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A NOTE ON THE WORST CASE APPROACH FOR A MARKET WITH A STOCHASTIC INTEREST RATE

Abstract. We solve a robust optimization problem and show an example of a market model for which the worst case measure is not a martingale measure. In our model the instantaneous interest rate is determined by the Hull–White model and the investor employs the HARA utility to measure his satisfaction. To protect against the model uncertainty he uses the worst case measure approach. The problem is formulated as a stochastic game between the investor and the market. PDE methods are used to find a saddle point and a precise verification argument is provided.

1. Introduction. We consider a portfolio problem embedded in a game-theoretic problem. We assume that the investor does not trust his model much and believes it is only the best guess based on existing data. In such a situation we say that the investor faces the *model uncertainty* (or the model ambiguity). In this work we would like to shed more light onto the portfolio optimization problem under the assumption that the short term interest rate exhibits some stochastic nature.

We consider a financial market consisting of n assets and a bank account. The interest rate on the bank account follows the Hull–White model, which is an extended version of the Vasicek model. The investor chooses between holding cash in a bank account and holding risky assets. The same model has been considered first by Korn and Kraft [4] but without the model uncertainty assumption. Instead of supposing that we have an exact model, we assume here the whole family of equivalent models, which will be described later. To determine robust investment controls the investor maximizes the

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total expected HARA utility of the final wealth after taking the infimum over all possible models. Robust optimization in the diffusion setting has been popularized especially by A. Schied and his coauthors (e.g. Schied [10] and references therein). The model ambiguity in the Vasicek model and its extensions has been considered by Flor and Larsen [2], Sun et al. [11], Munk and Rubtsov [6], and Wang and Li [12]. However, their objective function is different, because it includes an expression (along the lines of Maenhout [5]) which penalizes the expected utility for divergence from the reference probability measure. Our model is in fact their limiting model, when their ambiguity coefficients tend to $+\infty$ (0 respectively).

In the current paper the problem is formulated as a theoretic stochastic game between the market and the investor, and a saddle point of this game is determined, despite the fact that we do not include the penalizing term into the objective function. Moreover, in addition to the aforementioned papers we provide a correct and precise verification. First, we consider the full problem, without any constraints on the set of uncertainty measures. Further, we investigate what are the properties of the restricted model. To solve the game, we use the Hamilton–Jacobi–Bellman–Isaacs equation. After several substitutions we are able to solve the equation and use a suitable version of the verification theorem to justify the method.

Previously the same method has been used by Zawisza [13], [14], but in the model with a deterministic interest rate and with a different objective function. The major motivation for considering the present model is to provide an example in which the results of Øksendal and Sulem [7], [8] do not hold. In their papers they have considered the jump diffusion model but without assuming the stochastic nature of the interest rate, and have discovered that in that game the investor should always invest only in the bank account, and at the same time an optimal market strategy is to choose a martingale measure. This is interesting because the martingale measure plays a prominent role in derivative pricing. Our paper proves that in our framework the worst case measure is different from the martingale measure.

2. Model description. Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $(\mathcal{F}_t,\, 0 \leq t \leq \mathcal{T})$ (possibly enlarged to satisfy the usual assumptions) spanned by an n-dimensional Brownian motion $(W_t = (W_t^1, \ldots, W_t^n)^T,\, 0 \leq t \leq \mathcal{T})$. We have the initial measure P, but our investor contemplates model uncertainty, so the measure should be treated only as a proxy for the real life measure. Further, we will consider a whole class of equivalent measures, which will describe the model uncertainty. Our agent has access to a market with a bank account $(B_t,\, 0 \leq t \leq \mathcal{T})$ and risky assets $(S_t = (S_t^1, \ldots, S_t^n),\, 0 \leq t \leq \mathcal{T})$. Under the measure P the system is given by

$$(2.1)\qquad \begin{cases} dB_t = r_t B_t dt, \\ dS_t = \operatorname{diag}(S_t)[(r_t e + \Sigma_t \lambda_t^T)dt + \Sigma_t dW_t], \\ dr_t = (b_t - \kappa_t r_t)dt + a_t dW_t. \end{cases}$$

