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# A PENALTY-FREE NONSYMMETRIC NITSCHE-TYPE METHOD FOR THE WEAK IMPOSITION OF BOUNDARY CONDITIONS*

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**Abstract.** In this paper we show that the nonsymmetric version of Nitsche's method for the weak imposition of boundary conditions is stable without penalty term. For nonconforming elements we prove the same result for the symmetric formulation as well. We prove optimal $H^1$-error estimates and $L^2$-error estimates that are suboptimal with half an order in $h$. Both the pure diffusion and the convection–diffusion problems are discussed.

**Key words.** finite element methods, weak boundary conditions, convection dominated flow, stabilized methods

**AMS subject classifications.** 65N30, 65N12

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## 1. Introduction.

In his seminal paper from 1971 [16], Nitsche proposed a consistent penalty method for the weak imposition of boundary conditions. The formulation proposed was symmetric so as to reflect the symmetry of the underlying Poisson problem. Stability was obtained thanks to a penalty term, with a penalty parameter that must satisfy a lower bound to ensure coercivity.

A nonsymmetric version of Nitsche's method was later proposed by Freund and Stenberg [10], and it was noted that this method did not need the lower bound for stability. The penalty term, however, could not be omitted, since coercivity fails, and error estimates degenerate as the penalty parameter goes to zero. The nonsymmetric version of Nitsche's method was then proposed as a discontinuous Galerkin (DG) method by Oden, Babuška, and Baumann [17], and it was proved by Rivière, Wheeler, and Girault [18] and Larson and Niklasson [15] that the nonsymmetric version was stable for polynomial orders $k \geq 2$. In [15] stability for the penalty-free case was proved using an inf-sup argument that relies on the important number of degrees of freedom available in high order DG methods.

To the best of our knowledge no similar results have been proved for the nonsymmetric version of Nitsche's method for the imposition of boundary conditions when continuous approximation spaces are used. Indeed in this case the DG analysis does not work since polynomials may not be chosen independently on different elements because of the continuity constraints. Weak imposition of boundary conditions has been advocated by Bazilevs and Hughes for large eddy-type turbulence computations in [1]. They showed that the mean flow in the boundary layer was more accurately captured using weakly rather than strongly imposed boundary conditions. They noted that the nonsymmetric version of Nitsche's method appears stable without penalty (see also [14]).

In applications there is interest in reducing the number of free parameters used without increasing the number of degrees of freedom needed for the coupling; see [11] for a discussion. From this point of view a penalty-free Nitsche method is a

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welcome addition to the computational toolbox, in particular for flow problems where the system matrix is nonsymmetric anyway, because of the convection terms. It has no penalty parameter and does not make use of Lagrange multipliers.

Numerical evidence also suggests that the unpenalized nonsymmetric Nitsche-type method has some further interesting properties. When using iterative solution methods in domain decomposition it has been shown to have more favorable convergence properties compared to the symmetric method [9]. For the solution of Cauchy-type inverse problems using steepest descent–type algorithms it has been shown numerically to have superior convergence properties in the initial phase of the iterations compared to the symmetric version or strongly imposed conditions, in spite of the lack of dual consistency.

In view of this the question naturally arises whether the penalty-free method is sound or if it could fail under unfortunate circumstances.

In this paper we prove for the Poisson problem that the nonsymmetric form of Nitsche's method is indeed stable and optimally convergent in the $H^1$-norm for polynomial orders $k \geq 1$ on regular meshes. We also show that in this case, the convergence rate of the error in the $L^2$-norm is suboptimal with only half a power of $h$. Hence the nonoptimality due to the nonsymmetry is not as important for continuous Galerkin methods as it is for DG methods (see [17] and [12] for numerical evidence of the suboptimal behavior in this latter case).

We then show how the results may be applied in the case of convection–diffusion equations, considering first the streamline–diffusion method and then outlining how the results may be extended to the case of the continuous interior penalty method.

Nitsche's method, however, has some stabilizing properties of its own, in particular for outflow layers; this phenomenon was analyzed in [19] and is illustrated herein with a numerical example. This makes Nitsche's method on nonsymmetric form an appealing, parameter-free, method for flow problems where the system matrix is nonsymmetric and the use of stabilized methods usually also results in the loss of half a power of $h$. It should be noted, however, that the smallest error in the $L^2$-norm is obtained with the formulation using penalty on the boundary, as illustrated in the numerical section. So we do not claim that the penalty-free method is the most accurate.

We only prove the result in the case of the imposition of boundary conditions, but the extensions of the results to the domain decomposition case of [2] or the fictitious domain method of [4] are straightforward using techniques similar to those below. Also note that since the main aim of the present paper is the study of weak imposition of boundary conditions, we will assume that the reader has a basic understanding of the techniques for analyzing stabilized finite element methods, and thus some arguments are only sketched.

For the sake of clarity, we first prove the main result on the pure diffusion problem and then discuss the extension of our result to the case of convection–diffusion problems. We also show all arguments in the two-dimensional case only; the extension to three space dimensions is straightforward. Some numerical examples conclude the paper.

**2. The pure diffusion problem.** Let $\Omega$ be a bounded domain in $\mathbb{R}^2$, with polygonal boundary $\partial\Omega$. Wherever $H^2$-regularity of the exact solution is needed we also assume that $\Omega$ is convex. Let $\{\Gamma_i\}_i$ denote the faces of the polygonal such that $\partial\Omega = \cup_i \Gamma_i$. The Poisson equation that we propose as a model problem is given by

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$ or $g \in H^{3/2}(\partial\Omega)$.

We have the following weak formulation: find $u \in V_g$ such that

$$a(u, v) = (f, v)_\Omega \quad \forall v \in V_0, \tag{2.2}$$

where $(x, y)_\Omega$ denotes the $L^2$-scalar product over $\Omega$,

$$V_g := \{v \in H^1(\Omega) : v|_{\partial\Omega} = g\}$$

and

$$a(u, v) := (\nabla u, \nabla v)_\Omega.$$

This problem is well-posed by the Lax–Milgram lemma using the standard arguments to account for nonhomogeneous boundary conditions. The $H^1$-stability $\|u\|_{H^1(\Omega)} \leq C_{R1}(\|f\| + \|g\|_{H^{1/2}(\partial\Omega)})$ holds, and under the convexity assumption on $\Omega$ there holds $\|u\|_{H^2(\Omega)} \leq C_{R2}(\|f\| + \|g\|_{H^{3/2}(\partial\Omega)})$. Here we let $\|x\| := \|x\|_{L^2(\Omega)}$. Below, $C$ will be used as a generic constant that may change at each occasion and is independent of $h$, but not necessarily of the local mesh geometry. We will also use the notation $a \lesssim b$ for $a \leq Cb$.

**3. The finite element formulation.** Let $\{\mathcal{T}_h\}$ denote a family of quasi-uniform and shape regular triangulations fitted to $\Omega$, indexed by the mesh parameter $h$. The triangles of $\mathcal{T}_h$ will be denoted $K$ and their diameter $h_K := \operatorname{diam}(K)$. The interior of a set $P$ will be denoted $\mathring{P}$. For a given $\mathcal{T}_h$ the mesh parameter is determined by $h := \max_{K \in \mathcal{T}_h} h_K$. Shape regularity is expressed by the existence of a constant $c_\rho \in \mathbb{R}$ for the family of triangulations such that, with $\rho_K$ the radius of the largest ball inscribed in an element $K$, there holds

$$\frac{h_K}{\rho_K} \leq c_\rho \quad \forall K \in \mathcal{T}_h.$$

For technical reasons, and to avoid the treatment of special cases, we assume that for all $i$, $\Gamma_i$ contains no less than five element faces.

We introduce the standard finite element space of continuous piecewise polynomial functions,

$$V_h^k := \{v_h \in H^1(\Omega) : v_h|_K \in \mathbb{P}_k(K) \quad \forall K \in \mathcal{T}_h\},\ k \geq 1,$$

where $\mathbb{P}_k(K)$ denotes the space of polynomials of degree less than or equal to $k$ on the element $K$. The finite element formulation that we consider then takes the following form: find $u_h \in V_h^k$ such that

$$a_h(u_h, v_h) = (f, v_h)_\Omega + \langle g, \nabla v_h \cdot n \rangle_{\partial\Omega} \quad \forall v_h \in V_h^k, \tag{3.1}$$

where $\langle x, y \rangle_{\partial\Omega}$ denotes the $L^2$-scalar product over the boundary of $\Omega$ and

$$a_h(u_h, v_h) := a(u_h, v_h) - \langle \nabla u_h \cdot n, v_h \rangle_{\partial\Omega} + \langle u_h, \nabla v_h \cdot n \rangle_{\partial\Omega}. \tag{3.2}$$

Note that in the classical nonsymmetric version of Nitsche's method we also add a penalty term of the form

$$\sum_K \left\langle \gamma h_K^{-1} u_h, v_h \right\rangle_{\partial\Omega \cap \partial K} \tag{3.3}$$

and modify the second term on the right-hand side accordingly:

$$\sum_K \left\langle g, \gamma h_K^{-1} v_h + \nabla v_h \cdot n \right\rangle_{\partial\Omega \cap \partial K}.$$

The key observation of the present work is that the penalty parameter $\gamma$ may be chosen to be zero without loss of either stability or accuracy.

Inserting the exact solution $u$ into the formulation (3.1) and integrating by parts immediately leads to the following consistency relation.

LEMMA 3.1. *If $u$ is the solution of* (2.1) *and $u_h$ is the solution of* (3.1), *then there holds*

$$a_h(u - u_h, v_h) = 0.$$

For future reference we here recall the classical trace and inverse inequalities satisfied by the spaces $V_h^k$.

