Relation between linear and nonlinear $N = 3, 4$ supergravity theories

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The effective actions for $d = 2, N = 3, 4$ chiral supergravities with a linear and a nonlinear gauge algebra are related to each other by a quantum reduction; the latter is obtained from the former by putting quantum currents equal to zero. This implies that the renormalization factors for the quantum actions are identical.

I. INTRODUCTION

When a Lie algebra is generalized to a commutator algebra that is not linear in its generators, but contains quadratic or higher order polynomials as well, one obtains a nonlinear algebra. Especially the infinite-dimensional variety showing up in conformal field theory have recently been studied intensively, the most celebrated class being the W algebras (for a review, see [1]), and in particular the W_3 algebra [2].

Among the properties that the W_3 algebra, for one, shares with linear algebras is a remarkable renormalization property of the quantum theory that arises when one couples the currents J generating these algebras to gauge fields A. The resulting induced action for the gauge fields is nonzero due to anomalies (central terms, quantum corrections) in the current commutators as compared to the classical Poisson brackets, and one can take this induced action as a starting point to quantize the gauge fields. For linear current algebras such as affine Lie algebras and the Virasoro algebra this induced action Γ is proportional to the central charge, $\Gamma_{\mathrm{ind}}[A] = c \Gamma^{(0)}[A]$. Also, the effective action $\Gamma_{\mathrm{eff}}[A]$, or equivalently W, the generator of connected Green functions[1] for A, is related [3,4] to the same basic functional by a field and coupling renormalization $\Gamma_{\mathrm{eff}}[A] = Z_\Gamma \Gamma^{(0)}[Z_A A]$. There are several methods to compute these Z factors [4,5], with general agreement for Z_Γ and varying proposals for Z_A; for a discussion see [6]. For nonlinear algebras on the other hand, the dependence of the induced action on the central charge is not simply proportionality, but instead it can be expanded in powers of $1/c$,

$$\Gamma_{\mathrm{ind}}[A] = \sum_{i \geq 0} c^{1-i} \Gamma^{(i)}(A) .$$

It is remarkable that, nevertheless, for the quantum theory based on this action, the renormalization property still holds: the effective action is still equal, up to renormalization factors, to the "classical" (i.e., lowest order in c) term $\Gamma^{(0)}[A]$ of the induced action. This was shown for W_3 to first order (and conjectured to be true to all orders) in [7]. This was recently proved in [8], and extended in [9] to arbitrary extensions of the Virasoro algebra that can be obtained from a Drinfeld-Sokolov reduction [10] of Wess-Zumino-Witten (WZW) models.

In this paper we point out that there are a few cases, viz., $N = 3, 4$ supergravities where the renormalization of the linear and nonlinear effective actions is intimately related, due to the simple relation that exists between the $N = 3, 4$ linear [11,12] and nonlinear [13,14] superconformal algebras. Namely, we will show in both cases that the effective action W of the nonlinear theory results from that of the linear theory by putting to zero an appropriate set of currents (or integrating out an appropriate set of fields for Γ). By the same token we will then have shown that for $N = 3, 4$ chiral supergravity the same type of cancellations occur, as referred to above for W_3. Namely, nonleading terms in the central charge in $\Gamma_{\mathrm{ind}}[A]$ cancel with quantum contributions to Γ_{eff}. The identity of the renormalization factors also follows.

II. $N = 3$ SUPERGRAVITY

Both $N = 3$ superconformal algebras contain the energy-momentum tensor T, 3 supercharges G^a, $a \in \{1, 2, 3\}$ and an so(3) affine Lie algebra, U^a, $a \in \{1, 2, 3\}$. The linear one [11] contains in addition a dimension $\frac{1}{2}$ fermion Q. The operator product expansions (OPE's) of the generators are (we use tildes for the nonlinear algebra)

$$TT = \frac{c}{2}[1], \quad \tilde{T}\tilde{T} = \frac{\tilde{c}}{2}[1] ,$$

$$T\Phi = h_\Phi[\Phi], \quad \tilde{T}\tilde{\Phi} = h_\Phi[\tilde{\Phi}] ,$$

$$G^a G^b = \delta^{ab} \frac{2c}{3}[1] - \varepsilon^{abc} 2[U^c] ,$$

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[1] Hereafter also called "effective action" for brevity. The symbol used should resolve possible doubts on which functional is meant.

$$\widetilde{G}^a \widetilde{G}^b = \delta^{ab} \frac{2(\widetilde{c}-1)}{3}[1] - \frac{2(\widetilde{c}-1)}{\widetilde{c}+1/2} \varepsilon^{abc}[\widetilde{U}^c]$$
$$+ \frac{3}{\widetilde{c}+1/2}[\widetilde{U}^{(a}\widetilde{U}^{b)} - \frac{2\widetilde{c}+1}{3\widetilde{c}}\delta^{ab}\widetilde{T}] ,$$

$$U^a U^b = -\frac{c}{3}\delta^{ab}[1] + \varepsilon^{abc}[U^c] ,$$

$$\widetilde{U}^a \widetilde{U}^b = -\frac{\widetilde{c}+1/2}{3}\delta^{ab}[1] + \varepsilon^{abc}[\widetilde{U}^c] ,$$

$$U^a G^b = \delta^{ab}[Q] + \varepsilon^{abc}[G^c], \quad \widetilde{U}^a \widetilde{G}^b = \varepsilon^{abc}[\widetilde{G}^c] ,$$

$$QG^a = [U^a], \quad QQ = -\frac{c}{3}[1] ,$$

(2.1)

where $h_\Phi = h_{\widetilde{\Phi}} = \frac{3}{2}, 1, \frac{1}{2}$ for $\Phi = G^a, U^a, Q$.

