i \to H$, such that $h_j \circ f = h_i$, $\forall f \in F_{i,j}$, then there exists a unique homomorphism $h : G \to H$ that satisfy $h \circ f_i = h_i$, $\forall i \in I$.*

Furthermore, G and $\{f_i\}_{i\in I}$ are unique up to unique isomorphism.

$$\begin{tikzcd}
G_i \arrow[dd, "f"'] \arrow[dr, "f_i"] \arrow[drr, bend left, "h_i"] & & \\
 & G \arrow[r, dashed, "h"] & H \\
G_j \arrow[ur, "f_j"'] \arrow[urr, bend right, "h_j"'] & &
\end{tikzcd}$$

Proof. For the existence, take a set of generators S_i of each G_i, then take the disjoint union $\sqcup_{i\in I} S_i$ as the set of generators for G. The relations will be the disjoint union of the relations for each G_i together with $xy^{-1} = e$, whenever $f(x) = y$ for some $f \in F_{i,j}$, with $x \in G_i$ and $y \in G_j$. The f_i are just the inclusions.

The uniqueness is proved using the universal property: Suppose $G, \{f_i\}_{i\in I}$ and $G', \{f'_i\}_{i\in I}$ are such groups, then two homomorphisms $f' : G \to G'$ and $f : G' \to G$ are obtained; the compositions $f \circ f'$ and $f' \circ f$ must be the identity maps on G and G' respectively, so f and f' are isomorphisms. □

G is called the **direct limit** of the G_i relative to the $F_{i,j}$.

Now consider the case where there is a group A and a collection of groups $\{G_i\}_{i\in I}$ with a collection of injective homomorphisms $\{\alpha_i : A \to G_i\}_{i\in I}$, so A is identified with a subgroup of each G_i. The group obtained as the direct limit of $\{A\} \cup \{G_i\}_{i\in I}$ together with just the given homomorphisms $\{\alpha_i\}_{i\in I}$ is denoted as $*_A G_i$ and is called the **product of the G_i with A amalgamated**.

As before, let $G = *_A G_i$ and $f : A \to G$ and $\{f_i : G_i \to G\}_{i\in I}$ be the homomorphisms obtained in the *amalgamation*. In the next proposition, we give a unique description for each element of G. We introduce some notation first.

Let S_i be a set of representatives of G_i modulo A, with $e \in S_i$. For $n \geq 0$, consider a sequence $\vec{i} = (i_1, \ldots, i_n)$ of elementes of I such that $i_k \neq i_{k+1}$ for all $1 \leq k \leq n-1$. Now, to form a sequence of elements of the groups, choose an $a \in A$ and an element $s_k \neq e$ in S_{i_k}, for $1 \leq k \leq n$; then we get a sequence $\vec{m} = (a; s_1, \ldots, s_n)$, which is called a **reduced word** of type $\vec{i}$.

Proposition 2.8. *For each $g \in G$, there exist a unique $\vec{i} = (i_1, \dots, i_n)$, for some $n \geq 0$, and a unique reduced word of type $\vec{i}$, $\vec{m} = (a; s_1, \dots, s_n)$ as described above, such that*

$$g = f(a) f_{i_1}(s_1) \cdots f_{i_n}(s_n). \tag{1}$$

Furthermore, the f and f_i must be injective homomorphisms.

Proof. For each sequence $\vec{i}$, let $X_{\vec{i}}$ be the set of all the reduced words of type $\vec{i}$, and let X be the union of all the $X_{\vec{i}}$. We will see that there is an action of G on X. For this it is sufficient to make each G_j, $j \in I$, act on X and then check that the induced action of A does not depend on j.

First, for $j \in I$, let Y_j be the subset of X with the reduced words of the form $(e; s_1, \dots, s_n)$ with $i_1 \neq j$. Consider the functions

$$A \times Y_j \longrightarrow X, \quad (a, (e; s_1, \dots, s_n)) \mapsto (a; s_1, \dots, s_n), \quad \text{and}$$

$$A \times (S_j - \{e\}) \times Y_j \longrightarrow X, \quad (a, s, (e; s_1, \dots, s_n)) \mapsto (a; s, s_1, \dots, s_n).$$

These induce a bijection $(A \cup (A \times S_j - \{e\})) \times Y_j = (A \times Y_j) \cup (A \times (S_j - \{e\}) \times Y_j) \to X$, but $A \cup (A \times S_j - \{e\})$ can be identified with G_j, so there is a bijection

$$\theta_j : G_j \times Y_j \to X.$$

Thus, since G_j acts on $G_j \times Y_j$ by multiplication, $g_1 \cdot (g_2, y) = (g_1 g_2, y)$, we obtain an action of G_j on X via θ_j. The restriction on A is the same for any j:

$$a_1 \cdot (a_2; s_1, \dots, s_n) = (a_1 a_2; s_1, \dots, s_n).$$

With the resulting action of G on X, let $\alpha : G \to X$ denote the function "acting on $\vec{e} := (e;\ )$" (this is for the empty sequence $\vec{i} = \emptyset$) and let $\beta : X \to G$ be the association as in (1). We can see that $\alpha \circ \beta = id : X \to X$. Indeed, a reduced word $(a; s_1, \dots, s_n)$ is taken to a group element of the form (1), and the result of this, acting on $\vec{e}$ is the same as the result of the successive actions of $f_{i_n}(s_n)$, $f_{i_{n-1}}(s_{n-1})$, and so on, which act as $s_n \in G_{i_n}$, $s_{n-1} \in G_{i_{n-1}}$, etc.; then the initial reduced word is formed again.

The above implies that β must be injective, hence we have that a decomposition in reduced word is unique, which proves the injectivity of f and the f_j. We can identify $X = \beta(X) \subset G$, besides X contains the identity element $\beta(\vec{e}) = e \in G$. Now note that for any $j \in I$, $g \in G_j$, and $x \in X$ we have that (just a game of notation)

$$gx = f_j(g)\beta(x) = f_j(g) \cdot x = g \cdot x \ \in \ X = \beta(X),$$

so $g = ge = \beta(\alpha(g)) = (\beta \circ \alpha)(g)$, then this generalizes to all G and implies the other inclusion; finally we have the identification $X = G$. $\square$

In the case of three groups A, G_1, G_2, the **amalgam** is denoted as $G_1 *_A G_2$ and we obtain the respective **amalgamation diagram** as shown below.

$$\begin{tikzcd} A \arrow[r, "\alpha_1"] \arrow[d, "\alpha_2"'] & G_1 \arrow[d] \\ G_2 \arrow[r] & G_1 *_A G_2 \end{tikzcd}$$

And, in view of Proposition 2.8, we can write any $g \in G$ uniquely as a **word** of, first, an element in A and then n interleaved elements of a set of representatives of G_1 and G_2 modulo A, for $n \geq 0$.

Furthermore, we have the presentation

$$G_1 *_A G_2 = \langle\, G_1,\ G_2 \mid \text{ Relations of } G_1, \ \text{ Relations of } G_2, \ \alpha_1(a) = \alpha_2(a),\ \forall a \in A\rangle.$$

We will see several examples in the rest of the document.

2.4 A useful theorem

The following theorem [16, Chapter I, Theorem 6] will be used to proof the amalgam decomposition of the group $SL_2(\mathbb{Z})$.

Theorem 2.9. *Let G be a group acting on a graph X, let $T = P\bullet\!\xrightarrow{Y}\!\bullet Q$ be a fundamental domain of $X \bmod G$, and let G_P, G_Q and G_Y be the stabilizers of the vertices P, Q and of the edge Y, respectively. Then X is a tree if and only if the induced homomorphism $G_P *_{G_Y} G_Q \to G$ in an isomorphism.*

This theorem is direct consequence of the next two lemmas.

Lemma 2.10. *X is connected if and only if G is generated by $G_P \cup G_Q$.*

Proof. Take X' to be the connected component of X that contains T, take G_1 as the subgroup $\{g \in G : gX' = X'\}$ and G_2 as the subgroup generated by $G_P \cup G_Q$.

If $h \in G_P \cup G_Q$, then $hT \subset X'$, so $hX' = X'$ (because the action of G preserves connectedness, so $hX' \subset X'$ and $h^{-1}X' \subset X'$); this implies that $G_2 \subset G_1$. Now, by hypothesis we have $GT = X$, and also X is the disjoint union $G_2T \sqcup (G - G_2)T$, because if there were a point in the intersection, then we would have $P = h^{-1}g_2P$ for some $h \notin G_P$ and $g_2 \in G_2$, or the same for Q, this is a contradiction. However, $X' \subset G_2T$, because G_2 contains the identity, so $(G - G_2) \cap G_1 = \emptyset$ and the other inclusion is obtained: $G_1 \subset G_2$. Thus $G_1 = G_2$.

Now, we have $G = G_1$, that is, that X is connected, if and only if $G = G_2$. □

Lemma 2.11. *X has no circuits if and only if $G_P *_{G_Y} G_Q \to G$ is injective.*

Proof. A circuit in X is a path $c = (w_0, ..., w_n)$ with $o(c) = t(c)$ and without backtracing. Suppose we have just a path c without backtracing, then each w_i can be written as $h_i y_i$, where $h_i \in G$ and $y_i \in \{Y, \overline{Y}\}$, and also with $y_{i-1} = \overline{y_i}$, because $G \backslash X \cong T$.

For $i = 0, ..., n$, let $p_i = o(y_i) \in \{P, Q\}$. With $i \neq 0$, we have

$$h_i p_i = h_i o(y_i) = o(h_i y_i) = t(h_{i-1} y_{i-1}) = h_{i-1} t(y_{i-1}) = h_{i-1} p_i,$$

then $h_i = h_{i-1} g_i$, for some $g_i \in G_{p_i}$. Besides, $\overline{w_i} = h_i \overline{y_i} = h_i y_{i-1}$ is different from $w_{i-1} = h_{i-1} y_{i-1}$, since the path has no backtracing, thus $h_{i-1}^{-1} h_i = g_i \notin G_Y$.

So, if c were a circuit, namely $o(w_0) = t(w_n)$, we would have $h_n^{-1} h_0 = g_0 \in (G_{p_0} - G_Y)$, and $h_n = h_{n-1} g_n = ... = h_0 g_1 \cdots g_n = h_n g_0 \cdots g_n$, thus $g_0 g_1 \cdots g_n = e$. Then the word $g_0 \cdot g_1 \cdot ... \cdot g_n \in G_P *_{G_Y} G_Q$ will make the homomorphism not injective.

Reciprocally, if there is a word that becomes e taking the multiplication, we could build a circuit in X taking $h_0 = e$ and an appropriate y_0. □

Now it is clear, for the proof of Theorem 2.9, that the surjectivity and injectivity of $G_P *_{G_Y} G_Q \to G$ are equivalent, respectively, to the connectedness and lack of circuits of the graph X.

3 Topology

First we recall some definitions.

Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of topological spaces such that, for every $\alpha, \beta \in A$, $X_\alpha \cap X_\beta$ is open in both X_α and X_β and that its induced topologies from X_α and X_β are the same. Then we can define a space $X = \bigcup_{\alpha \in A} X_\alpha$ obtained from ***gluing*** the X_α. A set $U \subset X$ will be open in X if and only if each $U \cap X_\alpha$ is open in X_α. It is easy to see that this topology on X is unique for the needed properties; it is called the **weak topology**.

In the case of two spaces X_1 and X_2, with $X_1 \cap X_2 = A$, the space X is denoted by $X_1 \cup_A X_2$.

In a topological space X, a **Δ-complex** structure is a collection of continuous maps $\sigma_\alpha : \Delta_{n_\alpha} \mapsto X$, such that the following conditions are satisfied:

- each restriction $\sigma_\alpha | \mathrm{int}(\Delta_{n_\alpha})$ is injective, and each point of X is in the image of exactly one such restriction;
- any restriction of σ_α to a face of Δ_{n_α} is one of the maps $\sigma_\beta : \Delta_{n_\alpha - 1} \to X$;
- a set $A \subset X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ_{n_α} for each α.

For a space X, its **fundamental group** based in $x_0 \in X$ will be denoted as $\pi_1(X, x_0)$, and $\pi_1(X)$ if the space is path-connected (so the basepoint is not important). The notation $f : (X, x_0) \to (Y, y_0)$ will be used to say that f is a map from X to Y which maps x_0 to y_0. Any of these maps induces a group homomorphism (by composing the loops with the map)

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0).$$

Therefore, π_1 is a covariant functor from the category of pointed spaces (spaces with a particular point), where the morphisms are continuous maps as described above, to the category of groups.

A pair of spaces $A \subset X$ for which there is a neighbourhood of A in X that is a deformation retract of A is called a **good pair**.

3.1 Covering spaces

The results with covering spaces are necessary for the construction of classifying spaces; these will not be explained in full detail. For a complete, detailed development of the subject, see [12].

A **covering space** for X is a pair consisting of a space $\widetilde{X}$ and a continuous map $p : \widetilde{X} \to X$ such that for every point $x \in X$ there is an open neighbourhood $U \subset X$ of x such that $p^{-1}(U) \subset \widetilde{X}$ is a disjoint union of open subsets, where each of them is homeomorphic to U (via p). Two covering spaces $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$ are said to be **isomorphic** if there is a homeomorphism $f : \widetilde{X}_1 \to \widetilde{X}_2$ such that $p_2 \circ f = p_1$.

Any covering space map induces an injective homomorphism in the fundamental groups.

The following proposition contains some facts about covering spaces and subgroups of the fundamental group of some space X.

Proposition 3.1. *Let $x_0 \in X$. If X is path-connected and locally path-connected, the following are true:*

(a) *For every subgroup $H \subset \pi_1(X, x_0)$, there is a covering space $p : X_H \to X$ such that $p_*(\pi_1(X_H, \widetilde{x}_0)) = H$ for an appropriate basepoint $\widetilde{x}_0 \in p^{-1}(x_0)$.*

(b) *Two path-connected covering spaces $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$ are isomorphic via $f : (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$, with $\widetilde{x}_i \in p_i^{-1}(x_0)$, if and only if $p_{1*}(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = p_{2*}(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$.*

(c) *If X is also semi-locally simply connected, there is a bijective correspondence between isomorphism classes of path-connected covering spaces and conjugacy classes of subgroups of $\pi_1(X, x_0)$.*

Here, Proposition 3.1(c) is ignoring the basepoints, because changing a basepoint in the covering space is equivalent to taking a conjugate of the corresponding subgroup.

The covering space corresponding to the trivial subgroup, which is simply-connected and is unique up to isomorphism, is called the **universal cover** of X.

Given the concept of a covering space isomorphism, for any covering space $p : \widetilde{X} \to X$, we define $G(\widetilde{X})$ as the group obtained with all the isomorphisms $\widetilde{X} \to \widetilde{X}$, which are called **deck transformations**. If $\widetilde{X}$ is path-connected, a deck transformation is completely determined by where it sends a single point.

Proposition 3.2. *Let X be path-connected and locally path-connected with a path-connected covering space $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$, and let H be the corresponding subgroup of $\pi_1(X, x_0)$. Then the group $G(\widetilde{X})$ is isomorphic to the quotient $N(H)/H$, where $N(H)$ denotes the normalizer of H in $\pi_1(X, x_0)$. In particular, for the universal cover, we have $G(\widetilde{X}) \cong \pi_1(X)$.*

Since changing the basepoint from $\widetilde{x}_0$ to $\widetilde{x}_1 \in p^{-1}(x_0)$ is the same as conjugating by an element $[\gamma]$, where $p^{-1}(\gamma)$ is a path in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$, $[\gamma]$ will be in the normalizer of H if and only if there is a deck transformation mapping $\widetilde{x}_0$ to $\widetilde{x}_1$. Thus we can define a surjective homomorphism $N(H) \to G(\widetilde{X})$ which sends $[\gamma]$ to the deck transformation that maps $\widetilde{x}_0$ to $\widetilde{x}_1$; the kernel will be H, and the isomorphism is obtained.

Now, consider a group G acting on X. (This means the only thing that it should mean: a group homomorphism from G to the group of homeomorphisms of X.) An action satisfying that for each point $x \in X$ there is a neighbourhood U of x such that $g_1 U$ and $g_2 U$ are disjoint for any $g_1 \neq g_2$ in G, is called a **covering space action**. The reason for the terminology is the attainment of a covering space $p : X \to X/G$, although this name is not standard. The covering space is, indeed, obtained with p as the quotient map sending a point to its orbit; the topology of X/G is determined by this map and, as an extra property (supporting the terminology), any two points in $p^{-1}(x)$, with $x \in X/G$, will be linked by a deck transformation.

