Two Nonlinear Systems from Mathematical Physics

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PhD Dissertation

Dedicated to all the people that believed in me

Aknowledgments

Le savant doit ordonner;
on fait la science avec des faits comme une maison avec des pierres;
mais une accumulation de faits n'est pas plus une science qu'un tas de pierres n'est une maison.

The Scientist must set in order.
Science is built up with facts, as a house is with stones.
But a collection of facts is no more a science than a heap of stones is a house.

Jules Henri Poincaré

Finally, this is the work that I present for my PhD graduation. It was a huge effort for me, but made very light by all the support of professors, family, friends and colleagues. It was such a great pleasure to take part in the world of mathematical research, meet other students, participate to conferences, schools and attend courses. The path of the PhD is full of people, so I will try to have a short nice word for all of them.

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Introduction

This thesis is about the results obtained in [13] on the N-Vortex problem and other results on a system of Schrödinger equations. Both adressed problems comes from mathematical physics, as the title of this work suggests. This introduction is just a shortcut for a general overview on these pages.

The thesis is divided in two chapters.

In Chapter 1, we consider the Hamiltonian system

$$\kappa_k \dot{z}_k = J\nabla_{z_k} H_\Omega(z_1, z_2), \quad k = 1, 2,$$

for two point vortices $z_1, z_2 \in \Omega$ in a domain $\Omega \subset \mathbb{R}^2$, $\kappa_1, \kappa_2 \in \mathbb{R}$ with same sign. The Hamiltonian H_Ω is of the form

$$H_\Omega(z_1, z_2) = -\frac{\kappa_1\kappa_2}{\pi} \log|z_1 - z_2| - 2\kappa_1\kappa_2 g(z_1, z_2) - \kappa_1^2 h(z_1) - \kappa_2^2 h(z_2),$$

where $g : \Omega \times \Omega \to \mathbb{R}$ is the regular part of a hydrodynamic Green's function in Ω, and $h : \Omega \to \mathbb{R}$ is the Robin function: $h(z) = g(z, z)$. The system is singular and not integrable, except when Ω is a disk or an annulus. We will prove the existence of infinitely many periodic solutions with minimal period T which are a superposition of a slow motion of the center of vorticity along a level line of h and of a fast rotation of the two vortices around their center of vorticity. These vortices move in a prescribed subset $\mathcal{A} \subset \Omega$, which is an anular shaped region, whose boundary curves are star-shaped and level sets of the Robin function, with no critical points. The minimal period can be any T in an interval $I(\mathcal{A}) \subset \mathbb{R}$, depending on the subset $\mathcal{A}$. With these assumptions we can prove the following Theorem.

Theorem. *For any $T \in I(\mathcal{A})$ and any $a_0 > 0$ there exist $0 < a_1 < b_1 < a_0$ such that the Hamiltonian system describing the motion of 2 vortices has a T-periodic solution $(z_1(t), z_2(t))$ satisfying*

$$z_1(t), z_2(t) \in \mathcal{A} \quad \textit{for all } t \in \mathbb{R}, \ \textit{and} \quad |z_1(0) - z_2(0)| \in (a_1, b_1).$$

Our result can be applied in any generic bounded domain. The proofs are based on a recent higher dimensional version of the Poincaré-Birkhoff theorem due to Fonda and Ureña.

In Chapter 2, we study bifurcations of an elliptic system of the form:

$$\begin{cases} -\Delta u_i - u_i = \mu_i u_i^3 + \beta_1 \sum_{k=1,k\neq i}^n u_k^2 u_i - \gamma_1 \sum_{k=1}^m v_k^2 u_i & \text{in } \Omega \\ -\Delta v_j - v_j = \nu_j v_j^3 + \beta_2 \sum_{k=1,k\neq j}^m v_k^2 v_j - \gamma_2 \sum_{k=1}^n u_k^2 v_j & \text{in } \Omega \\ u_i = v_j = 0 \text{ on } \partial\Omega, u_i, v_j > 0 \text{ in } \Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth and bounded domain, $n, m \geq 2$, $\mu_i, \nu_j, \beta_1, \beta_2, \gamma_1, \gamma_2$ are real constants. β_1, β_2 represents the interaction between particles of the same component, while γ_1, γ_2 represents the competition with particles of the other component. γ_1, γ_2 will be positive parameters. Considering the well studied system:

$$\begin{cases} -\Delta\omega_1 - \omega_1 = -\omega_1^3 - \gamma\omega_2^2\omega_1 \\ -\Delta\omega_2 - \omega_2 = -\omega_2^3 - \gamma\omega_1^2\omega_2 \end{cases}$$

with $\omega_1, \omega_2 \in H_0^1(\Omega)$. We construct a solution branch synchronized to a positive solution (ω_1, ω_2) of the form

$$\{(\beta_1, \beta_2, \alpha_1^1\omega_1, \ldots, \alpha_1^n\omega_1, \alpha_2^1\omega_2, \ldots, \alpha_2^m\omega_2) \text{ with } \beta_1 \in I_1, \beta_2 \in I_2\} \subset \mathbb{R}^2 \times \left(H_0^1(\Omega)\right)^{n+m},$$

where I_1, I_2 are intervals, $\alpha_1^i, \alpha_2^j \in \mathbb{R}$ are positive, all depending from the parameters of the system. If we consider the values $\beta_{1,k}, \beta_{2,l}$ solutions with respect to β_1, β_2 of

$$-\frac{2}{1 + \beta_1 \sum_{i=1}^n \frac{1}{\mu_i - \beta_1}} - 1 = \lambda_{1,\gamma}^k \qquad \text{and} \qquad -\frac{2}{1 + \beta_2 \sum_{j=1}^m \frac{1}{\nu_j - \beta_2}} - 1 = \lambda_{2,\gamma}^k$$

From this branch, we find a sequence of local bifurcation values in the one dimensional case, the main result of the Chapter.

Theorem. *Suppose $N = 1$, $\mu_1 \leq \ldots \leq \mu_n$, $\nu_1 \leq \ldots \leq \nu_m$ and γ sufficiently large.*

- *If $\mu_1 > 0$ or $\mu_n < 0$, then couples of the form $(\beta_{1,k}, \beta_2)_{k=1,2,\ldots}$, $\beta_2 \neq \beta_{2,l}$ are local bifurcation values;*
- *if $\nu_1 > 0$ or $\nu_m < 0$, then couples of the form $(\beta_1, \beta_{2,l})_{l=1,2,\ldots}$, $\beta_1 \neq \beta_{1,k}$ are local bifurcation values;*
- *if $\mu_1 > 0$ and $\nu_1 > 0$, then couples of the form $(\beta_{1,k}, \beta_{2,l})_{k,l=1,2,\ldots}$ are local bifurcation values;*
- *if $\mu_n < 0$ and $\nu_m < 0$, then couples of the form $(\beta_{1,k}, \beta_{2,l})_{k,l=1,2,\ldots}$ are local bifurcation values.*

In all the other cases not covered by this Theorem, the sufficient condition for bifurcation may fail.

Contents

Chapter 1

Periodic solutions with prescribed minimal period of the 2-vortex problem in domains

1.1 Vorticity in Fluid Mechanics

The main task of Fluid Mechanics is the study of the behaviour of fluids, liquids (hydrodynamics) and gases (aerodynamics). This branch of physics has several applications like the prediction of weather patterns, the determination of the flow rate of petroleum through pipelines, the knowledge of nebulae in interstellar space. The standard process is the calculation of the fluid properties, which are temperature, density, velocity, pressure as functions of space and time. We will investigate the behaviour of fluids from a macroscopic point of view, not taking into account the single molecules. In particular we are interested in the velocity vector field of the fluid

$$v : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^3$$

which associates to every point of the fluid, contained in the domain Ω, its velocity vector. In applications usually $n = 3$, but one can also have $n = 2$, which is the case studied in this chapter. In this case we refer to the fluid as a planar one. For the sake of applications, planar means that one dimension is negligeable with respect to the other two, like for example the atmosphere. Thus we describe a mathematical model of an incompressible inviscid fluid, with normalized and constant density $\rho = 1$, under conservative body forces. The model is given by the Euler equations

$$\begin{cases} v_t + (v \cdot \nabla)v = -\nabla P & \text{in } \Omega \\ \nabla \cdot v = 0 & \text{in } \Omega, \end{cases}$$

in which P represents the pressure field, which is the only force acting on the fluid. The first equation is nothing else than Newton's law, while the second one is the incompressibility condition. From the velocity field we can pass to the vorticity ω taking the curl

$$\omega = \nabla \times v.$$

Up to the actual knowledge in this kind of equations, the mathematical theory of a genuine three-dimensional flow is still poor compared with the rather rich analysis of the two-dimensional case, to which we address our efforts. The first semplification in the two-dimensional case is that the curl of a vector field is a scalar, thus the vorticity is given by

$$\omega = \partial_1 v_2 - \partial_2 v_1.$$

Moreover we consider vorticity as distribute in a precise way, concentrating in some points $z_1, \dots, z_N$, making use of the so-called point vortex representation

$$\omega = \sum_{k=1}^{N} \kappa_k \delta_{z_k}, \tag{1.1.1}$$

where δ_{z_k} is the Dirac delta. From now on, every point $z_1, \dots, z_N$ will be adressed as a *vortex* or *point vortex*, while the real numbers $\kappa_1, \dots, \kappa_N$ are the respective *strength* or *intensity*.

Taking the curl of the Euler equation, we immediately get

$$\frac{D\omega}{Dt} = \omega_t + (v \cdot \nabla)\omega = 0,$$

which means that the vorticity is conserved.
Using the incompressibility condition in two dimensional fluids, one get that the velocity is determined by a scalar streamfunction ψ which then plays the role of the Hamiltonian, in fact

$$\dot{z}_k = v(z_k) = -i\nabla\psi(z_k) \ , \ k = 1, \dots, N. \tag{1.1.2}$$

Again, taking the curl of (1.1.2), we obtain that ψ satisfies a Poisson equation

$$-\Delta\psi = \omega \tag{1.1.3}$$

with the condition of no flow through the boundary of the domain. Thus we can use the Green's function for the laplacian to have an explicit formulation of ψ.
We just recall that the Green's function $G : \Omega^2 \to \mathbb{R}$ such that

$$\begin{cases} \Delta_x G(x, y) = \delta(y - x) & \text{for } x, y \in \Omega \\ G(x, y) = 0 & \text{for } x \in \partial\Omega, y \in \Omega, \end{cases}$$

which, for a domain $\Omega \subseteq \mathbb{R}^2$, has the shape

$$G(x, y) = -\frac{1}{2\pi} \log|x - y| - g(x, y).$$

The first term is the Green's function for the unbounded plane, while the second term $g \in \mathcal{C}^2(\Omega^2)$ is the regular part, which is harmonic in Ω. Note that if Ω is unbounded, one should impose some conditions on the behaviour of G at infinity.

In general, instead of the standard Green's function, one can use the so called *Hydrodynamic Green's functions* which, according to [28, Chapter 15], are defined as a solution G_z of the problem

$$\begin{array}{rcll} -\Delta G_z & = & \delta_z & \text{in } \Omega \\ \partial_{J\nu} G_z & = & 0 & \text{on } \partial\Omega \\ \int_{\Gamma_k} \partial_\nu G_z & = & \gamma_k & \text{for every } k \\ \int_{\partial\Omega} G_z \partial_\nu G_\zeta & = & 0 & \text{for every } z, \zeta \in \Omega \end{array}$$

where δ_z is the ususal Dirac Delta, ν is the exterior normal to $\partial\Omega$, which in the second condition is rotated by 90 degrees via the symplectic matrix J; $\partial\Omega$ is bounded by the curves Γ_k, $k = 1, \dots, K$ and the real numbers $\gamma_1, \dots, \gamma_K$ are the periods of the function, subject to $\sum \gamma_k = -1$.

For our purposes we will use the "standard" Green's function.

So ψ can be written as

$$\psi(z; z_1, \dots, z_N) = \int_\Omega G(z, w)\omega(dw) = \sum_{k=1}^{N} \kappa_k G(z, z_k).$$

In view of (1.1.2), we calculate

$$\nabla\psi(z) = \sum_{k=1}^{N} -\frac{\kappa_k}{2\pi} \frac{z - z_k}{|z - z_k|^2} - \kappa_k \nabla_x g(z, z_k).$$

Note that in this representation of $\nabla\psi$ there can be some singularities, when evaluated in the vortices $z_1, \dots, z_N$. We we will avoid the singularities by physical considerations. Consider $\psi(z_j)$, $j = 1, \dots, N$; we divide it in two kind of terms. The first ones are of the form

$$\sum_{\substack{k=1 \\ k \neq j}}^{N} -\frac{\kappa_k}{2\pi} \frac{z_j - z_k}{|z_j - z_k|^2} - \kappa_k \nabla_x g(z_k, z_j)$$

and they represent the vortex-vortex interaction; the second kind of terms represents the vortex-boundary interaction and it contains a singularity, which is not reasonable from the physical point of view, because a single vortex, when it is very far away from the boundary, is almost at rest. So we simply skip it and it remains just

$$-\kappa_j \nabla_x g(z_j, z_j).$$

In the future development of this system, it will be useful to refer to the function $h(z) = g(z, z)$ as the Robin function. Using the shape of $\nabla\psi$ evaluated in the vortices and plugged in (1.1.2)

$$\dot{z}_j = -i\nabla\psi(z_j) = \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{i\kappa_k}{2\pi} \frac{z_j - z_k}{|z_j - z_k|^2} + i\kappa_k \nabla_x g(z_j, z_k) + \frac{i\kappa_j}{2} \nabla h(z_j).$$

What we gave here was just a short derivation of the model in order to understand a little better where does it comes from. For sure one can spend a lot more time in explaining this motivation, but it is not the aim of this work. For the interested reader, we refer to [38, 39, 42, 48] for more detail about the modern presentations of the point vortex method.