The relation between the two algebras is [14] that Q commutes with the combinations that constitute the nonlinear algebra

$$\widetilde{T} \equiv T - \frac{3}{2c}Q\partial Q ,$$

$$\widetilde{G}^a \equiv G^a + \frac{3}{c}U^a Q ,$$

$$\widetilde{U}^a \equiv U^a ,$$

(2.2)

while the central charges are related by $\widetilde{c} = c - \frac{1}{2}$.

The induced action Γ is defined by

$$Z[h,\psi,A,\eta] = \exp[-\Gamma[h,\psi,A,\eta]] = \left\langle \exp\left[-\frac{1}{\pi}\int d^2x(h(x)T(x) + \psi_a(x)G^a(x) + A_a(x)U^a(x) + \eta(x)Q(x))\right]\right\rangle , \quad (2.3)$$

and similarly, without the η field, for the nonlinear induced action $\widetilde{\Gamma}$. These actions are completely determined by considering their transformation properties under $N=3$ supergravity transformations. These transformations read, for the linear case,

$$\delta h = \overline{\partial}\epsilon + \epsilon\partial h - \partial\epsilon h + 2\theta^a\psi_a, \quad \delta\psi^a = \overline{\partial}\theta^a + \epsilon\partial\psi^a - \frac{1}{2}\partial\epsilon\psi^a + \frac{1}{2}\theta^a\partial h - \partial\theta^a h - \varepsilon^{abc}(\theta^b A^c + \omega^b\psi^c) ,$$

$$\delta A^a = \overline{\partial}\omega^a + \epsilon\partial A^a - \varepsilon^{abc}(\partial\theta^b\psi^c - \theta^b\partial\psi^c) + \theta^a\eta - \varepsilon^{abc}\omega^b A^c - \partial\omega^a h + \tau\psi^a , \quad (2.4)$$

$$\delta\eta = \overline{\partial}\tau + \epsilon\partial\eta + \frac{1}{2}\partial\epsilon\eta + \theta^a\partial A_a - \partial\omega^a\psi_a - \frac{1}{2}\tau\partial h - \partial\tau h .$$

For the nonlinear case, they are the same, except that there is of course no field η and no parameter τ, and δA_a contains a $\widetilde{c}$-dependent extra term

$$\delta_{\text{extra}} A^a = \frac{3}{2\widetilde{c}}\varepsilon^{abc}(\partial\theta^b\psi^c - \theta^b\partial\psi^c) . \quad (2.5)$$

The anomaly for the linear theory is

$$\delta\Gamma[h,\psi,A,\eta] = -\frac{c}{12\pi}\int\epsilon\partial^3 h - \frac{c}{3\pi}\int\theta^a\partial^2\psi_a + \frac{c}{3\pi}\int\omega^a\partial A_a + \frac{c}{3\pi}\int\tau\eta . \quad (2.6)$$

Defining[2]

$$t = \frac{12\pi}{c}\frac{\delta\Gamma}{\delta h}, \quad g^a = \frac{3\pi}{c}\frac{\delta\Gamma}{\delta\psi^a}, \quad u^a = -\frac{3\pi}{c}\frac{\delta\Gamma}{\delta A^a}, \quad q = -\frac{3\pi}{c}\frac{\delta\Gamma}{\delta\eta} , \quad (2.7)$$

we obtain the Ward identities for the linear theory by combining Eqs. (2.4) and (2.6):

$$\partial^3 h = \overline{\nabla}t - (2\psi_a\partial + 6\partial\psi_a)g^a + 4\partial A_a u^a - (2\eta\partial - 2\partial\eta)q ,$$

$$\partial^2\psi_a = \overline{\nabla}g^a - \frac{1}{2}\psi^a t + \varepsilon^{abc}A_b g^c + \eta u^a + \varepsilon^{abc}(2\partial\psi_b + \psi_b\partial)u^c + \partial A_a q , \quad (2.8)$$

$$\partial A_a = \overline{\nabla}u^a - \varepsilon^{abc}\psi_b g^c + \varepsilon^{abc}A_b u^c - (\psi_a\partial + \partial\psi_a)q, \quad \eta = \overline{\nabla}q - \psi_a u^a ,$$

where

$$\overline{\nabla}\Phi = (\overline{\partial} - h\partial - h_\Phi(\partial h))\Phi , \quad (2.9)$$

with $h_\Phi = 2, \frac{3}{2}, 1, \frac{1}{2}$ for $\Phi = t, g^a, u^a, q$.

The Ward identities provide us with a set of functional differential equations for the induced action. Since these have no explicit dependence on c, the induced action can be written as

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[2]All functional derivatives are left derivatives.

$$\Gamma[h,\psi,A,\eta]=c\,\Gamma^{(0)}[h,\psi,A,\eta]\ ,\tag{2.10}$$

where $\Gamma^{(0)}$ is c independent.