Proposition 3.3. *If G acts on a path-connected and locally path-connected space X as a covering space action, then G equals the group of deck transformations of $p : X \to X/G$ and it is isomorphic to $\pi_1(X/G)/p_*(\pi_1(X))$.*

Proof. G is contained in the group of deck transformations, and with any transformation τ, we see that x and τx must be in the same orbit for every $x \in X$, so there is a $g \in G$ such that $gx = \tau x$, but then $\tau = g$ because a deck transformation is determined by the image of one point. Now, the property of $p : X \to X/G$ explained before the proposition will make its corresponding subgroup of $\pi_1(X/G)$ to be a normal subgroup, so the isomorphism is obtained from Proposition 3.2. □

3.2 CW-complexes

The definition of a CW-complex will be given with a construction.

Let $X^{(0)}$ be a discrete set of points. These points are called the **0-cells**.

Suppose that $X^{(n-1)}$ has been defined. Let $\{f_\sigma\}_{\sigma\in\Sigma}$ be a collection of continuous maps $f_\sigma : \mathbb{S}^{n-1} \to X^{(n-1)}$. Consider the sets of disjoint unions

$$Y = \bigsqcup_{\sigma\in\Sigma} \mathbb{D}^n_\sigma \qquad \text{and} \qquad B = \bigsqcup_{\sigma\in\Sigma} \mathbb{S}^{n-1}_\sigma$$

with $\mathbb{D}^n_\sigma = \mathbb{D}^n$ and $\mathbb{S}^{n-1}_\sigma = \text{boundary}(\mathbb{D}^n_\sigma)$. Putting together the f_σ we obtain a map $f : B \to X^{(n-1)}$ and we define

$$X^{(n)} = X^{(n-1)} \cup_f Y;$$

the notation with f means that $X^{(n)}$ is the quotient of the union, identifying $B \subset Y$ with $f(B)$ (point by point).

With $X^{(n)}$ defined for all $n \geq 0$, let X be the union $\bigcup_n X^{(n)}$ obtained from *gluing* the $X^{(n)}$. Then a subset of X would be open in X if and only if its intersection with each $X^{(n)}$ is open in $X^{(n)}$. (And the same for closed sets.)

The obtained X is called a **CW-complex**, whose n**-cells** are the images of the maps f_σ for each n. By a **subcomplex** $A \subset X$ we mean a closed subspace of X which is the union of some cells of X; it is not hard to see that an A of this kind has a CW-complex structure.

Lemma 3.4. *Let $X' \hookrightarrow X$ be an inclusion of connected CW-complexes such that the induced homomorphism $\pi_1(X') \to \pi_1(X)$ is injective. Let $p : \widetilde{X} \to X$ be the universal cover of X. Then each connected component of $p^{-1}(X')$ is simply connected (so it is a copy of the universal cover of X').*

Proof. Take any basepoint $x_0 \in p^{-1}(X') \subset \widetilde{X}$. The covering space map p induces injective homomorphisms $\pi_1(p^{-1}(X'), x_0) \to \pi_1(X', p(x_0))$ and $\pi_1(\widetilde{X}, x_0) \to \pi_1(X, p(x_0))$ (by composing the loops with p). Using the inclusions, the following commutative diagram is obtained.

$$\begin{tikzcd}
\pi_1(p^{-1}(X'), x_0) \arrow[r] \arrow[d] & \pi_1(\widetilde{X}, x_0) \arrow[d] \\
\pi_1(X', p(x_0)) \arrow[r] & \pi_1(X, p(x_0))
\end{tikzcd}$$

The bottom arrow is also injective (by hypothesis), and since $\widetilde{X}$ is simply connected, we have that $\pi_1(p^{-1}(X'), x_0)$ must be the trivial group. This is for all $x_0 \in p^{-1}(X')$.

$\square$

We need another facts about CW-complexes. First, regarding the cells of X, $X^{(n)}/X^{(n-1)}$ is a wedge sum of n-spheres. This can be seen directly from the construction, using the fact that the n-disk quotiented by its boundary is homeomorphic to the n-sphere.

Also, we have that for any subcomplex $A \subset X$, there is a neighbourhood of A that is a deformation retract of A. See [12] for a proof. We therefore have that $(X^{(n)}, X^{(n-1)})$ is a good pair, for all n.

Finally, we state the well-known Seifert–Van Kampen theorem, for the case of CW-complexes.

Theorem 3.5. *Let X be a CW-complex, which is the union of two connected subcomplexes X_1 and X_2, whose intersection Y is connected and non-empty. Then the square*

$$\begin{tikzcd} \pi_1(Y) \arrow[r] \arrow[d] & \pi_1(X_1) \arrow[d] \\ \pi_1(X_2) \arrow[r] & \pi_1(X) \end{tikzcd}$$

is an amalgamation diagram, where all fundamental groups are computed at a fixed vertex $y \in Y$ and all maps are induced by inclusions. Thus

$$\pi_1(X) = \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2).$$

For a general proof, see [2].

3.3 Homology and cohomology

In this subsection we explain both the singular and cellular homology of a space, explaining also how the cohomology is obtained from these.

First, we introduce the notion of a **chain complex**. This consists in a sequence of abelian groups $\ldots, A_0, A_1, A_2 \ldots$ (or modules over some ring) where there are morphisms $d_n : A_n \to A_{n-1}$, called **differentials** or boundary maps, such that the composition of any two consecutive maps is the zero map. For this we will use diagrams of the form

$$\cdots \to A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \to \cdots \to A_1 \to A_0 \to \cdots .$$

The condition $d_n \circ d_{n+1} = 0$ amounts to say that Im $d_{n+1} \subset$ Ker d_n. Hence, for each n, we can take the quotient inside A_n to obtain the **n-th homology groups** associated to the chain complex:

$$H_n(A_*, d_*) = \text{Ker } d_n \; / \; \text{Im } d_{n+1}.$$

The chain may also be increasing; in this case the subindices are changed by superindices and we call it a **cochain complex**.

The indices in the differentials are usually omitted. Also, the indices could vary on any subset of the integers; we will mostly restrict to non-negative indices. In these cases, we take the 0-th homology group as $A_0/\mathrm{Im}\ d_1$, or, if it is a cochain, as $\mathrm{Ker}\ d_0$.

A **standard n-simplex** is defined as

$$\Delta_n = \{(\lambda_0, \dots, \lambda_n) \in \mathbb{R}^{n+1}\ :\ \sum_{i=0}^{n} \lambda_i = 1 \text{ and } \lambda_i \geq 0 \text{ for all } i\},$$

with the induced topology as a subspace of $\mathbb{R}^{n+1}$; given some $v_0, \dots, v_n \in \mathbb{R}^N$, the **affine singular n-simplex** associated is the map $[v_0, \dots, v_n] : \Delta_n \to \mathbb{R}^N$, $(\lambda_0, \dots, \lambda_n) \mapsto \sum_i \lambda_i v_i$.

A **singular n-simplex** in X is a continuous map $\sigma : \Delta_n \to X$, and the **singular n-chain group** $\Delta_n(X)$ is defined as the free abelian group generated by all the singular n-simplices in X. There is a homomorphism

$$\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$$

given in the basis by $\partial_n(\sigma) = \sum_i (-1)^i \sigma^{(i)}$, where $\sigma^{(i)}$ denotes the **i-th face** of σ (which is just the $(n-1)$-simplex obtained by taking out the i-th coordinate in Δ_n). We have $\partial_{n-1} \circ \partial_n = 0$, so we obtain a chain complex, called the **singular chain complex** of X.

The **n-th singular homology group** of X is defined as $H_n(X) = \mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n+1})$; these groups are usually referred simply as the homology of X.

With this definition, if we take X as a single point we see that every group $\Delta_n(X)$ would be isomorphic to $\mathbb{Z}$, since there is only one option for the maps $\Delta_n \to X$. The differentials ∂_n are just a sum of ± 1's, thus they are the identity for n odd and the zero map for the others. Then the homology groups of a point are trivial for $n > 0$.

But sometimes one would like to have a totally trivial homology for a point (or contractible spaces), so in order to have a trivial homology in $n = 0$, we can extend the chain complex with the morphism $\varepsilon : \Delta_0(X) \to \mathbb{Z}$, sending any 0-simplex to $1 \in \mathbb{Z}$. The chain obtained

$$\cdots \to \Delta_2(X) \xrightarrow{\partial_2} \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$$

is usually called the **augmented** singular chain complex and the homology groups obtained from this chain are called the **reduced homology** of X; these are denoted as $\widetilde{H}_*(X)$. Clearly $\widetilde{H}_n(X) = H_n(X)$ for $n > 0$.

Now, let G be any group. We will construct a cochain complex applying the functor $\mathrm{Hom}(-, G)$ to the singular chain complex of X. We obtain other groups and morphisms:

$$\Delta^n(X; G) := \mathrm{Hom}(\Delta_n(X), G) \quad \text{and} \quad \delta^n : \Delta^n(X; G) \to \Delta^{n+1}(X; G)$$

where $\delta^n(\varphi) = \varphi \circ \partial_n$, with $\varphi \in \Delta^n(X;G)$. Then the condition $\delta^{n+1} \circ \delta^n = 0$ is satisfied as well. With this, the **n-th singular cohomology group** of X with coefficients in G is defined as the n-th homology group of the cochain $(\Delta^*(X;G), \delta^*)$. These groups are also referred just as the cohomology of X, and are denoted as $H^n(X;G)$. When the group G is not mentioned or the notation $H^n(X)$ is used it means we are assuming $G = \mathbb{Z}$.

We will proceed now to the construction of another homology groups associated to a space, in this case, a CW-complex. This homology, called cellular homology, will lead to groups isomorphic to the homology groups defined above, although the chain complex from which they come is easier to compute.

First, for a pair of spaces $B \subset A$, note that there are inclusions $\Delta_n(B) \to \Delta_n(A)$, for each n, induced by the inclusion $B \hookrightarrow A$. Now, define $\Delta_n(A,B) := \Delta_n(A)/\Delta_n(B)$ to make a short exact sequence

$$0 \to \Delta_n(B) \xrightarrow{i_n} \Delta_n(A) \xrightarrow{j_n} \Delta_n(A,B) \to 0$$

for each n. Also note that, since $\partial_n(\Delta_n(A)) \subset \Delta_{n-1}(A)$, we can derive the differentials $\partial_n : \Delta_n(X,A) \to \Delta_{n-1}(X,A)$ that make $(\Delta_*(A,B), \partial_*)$ a chain complex. It is then easy to see that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \Delta_n(B) \arrow[r, "i_n"] \arrow[d, "\partial_n"'] & \Delta_n(A) \arrow[r, "j_n"] \arrow[d, "\partial_n"'] & \Delta_n(A,B) \arrow[r] \arrow[d, "\partial_n"'] & 0 \\
0 \arrow[r] & \Delta_{n-1}(B) \arrow[r, "i_{n-1}"] & \Delta_{n-1}(A) \arrow[r, "j_{n-1}"] & \Delta_{n-1}(A,B) \arrow[r] & 0
\end{tikzcd}$$

is commutative. Putting these sequences together, we get a short exact sequence of chain complexes. A well-known algebraic construction (that will not be explained here) gives a long exact sequence with the homology groups of each chain complex. Hence we have

$$\cdots \longrightarrow H_n(B) \xrightarrow{i_n} H_n(A) \xrightarrow{j_n} H_n(A,B) \xrightarrow{\partial_n} H_{n-1}(B) \longrightarrow \cdots \longrightarrow H_0(A,B),$$

where $H_n(A,B)$ is defined to be the n-th homology group of the chain complex $\Delta_*(A,B)$, called the **relative homology** of A and B. The notation is the same, but the morphisms ∂_n are a result of that construction.

Applying the previous procedure with the cells of a CW-complex X, we obtain a long exact sequence for each pair $(X^{(n)}, X^{(n-1)})$.

We need two facts: (these are all completely developed in [12])

(1) For a good pair (X, A), the quotient map $q : (X, A) \to (X/A, A/A)$ induces an isomorphism $q_* : H_n(X, A) \to H_n(X/A, A/A) \cong \widetilde{H}_n(X/A)$.

(2) For a wedge sum $X = \vee_\alpha X_\alpha$, the inclusions $i_\alpha : X_\alpha \to X$ induce an isomorphism $\oplus_\alpha i_{\alpha *} : \oplus_\alpha \widetilde{H}_n(X_\alpha) \to \widetilde{H}_n(X)$, provided that the wedge sum is formed at basepoints $x_\alpha \in X_\alpha$ such that (X_α, x_α) are good pairs.

We will also take the part (c) of the following proposition as a fact.

Proposition 3.6. *Let X be a CW-complex. We have the following:*

(a) *$H_k(X^{(n)}, X^{(n-1)})$ is zero for $k \neq n$ and is free abelian for $k = n$, with a basis in correspondence with the n-cells of X.*

(b) *$H_k(X^{(n)})$ is zero for $k > n$.*

(c) *The map $H_k(X^{(n)}) \to H_k(X)$ induced by the inclusion is an isomorphism for $k < n$.*

Proof. By the fact (1), we have that $H_k(X^{(n)}, X^{(n-1)})$ is the same as $\widetilde{H}_k(X^{(n)}/X^{(n-1)})$, and since $X^{(n)}/X^{(n-1)}$ is a wedge sum on n-spheres, by (2) we obtain that $H_k(X^{(n)}, X^{(n-1)})$ is zero for $k \neq n$ and is a sum of $\mathbb{Z}$'s when $k = n$, where each $\mathbb{Z}$ is given by an n-cell of X.

For (b), we have the part of the long exact sequence for $(X^{(n)}, X^{(n-1)})$

$$H_{k+1}(X^{(n)}, X^{(n-1)}) \longrightarrow H_k(X^{(n-1)}) \longrightarrow H_k(X^{(n)}) \longrightarrow H_k(X^{(n)}, X^{(n-1)}),$$

then, using part (a), we see that $H_k(X^{(n-1)}) \to H_k(X^{(n)})$ is an isomorphism when k is different from both n and $n-1$. So we obtain that the chain of morphisms induced by inclusions

$$H_k(X^{(0)}) \longrightarrow H_k(X^{(1)}) \longrightarrow \cdots \longrightarrow H_k(X^{(n)})$$

is a chain of isomorphisms for $k > n$; the result follows from the fact that $H_k(X^{(0)})$ is zero for $k > 0$. $\square$

Using the previous proposition we are able to construct the following diagram, where the maps d_n are just $j_n \circ \partial_n$.

$$\begin{tikzcd}
 & & & & 0 & & \\
 & 0 \arrow[dr] & & H_n(X^{(n+1)}) \cong H_n(X) \arrow[ur] & & & \\
 & & H_n(X^{(n)}) \arrow[ur, "i_{n+1}"] \arrow[dr, "j_n"] & & & & \\
\cdots \arrow[r] & H_{n+1}(X^{(n+1)}, X^{(n)}) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"'] & & H_n(X^{(n)}, X^{(n-1)}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"'] & & H_{n-1}(X^{(n-1)}, X^{(n-2)}) \arrow[r] & \cdots \\
 & & & & H_{n-1}(X^{(n-1)}) \arrow[ur, "j_{n-1}"'] & & \\
 & & & 0 \arrow[ur] & & &
\end{tikzcd}$$

The **cellular chain complex** of X is defined to be the horizontal chain obtained, where each of these $C_n(X) := H_n(X^n, X^{n-1})$ can be thought as the free abelian group generated by the n-cells of X. The **cellular homology** groups of X are defined to be the homology groups of this chain complex.

It is clear that the groups $C_n(X)$ are much simpler than the groups $\Delta_n(X)$ since, for example, the later are not finitely generated. However, we have that both chains give the same homology groups.

Proposition 3.7. *Let X be a CW-complex and $\{C_*(X), d_*\}$ its cellular chain complex. We have isomorphisms $H_n(X) \cong Ker(d_n)/Im(d_{n+1})$ for all n.*

Proof. Take $n > 0$. We use the diagram above, where the *diagonals* are exact. First, note that, since i_{n+1} is surjective, we have $H_n(X) \cong H_n(X^n)/\mathrm{Ker}(i_{n+1}) \cong H_n(X^n)/\mathrm{Im}(\partial_{n+1})$. But, because j_n is injective, $\mathrm{Im}(\partial_{n+1}) \cong \mathrm{Im}(d_{n+1})$ and $H_n(X^n) \cong \mathrm{Im}(j_n) = \mathrm{Ker}(\partial_n)$. Finally, since j_{n-1} is also injective, $\mathrm{Ker}(\partial_n) = \mathrm{Ker}(d_n)$, and the result is obtained. □

Also, the **cellular cochain complex** of a CW-complex can be defined, constructing the same diagram but using the cohomology and relative cohomology of the cells of the space; that cochain will give the same cohomology groups.