1.2 N-Vortex problem

In this section we formulate the problem from the mathematical point of view and discuss previous works on this problem. As I have seen, sometimes in the litterature we find the problem formulated with different constants. In this chapter we state exactly the hamiltonian that we use and we will stay coherent throughout the whole chapter.

Before starting, it is worthwhile to mention that systems like (1.2.1) also arise in other contexts from mathematical physics, e.g. in models from superconductivity (Ginzburg-Landau-Schrödinger equation), or in equations modeling the dynamics of a magnetic vortex system in a thin ferromagnetic film (Landau-Lifshitz-Gilbert equation); see [10] for references to the literature.

We formulate the problem for a region in the plane, but the domain can also be a subset of a two-dimensional surface.

Given a domain $\Omega \subset \mathbb{R}^2$, the dynamics of N point vortices $z_1(t), \dots, z_N(t) \in \Omega$ with vortex strengths $\kappa_1, \dots, \kappa_N \in \mathbb{R}$ is described by a Hamiltonian system

$$\kappa_k \dot{z}_k = J\nabla_{z_k} H_\Omega(z_1, \dots, z_N), \quad k = 1, \dots, N; \tag{1.2.1}$$

here $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the standard symplectic matrix in $\mathbb{R}^2$. The Hamiltonian is of the form

$$H_\Omega(z_1, \dots, z_N) = -\frac{1}{2\pi} \sum_{\substack{j,k=1 \\ j\neq k}}^{N} \kappa_j \kappa_k \log|z_j - z_k| - F(z_1, \dots, z_N)$$

where $F : \Omega^N \to \mathbb{R}$ is a function of class $\mathcal{C}^2$. The Hamiltonian is defined on the configuration space

$$\mathcal{F}_N\Omega = \left\{(z_1, \dots, z_N) \in \Omega^N : z_j \neq z_k \text{ for } j \neq k\right\}.$$

Observe that the system is singular, but of a very different type than the singular second order equations from celestial mechanics.

Classically the point vortex equations (1.2.1) were first derived by Kirchhoff in [33], who considered the case where $\Omega = \mathbb{R}^2$ is the whole plane. In this case the function F in the Hamiltonian is identically zero. On the other hand, when $\Omega \neq \mathbb{R}^2$, one has to take account of the boundaries of the domain which leads to

$$F(z_1, \dots, z_N) = \sum_{j,k=1}^{N} \kappa_j \kappa_k g(z_j, z_k)$$

where $g : \Omega \times \Omega \to \mathbb{R}$ is the regular part of a hydrodynamic Green's function in Ω. An important role plays the Robin function $h : \Omega \to \mathbb{R}$ defined by $h(z) = g(z, z)$. In fact, a single vortex $z(t) \in \Omega$ moves along level lines of h according to the Hamiltonian system $\dot{z} = \kappa J\nabla h(z)$. This goes back to work of Routh [46] and Lin [36, 37]. Except in a few special cases, the Hamiltonian H_Ω is not explicitly known, it is not bounded from above or below, its level sets are not compact, and the system (1.2.1) is not integrable.

Many authors worked on this problem, mostly in the case $\Omega = \mathbb{R}^2$ with F=0, see [3] for a review about the system in the plane.

In fact one can easily note that the Hamiltonian in the plane is not the only first integral, but there are three conserved quantities due to the invariance of the system under translations and rotations. Thus by Liouville's Theorem of classical mechanics one immediately infer the integrability of the N-vortex problem for $N \leq 3$. When passing to the four vortex problem one can make a "restricted four-vortex problem" making little additional assumptions in order to consider one vortex as a neutral one, in analogy with the restricted three-body problem.

Many results in the plane are just a transposition of results for the N-body problem into this close context.

In the presence of boundaries much less is known, except in the case of special domains like the half plane or a radially symmetric domain, i.e. disk or annulus, when the Green's function is explicitly known. In the case of two vortices and $\kappa_1\kappa_2 < 0$ the Hamiltonian is bounded from above and satisfies $H_\Omega(z_1, z_2) \to -\infty$ as $z = (z_1, z_2) \to \partial\mathcal{F}_N\Omega$. Consequently all level surfaces of H_Ω are compact, and standard results about Hamiltonian systems apply. In particular, by a result of Struwe [55] almost every level surface contains periodic solutions. Another simple setting is the case of Ω being radially symmetric and $N = 2$ whence the system (1.2.1) is integrable and can be analyzed in detail. For Ω being a disk this has been done in [23].

Except in the above mentioned special cases even the existence of equilibrium solutions of (1.2.1) is difficult to prove; see [11, 12]. The problem of finding periodic solutions in a general domain has only recently been addressed in the papers [6, 7, 10] where several one parameter families of periodic solutions of the general N-vortex problem (1.2.1) have been found. These solutions rotate around their center of vorticity, which is situated near a stable critical point of the Robin function h. The periods tend to zero as the solutions approach the critical point of h. Recall that $h(z) \to \infty$ as $z \to \partial\Omega$, hence h always has a minimum. It may have arbitrarily many critical points. For a generic domain all critical points are non-degenerate (see [41]), hence in this case the results from [6, 7, 10] produce many one-parameter families of periodic solutions. Moreover, these solutions lie on global continua that are obtained via an equivariant degree theory for gradient maps. A different type of periodic solutions has been discovered in [8] on a simply connected domain Ω. There the solutions are choreographies where the vortices move near the boundary $\partial\Omega$ almost following a level line $h^{-1}(c)$ with $c \gg 1$.

In the present chapter we consider (1.2.1) in a domain $\Omega \subsetneq \mathbb{R}^2$. We find a new type of solutions that are not (necessarily) located near an equilibrium of h but lie in a prescribed annular shaped region $\mathcal{A}$ whose boundary curves are level lines of h. We require assumptions on $\mathcal{A}$ but no further assumptions on Ω, in particular we need not be close to an integrable setting. We find an interval $I = I(\mathcal{A}) \subset \mathbb{R}$ such that for every $T \in I$ the system has infinitely many periodic solutions in $\mathcal{A}$ with minimal period T. The solutions that we obtain are essentially superpositions of a slow motion of the center of vorticity along some level line $h^{-1}(c)$ of h, and of a fast rotation of the two vortices around their center of vorticity. This will be described in detail. These solutions are of a very different nature from those obtained in [6,7,10]. We also give several classes of domains Ω for which one can find such regions $\mathcal{A}$. In particular we can find $\mathcal{A}$ in any generic bounded domain. Our proofs are based on a recent generalization of the Poincaré-Birkhoff theorem due to Fonda-Ureña [30].

The chapter is organized as follows. In Section 1.3 we state and discuss our results about the existence and shape of periodic solutions of (1.2.1). In Section 1.4 we state the generalized Poincaré-Birkhoff theorem, [30, Theorem 1.2], which will be useful to prove our result. In Section 1.5 we prove the main Theorem 1.3.2 about the existence of a periodic solution. This requires the computation of certain rotation numbers which will be done in Section 1.6. In Section 1.7 we prove the various consequences of Theorem 1.3.2. In the last Section 1.8 we will discuss remarks and open problems.

1.3 Statement of results

We consider the Hamiltonian system

$$\kappa_k \dot{z}_k = J\nabla_{z_k} H_\Omega(z_1, z_2), \quad k = 1, 2, \tag{1.3.1}$$

on a domain $\Omega \subset \mathbb{R}^2$ with Hamilton function

$$H_\Omega(z_1, z_2) = -\frac{1}{\pi}\kappa_1\kappa_2 \log|z_1 - z_2| - 2\kappa_1\kappa_2 g(z_1, z_2) - \kappa_1^2 h(z_1) - \kappa_2^2 h(z_2)$$

where $\kappa_1, \kappa_2 \in \mathbb{R}$ with $\kappa_1\kappa_2 > 0$, $g : \Omega \times \Omega \to \mathbb{R} \in C^2$ is any symmetric function and $h : \Omega \to \mathbb{R}$ is the function: $h(z) = g(z, z)$. Thus the result will be valid in a slightly more general situation. For simplicity we assume that Ω satisfies the uniform exterior ball condition, which implies that the flow associated to (1.3.1) is defined for all $t \in \mathbb{R}$; see Proposition 1.5.1.

If $C \subset h^{-1}(a)$ is a compact connected component of $h^{-1}(a)$ not containing a critical point of h then the Hamiltonian system

$$\dot{z} = -(\kappa_1 + \kappa_2)J\nabla h(z) \tag{1.3.2}$$

has a periodic solution with trajectory C. Let $T(C)$ be the minimal period of this solution. Observe that system (1.3.2) describes the motion of one vortex in Ω with strength $\kappa = \kappa_1 + \kappa_2$.

We need one geometric assumption on h.

Assumption 1.3.1. *There exists an open bounded annular shaped region $\mathcal{A} \subset \Omega$ bounded by two closed curves Γ_1, Γ_2, each Γ_k being strictly star-shaped with respect to a point $z_0 \in \mathbb{R}^2$, and each being a connected component of some level set of h. Moreover h does not have a critical point in $\partial\mathcal{A} = \Gamma_1 \cup \Gamma_2$.*

Now we can state our main result.

Theorem 1.3.2. *Suppose that Assumption 1.3.1 holds and that $T(\Gamma_1) \neq T(\Gamma_2)$. Let $I = I(\mathcal{A}) \subset \mathbb{R}$ be the open interval with end points $T(\Gamma_1), T(\Gamma_2)$. Then for any $T \in I$ and any $a_0 > 0$ there exist $0 < a_1 < b_1 < a_0$ such that system (1.3.1) has a T-periodic solution satisfying*

$$z_1(t), z_2(t) \in \mathcal{A} \quad \text{for all } t \in \mathbb{R}, \text{ and } \quad |z_1(0) - z_2(0)| \in (a_1, b_1). \tag{1.3.3}$$

As a consequence we immediately obtain the existence of infinitely many T-periodic solutions.

Corollary 1.3.3. *Under the assumptions of Theorem 1.3.2, for every $T \in I$ there exists a sequence $z^{(n)}(t)$ of T-periodic solutions with trajectories in $\mathcal{A}$ and satisfying $z_1^{(n)}(0) - z_2^{(n)}(0) \to 0$ as $n \to \infty$.*

We can also describe the shape of the solutions of Theorem 1.3.2 in the limit $a_0 \to 0$.