The nonlinear theory can be treated in a parallel way. The anomaly is now

$$\delta\tilde{\Gamma}[h,\psi,A]=-\frac{\tilde{c}}{12\pi}\int\epsilon\partial^3 h-\frac{\tilde{c}-1}{3\pi}\int\theta^a\partial^2\psi_a+\frac{\tilde{c}+1/2}{3\pi}\int\omega^a\partial A_a-\frac{3}{\pi(\tilde{c}+1/2)}\int\theta_a\psi_b\left[U^{(a}U^{b)}\right]_{\mathrm{eff}}.\tag{2.11}$$

The last term, which is due to the nonlinear term in the algebra equation (2.1), can further be rewritten as

$$\left[U^{(a}U^{b)}\right]_{\mathrm{eff}}(x)=\left\langle\tilde{U}^{(a}\tilde{U}^{b)}(x)\exp\left[-\frac{1}{\pi}\int\left[h\tilde{T}+\psi_a\tilde{G}^a+A_a\tilde{U}^a\right]\right]\right\rangle\Big/\exp\left[-\tilde{\Gamma}\right]$$

$$=\left[\frac{\tilde{c}+1/2}{3}\right]^2 u^a(x)u^b(x)+\frac{(\tilde{c}+1/2)\pi}{6}\lim_{y\to x}\left[\frac{\partial u^a(x)}{\partial A_b(y)}-\frac{\partial}{\partial}\delta^{(2)}(x-y)\delta^{ab}+a\rightleftharpoons b\right].\tag{2.12}$$

The limit in the last term of Eq. (2.13) reflects the point-splitting regularization of the composite terms in the $\tilde{G}\tilde{G}$ OPE (2.1). One notices that in the limit $\tilde{c}\to\infty$, u becomes $\tilde{c}$ independent and one has simply

$$\lim_{\tilde{c}\to\infty}\left[\frac{3}{\tilde{c}+1/2}\right]^2\left[U^{(a}U^{b)}\right]_{\mathrm{eff}}(x)=u^a(x)u^b(x)\ .\tag{2.13}$$

Using Eq. (2.13) we find that Eq. (2.11) can be rewritten as

$$\delta\tilde{\Gamma}[h,\psi,A]=-\frac{\tilde{c}}{12\pi}\int\epsilon\partial^3 h-\frac{\tilde{c}-1}{3\pi}\int\theta^a\partial^2\psi_a+\frac{\tilde{c}+1/2}{3\pi}\int\omega^a\partial A_a-\frac{\tilde{c}+1/2}{3\pi}\int\theta_a\psi_b u^a u^b$$

$$-\lim_{y\to x}\int\theta^{(a}\psi^{b)}\left[\frac{\partial u^a(x)}{\partial A_b(y)}-\frac{\partial}{\partial}\delta^{(2)}(x-y)\delta_{ab}\right],\tag{2.14}$$

where the last term disappears in the large $\tilde{c}$ limit. The term proportional to $\int\theta_a\psi_b u^a u^b$ in Eq. (2.14) can be absorbed by adding a field-dependent term in the transformation rule for A:

$$\delta^{\mathrm{nl}}_{\mathrm{extra}}A_a=-\theta_a\psi_b u^b\ .\tag{2.15}$$

Doing this we find that in the large $\tilde{c}$ limit, the anomaly reduces to the minimal one. Combining the nonlinear transformations with Eq. (2.14), and defining

$$\tilde{t}=\frac{12\pi}{\tilde{c}}\frac{\delta\tilde{\Gamma}}{\delta h}\ ,\qquad\tilde{g}^a=\frac{3\pi}{\tilde{c}-1}\frac{\delta\tilde{\Gamma}}{\delta\psi_a}\ ,\qquad\tilde{u}^a=-\frac{3\pi}{\tilde{c}+1/2}\frac{\delta\tilde{\Gamma}}{\delta A_a}\ ,\tag{2.16}$$

we find the Ward identities for $\tilde{\Gamma}[h,\psi,A]$ (they can also be found in [15]):

$$\partial^3 h=\bar{\nabla}\tilde{t}-\left[1-\frac{1}{\tilde{c}}\right](2\psi_a\partial+6\partial\psi_a)\tilde{g}^a+4\left[1+\frac{1}{2\tilde{c}}\right]\partial A_a\tilde{u}^a\ ,$$

$$\partial^2\psi_a=\bar{\nabla}\tilde{g}^a-\left[\frac{1}{2}+\frac{1}{2\tilde{c}-2}\right]\psi_a\tilde{t}+\varepsilon^{abc}A_b\tilde{g}^c+\varepsilon^{abc}(2\partial\psi_b+\psi_b\partial)\tilde{u}^c-\left[1+\frac{3}{2\tilde{c}-2}\right]\left[\frac{3}{\tilde{c}+1/2}\right]^2\psi_b\left[U^{(a}U^{b)}\right]_{\mathrm{eff}},\tag{2.17}$$

$$\partial A_a=\bar{\nabla}\tilde{u}^a-\left[1-\frac{3}{2\tilde{c}+1}\right]\varepsilon^{abc}\psi_b\tilde{g}^c+\varepsilon^{abc}A_b\tilde{u}^c\ .$$

The normalization of the currents has been chosen so that the anomalous terms on the left-hand side (LHS) have coefficient unity. The explicit $\tilde{c}$ dependence of the Ward identities arises from several sources: the fact that in the nonlinear algebra Eq. (2.1) some couplings are explicitly $\tilde{c}$ dependent, the $\tilde{c}$ dependence of the transformation Eq. (2.5), and the field-nonlinearity. The dependence implies that the induced action is given by a $1/\tilde{c}$ expansion:

$$\tilde{\Gamma}[h,\psi,A]=\sum_{i\geq 0}\tilde{c}^{1-i}\tilde{\Gamma}^{(i)}[h,\psi,A]\ .\tag{2.18}$$

This is familiar from W_3 [7].