We assume that $e = (1, \dots, 1)$, the coefficients κ_t, b_t, $\lambda_t = (\lambda_t^1, \dots, \lambda_t^n)$, $a_t = (a_t^1, \dots, a_t^n)$, $\Sigma_t = [\sigma_t^{i,j}]_{i,j=1}^n$ are continuous deterministic functions, and in addition Σ_t is invertible. For notational convenience we omit the term $a_t \lambda_t^T dt$ in the dynamics for r, and we assume it is already included in the $b_t dt$ term. A representative example for the process $(S_t, t \in [0, \mathcal{T}])$ is the mixed stock-bond model (e.g. Korn and Kraft [4, Section 2.2]):

$$\begin{cases} dS_t^1 = (r_t + \lambda_t^1 \sigma_t^{1,1} + \lambda_t^2 \sigma_t^{1,2}) S_t^1 dt + \sigma_t^{1,1} S_t^1 dW_t^1 + \sigma_t^{1,2} S_t^1 dW_t^2, \\ dS_t^2 = (r_t + \lambda_t^2 \sigma_t^{2,2}) S_t^2 dt + \sigma_t^{2,2} S_t^2 dW_t^2, \\ dr_t = (b_t - \kappa r_t)dt + a_t dW_t^2. \end{cases}$$

Here $\{S_t^2\}$ is the price of the bond in the Vasicek model with maturity $\mathcal{T}' > \mathcal{T}$, which means that $\sigma_t^{2,2} = \frac{a}{\kappa}(1 - e^{-\kappa(\mathcal{T}' - t)})$.

The portfolio process evolves according to

$$dX_t^\pi = r_t X_t^\pi dt + \pi_t \Sigma_t \lambda_t^T X_t^\pi dt + X_t^\pi \pi_t \Sigma_t dW_t.$$

The symbol $\mathcal{A}_t$ denotes the class of progressively measurable processes $\pi = (\pi^1, \dots, \pi^n)$ such that

$$\int_t^{\mathcal{T}} |\pi_s|^2 \, ds < \infty \qquad \text{a.s.}$$

To describe the model uncertainty or model ambiguity issues we assume that the probability measure is not precisely known and the investor considers a whole class of possible measures. We follow the approach of Øksendal and Sulem [7] or Schied [10] in defining the set

$$(2.2)\qquad \mathcal{Q}_{\mathcal{T}} := \left\{ Q_{\mathcal{T}}^\eta \sim P \;\middle|\; \frac{dQ_{\mathcal{T}}^\eta}{dP} = \mathcal{E}\left(\int \eta_t \, dW_t \right)_{\mathcal{T}}, \ \eta \in \mathcal{M} \right\},$$

where $\mathcal{E}(\cdot)_t$ denotes the Doléans–Dade exponential and $\mathcal{M}$ denotes the set of all progressively measurable processes $\eta = (\eta^1, \dots, \eta^n)$ such that

$$\mathbb{E}\left[\frac{dQ_{\mathcal{T}}^\eta}{dP} \right]^2 < \infty.$$

In the last section we assume that the process η takes values in a fixed compact and convex set Γ. It is convenient to use the $Q_{\mathcal{T}}^\eta$-dynamics of the stochastic system (X_t, r_t), i.e.

$$\begin{cases} dX_t^\pi = r_t X_t^\pi dt + \pi_t \Sigma_t(\lambda_t^T + \eta_t^T) X_t^\pi dt + \pi_t \Sigma_t X_t^\pi dW_t^\eta, \\ dr_t = [(b_t - \kappa_t r_t) + a_t \eta_t^T]dt + a_t dW_t^\eta. \end{cases} \tag{2.3}$$

Our investor takes into account the model ambiguity and has worst case preferences (Gilboa and Schmeidler [3]), i.e. his aim is to maximize

$$\mathcal{J}^{\pi,\eta}(x,r,t) = \inf_{\eta \in \mathcal{M}} \mathbb{E}_{x,r,t}^\eta U(X_\mathcal{T}^\pi). \tag{2.4}$$

The symbol $\mathbb{E}_{x,r,t}^\eta$ is used to denote the expected value under the measure $Q_\mathcal{T}^\eta$ when the system starts at (x,r,t). Here we assume that $U(x) = x^\gamma/\gamma$ with $0 < \gamma < 1$. The solution for $\gamma < 0$ will be the same but due to the fact that U has negative values, one has to use a few more restrictions and technicalities to complete the proof.