Theorem 1.3.4. *Let $z^{(n)}(t)$ be a sequence of solutions of (1.3.1) satisfying $z_1^{(n)}(0), z_2^{(n)}(0) \to C_0 \in \Omega$ and such that the solution of*

$$\dot{C}(t) = -(\kappa_1 + \kappa_2) J \nabla h\big(C(t)\big), \quad C(0) = C_0, \tag{1.3.4}$$

is non-stationary periodic. Then the following holds.

a) The center of vorticity $C^{(n)}(t) := \dfrac{\kappa_1 z_1^{(n)}(t) + \kappa_2 z_2^{(n)}(t)}{\kappa_1 + \kappa_2}$ converges as $n \to \infty$ uniformly in t towards the solution $C(t)$ of (1.3.4). Setting $\Gamma_0 := \{C(t) : t \in \mathbb{R}\}$ the minimal period of $C^{(n)}(t)$ converges towards $T(\Gamma_0)$ as $n \to \infty$.

b) Consider the difference $D^{(n)}(t) := z_1^{(n)}(t) - z_2^{(n)}(t) = \rho^{(n)}(t)(\cos\theta^{(n)}(t), \sin\theta^{(n)}(t))$ in polar coordinates and set $d_n = \big|z_1^{(n)}(0) - z_2^{(n)}(0)\big|$. Then the angular velocity $\dot{\theta}^{(n)}$ satisfies

$$d_n^2 \dot{\theta}^{(n)}(t) = \frac{\kappa_1 \kappa_2}{\pi} + o(1) \quad \text{as } n \to \infty \qquad \text{uniformly in } t \in \mathbb{R}.$$

Remark 1.3.5. *a) This result can be interpreted as follows, using the notation of Theorem 1.3.4. In the limit $n \to \infty$*

$$z_1^{(n)}(t) = C^{(n)}(t) + K_1 D^{(n)}(t) \quad \text{and} \quad z_2^{(n)}(t) = C^{(n)}(t) - K_2 D^{(n)}(t)$$

with K_1, K_2 positive constants, the solutions are superpositions of a slow motion of the center of vorticity along a level line of h with minimal period approaching $T(\Gamma_0)$, and of a fast rotation of the two vortices around their center of vorticity. The angular velocity of the two vortices around their center of vorticity is asymptotic to $\frac{\kappa_1 \kappa_2}{d_n^2 \pi}$ as $d_n \to 0$ where d_n is the distance of the initial positions of the two vortices. The rotation number of $z_1^{(n)}(t) - z_2^{(n)}(t)$ in $[0, T]$ is asymptotic to $(\kappa_1 \kappa_2) \frac{T}{\pi d_n^2}$ and tends to infinity as $d_n \to 0$.

b) Suppose $C_0 \in \mathcal{A}$ is such that the solution of (1.3.4) is periodic with minimal period $T(\Gamma_0) \in I(\mathcal{A})$ where Γ_0 is the trajectory of (1.3.4) as in Theorem 1.3.4 a). Then we obtain solutions with minimal period T near $T(\Gamma_0)$

c) If the solution of (1.3.4) is not periodic then the behavior of $z^{(n)}(t)$ as $n \to \infty$ can be very different from the one described in Theorem 1.3.4. Of course, if $C_0 \in \mathcal{A}$ and if h does not have a critical point in $\mathcal{A}$ then the solution of (1.3.4) is periodic.

d) Suppose that for some $c_0 \in \mathbb{R}$ the level set $h^{-1}(c_0)$ contains a connected component $\Gamma(c_0) \subset h^{-1}(c_0)$ which is strictly star-shaped with respect to some $z_0 \in \mathbb{R}^2$, and which does not contain a critical point of h. Then for $c \in [c_0 - \delta, c_0 + \delta]$ close to c_0 there exists such a component $\Gamma(c) \subset h^{-1}(c)$ close to $\Gamma(c_0)$. Hence assumption 1.3.1 holds for

$\mathcal{A} = \bigcup_{c\in(a,b)} \Gamma(c)$ *for any* $c_0 - \delta \le a < b \le c_0 + \delta$. *Below we shall produce several examples of this kind.*

e) The assumption that Ω *satisfies the uniform exterior ball condition can be dropped. We stayed with the explicit setting of vortex dynamics because we use results from [28] that we would otherwise have to reprove in the more general setting. More precisely, we would need a substitute for Proposition 1.5.1 below. The full strength of this proposition is not necessary, however. We will say something more at the end of the chapter.*

We shall now present several examples where the assumptions of Theorem 1.3.2 can be verified. Let us begin with the case of a bounded convex domain Ω. Clearly the uniform exterior ball condition is automatically satisfied for convex domains. It is well known that the Robin function $h : \Omega \to \mathbb{R}$ is strictly convex and that it has a unique non-degenerate minimum (see [17]). Moreover $h(z) \to \infty$ as $z \to \partial\Omega$. We may assume without loss of generality that $0 \in \Omega$ and that the minimum of h is at the origin. We set $m := h(0) = \min h$. Obviously the level sets $h^{-1}(c)$ with $c > m$ are connected and strictly star-shaped with respect to the origin. For $c > m$ we may therefore define $T_c = T\left(h^{-1}(c)\right)$ to be the minimal period of the solution of (1.3.2) with trajectory $h^{-1}(c)$. The following lemma shows that the assumptions of Theorem 1.3.2 are satisfied for $\mathcal{A} = \mathcal{A}(a,b) = \{z \in \Omega : a \le h(z) \le b\}$, any $m < a < b < \infty$; the boundary of $\mathcal{A}$ consists of the two curves $\Gamma_1 = h^{-1}(a)$ and $\Gamma_2 = h^{-1}(b)$.

Lemma 1.3.6. *For a bounded convex domain* Ω *the function* $(m,\infty) \to \mathbb{R}$, $c \mapsto T_c$, *defined above is strictly decreasing with* $T_m := \lim_{c\to m} T_c = \dfrac{2\pi}{|\kappa_1+\kappa_2|\sqrt{\det h''(0)}}$ *and* $T_c \to 0$ *as* $c \to \infty$.

The lemma will be proved in Section 1.7 below. As a consequence of this lemma we can apply Theorem 1.3.2 in an arbitrary bounded convex domain for any $\mathcal{A} = \mathcal{A}(a,b)$:

Corollary 1.3.7. *For all* $m < a < b < \infty$, *for every* $T \in (T_b, T_a)$ *and for every* $a_0 > 0$ *there exist* $0 < a_1 < b_1 < a_0$ *such that system* (1.3.1) *has a* T*-periodic solution satisfying*

$$z_1(t), z_2(t) \in \mathcal{A}(a,b) \quad \textit{and} \quad |z_1(0) - z_2(0)| \in (a_1, b_1).$$

In particular there exist infinitely many T*-periodic solutions of* (1.3.1) *in* $\mathcal{A}(a,b)$.

Now we get back to a general domain Ω. Here we obtain solutions near a non-degenerate local minimum.

Corollary 1.3.8. *Let* z_0 *be a non-degenerate local minimum of* h *and set* $m := h(z_0)$, $T_m := \frac{2\pi}{|\kappa_1+\kappa_2|\sqrt{\det h''(0)}}$. *Then for any neighborhood* U *of* z_0 *there exists* $T(U) < T_m$ *such that for any* $T(U) < T < T_m$ *and for every* $a_0 > 0$ *there exist* $0 < a_1 < b_1 < a_0$ *such that system* (1.3.1) *has a* T*-periodic solution satisfying*

$$z_1(t), z_2(t) \in U \quad \textit{and} \quad |z_1(0) - z_2(0)| \in (a_1, b_1).$$

There exist infinitely many periodic solutions of (1.3.1) *with minimal period* T *and with trajectory in* U.

Remark 1.3.9. *a) Since, in the case of the N-Vortex problem, the Robin function satisfies $h(z) \to \infty$ as $z \to \partial\Omega$ in a bounded domain there always exists a minimum. It is not difficult to produce examples of domains so that the associated Robin function has many local minima. Moreover, for a generic domain all critical points are non-degenerate; see [41]. Therefore Corollary 1.3.8 applies to generic domains.*

b) Corollary 1.3.8 in particular yields solutions $z^{(n)}(t)$ approaching the local minimum z_0 of h, i. e. $z_k^{(n)}(t) \to z_0$ as $n \to \infty$, $k = 1, 2$. The minimal periods of these solutions converge towards $T_m = \frac{2\pi}{|\kappa_1+\kappa_2|\sqrt{\det h''(0)}}$. In [6, 7, 10] the authors also obtained periodic solutions of the N-Vortex problem converging towards z_0. More precisely, they produced a family of T_r-periodic solutions $z^{(r)}(t)$, parameterized over $r \in (0, r_0)$ with $z_k^{(r)}(t) \to z_0$ and $T_r \to 0$ as $r \to 0$. Therefore these solutions are different from those obtained in the present paper. Also the method of proof is very different. In [6, 7, 10] variational methods or degree methods were used whereas we apply a multidimensional version of the Poincaré-Birkhoff theorem. Therefore here we do not obtain continua of periodic solutions. Instead we obtain infinitely many periodic solutions with prescribed period.

In our last corollary we consider the case when $\partial\Omega$ has a component that is strictly star-shaped.

Corollary 1.3.10. *Suppose $\partial\Omega$ has a compact component Γ_0 that is of class $\mathcal{C}^2$ and strictly star-shaped with respect to some point $z_0 \in \mathbb{R}^2$. Then for any neighborhood U of Γ_0 there exists $T(U) > 0$ such that for any $T < T(U)$ and for any $a_0 > 0$ there exist $0 < a_1 < b_1 < a_0$ such that system* (1.3.1) *has a T-periodic solution satisfying*

$$z_1(t), z_2(t) \in U \quad \text{and} \quad |z_1(0) - z_2(0)| \in (a_1, b_1).$$

In particular there exist infinitely many periodic solutions of (1.3.1) *with minimal period T and with trajectory in U.*

Remark 1.3.11. *In [8] the authors also obtain periodic solutions of the N-Vortex problem near the boundary. There Ω has to be bounded and simply connected, hence $\partial\Omega$ consists of just one (connected) curve. On the other hand it is not required that Ω is star-shaped, and the authors could deal with $N \geq 2$ vortices. For $T > 0$ small they obtain T-periodic solutions where the vortices $z_1, \dots, z_N$ all follow the same trajectory $\Gamma = \{z_1(t) : t \in \mathbb{R}\}$ with a time shift: $z_k(t) = z_1(t + \frac{(k-1)T}{N})$. Moreover for $T \to 0$ the trajectory Γ approaches $\partial\Omega$. These solutions are very different from those obtained in Corollary 1.3.10, however.*

1.4 The generalized Poincaré-Birkhoff theorem

A classical tool to find periodic solutions is the Poincaré-Birkhoff fixed point theorem, also called Poincaré's last geometric theorem. It states the existence of at least two fixed points for area-preserving homeomorphisms of the planar annulus keeping both boundary circles invariant and twisting them in opposite directions. This theorem was conjectured by Poincaré before his death in 1912. He proved the theorem in some special cases but also provided two examples of applications in Dynamics, namely the search of closed geodesic lines on a convex surface, and the study of periodic solutions in the restricted three body problem. The full proof of the theorem is due to Birkhoff, who was also motivated by its applications to the search of periodic solutions of conservative dynamical systems. Birkhoff himself was conscious of the importance of this problem and its generalizations to higher dimensions.

Switching to Hamiltonian system, we search for fixed points of the so called Poincaré time map. We will use the result stated in [30], a multi-dimensional version of the Poincar-Birkhoff Theorem, which we state here for the sake of completeness. If I_N is the $N \times N$ identity matrix, define the standard symplectic matrix

$$J_N = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}.$$

Given $z = (x, y) \in \mathbb{R}^{2N}$, consider the hamiltonian system

$$\dot{z} = J_N \nabla H(t, z) \tag{1.4.1}$$

where $H : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is an admissible Hamiltonian, i. e. it is a continuous function, T-periodic in its first variable, with a continuously defined gradient with respect to z.
For each $i = 1, \ldots, N$, two planar strictly star-shaped Jordan curves around the origin $\Gamma_1^i, \Gamma_2^i \subset \mathbb{R}^2$ are given, such that

$$0 \in \mathcal{D}(\Gamma_1^i) \subseteq \overline{\mathcal{D}(\Gamma_1^i)} \subseteq \mathcal{D}(\Gamma_2^i),$$

where $\mathcal{D}(\Gamma)$ stands for the planar open bounded region delimited by the Jordan curve Γ. Consider the anular region

$$\mathcal{A} = \left[\overline{\mathcal{D}(\Gamma_2^1)} \setminus \mathcal{D}(\Gamma_1^1)\right] \times \ldots \times \left[\overline{\mathcal{D}(\Gamma_2^N)} \setminus \mathcal{D}(\Gamma_1^N)\right] \subseteq \mathbb{R}^{2N}.$$

Assuming that every solution $z(t) = (z_1(t), \ldots, z_N(t))$ of (1.4.1) starting from the initial point $z(0) \in \mathcal{A}$ is defined for $t \in [0, T]$ and satisfies

$$z_i(t) \neq 0, \text{ for every } t \in [0, T] \text{ and } i = 1, \ldots, N.$$

The fact that we never reach the singularity for the polar coordinates allows us to choose continuous determinations of the argument of $z_i(t)$ along the solution curves and we denote the rotation numbers by

$$\mathrm{Rot}(z_i(t); [0, T]) = \frac{\arg(z_i(T)) - \arg(z_i(0))}{2\pi}.$$

We are not supposing the curves $z_i(t)$ to be closed, so the rotation numbers are in general not integers, but they can assume any real value as we choose a different initial data in $\mathcal{A}$. Suppose that there exist integer numbers $\nu_1, \ldots, \nu_N \in \mathbb{Z}$ such that for $i = 1, \ldots, N$ either

$$\mathrm{Rot}(z_i(t); [0, T]) \begin{cases} > \nu_i, & \text{if } z_0 \in \Gamma_1^i \\ < \nu_i, & \text{if } z_0 \in \Gamma_2^i, \end{cases}$$

or

$$\mathrm{Rot}(z_i(t); [0, T]) \begin{cases} < \nu_i, & \text{if } z_0 \in \Gamma_1^i \\ > \nu_i, & \text{if } z_0 \in \Gamma_2^i. \end{cases}$$

Theorem 1.4.1 (Fonda - Ureña). *Let the Hamiltonian function $H : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ be admissible and all the assumptions above be satisfied. Then system (1.4.1) has at least $N+1$ distinct T-periodic solutions $z^{(0)}, \dots, z^{(N)}$ with $z^{(k)}(0) \in \mathcal{A}$ and such that*

$$\mathrm{Rot}(z_i^{(k)}(t); [0,T]) = \nu_i, \text{ for } i = 1, \dots, N$$

for every $k = 0, \dots, N$

We are not going to prove here this theorem, but we just want to say what do we mean by distinct in the statement. For the proof, it is needed to use a time dependent transformation to polar coordinates (ρ_i, θ_i)

$$x_i = \sqrt{2\rho_i} \cos\left(\theta_i - \frac{2\pi}{T}\nu_i t\right), \; y_i = -\sqrt{2\rho_i} \sin\left(\theta_i - \frac{2\pi}{T}\nu_i t\right)$$

for $i = 1, \dots, N$. Thus in the case of a time dependent hamiltonian $\tilde{H}(t, \rho, \theta)$ distinct means that different periodic solutions are not obtained by shifting one of the θ-variables by 2π.