Turning back to the Ward identities for the linear theory equation (2.8), we observe a remarkable relation with the nonlinear ones. If we take $\tilde{c}=c+\frac{1}{2}$ and put $q=0$ we find from the last identity in Eq. (2.8) that $\eta=-\psi_a u^a$. Substituting this back into the first three

identities in Eq. (2.8) yields precisely the Ward identities for the nonlinear theory equation (2.17) in the $c \rightarrow \infty$ limit. Also, the extra term in the nonlinear δA_a [Eq. (2.5)] that was added to bring the anomaly to a minimal form now effectively reinserts the $\theta^a \eta$ term that dropped out of the linear transformation, Eq. (2.4). This strongly sug-

gests that the relation between the effective theories should be obtained by putting the current q equal to zero on the quantum level. We will now derive this fact.

First we rewrite Eq. (2.3) using Eq. (2.2), the crucial ingredient being that Q commutes with the nonlinear algebra, thus factorizing the averages:

$$Z[h,\psi,A,\eta] = \left\langle \exp\left[-\frac{1}{\pi} \int (h\tilde{T} + \psi_a \tilde{G}^a + A_a \tilde{U}^a) \right] \left\langle \exp\left[-\frac{1}{\pi} \int (hT_Q + \hat{\eta}Q) \right] \right\rangle_Q \right\rangle$$

$$= \left\langle \exp\left[-\frac{1}{\pi} \int (h\tilde{T} + \psi_a \tilde{G}^a + A_a \tilde{U}^a) - \Gamma[h,\hat{\eta}] \right] \right\rangle , \tag{2.19}$$

where

$$T_Q = \frac{3}{2c} Q\partial Q, \quad \hat{\eta} = \eta - \frac{1}{3c}\psi_a U^a . \tag{2.20}$$

The Q integral can easily be expressed in terms of the Polyakov action:

$$\Gamma[h,\hat{\eta}] = \frac{1}{48\pi}\Gamma_{\text{Pol}}[h] - \frac{c}{6\pi}\int \hat{\eta}\frac{1}{\bar{\nabla}}\hat{\eta} , \tag{2.21}$$

where $\bar{\nabla} = \bar{\partial} - h\partial - \frac{1}{2}\partial h$ and

$$\Gamma_{\text{Pol}}[h] = \int \partial^2 h \frac{1}{\bar{\partial}} \frac{1}{1 - h\frac{\partial}{\bar{\partial}}} \frac{1}{\partial} \partial^2 h . \tag{2.22}$$

Using Eqs. (2.19) and (2.21) we find

$$\exp[-\tilde{\Gamma}[h,\psi,A]] = \exp\left[\Gamma\left[h, \hat{\eta} = \eta + \frac{\pi}{3c}\psi_b\frac{\delta}{\delta A_b} \right] \right] \exp[-\Gamma[h,\psi,A,\eta]] . \tag{2.23}$$

The double functional derivative in the exponential is well defined due to the presence of the nonlocal operator $\bar{\nabla}^{-1}$. This formula was checked explicitly on the correlation functions using [17]. Introducing the Fourier transform of Γ with respect to A,

$$\exp[-\Gamma[h,\psi,A,\eta]] = \int [du] \exp\left[-\Gamma[h,\psi,u,\eta] + \frac{c}{3\pi}\int u^a A_a \right] , \tag{2.24}$$

Eq. (2.23) further reduces to

$$\exp[-\tilde{\Gamma}[h,\psi,A]]$$

$$= \exp\left[\frac{1}{48\pi}\Gamma_{\text{Pol}}[h] \right] \int [du] \exp\left[-\Gamma[h,\psi,u,\eta] - \frac{c}{6\pi}\int \left[\eta + \psi_a u^a \right]\frac{1}{\bar{\nabla}}\left[\eta + \psi_b u^b \right] + \frac{c}{3\pi}\int u^a A_a \right] . \tag{2.25}$$

As the LHS of Eq. (2.25) is η independent the RHS should also be. We can integrate both sides over η with a measure chosen such that the integral is equal to one:

$$\exp\left[-\frac{1}{48\pi}\Gamma_{\text{Pol}}[h] \right] \int [d\eta] \exp\left[\frac{c}{6\pi}\int \left[\eta + \psi_a u^a \right]\frac{1}{\bar{\nabla}}\left[\eta + \psi_b u^b \right] \right] = 1. \tag{2.26}$$

Combining this with Eq. (2.25) we obtain finally a very simple expression for $\tilde{\Gamma}[h,\psi,A]$ in terms of $\Gamma[h,\psi,u,\eta]$:

$$\exp[-\tilde{\Gamma}[h,\psi,A]] = \int [d\eta] \exp[-\Gamma[h,\psi,u,\eta]] . \tag{2.27}$$