Here, we will just use the fact that another way to obtain the cohomology of a space is using the cochain obtained from applying $\mathrm{Hom}(-, G)$ directly to the cellular chain complex. We can prove easily that this cochain gives the same cohomology using the **Universal Coefficient Theorem** for cohomology, stated below.

Theorem 3.8. *If a chain complex C of free abelian groups has homology groups $H_n(C)$, then the cohomology groups $H^n(C;G)$ of the cochain complex $\mathrm{Hom}(C_n, G)$ are determined be split exact sequences*

$$0 \longrightarrow \mathrm{Ext}(H_n(C), G) \longrightarrow H^n(C;G) \longrightarrow \mathrm{Hom}(H_n(C), G) \longrightarrow 0.$$

We just need to use the theorem twice with the chains $\Delta_*(X)$ and $C_*(X)$ and use the fact that both chains have isomorphic homology groups.

3.4 Eilenberg–Maclane spaces

We define a special type of complexes. Let G be a group.

Definition 3.9. *A CW-complex Y is called an* ***Eilenberg–MacLane space*** *of type $(G, 1)$, or simply a $K(G,1)$**-complex** if the following three conditions are satisfied:*

(i) *Y is connected.*

(ii) *$\pi_1(Y) \cong G$.*

(iii) *$H_i(X) = 0$ for $i \geq 2$, where X is the universal cover of Y.*

In this case Y is also called a **classifying space** of the group G. It can be shown that condition (iii) can be replaced by any of the following conditions:

(iii)′ The universal cover of Y is contractible.

(iii)″ $\pi_i(Y) = 0$ for $i \geq 2$.

We will use condition (iii)′ to justify the following construction.

Proposition 3.10. *For any group G, there is a $K(G,1)$-complex.*

Proof. Let EG be the Δ-complex whose n-simplices are made up by $(n+1)$-tuples of the form $[g_0, \dots, g_n]$, with $g_i \in G$. To view this, think of an abstract, infinite-dimensional space where for any tuple of $n+1$ group elements, or points, (not necessarily different) there is a singular n-simplex joining these points. EG is path-connected and locally path-connected.

Now, any point $x \in EG$ lies inside the interior of one simplex $[g_0, \dots, g_n]$; this simplex is a face of the simplex $[e, g_0, \dots, g_n]$, and inside this simplex, which is convex, there is a line joining the point x and the vertex $[e]$, so there is a continuous function (of $t \in [0,1]$) $h_t(x)$ with $h_0(x) = x$ and $h_1(x) = e$. Doing this for every point we obtain a desired homotopy, thus EG is a contractible space.

Clearly the group G acts on EG by left multiplication:

$$g \cdot [g_0, \dots, g_n] = [gg_0, \dots, gg_n].$$

This action is a covering space action (see Section 3.1), since every point of EG is uniquely contained in the interior of one $[g_0, \dots, g_n]$, and multiplying this simplex by any $g \neq e$ gives a different simplex in EG whose interior is disjoint from the interior of $[g_0, \dots, g_n]$. Then, with the quotient, denoted by $BG = EG/G$, we obtain a covering space $p : EG \to BG$.

So, we have that BG has a contractible universal cover and that $\pi_1(BG) \cong G$, using Proposition 3.3. □

We continue with some properties of Eilenberg–Maclane spaces in the next section.

4 Homology and cohomology of groups

In this section we define the cohomology of a group with coefficients in some module, then we describe the cohomology of a finite cyclic group.

4.1 Definitions

Let R be a ring with identity and M a left R-module. A **resolution** of M over R is an exact sequence of R-modules

$$\cdots \longrightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

The notation for this will be $F \xrightarrow{\varepsilon} M$, with ∂ for a general morphism in the sequence, and ε is called the augmentation. If each F_i is free (respectively projective), then it is called a **free resolution** (respectively projective resolution).

Let G be a group. The group ring $\mathbb{Z}G$ denotes the free abelian group generated by the elements of G, where there is a multiplication induced from the operation in G and the multiplication in $\mathbb{Z}$. We call an abelian group A a (left) **$\mathbb{Z}G$-module**, or a **G-module**, if there is an action of G on A, which is extended linearly to an action of $\mathbb{Z}G$.

For a G-module M, the group of **co-invariants** of M is denoted as M_G. This group is obtained as the quotient group M/S, where S is the additive subgroup generated by the elements of the form $gm - m$, for $g \in G$ and $m \in M$. Also, it is not hard to see the isomorphism

$$M_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} M,$$

where $\mathbb{Z}$ is considered as a right $\mathbb{Z}G$-module with the trivial action, using the map $\overline{m} \mapsto 1 \otimes m$ with inverse $a \otimes m \mapsto a\overline{m}$.

With that isomorphism, if M is a free $\mathbb{Z}G$-module with basis $\{e_i\}$, we have that M_G is a free $\mathbb{Z}$-module with basis $\{\overline{e}_i\}$.

Let $F \to \mathbb{Z}$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, we define the **n-th homology group** of G, denoted $H_n(G)$, as the n-th homology group of the chain complex $(F_*)_G$. It can be proved that this does not depend on the resolution; see [3].

In general, for a ring R and two R-modules A and B, there is an abelian group $\mathrm{Hom}_R(A, B)$, which consists of the group homomorphisms $f : A \to B$ such that for all $r \in R$, $a \in A$, we have $f(r \cdot a) = r \cdot f(a)$. For $R = \mathbb{Z}G$, these are **homomorphisms of G-modules** and the group is denoted by $\mathrm{Hom}_G(A, B)$.

Now, with the notation as above, will apply $\mathrm{Hom}_G(-, M)$ to the resolution F. Then we have differentials $\delta^n : \mathrm{Hom}_G(F_n, M) \to \mathrm{Hom}_G(F_{n+1}, M)$ and a cochain complex, since $(\delta^{n+1} \circ \delta^n)(\varphi) = \varphi \circ \partial_n \circ \partial_{n+1} = 0$. The n-th cohomology group of the cochain complex $\mathrm{Hom}_G(F_*, M)$ is defined to be the **n-th cohomology group** of G with coefficients in M, denoted as $H^n(G; M)$.

We will later use the fundamental lemma of homological algebra, stated below. For this, we define chain maps and homotopy.

If (C_*, d_*) and (C'_*, d'_*) are chain complexes, a **chain map** from C_* to C'_* is a morphism $f_n : C_n \to C'_n$ for each n such that $d'_n \circ f_n = f_{n-1} \circ d_n$; this is always denoted simply as $f : C \to C'$. A **homotopy** h from a chain map f to a chain map g, both from C_* to C'_*, is a morphism $h_n : C_n \to C'_{n+1}$ for each n such that $d'_{n+1} \circ h_n + h_{n-1} \circ d_n = f_n - g_n$, or simply $d' \circ h + h \circ d = f - g$.

Note that this is consistent in a way with the topological concept of homotopy, since if for some chain complex there is a homotopy from the identity map to the zero map, then the homology of the chain complex is trivial.

Lemma 4.1. *Let R be any ring. Let $\{C, \partial\}$ and $\{C', \partial'\}$ be chain complexes of R-modules and let k be an integer. Let $\{f_i : C_i \to C'_i\}_{i \leq k}$ be a family of maps such that $\partial'_i \circ f_i = f_{i-1} \circ \partial_i$ for $i \leq k$. If C_i is projective for $i > k$ and $H_i(C') = 0$ for $i \geq k$, then $\{f_i\}_{i \leq k}$ extends to a chain map $f : C \to C'$, and f is unique up to homotopy.*

Given a morphism of groups $\alpha : G \to G'$, with C and C' resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$ and $\mathbb{Z}G'$, respectively, we can make C' to be a chain of $\mathbb{Z}G$-modules via α. Then, using this lemma, we can construct a chain map $C \to C'$ extending $id : \mathbb{Z} \to \mathbb{Z}$, which will be uniquely determined up to homotopy by α. Now, we obtain induced morphisms $C_G \to C'_{G'}$ and $\mathrm{Hom}_{G'}(C', \mathbb{Z}) \to \mathrm{Hom}_G(C, \mathbb{Z})$, which then lead to two **induced morphisms**, for homology and cohomology:

$$\alpha_* : H_*(G) \to H_*(G'), \qquad \alpha^* : H^*(G'; \mathbb{Z}) \to H^*(G; \mathbb{Z}).$$

These maps are unique given α, since the induced chain maps are unique up to homotopy.

4.2 Cyclic groups

This is a simple example of the computation of cohomology groups, besides, both the construction and the result will be useful when we compute the cohomology groups of $SL_2(\mathbb{Z})$ and later in the construction of a spectral sequence that converges to the cohomology of the symmetric group S_3.

Let G be the cyclic group of order n with t as generator and let M be a G-module. There is a projective resolution

$$\cdots \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \to \mathbb{Z} \to 0$$

where $N = \sum_{i=0}^{n-1} t^i$, and the morphisms are by multiplication.

Now we simplify to $M = \mathbb{Z}$, where the action of G on $\mathbb{Z}$ is trivial. Applying $\mathrm{Hom}(-, \mathbb{Z})$ we obtain a cochain

$$\mathbb{Z} \xrightarrow{t-1} \mathbb{Z} \xrightarrow{N} \mathbb{Z} \xrightarrow{t-1} \mathbb{Z} \xrightarrow{N} \cdots$$

where, similarly, the maps are given by the action of $t-1$ and N, but since the action is trivial, acting by $t-1$ is the zero map, and acting by $N = \sum_{i=0}^{n-1} t^i$ is to add n times. Therefore we have

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \cdots$$

and we get

$$H^k(G; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k = 0; \\ 0, & k \text{ odd}; \\ \mathbb{Z}/n\mathbb{Z}, & k > 0 \text{ even}. \end{cases}$$

4.3 Group cohomology as the cellular cohomology of some CW-complex

We already defined two homology and cohomology groups, for topological spaces and for groups, so the purpose of this subsection is Proposition 4.6, which states the isomorphism between the homology and cohomology of a group and a topological space, that is the classifying space of the group.

For a group G, we call a CW-complex X a **G-complex** if there is an action of G on X which permutes the cells. Having the cellular chain complex of X, $C_*(X)$, this action induces an action on each group $C_n(X)$, so the chain complex $C_*(X)$ would become a chain of G-modules. Also, the augmentation $\varepsilon : C_0(X) \to \mathbb{Z}$, which sends the elements of the basis (the 0-cells) to 1, is a homomorphism of G-modules. It satisfies $\varepsilon \circ d = 0$, hence we obtain a chain complex $C_*(X) \to \mathbb{Z}$, this is called the **augmented cellular chain**

complex. Note that the homology of this chain will be equal to that of the augmented singular chain complex, which gives the reduced holomology of X.

We say that X is a **free** G-complex if the action of G is free; the action on the basis of $C_n(X)$ would be free as well. The following lemma let us conclude that $C_n(X)$ becomes a free $\mathbb{Z}G$-module which has any set of representatives of the G-orbits of n-cells as a basis.

Lemma 4.2. *If X is a set with a free action of a group G, and $E \subset X$ is a set of representatives of the G-orbits, then $\mathbb{Z}X$ is a free $\mathbb{Z}G$-module with basis E.*

Proof. Clearly E is a generating set for $\mathbb{Z}X$, since it is a complete set of representatives of the orbits. The condition of linear independence is obtained from the fact that two different elements of G cannot send an element of X to the same element. □

Consider again the $\mathbb{Z}G$-module $\mathbb{Z}$ with trivial action of G.

Proposition 4.3. *Let X be a contractible free G-complex. The augmented cellular chain complex of X is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.*

Proof. Since X is contractible, it has the homology of a point, then it has trivial reduced homology, so the sequence $C_*(X) \to \mathbb{Z}$ must be exact. Besides we know already that the $C_n(X)$ are free $\mathbb{Z}G$-modules. □

Proposition 4.4. *Let X be a free G-complex and let Y be the orbit complex X/G. We have $C_*(Y) \cong C_*(X)_G$ and $\mathrm{Hom}(C_*(Y), M) \cong \mathrm{Hom}_G(C_*(X), M)$ (the first is homomorphisms of groups and the second is homomorphisms of G-modules) for any G-module M.*

Proof. The projection $X \to Y$ induces a projection $C_*(X) \to C_*(Y)$, which also induces a map $\varphi : C_*(X)_G \to C_*(Y)_G = C_*(Y)$ (these are equal because G acts trivially on Y). Then, by the observations in Section 4.1, $C_*(X)_G$ has a $\mathbb{Z}$-basis in correspondence with the G-orbits of cells of X, but these orbits are precisely the basis of the orbit complex Y, and φ is sending the cells to their orbits, so φ in an isomorphism.

Similarly, since $C_*(X)$ is free, a G-module homomorphism $C_*(X) \to M$ is determined by the image of the G-orbits in the basis of $C_*(X)$ (actually, in each $C_n(X)$), but these are precisely the basis for $C_*(Y)$, which determine any group homomorphism $C_*(Y) \to M$. □

Proposition 4.5. *If Y is a classifying space of G (BG), then the augmented cellular chain complex of the universal cover of Y (EG) is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.*

Proof. Let X be the universal cover of Y. By Proposition 3.2, we know that the group of deck transformations of X, which acts freely on it, is isomorphic to G, so X is a contractible free G-complex, and the result follows from Proposition 4.3. □

Finally, we arrive to the relation between the homology (cohomology) of a group and the homology (cohomology) of its classifying space.

Proposition 4.6. *If Y is a classifying space of G, then we have the chain isomorphism $H_*(Y) \cong H_*(G)$ and the cochain isomorphism $H^*(Y; M) \cong H^*(G; M)$, for any $\mathbb{Z}G$-module M.*

Proof. Let X be the universal cover of Y. $H_*(Y)$ is the homology of $C_*(Y)$, and $H_*(G)$ can be obtained as the homology of $C_*(X)_G$, because of Proposition 4.5. The isomorphism is clear from Proposition 4.4.

At the end of Section 3.3 we saw that $H^*(Y; M)$ can be obtained as the homology of the cochain $\mathrm{Hom}(C_*(Y), M)$, which from Proposition 4.4 is isomorphic to a cochain from which $H^*(G; M)$ is obtained. $\square$

4.4 Results with amalgams

Lemma 4.7. *Any injective diagram $G_1 \leftarrow A \rightarrow G_2$ of groups can be realized by a diagram $X_1 \hookleftarrow Y \hookrightarrow X_2$ of the respective classifying spaces.*

Proof. Using the construction in Proposition 3.10, define the spaces $Y = BA$, $X_1 = BG_1$, and $X_2 = BG_2$. Via the given monomorphism, we can identify A as a subgroup of G_1, then we can identify EA as a subspace of EG_1, and $Y = EA/A$ would be homeomorphic to the image of EA inside $X_1 = EG_1/G_1$. The same works for G_2 and the result is obtained. $\square$

Theorem 4.8. *Any amalgamation diagram (left), with α_1 and α_2 injective, can be realized by a diagram of $K(-,1)$-complexes (right) with $X = X_1 \cup_Y X_2$.*

$$\begin{array}{ccc} A & \xrightarrow{\alpha_1} & G_1 \\ {\scriptstyle \alpha_2}\downarrow & & \downarrow \\ G_2 & \longrightarrow & G \end{array} \qquad\qquad \begin{array}{ccc} Y & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X \end{array}$$

Proof. Let $X_1 \hookleftarrow Y \hookrightarrow X_2$ be as obtained in Lemma 4.7, and let $X = X_1 \cup_Y X_2$. By Theorem 3.5 we know that $\pi_1(X) = G_1 *_A G_2 = G$, where A, G_1, and G_2 can be considered subgroups of G, because of Proposition 2.8. Then, from Lemma 3.4 we obtain that the preimages of Y, X_1, and X_2 in the universal cover of X are copies of the universal covers of each one, and since these are $K(-,1)$-complexes, they have all trivial homology after dimension 1. The homology for $\widetilde{X}$ will be trivial as well, using a Mayer–Vietoris long exact sequence for the injective diagram

$$\begin{tikzcd}
\widetilde{Y} \arrow[r] \arrow[d] & \widetilde{X}_1 \arrow[d] \\
\widetilde{X}_2 \arrow[r] & \widetilde{X}.
\end{tikzcd}$$

So X is a $K(G,1)$-complex. $\square$

Following the previous theorem, there is a Mayer–Vietoris sequence for homology associated to the space $X = X_1 \cup_Y X_2$

$$\cdots \longrightarrow H_n(Y) \longrightarrow H_n(X_1) \oplus H_n(X_2) \longrightarrow H_n(X) \longrightarrow H_{n-1}(Y) \longrightarrow \cdots .$$

Also, for cohomology with coefficients in a group M,

$$\cdots \longrightarrow H^{n-1}(Y;M) \longrightarrow H^n(X;M) \longrightarrow H^n(X_1;M) \oplus H^n(X_2;M) \longrightarrow H^n(Y;M) \longrightarrow \cdots .$$

Both are the associated to the short exact sequences for the cellular chain complexes

$$0 \to C_*(Y) \to C_*(X_1) \oplus C_*(X_2) \to C_*(X) \to 0$$

and

$$0 \to \mathrm{Hom}(C_*(X), M) \to \mathrm{Hom}(C_*(X_1) \oplus C_*(X_2), M) \to \mathrm{Hom}(C_*(Y), M) \to 0.$$

Using Proposition 4.6, we obtain the next proposition.