In our case, the N-Vortex problem, the Hamiltonian is autonomous. Thus, two solutions can be distinct for the theorem, but actually coincident: it is just necessary to take for the second solution initial data equals to the first solution at some point $\bar{t} > 0$. So, for us, this notion of distinctness is not meaningful, that's why in our theorem we state the existence of just one periodic solution.

1.5 Proof of Theorem 1.3.2

We begin with a few known facts about the 2-vortex problem. The following result concerns the global existence of solutions and some continuous dependence of the length of the difference on its initial value. This is a consequence of [28, Chapter 15].

Proposition 1.5.1. *Consider* (1.2.1) *for $N = 2$ and suppose that the domain Ω satisfies the uniform exterior ball condition. Then the following hold:*

a) All solutions exist for all times $t \in \mathbb{R}$.

b) There exists a constant C_Ω such that $|z_1(t) - z_2(t)| \leq C_\Omega |z_1(0) - z_2(0)|$ for all solutions and all $t \in \mathbb{R}$.

Remark 1.5.2. *Proposition 1.5.1 has been proved in [28] for g being the regular part of a hydrodynamic Green's function and h the Robin function. It holds for much more general classes of functions g and associated $h(z) = g(z,z)$. In fact, for our purpose we do not even need the full strength of Proposition 1.5.1, and we can deal with very general $\mathcal{C}^2$ maps $g : \mathcal{F}_2(\Omega) \to \mathbb{R}$ in H_Ω. We do need that g is symmetric and that $h(z) = g(z,z)$. We will talk about these generalizations at the end of the chapter.*

For the proof of Theorem 1.3.2 we may assume that the center of the star shaped curves Γ_1, Γ_2 is in the origin: $z_0 = 0$. We may also assume $T(\Gamma_1) < T(\Gamma_2)$. From now on we fix $T \in I = (T(\Gamma_1), T(\Gamma_2))$. The following lemma is an immediate consequence of the assumptions of Theorem 1.3.2.

Lemma 1.5.3. *There exists an open annular shaped region $\mathcal{A}' \subset \Omega$ with the following properties.*

(i) $\mathcal{A}'$ is compactly contained in $\mathcal{A}$: $\overline{\mathcal{A}'} \subset \mathcal{A}$.

(ii) The boundary of $\mathcal{A}'$ consists of two closed curves Γ'_1, Γ'_2 that are strictly star-shaped with respect to $z_0 = 0$, and that are components of level sets of h. Moreover, h does not have a critical point in $\partial\mathcal{A}' = \Gamma'_1 \cup \Gamma'_2$.

(iii) $T(\Gamma'_1) < T(\Gamma'_2)$ where $T(\Gamma'_k)$ denotes the minimal period of the solution of (1.3.2) *with trajectory Γ'_k. Moreover $T \in (T(\Gamma'_1), T(\Gamma'_2))$.*

Proof. It is sufficient to take the two curves Γ'_1 close to Γ_1 and Γ'_2 close to Γ_2 which are level curves of h in such a way that:

- the periods $T(\Gamma'_1), T(\Gamma'_2)$ preserve the inequality between $T(\Gamma_1), T(\Gamma_2)$ which, according to our simplifying assumption is
$$T(\Gamma'_1) < T(\Gamma'_2);$$
- we have $T \in (T(\Gamma'_1), T(\Gamma'_2))$;
- the two curves Γ'_1, Γ'_2 don't contain critical points of h;
- the curves Γ'_1, Γ'_2 are star shaped.

This can be done because Γ_1, Γ_2 don't contain critical points of h. It is clear that the annulus
$$\mathcal{A}' = \mathcal{D}(\Gamma'_2) \setminus \mathcal{D}(\Gamma'_1)$$
has compact closure and it is contained in $\mathcal{A}$.

□

We need to transform a little our original system into a more fitting one. In fact, the original system (1.3.1) have also coefficients in front of $\dot{z}_k$. We want to get rid of these coefficients and use new variable, in which we have the difference $z_1 - z_2$ and the center of vorticity, up to constants.

So, first, in order to get rid of the coeefficients κ_1, κ_2 in front of $\dot{z}_1, \dot{z}_2$ and put them directly into the hamiltonian, as was noted in [23], Section 2.1 in the case of the disk, we transform $z_k = x_k + iy_k$ with

$$\begin{cases} q_k = sgn(\kappa_k)\sqrt{|\kappa_k|}x_k \\ p_k = \sqrt{|\kappa_k|}y_k \end{cases} \qquad k = 1, 2.$$

Even more, this is valid for all hamiltonian system of the shape

$$\kappa_k \dot{z}_k = -J\nabla_{z_k} H(z_1, \ldots, z_N), \qquad k = 1, \ldots, N.$$

Call $s_k = sgn(\kappa_k)$.

After these considerations, we are searching a transformation of the form

$$
\left\{
\begin{aligned}
w_1 &= a\left(\frac{s_1 q_1}{\sqrt{|\kappa_1|}} + i\frac{p_1}{\sqrt{|\kappa_1|}} - \frac{s_2 q_2}{\sqrt{|\kappa_2|}} - i\frac{p_2}{\sqrt{|\kappa_2|}}\right)\\
w_2 &= b\left(\frac{s_1 q_1}{\sqrt{|\kappa_1|}} + i\frac{p_1}{\sqrt{|\kappa_1|}}\right) + c\left(\frac{s_2 q_2}{\sqrt{|\kappa_2|}} + i\frac{p_2}{\sqrt{|\kappa_2|}}\right)
\end{aligned}
\right. \tag{1.5.1}
$$

in which the parameters a, b, c have to be found in such a way that the transformation is canonical.

Definition 1.5.4. *Given a Hamiltonian system*

$$
\left\{
\begin{aligned}
\dot{q}_k &= \frac{\partial}{\partial p_k} H(q,p)\\
\dot{p}_k &= -\frac{\partial}{\partial q_k} H(q,p)
\end{aligned}
\right.
$$

a coordinate transformation

$$Q : q_k \mapsto Q_k(q,p), \qquad P : p_k \mapsto P_k(q,p)$$

is canonical if its Jacobian $Jac = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix}$ *satisfies*

$$Jac J Jac^T = J,$$

where the matrix J is the usual symplectic matrix, as stated in the previous sections.

Calculating the Jacobian matrix of our transformation (1.5.1) and verifying the condition for canonical transformations, we get that

$$a = \frac{\sqrt{|\kappa_1 \kappa_2|}}{\sqrt{s_1|\kappa_1| + s_2|\kappa_2|}}$$

From this we need to set a condition on s_1, s_2. For simplicity we will consider the case $s_1 = s_2 = 1$, which means that the strengths are both positive. We will keep this assumption till the end of the proof of the main result. Just note that in the case of both negative strengths $s_1 = s_2 = -1$, the hamiltonian is exactly the same, just the velocity has opposite sign. Another way, one can rewrite the system in order to have positive strengths and obtain the hamiltonian with opposite sign. Hence, even in this case everything can be proved in the same way, just adjusting a sign.

So the three parameters a, b, c are given by

$$a = \frac{\sqrt{\kappa_1 \kappa_2}}{\sqrt{\kappa_1 + \kappa_2}}, \qquad b = \frac{\kappa_1}{\sqrt{\kappa_1 + \kappa_2}}, \qquad c = \frac{\kappa_2}{\sqrt{\kappa_1 + \kappa_2}}.$$

So we our transformation from z_1, z_2 to w_1, w_2 becomes

$$
\left\{
\begin{aligned}
w_1 &= \frac{\sqrt{\kappa_1 \kappa_2}}{\sqrt{\kappa_1 + \kappa_2}}(z_1 - z_2)\\
w_2 &= \frac{\kappa_1 z_1 + \kappa_2 z_2}{\sqrt{\kappa_1 + \kappa_2}}
\end{aligned}
\right. \tag{1.5.2}
$$

with inverse transformation given by

$$\begin{cases} z_1 = \dfrac{1}{\sqrt{\kappa_1+\kappa_2}}(\sqrt{\dfrac{\kappa_2}{\kappa_1}}w_1 + w_2) \\ z_2 = \dfrac{1}{\sqrt{\kappa_1+\kappa_2}}(-\sqrt{\dfrac{\kappa_1}{\kappa_2}}w_1 + w_2). \end{cases}$$

We will denote the transformation with the matrix A

$$A = \frac{1}{\sqrt{\kappa_1+\kappa_2}} \begin{pmatrix} \sqrt{\kappa_1\kappa_2} & -\sqrt{\kappa_1\kappa_2} \\ \kappa_1 & \kappa_2 \end{pmatrix}$$

So $w = Az$ and $z = A^{-1}w$. The system (1.3.1) transforms to

$$\dot{w}_k = J\nabla_{w_k} H_1(w_1, w_2) \quad \text{for } k = 1, 2, \tag{1.5.3}$$

with Hamiltonian

$$\begin{aligned} H_1(w_1, w_2) = &-\frac{\kappa_1\kappa_2}{\pi}\log|w_1| \\ &- 2\kappa_1\kappa_2 g\left(\frac{1}{\sqrt{\kappa_1+\kappa_2}}(\sqrt{\frac{\kappa_2}{\kappa_1}}w_1 + w_2), \frac{1}{\sqrt{\kappa_1+\kappa_2}}(-\sqrt{\frac{\kappa_2}{\kappa_1}}w_1 + w_2)\right) \\ &- \kappa_1^2 h\left(\frac{1}{\sqrt{\kappa_1+\kappa_2}}(\sqrt{\frac{\kappa_2}{\kappa_1}}w_1 + w_2)\right) - \kappa_2^2 h\left(\frac{1}{\sqrt{\kappa_1+\kappa_2}}(-\sqrt{\frac{\kappa_2}{\kappa_1}}w_1 + w_2)\right) \end{aligned}$$

defined on $A\mathcal{F}_2\Omega = A(\mathcal{F}_2\Omega)$. Note that $w_2 \in \sqrt{\kappa_1+\kappa_2}\Omega$ provided $|z_1 - z_2| < \operatorname{dist}(z_2, \partial\Omega)$, and that given a compact subset $K \subset \sqrt{\kappa_1+\kappa_2}\Omega$ there exists $\delta > 0$ so that $\big(B_\delta(0) \setminus \{0\}\big) \times K \subset A\mathcal{F}_2\Omega$. Here $B_\delta(0)$ denotes the closed disk around 0 with radius δ.

This new coordinates allow us to focus the singularity just on one variable, w_1, and let the other one be more regular. Thus, when taking the gradient with respect to w_2, we have a regular function and next Lemma gives us information about the behaviour when w_1 is close to 0.

Lemma 1.5.5. *The gradient of H_1 with respect to w_2 satisfies*

$$\nabla_{w_2} H_1(w) = -(\kappa_1+\kappa_2)\sqrt{\kappa_1+\kappa_2}\nabla h\left(\frac{w_2}{\sqrt{\kappa_1+\kappa_2}}\right) + Q(w),$$

with $Q(w) = o(1)$ as $w_1 \to 0$ uniformly for w_2 in compact subsets of $\sqrt{\kappa_1+\kappa_2}\Omega$.