Introducing the generating functionals W,

$$\exp[-W[t,g,u,q]] = \int [dh][d\psi][dA][d\eta] \exp\left[-\Gamma[h,\psi,A,\eta] + \frac{1}{12\pi}\int (ht + 4\psi^a g_a - 4A_a u^a - 4\eta q) \right] \tag{2.28}$$

and similarly for $\widetilde{W}$ (without the η term), one finds by combining Eqs. (2.27) and (2.28), an extremely simple expression of the relation between the quantum theories of induced $N=3$ supergravities based on the linear and nonlinear algebras:

$$\widetilde{W}[t,g,u]=W[t,g,u,q=0] \ . \tag{2.29}$$

Therefore, the two theories are related by a *quantum* Hamiltonian reduction.

III. $N=4$ SUPERGRAVITY

Now we extend the method applied for $N=3$ to the case of $N=4$. Again, there is a linear $N=4$ algebra and a nonlinear one, obtained [14] by decoupling 4 free fermions and a U(1) current. In the previous case we made use in the derivation of the explicit form of the action induced by integrating out the fermions. In the present case no explicit expression is available for the corresponding quantity, but we will see that in fact it is not needed.

The $N=4$ superconformal algebra [12] is generated by the energy-momentum tensor T, 4 supercharges G^a, $a \in \{1,2,3,4\}$, an so(4) affine Lie algebra, $U^{ab}=-U^{ba}$, $a,b \in \{1,2,3,4\}$, 4 free fermions Q^a and a U(1) current P. The two su(2) algebras have levels k_+ and k_-. The supercharges G^a and the dimension $\frac{1}{2}$ fields Q^a form two (2,2) representations of SU(2)$\otimes$SU(2). The central charge is given by

$$c=\frac{6k_+k_-}{k_++k_-} \ . \tag{3.1}$$

The OPE's are (we omit the OPE's of T)

$$G^aG^b=\frac{3c}{2}\delta^{ab}[1]+[-2U^{ab}+\zeta\varepsilon^{abcd}U^{cd}] \ ,$$

$$U^{ab}U^{cd}=\frac{k}{2}\left[\delta^{ad}\delta^{bc}-\delta^{ac}\delta^{bd}-\zeta\varepsilon^{abcd}\right][1]$$
$$+[\delta^{bd}U^{ac}-\delta^{bc}U^{ad}-\delta^{ad}U^{bc}+\delta^{ac}U^{bd}] \ ,$$

$$U^{ab}G^c=-\zeta\left[\delta^{bc}[Q^a]-\delta^{ac}[Q^b]\right]+\varepsilon^{abcd}[Q^d]$$
$$-(\delta^{bc}[G^a]-\delta^{ac}[G^b]) \ ,$$

$$Q^aG^b=\delta^{ab}[P]-\frac{1}{2}\varepsilon^{abcd}[U^{cd}] \ , \tag{3.2}$$

$$Q^aU^{bc}=\delta^{ac}[Q^b]-\delta^{ab}[Q^c], \quad PG^a=[Q^a] \ ,$$

$$PP=-\frac{k}{2}[1], \quad Q^aQ^b=-\frac{k}{2}\delta^{ab}[1] \ ,$$

where $k=k_++k_-$ and $\zeta=(k_+-k_-)/k$.

The induced action $\Gamma[h,\psi,A,b,\eta]$ is defined as in (2.3). All the structure constants of the linear algebra (3.2) depend only on the ratio k_+/k_-. Apart from this ratio, k enters as a proportionality constant for *all* two-point functions. As a consequence, Γ depends on that ratio in a nontrivial way, but its k dependence is simply an overall factor k.

Using the definitions

$$t=\frac{12\pi}{c}\frac{\delta\Gamma}{\delta h}, \quad g^a=\frac{3\pi}{c}\frac{\delta\Gamma}{\delta\psi_a} \ ,$$

$$u^{ab}=-\frac{\pi}{k}\frac{\delta\Gamma}{\delta A_{ab}} \ , \tag{3.3}$$

$$q^a=-\frac{2\pi}{k}\frac{\delta\Gamma}{\delta\eta_a}, \quad p=-\frac{2\pi}{k}\frac{\delta\Gamma}{\delta b} \ ,$$

and $\gamma=6k/c$, the Ward identities are

$$\partial^3h=\overline{\nabla}t-2(\psi_a\partial+3\partial\psi_a)g^a+2\gamma\partial A_{ab}u^{ab}+\frac{\gamma}{2}(\partial\eta_a-\eta_a\partial)q^a+\gamma\partial bp \ ,$$

$$\partial^2\psi_a=\overline{\nabla}g^a-2A_{ab}g^b-\frac{1}{2}\psi_at+\frac{\gamma}{4}\varepsilon_{abcd}\eta_bu^{cd}+\frac{\gamma}{4}(\psi_b\partial+2\partial\psi_b)(2u^{ab}-\zeta\varepsilon_{abcd}u^{cd})$$

$$+\frac{\gamma}{4}\partial bq^a+\frac{\gamma}{2}\zeta\partial A_{ab}q^b+\frac{\gamma}{4}\varepsilon_{abcd}\partial A_{bc}q^d+\frac{\gamma}{4}\eta_ap \ ,$$