LEMMA 3.2 (trace inequality). *There exists $C_T \in \mathbb{R}$ such that for all $v_h \in \mathbb{P}_k(K)$ and for all $K \in \mathcal{T}_h$ there holds*

$$\|v_h\|_{L^2(\partial K)} \leq C_T(h_K^{-\frac{1}{2}} \|v_h\|_{L^2(K)} + h_K^{\frac{1}{2}} \|\nabla v_h\|_{L^2(K)}).$$

LEMMA 3.3 (inverse inequality). *There exists $C_I \in \mathbb{R}$ such that for all $v_h \in \mathbb{P}_k(K)$ and for all $K \in \mathcal{T}_h$ there holds*

$$\|\nabla v_h\|_{L^2(K)} \leq C_I h_K^{-1} \|v_h\|_{L^2(K)}.$$

**4. Stability.** Testing (3.1) with $v_h = u_h$ immediately gives control of the $H^1$-seminorm of $u_h$. In order for the formulation to be well-posed this is not sufficient. Indeed well-posedness is a consequence of the Poincaré inequality that holds, provided we have sufficient control of the trace of $u_h$ on $\partial\Omega$. This is the role of the penalty term (3.3); it ensures that the following Poincaré inequality is satisfied:

$$\|u_h\| \leq C_P \|u_h\|_{1,h}, \text{ where } \|u_h\|_{1,h}^2 := \|\nabla u_h\|^2 + \|u_h\|_{\frac{1}{2},h,\partial\Omega}^2$$

with

$$\|u_h\|_{\frac{1}{2},h,\partial\Omega}^2 := \sum_K \left\langle h_K^{-1} u_h, u_h \right\rangle_{\partial\Omega \cap \partial K}.$$

Since we have omitted the penalty term, boundary control of $u_h$ is not an immediate consequence of testing with $v_h = u_h$. What we will show below is that control of the boundary term can be recovered by proving an inf-sup condition. Indeed the nonsymmetric version of Nitsche's method can be interpreted as a Lagrange multiplier method where the Lagrange multiplier $\lambda_h$ has been replaced by the normal gradient of the solution: $\nabla u_h \cdot n$. This interpretation of Nitsche's method was originally proposed in [21], however, without considering the inf-sup condition. The DG framework was considered in [8], where equivalence was shown between a certain Lagrange multiplier method and a certain DG-method. When Lagrange multipliers are used to impose continuity, the system has a saddle point structure and the inf-sup condition is the standard way of proving well-posedness. Here we will follow a similar procedure, the only difference being that the solution space and the multiplier space are strongly coupled, since the latter consists simply of the normal gradients of the former. A

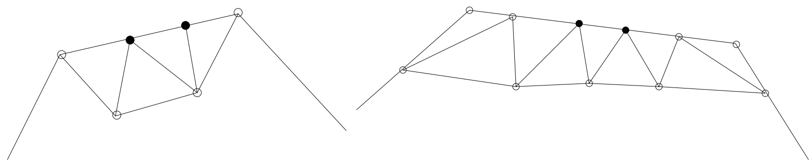

FIG. 1. *Example of boundary patches $P_j$. Left, the smallest possible patch; right, the worst-case scenario, with elements with two sides on the boundary. The function $\tilde{\varphi}_j$ takes the value 1 in filled nodes and zero in the other nodes.*

key result is given in the following lemma, where we construct a function in the test space that will allow us to control certain averages of the solution on the boundary. To this end regroup the boundary elements, i.e., the elements with either a face or a vertex on the boundary, in (closed) patches $P_j$, with boundary $\partial P_j$, $j = 1, \ldots, N_P$. Let $F_j := \partial P_j \cap \partial\Omega$. We assume that the $P_j$ are designed such that each $F_j$ has at least two inner nodes, but in some cases they may need up to four inner nodes (this is necessary only if both end vertices of $P_j$ belong to corner elements with all their vertices on the boundary; see Figure 1, right). Under our assumptions on the mesh, every $\Gamma_i$ contains at least one patch $P_j$ and there exist $c_1, c_2$ such that for all $j$

$$c_1 h \leq \operatorname{meas}(F_j) \leq c_2 h. \tag{4.1}$$

The average value of a function $v$ over $F_j$ will be denoted by $\bar{v}^j$. First we prove the lemma under a weakly acute assumption on the patches $P_i$, and then we will discuss the extension to the general case. We only give the proof for the left situation of Figure 1; the extension to the right case is immediate by considering the acute condition on the support of the function instead.

LEMMA 4.1. *Assume that, for all $P_j$, $\partial P_j$ meets $\partial\Omega$ at an angle $\leq \frac{\pi}{2}$. For any given vector $(r_j)_{j=1}^{N_P} \in \mathbb{R}^{N_P}$ there exists $\varphi_r \in V_h^1$ such that for all $1 \leq j \leq N_P$ there holds*

$$\operatorname{meas}(F_j)^{-1} \int_{F_j} \nabla \varphi_r \cdot n \; ds = r_j, \tag{4.2}$$

*and, if $r(x) : \partial\Omega \mapsto \mathbb{R}$ denotes the function such that $r|_{F_i} = r_i$,*

$$\|\varphi_r\|_{1,h} \lesssim \left( \sum_{j=1}^{N_P} \|h^{\frac{1}{2}} r\|^2_{L^2(F_j)} \right)^{1/2}. \tag{4.3}$$

*Proof.* We first construct a function $\tilde{\varphi}_j$ taking the value 1 in the interior nodes of $\partial\Omega \cap \partial P_j$ and zero elsewhere; see Figure 1. Fix $j$ and let $\tilde{\varphi}_j \in V_h^1$ be defined, in each vertex $x_i \in \mathcal{T}_h$, by

$$\tilde{\varphi}_j(x_i) = \begin{cases} 0 & \text{for } x_i \in \Omega \setminus \mathring{P}_j \\ & \text{and for } x_i \text{ in a triangle } K \text{ that has three vertices on } \partial\Omega; \\ 1 & \text{for } x_i \in \mathring{F}_j. \end{cases}$$

Let

$$\Xi_j := \operatorname{meas}(F_j)^{-1} \int_{F_j} \nabla \tilde{\varphi}_j \cdot n \, \mathrm{d}s$$

and define the normalized function $\varphi_j$ by

$$\varphi_j := \Xi_j^{-1}\tilde{\varphi}_j.$$

This quantity is well defined thanks to the following lower bound that holds uniformly in $j$ and $h$:

$$0 < C_\Xi \leq \Xi_j h.$$

The constant $C_\Xi$ depends only on the local geometry of the patches $P_j$. By definition there holds

$$\text{meas}(F_j)^{-1}\int_{F_j} \nabla\varphi_j \cdot n \, \mathrm{d}s = 1, \tag{4.4}$$

and using the standard inverse inequality (Lemma 3.3) we obtain

$$\|\nabla\varphi_j\| \lesssim C_I h^{-1}\Xi_j^{-1}\|\tilde{\varphi}_j\|_{L^2(P_j)} \lesssim C_I h^{-1}\Xi_j^{-1}\text{meas}(P_j)^{1/2} \lesssim C_I C_\Xi^{-1} h. \tag{4.5}$$

Now defining

$$\varphi_r := \sum_{j=1}^{N_P} r_j \varphi_j$$

we immediately see that condition (4.2) is satisfied by (4.4). The upper bound (4.3) follows from (4.5), relation (4.1), and using that

$$\begin{aligned}\|\varphi_r\|^2_{\frac{1}{2},h,\partial\Omega} &:= \sum_{j=1}^{N_P} \|h^{-\frac{1}{2}} r_j \varphi_j\|^2_{L^2(F_j)} \\ &\lesssim \sum_{j=1}^{N_P} h^{-1} r_j^2 \Xi_j^{-2}\|\tilde{\varphi}_j\|^2_{L^2(F_j)} \lesssim C_\Xi^{-2}\sum_{j=1}^{N_P}\|h^{\frac{1}{2}} r\|^2_{L^2(F_j)}. \qquad \square\end{aligned}$$

*Remark* 1. If the weakly acute condition is violated, patches may be constructed such that (4.1)–(4.2) fail. However, for a fixed $c_\rho$ the result of Lemma 4.1 can always be made to hold uniformly by including a sufficient number of elements in each $F_j$.

With the help of this technical lemma it is straightforward to prove the inf-sup condition for the formulation (3.1).

THEOREM 4.2. *There exists $c_s > 0$ such that for all functions $v_h \in V_h^k$ there holds*

$$c_s \|v_h\|_{1,h} \leq \sup_{w_h \in V_h^k} \frac{a_h(v_h, w_h)}{\|w_h\|_{1,h}}.$$

*Proof.* Recall that

$$a_h(v_h, w_h) = (\nabla v_h, \nabla w_h)_\Omega - \langle \nabla v_h \cdot n, w_h\rangle_{\partial\Omega} + \langle v_h, \nabla w_h \cdot n\rangle_{\partial\Omega}.$$

Taking $w_h = v_h$ gives

$$a_h(v_h, v_h) = \|\nabla v_h\|^2.$$

To recover control over the boundary integral we let

$$r_j = h^{-1}\bar{v}^j := h^{-1}\text{meas}(F_j)^{-1}\int_{F_j} v_h \; \mathrm{d}s \tag{4.6}$$

in the construction of $\varphi_r$ in Lemma 4.1 and note that

$$\langle v_h, \nabla\varphi_r\cdot n\rangle_{\partial\Omega} = \sum_{j=1}^{N_P}\left(\|h^{-1/2}\bar{v}^j\|^2_{L^2(F_j)} + \big\langle (v_h - \bar{v}^j), \nabla\varphi_r\cdot n\big\rangle_{F_j}\right).$$

Using standard approximation,

$$\|v_h - \bar{v}^j\|_{L^2(F_j)} \lesssim h\|\nabla v_h \times n\|_{L^2(F_j)}, \tag{4.7}$$

and by the trace and inverse inequalities of Lemmas 3.2 and 3.3 we have

$$\big\langle (v_h - \bar{v}^j), \nabla\varphi_r\cdot n\big\rangle_{F_j} \lesssim C_T^2(1+C_I)\|\nabla v_h\|_{L^2(P_j)}\|\nabla\varphi_r\|_{L^2(P_j)}.$$

Moreover, since by the Cauchy–Schwarz inequality and the trace inequality

$$|(\nabla v_h, \nabla w_h)_\Omega - \langle\nabla v_h\cdot n, w_h\rangle_{\partial\Omega}| \lesssim \|\nabla v_h\|\|w_h\|_{1,h},$$

we deduce using the stability (4.3) that

$$\begin{aligned} a_h(v_h,\varphi_r) &\geq \sum_{j=1}^{N_P}\|h^{-1/2}\bar{v}^j\|^2_{L^2(F_j)} - C\|\nabla v_h\|\|\varphi_r\|_{1,h} \\ &\geq \sum_{j=1}^{N_P}\|h^{-1/2}\bar{v}^j\|^2_{L^2(F_j)} - C_s\|\nabla v_h\|\left(\sum_{j=1}^{N_P}\|h^{-1/2}\bar{v}^j\|^2_{L^2(F_j)}\right)^{1/2}. \end{aligned}$$