Proof. With the notation settled before $z := A^{-1}w$, we obtain

$$\nabla_{w_2} H_1(w) = -\frac{1}{\sqrt{\kappa_1+\kappa_2}}\big(2\kappa_1\kappa_2\nabla_{z_1}g(z) + 2\kappa_1\kappa_2\nabla_{z_2}g(z) + k_1^2\nabla h(z_1) + \kappa_2^2\nabla h(z_2)\big).$$

The Taylor expansion for h near $\frac{w_2}{\sqrt{\kappa_1+\kappa_2}}$ yields

$$\nabla h(z_1) = \nabla h\left(\frac{w_2}{\sqrt{\kappa_1+\kappa_2}}\right) + o(1) \quad \text{as } w_1 \to 0,$$

and

$$\nabla h(z_2) = \nabla h\left(\frac{w_2}{\sqrt{\kappa_1+\kappa_2}}\right) + o(1) \quad \text{as } w_1 \to 0.$$

This implies

$$\kappa_1^2 \nabla h(z_1) + \kappa_2^2 \nabla h(z_2) = (\kappa_1^2+\kappa_2^2)\nabla h\left(\frac{w_2}{\sqrt{\kappa_1+\kappa_2}}\right) + o(1) \quad \text{as } w_1 \to 0.$$

Using the symmetry of $g(z_1, z_2)$ and $h(z) = g(z, z)$ we obtain analogously

$$\nabla_{z_1} g(z) + \nabla_{z_2} g(z) = \nabla h\left(\frac{w_2}{\sqrt{\kappa_1+\kappa_2}}\right) + o(1) \quad \text{as } w_1 \to 0.$$

This yields $Q(w) = o(1)$ as $w_1 \to 0$, and since all functions are of class $\mathcal{C}^2$ the convergence is uniform for w_2 in a compact subset of $\sqrt{\kappa_1+\kappa_2}\Omega$. □

Now let $W(t, w) \in A\mathcal{F}_2\Omega$ be the solution of the initial value problem for (1.5.3) with initial condition $W(0, w) = w$. Recall that it is defined for all $t \in \mathbb{R}$ by Proposition 1.5.1. The following lemma concerns $W_2(t, w)$ as $w_1 \to 0$. We use the notation

$$\mathcal{A}_2 = \sqrt{\kappa_1+\kappa_2}\mathcal{A} \qquad \text{and} \qquad \mathcal{A}_2' = \sqrt{\kappa_1+\kappa_2}\mathcal{A}'.$$

Lemma 1.5.6. *The solution $W_2(t, w)$ converges towards $Z(t, w_2)$ as $w_1 \to 0$ uniformly in $t \in [0, T]$, $w_2 \in \mathcal{A}_2'$. The function $Z(t, w_2)$ solves the initial value problem*

$$\dot{Z}(t, w_2) = -(\kappa_1+\kappa_2)\sqrt{\kappa_1+\kappa_2}J\nabla h\left(\frac{Z(t,w)}{\sqrt{\kappa_1+\kappa_2}}\right), \quad Z(0, w_2) = w_2. \tag{1.5.4}$$

Proof. Set $\varepsilon := \frac{1}{2}\operatorname{dist}(\mathcal{A}_2', \partial\mathcal{A}_2)$, choose $\delta_0 > 0$ such that $\big(B_{\delta_0}(0)\setminus\{0\}\big)\times\overline{\mathcal{A}_2} \subset A\mathcal{F}_2\Omega$ and set

$$C := \sup_{\substack{0<|w_1|\le\delta_0 \\ w_2\in\overline{\mathcal{A}_2}}} |\nabla_{w_2} H_1(w_1, w_2)|.$$

Note that $0 < C < \infty$ because $\nabla_{w_2} H_1$ is defined also for $|w_1| = 0$. Let

$$\mathcal{U}_\varepsilon(\mathcal{A}_2') = \{w \in \mathcal{A}_2 : \operatorname{dist}(w, \mathcal{A}_2') < \varepsilon\}$$

be the ε-neighborhood of $\mathcal{A}_2'$. So C is the maximum speed of W_2 and the shortest distance to get out from $\mathcal{A}_2$ starting with $w_2 \in \overline{\mathcal{U}_\varepsilon(\mathcal{A}_2')}$ is ε: this means that the W_2 should take at least more time than $\frac{\varepsilon}{C}$. Mathematically, if $W_2(t, w) \in \partial\mathcal{A}_2$ for some $t > 0$, $0 < |w_1| \le \frac{\delta_0}{C_\Omega}$ and $w_2 \in \overline{\mathcal{U}_\varepsilon(\mathcal{A}_2')}$, then $t \ge \frac{\varepsilon}{C} =: t_0$.

Here is the plan of the proof. It is divided into two steps: in the first one we prove that until this maximum time t_0 we have a good convergence of W_2 to Z; in the second step we prove that when the initial datum w_1 is close to zero, then W_2 stays in $\mathcal{A}_2$. Combining these two steps together we get the statement of the Lemma.

Step 1: If $w_1^{(n)} \to 0$ and $w_2^{(n)} \in \mathcal{U}_\varepsilon(\mathcal{A}_2')$ with $w_2^{(n)} \to w_2$, $w_2 \in \overline{\mathcal{U}_\varepsilon(\mathcal{A}_2')}$, then $W_2(t, w^{(n)}) \to Z(t, w_2)$, uniformly for $t \in [0, t_0]$.

In fact, using the equation for w_2 in integral form we have for $t \in [0, t_0]$:

$$\begin{aligned}&\big|W_2(t, w^{(n)}) - W_2(t, w^{(m)})\big| \\ &\qquad \le \big|w_2^{(n)} - w_2^{(m)}\big| + \int_0^t \big|\nabla_{w_2} H_1(W(s, w^{(n)})) - \nabla_{w_2} H_1(W(s, w^{(m)}))\big| ds.\end{aligned}$$

Note that $\Big\{W(t, w) : t \in [0, t_0],\ w \in \big(B_{\delta_0/C_\Omega}(0) \setminus \{0\}\big) \times \overline{\mathcal{U}_\varepsilon(\mathcal{A}_2')}\Big\} \subset A\mathcal{F}_2\Omega$ is a relatively compact subset in $\Omega \times \Omega$. Since $\nabla_{w_2} H_1$ is defined on $\Omega \times \Omega$ and is Lipschitz continuous on compact sets there exists $k > 0$ such that

$$\begin{aligned}&\big|W_2(t; w^{(n)}) - W_2(t; w^{(m)})\big| \\ &\quad \le \big|w_2^{(n)} - w_2^{(m)}\big| + k \int_0^t \big|W_1(s, w^{(n)}) - W_1(s, w^{(m)})\big| + \big|W_2(s, w^{(n)}) - W_2(s, w^{(m)})\big| ds \\ &\quad \le \big|w_2^{(n)} - w_2^{(m)}\big| + kC_\Omega t_0 \left(\big|w_1^{(n)}\big| + \big|w_1^{(m)}\big|\right) + k \int_0^t \big|W_2(s, w^{(n)}) - W_2(s, w^{(m)})\big| ds.\end{aligned}$$

Now Gronwall's Lemma yields for $t \in [0, t_0]$:

$$|W_2(t, w^{(n)}) - W_2(t, w^{(m)})| \le \left(\big|w_2^{(n)} - w_2^{(m)}\big| + kC_\Omega t_0 \left(|w_1^{(n)}| + |w_1^{(m)}|\right)\right) e^{kt_0}.$$

This implies that $W_2(t, w^{(n)})$ converges as $n \to \infty$ uniformly for $t \in [0, t_0]$. The limit $Z(t, w_2)$ satisfies the equation (1.5.4) because

$$\nabla_{w_2} H_1(W(t, w^{(n)})) \to -(\kappa_1 + \kappa_2)\sqrt{\kappa_1 + \kappa_2} \nabla h \left(\frac{Z(t, w_2)}{\sqrt{\kappa_1 + \kappa_2}}\right) \quad \text{as } n \to \infty;$$

see Lemma 1.5.5. This proves Step 1.

Step 2: There exists δ_1 with $0 < \delta_1 \le \frac{\delta_0}{C_\Omega}$ such that if $0 < |w_1| \le \delta_1$ and $w_2 \in \mathcal{A}_2'$ then $W_2(t, w) \in \mathcal{A}_2$, for all $t \in [0, T]$.

Arguing by contradiction, suppose there exist $w_1^{(n)} \to 0$, $w_2^{(n)} \to w_2 \in \overline{\mathcal{A}_2'}$ and $t_n \in [0, T]$ such that $W_2(t_n, w^{(n)}) \in \partial\mathcal{A}_2$. It is immediate to see that $t_n \ge t_0$ for all n. Moreover, by Step 1, $W_2(t, w^{(n)}) \to Z(t, w_2)$ as $n \to \infty$ uniformly on $[0, t_0]$. Then there exists n_1 such that for all $n \ge n_1$ we have $W_2(t_0, w^{(n)}) \in \mathcal{U}_\varepsilon(\mathcal{A}_2')$. This implies that $t_n \ge 2t_0$ for all $n \ge n_1$. So we can apply again Step 1 and obtain that $W_2(t, w^{(n)}) \to Z(t, w_2)$ uniformly on $[0, 2t_0]$. Proceeding as before, we can find $n_2 \ge n_1$ such that for all $n \ge n_2$ we have $W_2(2t_0, w^{(n)}) \in \mathcal{U}_\varepsilon(\mathcal{A}_2')$. By induction the procedure continues until we obtain in a finite number of steps that $W_2(t, w^{(n)}) \to Z(t, w_2)$ uniformly on $[0, T]$, which gives the contradiction and proves Step 2.

In order to complete the proof, one argues as in Step 1 using that

$$\{W(t, w) : t \in [0, T], 0 < |w_1| \le \delta_1, w_2 \in \mathcal{A}_2'\} \subset A\mathcal{F}_2\Omega$$

is a relatively compact subset of $\Omega \times \Omega$ as a consequence of Step 2. □

Step 2 in the proof is actually a very useful result by itself, thus we can immediately state a corollary for this Lemma.

Corollary 1.5.7. *There exists* $0 < \delta_1 \le \delta_0$ *such that* $W_2(t,w) \in \mathcal{A}_2 = \sqrt{\kappa_1+\kappa_2}\mathcal{A}$ *for all* $t \in [0,T]$*, provided* $0 < |w_1| \le \delta_1$*,* $w_2 \in \mathcal{A}_2' = \sqrt{\kappa_1+\kappa_2}\mathcal{A}'$*.*

Corollary 1.5.7 and Proposition 1.5.1 imply that the first statement in (1.3.3) of Theorem 1.3.2 is a consequence of the second provided b_1 is small and provided the initial conditions $z_1(0), z_2(0)$ lie in $\mathcal{A}'$.

Clearly $\mathcal{A}_2' = \sqrt{\kappa_1+\kappa_2}\mathcal{A}'$ is bounded by the strictly star-shaped curves $\sqrt{\kappa_1+\kappa_2}\Gamma_k'$, $k = 1,2$. Now we let δ_1 be as in Corollary 1.5.7. For $0 < a_1 < b_1$ we define the annulus

$$\mathcal{A}_1(a_1,b_1) := \{w_1 \in \mathbb{R}^2 : a_1 < |w_1| < b_1\}.$$

We want to find $0 < a_1 < b_1 < \min\{a_0, \delta_1\}$ and a T-periodic orbit of the map $W(t,w)$ with $w \in \mathcal{A}_1(a_1,b_1) \times \mathcal{A}_2'$.

Observe that $W_1(t,w) \ne 0$ for any $w \in A\mathcal{F}_2\Omega$ and any $t \in \mathbb{R}$ by Proposition 1.5.1. Therefore there exists a continuous choice of the argument of $W_1(t,w)$ and we may define the rotation number

$$\mathrm{Rot}(W_1(t,w);[0,T]) := \frac{1}{2\pi}\big(\arg(W_1(T,w)) - \arg(w_1)\big) \in \mathbb{R}.$$

Moreover, Corollary 1.5.7 implies that $W_2(t,w) \ne 0$ for $w \in \mathcal{A}_1(a_1,b_1) \times \mathcal{A}_2'$ and $t \in [0,T]$ provided $0 < a_1 < b_1 < \delta_1$. Thus we may also define the rotation number

$$\mathrm{Rot}(W_2(t,w);[0,T]) := \frac{1}{2\pi}\big(\arg(W_2(T,w)) - \arg(w_2)\big) \in \mathbb{R}.$$

In the next section we shall prove the following result.

Proposition 1.5.8. *For every* $a_0 > 0$ *there exist* $0 < a_1 < b_1 < \min\{a_0, \delta_1\}$ *and* $\nu \in \mathbb{Z}$ *such that the following holds for* $w \in \mathcal{A}_1(a_1,b_1) \times \mathcal{A}_2'$*.*

a) $\mathrm{Rot}(W_1(t,w);[0,T]) \begin{cases} > \nu, & \text{if } |w_1| = a_1 \\ < \nu, & \text{if } |w_1| = b_1. \end{cases}$

b) $\mathrm{Rot}(W_2(t,w);[0,T]) \begin{cases} > 1, & \text{if } \frac{w_2}{\sqrt{|\kappa_1|+|\kappa_2|}} \in \Gamma_1' \\ < 1, & \text{if } \frac{w_2}{\sqrt{|\kappa_1|+|\kappa_2|}} \in \Gamma_2'. \end{cases}$

Thus for any $w_2 \in \mathcal{A}_2'$ the rotation number of $W_1(t,w)$ in the interval $[0,T]$ changes from bigger than ν to less than ν as w_1 passes from the inner boundary of $\mathcal{A}_1(a_1,b_1)$ to the outer boundary of $\mathcal{A}_1(a_1,b_1)$. Similarly, for any $w_1 \in \mathcal{A}_1(a_1,b_1)$ the rotation number of $W_2(t,w)$ in the interval $[0,T]$ changes from bigger than 1 to less than 1 as w_2 passes from the boundary curve $\sqrt{\kappa_1+\kappa_2}\Gamma_1'$ of $\mathcal{A}_2'$ to the boundary curve $\sqrt{\kappa_1+\kappa_2}\Gamma_2'$ of $\mathcal{A}_2'$.