$$\partial A_{ab}+\frac{\zeta}{2}\varepsilon_{abcd}\partial A_{cd}=\overline{\nabla}u^{ab}-4A_{c[a}u^{b]c}-\frac{4}{\gamma}\psi_{[a}g^{b]}+\eta_{[a}q^{b]}-\zeta\left[\psi_{[a}\partial+\partial\psi_{[a}\right]q^{b]}-\frac{1}{2}\varepsilon_{abcd}(\psi_c\partial+\partial\psi_c)q^d \ , \tag{3.4}$$

$$\eta_a=\overline{\nabla}q^a-2A_{ab}q^b-\psi_aP+\varepsilon_{abcd}\psi_bu^{cd}, \quad \partial b=\overline{\nabla}p-(\psi_a\partial+\partial\psi_a)q^a \ .$$

The nonlinear $N=4$ superconformal algebra has the same structure as (3.2) but there is no P and Q^a. The central charge is related to the su(2) levels by $\widetilde{c}=[3(\widetilde{k}+2\widetilde{k}_+\widetilde{k}_-)]/(2+\widetilde{k})$ where $\widetilde{k}=\widetilde{k}_++\widetilde{k}_-$. We only give the $\widetilde{G}\widetilde{G}$ OPE explicitly:

$$\widetilde{G}^a\widetilde{G}^b=\frac{4\widetilde{k}_+\widetilde{k}_-}{\widetilde{k}+2}\delta^{ab}[1]-\frac{2\widetilde{k}}{\widetilde{k}+2}[\widetilde{U}^{ab}]+\frac{\widetilde{k}_+-\widetilde{k}_-}{\widetilde{k}+2}\varepsilon_{abcd}[\widetilde{U}^{cd}]$$

$$+\left[\frac{2\widetilde{k}}{\widetilde{k}+2\widetilde{k}_+\widetilde{k}_-}\delta^{ab}\widetilde{T}+\frac{1}{4(\widetilde{k}+2)}\varepsilon_{acdg}\varepsilon_{befg}(\widetilde{U}^{cd}\widetilde{U}^{ef}+\widetilde{U}^{ef}\widetilde{U}^{cd})\right] \ . \tag{3.5}$$

To write down the Ward identities in this case we define

$$\widetilde{t}=\frac{12\pi}{\widetilde{c}}\frac{\delta\widetilde{\Gamma}}{\delta h}, \quad \widetilde{g}^a=\frac{(\widetilde{k}+2)\pi}{2\widetilde{k}_++\widetilde{k}_-}\frac{\delta\widetilde{\Gamma}}{\delta\psi_a}, \quad \widetilde{u}^{ab}=-\frac{\pi}{\widetilde{k}}\frac{\delta\widetilde{\Gamma}}{\delta A_{ab}} \ , \tag{3.6}$$

and

$$\tilde{\gamma} = \frac{\tilde{k}(\tilde{k}+2)}{\tilde{k}_+ \tilde{k}_-}, \quad \tilde{\kappa} = \frac{6\tilde{k}}{\tilde{c}}, \quad \tilde{\zeta} = \frac{\tilde{k}_+ - \tilde{k}_-}{\tilde{k}}, \tag{3.7}$$

$$\partial^3 h = \overline{\nabla}\tilde{t} - \frac{2\tilde{\kappa}}{\tilde{\gamma}}(\psi_a \partial + 3\partial\psi_a)\tilde{g}^a + 2\tilde{\kappa}\partial A_{ab}\tilde{u}^{ab},$$

$$\partial^2 \psi_a = \overline{\nabla}\tilde{g}^a - 2A_{ab}\tilde{g}^b - \frac{\tilde{\gamma}}{2\tilde{\kappa}}\psi_a \tilde{t} - \frac{\tilde{\gamma}}{4\tilde{k}(\tilde{k}+2)}\varepsilon_{acdg}\varepsilon_{befg}\psi_b((\tilde{U}^{cd}\tilde{U}^{ef})_{\mathrm{eff}} + (\tilde{U}^{ef}\tilde{U}^{cd})_{\mathrm{eff}})$$

$$\qquad + \frac{\tilde{\gamma}\tilde{k}}{4(\tilde{k}+2)}(\psi_b \partial + 2\partial\psi_b)(2\tilde{u}^{ab} - \tilde{\zeta}\varepsilon_{abcd}\tilde{u}^{cd}), \tag{3.8}$$

$$\partial A_{ab} + \frac{\tilde{\zeta}}{2}\varepsilon_{abcd}\partial A_{cd} = \overline{\nabla}\tilde{u}^{ab} - 4A_{c[a}\tilde{u}^{b]c} - \frac{4}{\tilde{\gamma}}\psi_{[a}\tilde{g}^{b]}.$$

As in the previous section we will use the results of [14] on the construction of a nonlinear algebra by eliminating free fermion fields. In the present case it turns out that, at the same time, one can also eliminate the U(1)-field P. The new currents are

$$\tilde{T} = T + \frac{1}{k}PP + \frac{1}{k}\partial Q^c Q^c,$$

$$\tilde{G}^a = G^a + \frac{2}{k}PQ^a + \varepsilon_{abcd}\left[\frac{2}{3k^2}Q^b Q^c Q^d + \frac{1}{k}Q^b \tilde{U}^{cd}\right],$$

$$\tag{3.9}$$

$$\tilde{U}^{ab} = U^{ab} - \frac{2}{k}Q^a Q^b,$$

and the constants in the algebras are related by $\tilde{k}_\pm = k_\pm - 1$, and thus $\tilde{c} = c - 3$. Again, we find agreement between the large k limit of the Ward identities putting q^a and p to zero, and solve η_a and ∂b from the two last identities of (3.4). bote that the b field is present only as a derivative in Eq. (3.4). Thus again the suggestion presents itself that the nonlinear action can be obtained by putting some currents to zero.