Here we are interested not only in the optimal portfolio π^*, but also in the measure $Q_\mathcal{T}^{\eta^*}$ for which the infimum is attained. Therefore, we are looking for a saddle point (π^*, η^*), i.e.

$$\mathcal{J}^{\pi,\eta^*}(x,r,t) \le \mathcal{J}^{\pi^*,\eta^*}(x,r,t) \le \mathcal{J}^{\pi^*,\eta}(x,r,t), \qquad \pi \in \mathcal{A}_t,\ \eta \in \mathcal{M}.$$

3. The solution. To solve the problem we will use the Hamilton–Jacobi–Bellman–Isaacs operator given by

$$\begin{aligned} \mathcal{L}^{\pi,\eta}V(x,r,t) := V_t + \tfrac{1}{2}|a_t|^2 V_{rr} + \tfrac{1}{2}\pi\Sigma_t\Sigma_t^T\pi^T x^2 V_{xx} + \pi\Sigma_t a_t x V_{xr} \\ + \pi\Sigma_t(\lambda_t^T + \eta^T)xV_x + \eta a_t^T V_r + (b_t - \kappa_t r)V_r + rxV_x. \end{aligned} \tag{3.1}$$

It should be considered together with the verification theorem. The reasoning behind its proof is standard (see for instance Zawisza [13, Theorem 3.1]). Here we only present a short sketch, just to emphasize some minor differences.

THEOREM 3.1 (Verification Theorem). *Suppose there exists a positive function*

$$V \in \mathcal{C}^{2,2,1}((0,\infty) \times \mathbb{R} \times [0,\mathcal{T})) \cap \mathcal{C}([0,\infty) \times \mathbb{R} \times [0,\mathcal{T}])$$

and a Markov control

$$(\pi^*(x,r,t), \eta^*(x,r,t)) \in \mathcal{A}_t \times \mathcal{M}$$

such that

$$\mathcal{L}^{\pi^*(x,r,t),\eta}V(x,r,t) \ge 0, \tag{3.2}$$

$$\mathcal{L}^{\pi,\eta^*(x,r,t)}V(x,r,t) \le 0, \tag{3.3}$$

$$\mathcal{L}^{\pi^*(x,r,t),\eta^*(x,r,t)}V(x,r,t) = 0, \tag{3.4}$$

$$V(x,r,T) = x^\gamma/\gamma \tag{3.5}$$

for all $\eta \in \mathbb{R}$, $\pi \in \mathbb{R}$, $(x,r,t) \in (0,\infty) \times \mathbb{R} \times [0,\mathcal{T})$, *and*

$$\mathbb{E}^{\eta}_{x,r,t}\Big[\sup_{t \le s \le \mathcal{T}} |V(X^{\pi^*}_s, r_s, s)|\Big] < \infty \tag{3.6}$$

for all $(x,r,t) \in [0,\infty) \times \mathbb{R} \times [0,\mathcal{T}]$, $\pi \in \mathcal{A}_t$, $\eta \in \mathcal{M}$. *Then*

$$J^{\pi,\eta^*}(x,r,t) \le V(x,r,t) \le J^{\pi^*,\eta}(x,r,t) \quad \textit{for all } \pi \in \mathcal{A}_t,\ \eta \in \mathcal{M},$$

and

$$V(x,r,t) = J^{\pi^*,\eta^*}(x,r,t).$$

Proof. Let us fix first $\pi \in \mathcal{A}_t$. Consider the $Q^{\eta^*}_{\mathcal{T}}$-dynamics of the system (X_t, r_t) and apply the Itô formula using the function V. By using inequality (3.3) and taking the expectation of both sides, we obtain