We now fix $w_h = v_h + \eta\varphi_r$ and note that

$$\begin{aligned} a_h(v_h,w_h) &\geq \|\nabla v_h\|^2 + \eta\sum_{j=1}^{N_P}\|h^{-1/2}\bar{v}^j\|^2_{L^2(F_j)} \\ &\quad - C_s\|\nabla v_h\|\eta\left(\sum_{j=1}^{N_P}\|h^{-1/2}\bar{v}^j\|^2_{L^2(F_j)}\right)^{1/2} \\ &\geq (1-\epsilon)\|\nabla v_h\|^2 + \eta(1 - C_s^2\eta/(4\epsilon))\sum_{j=1}^{N_P}\|h^{-1/2}\bar{v}^j\|^2_{L^2(F_j)}. \end{aligned} \tag{4.8}$$

It follows, using once again the approximation properties of the $L^2$-projection on the piecewise constants (4.7), that for any $\epsilon < 1$ we may take $\eta$ sufficiently small so that there exists $c_{\eta,\epsilon}$ such that

$$c_{\eta,\epsilon}\|v_h\|^2_{1,h} \leq Cc_{\eta,\epsilon}\left(\|\nabla v_h\|^2 + \sum_{j=1}^{N_P}\|h^{-1/2}\bar{v}^j\|^2_{L^2(F_j)}\right) \leq a_h(v_h,w_h).$$

We may conclude by noting that by (4.3), our choice of $r_j$, and the stability of the $L^2$-projection on piecewise constants there holds

$$\|w_h\|_{1,h} \leq \|v_h\|_{1,h} + \eta\|\varphi_r\|_{1,h} \leq C_\eta\|v_h\|_{1,h}. \qquad \square \tag{4.9}$$

**5. A priori error estimates.** The stability estimate proved in the previous section together with the Galerkin orthogonality of Lemma 3.1 leads to error estimates in the $\|\cdot\|_{1,h}$-norm in a straightforward manner. First we will prove an auxiliary lemma for the continuity of $a_h(\cdot,\cdot)$. To this end we introduce the norm

$$\|u\|_* := \|u\|_{1,h} + \|h^{\frac{1}{2}}\nabla u \cdot n\|_{L^2(\partial\Omega)}.$$

LEMMA 5.1. *Let $u \in H^2(\Omega) + V_h^k$ and $v_h \in V_h^k$. Then the bilinear form $a_h(\cdot,\cdot)$ defined by* (3.2) *satisfies*

$$a_h(u, v_h) \lesssim \|u\|_* \|v_h\|_{1,h}.$$

*Proof.* The result is immediate by application of the Cauchy–Schwarz inequality and the inequalities of Lemmas 3.2 and 3.3. □

PROPOSITION 5.2. *Let $u \in H^{k+1}(\Omega)$ be the solution of* (2.1) *and $u_h$ the solution of* (3.1)*. Then there holds*

$$\|u - u_h\|_{1,h} \lesssim h^k |u|_{H^{k+1}(\Omega)}.$$

*Proof.* Let $i_{\mathsf{SZ}}^k u$ denote the Scott–Zhang interpolant of $u$ [20]. Using the approximation properties of the interpolant, it is straightforward to show that

$$\|u - i_{\mathsf{SZ}}^k u\|_{1,h} + \|u - i_{\mathsf{SZ}}^k u\|_* \lesssim h^k |u|_{H^{k+1}(\Omega)}.$$

We therefore use the triangle inequality to obtain

$$\|u - u_h\|_{1,h} \leq \|u - i_{\mathsf{SZ}}^k u\|_{1,h} + \|u_h - i_{\mathsf{SZ}}^k u\|_{1,h},$$

where only the second term needs to be bounded. To this end we apply the result of Theorem 4.2 followed by the consistency of Lemma 3.1:

$$c_s \|u_h - i_{\mathsf{SZ}}^k u\|_{1,h} \leq \sup_{w_h \in V_h^k} \frac{a_h(u_h - i_{\mathsf{SZ}}^k u, w_h)}{\|w_h\|_{1,h}} = \sup_{w_h \in V_h^k} \frac{a_h(u - i_{\mathsf{SZ}}^k u, w_h)}{\|w_h\|_{1,h}}.$$

By the continuity of Lemma 5.1 and the approximation properties of $i_{\mathsf{SZ}}^k u$ we conclude

$$c_s \|u_h - i_{\mathsf{SZ}}^k u\|_{1,h} \lesssim \|u - i_{\mathsf{SZ}}^k u\|_* \lesssim h^k |u|_{H^{k+1}(\Omega)}. \qquad \square$$

For DG methods it is well known that the nonsymmetric version may suffer from suboptimality in the convergence of the error in the $L^2$-norm due to the lack of adjoint consistency. This is true also for the nonsymmetric version of Nitsche's method considered here; however, since the method is used on the scale of the domain and not of the element, the suboptimality may be reduced to $h^{\frac{1}{2}}$, as we prove below.

PROPOSITION 5.3. *Let $u \in H^{k+1}(\Omega)$ be the solution of* (2.1) *and $u_h$ the solution of* (3.1)*. Then*

$$\|u - u_h\| \leq C h^{k+\frac{1}{2}} |u|_{H^{k+1}(\Omega)}.$$

*Proof.* Let $z$ satisfy the adjoint problem

$$\begin{cases} -\Delta z = u - u_h & \text{in } \Omega, \\ \quad z = 0 & \text{on } \partial\Omega. \end{cases}$$

Under the assumptions on $\Omega$ we know that $\|z\|_{H^2(\Omega)} \le C_{R2}\|u-u_h\|$. It follows that

$$\begin{aligned}\|u-u_h\|^2 &= (u-u_h, -\Delta z)_\Omega = (\nabla(u-u_h), \nabla z)_\Omega - \langle u-u_h, \nabla z\cdot n\rangle_{\partial\Omega} \\ &= a_h(u-u_h, z) + 2\,\langle u-u_h, \nabla z\cdot n\rangle_{\partial\Omega}.\end{aligned}$$

By Lemma 3.1 and a continuity argument similar to that of Lemma 5.1, using that $(z-i^1_{\mathsf{SZ}}z)|_{\partial\Omega}\equiv 0$, it follows that

$$\begin{aligned}a_h(u-u_h, z) &= a_h(u-u_h, z-i^1_{\mathsf{SZ}}z) \\ &= (\nabla(u-u_h), \nabla(z-i^1_{\mathsf{SZ}}z))_\Omega - \left\langle u-u_h, \nabla(z-i^1_{\mathsf{SZ}}z)\cdot n\right\rangle_{\partial\Omega} \\ &\lesssim \|u-u_h\|_{1,h}\|z-i^1_{\mathsf{SZ}}z\|_* \\ &\lesssim h\|u-u_h\|_{1,h}|z|_{H^2(\Omega)}.\end{aligned} \tag{5.1}$$

We also have, using the global trace inequality

$$\|\nabla z\cdot n\|_{L^2(\partial\Omega)} \lesssim \|z\|_{H^2(\Omega)},$$

that

$$|\,\langle u-u_h, \nabla z\cdot n\rangle_{\partial\Omega}\,| \lesssim h^{1/2}\|u-u_h\|_{\frac{1}{2},h,\partial\Omega}\|z\|_{H^2(\Omega)}. \tag{5.2}$$

Collecting inequalities (5.1) and (5.2), we arrive at the estimate

$$\|u-u_h\|^2 \lesssim (h+h^{1/2})h^k|u|_{H^{k+1}(\Omega)}\|z\|_{H^2(\Omega)}$$

and conclude by applying the regularity estimate $\|z\|_{H^2(\Omega)} \le C_{R2}\|u-u_h\|$. ∎

**6. A penalty-free symmetric Nitsche-type method.** Optimal convergence in the $L^2$-norm would be obtained if the symmetric form of Nitsche's method were used. One may ask if the above stability argument could be extended to the symmetric form without penalty, in the spirit of [7]. In general the answer to this question appears to be no, the spaces of $H^1$-conforming elements are simply too small to satisfy all the required patch tests. For the nonconforming method using piecewise affine approximation (the Crouzeix–Raviart element), on the other hand, it is easy to prove the result. For simplicity we assume that no element has more than one face on the boundary of $\Omega$. Let

$$[\nabla u_{nc}\cdot n_{\partial K}] := \lim_{\epsilon\to 0^+}(\nabla u_{nc}(x-\epsilon n_{\partial K})\cdot n_{\partial K} - \nabla u_{nc}(x+\epsilon n_{\partial K})\cdot n_{\partial K}).$$

For jumps of scalar quantities without normal vector, the orientation is irrelevant. Let $\{u_{nc}\}|_F$ denote the average of $u_{nc}$ across the face $F$ and let $\mathcal{F}_{in}$ denote the set of interior faces in $\mathcal{T}_h$.

$$V^1_{nc} := \left\{ v\in L^2(\Omega) : v|_K \in \mathbb{P}_1(K)\,\forall K\in\mathcal{T}_h \text{ and } \int_F [v]\,\mathrm{d}s = 0, \forall F\in\mathcal{F}_{in}\right\}.$$

To account for the nonconformity, we redefine the discrete norm as follows:

$$\|v_{nc}\|^2_{1,h} := \sum_{K\in\mathcal{T}_h}\|\nabla u_{nc}\|^2_{L^2(K)} + \|u_{nc}\|^2_{\frac{1}{2},h,\partial\Omega}.$$

The nonconforming formulation then reads: find $u_{nc}\in V_h^{nc}$ such that

$$a_{nc}(u_{nc}, v_{nc}) = (f, v_{nc})_\Omega - \langle g, \nabla v_{nc}\cdot n\rangle_{\partial\Omega} \quad \forall v_{nc}\in V^1_{nc},$$

where

$$a_{nc}(u_{nc}, v_{nc}) := \sum_{K \in \mathcal{T}_h} (\nabla u_{nc}, \nabla v_{nc})_K - \langle \nabla u_{nc} \cdot n, v_{nc} \rangle_{\partial\Omega} - \langle u_{nc}, \nabla v_h \cdot n \rangle_{\partial\Omega}.$$

Let $\xi_{nc} \in V_{nc}^1$ be a function such that for each element with one face on the boundary $\nabla \xi_{nc} \cdot n_{\partial\Omega} = h_K^{-1} \pi_0 u_{nc}$ and $\int_F \xi_{nc} \, \mathrm{d}s = 0$ for interior faces $F$. By an integration by parts and using the second design criterion of $\xi_{nc}$, we see that

$$a_{nc}(u_{nc}, \xi_{nc}) = \sum_K \int_{\partial K \setminus \partial\Omega} [\nabla u_{nc} \cdot n_{\partial K}] \{\xi_{nc}\} \, \mathrm{d}s + \|\pi_0 u_{nc}\|^2_{\frac{1}{2},h,\partial\Omega} = \|\pi_0 u_{nc}\|^2_{\frac{1}{2},h,\partial\Omega}.$$