We now set out to write the nonlinear effective action in terms of the effective action of the linear theory. In analogy with the $N = 3$ case, it would seem that, again, the operators of the nonlinear theory can be written as the difference of the operators of the linear theory, and a realization of the linear theory given by the free fermions. In the present case, this simple linear combination works for the integer spin currents $\tilde{T}$ and $\tilde{U}$, but not for $\tilde{G}$. A second complication is that, due to the presence of a trilinear term (in Q) in the relation between $\tilde{G}$ and G, integrating out the Q fields is more involved. Nevertheless, we can still obtain an explicit formula relating the effective actions.

There is a variety of ways to derive this relation, starting by rewriting the decompositions of (3.9) in different ways. We will use the form

$$G^a + \frac{1}{k}\varepsilon_{abcd}Q^b U^{cd} = \tilde{G}^a - \frac{2}{k}PQ^a + \frac{4}{3k^2}\varepsilon_{abcd}Q^b Q^c Q^d.$$

$$\tag{3.10}$$

This leads immediately to[3]

$$\left\langle \exp\left[-\frac{1}{\pi}\int\left[hT + \psi_a G^a + A_{ab}U^{ab} + bP + \eta_a Q^a + \frac{1}{k}\varepsilon_{abcd}\psi_a Q^b U^{cd}\right]\right]\right\rangle$$

$$= \left\langle \exp\left[-\frac{1}{\pi}\int\left[\left[h\tilde{T} + \psi_a \tilde{G}^a + A_{ab}\tilde{U}^{ab}\right]\right.\right.\right.$$

$$\left.\left.\left. + \frac{1}{k}\left[-hP^2 - h\partial Q^a Q_a - 2\psi_a PQ^a + 2A_{ab}Q^a Q^b + bP + \eta_a Q^a\right] + \frac{4}{3k^2}\varepsilon_{abcd}\psi_a Q^b Q^c Q^d\right]\right]\right\rangle. \tag{3.11}$$

Again the crucial step is that in the RHS, the expectation value factorizes: the average over Q^a and P can be computed separately, since these fields commute with the nonlinear supersymmetric (SUSY) algebra. This average is in fact closely related to the partition function for the linear $N = 4$ algebra with $k_+ = k_- = 1$ and $c = 3$, up to the renormalization of some coefficients. We have

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[3]Note that there are no normal ordering problems as the OPE's of the relevant operators turn out to be nonsingular (e.g., the term cubic in the Q^a is an antisymmetric combination).

$$Z^{c=3}[h,\psi,A,b,\eta]=\left\langle \exp\left[-\frac{1}{\pi}\int\left[-\frac{h}{2}\left(\hat{P}\hat{P}+\partial\hat{Q}_a\hat{Q}^a\right)-\psi_a\left(\hat{P}\hat{Q}^a+\tfrac{1}{6}\varepsilon_{abcd}\hat{Q}^b\hat{Q}^c\hat{Q}^d\right)+A_{ab}\hat{Q}^a\hat{Q}^b+b\hat{P}+\eta_a\hat{Q}^a\right]\right]\right\rangle,$$

$$(3.12)$$

where the average value is over free fermions $\hat{Q}^a$ and a free U(1)-current $\hat{P}$. These are normalized in a k-independent fashion,

$$\hat{P}\hat{P}=-[1],\quad \hat{Q}^a\hat{Q}^b=-\delta^{ab}[1]\,,\tag{3.13}$$

and the explicit form [16,12] of the currents making up the $c=3$ algebra has been used. The average can be represented as a functional integral with the measure

$$[d\hat{Q}][d\hat{P}]\exp\left[-\frac{1}{2\pi}\left(\hat{P}\frac{\bar{\partial}}{\partial}\hat{P}+\hat{Q}^a\bar{\partial}\hat{Q}_a\right)\right].\tag{3.14}$$

The (nonlocal) form of the free action for $\hat{P}$ follows from its two-point function: it is the usual (local) free scalar field action if one writes $\hat{P}=\partial\phi$. The connection between the linear theory, the nonlinear theory, and the $c=3$ realization is then

$$\exp\left[-\frac{\pi}{k}\varepsilon^{abcd}\psi_a\frac{\delta}{\delta\eta_b}\frac{\delta}{\delta A_{cd}}\right]Z[\psi,A,\eta,b]$$

$$=\tilde{Z}[\psi,A]\exp\left[\frac{\pi^2}{3k^2}(4+\sqrt{2k})\varepsilon^{abcd}\psi_a\frac{\delta}{\delta\eta_b}\frac{\delta}{\delta\eta_c}\frac{\delta}{\delta\eta_d}\right]Z^{c=3}[h,\psi,A,\eta\sqrt{k/2},b\sqrt{k/2}]\,.\tag{3.15}$$