This is precisely the setting of the generalized Poincaré-Birkhoff Theorem 1.4.1. As a consequence we deduce that the Hamiltonian system (1.5.3) has a T-periodic solution with initial conditions $w \in \mathcal{A}_1(a_1,b_1) \times \mathcal{A}_2'$. For the proof of Theorem 1.3.2 it therefore remains to prove Proposition 1.5.8.

1.6 Proof of Proposition 1.5.8

It will be useful to introduce polar coordinates for W_1, W_2. Recall that any solution of (1.5.3) with initial condition $w \in \mathcal{A}_1(a_1, b_1) \times \mathcal{A}_2'$ satisfies $W_k(t, w) \neq 0$ for $t \in [0, T]$, $k = 1, 2$. We set

$$e(\theta) = (\cos\theta, \sin\theta) \tag{1.6.1}$$

and fix initial conditions $w_1 = \rho_1 e(\theta_1)$, $w_2 = \rho_2 e(\theta_2)$. Then setting $\rho = (\rho_1, \rho_2)$ and $\theta = (\theta_1, \theta_2)$ we define $R_k(t, \rho, \theta) = \big|W_k(t, \rho_1 e(\theta_1), \rho_2 e(\theta_2))\big|$ and let $\Theta_k(t, \rho, \theta)$ be a continuous choice of the argument of $W_k(t, \rho_1 e(\theta_1), \rho_2 e(\theta_2))$. Thus we can write

$$W_k(t, w) = R_k(t, \rho, \theta) e(\Theta_k(t, \rho, \theta)) \text{ for } k = 1, 2.$$

We will also write $R(t, \rho, \theta) = (R_1, R_2)(t, \rho, \theta)$ and $\Theta(t, \rho, \theta) = (\Theta_1, \Theta_2)(t, \rho, \theta)$.

Next we describe the radial component of the boundary curves of $\mathcal{A}_2'$ as a function of the angle, obtaining functions $r_k : \mathbb{R} \to (0, \infty)$ defined by $r_k(\theta)e(\theta) \in \sqrt{\kappa_1 + \kappa_2}\Gamma_k'$. Since Γ_k' is strictly star-shaped with respect to the origin, r_k is well defined. Clearly r_k is 2π-periodic and there holds

$$\sqrt{\kappa_1 + \kappa_2}\Gamma_k' = \{r_k(\theta)e(\theta) : \theta \in \mathbb{R}\}.$$

We also set

$$\mathcal{A}_2^{pol} := \{(\rho_2, \theta_2) \in \mathbb{R}^+ \times \mathbb{R} : \rho_2 e(\theta_2) \in \mathcal{A}_2'\}.$$

Proposition 1.5.8 is now equivalent to the following result.

Proposition 1.6.1. *For every $a_0 > 0$ there exist $0 < a_1 < b_1 < a_0$ and $\nu \in \mathbb{Z}$ such that the following holds for $w \in \mathcal{A}_1(a_1, b_1) \times \mathcal{A}_2'$.*

$$a)\ \Theta_1(T, \rho_1, \rho_2, \theta_1, \theta_2) - \theta_1 \begin{cases} > 2\pi\nu, & \text{if } \rho_1 = a_1,\ (\rho_2, \theta_2) \in \mathcal{A}_2^{pol}, \\ < 2\pi\nu, & \text{if } \rho_1 = b_1,\ (\rho_2, \theta_2) \in \mathcal{A}_2^{pol}. \end{cases}$$

$$b)\ \Theta_2(T, \rho_1, \rho_2, \theta_1, \theta_2) - \theta_2 \begin{cases} > 2\pi, & \text{if } \rho_1 \in [a_1, b_1],\ \rho_2 = r_1(\theta_2), \\ < 2\pi, & \text{if } \rho_1 \in [a_1, b_1],\ \rho_2 = r_2(\theta_2). \end{cases}$$

Proof. We begin with the proof of part b) because this determines the choice of b_1 which will then be used in the proof of part a) where we choose a_1. For $\rho_2 = r_1(\theta_2)$, that is

$$w_2 = \rho_2 e(\theta_2) \in \sqrt{\kappa_1 + \kappa_2}\Gamma_1' \subset \partial\mathcal{A}_2' = \sqrt{\kappa_1 + \kappa_2}\partial\mathcal{A}',$$

the solution $Z(t, w_2)$ of the initial value problem (1.5.4) has the period $T(\Gamma_1')$. Now Corollary 1.5.7 implies that $W_2(T, w) \to Z(T, w_2)$ as $w_1 \to 0$. Since $T(\Gamma_1') < T$ the argument Θ_2 of W_2 satisfies

$$\Theta_2(T, \rho_1, \rho_2, \theta_1, \theta_2) - \theta_2 > 2\pi \tag{1.6.2}$$

for $\rho_1 = |w_1|$ small. Similarly, for $\rho_2 = r_2(\theta_2)$, that is

$$w_2 = \rho_2 e(\theta_2) \in \sqrt{\kappa_1 + \kappa_2}\Gamma_2' \subset \partial\mathcal{A}_2' = \sqrt{\kappa_1 + \kappa_2}\partial\mathcal{A}',$$

the solution $Z(t,w_2)$ of the initial value problem (1.5.4) has the period $T(\Gamma_2')>T$, so $W_2(T,w)\to Z(T,w_2)$ as $w_1\to 0$ implies

$$\Theta_2(T,\rho_1,\rho_2,\theta_1,\theta_2)-\theta_2<2\pi \tag{1.6.3}$$

for $\rho_1=|w_1|$ small. Part b) follows provided we choose b_1 so small that (1.6.2) and (1.6.3) hold for $\rho_1=|w_1|<b_1$.

Now we can prove part a). The proof of this part is similar to the proof of the main result in [16]. With b_1 determined above we choose $\nu\in\mathbb{Z}$ satisfying

$$2\pi\nu>\max\left\{\Theta_1(T;b_1,\rho_2,\theta_1,\theta_2)-\theta_1:\theta_1\in[0,2\pi],\ (\rho_2,\theta_2)\in\overline{\mathcal{A}_2^{pol}}\right\}. \tag{1.6.4}$$

Setting

$$\begin{aligned}z_1(R,\Theta)&=\frac{1}{\sqrt{\kappa_1+\kappa_2}}\left(\sqrt{\frac{\kappa_2}{\kappa_1}}R_1e(\Theta_1)+R_2e(\Theta_2)\right),\\ z_2(R,\Theta)&=\frac{1}{\sqrt{\kappa_1+\kappa_2}}\left(-\sqrt{\frac{\kappa_1}{\kappa_2}}R_1e(\Theta_1)+R_2e(\Theta_2)\right)\end{aligned}$$

and

$$\begin{aligned}k(R,\Theta)=2\left(\kappa_2\sqrt{\kappa_1\kappa_2}\nabla_{z_1}-\kappa_1\sqrt{\kappa_1\kappa_2}\nabla_{z_2}\right)g(z_1(R,\Theta),z_2(R,\Theta))\\ +\kappa_1\sqrt{\kappa_1\kappa_2}\nabla h(z_1(R,\Theta))-\kappa_2\sqrt{\kappa_1\kappa_2}\nabla h(z_2(R,\Theta)),\end{aligned}$$

the equations for R_1,Θ_1 are given by

$$\begin{cases}\dot R_1=\dfrac{1}{\sqrt{\kappa_1+\kappa_2}}\langle -Jk(R,\Theta),e(\Theta_1)\rangle\\ \dot\Theta_1=\dfrac{\kappa_1\kappa_2}{\pi R_1^2}+\dfrac{1}{\sqrt{\kappa_1+\kappa_2}R_1}\langle k(R,\Theta),e(\Theta_1)\rangle=:f(R_1,R_2,\Theta_1,\Theta_2).\end{cases} \tag{1.6.5}$$

Observe that

$$\lim_{R_1\to0}f(R_1,R_2,\Theta_1,\Theta_2)=+\infty$$

because

$$\lim_{R_1\to0}\frac{1}{\sqrt{\kappa_1+\kappa_2}R_1}\langle k(R,\Theta),e(\Theta_1)\rangle=(\kappa_1+\kappa_2)^{\frac32}\left\langle D^2h\left(\frac{R_2}{\sqrt{\kappa_1+\kappa_2}}e(\Theta_2)\right)e(\Theta_1),e(\Theta_1)\right\rangle.$$

Thus we can choose $0<\tilde a_1<b_1$ such that

$$f(R,\Theta)>\frac{2\pi\nu}{T}\quad\text{for every }0<R_1\le\tilde a_1,\ \Theta_1\in\mathbb{R},\ (R_2,\Theta_2)\in\overline{\mathcal{A}_2^{pol}}. \tag{1.6.6}$$

Then, by Proposition 1.5.1, there exists $0<a_1<\tilde a_1$ such that

$$R_1(t;a_1,\rho_2,\theta_1,\theta_2)\le\tilde a_1\quad\text{for every }t\in[0,T],\ \theta_1\in\mathbb{R},\ (\rho_2,\theta_2)\in\overline{\mathcal{A}_2^{pol}}.$$

Now integrating (1.6.6) on $[0,T]$ gives

$$\Theta_1(T;a_1,\rho_2,\theta_1,\theta_2)-\theta_1=\int_0^T f\left(R(t,a_1,\rho_2,\theta_1,\theta_2),\Theta(t,a_1,\rho_2,\theta_1,\theta_2)\right)dt>2\pi\nu \tag{1.6.7}$$

for all $\theta_1\in\mathbb{R}$, all $(\rho_2,\theta_2)\in\mathcal{A}_2^{pol}$. Now (1.6.4) and (1.6.7) imply a). $\square$

1.7 Proof of the remaining results

In this section we prove the remaining results, namely:

- Theorem 1.3.4 concerning the shape of any periodic solutions near the center of vorticity;
- Lemma 1.3.6 about the behaviour of the period of the one vortex problem in convex domains when the vortex is placed close to a minimum of h;
- Corollary 1.3.8 which shows the applicability of our main result in the case of a vortex close to a minimum of h;
- Corollary 1.3.10 about how our theorem works when the vortices are close to a star-shaped boundary.

Proof of Theorem 1.3.4. Consider solutions $z^{(n)}(t)$ with $z_1^{(n)}(0), z_2^{(n)}(0) \to C_0 \in \Omega$ and such that the solution of (1.3.4) is non-stationary periodic. It follows from Proposition 1.5.1 that

$$w_1^{(n)}(t) = \frac{\sqrt{\kappa_1\kappa_2}}{\sqrt{|\kappa_1+\kappa_2|}}\left(z_1^{(n)}(t) - z_2^{(n)}(t)\right) \overset{n\to\infty}{\longrightarrow} 0 \quad \text{uniformly in } t \in \mathbb{R}.$$

Lemma 1.5.6 now implies that

$$w_2^{(n)}(t) = \frac{1}{\sqrt{|\kappa_1+\kappa_2|}}\left(\kappa_1 z_1^{(n)}(t) + \kappa_2 z_2^{(n)}(t)\right) \overset{n\to\infty}{\longrightarrow} Z(t, \sqrt{|\kappa_1+\kappa_2|}C_0) \quad \text{uniformly in } t \in \mathbb{R}$$

where $Z(t, \sqrt{|\kappa_1+\kappa_2|}C_0)$ solves the initial value problem (1.5.4) with initial condition $w_2 = \sqrt{|\kappa_1+\kappa_2|}C_0$. This is equivalent to part a) from Theorem 1.3.4 because the centers of vorticity satisfies $C^{(n)}(t) = \frac{1}{\sqrt{|\kappa_1+\kappa_2|}}w_2^{(n)}(t)$ and $C(t) = \frac{1}{\sqrt{|\kappa_1+\kappa_2|}}Z(t)$.

For the proof of part b) we define

$$u_n(s) := \frac{1}{d_n}D^{(n)}(d_n^2 s) = \rho^{(n)}\big(e(\theta^{(n)}(d_n^2 s))\big)$$

where $d_n = \left|z_1^{(n)}(0) - z_2^{(n)}(0)\right|$ and $e(\theta)$ is as in (1.6.1). Then u_n satisfies

$$\dot{u}_n = -\frac{\kappa_1\kappa_2}{\pi}J\frac{u_n}{|u_n|^2} - o(1) \qquad \text{as } n \to \infty, \text{ uniformly in } [0,T].$$

Note that $|u_n(0)| = 1$ for all n, so up to a subsequence $u_n(0) \to \bar{u}$ with $|\bar{u}| = 1$. By a straightforward calculation we obtain that $\frac{d}{ds}|u_n(s)|^2 = o(1)$ as $n \to \infty$, uniformly in $[0,T]$. Thus there exists $\varepsilon > 0$ such that for n sufficiently large we have $|u_n(s)| \geq \varepsilon$ uniformly for $s \in [0,T]$. Next let u_∞ be the solution of the initial value problem

$$\begin{cases} \dot{u}_\infty = -\dfrac{\kappa_1\kappa_2}{\pi}J\dfrac{u_\infty}{|u_\infty|^2} \\ u_\infty(0) = \bar{u}. \end{cases}$$

We now deduce easily that $u_n \to u_\infty$ uniformly on $[0,T]$. Note that $\frac{d}{ds}arg(u_\infty(s)) = \frac{\kappa_1\kappa_2}{\pi}$, which implies $d_n^2\dot{\theta}^{(n)}(s) \to \frac{\kappa_1\kappa_2}{\pi}$. □

Proof of Lemma 1.3.6. Suppose $\kappa_1, \kappa_2 > 0$. First we transform the equation (1.3.2) using the canonical coordinate change $(\rho,\theta) \mapsto \sqrt{2\rho}e(\theta)$. Setting $h_1(\rho,\theta) = (\kappa_1+\kappa_2)h(\sqrt{2\rho}e(\theta))$ this leads to the system

$$\begin{cases} \dot{\rho} = -\dfrac{\partial}{\partial\theta}h_1(\rho,\theta) \\ \dot{\theta} = \dfrac{\partial}{\partial\rho}h_1(\rho,\theta). \end{cases}$$

In convex domains we have that h is strictly convex by [17]. This allows us to prove that the minimal period T_c is decreasing with respect to c.