Proposition 4.9. *Let $G = G_1 *_A G_2$, with $A \hookrightarrow G_1$ and $A \hookrightarrow G_2$. We have a long exact sequence for homology groups*

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(G_1) \oplus H_n(G_2) \longrightarrow H_n(G) \longrightarrow H_{n-1}(A) \longrightarrow \cdots$$

and a long exact sequence for cohomology groups

$$\cdots \longrightarrow H^{n-1}(A;M) \longrightarrow H^n(G;M) \longrightarrow H^n(G_1;M) \oplus H^n(G_2;M) \longrightarrow H^n(A;M) \longrightarrow \cdots ,$$

where M is a G-module.

5 $SL_2(\mathbb{Z})$

In this section we will prove the group isomorphism $SL_2(\mathbb{Z}) \cong C_4 *_{C_2} C_6$, where $C_n = \mathbb{Z}/n\,\mathbb{Z}$ is the cyclic group of order n, by constructing a tree in the upper complex half plane using an action of the group.

5.1 Generators of $SL_2(\mathbb{Z})$

First, we will show some pair of generators for the group.

Recall the definition of the group:

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \right\}.$$

Define $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. There is a simple algebraic proof to show that S and T generate $SL_2(\mathbb{Z})$. First, we see that S has order 4 and T has infinite order, so $S^2 = -I$ and $T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for any integer n. Now, take any element of the group, say $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We want to multiply γ by S and T to reduce it to I or $-I$. We have

$$S \cdot \gamma = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \qquad \text{and} \qquad T^n \cdot \gamma = \begin{pmatrix} a + cn & b + dn \\ c & d \end{pmatrix}.$$

Suppose $|a| \geq |c|$, because if $|c| > |a|$, take γ as $S\gamma$. And, if $c = 0$, we would have $\gamma = \pm T^{\pm b}$, so suppose $c \neq 0$. By euclidean division, there are integers q and r, with $0 \leq r < |c|$, such that $a = cq + r$. Let $n = -q$, then $T^n\gamma$ is a matrix whose entry $(1, 1)$ is strictly smaller than the absolute value of its entry $(2, 1)$. Now, if the entry $(2, 1)$ of $\gamma' = ST^n\gamma$ is zero, we are done, because γ' would be a power of T multiplied by $\pm I$. If it is not zero, we can use euclidean division again, and because the inequality $r < |c|$ is strict, we should arrive to $c = 0$ (the entry $(2, 1)$ of the matrix) in a finite number of iterations. This concludes the proof, so we have the following proposition.

Let $R = TS = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$.

Proposition 5.1. *The sets $\{S, T\}$ and $\{S, R\}$ are both sets of generators for the group $SL_2(\mathbb{Z})$.*

Proof. We already have this for $\{S, T\}$. Now, T can be written as $RS^{-1} = RS^3$, so the subgroup generated by $\{S, R\}$ contains T, hence contains the subgroup generated by $\{S, T\}$, which is already the whole group. □

Note that R have finite order equal to 6, since $R^3 = -I$, so $SL_2(\mathbb{Z})$ is generated by two elements of finite order.

This group acts on the half plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ by Möbius transformations, that is,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

Note that $-I$ acts trivially.

Using this action, there is another way to prove that $SL_2(\mathbb{Z})$ is generated by S and T; see [6].

5.2 $SL_2(\mathbb{Z})$ as an amalgam

Some of these procedures are taken from [7].

With $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, let $C = \mathbb{S}^1 \cap \mathbb{H}$. We have that $\gamma(C)$ is the set

$$\left\{ z \in \mathbb{H} : \quad \left| z - \frac{ac-bd}{c^2-d^2} \right| = \frac{1}{|c^2-d^2|} \right\} \quad \text{if } c^2 \neq d^2, \quad \text{or}$$

$$\left\{ z \in \mathbb{H} : \quad \operatorname{Re}(z) = ac - \frac{cd}{2} \right\} \quad \text{if } c^2 = d^2 = 1, \quad \text{with } cd = \pm 1.$$

We can use the limits to $\gamma(1)$ and $\gamma(-1)$ together with the properties of a Möbius transformation to proof this.

Indeed, if $c^2 \neq d^2$, we have that $\gamma(1) = \frac{a+b}{c+b}$ and $\gamma(-1) = \frac{-a+b}{-c+b}$, so the midpoint and distance are

$$\mu := \frac{\gamma(1)+\gamma(-1)}{2} = \frac{ac-bd}{c^2-d^2}, \qquad \text{and} \qquad \delta := \left| \frac{\gamma(1)-\gamma(-1)}{2} \right| = \left| \frac{2}{c^2-d^2} \right|.$$

Also,

$$\gamma(i) = \frac{ai+b}{ci+d} = \frac{b+ia}{d+ic} \cdot \frac{d-ic}{d-ic} = \frac{ac+bd+i}{c^2+d^2},$$

then

$$\left| \gamma(i) - \frac{ac-bd}{c^2-d^2} \right| = \left| \frac{((ac+bd)(c^2-d^2)-(ac-bd)(c^2+d^2))}{(c^2+d^2)(c^2-d^2)} + \frac{i}{c^2+d^2} \right|$$

$$= \left| \frac{-2cd + i(c^2-d^2)}{(c^2+d^2)(c^2-d^2)} \right| = \left| \frac{1}{c^2-d^2} \right| = \frac{\delta}{2}.$$

So we have here three points of $\gamma(\mathbb{S}^1)$ that are contained in the circle centred in μ with radius $\delta/2$, but Möbius transformations carry generalized circles to generalized circles, thus, in particular, $\gamma(C)$ must be the half circle in $\mathbb{H}$ centred in μ with radius $\delta/2$.

If $c^2 = d^2$, then $1 = ad - bc = c(\pm a - b)$, so c, and also d, must be 1 or -1. We have, with $cd = \pm 1$ and $bd = ac - (\pm 1)$,

$$\gamma(z) = \frac{az+b}{cz+d} \cdot \frac{c\bar{z}+d}{c\bar{z}+d} = \frac{ac+adz+bc\bar{z}+bd}{c^2+d^2+cd(z+\bar{z})} = \frac{ac(1 \pm z)+bd(1 \pm \bar{z})}{2(1 \pm \operatorname{Re}(z))}$$

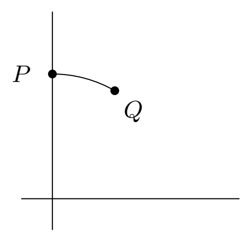
$$= \frac{2ac(1 \pm \mathrm{Re}(z)) - (\pm 1 + \bar{z})}{2(1 \pm \mathrm{Re}(z))} = ac - \frac{\pm 1}{2} + \frac{\mathrm{Im}(z)}{2(1 \pm \mathrm{Re}(z))} i,$$

where all the $\pm$ represent the same sign. Also, the limits $z \to 1$ and $z \to -1$ give the limits $\gamma(z) \to (ac - (\pm 1/2))$ and $\gamma(z) \to \infty$ (in some order), because

$$\lim_{x \to 1^-} \frac{\sqrt{1 - x^2}}{1 - x} = \lim_{x \to 1^-} \frac{\sqrt{1 + x}}{\sqrt{1 - x}} = +\infty.$$

So $\gamma(C)$ is the whole line $\{z \in \mathbb{H} : \mathrm{Re}(z) = ac - (\pm 1/2)\}$.

Now, consider the segment $Y = \{z = e^{i\theta} : \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}\}$ of the unit circle, together with its extremities $P = e^{i\pi/2} = i$ and $Q = e^{i\pi/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, as the realization of a graph, where P and Q are the vertices of the graph, connected by the edge Y, as shown below.



Let us see which are the stabilizers of P and Q. For P, we would have

$$\frac{ai + b}{ci + d} = i \iff b + ia = -c + id \iff b = -c \text{ and } a = d.$$

And since $ad - bc = 1$, these are exactly the four matrices of the cyclic subgroup generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

For Q, we would have

$$\frac{ae^{i\pi/3} + b}{ce^{i\pi/3} + d} = e^{i\pi/3} \iff \left(\frac{a}{2} + b\right) + i\left(a\frac{\sqrt{3}}{2}\right) = \left(-\frac{c}{2} + \frac{d}{2}\right) + i\frac{\sqrt{3}}{2}(c + d)$$

$$\iff a + 2b = d - c \text{ and } a = c + d \iff b = -c \text{ and } a(a - c) + c^2 = 1.$$

Solving for $a^2 - ca + (c^2 - 1) = 0$, the discriminant must be nonnegative, so $c^2 - 4(c^2 - 1) \geq 0$, and then $|c| \leq 2/\sqrt{3}$; this gives the possibilities $c \in \{0, 1, -1\}$. Each value for c gives two values for a, and b and d are determined by c and a, respectively. So we have six matrices, which are the elements of the subgroup generated by $R = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$.

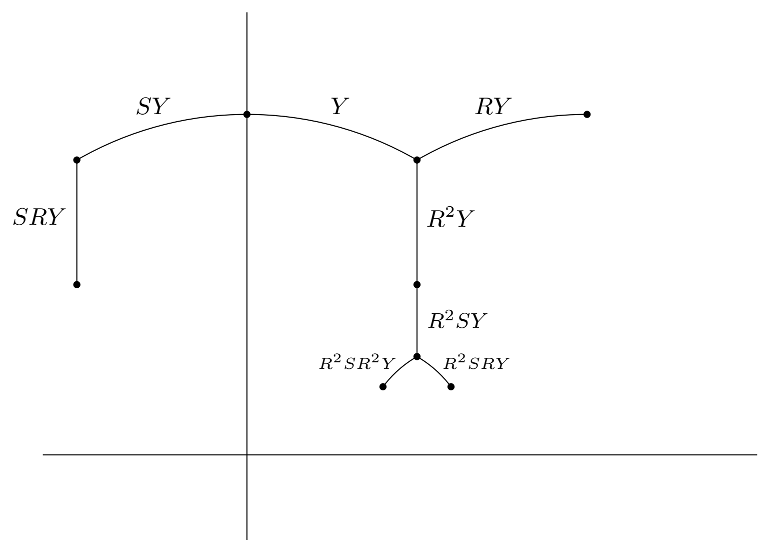
The stabilizers of P and Q have different orders, so in particular the subgroups cannot be conjugate and P and Q must be in distinct orbits. Also, the stabilizer of Y should be the intersection of these subgroups, that is, $\{I, -I\}$, which acts trivially.

Let X be the set $SL_2(\mathbb{Z}) \cdot Y$. Using the description of the translates of S, we see that the only way in which Y intersects some $\gamma(Y)$ is for it to be P or Q intersected by $\gamma(P)$ or $\gamma(Q)$, so it can be obtained that X is the realization of a graph.

Indeed, suppose $\gamma(Y)$ (with the usual notation for its entries) intersects Y. First, if $c^2 \neq d^2$, the radius $|\frac{1}{c^2-d^2}|$ is needed to be 1, so either c or d are zero, and the centre is needed to be 0 or 1. If it is 0, this gives a translation, which can only be $\pm I$, or gives $\gamma = S$, because a must be 0; for $\gamma = S$, $\gamma(Y)$ is $-\overline{Y}$, the reflection on the y-axis. If the centre is 1, $d = 0$ and $ac = 1$, which would fix Q ($\gamma = \pm R$). And, in the case $c^2 = d^2 = 1$, the real part $c(a - d/2)$ must be $1/2$, so $a \in \{0, 1, -1\}$; if $a = 0$, $\gamma = \pm R^2$ (so it fixes Q), and with $a = \pm 1$ the intersection would not happen.

Now, we know from Proposition 5.1 that $SL_2(\mathbb{Z})$ is generated by S and R, the stabilizers of P and Q (and they have finite order), so any edge of X, that is, a translation of Y, is equal to the translation of Y by a finite sequence of S's and R's. Clearly, Y is connected to $S(Y)$ and $R(Y)$, so an S- or T-translation of $Y' = S(Y) \cup Y \cup R(Y)$ is connected to Y', and then an S- or T-translation of $Y'' = S(Y') \cup Y' \cup R(Y')$ is also connected to Y' and Y. Repeating this we see that all X is a connected graph.

See below an example of how the graph is constructed.



By last, we know from the form of the translations of $C = \mathbb{S}^1 \cap \mathbb{H}$ that the only intersection of X with the y-axis is $P = i$, then the only edges that intersect the y-axis are

Y and $S(Y)$. Now, since any path in X can be translated so that it intersects the axis and then any circuit should cross the axis twice, a circuit in X is not possible.

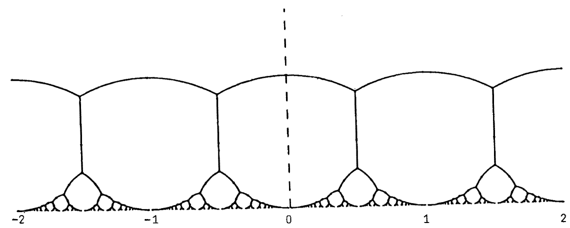


Figure 1: Edited from [7].

Finally, we obtain that X is a tree. Furthermore, Y (together with P and Q) is a fundamental domain for X, and the stabilizers of P, Q, and Y are clearly isomorphic to $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$, and $\mathbb{Z}/2\mathbb{Z}$, respectively. Then, by Theorem 2.9, we have the group isomorphism

$$SL_2(\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$$

and, therefore, the presentation

$$SL_2(\mathbb{Z}) = \langle R, S \mid R^6 = S^4 = 1,\ R^3 = S^2 \rangle.$$

Also, since the group $PSL_2(\mathbb{Z})$ is $SL_2(\mathbb{Z})$ mod $\pm I$, we have

$$PSL_2(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$$

and

$$PSL_2(\mathbb{Z}) = \langle r, s \mid r^3 = s^2 = 1 \rangle = \langle s, t \mid s^2 = (st)^3 = 1 \rangle.$$

5.3 Cohomology of $SL_2(\mathbb{Z})$

For simplicity, let C_n be the cyclic group of order n.

Let r, s, and t be generators for C_2, C_4, and C_6, respectively. We know that the morphisms used for the amalgamation are

$$\alpha_1 : C_2 \to C_4, \quad r \mapsto s^2, \qquad \text{and} \qquad \alpha_2 : C_2 \to C_6, \quad r \mapsto t^3.$$

These morphisms let us consider $\mathbb{Z}C_4$ and $\mathbb{Z}C_6$ as $\mathbb{Z}C_2$-modules, with r acting as multiplication by $\alpha_1(r)$ and $\alpha_2(r)$, respectively, and extending linearly.