Taking $v_{nc} := u_{nc} + \eta \xi_{nc}$ with $\eta \in \mathbb{R}$ a coefficient to be fixed, we have

$$\sum_{K \in \mathcal{T}_h} \|\nabla u_{nc}\|^2_{L^2(K)} + \eta \|\pi_0 u_{nc}\|^2_{\frac{1}{2},h,\partial\Omega} - 2 \langle \nabla u_{nc} \cdot n, u_{nc} \rangle_{\partial\Omega} = a_{nc}(u_{nc}, u_{nc} + \eta \xi_{nc}).$$

The left-hand side is controlled in the standard fashion using

$$\langle \nabla u_{nc} \cdot n, u_{nc} \rangle_{\partial\Omega} \leq \eta^{-1} C_T^2 \sum_{K \in \mathcal{T}_h} \|\nabla u_{nc}\|^2_{L^2(K)} + \eta/4 \|\pi_0 u_{nc}\|^2_{\frac{1}{2},h,\partial\Omega}$$

and choosing $\eta > 2C_T^2$. This leads to

$$(1 - 2\eta^{-1} C_T^2) \sum_{K \in \mathcal{T}_h} \|\nabla u_{nc}\|^2_{L^2(K)} + \frac{1}{2} \eta \|\pi_0 u_{nc}\|^2_{\frac{1}{2},h,\partial\Omega} \leq a_{nc}(u_{nc}, u_{nc} + \eta \xi_{nc}).$$

It is straightforward to show that

$$\sum_{K \in \mathcal{T}_h} \|\nabla (u_{nc} + \eta \xi_{nc})\|^2_{L^2(K)} + \eta \|\pi_0 u_{nc} + \eta \xi_{nc}\|^2_{\frac{1}{2},h,\partial\Omega} \lesssim \sum_{K \in \mathcal{T}_h} \|\nabla u_{nc}\|^2_{L^2(K)} + \eta \|\pi_0 u_{nc}\|^2_{\frac{1}{2},h,\partial\Omega}$$

and that

$$\|u_{nc}\|^2_{1,h} \lesssim \sum_{K \in \mathcal{T}_h} \|\nabla u_{nc}\|^2_{L^2(K)} + \|\pi_0 u_{nc}\|^2_{\frac{1}{2},h,\partial\Omega}.$$

We have proved the following.

PROPOSITION 6.1. *There exists $c_s > 0$ such that for all functions $v_{nc} \in V_{nc}^1$ there holds*

$$c_s \|v_{nc}\|_{1,h} \leq \sup_{w_{nc} \in V_{nc}^1} \frac{a_{nc}(v_{nc}, w_{nc})}{\|w_{nc}\|_{1,h}}.$$

Optimal a priori error estimates follow in the standard fashion using Strang's lemma.

*Remark* 2. Since the system matrix corresponding to the symmetric method without penalty is indefinite, certain constraints on the time step apply for transient flow problems as discussed in [8].

**7. The convection–diffusion problem.** Since the method we discuss leads to a nonsymmetric system matrix, the main interest of the method is for solving flow problems where an advection term makes the problem nonsymmetric anyway. Note that there appears to be no analysis that is robust with respect to the Péclet number, even in the case of the nonsymmetric DG method.

We will therefore now show how the above analysis can be extended to the case of convection–diffusion equations yielding optimal stability and accuracy in both the convection- and the diffusion-dominated regime. We will consider the convection–diffusion–reaction equation

$$\sigma u + \beta \cdot \nabla u - \varepsilon \Delta u = f \text{ in } \Omega \tag{7.1}$$

and homogeneous Dirichlet boundary conditions. We assume that $\beta \in [W^1_\infty(\Omega)]^2$, $\sigma \in \mathbb{R}$,

$$\sigma - \frac{1}{2}\nabla \cdot \beta \geq c_\sigma \geq 0,$$

and $\varepsilon \in \mathbb{R}^+$. In this case the formulation is written as follows: find $u_h \in V_h$ such that

$$\begin{aligned} A_h(u_h, v_h) := (\sigma u_h + \beta \cdot \nabla u_h, v_h)_\Omega - \langle \beta \cdot n, u_h, v_h\rangle_{\partial\Omega^-} \\ + \varepsilon a_h(u_h, v_h) = (f, v_h)_\Omega \quad \forall v_h \in V_h, \end{aligned} \tag{7.2}$$

where $\partial\Omega^\pm := \{x \in \partial\Omega : \pm\beta \cdot n > 0\}$. First note that the positivity of the form now reads

$$A_h(u_h, u_h) \geq \frac{1}{2}\||\beta \cdot n|^{\frac{1}{2}} u_h\|^2_{\partial\Omega} + \|\varepsilon^{\frac{1}{2}} \nabla u_h\|^2; \tag{7.3}$$

hence provided $|\beta \cdot n| > 0$ on some portion of the boundary with nonzero measure, the matrix is invertible. In the diffusion-dominated case we make no such assumptions on $\beta$, whereas when convection dominates we assume that $|\beta \cdot n| > 0$ on some subset of $\partial\Omega$ with nonzero measure. To prove optimal error estimates in general, we require stronger stability results of the type proved above to hold. It appears difficult to prove these stronger results independently of the flow regime. Indeed it is convenient to characterize the flow using the local Péclet number:

$$\mathrm{Pe} := \frac{|\beta| h}{\varepsilon}.$$

If $\mathrm{Pe} < 1$, the flow is said to be diffusion dominated, and if $\mathrm{Pe} > 1$, we say that it is convection dominated. We will now treat these two cases separately.

In view of equality (7.3) we introduce the following strengthened norm:

$$\|v_h\|^2_{1,h,\beta} := \varepsilon \|v_h\|^2_{1,h} + \frac{1}{2}\||\beta \cdot n|^{\frac{1}{2}} v_h\|^2_{\partial\Omega}.$$

This norm is suitable in the diffusion-dominated regime, but will be modified by the introduction of stabilization when the convection-dominated regime is considered.

**7.1. Diffusion-dominated regime Pe < 1.** In this case we may prove an inf-sup condition similar to that of Theorem 4.2. For simplicity we assume that $\sigma = 0$.

PROPOSITION 7.1 (inf-sup for convection–diffusion, $\mathrm{Pe} < 1$). *For all functions $v_h \in V_h^k$ there holds*

$$c_s \|v_h\|_{1,h,\beta} \leq \sup_{w_h \in V_h^k} \frac{A_h(v_h, w_h)}{\|w_h\|_{1,h,\beta}}. \tag{7.4}$$

Clearly, compared to the proof of Theorem 4.2 we only need to show how to handle the term

$$(\beta \cdot \nabla v_h, \varphi_r)_\Omega - \langle \beta \cdot n\, v_h, \varphi_r \rangle_{\partial\Omega^-}\,.$$

The necessary bound on this term is given in the following lemma.

Lemma 7.2. *Let $\varphi_r$ be the function of Lemma* 4.1 *with $r$ chosen as in* (4.6)*. Then for* $\mathrm{Pe} < 1$ *there holds for all $\mu > 0$ that*

$$\begin{aligned}(\beta \cdot \nabla v_h, \eta\varphi_r)_\Omega &- \langle \beta \cdot n\, v_h, \eta\varphi_r \rangle_{\partial\Omega^-} \\ &\leq \mu(\varepsilon\|\nabla v_h\|^2 + \||\beta \cdot n|^{\frac{1}{2}} v_h\|^2_{L^2(\partial\Omega)}) + C_\partial^2 (2\mu)^{-1}\eta^2 \varepsilon \|v_h\|^2_{\frac{1}{2},h,\partial\Omega}.\end{aligned}$$

*Proof.* Let

$$(\beta \cdot \nabla v_h, \eta\varphi_r)_\Omega - \langle \beta \cdot n\, v_h, \eta\varphi_r \rangle_{\partial\Omega^-} = T_1 + T_2.$$

By the definition of the Péclet number and the Cauchy–Schwarz inequality, we have

$$T_1 \leq \mathrm{Pe}\varepsilon^{\frac{1}{2}} \|\nabla v_h\| \eta \varepsilon^{\frac{1}{2}} \|h^{-1}\varphi_r\|.$$

From the construction of $\varphi_r$, a scaling argument, the stability (4.3), and the choice of $r$ (4.6) we deduce that

$$\|h^{-1}\varphi_r\| \lesssim \|\nabla \varphi_r\| \leq C_\partial \|v_h\|_{\frac{1}{2},h,\partial\Omega}.$$

Using the arithmetic-geometric inequality we have

$$T_1 \leq \mu\varepsilon\|\nabla v_h\|^2 + C_\partial^2 (4\mu)^{-1} \mathrm{Pe}^2 \eta^2 \varepsilon \|v_h\|^2_{\frac{1}{2},h,\partial\Omega}.$$

For $T_2$ we have, using a Cauchy–Schwarz inequality, the definition of the Péclet number, and the stability (4.3)

$$T_2 \leq \||\beta \cdot n|^{\frac{1}{2}} v_h\|_{L^2(\partial\Omega)} \mathrm{Pe}^{\frac{1}{2}} \eta \varepsilon^{\frac{1}{2}} \|\varphi_r\|_{\frac{1}{2},h,\partial\Omega} \leq C_\partial \||\beta \cdot n|^{\frac{1}{2}} v_h\|_{L^2(\partial\Omega)} \eta \varepsilon^{\frac{1}{2}} \|v_h\|_{\frac{1}{2},h,\partial\Omega}.$$

We apply the arithmetic-geometric inequality once again to conclude. ∎

*Proof of Proposition* 7.1. The inf-sup stability (7.4) now follows by taking $w_h := v_h + \eta\varphi_r$ and proceeding as in (4.8) using (7.3) and Lemma 7.2 in the following fashion:

$$\begin{aligned}A_h(v_h, v_h + \eta\varphi_r) \geq (1 - \epsilon - \mu)\varepsilon\|\nabla v_h\|^2 &+ \left(\frac{1}{2} - \mu\right)\||\beta \cdot n|^{\frac{1}{2}} v_h\|^2_{L^2(\Omega)} \\ &+ \eta(1 - C_s^2\eta/(4\epsilon) - C_\partial^2\eta/(2\mu))\varepsilon\|v_h\|^2_{\frac{1}{2},h,\partial\Omega}.\end{aligned}$$

We may now choose $\epsilon = 1/4$ and $\mu = 1/4$ and then $\eta$ small enough so that positivity is ensured. Then

$$A_h(v_h, v_h + \eta\varphi_r) \geq C_\eta \|v_h\|^2_{1,h,\beta}.$$

We conclude as in Theorem 4.2, but now using the norm $\|\cdot\|_{1,h,\beta}$,

$$\begin{aligned}\|w_h\|_{1,h,\beta} &\leq \|v_h\|_{1,h,\beta} + \eta\|\varphi_r\|_{1,h,\beta} \leq \|v_h\|_{1,h,\beta} + \eta C\|v_h\|_{1,h,\beta} + \eta\||\beta \cdot n|^{\frac{1}{2}}\varphi_r\|_{L^2(\partial\Omega)} \\ &\leq C\|v_h\|_{1,h,\beta} + \mathrm{Pe}^{\frac{1}{2}}\eta\varepsilon^{\frac{1}{2}}\|\varphi_r\|_{1,h} \leq C_{Pe,\eta}\|v_h\|_{1,h,\beta}.\end{aligned}$$ ∎

Proceeding as in Proposition 5.2, this leads to optimal a priori estimates in the norm $\|\cdot\|_{1,h}$ for $Pe < 1$.