Of course this is true close to the boundary, because as a consequence of $|\nabla h(z)| \to \infty$ as $z \to \partial\Omega$ we have $T_c \to 0$ as $c \to \infty$.

Suppose by contradiction that for $c < d$ we have $T_c = T_d$. This condition tells us that the two solutions with initial conditions s.t. $h(\sqrt{2R_0}e^{i\varphi_0}) = c$ and $h(\sqrt{2R_0}e^{i\varphi_0}) = d$ can behave only in one of these two ways:

- or they rotate with the same angular velocity;
- or one solution rotates faster than the other one but at some point they switch this behaviour.

But both these two possibilities cannot happen.

In fact, because h is strictly convex, the one variable function obtained by restricting h on every segment pointing out from the origin in Ω is strictly convex. So, fixing a direction $\overline{\varphi}$ we can consider

$$\begin{array}{rccl} \overline{h}: & [0, rad(\overline{\varphi})[& \to & \mathbb{R}^+ \\ & R & \mapsto & h(\sqrt{2R}e(\overline{\varphi})) \end{array}$$

where $rad(\overline{\varphi}) = \frac{|p|^2}{2}$ and p is the only point in the intersection $\mathbb{R}^+e(\overline{\varphi}) \cap \partial\Omega$. $\overline{h}$ satisfies $\overline{h}(R)'' > 0$, thus

$$\frac{\partial}{\partial R}\dot{\theta}(R,\overline{\varphi}) = (\kappa_1+\kappa_2)\frac{\partial^2}{\partial^2 R}h(\sqrt{2R}e^{i\overline{\varphi}}) > 0.$$

This means that the angular velocity in any fixed radial direction is strictly increasing with respect to the radius, which exclude both possibilities, leading to a contradiction. Hence T_c is strictly decreasing.

Finally, since the origin is a nondegenerate critical point of h, we can apply Hartman-Grobman Theorem, which tells us that the flow of the system near the hyperbolic critical point is topologically equivalent to the flow of the linearized system. So, by the Taylor expansion $\nabla h(z) = h''(0)[z] + o(|z|)$ at 0, we need to investigate the behaviour of the system

$$\dot{x} = -(\kappa_1+\kappa_2)Jh''(0)x \ , \ x \in U$$

where $U \in \mathbb{R}^2$ is a neighborhood of 0. We can suppose that $h''(0)$ is in diagonal form

$$h''(0) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

so the linearized system is

$$\begin{cases} \dot{x} = -\lambda_1(\kappa_1+\kappa_2)y \\ \dot{y} = \lambda_2(\kappa_1+\kappa_2)x \end{cases}, \ (x,y) \in U,$$

where $\lambda_1, \lambda_2 > 0$, which is nothing else than the harmonic oscillator

$$\ddot{x} = -\lambda_1\lambda_2(\kappa_1+\kappa_2)^2 x = -\det h''(0)(\kappa_1+\kappa_2)^2 x.$$

The solution of the harmonic oscillator is periodic with period

$$T = \frac{2\pi}{|\kappa_1+\kappa_2|\sqrt{\det h''(0)}}.$$

So

$$T_c \to T_m := \frac{2\pi}{|\kappa_1+\kappa_2|\sqrt{\det h''(0)}} \qquad \text{as } c \to m$$

because T_m is the minimal period of the nontrivial solutions of $\dot{z} = -(\kappa_1+\kappa_2)Jh''(0)[z]$. □

Proof of 1.3.8. Since $h''(z_0)$ is positive definite the Robin function is strictly convex in a neighborhood U of z_0. Therefore the level lines $h^{-1}(c) \cap U$ for $c > c_0 = h(z_0)$ close to c_0 are convex. As in the proof of Lemma 1.3.6 the period T_c of the solution of (1.3.2) with trajectory $h^{-1}(c) \cap U$ is strictly decreasing in c. The corollary follows now from Theorem 1.3.2. □

Proof of 1.3.10. Let $U \subset \mathbb{R}^2$ be a tubular neighborhood of Γ_0 and $p : U \to \Gamma_0$ be the orthogonal projection. Moreover let $\nu : \Gamma_0 \to \mathbb{R}^2$ be the exterior normal. It is well known that

$$\nabla h(z) = \frac{\nu(p(z))}{2\pi d(z,\Gamma_0)} + O(1) \quad \text{as } d(z,\Gamma_0) = \operatorname{dist}(z,\Gamma_0) \to 0; \tag{1.7.1}$$

see [5]. Therefore the level lines $h^{-1}(c) \cap U$ for $c > c_0$ are also strictly star-shaped with respect to z_0, if c_0 is large enough. Moreover the period T_c of the solution of (1.3.2) with trajectory $h^{-1}(c) \cap U$ is strictly decreasing in c due to (1.7.1). Consequently the corollary follows from Theorem 1.3.2. □

1.8 Remarks and open problems

We end this chapter by some remarks and stating open problems related to this work.

Firstly, about Proposition 1.5.1, we can actually prove some facts useful to skip it. We just state it here in an informal way and leave the details to the interested reader.

This first Lemma is an argument provided by Alberto Boscaggin and allows us to state the uniform convergence of W_1 to 0 as $|w_1| \to 0$, which is helpful in order not to use statment b) of Proposition 1.5.1.

Lemma 1.8.1. *For all $T > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ depending on δ such that if $|w_1| < \delta$ then*

$$|W_1(t, w)| \leq \varepsilon \text{ for every } t \in [0, T], w_2 \in \mathcal{A}_2.$$

Proof. By contradiction, suppose that for some $T > 0$, $\varepsilon > 0$ there exist sequences $t_n, w_n = (w_{1,n}, w_{2,n})$ with $t_n \in [0, T]$, $|w_{1,n}| \to 0$ as $n \to +\infty$, $w_{2,n} \in \mathcal{A}_2$ and

$$|W_1(t_n, w_n)| \geq \varepsilon. \tag{1.8.1}$$

Then considering that the Hamiltonian is constant along a solution we have

$$H_1(W_1(t_n, w_n), W_2(t_n, w_n)) = H_1(w_{1,n}, w_{2,n}).$$

But in this last equality we have that the left hand side is bounded from above because of (1.8.1) while, letting $n \to +\infty$, the right hand side tends to $+\infty$. □

We can also say something about existence till time T of the solution in order to get rid of statement a) of Proposition 1.5.1.

Lemma 1.8.2. *For all $T > 0$, there exists $\delta > 0$ such that if $0 < |w_1| < \delta$ and $w_2 \in \mathcal{A}_2'$ then $W(\cdot, w_1, w_2)$ exists till time T.*

Proof. First note that by energy constraints the solution $W(t, w_1, w_2)$ exists until the two vortices reach the boundary together in the same point and at the same time.

Thus, in the setting of Lemma 1.5.6, the solution starting from $w = (w_1, w_2)$ takes at least time $t_0 > 0$ to exit from $\mathcal{A}_2$ and on $[0, t_0]$ we have uniform convergence of $W_2(\cdot, w)$ to $Z(\cdot, w_2)$. We can say this withou using Proposition 1.5.1, but just using previous Lemma.

Then, the second part of Lemma 1.5.6 says that taking w_1 sufficiently close to 0 our solution exit from $\mathcal{A}_2$ after time T, thus we have existence. □

With these two Lemmas, it's easy to generalize dropping some condition in these chapter. In fact we can skip Ω satisfying the uniform exterior ball condition. With this new point of view, it's just the same calculations if one takes g any $C^2(\Omega \times \Omega)$ symmetric function.

It is an interesting problem whether it is possible to weaken or to drop the condition that Γ_1, Γ_2 are strictly star-shaped. We refer the reader to [27,34,40] for results and discussions of this delicate issue in the setting of the Poincaré-Birkhoff fixed point theorem for nonautonomous one degree of freedom Hamiltonian systems. Although star-shapedness is essential for the multidimensional Poincaré-Birkhoff fixed point theorem [30, Theorem 1.2] we believe that it is not essential in our special case; see also [29].

Another natural question is a possible generalization to the N-Vortex case. Of course in this case a lot new difficulties arise. The first one is the existence of solutions, because it can happen that some vortices collide, some of them go to the boundary in different points and some of them may also stay quiet in the domain. The second difficulty is to find a correct canonical trasformation to transpose the problem into the setting of the Poincaré-Birkhoff Theorem, because transformation (1.5.2) is valid just for the case of two vortices. Even if we knew a suitable transformation, it is probably not easy to calculate the rotation numbers for the new variable.

So the generalization to the full case of N vortices is not genuine, but it will require new ideas. This will be part of future work.

Chapter 2

Bifurcation for a multi-component Schrödinger system

2.1 Bose-Einstein condensation

At the end of the 20th century, a physical phenomenon called Bose-Einstein condensation, which happens when most of the particles of a gas of bosons occupy the lowest energy quantum state, was experimentally observed by cooling a gas of bosons very close to the absolute zero. This phenomenon was already theorically predicted by Satyendra Nath Bose and Albert Einstein in the twenties. This state was reached experimentally in 1995, but in the preceding years some mathematical models were already developed, especially by Eugene P. Gross and Lev Petrovich Pitaevskii. We will follow the introduction book by Pitaevskii and Stringari, see [45]. Leaving the details and the deep physical theory which the interested reader can find in the book, we will give just a sketch introduction. Consider the wave function of the condensate $\Psi(t,x)$. As a consequence of the diluteness of the gas, one can ignore correlations among particles and write the many-body wave function of the system in the symmetrized form, making use of the Hartree-Fock approximation. This means that the total wave function $\Psi(t,x)$ of a system with d bosons is given by the product of every single wave function

$$\Psi(t,x) = \prod_{j=1}^{d} \psi(t,x_i),$$

From the equation that rules the wave function, we make use of the pseudopotential interaction model, given by

$$H(x) = \left(-\frac{\hbar^2}{2m}\Delta + V(x)\right) + \sum_{1\le i<j\le d} \frac{4\pi\hbar^2}{m}\omega\delta(x_i - x_j).$$

This is the Hamiltonian of the equation model, in which $\hbar$ is the reduced Plank constant, m is the atom mass of the particles, V is the potential and δ is the Dirac delta distribution. So, if the single particle wave function ψ solves up to multiplicative constants the so called Gross-Pitaevskii equation

$$-i\frac{\partial}{\partial t}\psi(t,x) = \Delta\psi(t,x) + V(x)\psi(t,x) + \beta|\psi(t,x)|^2\psi(t,x)$$

and it is normalized, then the total wave function minimizes the expectation value of the Hamiltonian. If the condensation happens in a mixture of gases, the equation becomes a system of nonlinear Schrödinger equations

$$\begin{cases} -i\frac{\partial}{\partial t}\psi_j(t,x) = \Delta\psi_j(t,x) + V(x)\psi(t,x) + \sum_{k=1}^{d}\beta_{ij}|\psi_k(t,x)|^2\psi_j(t,x) & (t,x)\in\mathbb{R}^+\times\Omega \\ \psi_j \in H_0^1(\Omega;\mathbb{C}), \forall t>0 \end{cases}$$

for $j = 1,\dots,d$ where β_{ij}, $i \neq j$ describe the interaction between particles in different condensates, while β_{jj} describes the interaction between particles of the same condensate. If the sign of β_{jj} is negative, this means attraction, otherwise the behaviour is repulsive.

As a standard physical approach, we impose that wave functions are localized, i.e.

$$\psi_j(t,x) = \mathrm{e}^{-i\lambda_j t}u_j(x),$$

for some $\lambda_j > 0$ and for every $j = 1,\dots,d$. The system arising after some semplifications will be stated in next section.