Using Lemma 4.1 with the resolutions for C_2 and C_4, we have a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{Z}C_2 \arrow[r, "1+r"] \arrow[d, "f_{n+1}"] & \mathbb{Z}C_2 \arrow[r, "r-1"] \arrow[d, "f_n"] & \mathbb{Z}C_2 \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \arrow[r] & \mathbb{Z}C_2 \arrow[r, "\varepsilon"] \arrow[d, "f_0"] & \mathbb{Z} \arrow[r] \arrow[d, "id"] & 0 \\
\cdots \arrow[r] & \mathbb{Z}C_4 \arrow[r, "1+s+s^2+s^3"] & \mathbb{Z}C_4 \arrow[r, "s-1"] & \mathbb{Z}C_4 \arrow[r] & \cdots \arrow[r] & \mathbb{Z}C_4 \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

where the f's are morphisms of $\mathbb{Z}C_2$-modules which extend f_0, defined as the linear extension of α_1. Here n is odd.

The f's are uniquely determined by the image of 1. So we have $(1+s+s^2+s^3)f_{n+1}(1) = f_n((1+r)(1)) = (1+\alpha_1(r))f_n(1) = (1+s^2)f_n(1)$, but

$$0 = (s-1)(1+s+^2+s^3)f_{n+1}(1) = (s-1)(1+s^2)f_n(1) = (-1+s-s^2+s^3)f_n(1),$$

then we can take $f_n(1) = 1+s$. Now, $(1+s+s^2+s^3)f_{n+1}(1) = (1+s^2)(1+s) = 1+s+s^2+s^3$, so $f_{n+1}(1) = 1$.

Next, we apply $\mathrm{Hom}_{C_2}(-,\mathbb{Z})$ and $\mathrm{Hom}_{C_4}(-,\mathbb{Z})$ to the first and second row in the diagram, respectively, (not including the right part after the column of f_0) to obtain the following.

$$\begin{tikzcd}
\cdots & \mathbb{Z} \arrow[l] & \mathbb{Z} \arrow[l, "2"'] & \mathbb{Z} \arrow[l, "0"'] & \mathbb{Z} \arrow[l, "2"'] & \mathbb{Z} \arrow[l, "0"'] \\
\cdots & \mathbb{Z} \arrow[l] \arrow[u, "f_4^*"'] & \mathbb{Z} \arrow[l, "4"'] \arrow[u, "f_3^*"'] & \mathbb{Z} \arrow[l, "0"'] \arrow[u, "f_2^*"'] & \mathbb{Z} \arrow[l, "4"'] \arrow[u, "f_1^*"'] & \mathbb{Z} \arrow[l, "0"'] \arrow[u, "f_0^*"']
\end{tikzcd}$$

In the diagram, the f^*'s are now morphisms of groups given by $f_n^*(1) = 1$ if n is even and $f_n^*(1) = 2$ if n is odd. This is because an element in $\mathrm{Hom}_{C_k}(\mathbb{Z}C_k, \mathbb{Z})$ is uniquely determined by the image of 1, and the action of C_k on $\mathbb{Z}$ is trivial, for $k = 2, 4$.

Passing to cohomology, knowing that $H^n(C_k;\mathbb{Z}) = C_k$ for $n > 0$ even, we obtain morphisms $\alpha_1^* : H^n(C_4;\mathbb{Z}) \to H^n(C_2;\mathbb{Z})$ given as the identity for $n = 0$, as zero for n odd, and by $s \mapsto r$ for $n > 0$ even.

We can do the same to obtain morphisms $\alpha_2^* : H^n(C_6;\mathbb{Z}) \to H^n(C_2;\mathbb{Z})$ given in the same way, with $t \mapsto r$ for $n > 0$ even.

With the discussion following Lemma 4.1, we know that these induced morphisms are the same to those in the long exact sequence obtained as in Proposition 4.9. We have

$$\cdots \to H^{n-1}(C_2) \xrightarrow{\psi} H^n(SL_2(\mathbb{Z})) \xrightarrow{\varphi} H^n(C_4) \oplus H^n(C_6) \xrightarrow{\alpha_1^*+\alpha_2^*} H^n(C_2) \to \cdots,$$

where, abusing the notation, $\alpha_1^* + \alpha_2^* : (a,b) \mapsto a+b$ for all n (including the case $n = 0$). Clearly these morphisms are surjective, so $\psi = 0$ and the φ are injective. Counting and making the list, we see that the kernel of $\alpha_1^* + \alpha_2^*$, for $n > 0$ even, is the subgroup of order 12 generated by (s, t).

Now, for $n = 0$, we have

$$0 \to H^0(SL_2(\mathbb{Z})) \xrightarrow{\varphi} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to \dots,$$

so $H^0(SL_2(\mathbb{Z})) \cong \mathrm{Im}(\varphi) = \mathrm{Ker}(\alpha_1^* + \alpha_2^*) \cong \mathbb{Z}$; for $n > 0$ even,

$$\cdots \to 0 \to 0 \to H^n(SL_2(\mathbb{Z})) \xrightarrow{\varphi} C_4 \oplus C_6 \to C_2 \to \cdots,$$

so $H^n(SL_2(\mathbb{Z})) \cong \mathrm{Im}(\varphi) = \mathrm{Ker}(\alpha_1^* + \alpha_2^*) \cong C_12$; and for n odd,

$$\cdots \to H^{n-1}(C_4) \oplus H^{n-1}(C_6) \to H^{n-1}(C_2) \to H^n(SL_2(\mathbb{Z})) \xrightarrow{\varphi} 0 \to 0 \to \cdots,$$

so $H^n(SL_2(\mathbb{Z})) = 0$.

For the record, we have the following theorem.

Theorem 5.2.

$$H^n(SL_2(\mathbb{Z}); \mathbb{Z}) = \begin{cases} \mathbb{Z}, & n = 0; \\ 0, & n \textit{ odd}; \\ \mathbb{Z}/12\,\mathbb{Z}, & n > 0 \textit{ even}. \end{cases}$$

A next step would be to find out if the ring structure on this cohomology can be described.

Part II
The Bianchi Groups

In the first part, we studied the group $SL_2(\mathbb{Z})$; a next step could be to replace $\mathbb{Z}$ by some other object with similar properties. Since $\mathbb{Z}$ is the ring of integers of the field of rational numbers $\mathbb{Q}$, it is natural to consider the ring of integers of some algebraic extension of $\mathbb{Q}$.

Therefore, let d be a positive square-free integer, and let $\mathbb{O}_d$ be the ring of integers of the imaginary quadratic extension $\mathbb{Q}(\sqrt{-d})$, then the **Bianchi group** associated to d is defined as

$$\Gamma_d = PSL_2(\mathbb{O}_d).$$

In general, except from $d = 3$, these groups can be expressed as amalgamated products, but not all the factor groups are described easily.

This part contains, first, the definition of HNN extensions, which will be useful to describe some Bianchi groups. Then, a brief introduction to spectral sequences (in general and also two in particular) will be given, in order to compute the cohomology of one Bianchi group. Besides, we construct the amalgam decomposition of the Euclidean Bianchi groups.

6 HNN extensions

An HNN extension is a construction similar to that of an amalgamated product.

Let G be a group with a presentation. Suppose there is a collection $\{A_i\}_{i\in I}$ of subgroups of G together with a collection of injections $\{\varphi_i : A_i \to G\}_{i\in I}$. Then the **HNN extension** of G associated to the $\{A_i, \varphi_i\}$ is defined as the group with the presentation

$$G^* = \langle \quad G,\ \{t_i\}_{i\in I} \quad | \quad \text{Relations of } G,\ t_i a t_i^{-1} = \varphi_i(a),\ \text{for } i \in I,\ a \in A_i \quad \rangle.$$

G is called the **base**, $\{t_i\}_{i\in I}$ is called the **free part**, and the $\{A_i, \varphi_i(A_i)\}_{i\in I}$ are called the **associated subgroups**. The size of the set I is the **free part rank**.

Consequently, a group is called an **HNN group** if it is the HNN extension of some group with some associated subgroups.

We will only use HNN extensions associated to one subgroup and its inclusion, in this case, with $A \subset G$, we have

$$G^* = \langle \quad G,\ t \quad | \quad \text{Relations of } G,\ tat^{-1} = a,\ \text{for } a \in A \quad \rangle.$$

As in amalgamated products, there is a way to write uniquely each element of a HNN extension. This is known as the Britton's lemma.

Using the notation used in the initial definition, let S_i be a set of representatives of G modulo A_i and let R_i be a set of representatives of G modulo $\varphi_i(A_i)$. Then every $g \in G^*$ is written uniquely as

$$g_0\, t_{i_1}^{e_1}\, g_1\, t_{i_2}^{e_2} \cdots t_{i_k}^{e_k}\, g_k, \quad \text{with } e_j = \pm 1,$$

where $g_o \in G$, while $g_j \in S_{t_j}$ if $e_j = -1$ and $g_j \in R_{t_j}$ if $e_j = 1$; also, there is no subsequence $t^e \cdot 1 \cdot t^{-e}$.

An example of this is the group with the presentation

$$\langle s, t, u \mid s^2 = (st)^3 = [t, u] = 1 \rangle.$$

This group is both the amalgamated product $PSL_2(\mathbb{Z}) *_{\mathbb{Z}} (\mathbb{Z} \oplus \mathbb{Z})$ and the HNN extension of $PSL_2(\mathbb{Z}) = \langle s, t \mid s^2 = (st)^3 = 1 \rangle$ associated to the subgroup generated by t (together with the inclusion).

7 Spectral Sequences

Spectral sequences are algebraic objects, somewhat complicated, used mostly in algebraic topology. We will describe briefly what this objects are and some of their applications. For a complete introduction and development of this subject see [14].

7.1 Basic notions

A **differential bigraded module** over a ring R, is a collection of R-modules, $\{E^{p,q}\}_{p,q \in \mathbb{Z}}$, together with an R-linear mapping $d : E^{*,*} \to E^{*,*}$, of bidegree $(s, 1-s)$, or $(-s, s-1)$, for some $s \in \mathbb{Z}$, such that $d \circ d = 0$. Sometimes we use the same indices in d to specify the particular domain.

The mapping d is called the **differential**. The bidegree (m, n) means that d goes from $E^{p,q}$ to $E^{p+m,q+n}$, for each pair p, q. With this, we can take the homology of the differential bigraded module, with bidegree $(s, 1-s)$; we define it as

$$H^{p,q}(E^{*,*}, d) = \mathrm{Ker}(d : E^{p,q} \to E^{p+s,q+1-s}) \,/\, \mathrm{Im}(d : E^{p-s,q-1+s} \to E^{p,q}).$$

This gives another bigraded module $\{H^{p,q}(E^{*,*}, d)\}_{p,q\in\mathbb{Z}}$. The same can be done with a differential bigraded module of bidegree $(-s, s-1)$.

Now we can give the definition.

Definition 7.1. *A **spectral sequence** is a collection of differential bigraded R-modules $\{E_r^{*,*}\}_{r=1}^{\infty}$, such that the differentials are either all of bidegree $(-r, r-1)$ (for homological type) or all of bidegree $(r, 1-r)$ (for cohomological type) and, for all p, q, r, there is an isomorphism $E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}, d_r)$. The module $E_r^{*,*}$ is called the E_r**-term** of the spectral sequence.*

It is important to mention that $E_r^{*,*}$ and d_r determine the $E_{r+1}^{p,q}$, but not d_{r+1}, so one term of the spectral sequence is not enough to describe it all.

There is another way to describe an spectral sequence. Let $\{E_0^{p,q}\}_{p,q\in\mathbb{Z}}$ be a family of R-modules. Suppose that for each $p, q \in \mathbb{Z}$ there is a tower of submodules

$$B_0^{p,q} \subset B_1^{p,q} \subset \cdots \subset B_n^{p,q} \subset \cdots \subset Z_n^{p,q} \subset \cdots \subset Z_1^{p,q} \subset Z_0^{p,q} \subset E_0^{p,q}$$

together with short exact sequences

$$0 \longrightarrow Z_{n+1}^{p,q}/B_n^{p,q} \longrightarrow Z_n^{p,q}/B_n^{p,q} \longrightarrow B_{n+1}^{p+n+1,q-n}/B_n^{p+n+1,q-n} \longrightarrow 0.$$

These define a spectral sequence by setting $E_{n+1}^{p,q} = Z_n^{p,q}/B_n^{p,q}$ and

$$d_n^{p,q} : E_n^{p,q} = Z_{n-1}^{p,q}/B_{n-1}^{p,q} \longrightarrow B_n^{p+n,q+1-n}/B_{n-1}^{p+n,q+1-n} \subset E_n^{p+n,q+1-n},$$

taken from the exact sequences. See [14] for the complete explanation.

A spectral sequence is said to **collapse at the N-th term** if $d_r = 0$ for $r \geq N$. This would imply that $E_r^{*,*} = E_{r+1}^{*,*}$, then we define the **limit term**, E_∞, as $E_N^{*,*}$.

Let F^* be a **filtration** on an R-module M, that is, a family of submodules $\{F^pM\}_{p\in\mathbb{Z}}$, which could be decreasing, so $F^{p+1}M \subset F^pM$, or increasing, so $F^pM \subset F^{p+1}M$, such that

$$\bigcap_{p\in\mathbb{Z}} F^pM^* = 0 \qquad \text{and} \qquad \bigcup_{p\in\mathbb{Z}} F^pM^* = M^*.$$

Now, define its **associated graded module**, $E_o^*(M)$, as

$$E_0^p(M, F) = \begin{cases} F^pM/F^{p+1}M, & \text{for } F^* \text{ decreasing;} \\ F^pM/F^{p-1}M, & \text{for } F^* \text{ increasing.} \end{cases}$$

Also, if we have a graded R-module M^*, in order to examine its filtration on each degree, we define $F^pM^k = F^pM^* \cap M^k$ and the associated graded module as

$$E_0^{p,q}(M^*, F) = \begin{cases} F^pM^{p+q}/F^{p+1}M^{p+q}, & \text{for } F^* \text{ decreasing;} \\ F^pM^{p+q}/F^{p-1}M^{p+q}, & \text{for } F^* \text{ increasing.} \end{cases}$$

Using these notions, we say that a spectral sequence $\{E_r^{*,*}, d_r\}_r$ **converges** to a graded R-module M^* if there is a filtration F^* on M^* such that, for each p, q,

$$E_\infty^{p,q} \cong E_0^{p,q}(M^*, F).$$

If M has a differential structure, then we can construct a spectral sequence from it. We define that an R-module M is a **filtered differential graded module** if

- M is a direct sum of submodules, $M = \bigoplus_{n=0}^\infty M^n$;
- there is an R-linear mapping, $d : M \to M$, of degree 1 (so $d : M^n \to M^{n+1}$) or degree -1 (so $d : M^n \to M^{n-1}$) satisfying $d \circ d = 0$; and
- M has a filtration F^* and the differential d respects the filtration, which means that $d : F^pM \to F^pM$.

Then, we have the next theorem.

Theorem 7.2. *Each filtered differential graded module (M, d, F^*) determines a spectral sequence, $\{E_r^{*,*}, d_r\}_{r=1}^\infty$, with d_r of bidegree $(r, 1-r)$ and*

$$E_1^{p,q} \cong H^{p+q}(F^pM/F^{p+1}M).$$

If we suppose further that the filtration is bounded, that is, for each dimension n, there are values $s = s(n)$ and $t = t(n)$, so that

$$0 = F^sM^n \subset F^{s-1}M^n \subset \cdots \subset F^{t+1}M^n \subset F^tM^n = M^n,$$

then the spectral sequence converges to $H(M, d)$, that is,

$$E_\infty^{p,q} \cong F^pH^{p+q}(M, d)/F^{p+1}H^{p+q}(M, d).$$

The proof for this is a long one; see [14].

In general, there are much more different (weaker, stronger) algebraic structures on which spectral sequences can be used, such as taking R to be a graded ring, or M to be a graded vector space, a graded algebra, or simply a graded group (a direct sum of groups).

Spectral sequences are objects that usually are given with the purpose of discovering or describing some graded object M, although arriving to M may be impossible without enough information. One of the main obstacles could be the so-called *extension problem*. We will discuss this in the next paragraphs.

Suppose that there is a spectral sequence which converges to a graded object M and that we already know what is the limit term E_∞, but without any relation between the $E_\infty^{p,q}$. The convergence guarantees that there is a (decreasing) filtration F^* on M for which, for all p, q,

$$E_\infty^{p,q} \cong F^p M^{p+q} / F^{p+1} M^{p+q}.$$

This is the same as saying that there are short exact sequences

$$0 \longrightarrow F^{p+1} M^{p+q} \longrightarrow F^p M^{p+q} \longrightarrow E_\infty^{p,q} \longrightarrow 0 \tag{2}$$

for any p, q.

We cannot say much more without assuming something else.