PROPOSITION 7.3. *Let* $u \in H^{k+1}(\Omega)$ *be the solution of* (7.1) *and* $u_h$ *the solution of* (7.2) *and assume that* $\mathrm{Pe} < 1$. *Then*

$$\|u - u_h\|_{1,h} \le Ch^k |u|_{H^{k+1}(\Omega)}.$$

*Proof.* As in the proof of Proposition 5.2 we arrive at the following representation of the discrete error:

$$c_s \|u_h - i_{\mathsf{SZ}}^k u\|_{1,h,\beta} \le \sup_{w_h \in V_h^k} \frac{A_h(u_h - i_{\mathsf{SZ}}^k u, w_h)}{\|w_h\|_{1,h,\beta}} = \sup_{w_h \in V_h^k} \frac{A_h(u - i_{\mathsf{SZ}}^k u, w_h)}{\|w_h\|_{1,h,\beta}}.$$

By the continuity of Lemma 5.1 and an integration by parts in the convective term we obtain

$$\begin{aligned} A_h(u_h - i_{\mathsf{SZ}}^k u, w_h) &\lesssim \varepsilon \|u - i_{\mathsf{SZ}}^k u\|_* \|w_h\|_{1,h} \\ &\quad + (u - i_{\mathsf{SZ}}^k u, \beta \cdot \nabla w_h)_\Omega + \left\langle \beta \cdot n (u - i_{\mathsf{SZ}}^k u), w_h \right\rangle_{\partial\Omega^+}) \\ &\lesssim \varepsilon^{1/2} (\|u - i_{\mathsf{SZ}}^k u\|_* + \mathrm{Pe} \|h^{-1}(u - i_{\mathsf{SZ}}^k u)\| + \mathrm{Pe} \|u - i_{\mathsf{SZ}}^k u\|_{\frac12, h, \partial\Omega}) \|w_h\|_{1,h,\beta}. \end{aligned}$$

As a consequence

$$\begin{aligned} \varepsilon^{1/2} \|u_h - i_{\mathsf{SZ}}^k u\|_{1,h} &\le \|u_h - i_{\mathsf{SZ}}^k u\|_{1,h,\beta} \\ &\lesssim c_s^{-1} \varepsilon^{1/2} (\|u - i_{\mathsf{SZ}}^k u\|_* + \mathrm{Pe} \|h^{-1}(u - i_{\mathsf{SZ}}^k u)\| + \mathrm{Pe} \|u - i_{\mathsf{SZ}}^k u\|_{\frac12, h, \partial\Omega}). \end{aligned}$$

The claim follows by dividing through by $\varepsilon^{1/2}$, using approximation and the assumption $\mathrm{Pe} < 1$. ∎

**7.2. Convection-dominated regime: The streamline-diffusion method.** In the convection-dominated regime, when $\mathrm{Pe} > 1$, we need to add some stabilization in order to obtain a robust scheme. We will here first consider the simple case of streamline-diffusion (SD) stabilization and assume $\sigma = 0$. In the next section the results will be extended to include the continuous interior penalty (CIP) method.

The formulation now takes the following form: find $u_h \in V_h^k$ such that

$$\begin{aligned} A_{SD}(u_h, v_h) := &(\beta \cdot \nabla u_h, v_h + \delta \beta \cdot \nabla v_h)_\Omega \\ &- \sum_K (\varepsilon \Delta u_h, \delta \beta \cdot \nabla v_h)_K - \langle \beta \cdot n\, u_h, v_h \rangle_{\partial\Omega^-} \\ &+ \varepsilon a_h(u_h, v_h) = (f, v_h + \delta \beta \cdot \nabla v_h)_\Omega \quad \forall v_h \in V_h^k, \end{aligned} \tag{7.5}$$

where $\delta = \gamma_{SD} h / |\beta|$ when $\mathrm{Pe} > 1$ and $\delta = 0$ otherwise. At high Péclet numbers, the enhanced robustness of the stabilized method allows us to work in the stronger norm $|||u_h|||_{h,\delta}$ defined by

$$|||u_h|||_{h,\delta}^2 := \|\delta^{\frac12} \beta \cdot \nabla u_h\|^2 + \frac{1}{2} \||\beta \cdot n|^{\frac12} u_h\|_{L^2(\partial\Omega)}^2 + \varepsilon \|\nabla u_h\|^2. \tag{7.6}$$

We will also use the weaker form $|||u_h|||_{h,0}^2$ defined by (7.6) with $\delta = 0$, and for the convergence analysis we introduce the norm

$$|||u|||_*^2 := \|\delta^{-\frac12} u\|^2 + \varepsilon \|h^{\frac12} \nabla u \cdot n\|_{L^2(\partial\Omega)}^2 + \sum_K \|\delta^{\frac12} \varepsilon \Delta u\|_{L^2(K)}^2 + \varepsilon \|u\|_{\frac12, h, \partial\Omega}^2 + |||u|||_{h,\delta}^2.$$

Testing the formulation (7.5) with $v_h = u_h$ yields the positivity

$$c|||u_h|||_{h,\delta}^2 \leq A_{SD}(u_h, u_h) \tag{7.7}$$

in the standard way using an elementwise inverse inequality to absorb the second order term, i.e.,

$$\sum_K (\varepsilon \Delta u_h, \delta \beta \cdot \nabla u_h)_K \leq \frac{1}{2} C_I^2 \gamma_{SD} \mathrm{Pe}^{-1/2} \|\varepsilon^{\frac{1}{2}} \nabla u_h\|^2 + \frac{1}{2} \|\delta^{\frac{1}{2}} \beta \cdot \nabla u_h\|^2.$$

Clearly for $\gamma_{SD} < 1/(C_I^2)$ stability holds for $\mathrm{Pe} > 1$.

Unfortunately the norms proposed above seem too weak to allow for optimal error estimates. Indeed, since we do not control all of $\|u_h\|_{1,h}$, for general $u \in H^2 + V_h^k$, $v_h \in V_h^k$ there does not hold $A_{SD}(u, v_h) \leq |||u|||_* |||v_h|||_{h,\delta}$, (cf. Lemma 5.1) unless an assumption on the boundary velocity such as $|\beta \cdot n| h > \varepsilon$ is made. It also appears to be difficult to obtain an inf-sup condition similar to (7.4) in the high Péclet regime.

We therefore use another technique to prove optimal convergence directly. The idea is to construct an interpolation operator $\pi_\partial u$, such that the *interpolation error* $u - \pi_\partial u$ satisfies the continuity estimate

$$A_{SD}(u - \pi_\partial u, v_h) \lesssim |||u - \pi_\partial u|||_* |||v_h|||_{h,\delta}. \tag{7.8}$$

Assume that we have an interpolation operator $\pi_\partial : H^1(\Omega) \mapsto V_h^1$ such that the following hypothesis are satisfied.

(H1) Approximation:

$$\|\pi_\partial u - u\| + h\|\nabla(\pi_\partial u - u)\| \leq C h^{k+1} |u|_{H^{k+1}(\Omega)}. \tag{7.9}$$

(H2) Normal gradient:

$$\int_{F_i} \nabla(\pi_\partial u - u) \cdot n \, \mathrm{d}s = 0, \quad i = 1, \ldots, N_P, \tag{7.10}$$

where $F_i$ are the boundary segments introduced in section 4.

Under assumptions (H1) and (H2), we may prove the optimal convergence of the SD method.

PROPOSITION 7.4. *Let $u \in H^{k+1}(\Omega)$ be the solution of* (7.1) *and $u_h$ the solution of* (7.5). *Assume that there exists $\pi_\partial u \in V_h^k$ satisfying* (H1) *and* (H2). *Then*

$$|||u - u_h|||_{h,\delta} \lesssim h^{k+\frac{1}{2}}(1 + \mathrm{Pe}^{-\frac{1}{2}})|u|_{H^{k+1}(\Omega)}.$$

*Proof.* It follows from the approximation properties of $\pi_\partial$ that

$$|||u - \pi_\partial u|||_* \lesssim \|\beta\|_\infty^{\frac{1}{2}} h^{k+\frac{1}{2}}(1 + \mathrm{Pe}^{-1/2})|u|_{H^{k+1}(\Omega)}.$$

We now need to prove the continuity (7.8). Note that

$$\begin{aligned} A_{SD}(u - \pi_\partial u, v_h) &= (\delta^{\frac{1}{2}} \beta \cdot \nabla(u - \pi_\partial u) - \delta^{-\frac{1}{2}}(u - \pi_\partial u), \delta^{\frac{1}{2}} \beta \cdot \nabla v_h) \\ &- \sum_K (\delta^{\frac{1}{2}} \varepsilon \Delta(u - \pi_\partial u), \delta^{\frac{1}{2}} \beta \cdot \nabla v_h)_K + \langle \beta \cdot n \, (u - \pi_\partial u), v_h \rangle_{\partial\Omega^+} + \varepsilon a_h(u - \pi_\partial u, v_h) \\ &\lesssim |||u - \pi_\partial u|||_* |||v_h|||_{h,\delta} + \underbrace{\varepsilon a_h(u - \pi_\partial u, v_h)}_{I_1}. \end{aligned}$$

Consider now the term $I_1$. We will prove the continuity

$$\varepsilon a_h(u - \pi_\partial u, v_h) \leq |||u - \pi_\partial u|||_* |||v_h|||_{h,\delta}. \tag{7.11}$$