$$V(x,r,t) \ge \mathbb{E}^{\eta^*} V\big(X_{(\mathcal{T}-\varepsilon)\wedge\tau_n}, r_{(\mathcal{T}-\varepsilon)\wedge\tau_n}, (\mathcal{T}-\varepsilon)\wedge\tau_n\big),$$

where $(\tau_n, n \ge 0)$ is a localizing sequence of stopping times. The function V is positive, thus the Fatou lemma implies

$$V(x,r,t) \ge \mathbb{E}^{\eta^*}_{x,r,t} V(X^{\pi}_{\mathcal{T}}, r_{\mathcal{T}}, \mathcal{T}) = \mathbb{E}^{\eta^*}_{x,r,t} U(X^{\pi}_{\mathcal{T}}) = J^{\pi,\eta^*}(x,r,t).$$

To prove the reverse inequality we fix $\eta \in \mathcal{M}$ and consider the $Q^{\eta}_{\mathcal{T}}$-dynamics of the system (X_t, r_t). After applying the Itô rule we get

$$V(x,r,t) \le \mathbb{E}^{\eta}_{x,r,t} V\big(X^{\pi^*}_{(\mathcal{T}-\varepsilon)\wedge\tau_n}, r_{(\mathcal{T}-\varepsilon)\wedge\tau_n}, (\mathcal{T}-\varepsilon)\wedge\tau_n\big)$$

and the same is true with the equality

$$V(x,r,t) = \mathbb{E}^{\eta^*}_{x,r,t} V\big(X^{\pi^*}_{(\mathcal{T}-\varepsilon)\wedge\tau_n}, r_{(\mathcal{T}-\varepsilon)\wedge\tau_n}, (\mathcal{T}-\varepsilon)\wedge\tau_n\big).$$

Property (3.6) and the dominated convergence theorem finish the proof. ∎

Following Korn and Kraft [4] we predict that conditions (3.2)–(3.6) are satisfied by the function of the form

$$V(x,r,t) = \frac{x^{\gamma}}{\gamma} e^{f(t)r + g(t)}, \qquad f(\mathcal{T}) = 0,\ g(\mathcal{T}) = 0.$$

Substituting it into (3.2)–(3.4) and dividing the result by $\frac{x^{\gamma}}{\gamma} e^{f(t)r+g(t)}$, we get

$$H^{(\pi,\eta^*)}(r,t) \le H^{(\pi^*,\eta^*)}(r,t) = 0 \le H^{(\pi^*,\eta)}(r,t), \qquad \pi, \eta \in \mathbb{R}^n,$$

where

$$H^{(\pi,\eta)}(r,t) := f'(t)r + g'(t) + \tfrac{1}{2}|a_t|^2 f^2(t) + \tfrac{1}{2}\gamma(\gamma-1)\pi\Sigma_t\Sigma_t^T\pi^T + \gamma\pi\Sigma_t a_t^T f(t) + \gamma\pi\Sigma_t(\lambda_t^T + \eta^T) + \eta a_t^T f(t) + (b_t - \kappa_t r) f(t) + \gamma r.$$

Now, we can determine the saddle point. Suppose first that we already have the saddle point (π^*, η^*). Then

$$H^{(\pi,\eta^*)}(r,t) \le H^{(\pi^*,\eta^*)}(r,t), \qquad \pi \in \mathbb{R}^n,$$

and consequently

$$\pi_t^* = \frac{1}{1-\gamma}(\lambda_t + \eta^* + f(t)a_t)\Sigma_t^{-1}.$$

On the other hand,

$$H^{(\pi^*,\eta^*)}(r,t) \le H^{(\pi^*,\eta)}(r,t), \qquad \eta \in \mathbb{R}^n.$$

We notice first that H is a linear function of η. In that case, the only chance to find η^* is to delete the expression with η, i.e.