For the case when M is a graded vector space, it can be recovered taking the direct sum, for any n,

$$M^n \cong \bigoplus_{p \in \mathbb{Z}} F^p M^n / F^{p+1} M^n \cong \bigoplus_{p+q=n} E_\infty^{p,q},$$

since vector spaces are determined up to isomorphism just with their dimension.

Now, going back to the general case, suppose that for some n_0, we have $E_\infty^{p,q} = 0$ whenever $p + q = n_0$, that is, the (anti-)diagonal is zero. Then, from the exact sequences (2) we can conclude that $F^{p+1} M^{n_0} = F^p M^{n_0}$ for all p, and then clearly $M^{n_0} = 0$.

Now, suppose that $E_\infty^{p,q}$ is not zero only for a pair p_0, q_0. We have

$$0 = \cdots = F^{p_0+2} M^{n_0} = F^{p_0+1} M^{n_0} \subset F^{p_0} M^{n_0} = F^{p_0-1} M^{n_0} = \cdots = M^{n_0}$$

but also we have an exact sequence

$$0 \longrightarrow F^{p_0+1} M^{n_0} \longrightarrow F^{p_0} M^{n_0} \longrightarrow E_\infty^{p_0,q_0} \longrightarrow 0,$$

so we obtain the isomorphism

$$M^{n_0} \cong E_\infty^{p_0,q_0}.$$

Following with the idea, suppose that $E_\infty^{p,q}$ is not zero only for two pairs p_0, q_0 and p_1, q_1, with $p_0 > p_1$. Then,

$$0 = \cdots = F^{p_0+1} M^{n_0} \subset F^{p_0} M^{n_0} = \cdots = F^{p_1+1} M^{n_0} \subset F^{p_1} M^{n_0} = \cdots = M^{n_0},$$

so the exact sequences

$$0 \to F^{p_1+1} M^{n_0} \to F^{p_1} M^{n_0} \to E_\infty^{p_1,q_1} \to 0, \qquad 0 \to F^{p_0+1} M^{n_0} \to F^{p_0} M^{n_0} \to E_\infty^{p_0,q_0} \to 0$$

imply we have the exact sequence

$$0 \longrightarrow E_\infty^{p_0,q_0} \longrightarrow M^{n_0} \longrightarrow E_\infty^{p_1,q_1} \longrightarrow 0$$

and the isomorphism $E_\infty^{p_1,q_1} \cong M^{n_0}/E_\infty^{p_0,q_0}$. With this result, M^{n_0} is not determined yet, thus it will be a matter of particular cases.

We see that, after this, with more than two non zero terms in a diagonal n, the object M^n is hard to describe.

In the next subsections we will discuss two particular spectral sequences that will be useful for us: one to compute the cohomology of S_3 and the other to compute the cohomology of the Bianchi group Γ_1. Both spectral sequences will be **first-quadrant**; this means that $E_r^{p,q} = 0$ when $p < 0$ or $q < 0$.

7.2 The Lyndon–Hochschild–Serre spectral sequence

From the original paper of G. Hochschild and J-P. Serre [13], we have the following theorem.

Theorem 7.3. *Let G be a group, K a normal subgroup of G, and M a G-module. Then there exists a spectral sequence $\{E_r\}$ in which the term $E_2^{p,q}$ is isomorphic with $H^p(G/K; H^q(K; M))$, and E_∞ is isomorphic with the graduated group associated with $H^*(G; M)$, appropriately filtered.*

In other words, if there is a group extension, that is, a short exact sequence of groups,

$$0 \longrightarrow A \longrightarrow G \longrightarrow B \longrightarrow 0,$$

then, for a G-module M, there is a spectral sequence, whose E_2-term is given by

$$E_2^{p,q} = H^p(B; H^q(A; M)),$$

which converges to $H^*(G; M)$. This is called the **Lyndon–Hochschild–Serre spectral sequence**, associated to a group extension.

Note that here $H^q(A; M)$ is a B-module, where the action may not be trivial.

Our example will be to compute the cohomology groups of the symmetric group S_3. Although the formal computations happen to be quite long, this is a simple example, since there are no differentials needed in the spectral sequence (all of them are zero maps). We will use this construction to deduce the morphism $H^*(S_3; \mathbb{Z}) \to H^*(C_3; \mathbb{Z})$ induced by the inclusion.

Example 7.4. Consider the extension

$$0 \longrightarrow C_3 \longrightarrow S_3 \longrightarrow C_2 \longrightarrow 0,$$

where S_3 can be written as the semidirect product $C_3 \rtimes C_2$, with $C_2 = \langle s \rangle$ acting on $C_3 = \langle t \rangle$ by $s \cdot t = t^{-1} = t^2$.

According to the above, there is a spectral sequence converging to $H^*(S_3;\mathbb{Z})$, with trivial action on $\mathbb{Z}$, whose E_2-term is given by

$$E_2^{p,q} = H^p(C_2; H^q(C_3;\mathbb{Z})).$$

Now, we know that

$$H^q(C_3;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & q = 0; \\ 0, & q \text{ odd}; \\ C_3, & q > 0 \text{ even}. \end{cases}$$

But, to compute the cohomology of C_2 with coefficients in $H^q(C_3;\mathbb{Z})$, we need to know how C_2 acts on $H^q(C_3;\mathbb{Z})$.

The action on $H^q(C_3;\mathbb{Z})$ is simply the induced in cohomology from the action on C_3. That is, each element of C_2 corresponds to an automorphism of C_3, and therefore, an automorphism of $H^q(C_3;\mathbb{Z})$. In this case it is easy because we only need the induced automorphism associated to s.

So, recalling Lemma 4.1 and the discussion after it, we proceed with the computation. Consider the projective resolution of $\mathbb{Z}$ over $\mathbb{Z}C_3$

$$\cdots \xrightarrow{1+t+t^2} \mathbb{Z}C_3 \xrightarrow{t-1} \mathbb{Z}C_3 \xrightarrow{1+t+t^2} \mathbb{Z}C_3 \xrightarrow{t-1} \mathbb{Z}C_3 \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where the morphisms are given by multiplication. Denoting the automorphism $t \mapsto t^2$ by φ, we have the commutative diagram, for n odd,

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{Z}C_3 \arrow[r, "1+t+t^2"] \arrow[d, "f_{n+1}"] & \mathbb{Z}C_3 \arrow[r, "t-1"] \arrow[d, "f_n"] & \mathbb{Z}C_3 \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \arrow[r] & \mathbb{Z}C_3 \arrow[r, "\varepsilon"] \arrow[d, "f_0"] & \mathbb{Z} \arrow[r] \arrow[d, "id"] & 0 \\
\cdots \arrow[r] & \mathbb{Z}C_3 \arrow[r, "1+t+t^2"] & \mathbb{Z}C_3 \arrow[r, "t-1"] & \mathbb{Z}C_3 \arrow[r] & \cdots \arrow[r] & \mathbb{Z}C_3 \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

which extends the identity on $\mathbb{Z}$ and f_0 as the linear extension of φ, and where the f_i are morphisms of $\mathbb{Z}C_3$-modules, respecting the action φ. This means that

$$\begin{aligned} f_i(a+bt+ct^2) &= f_i(a) + f_i(bt) + f_i(ct^2) \\ &= f_i(a) + f_i(b)\varphi(t) + f_i(c)\varphi(t^2) \\ &= f_i(a) + f_i(b)t^2 + f_i(c)t \quad = \quad f_i(1)(a+ct+bt^2), \end{aligned}$$

hence, each morphism is determined by the image of 1.

We take $f_0(1) = 1$. Then, by commutativity we have

$$f_{n-1}(1)(t^2 - 1) = f_{n-1}(t - 1) = (t - 1)f_n(1),$$

so $f_n(1)$ could be $-t^2 f_{n-1}(1)$ (it also could be $(t+1)f_{n-1}(1)$, but the calculations would be more confusing and the final result is the same), and,

$$f_n(1)(1 + t + t^2) = f_n(1 + t + t^2) = (1 + t + t^2)f_{n+1}(1),$$

which allows the equality $f_n(1) = f_{n+1}(1)$. Gathering all, we have

$$f_0(1) = 1, \qquad f_{2k+1}(1) = -t^2 f_{2k}(1), \qquad f_{2k+2}(1) = f_{2k+1}(1).$$

We proceed to apply $\mathrm{Hom}_{C_3}(-, \mathbb{Z})$ to the diagram. Then, $\mathrm{Hom}_{C_3}(\mathbb{Z}C_3, \mathbb{Z})$ can be identified with $\mathbb{Z}$, since any homomorphism is determined by the image of 1. We obtain

$$\begin{tikzcd}
\cdots & \mathbb{Z} \arrow[l] & \mathbb{Z} \arrow[l, "3"'] & \mathbb{Z} \arrow[l, "0"'] & \cdots \arrow[l] & \mathbb{Z} \arrow[l] \\
\cdots & \mathbb{Z} \arrow[l] \arrow[u, "\tilde{f}_{n+1}"'] & \mathbb{Z} \arrow[l, "3"'] \arrow[u, "\tilde{f}_{n}"'] & \mathbb{Z} \arrow[l, "0"'] \arrow[u, "\tilde{f}_{n-1}"'] & \cdots \arrow[l] & \mathbb{Z} \arrow[l] \arrow[u, "\tilde{f}_{0}"']
\end{tikzcd}$$

where we claim that $\tilde{f}_k$ is multiplication by 1 (the identity) if $k \equiv 0, 3 \bmod 4$ and is multiplication by -1 if $k \equiv 1, 2 \bmod 4$.

Indeed, note that for $N \in \mathbb{Z}$, identified with the homomorphism $N : \mathbb{Z}C_3 \to \mathbb{Z}$, $1 \mapsto N$, we have

$$\tilde{f}_k(N) = N(f_k(1)) = N(\pm t^{r_k}) = \pm t^{r_k} \cdot N(1) = \pm N,$$

since the action is trivial. Thus, $\tilde{f}_k$ depends only on the sign of $f_k(1)$, which behaves just as we claimed before.

Taking quotients to obtain $H^q(C_3; \mathbb{Z})$, we see that, for $q > 0$ even, the automorphism $\varphi*$ of $H^q(C_3; \mathbb{Z}) = C_3 = \langle t \rangle$ is the identity for $q \equiv 0 \bmod 4$ and $t \mapsto t^{-1}$ for $q \equiv 2 \bmod 4$. We conclude that the action of C_2 on $H^q(C_3; \mathbb{Z})$ is trivial for $q \equiv 0 \bmod 4$ and by inversion for $q \equiv 2 \bmod 4$.

Now, to compute $H^p(C_2; H^q(C_3; \mathbb{Z}))$, we start with the resolution

$$\cdots \xrightarrow{1+s} \mathbb{Z}C_2 \xrightarrow{s-1} \mathbb{Z}C_2 \xrightarrow{1+s} \mathbb{Z}C_2 \xrightarrow{s-1} \mathbb{Z}C_2 \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

and we need to apply $\mathrm{Hom}_{C_2}(-, H^q(C_3; \mathbb{Z}))$, but $\mathrm{Hom}_{C_2}(\mathbb{Z}C_2, H^q(C_3; \mathbb{Z}))$ is identified with $H^q(C_3; \mathbb{Z})$ in any case, because the homomorphisms are determined (freely) by the image of 1.

For $q = 0$, the action on $H^q(C_3; \mathbb{Z}) = \mathbb{Z}$ is trivial, so $H^*(C_2; H^q(C_3; \mathbb{Z}))$ is just the cohomology of C_2.

For $q > 0$ even, and $\omega \in \mathrm{Hom}_{C_2}(\mathbb{Z}C_2, H^q(C_3;\mathbb{Z})) = \mathrm{Hom}_{C_2}(\mathbb{Z}C_2, C_3)$, we have

$$\begin{aligned} \omega(a+bs) &= \omega(a)\,\omega(bs) = \omega(1)^a\,\omega(s)^b \\ &= \omega(1)^a\,(s\cdot\omega(1))^b = \begin{cases} \omega(1)^{a-b}, & q \equiv 2 \bmod 4; \\ \omega(1)^{a+b}, & q \equiv 0 \bmod 4. \end{cases} \end{aligned}$$

With this in mind, we see that for $q \equiv 2 \bmod 4$, the multiplication by $s-1$ becomes the identity on C_3 and the multiplication by $1+s$ becomes the zero map. Therefore, the cochain obtained is

$$\cdots \xleftarrow{0} C_3 \xleftarrow{id} C_3 \xleftarrow{0} C_3 \xleftarrow{id} C_3,$$

and $H^*(C_2; H^q(C_3;\mathbb{Z}))$ would be completely trivial.

On the other hand, for $q \equiv 0 \bmod 4$, the multiplication by $s-1$ becomes the zero map on C_3 and the multiplication by $1+s$ becomes the inversion automorphism. Then we have the cochain

$$\cdots \xleftarrow{inv} C_3 \xleftarrow{0} C_3 \xleftarrow{inv} C_3 \xleftarrow{0} C_3,$$

and $H^p(C_2; H^q(C_3;\mathbb{Z}))$ would be C_3 for $p = 0$ and trivial for $p > 0$.

Putting together all the computations, we can give the E_2-term of the spectral sequence, which is already the limit term because there are no non-zero differentials (of any bidegree).

$$\begin{array}{c|cccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
4 & C_3 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & \mathbb{Z} & 0 & C_2 & 0 & C_2 & 0 & C_2 & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 &
\end{array}$$

Consequently, the convergence implies that there is a (decreasing) filtration F^* on $H^* = H^*(S_3;\mathbb{Z})$ such that

$$E_\infty^{p,q} \cong F^p H^{p+q} / F^{p+1} H^{p+q}.$$

From the comments made in the previous subsection, we see that

$$H^m(S_3;\mathbb{Z}) \cong E_\infty^{m,0}$$

when $m = 0$ or when $m > 0$ and $m \not\equiv 0 \bmod 4$.

For the other cases, let $k > 0$. We know that the filtration should be of the form

$$0 = \cdots = F^{4k+1}H^{4k} \subset F^{4k}H^{4k} = \cdots = F^1H^{4k} \subset F^0H^{4k} = \cdots = H^{4k}.$$

Also, we know that $F^{4k}H^{4k} \cong E_\infty^{4k,0}$, so the exact sequence

$$0 \longrightarrow F^1H^{4k} \longrightarrow F^0H^{4k} \longrightarrow E_\infty^{0,4k} \longrightarrow 0$$

becomes

$$0 \longrightarrow C_2 \longrightarrow H^{4k}(S_3;\mathbb{Z}) \longrightarrow C_3 \longrightarrow 0.$$

This sequence implies that $H^{4k}(S_3;\mathbb{Z})$ is an extension of C_2 by C_3, which can only be $C_2 \oplus C_3 = C_6$. Indeed, this is because, for instance, the only other option is S_3, but this group does not have a normal subgroup of order 2; also simply because $H^{4k}(S_3;\mathbb{Z})$ must be abelian.

Finally, we can conclude that

$$H^n(S_3;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & n = 0; \\ 0, & n \text{ odd}; \\ \mathbb{Z}/2\,\mathbb{Z}, & n \equiv 2 \bmod 4; \\ \mathbb{Z}/6\,\mathbb{Z}, & n \equiv 0 \bmod 4,\ n > 0. \end{cases}$$

Now, we know that from the inclusion $C_3 \to S_3$ we have an induced morphism in cohomology $H^*(S_3;\mathbb{Z}) \to H^*(C_3;\mathbb{Z})$. In particular, in degree $4k$, $k > 0$, the induced morphisms are the same (surjective) morphisms

$$C_6 = H^{4k}(S_3;\mathbb{Z}) \longrightarrow C_3 = E_\infty^{0,4k} = H^{4k}(C_3;\mathbb{Z})$$

obtained from the spectral sequence. See [8, Proposition 7.2.2].

7.3 The Mayer–Vietoris spectral sequence

The following spectral sequence obtains its name (although it is not standard) because it may be considered as the generalization of the Mayer–Vietoris long exact sequence relating the cohomology groups of two spaces and their union by a subspace (see Section 4.4). For a little more detailed construction and for alternative constructions see [15] and [17]. The spectral sequence is constructed in [17] for homology groups; here we will use the same method to obtain cohomology groups.

Here we will require a topological space that is a **simplicial complex**. These are the Δ-complexes (see Chapter 1, section 2) whose simplices are uniquely determined by their vertices; this is the same as saying that each n-simplex has $n+1$ different vertices, and that no other n-simplex has this same set of vertices. Anyway, we will not use this concepts in the construction of the spectral sequence (but the assumption should be necessary for everything in the background to work well).