Using the Cauchy–Schwarz inequality and a trace inequality, we show the continuity of the first and last terms of $I_1$:

$$\begin{aligned} I_1 &= \varepsilon(\nabla(u - \pi_\partial u), \nabla v_h)_\Omega - \varepsilon \left\langle \nabla(u - \pi_\partial u) \cdot n, v_h \right\rangle_{\partial\Omega} + \varepsilon \left\langle \nabla v_h \cdot n, (u - \pi_\partial u) \right\rangle_{\partial\Omega} \\ &\leq \varepsilon^{\frac{1}{2}} \|u - \pi_\partial u\|_* |||v_h|||_{h,0} - \varepsilon \left\langle \nabla(u - \pi_\partial u) \cdot n, v_h \right\rangle_{\partial\Omega}. \end{aligned}$$

For the remaining term we must exploit the orthogonality property (7.10) of $\pi_\partial u$ on the boundary. Indeed by decomposing the boundary integral on the $N_P$ subdomains $F_i$ we have, denoting by $\bar{v}_h^i$ the average of $v_h$ over the boundary segment $F_i$,

$$\begin{aligned} \varepsilon \left\langle \nabla(u - \pi_\partial u) \cdot n, v_h \right\rangle_{\partial\Omega} &= \varepsilon \sum_{i=1}^{N_P} \left\langle \nabla(u - \pi_\partial u) \cdot n, v_h - \bar{v}_h^i \right\rangle_{F_i} \\ &\leq \varepsilon \sum_{i=1}^{N_P} \|\nabla(u - \pi_\partial u) \cdot n\|_{L^2(F_i)} \|v_h - \bar{v}_h^i\|_{L^2(F_i)} \\ &\lesssim \varepsilon^{\frac{1}{2}} \|\nabla(u - \pi_\partial u) \cdot n\|_{-\frac{1}{2},h,\partial\Omega} \varepsilon^{\frac{1}{2}} \|\nabla v_h\| \\ &\lesssim \varepsilon^{\frac{1}{2}} \|u - \pi_\partial u\|_* |||v_h|||_{h,0} \end{aligned}$$

where we used the approximation properties of the local average and a trace inequality. Collecting the above estimates and noting that

$$\varepsilon^{\frac{1}{2}} \|u - \pi_\partial u\|_* \leq |||u - \pi_\partial u|||_*,$$

concludes the proof of (7.8).

Using the positivity (7.7), and the consistency of the method, we have, setting $e_h := u_h - \pi_\partial u$ and using that $\mathrm{Pe} > 1$,

$$\begin{aligned} |||e_h|||_{h,\delta}^2 \lesssim A_{SD}(e_h, e_h) = A_{SD}(u - \pi_\partial u, e_h) &\lesssim |||u - \pi_\partial u|||_* |||e_h|||_{h,\delta} \\ &\lesssim h^{k+\frac{1}{2}} \|\beta\|_\infty^{\frac{1}{2}} (1 + \mathrm{Pe}^{-\frac{1}{2}}) |u|_{H^{k+1}(\Omega)} |||e_h|||_{h,\delta}. \qquad \square \end{aligned}$$

We end this section by the following lemma establishing the existence of the interpolation $\pi_\partial$ with the required properties.

LEMMA 7.5. *The interpolation operator $\pi_\partial : H^1(\Omega) \mapsto V_h^1$ satisfying the properties* (H1) *and* (H2) *exists.*

*Proof.* Let $\pi_\partial u := i_{\mathrm{SZ}}^k u + \varphi_r$, where $\varphi_r$ is the function of Lemma 4.1 with the $r_j$ chosen such that

$$r_j = \overline{\nabla u \cdot n}^j - \overline{\nabla i_{\mathrm{SZ}}^k u \cdot n}^j.$$

Clearly by construction there holds

$$\begin{aligned} \int_{F_i} (\nabla \pi_\partial u \cdot n - \nabla u \cdot n) \, \mathrm{d}s &= \int_{F_i} (\nabla i_{\mathrm{SZ}}^k u \cdot n + \nabla \varphi_r \cdot n - \nabla u \cdot n) \, \mathrm{d}s \\ &= \int_{F_i} (\nabla i_{\mathrm{SZ}}^k u \cdot n + r_i - \nabla u \cdot n) \, \mathrm{d}s = 0. \end{aligned}$$

To prove the approximation results we decompose the error

$$\|u - \pi_\partial u\| \leq \|u - i_{\mathsf{SZ}}^k u\| + \|i_{\mathsf{SZ}}^k u - \pi_\partial u\| \leq C h^{k+1} |u|_{H^{k+1}(\Omega)} + \|\varphi_r\|.$$

Using local Poincaré inequalities and the stability (4.3) of $\varphi_r$ we get

$$\|\varphi_r\| \lesssim \|h \nabla \varphi_r\| \lesssim h^{\frac{3}{2}} \left( \sum_{i=1}^{N_P} \|r_i\|_{L^2(F_i)}^2 \right)^{\frac{1}{2}} = h^{\frac{3}{2}} \left( \sum_{i=1}^{N_P} \|\overline{\nabla u \cdot n}^i - \overline{\nabla i_{\mathsf{SZ}}^k u \cdot n}^i\|_{L^2(F_i)}^2 \right)^{\frac{1}{2}}.$$

Using the stability of the projection onto piecewise constants, elementwise trace inequalities, and finally approximation, we conclude

$$\begin{aligned} &\|\overline{\nabla u \cdot n}^i - \overline{\nabla i_{\mathsf{SZ}}^k u \cdot n}^i\|_{L^2(F_i)}^2 \leq \|\nabla u \cdot n - \nabla i_{\mathsf{SZ}}^k u \cdot n\|_{L^2(F_i)}^2 \\ &\leq 2 C_T^2 (h^{-1} \|\nabla (u - i_{\mathsf{SZ}}^k u)\|_{L^2(P_i)}^2 + h \sum_{K \in P_i} \|D^2 (u - i_{\mathsf{SZ}}^k u)\|_{L^2(K)}^2) \lesssim h^{2k-1} |u|_{H^{k+1}(P_i)}^2, \end{aligned}$$

where $D^2 u$ is the standard multi-index notation for all the second derivatives of $u$. We conclude that

$$\|\varphi_r\| \lesssim h^{\frac{3}{2}} \left( \sum_{i=1}^{N_P} \|\nabla u \cdot n - \nabla i_{\mathsf{SZ}}^k u \cdot n\|_{L^2(F_i)}^2 \right)^{\frac{1}{2}} \lesssim h^{k+1} |u|_{H^{k+1}(\Omega)}.$$

The estimate on the gradient is immediate by

$$\begin{aligned} \|\nabla (u - \pi_\partial u)\| &\leq \|\nabla (u - i_{\mathsf{SZ}}^k u)\| + \|\nabla (i_{\mathsf{SZ}}^k u - \pi_\partial u)\| \\ &\leq \|\nabla (u - i_{\mathsf{SZ}}^k u)\| + C_I h^{-1} \|i_{\mathsf{SZ}}^k u - \pi_\partial u\| \lesssim h^k |u|_{H^{k+1}(\Omega)}. \qquad \square \end{aligned}$$

**7.2.1. Convection-dominated regime: The continuous interior penalty method.** In this section we will sketch how the above results extend to symmetric stabilization methods assuming that $c_\sigma > 0$. To reduce technicalities we also assume that $\beta \in \mathbb{R}^2$. We give a full proof only in the case of piecewise affine finite elements. Recall that the CIP method is obtained by adding a penalty term on the jump of the gradient over element faces to the finite element formulation (7.2). The formulation can then be written as follows: find $u_h \in V_h^k$ such that

$$A_h(u_h, v_h) + J_h(u_h, v_h) = (f, v_h)_\Omega \quad \forall v_h \in V_h^k, \tag{7.12}$$

where

$$J_h(u_h, v_h) := \gamma_{CIP} \sum_{K \in \mathcal{T}_h} \sum_{F \in \partial K \setminus \partial \Omega} \int_F h_F^2 |\beta \cdot n_F| [\nabla u_h \cdot n_F][\nabla v_h \cdot n_F] \, \mathrm{d}s,$$

with $[x]$ denoting the jump of the quantity $x$ over the face $F$ and $n_F$ the normal to $F$; the orientation is arbitrary but fixed in both cases.

The analysis once again depends on the construction of a special interpolant $\pi_{CIP} u \in V_h^k$. This time $\pi_{CIP} u$ must satisfy both the optimal approximation error estimates of (7.9), the property (7.10) on the normal gradient, and the following additional design condition:

$$(u - \pi_{CIP} u, \beta \cdot \nabla v_h) \lesssim \|h^{-\frac{1}{2}} |\beta|^{\frac{1}{2}} (u - \pi_{CIP} u)\| \gamma_{CIP}^{-\frac{1}{2}} J_h(v_h, v_h)^{\frac{1}{2}} \quad \forall v_h \in V_h^k. \tag{7.13}$$

Once such an interpolant has been proved to exist, the technique of [3], combined with the analysis above, may be used to prove quasi-optimal $L^2$-convergence for $c_\sigma > 0$. Using a similarly designed interpolation operator, an inf-sup condition can be used to prove stability and error estimates in the norm $|||\cdot|||_{h,\delta}$ following [6, 5]. Here we will first prove the error estimate in the $L^2$-norm, assuming the existence of $\pi_{CIP}u$, and then show how to construct the interpolant in the special case $k = 1$.