$$\gamma\pi^*\Sigma_t + a_t f(t) = 0.$$

This means that

$$\pi_t^* = -\frac{f(t)}{\gamma} a_t \Sigma_t^{-1}.$$

So, we should have

$$\frac{f(t)}{1-\gamma} a_t \Sigma_t^{-1} + \frac{\lambda_t + \eta^*}{1-\gamma}\Sigma_t^{-1} = -\frac{f(t)}{\gamma} a_t \Sigma_t^{-1},$$

which yields

$$\eta_t^* = -\lambda_t - \frac{f(t)}{\gamma} a_t.$$

Substituting π^* and η^* into the equation and using the fact that the expression with η is equal to 0, we get

$$f'(t)r + g'(t) + \tfrac{1}{2}|a_t|^2 f^2(t) + \tfrac{1}{2}|a_t|^2 f^2(t)\frac{\gamma-1}{\gamma} - |a_t|^2 f(t) \\ - \lambda_t a_t^T f(t) + (b_t - \kappa_t r) f(t) + \gamma r = 0.$$

Thus,

$$f'(t) - \kappa_t f(t) + \gamma = 0,$$

$$g'(t) + \tfrac{1}{2}|a_t|^2 f^2(t) + \tfrac{1}{2}|a_t|^2 f^2(t)\frac{\gamma-1}{\gamma} - |a_t|^2 f(t) - \lambda_t a_t^T f(t) + b_t f(t) = 0.$$

More explicit forms are:

$$f(t) = \gamma e^{-\int_t^{\mathcal{T}} \kappa_s\, ds} \int_t^{\mathcal{T}} e^{\int_k^{\mathcal{T}} \kappa_s\, ds}\, dk,$$

$$g(t) = \int_t^{\mathcal{T}} \left[\frac{1}{2} f^2(s)|a_s|^2 + \frac{1}{2}|a_s|^2 f^2(s)\frac{\gamma-1}{\gamma} - |a_s|^2 f(s) - \lambda_s a_s^T f(s) + b_s f(s)\right] ds.$$

We can now summarize our preparatory calculations.

PROPOSITION 3.2. *The pair* (π^*, η^*) *given by*

$$\pi_t^* = -\frac{f(t)}{\gamma} a_t \Sigma_t^{-1}, \qquad \eta_t^* = -\lambda_t - \frac{f(t)}{\gamma} a_t$$

is a saddle point for problem (2.4).

Proof. Note that π_t^* and Σ_t are deterministic functions. To complete the proof we need only verify that

$$\mathbb{E}^{\eta}_{x,r,t}\Big[\sup_{t\le s\le \mathcal{T}} |V(X_s^{\pi^*}, r_s, s)|\Big] < \infty, \quad \eta \in \mathcal{M}.$$

We have

$$\mathbb{E}^{\eta}_{x,r,t}\Big[\sup_{t\le s\le \mathcal{T}} |V(X_s^{\pi^*}, r_s, s)|\Big] = \mathbb{E}_{x,r,t}\frac{dQ^\eta}{dP}\Big[\sup_{t\le s\le \mathcal{T}} V(X_s^{\pi^*}, r_s, s)\Big].$$

By the Cauchy–Schwarz inequality,

$$\mathbb{E}_{x,r,t}\frac{dQ^\eta}{dP}\Big[\sup_{t\le s\le \mathcal{T}} V(X_s^{\pi^*}, r_s, s)\Big] \le \bigg[\mathbb{E}\bigg[\frac{dQ^\eta}{dP}\bigg]^2\bigg]^{1/2}\Big[\mathbb{E}_{x,r,t}\Big[\sup_{t\le s\le \mathcal{T}} V^2(X_s^{\pi^*}, r_s, s)\Big]\Big]^{1/2}.$$

The explicit formula for the function V leads to

$$V(X_s^{\pi}, r_s, s) = \frac{1}{\gamma}\big[X_s^{\pi^*}\big]^{\gamma} e^{f(s)r_s+g(s)}.$$

The portfolio process X_t is a solution to the linear equation, so

$$X_s = x e^{\int_t^s [r_l + \pi_l^* \Sigma_l \lambda_l^T - \frac{1}{2}(\pi_l^* \Sigma_l \Sigma_l^T \pi_l^{T*})]\, dl + \int_t^s \pi_l^* \Sigma_l \, dW_l}.$$