Let X be a simplicial complex and suppose there is a covering $\mathcal{U} = \{U_i\}_{i \in I}$ for X, where I is an ordered set. We define the **nerve** of $\mathcal{U}$ as the family $N(\mathcal{U})$ of finite subsets $\sigma \subset I$ for which the subspace $X_\sigma := \bigcap_{i \in \sigma} U_i$ is not empty. ($N(\mathcal{U})$ is obtained to be an abstract simplicial complex.)

For each $k \geq 0$, take $N_k(\mathcal{U})$ as the set of the $\sigma \in N(\mathcal{U})$ of order k and define the (co)chain complex

$$C^k = \bigoplus_{\sigma \in N_k(\mathcal{U})} C^*(X_\sigma)$$

where $C^*(-)$ denotes the cellular cochain complex (see chapter 1, subsection 2.3). Now, for each $0 \leq i \leq k$ there is a boundary map $\partial_i : N_k(\mathcal{U}) \to N_{k-1}(\mathcal{U})$, given by

$$\partial_i \sigma = \partial_i \{j_0 < \cdots < j_k\} = \{j_0 < \cdots < \widehat{j_i} < \cdots < j_k\},$$

which gives the inclusions $X_\sigma \hookrightarrow X_{\partial_i \sigma}$ that induce morphisms $\partial_i : C^*(X_{\partial_i \sigma}) \to C^*(X_\sigma)$. Then, taking $\partial = \sum_{i=0}^{k} (-1)^i \partial_i$ and extending linearly over the direct sum we obtain a chain map $\partial : C^{k-1} \to C^k$, $k \geq 1$.

With this, define

$$E_1^{p,q} = H_q(C^p) = \bigoplus_{\sigma \in N_p(\mathcal{U})} H^q(X_\sigma),$$

together with the differentials $d : E_1^{p,q} \to E_1^{p+1,q}$ induced from ∂. Here, $H_q(C^p)$ denotes the homology on the q-th dimension of the cochain complex C^p.

Note that if we *restrict* to the image of d for some $\sigma \in N_p(\mathcal{U})$, the map

$$\bigoplus_{i=0}^{p} H^q(X_{\partial_i \sigma}) \to H^q(X_\sigma)$$

has to be the direct sum of the induced morphisms in cohomology from the inclusions $X_\sigma \hookrightarrow X_{\partial_i \sigma}$.

Then we can obtain the $E_2^{p,q}$, and it can be shown that $E_2^{*,*}$ converges to $H^*(X)$. This is called the **Mayer–Vietoris spectral sequence** for X associated to the covering $\mathcal{U}$.

The example for this spectral sequence will be presented in the last section; it is the computation of the cohomology $H^*(\Gamma_1;\mathbb{Z})$, where Γ_1 is a Bianchi group.

8 Bianchi Groups

As said before, the Bianchi group associated with a positive square-free integer d is defined as

$$\Gamma_d = PSL_2(\mathbb{O}_d)$$

where $\mathbb{O}_d$ is the ring of integers of the imaginary quadratic extension $\mathbb{Q}(\sqrt{-d})$.

We can describe these rings explicitly: With $\delta = \sqrt{-d}$ and $\eta = \frac{1}{2}(1+\delta)$, we have

$$\mathbb{O}_d = \mathbb{Z}[\delta] \quad \text{for} \quad d \equiv 1, 2 \mod 4, \quad \text{and}$$

$$\mathbb{O}_d = \mathbb{Z}[\eta] \quad \text{for} \quad d \equiv 3 \mod 4.$$

This is shown easily. See [1, Chapter 13].

8.1 Subgroup of elementary matrices

Let R be a ring. Let $x \in R$ be any element and $\mu \in R$ a unit. We define the matrices

$$E(x) = \begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad D(\mu) = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}.$$

The matrices $E(x)$ are called **elementary matrices**, and the group generated by them, $E_2(R)$, is called the 2×2 **elementary matrix group**.

A theorem of Cohn [4] provides a presentation of the group $E_2(R)$ for certain subrings of $\mathbb{C}$; it is as follows.

Theorem 8.1. *Let R be a subring of $\mathbb{C}$ with the usual absolute value, such that the range of values is well-ordered and if $\alpha \in R$ with $|\alpha|^2 < 4$ then $|\alpha|^2$ is an integer. Then, a presentation for $E_2(R)$ is given by the generators $E(x)$, $x \in R$, and the relations*

- $E(x)E(0)E(y) = -E(x+y), \quad x, y \in R;$
- $E(x)D(\mu) = D(\mu^{-1})E(\mu x \mu), \quad x, \mu \in R,\ \mu$ *unit;*
- $(E(\alpha)E(\bar{\alpha}))^p = -I, \quad$ *for all* $\alpha \in R$ *with* $|\alpha| = \sqrt{p}, \quad p \in \{2, 3\};$

- $E(\mu)E(\mu^{-1})E(\mu) = -D(\mu), \quad \mu \in R$ *unit.*

From the first relation we can deduce that $E(0)^2 = -I$, taking $x = 0$ and canceling the $E(y)$. Hence $E(0)^{-1} = -E(0)$.

Now, take ξ such that $\mathbb{O}_d = \mathbb{Z}[\xi]$. The rings $\mathbb{Z}[\xi]$ satisfy the hypotheses.

Let $x = a + b\xi$, with a and b positive integers. We see from the first relation that $E(x) = E(a)E(0)^{-1}E(b\xi)$, and then

$$\begin{aligned} E(x) &= E(a-1)E(0)^{-1}E(1)\cdot E(0)^{-1}\cdot E((b-1)\xi)E(0)^{-1}E(\xi) \\ &= \underbrace{E(1)E(0)^{-1}E(1)\dots E(1)E(0)^{-1}E(1)}_{a \text{ times}}E(0)^{-1}\underbrace{E(\xi)E(0)^{-1}E(\xi)\dots E(\xi)E(0)^{-1}E(\xi)}_{b \text{ times}}. \end{aligned}$$

Also, if a or b are negative we could use -1 or $-\xi$ respectively. So, we can reduce the set of generators to $\{E(0), E(1), E(-1), E(\xi), E(-\xi)\}$. We will see later that also $E(-1)$ and $E(-\xi)$ are not necessary.

8.2 The Euclidean Bianchi groups and their amalgam decompositions

The rings $\mathbb{O}_d$ are an Euclidean domain only when

$$d = 1, 2, 3, 7, 11.$$

For a proof see [11, Theorem 246]. For that reason, these Γ_d are called the **Euclidean Bianchi groups**.

P. M. Cohn [5, Theorems 6.1 and 9.3 and further discussions] states that for the Euclidean cases we have the equality

$$E_2(\mathbb{O}_d) = SL_2(\mathbb{O}_d).$$

In this way we will obtain finite presentations for the Euclidean Bianchi groups and then deduce their amalgam decomposition. The presentations for Γ_1 and Γ_7 are explained in [9]; here we will develop completely the presentation for the group Γ_2.

The units in $\mathbb{O}_2$ are just 1 and -1. Let $\delta = \sqrt{-2} = i\sqrt{2}$. So, for $E_2(\mathbb{O}_2) = SL_2(\mathbb{O}_2)$, we have that a complete set of relations in terms of the generators $E(x)$, $x \in \mathbb{O}_2$, and J is

(1) $E(x)E(0)E(y) = JE(x+y), \quad x, y \in \mathbb{O}_2;$

(2) $J^2 = I, \quad J$ central;

(3) $(E(\delta)E(-\delta))^2 = (E(1+\delta)E(1-\delta))^3 = J$;

(4) $E(1)^3 = J, \quad E(-1)^3 = I$.

As seen before, we can reduce the set of generators to $E(0), E(1), E(-1), E(\delta), E(-\delta)$, and J. In this way, we would have by *definition* that, for a positive integer n, $E(n)$ is written in terms of the generators as $E(1)E(0)^{-1}E(1)\ldots E(1)E(0)^{-1}E(1)$ (n times); the same for $E(-n), E(n\delta)$, and $E(-n\delta)$, and therefore for any $E(a+b\delta)$ written as $E(a)E(0)^{-1}E(b\delta)$.

Now, we claim that the generators $E(0), E(1), E(-1), E(\delta), E(-\delta)$, and J together with the relations

(a) $E(0)^2 = E(1)^3 = J$;

(b) $J^2 = I, \quad J$ central;

(c) $(E(\delta)E(-\delta))^2 = (E(1+\delta)E(1-\delta))^3 = J$;

(d) $E(1)E(0)E(\delta) = E(\delta)E(0)E(1), \quad E(0)E(1)E(0)E(-1) = E(0)E(\delta)E(0)E(-\delta) = I$;

(e) $E(1+\delta) = E(1)E(0)^{-1}E(\delta), \quad E(1-\delta) = E(1)E(0)^{-1}E(-\delta)$;

are equivalent to the previous presentation. Clearly the first presentation implies this one, so we see the other direction.

Since we already know what does any $E(x)$ mean in terms of the new generators, the relations (1) are obtained simply from the pseudo-commuting relation in (d), which can be written as $E(1)E(0)^{-1}E(\delta) = E(\delta)E(0)^{-1}E(1)$ (because $E(0) = JE(0)^{-1}$).

Indeed, if $x = a + b\delta$ and $y = c + d\delta$, with a, b, c, d positive, then we could take $E(x)E(0)^{-1}E(y)$ and move all the $E(1)$'s to the left and all the $E(\delta)$'s to the right to obtain $E(x+y) = E((a+c)+(b+d)\delta)$. For the cases where a, b, c, or d are negative, we should use the relations

$$
\begin{aligned}
E(1)E(0)^{-1}E(-\delta) &= E(-\delta)E(0)^{-1}E(1),\\
E(-1)E(0)^{-1}E(\delta) &= E(\delta)E(0)^{-1}E(-1),\\
E(-1)E(0)^{-1}E(-\delta) &= E(-\delta)E(0)^{-1}E(-1),\\
E(1)E(0)^{-1}E(-1) &= E(-1)E(0)^{-1}E(1) = E(0),\\
E(\delta)E(0)^{-1}E(-\delta) &= E(-\delta)E(0)^{-1}E(\delta) = E(0),
\end{aligned}
$$

which can be deduced from (d).

The only relation left to verify is $E(-1)^3 = I$, but this is obtained easily from (a) and the equation $E(-1) = E(0)^{-1}E(1)^{-1}E(0)^{-1}$, which comes from (d).

Now, we should make another reduction to the presentation for $E_2(\mathbb{O}_2)$. First, we see that (e) is unnecessary since we can remove it and replace that in (c); besides, (d) gives an expression for $E(-1)$ and $E(-\delta)$ in terms of $E(0), E(1)$, and $E(\delta)$, so we can remove both generators and replace the expressions where it is necessary.

Define

$$A = E(0)^{-1}, \quad T = E(0)E(1)^{-1}, \quad \text{and} \quad U = E(0)E(\delta)^{-1}.$$

With the generators A, T, U, and J, the last relations would be

(a) $(A^{-1})^2 = (T^{-1}A^{-1})^3 = J$;

(b) $J^2 = I, \quad J$ central;

(c) $(U^{-1}AUA)^2 = (T^{-1}U^{-1}A^{-1}T^{-1}AAUA)^3 = J$;

(d) $T^{-1}A^{-1}A^{-1}U^{-1}A^{-1} = U^{-1}A^{-1}A^{-1}T^{-1}A^{-1}$.

This is equivalent to the presentation

$$\langle A, T, U, J \mid J^2 = I, J \text{ central}, \;\; A^2 = (AT)^3 = (U^{-1}AUA)^2 = J, \;\; [T, U] = I\rangle.$$

The second relation in (c) is omitted because it can be obtained from the others. Indeed, we have, using $A^{-1} = JA$, $T^{-1}A^{-1}T^{-1} = ATA$, and other relations,

$$\begin{aligned}
&(T^{-1}U^{-1}A^{-1}T^{-1}AAUA)^3 \\
&\quad = (T^{-1}U^{-1}A^{-1}T^{-1}UA^{-1})^3 = (T^{-1}U^{-1}A^{-1}U(T^{-1}A^{-1}T^{-1})T)^3 \\
&\quad = (T^{-1}(U^{-1}A^{-1}UA)TAT)^3 = (JT^{-1}A^{-1}U^{-1}AUTAT)^3 \\
&\quad = J^3T^{-1}A^{-1}U^{-1}AUT(ATT^{-1}A^{-1})U^{-1}AUT(ATT^{-1}A^{-1})U^{-1}AUTAT \\
&\quad = JT^{-1}A^{-1}U^{-1}A(UTU^{-1})A(UTU^{-1})AUTAT \\
&\quad = JT^{-1}A^{-1}U^{-1}(ATATAT)UAT \\
&\quad = J^2T^{-1}A^{-1}U^{-1}UAT = I.
\end{aligned}$$

Finally, adding the relation $J = I$, we obtain what must be the simplest presentation for this Bianchi group:

$$\Gamma_2 = PSL_2(\mathbb{O}_2) = \langle a, t, u \mid a^2 = (at)^3 = (u^{-1}aua)^2 = [t, u] = 1\rangle.$$

Now, to deduce the amalgam decomposition, we have to do some changes to the last presentation.

Take $s = at$, $v = u^{-1}su$, and $m = u^{-1}au$. We may obtain

$$\begin{aligned}
\langle \; a, m, s, u, v \;\; \mid \;\; & a^2 = m^2 = s^3 = v^3 = (am)^2 = (sv^{-1})^2 = 1, \\
& am = sv^{-1}, \; m = u^{-1}au, \; v = u^{-1}su \; \rangle.
\end{aligned}$$

With this, define

$$G_1 = \langle a, m, u \mid a^2 = m^2 = (am)^2 = 1,\ m = u^{-1}au \rangle,$$

and

$$G_2 = \langle s, v, u \mid s^3 = v^3 = (sv^{-1})^2 = 1,\ v = u^{-1}su \rangle,$$

so Γ_2 is the free product of G_1 and G_2 with the identifications $u = u$ and $am = sv^{-1}$.

Note that G_1 is the HNN extension of the Klein group $C_2 \times C_2$ with the associated subgroups $\langle a = (1,0) \rangle$ and $\langle m = (0,1) \rangle$ (and the monomorphism $a \mapsto m$). And, G_2 is an HNN extension of the alternating group A_4 with the associated subgroups $\langle s = (123) \rangle$ and $\langle v = (134) \rangle$ (s and v could be any pair of generators for A_4 such that sv^{-1} is a product of two transpositions; G_2 will be the same).

We can see there is a common subgroup of G_1 and G_2, this is the group $\mathbb{Z} * C_2$. In G_1, it is the subgroup $\langle u \rangle * \langle am \rangle$, and in G_2 it is $\langle u \rangle * \langle sv^{-1} \rangle$. (The equalities $\langle u, am \rangle = \langle u \rangle * \langle am \rangle$ and $\langle u, sv^{-1} \rangle = \langle u \rangle * \langle sv^{-1} \rangle$ can be verified with the presentations.)

From all the above we can conclude the amalgam structure of Γ_2.

Proposition 8.2. *We have*

$$\Gamma_2 \cong G_1 *_{(\mathbb{Z} * C_2)} G_2,$$

where G_1 is the HNN extension of $C_2 \times C_2$ associating two generators and G_2 is the HNN extension of A_4 associating two 3-cycles.

For the rest of the Euclidean Bianchi groups, except Γ_3, we have the following amalgam decompositions:

$$\Gamma_1 = \big(A_4 *_{C_3} S_3\big) *_{PSL_2(\mathbb{Z})} \big(S_3 *_{C_2} D_2\big);$$
$$\Gamma_7 = \big(\,\mathbb{Z} * C_2\big) *_{(\mathbb{Z} * C_2 * C_2)} G,$$

where G is the HNN extension of $S_3 *_{C_2} S_3$ associating a 3-cycle with itself; and

$$\Gamma_{11} = \big(\,\mathbb{Z} * C_3\big) *_{(\mathbb{Z} * C_3 * C_3)} G,$$

where G is the HNN extension of $A_4 *_{C_3} A_4$ associating a 3-cycle with itself. These can be proved exactly the same way.

Since the only group that does not contain an HNN extension as factor group is Γ_1, we will be able to compute its cohomology groups *directly* using a Mayer–Vietoris spectral sequence associated to the classifying space.