PROPOSITION 7.6. *Assume that $\pi_{CIP}u \in V_h^k$, satisfying* (7.9), (7.10), *and* (7.13), *exists. Let $u \in H^{k+1}(\Omega)$ be the solution to* (7.1), *with $c_\sigma > 0$, and let $u_h$ be the solution to* (7.12). *Then*

$$\|u - u_h\| \lesssim c_\sigma^{-1/2}(\sigma^{\frac{1}{2}} h^{\frac{1}{2}} + |\beta|^{\frac{1}{2}}(1 + \mathrm{Pe}^{-\frac{1}{2}}))h^{k+\frac{1}{2}}|u|_{H^{k+1}(\Omega)}.$$

*Proof.* Let $e_h := u_h - \pi_{CIP}u$. There holds, with $c_\sigma > 0$,

$$c_\sigma\|e_h\|^2 + |||e_h|||_{h,0}^2 + J_h(e_h, e_h) \leq A_h(e_h, e_h) + J_h(e_h, e_h).$$

By the consistency of the method, we have

$$c_\sigma\|e_h\|^2 + |||e_h|||_{h,0}^2 + J_h(e_h, e_h) \leq A_h(u - \pi_{CIP}u, e_h) - J_h(\pi_{CIP}u, e_h).$$

Finally by the continuity (7.11), which holds thanks to property (7.10), we have

$$\begin{aligned}
(7.14)\quad & A_h(u - \pi_{CIP}u, e_h) - J_h(\pi_{CIP}u, e_h) \\
&= (\sigma(u - \pi_{CIP}u), e_h) + (u - \pi_{CIP}u, \beta \cdot \nabla e_h) - \int_{\partial\Omega} \beta \cdot n(u - \pi_{CIP}u)e_h \,\mathrm{d}s \\
&\quad + \varepsilon a_h(u - \pi_{CIP}u, e_h) + J_h(\pi_{CIP}u, e_h) \\
&\leq ((\sigma^{\frac{1}{2}} h^{\frac{1}{2}} + C|\beta|^{\frac{1}{2}}\gamma_{CIP}^{-\frac{1}{2}})\|h^{-\frac{1}{2}}(u - \pi_{CIP}u)\| + \|u - \pi_{CIP}u\|_{1,h,\beta} \\
&\quad + \varepsilon^{\frac{1}{2}}\|u - \pi_{CIP}u\|_* + J_h(\pi_{CIP}u, \pi_{CIP}u)^{\frac{1}{2}}) \\
&\quad \times(\sigma\|e_h\|^2 + |||e_h|||_{h,0}^2 + J_h(e_h, e_h))^{\frac{1}{2}},
\end{aligned}$$

and we end the proof by applying approximation estimates. □

We will now prove the existence of the interpolant $\pi_{CIP}u$ in the case of piecewise affine continuous finite element approximation.

LEMMA 7.7. *There exists a function $\pi_{CIP}u \in V_h^1$, satisfying* (7.9), (7.10), *and* (7.13).

*Proof.* We write $\pi_{CIP}u := \pi_h u + \varphi_{CIP}$, where $\pi_h u$ denotes the $L^2$-projection on $V_h^1$ and $\varphi_{CIP} \in V_h^1$ is a function defined on patches $P_i$ that satisfies the inequalities (4.2) and (4.3), but also has the property

$$\int_{P_i} \varphi_{CIP} \,\mathrm{d}x = 0, \quad i = 1, \ldots, N_P.$$

Clearly for this to hold we must modify the definition of the patches on the faces $F_i$ to include interior nodes in the domain. For simplicity we assume that any element containing a node that connects to two nodes in the boundary segment $\bar{F}_i$ (through edges that may be associated to other elements) is included in the patch $P_i$ (see Figure 2). Define two functions $w_I$ and $w_F$ on $P_i$ (also illustrated in Figure 2) such that

$$w_I := \begin{cases} 1 \text{ in all nodes } x \in \mathring{P}_i, \\ 0 \text{ in all nodes } x \in \Omega \setminus \mathring{P}_i, \end{cases} \qquad w_F := \begin{cases} 1 \text{ in all nodes } x \in \mathring{F}_i, \\ 0 \text{ in all nodes } x \in \bar{\Omega} \setminus \mathring{F}_i. \end{cases}$$

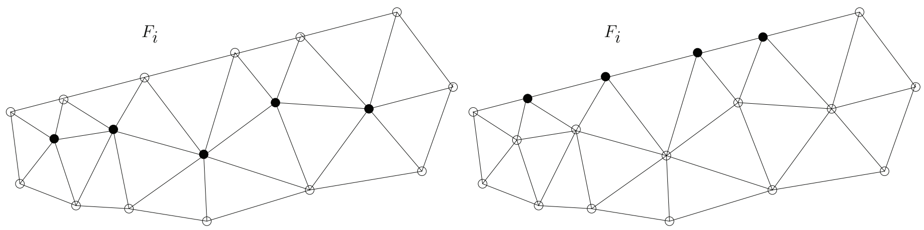


FIG. 2. *Example of a boundary patch $P_i$, with the functions $w_I$ (left) and $w_F$ (right). The functions take the value 1 in filled nodes and zero in the other nodes.*

We must now show that there exists a function $\varphi_i = aw_I + bw_F$ satisfying the two constraints

$$\int_{P_i} \varphi_i \, \mathrm{d}x = 0, \quad \overline{\nabla \varphi_i \cdot n}^i = r_i. \tag{7.15}$$

The construction of $\pi_{CIP} u$ is obtained by choosing $r_i = \overline{\nabla u \cdot n}^i - \overline{\nabla \pi_h u \cdot n}^i$ in the system (7.15) above and then defining $\varphi_{CIP}|_{P_i} := \varphi_i$.

To study $\varphi_i$, first map the patch $P_i$ to the reference patch $\hat{P}_i$, obtained by mapping $F_i$ to the unit interval using the same scaling in the direction orthogonal to $F_i$. Consider the linear system for $v := (a, b)^T \in \mathbb{R}^2$ of the form

$$\mathcal{A}v := \begin{bmatrix} \int_{\hat{P}_i} \hat{w}_I \, \mathrm{d}\hat{x} & \int_{\hat{P}_i} \hat{w}_F \, \mathrm{d}\hat{x} \\ \int_{\hat{F}_i} \nabla \hat{w}_I \cdot \hat{n} \, \mathrm{d}\hat{s} & \int_{\hat{F}_i} \nabla \hat{w}_F \cdot \hat{n} \, \mathrm{d}\hat{s} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ \int_{\hat{F}_i} \nabla (\hat{u} - \pi_h \hat{u}) \cdot \hat{n} \, \mathrm{d}\hat{s} \end{bmatrix} =: \hat{f}.$$

We must prove that the matrix $\mathcal{A}$ is invertible, but this is immediate noting that the two coefficients in the first line of the matrix are both strictly positive, whereas in the second line the coefficient in the first column is negative by construction and that in the right column is positive. The stability estimate (4.3) now follows from a scaling argument back to the physical patch $P_i$. Indeed since the matrix $\mathcal{A}$ is invertible we have

$$|v| \lesssim \sup_{w \in \mathbb{R}^2} \frac{w^T \mathcal{A} v}{|w|} = \sup_{w \in \mathbb{R}^2} \frac{w^T \hat{f}}{|w|} = |\hat{f}|.$$

By norm equivalence we have

$$\|\hat{\varphi}_i\|_{\hat{P}_i} \lesssim \|\nabla \hat{\varphi}_i\|_{\hat{P}_i} \lesssim |v| \lesssim |\hat{f}|.$$

After scaling back to the physical element we get

$$h^{-1} \|\varphi_i\|_{P_i} \lesssim \|\nabla \varphi_i\|_{P_i} \lesssim |f| \lesssim \|h^{\frac{1}{2}} \overline{\nabla (u - \pi_h u) \cdot n}^i\|_{F_i}, \tag{7.16}$$

which proves (4.3).

The approximation error estimates are proved in the same way as in Lemma 7.5. Indeed, by a decomposition similar to that of the error, we have for this case

$$\|u - \pi_{CIP} u\| \leq \|u - \pi_h u\| + \|\pi_h u - \pi_{CIP} u\| \lesssim h^2 |u|_{H^2(\Omega)} + \|\varphi_{CIP}\|,$$

and for $\varphi_{CIP}$ we may conclude using the proof of Lemma 7.5, together with (7.16).

It remains to prove the continuity (7.13). This follows from

$$(u-\pi_{CIP}u,\beta\cdot\nabla v_h)=(u-\pi_h u,\beta\cdot\nabla v_h)+\sum_{i=1}^{N_P}(\varphi_i,\beta\cdot\nabla v_h)$$
$$=(u-\pi_h u,\beta\cdot\nabla v_h-I_{CIP}\beta\cdot\nabla v_h)+\sum_{i=1}^{N_P}(\varphi_i,(\beta\cdot\nabla v_h-\pi_{0,P_i}\beta\cdot\nabla v_h)).$$

Here $I_{CIP}$ denotes a particular quasi-interpolation operator defined using averages of $\beta\cdot\nabla v_h$ in each node (see [3]), and $\pi_{0,P_i}$ denotes the projection on piecewise constant functions on $P_i$. Using norm equivalence on discrete spaces and mapping from the reference patch, we observe that

$$\|h^{\frac{1}{2}}|\beta|^{-\frac{1}{2}}(\beta\cdot\nabla v_h-I_{CIP}\beta\cdot\nabla v_h)\|^2\lesssim\gamma_{CIP}^{-1}J_h(v_h,v_h)$$

and

$$\sum_{i=1}^{N_P}\|h^{\frac{1}{2}}|\beta|^{-\frac{1}{2}}(\beta\cdot\nabla v_h-\pi_{0,P_i}\beta\cdot\nabla v_h)\|_{P_i}^2\lesssim\gamma_{CIP}^{-1}J_h(v_h,v_h).$$

The first claim was proved in [3], and the second holds since $\beta\cdot\nabla v_h$ is constant on each element. ▯

*Remark* 3. For high order elements the construction of the interpolant $\pi_{CIP}u$ is much more technical and beyond the scope of the present work. Indeed it is no longer sufficient to prove orthogonality of $\varphi_i$ against a constant on $P_i$, but it must be shown to be orthogonal to the continuous finite element space of order $k-1$ on $P_i$. On the other hand the patches $P_i$ can be chosen freely, provided $diam(P_i)=O(h)$.

**8. Numerical examples.** We study two different numerical examples, both have been computed using the package FreeFem++ [13]. First we consider a simple problem with smooth exact solution, then we consider a convection–diffusion problem and show the stabilizing effect of the Nitsche-type weak boundary condition for convection-dominated flow.

**8.1. Problem with smooth solution.** We consider (2.1) in the unit square, with $f=5\pi^2\sin(\pi x)\sin(2\pi y)$ and $g=0$. The mesh is unstructured with $N=10,20,40,80$ elements per side. The exact solution is then given by $u=\sin(\pi x)\sin(2\pi y)$. We give the convergence in both the $L^2$-norm and the $H^1$-norm for piecewise affine approximation in Table 1. The case of quadratic approximation is considered in Table 2. The order $p$ in $O(h^p)$ is given in parentheses next to the error.