Note that the process $\zeta_s = e^{\int_t^s \kappa_l \, dl} r_s$ has the dynamics

$$d\zeta_s = e^{\int_t^s \kappa_l \, dl} b_s \, ds + e^{\int_t^s \kappa_l \, dl} a_s \, dW_s.$$

We have

$$r_s = e^{-\int_t^s \kappa_l \, dl}\Big[r + \int_t^s b_l \, dl + \int_t^s a_l \, dW_l\Big].$$

By the stochastic Fubini theorem, the expression $V^2(X_s^\pi, r_s, s)$ can be rewritten in the form

$$V^2(X_s^\pi, r_s, s) = x Z_s e^{\beta(s) r_s + \xi(s)},$$

where the process Z_s is a square integrable martingale, and β, ξ are bounded and deterministic functions.

After applying the Cauchy–Schwarz inequality once more it is now sufficient to prove that for any continuous deterministic function $\hat\beta$ we have

$$\mathbb{E}_{r,t} \sup_{t\le s\le \mathcal{T}} e^{\hat\beta(s)\zeta_s} < \infty. \tag{3.7}$$

Note that

$$e^{\hat\beta(s)\zeta_s} \le e^{\hat\beta_{\max}\zeta_s} + e^{\hat\beta_{\min}\zeta_s},$$

where

$$\hat\beta_{\max} = \max_{t\le s\le \mathcal{T}} \hat\beta(s), \qquad \hat\beta_{\min} = \min_{t\le s\le \mathcal{T}} \hat\beta(s).$$

Both processes $e^{\hat{\beta}_{\max}\zeta_s}$, $e^{\hat{\beta}_{\min}\zeta_s}$ are solutions to linear equations with bounded coefficients, and thus the usual Lipschitz and linear growth conditions are satisfied. Property (3.7) follows from standard estimates for stochastic differential equations (see Pham [9, Theorem 1.3.16]). ■

Concluding remarks. To complement Proposition 3.2 we show that the measure $Q_{\mathcal{T}}^{\eta^*}$ is not a martingale measure, i.e. the process $S_t e^{-\int_0^t r_s\,ds}$ is not a $Q_{\mathcal{T}}^{\eta^*}$-martingale. To see this, it is sufficient to write the $Q_{\mathcal{T}}^{\eta^*}$-dynamics of S_t:

$$dS_t = \operatorname{diag}(S_t)\left[\left[r_t e - \frac{f(t)}{\gamma}\Sigma_t a_t^T\right] dt + \Sigma_t\, dW_t\right].$$

Finally, it is worth to compare the robust investment strategy

$$\pi_t^* = \frac{1}{1-\gamma}(\lambda_t + \eta_t^* + f(t)a_t)\Sigma_t^{-1}, \qquad \eta_t^* = -\lambda_t - \frac{f(t)}{\gamma}a_t$$

with the solution to the traditional utility maximization problem

$$\pi_t^* = \frac{1}{1-\gamma}(\lambda_t + f(t)a_t)\Sigma_t^{-1}.$$

Notice also that π^* can be rewritten as

$$\pi_t^* = -\frac{f(t)}{\gamma}a_t\Sigma_t^{-1} = -e^{-\int_t^{\mathcal{T}} \kappa_s\,ds}\int_t^{\mathcal{T}} e^{\int_k^{\mathcal{T}} \kappa_s\,ds}\,dk\,\left[a_t\Sigma_t^{-1}\right],$$

and it does not depend on the risk aversion coefficient γ. The same property is true for η^*.

4. Model uncertainty with restrictions.

From the practitioner's point of view, it might be interesting to solve the problem with some restrictions imposed on the uncertainty set $\mathcal{M}$. In this section we assume that the class $\mathcal{M}$ consists of all progressively measurable processes taking values in a fixed compact and convex set $\Gamma \subset \mathbb{R}^n$.