Contrary to the $N=3$ case, where the Polyakov partition function was obtained very explicitly, this connection is not particularly useful, but the representation (3.14) of $Z^{c=3}$ as a functional integral can be used effectively. Indeed, when we take the Fourier transform of Eq. (3.15), i.e., we integrate (3.15) with

$$\int [dh][d\psi][dA][db][d\eta]\exp\left[\frac{1}{\pi}\int\left(ht+\psi_a g^a+A_{ab}u^{ab}+bp+\eta_a q^a\right)\right],\tag{3.16}$$

we obtain, using Eqs. (3.12) and (3.14),[4]

$$\exp\left[-W\left[t,g^a-\frac{1}{k}\varepsilon_{abcd}q^b u^{cd},u,p,q\right]\right]$$

$$=\exp\left[-\frac{1}{\pi k}\left(p\frac{\bar{\partial}}{\partial}p+q^a\bar{\partial}q_a\right)\right]\exp\left[-\tilde{W}\left[t+\frac{1}{k}\left(p^2+\partial qq\right),g^a+\frac{2}{k}pq^a-\frac{4}{3k^2}\varepsilon_{abcd}q^b q^c q^d,u^{ab}-\frac{2}{k}q^a q^b\right]\right],\tag{3.17}$$

giving the concise relation

$$\tilde{W}[t,g^a,u^{ab}]+\frac{1}{\pi k}\left(p\frac{\bar{\partial}}{\partial}p+q^a\bar{\partial}q_a\right)$$

$$=W\left[t-\frac{1}{k}\left(p^2+\partial q^a q_a\right),g^a-\frac{2}{k}pq^a-\frac{1}{k}\varepsilon_{abcd}q^b u^{cd}-\frac{2}{3k^2}\varepsilon_{abcd}q^b q^c q^d,u^{ab}+\frac{2}{k}q^a q^b,p,q^a\right].\tag{3.18}$$

Putting the free p and q^a currents equal to zero, one obtains the equality of effective actions:

$$\tilde{W}[t,g^a,u^{ab}]=W[t,g^a,u^{ab},p=0,q^a=0]\,.\tag{3.19}$$

IV. DISCUSSION

We take for granted that the linear theory is given by simple renormalizations of the "classical" theory, as described in the Introduction. Then Eqs. (2.29) and (3.19) immediately transfer this property to the nonlinear theory. Moreover, since the "classical" parts are equal also (as implied by the $c\to\infty$ limit of the Ward identities) the renormalization factors for both theories are the same (for couplings as well as for fields) if one takes into account the shifts in the values for the central extensions c, k_+, and k_-. This fact can be confirmed by looking at

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[4]The effective action W is defined by the Fourier transform of Z, and similarly for $\tilde{W}$.

explicit calculations of these renormalization factors. For $N=3$, a semiclassical approximation to the nonlinear renormalization factors was set up in [15]. This calculation was amended in [9], which also contains an all-order calculation of these factors. On the other hand, [9] also contains a semiclassical derivation of the factors for the *linear* algebras, and the $N=4$ factors as well. The results are

	Nonlinear all-order	Linear semiclassical
$N=3$:	$Z_\Gamma = \dfrac{2\tilde{c}+1}{2} - 3$	$Z_\Gamma = c - 3$
	$Z_h = \dfrac{2\tilde{c}+1}{2\tilde{c}-5}$	$Z_h = \dfrac{c}{c-3}$
$N=4$:	$Z_\Gamma = \tilde{c} + 3$	$Z_\Gamma = c$
	$Z_h = 1$	$Z_h = 1$

and the other field renormalization factors (Z_ψ, Z_u, Z_q, Z_p) for the effective action are the same. Clearly, these results coincide if one takes into account that $c = \tilde{c} + \frac{1}{2}$ for $N=3$ and $c = \tilde{c} + 3$ for $N=4$.

The remarkable property that the values of the central extensions of the Virasoro and affine algebras are related by $c = 6k + n$ with $n \in \mathbb{Z}$, is shared by a number of other nonlinear superconformal and quasisuperconformal algebras. These contain only one dimension-2 field, a number of (bosonic and fermionic) dimension-$\frac{3}{2}$ fields, and an affine superalgebra by which we will identify them. There are the osp($m|2n$) cases with $|m - 2n - 3| \leq 1$ (with $m = 3,4$ and $n = 0$ treated in this paper, and $m = 2, n = 0$ the ordinary $N=2$ superalgebra) and the u($n|m$) cases with $|n - m - 2| \leq 1$ from the series in [18], and further the osp($n|2m$)$\oplus$sl$_2$ algebras with $|n - 2m + 3| \leq 1$ from [19]. These same algebras arise also (among others) by quantum Drinfeld-Sokolov reduction from the list [9] where there are no corrections to the coupling beyond one loop. This is reminiscent of $N=2$ supergravity, and consequently also of supersymmetry nonrenormalization theorems [20], but the evidence is not conclusive. It would be interesting to investigate whether one can extend the analysis of the present paper in this direction.

ACKNOWLEDGMENTS

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