8.3 The non-Euclidean Bianchi groups

Theorem 8.3. *For any $d \neq 1, 2, 3, 7, 11$, we have the presentation*

$$PE_2(\mathbb{O}_d) = \langle\, a, t, u \mid a^2 = (at)^3 = [t, u] = 1 \,\rangle.$$

Proof. The rings $\mathbb{O}_d$ have no units apart from ± 1, and there are no elements α such that $|\alpha| < 2$. So from the theorem of Cohn, $E_2(\mathbb{O}_d)$ has the generators J and $E(x)$, for $x \in \mathbb{O}_d$, with the relations

- $E(x)E(0)E(y) = JE(x + y), \quad x, y \in R;$
- $J^2 = I, \quad J$ central;
- $E(1)^3 = J, \quad E(-1)^3 = I.$

As we did with Γ_2, we may reduce to

$$E_2(\mathbb{O}_d) = \langle\, J, E(0), E(1), E(\xi) \mid E(0)^2 = E(1)^3 = J,\ J \text{ central},\ J^2 = I,$$
$$E(1)E(0)E(\xi) = E(\xi)E(0)E(1) \,\rangle.$$

Then, letting $A = E(0)^{-1}$, $T = E(0)E(1)^{-1}$, $U = E(0)E(\xi)^{-1}$ we have

$$E_2(\mathbb{O}_d) = \langle\, J, A, T, U \mid A^2 = (AT)^3 = J,\ J \text{ central},\ J^2 = [T, U] = I \,\rangle.$$

Identifying a, t, u with A, T, U after making $J = I$ the result is obtained. $\square$

Note that we have already seen this group; this was the example mentioned as a group that is both an amalgamated product and an HNN extension. So we have the isomorphism

$$PE_2(\mathbb{O}_d) \cong PSL_2(\mathbb{Z}) *_{\mathbb{Z}} (\mathbb{Z} \oplus \mathbb{Z})$$

and the fact that $PE_2(\mathbb{O}_d)$ is also the HNN extension of $PSL_2(\mathbb{Z})$ by an infinite cyclic subgroup.

Furthermore, B. Fine [9] exhibited an amalgam decomposition for all the non-Euclidean Bianchi groups as

$$\Gamma_d \cong PE_2(\mathbb{O}_d) *_H G_d$$

where H is an amalgam of two copies of $PSL_2(\mathbb{Z})$ and G_d is a particular group depending on d. This is proved with Poincaré polygons and polyhedrons, using that the Γ_d act on the non-Euclidean hyperbolic 3-space, where the action define some particular regions (polygons/polyhedrons) that lead to a construction of presentations for the Γ_d.

9 Group Cohomology of Γ_1

The method used in this section can be applied to the cases of the other Bianchi groups with amalgam decompositions, but we will only include this case for a matter of time.

From the previous section we have the isomorphism

$$\Gamma_1 \cong \big(A_4 *_{C_3} S_3\big) *_{PSL_2(\mathbb{Z})} \big(S_3' *_{C_2} D_2\big),$$

with the intersections $A_4 \cap S_3' \cong C_3'$, $A_4 \cap D_2 = \{1\}$, $S_3 \cap S_3' = \{1\}$, and $S_3 \cap D_2 \cong C_2'$. These intersections can be seen easily from the presentation

$$\Gamma_2 = \langle\, a, b, c, d \mid a^3 = b^2 = c^3 = d^2 = (ac)^2 = (ad)^2 = (bd)^2 = (bc)^2 = 1 \,\rangle$$

that is equivalent to the amalgam decomposition. Here $A_4 = \langle a, c\rangle$, $S_3 = \langle a, d\rangle$, $S_3' = \langle b, c\rangle$, and $D_2 = \langle b, d\rangle$.

Let X be a model for $B\Gamma_1$, a classifying space for Γ_1, and let $X_{11} = BA_4$, $X_{12} = BS_3$, $X_{21} = BS_3'$, $X_{22} = BD_2$, $Y_1 = BC_3$, $Y_2 = BC_2$, and $Z = BPSL_2(\mathbb{Z})$, so we have

$$X \cong \big(X_{11} \cup_{Y_1} X_{12}\big) \cup_Z \big(X_{21} \cup_{Y_2} X_{22}\big).$$

With this, we obtain a covering $\{X_{11}, X_{12}, X_{21}, X_{22}\}$ for a classifyng space of Γ_1. We can use this to construct a Mayer–Vietoris spectral sequence (presented before) whose E_1-term is given by

$$\begin{aligned} E_1^{0,q} &= H^q(X_{11}) \oplus H^q(X_{12}) \oplus H^q(X_{21}) \oplus H^q(X_{22}) \\ &\cong H^q(A_4) \oplus H^q(S_3) \oplus H^q(S_3') \oplus H^q(D_2), \end{aligned}$$

$$\begin{aligned} E_1^{1,q} &= H^q(X_{11} \cap X_{12}) \oplus H^q(X_{11} \cap X_{21}) \oplus H^q(X_{12} \cap X_{22}) \oplus H^q(X_{21} \cap X_{22}) \\ &\cong H^q(C_3) \oplus H^q(C_3') \oplus H^q(C_2') \oplus H^q(C_2), \end{aligned}$$

for $q \geq 0$, and $E_1^{p,q}$ trivial for $p \geq 2$, where the differentials of bidegree $(1,0)$ are all induced by inclusions. This spectral sequence converges to $H^*(B\Gamma_1) = H^*(\Gamma_1)$.

We computed the cohomology for the finite cyclic groups and for S_3. Also, it is known that

$$H^n(D_2; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & n = 0; \\ (\mathbb{Z}/2\,\mathbb{Z})^{(p-1)/2}, & n \text{ odd}; \\ (\mathbb{Z}/2\,\mathbb{Z})^{(p+2)/2}, & n \text{ even},\ n > 0. \end{cases}$$

This can be proved with the Künneth formula (see [12] for the definition). And, using [10], the first cohomology groups of A_4 are

$$H^0(A_4) = \mathbb{Z}, \quad H^1(A_4) = 0, \quad H^2(A_4) = C_3,$$

$$H^3(A_4) = C_2, \quad H^4(A_4) = C_6, \quad H^5(A_4) = 0.$$

Then the E_1-term looks like this:

$$\begin{array}{c|ccccc}
 & \vdots & & \vdots & & \\
5 & C_2^4 & \longrightarrow & 0 & \longrightarrow & 0 \ \cdots \\
4 & C_6^3 \oplus C_2^3 & \xrightarrow{d_1^{0,4}} & C_3^2 \oplus C_2^2 & \longrightarrow & 0 \ \cdots \\
3 & C_2 \oplus C_2 & \longrightarrow & 0 & \longrightarrow & 0 \ \cdots \\
2 & C_3 \oplus C_2^4 & \xrightarrow{d_1^{0,2}} & C_3^2 \oplus C_2^2 & \longrightarrow & 0 \ \cdots \\
1 & 0 & \longrightarrow & 0 & \longrightarrow & 0 \ \cdots \\
0 & \mathbb{Z}^4 & \xrightarrow{d_1^{0,0}} & \mathbb{Z}^4 & \longrightarrow & 0 \ \cdots \\
\hline
 & 0 & & 1 & & 2
\end{array}$$

With all the corresponding inclusions, we define the group homomorphims

$$\begin{aligned}
\alpha_1 : H^*(A_4) \to H^*(C_3) \quad &\text{and} \quad \alpha_2 : H^*(A_4) \to H^*(C_3'),\\
\beta_1 : H^*(S_3) \to H^*(C_3) \quad &\text{and} \quad \beta_2 : H^*(S_3) \to H^*(C_2'),\\
\gamma_1 : H^*(S_3') \to H^*(C_3') \quad &\text{and} \quad \gamma_2 : H^*(S_3') \to H^*(C_2),\\
\delta_1 : H^*(D_2) \to H^*(C_2') \quad &\text{and} \quad \delta_2 : H^*(D_2) \to H^*(C_2),
\end{aligned}$$

induced in cohomology, so we have

$$d_1^{0,q} = \alpha_1 + \beta_1 \oplus \alpha_2 + \gamma_1 \oplus \beta_2 + \delta_1 \oplus \gamma_2 + \delta_2 \ : \ E_1^{0,q} \longrightarrow E_1^{1,q}.$$

The $\oplus$ are used to separate the components in each direct sum; the $+$ denote the (abelian) operation in each component. Go back to the direct sum decomposition of $E_1^{0,q}$ and $E_1^{1,q}$ to make this clear.

We deal with these morphisms by separated cases in order to confirm that the induced homomoprhisms are not trivial as long as they can be not trivial. Since all the homomorphisms listed go to cyclic groups, the non-triviality will leave just one other option (where the final results do not change).

Using the previous notation for $A_4 = \langle a, c\rangle$, note that there is a group homomorphism $j : A_4 \to C_3 = \langle a\rangle$ given by $j(a) = a$ and $j(c) = a^2$. Then, the composition $j \circ i$ with the inclusion into A_4 is the identity map on C_3; this implies that the induced morphism

$$(j \circ i)^* = i^* \circ j^* : H^*(C_3) \to H^*(C_3)$$

is the identity map as well, which means that $i^* = \alpha_1$ must be not trivial whenever $H^*(C_3)$ is not trivial. The same is obtained for α_2.

For the group S_3, the fact that it is an extension of C_3 by C_2 gives a homomorphism $S_3 \to C_2$ which becomes the identity on C_2 when composed with the inclusion. Using the same argument we obtain that β_2 and γ_2 are trivial only when they *must* be trivial.

For C_3, this is not immediate. We already made this comment at the end of Section 7.2 to conclude that the morphism is non-trivial in even cohomology groups.

At last, the group D_2 is isomorphic to the direct product $C_2 \oplus C_2$, hence there is a homomorphism $D_2 \to C_2$ (the projection) that, composed with the inclusion, is equal to the identity map on C_2. As before, δ_1 and δ_2 are then not trivial whenever they are not forced to be trivial.

Now we can give explicitly the differentials in the E_1-term.

For $d_1^{0,0}$, all the homomorphisms are identity maps between $\mathbb{Z}$'s, then

$$d_1^{0,0} : (a, b, c, d) \mapsto (a + b, a + c, b + d, c + d),$$

and we have

$$\mathrm{Ker}(d_1^{0,0}) \cong \mathbb{Z}, \qquad \mathrm{Im}(d_1^{0,0}) \cong \mathbb{Z}^3, \qquad \text{and} \qquad E_1^{1,0}/\mathrm{Im}(d_1^{0,0}) \cong \mathbb{Z}\,.$$

For $d_1^{0,2}$,

$$d_1^{0,2} : (a, b, c, d_1, d_2) \mapsto (a, a, b + d_1, c + d_1),$$

(in the image, d_1 may be replaced by d_2 or $d_1 + d_2$; the result is the same) then

$$\mathrm{Ker}(d_1^{0,2}) \cong C_2 \oplus C_2, \qquad \mathrm{Im}(d_1^{0,2}) \cong C_3 \oplus C_2 \oplus C_2, \qquad \text{and} \qquad E_1^{1,2}/\mathrm{Im}(d_1^{0,2}) \cong C_3.$$

For $d_1^{0,4}$,

$$d_1^{0,4} : (a, b, c, d_1, d_2, d_3) \mapsto (a_{(3)} + b_{(3)}, a_{(3)} + c_{(3)}, b_{(2)} + d_1, c_{(2)} + d_1),$$

(in the image, d_1 may be replaced by any other nontrivial sum of d_i's; the result is the same). Doing the computations we obtain

$$\mathrm{Ker}(d_1^{0,4}) \cong C_6 \oplus C_2^3, \qquad \mathrm{Im}(d_1^{0,4}) = E_1^{1,4}, \qquad \text{and} \qquad E_1^{1,4}/\mathrm{Im}(d_1^{0,4}) = 0.$$

Then E_2 looks like this:

$$
\begin{array}{c|cccc}
 & \vdots & \vdots & \vdots & \\
5 & C_2^4 & 0 & 0 & \cdots \\
4 & C_6 \oplus C_2^3 & 0 & 0 & \cdots \\
3 & C_2 \oplus C_2 & 0 & 0 & \cdots \\
2 & C_2 \oplus C_2 & C_3 & 0 & \cdots \\
1 & 0 & 0 & 0 & \cdots \\
0 & \mathbb{Z} & \mathbb{Z} & 0 & \cdots \\
\hline
 & 0 & 1 & 2 &
\end{array}
$$

From the comments made before, since the 4-th diagonal is the only one where there is more than one non-trivial factor, we have the group extension

$$0 \longrightarrow C_3 \longrightarrow H^4(\Gamma_1) \longrightarrow C_2 \oplus C_2 \longrightarrow 0.$$

But the only abelian extension of these groups is their direct product.

Finally, we got the first six cohomology groups for Γ_1:

$$H^0(\Gamma_1) = \mathbb{Z}, \qquad H^1(\Gamma_1) = \mathbb{Z},$$

$$H^2(\Gamma_1) = C_2^2, \qquad H^3(\Gamma_1) = C_2^2 \oplus C_3,$$

$$H^4(\Gamma_1) = C_2^4 \oplus C_3, \qquad H^5(\Gamma_1) = C_2^4.$$

The only non-periodic cohomology is that of A_4, so we can compute the cohomology of Γ_1 as far as we can compute the cohomology of A_4.

More explicitly, we have the E_1-term, for $k > 0$, as

$$
\begin{array}{c|ccccccc}
4k+3 & H^*(A_4) \oplus C_2^{2k+1} & \longrightarrow & 0 & \longrightarrow & 0 & \cdots \\
4k+2 & H^*(A_4) \oplus C_2^2 \oplus C_2^{2k+2} & \longrightarrow & C_3^2 \oplus C_2^2 & \longrightarrow & 0 & \cdots \\
4k+1 & H^*(A_4) \oplus C_2^{2k} & \longrightarrow & 0 & \longrightarrow & 0 & \cdots \\
4k & H^*(A_4) \oplus C_6^2 \oplus C_2^{2k+1} & \longrightarrow & C_3^2 \oplus C_2^2 & \longrightarrow & 0 & \cdots \\
\hline
 & 0 & & 1 & & 2 &
\end{array}
$$

The differential $d_1^{0,4k}$ is surjective always due to the surjectivity of β_1 and β_2. Its kernel is isomorphic to $H^{4k}(A_4) \oplus C_2^{2k+1}$; this can be seen by thinking that for each element in $H^{4k}(A_4)$, the options to go to zero are given by a C_2^{2k+1}.

Conversely, for $d_1^{0,4k+2}$, since β_1 and β_2 are zero maps, we have

$$\mathrm{Ker}(d_1^{0,4k+2}) \cong \mathrm{Ker}(\alpha_1) \oplus C_2^{2k+2} \quad \text{and} \quad \mathrm{Im}(d_1^{0,4k+2}) \cong \mathrm{Im}(\alpha_1) \oplus C_2^2.$$

In the quotient $E_1^{1,4k+2}/\mathrm{Im}\ d_1$ the components with C_2 become trivial.

These pairs of kernels and images can be verified using the first isomorphism theorem.

We have then the E_2-term:

$$
\begin{array}{c|ccc}
4k+3 & H^*(A_4) \oplus C_2^{2k+1} & 0 & 0 \\
4k+2 & \mathrm{Ker}(\alpha_1) \oplus C_2^{2k+2} & C_3^2/\mathrm{Im}(\alpha_1 \oplus \alpha_2) & 0 \\
4k+1 & H^*(A_4) \oplus C_2^{2k} & 0 & 0 \\
4k & H^*(A_4) \oplus C_2^{2k+1} & 0 & 0 \\
\hline
 & 0 & 1 & 2
\end{array}
$$

With this we can conclude that, for $q \not\equiv 3 \bmod 4$,

$$H^q(\Gamma_1) = E_\infty^{0,q} = E_2^{0,q}.$$

For $q \equiv 3 \bmod 4$, there is an exact sequence

$$0 \longrightarrow C_3^2/\mathrm{Im}(\widetilde{\alpha}) \longrightarrow H^q(\Gamma_1) \longrightarrow H^q(A_4) \oplus C_2^{(q-1)/2} \longrightarrow 0,$$

where $\widetilde{\alpha}$ is the morphism $H^{q-1}(A_4) \to C_3^2$ given by (the direct sum of) two copies of the induced morphism $H^{q-1}(A_4) \to H^{q-1}(C_3) = C_3$.

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