We have not managed to construct an example exhibiting the suboptimal convergence order of the Nitsche method. Some cases with nonhomogeneous boundary

Table 1

*Comparison of errors between the nonsymmetric version of Nitsche's method and standard strongly imposed boundary conditions, using piecewise affine approximation on unstructured meshes.*

| $N$ | Nitsche $H^1$ | Strong $H^1$ | Nitsche $L^2$ | Strong $L^2$ |
|---|---|---|---|---|
| 10 | 7.0E-1 (—) | 6.7E-1 (—) | 2.4E-2 (—) | 2.0E-2 (—) |
| 20 | 3.5E-1 (1.0) | 3.5E-1 (0.94) | 5.5E-3 (2.1) | 5.5E-3 (1.9) |
| 40 | 1.7E-1 (1.0) | 1.7E-1 (1.0) | 1.3E-3 (2.1) | 1.3E-3 (2.1) |
| 80 | 8.2E-2 (1.1) | 8.2E-2 (1.1) | 3.3E-4 (2.0) | 3.1E-4 (2.1) |

TABLE 2

*Comparison of errors between the nonsymmetric version of Nitsche's method and standard strongly imposed boundary conditions, using piecewise quadratic approximation on unstructured meshes.*

| $N$ | Nitsche $H^1$ | Strong $H^1$ | Nitsche $L^2$ | Strong $L^2$ |
|---|---|---|---|---|
| 10 | 5.3E-2 (—) | 5.1E-2 (—) | 1.7E-3 (—) | 6.5E-4 (—) |
| 20 | 1.4E-2 (1.9) | 1.4E-2 (1.9) | 2.2E-4 (2.9) | 9.6E-5 (2.8) |
| 40 | 3.5E-3 (2.0) | 3.5E-3 (2.0) | 2.1E-5 (3.4) | 1.1E-5 (3.1) |
| 80 | 8.6E-4 (2.0) | 8.6E-4 (2.0) | 2.5E-6 (3.1) | 1.4E-6 (3.0) |

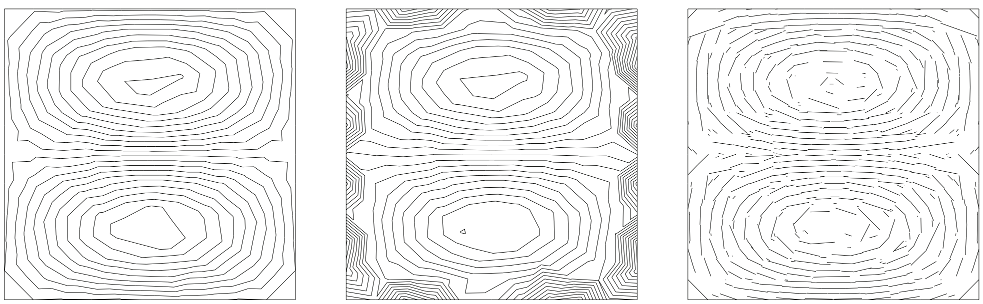

FIG. 3. *Comparison of the contourplots of the unstabilized nonsymmetric method (left), symmetric method with piecewise affine conforming approximation (middle), and symmetric method with piecewise affine nonconforming approximation (right), $N = 10$.*

conditions, not reported here, were computed both with affine and quadratic elements. They all had optimal convergence on the finer meshes. For $H^1$-conforming spaces the theoretical results do not extend to the symmetric version of Nitsche's method and stability is unlikely to hold on general meshes. Applying the symmetric method to the proposed numerical example yields a solution with clear boundary oscillations on the coarse meshes; see Figure 3. On finer meshes these oscillations vanish and the performance is similar to that of the nonsymmetric method. The solutions of the stable nonsymmetric method using piecewise affine $H^1$-conforming approximation and the symmetric method using piecewise affine nonconforming approximation are also presented for comparison. Note that although the convergence of the Nitsche method is optimal in this case, the error constant of the nonsymmetric method in the $L^2$-norm is a factor two larger than that of the strongly imposed boundary conditions for piecewise quadratic approximation. The same computations were made on structured meshes (not reported here), and this effect was slightly larger in this case, with a factor two in the affine case and four in the quadratic case. The errors in the $H^1$-norm, on the other hand, are of comparable size for the two methods.

This motivates a study of how the error depends on the penalty parameter $\gamma$ in (3.3). We therefore run a series of computations with $\gamma = 0, 10, 20, 40, 80$. In Table 3 we report the results for piecewise affine approximation and in Table 4 the results for piecewise quadratic approximation. We note that there is a visible, but negligible, effect on the error measured in the $L^2$-norm, but no effect on the error in the $H^1$-norm.

**8.2. Problem with outflow layer.** For this case we only compare the solutions qualitatively. We consider the problem with a convection term (7.1). To create an outflow layer we have chosen $f := 1$, $\beta := (0.5, 1)$, $\sigma := 0$ in $\Omega$. We discretized $\Omega$ with a

Table 3
*Study of the dependence of the accuracy on the penalty parameter, piecewise affine approximation, unstructured mesh, $N = 80$.*

| Error norm | $\gamma = 0$ | $\gamma = 10$ | $\gamma = 20$ | $\gamma = 40$ | $\gamma = 80$ |
|---|---|---|---|---|---|
| $\|u - u_h\|_{L^2}$ | 3.3E-4 | 2.9E-4 | 3.0E-4 | 3.0E-4 | 3.0E-4 |
| $\|u - u_h\|_{H^1}$ | 8.2E-2 | 8.2E-2 | 8.2E-2 | 8.2E-2 | 8.2E-2 |

Table 4
*Study of the dependence of the accuracy on the penalty parameter, piecewise quadratic approximation, unstructured mesh, $N = 40$.*

| Error norm | $\gamma = 0$ | $\gamma = 10$ | $\gamma = 20$ | $\gamma = 40$ | $\gamma = 80$ |
|---|---|---|---|---|---|
| $\|u - u_h\|_{L^2}$ | 2.1E-5 | 1.3E-5 | 1.2E-5 | 1.2E-5 | 1.2E-5 |
| $\|u - u_h\|_{H^1}$ | 3.5E-3 | 3.5E-3 | 3.5E-3 | 3.5E-3 | 3.5E-3 |

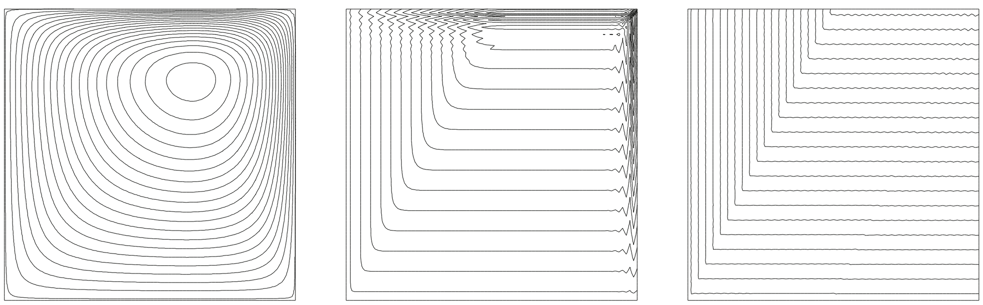

Fig. 4. *Convection–diffusion equation discretized using the nonsymmetric Nitsche-type boundary condition, no stabilization, $N = 80$, piecewise affine approximation, from left to right: $\varepsilon = 0.1$, $\varepsilon = 0.001$, $\varepsilon = 0.00001$.*

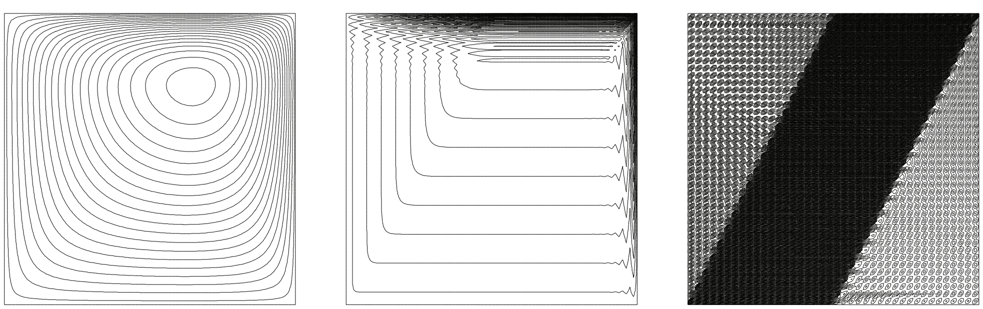

Fig. 5. *Convection–diffusion equation discretized using strongly imposed boundary condition, no stabilization, $N = 80$, piecewise affine approximation, from left to right: $\varepsilon = 0.1$, $\varepsilon = 0.001$, $\varepsilon = 0.00001$.*

structured mesh having 80 piecewise affine elements on each side. The contourplots for $\varepsilon = 0.1, 0.001, 0.00001$ are reported in Figure 4 for Nitsche's method and in Figure 5 for the strongly imposed boundary conditions. Note that no stabilization has been added in either case. This computation illustrates the strong stabilizing effect of the weakly imposed boundary condition. A theoretical explanation of this phenomenon was given in [19]. Finally we consider the effect of adding stabilization to the computation. In this case we take $N = 80$ with piecewise quadratic approximation. We report the

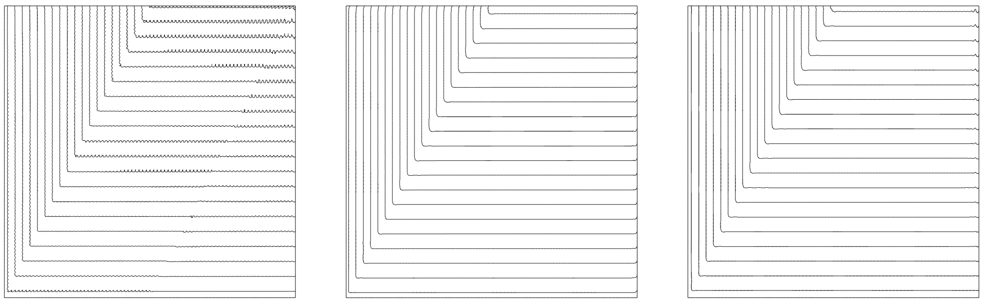

FIG. 6. *Convection–diffusion equation discretized using the nonsymmetric Nitsche-type boundary condition, $N = 80$, $\varepsilon = 0.00001$, piecewise quadratic approximation, from left to right: no stabilization, SD stabilization ($\gamma_{SD} = 0.5$), CIP stabilization ($\gamma_{CIP} = 0.005$).*

results of a computation without stabilization, with the SD method ($\gamma_{SD} = 0.2$) and with the CIP method ($\gamma_{CIP} = 0.005$) in Figure 6. Note that the stabilized methods clean up the remaining spurious oscillations in both cases.

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