We can use the same function H:

$$H^{(\pi,\eta)}(r,t) = f'(t)r + g'(t) + \tfrac{1}{2}|a_t|^2 f^2(t) + \tfrac{1}{2}\gamma(\gamma-1)\pi\Sigma_t\Sigma_t^T\pi^T + \gamma\pi\Sigma_t a_t^T f(t) + \gamma\pi\Sigma_t(\lambda_t^T + \eta^T) + \eta a_t^T f(t) + (b_t - \kappa_t r)f(t) + \gamma r.$$

To find an explicit saddle point for the function H, we start by solving the upper Isaacs equation

$$\min_{\eta\in\Gamma}\max_{\pi\in\mathbb{R}^n} H^{(\pi,\eta)}(r,t) = 0. \tag{4.1}$$

We use a max-min theorem (Fan [1, Theorem 2]) to see that

$$\min_{\eta\in\Gamma}\max_{\pi\in\mathbb{R}^n} H^{(\pi,\eta)}(r,t) = \max_{\pi\in\mathbb{R}^n}\min_{\eta\in\Gamma} H^{(\pi,\eta)}(r,t).$$

We can determine a saddle point candidate (π^*, η^*) by finding a Borel measurable function η^* such that

$$\min_{\eta\in\Gamma}\max_{\pi\in\mathbb{R}} H^{(\pi,\eta)}(r,t) = \max_{\pi\in\mathbb{R}} H^{(\pi,\eta^*)}(r,t),$$

and a Borel measurable function π^* such that

$$\min_{\eta\in\Gamma}\max_{\pi\in\mathbb{R}} H^{(\pi,\eta)}(r,t) = \min_{\eta\in\Gamma} H^{(\pi^*,\eta)}(r,t).$$

Because the variable η is separated from r, equation (4.1) can be split into two equations (the first one has already been solved):

$$f(t) = \gamma e^{-\int_t^{\mathcal{T}} \kappa_s\, ds} \int_t^{\mathcal{T}} e^{\int_k^{\mathcal{T}} \kappa_s\, ds}\, dk$$

and

$$g'(t) + \tfrac{1}{2}|a_t|^2 f^2(t) + b_t f(t) + \min_{\eta\in\Gamma}\bigg[-\frac{1}{2}\frac{\gamma}{1-\gamma}|\lambda_t + \eta + f(t)a_t|^2 + \frac{\gamma}{1-\gamma}(\lambda_t + \eta + f(t)a_t)(\lambda_t+\eta)^T + f(t)a_t\eta^T\bigg] = 0.$$

Therefore, to find η^*, it is sufficient to determine any Borel measurable minimizer to the expression

$$-\frac{1}{2}\frac{\gamma}{1-\gamma}|\lambda_t+\eta+f(t)a_t|^2 + \frac{\gamma}{1-\gamma}(\lambda_t+\eta+f(t)a_t)(\lambda_t+\eta)^T + f(t)a_t\eta^T. \tag{4.2}$$

Now, let π^* be a Borel measurable maximizer of the function

$$\min_{\eta\in\Gamma} H^{(\pi,\eta)}(r,t).$$

Then (π^*, η^*) is a saddle point for the function $H^{(\pi,\eta)}(r,t)$. In particular,

$$H^{(\pi,\eta^*)}(r,t) \le H^{(\pi^*,\eta^*)}(r,t), \qquad \pi \in \mathbb{R}^n.$$

The unique function π^* which satisfies the above condition is given by

$$\pi_t^* = \frac{1}{1-\gamma}(\lambda_t + \eta_t^* + f(t)a_t)\Sigma_t^{-1}.$$

PROPOSITION 4.1. *Suppose that η^* is a minimizer of* (4.2) *and*

$$\pi_t^* = \frac{1}{1-\gamma}(\lambda_t + \eta_t^* + f(t)a_t)\Sigma_t^{-1}.$$

Then the pair (π^*, η^*) is a saddle point for problem (2.4) *with the restrictions imposed by the set Γ.*

The proof is omitted because it is the repetition of the steps from the proof of Proposition 3.2.

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