2.2 The mathematical model

After the short physical introduction, we state here the system of Schrödinger equations. Given a smooth domain $\Omega \in \mathbb{R}^N$

$$\begin{cases} -\Delta u_i + a_i(x)u_i = \mu_i u_i^3 + \sum_{k=1,k\neq i}^{n}\beta_{ki}u_k^2u_i & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega \quad \text{for } i=1,\dots,n, \\ u_i > 0 & \text{in } \Omega \end{cases}$$

where a_i plays the role of the potential, μ_i, β_{ki} are parameters for the system. If we have the symmetry $\beta_{ki} = \beta_{ik}$, then the system has a variational structure. Note that in the case of two equations $n = 2$, we can always rescale $u = (u_1, u_2)$ to (s_1u_1, s_2u_2) in order to obtain this symmetry, just need to choose s_1, s_2 such that $s_1^2\beta_{12} = s_2^2\beta_{21}$. Consider the principal eigenvalue $\Lambda_1(-\Delta, H_0^1(\Omega))$, we say that the system is definite if the operator $\Lambda_1 + a_i > 0$ for all $i = 1,\dots,n$, indefinite if $\Lambda_1 + a_i \leq 0$ for at least one i between 1 and n. The energy functional is well defined if $N \leq 3$, because in this case the continuous embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$ holds (for higher dimensional systems one should replace the cubic nonlinearities with subcritical ones). So the solutions of (2.2) are critical points of the functional

$$J(u_1,\dots,u_d) = \frac{1}{2}\sum_{j=1}^{d}\int_\Omega (|\nabla u_j|^2 + a_j(x)u_j^2) - \frac{1}{4}\sum_{i,j=1}^{d}\int_\Omega \beta_{ij}u_j^2u_i^2,$$

where $\beta_{ii} = \mu_i$ for $i = 1,\dots,n$.

There is now a rich litterature about this problem, but we state here just a few results to underline what will be useful in this chapter.

In [49], the system has been studied in the case of two equations, $\Omega = \mathbb{R}^N$ and a_i constantly equal to 1. They proved that there always exist ranges of positive parameters for which this system has a least energy solution, and ranges of positive parameters for which the functional cannot be minimized on the natural set where the eventual solutions lie. The conditions on the parameters are the following: if we set $\beta = \beta_{12} = \beta_{21}$, then $\beta < C_1$ or $\beta > C_2$ with $C_1 \leq C_2$ contants depending on β_{11}, β_{22}, otherwise solutions with no zero components don't exist. In [50] the author considered a more general situation, Ω a bounded domain or $\mathbb{R}^N$, where also the β_{ij} are functions of x,with $a_i, \beta_{ij} \in L^\infty(\Omega)$, $\beta_{ij} = \beta_{ji}$, $a_i \geq 0$, $\beta_{ii} \geq \mu_i > 0$ almost everywhere for some $\mu_i > 0$ and for every i, j, and they obtain results both for existence and for non existence, but with a natural constraint on suitable subset of $(H_0^1(\Omega))^d$. In [43] the authors dealt with the system with two equations and with $\beta_{12} = \beta_{21} = \beta$, $a_1 \equiv \lambda_\beta$, $a_2 \equiv \mu_\beta$ to obtain estimates and regularity for the solutions and they proved that uniform L^∞ boundedness implies $C^{0,\alpha}$ boundedness, uniformly as $\beta \to +\infty$, for all $\alpha \in (0, 1)$. Moreover, for this kind of system with two equations, it is well known that phase separation occurs: the solution (u_1, u_2) tends to (u, v) as $\beta \to +\infty$, with $u \cdot v \equiv 0$ in Ω that satisfies

$$\begin{cases} -\Delta u + \lambda u = \omega_1 u^3 & \text{in } \{u > 0\} \\ -\Delta v + \mu v = \omega_2 v^3 & \text{in } \{v > 0\}, \end{cases}$$

with $\lambda = \lim \lambda_\beta$, $\mu = \lim \mu_\beta$. The positivity domains $\{u > 0\}, \{v > 0\}$ are composed of a finite number of disjoint connected components with positive Lebesgue measure. Then, the natural step in the analysis of this kind of systems is the regularity of the limiting profile and of the free boundary, as well as uniform bounds in suitable functional spaces for the family $\{(u_{1,\beta}, u_{2,\beta})\}$. In [56] the authors dealt with a class of Lipschitz vector functions with nonnegative components, disjointly supported and verifying an elliptic equation on each support. They proved that the nodal set is a collection of $C^{1,\alpha}$ hyper-surfaces for every $0 < \alpha < 1$ up to a residual set with small Hausdorff dimension. Moreover, if we take a point x_0 in this regular part of the nodal set, we have

$$\lim_{x \to x_0^+} |\nabla \omega_1(x)| = \lim_{x \to x_0^-} |\nabla \omega_2(x)|,$$

where $x \to x_0^\pm$ means that the functions are approaching the hyper-surfaces from opposite sides with the same slope. This result is stated in [56] is true in the setting of this chapter, but it is valid in a more general setting, too.

Many authors performed a blow-up analysis on the interface between $u_{1,\beta}$ and $u_{2,\beta}$. Let us consider the set

$$\Omega_u = \{x \in \Omega : \ u_1(x) = u_2(x)\}$$

we wish to scale the equations close to this set in order to deduce some information about the limit configuration. At least in dimension $N = 1$, Berestycki and others in [15] showed that if $x_\beta \in \Omega_u$ with $m_\beta = u_1(x_\beta) = u_2(x_\beta)$ then

$$m_\beta^4 \beta \to C \text{ as } \beta \to \infty.$$

This asymptotics let us choose the correct scaling rate for the blow-up, which is

$$U_\beta = \frac{1}{m_\beta} u_1(m_\beta x + x_\beta) \ , \ V_\beta = \frac{1}{m_\beta} u_2(m_\beta x + x_\beta)$$

and gives us convergence of the sequence in $C^2_{loc}(\Omega)$ to a positive solution of

$$\begin{cases} U'' = UV^2 & \text{in } \Omega \\ V'' = U^2 V & \text{in } \Omega. \end{cases}$$

Then, naturally, many authors studied solutions to this limiting system, see [52] as latest result and the references therein for more informations about these solutions. Some generalizations to higher dimensions are natural, too, see [43], [57].

Another important study in this kind of system is the bifurcation arising when parameters change. This was studied by [9] in the definite case and in [14] in the indefinite case, when $N \le 3$, with parameter $\beta = \beta_{ij}$ for $i \ne j$. The two papers are about the existence of a branch of positive locked solution for the system, i. e. with $\frac{u_i}{u_j}$ being constant for all $i, j \in \{1, \dots, N\}$. The authors also proved results for partially sinchronized solutions and non existence results.

2.3 Statement of the results

Following the idea in the last references [9], [14], we study bifurcation of the following system

$$(2.3.1) \qquad \begin{cases} -\Delta u_i - u_i = \mu_i u_i^3 + \beta_1 \sum_{k=1, k\ne i}^n u_k^2 u_i^j - \frac{\gamma'}{t} \sum_{k=1}^m v_k^2 u_i & \text{in } \Omega \\ -\Delta v_j - v_j = \nu_j v_j^3 + \beta_2 \sum_{k=1, k\ne j}^m v_k^2 v_j - \gamma' \sum_{k=1}^n u_k^2 v_j & \text{in } \Omega \\ u_i = v_j = 0 & \text{on } \partial\Omega \\ u_i, v_j > 0 & \text{in } \Omega \end{cases}$$

for $i = 1, \dots, n, j = 1, \dots, m$, where $\Omega \subset \mathbb{R}^N$, $N \le 3$, is a smooth and bounded domain, $\mu_1, \dots, \mu_n$, $\nu_1, \dots, \nu_m$, $\beta_1, \beta_2, \gamma', t$ are real parameters, $n, m \ge 2$. We can assume $\mu_1 \le \dots \le \mu_n$ and $\nu_1 \le \dots \le \nu_m$. If β_1 is postive we will say that there is cooperation in the species u_i; when negative there is competition. The same holds for β_2. γ' will be positive and representing competition between $u = (u_i)_{i=1,\dots,n}$ and $v = (v_j)_{j=1,\dots,m}$. The parameter t allow us to have more freedom with the energy of the system. We call a family of equations *focusing* if the parameters in front of the cubic terms are positive, *defocusing* if they are negative and *mixed* in all other cases.

Definition 2.3.1. *About solutions of the equations in (2.3.1)*

- *The trivial solution is the solution with all components equal to 0.*
- *A semitrivial solutions is a solution with some zero and some nonzero components.*
- *A nontrivial solutions is a solution with all nonzero components.*

Definition 2.3.2. *Given* $\omega_1, \omega_2 > 0$ *solutions of*

$$(2.3.2) \qquad \begin{cases} -\Delta\omega_1 - \omega_1 = -\omega_1^3 - \gamma\omega_2^2\omega_1 \\ -\Delta\omega_2 - \omega_2 = -\omega_2^3 - \gamma\omega_1^2\omega_2 \end{cases}$$

with $\omega_1, \omega_2 \in H_0^1(\Omega)$ *and* γ *positive parameter, we say that a solution* (u, v) *of (2.3.1) is synchronized to* (ω_1, ω_2) *if*

$$(2.3.3) \qquad u_i = \alpha_1^i \omega_1 \ , \ v_j = \alpha_2^j \omega_2$$

for $i = 1, \dots, n$, $j = 1, \dots, m$.

Proceeding as in [14], substituting (2.3.3) in system (2.3.1), we find that

$$\alpha_1^i = ((\beta_1 - \mu_i)g_1(\beta_1))^{-\frac{1}{2}}, \quad \alpha_2^j = ((\beta_2 - \nu_j)g_2(\beta_2))^{-\frac{1}{2}}, \quad t = \frac{\sum_k (\alpha_2^k)^2}{\sum_k (\alpha_1^k)^2}, \quad \gamma' = \frac{\gamma}{\sum_k (\alpha_1^k)^2}$$

with

$$g_1(\beta_1) = 1 + \beta_1 \sum_{k=1}^{n} \frac{1}{\mu_k - \beta_1}\ , \ g_2(\beta_2) = 1 + \beta_2 \sum_{k=1}^{m} \frac{1}{\nu_k - \beta_2}.$$

Notice that t is positive, otherwise we would not have competition. Thus γ' has the same sign of γ, so it is always a competition parameter. Both t and γ' depends on β_1, β_2. The two functions g_1, g_2 tend to $1 - n$ as $\beta_1, \beta_2 \to -\infty$ respectively, $g_1(0) = g_2(0) = 1$ and have vertical asymptotes at the parameters μ_i, ν_j respectively. Moreover in the focusing case they are increasing and thus we have negative zeroes $\bar{\beta}_1 \leq \mu_1, \bar{\beta}_2 \leq \nu_1$ respectively, while in the defocusing case they are decreasing and have positive zeroes $\bar{\beta}_1 \geq \mu_n, \bar{\beta}_2 \geq \nu_m$ respectively. See [14] for more details. Thus we have the following result.

Proposition 2.3.3. *System (2.3.1) has a branch of synchronized solutions*

$$\tau = \{(\beta_1, \beta_2, \alpha_1^1\omega_1, \ldots, \alpha_1^n\omega_1, \alpha_2^1\omega_2, \ldots, \alpha_2^m\omega_2) \text{ with } \beta_1 \in I_1, \beta_2 \in I_2\} \subset \mathbb{R}^2 \times \left(H_0^1(\Omega)\right)^{n+m}$$

with

$$I_1 = \begin{cases} (-\infty, \bar{\beta}_1) & \text{in the focusing case} \\ (-\infty, \mu_1) \cup (\mu_n, \bar{\beta}_1) & \text{in the defocusing case} \\ (-\infty, \mu_1) & \text{in the mixed case} \end{cases}$$

(here focusing, defocusing and mixed refers to the parameters μ_i) and

$$I_2 = \begin{cases} (-\infty, \bar{\beta}_2) & \text{in the focusing case} \\ (-\infty, \nu_1) \cup (\nu_n, \bar{\beta}_2) & \text{in the defocusing case} \\ (-\infty, \nu_1) & \text{in the mixed case} \end{cases}$$

(here focusing, defocusing and mixed refers to the parameters ν_j).

In order to see if this branch has bifurcation values, in section 2.5 we investigate the linearized system of (2.3.1). It would be useful to make a change of coordinates to write the system in the form of the next preposition.

Proposition 2.3.4. *The linearized system of (2.3.1), see (2.5.1) below, is equivalent to the system composed by the following three equations*

$$-\Delta\varphi' - \varphi' + (\frac{2}{g_1(\beta_1)} + 1)\omega_1^2\varphi' + \gamma\omega_2^2\varphi' = 0, \tag{2.3.4}$$

$$-\Delta\psi' - \psi' + (\frac{2}{g_2(\beta_2)} + 1)\omega_2^2\psi' + \gamma\omega_1^2\psi' = 0 \tag{2.3.5}$$

and

$$\begin{cases} (-\Delta\varphi' - \varphi' + 3\omega_1^2\varphi' + \gamma\omega_2^2\varphi')t + 2\gamma\sqrt{t}\omega_1\omega_2\psi' = 0 \\ -\Delta\psi' - \psi' + 3\omega_2^2\psi' + \gamma\omega_1^2\psi' + 2\gamma\sqrt{t}\omega_1\omega_2\varphi' = 0 \end{cases} \tag{2.3.6}$$

In order to study equation (2.3.4), we compare with the eigenvalue problem

$$-\Delta\phi - \phi + \gamma\omega_{2,\gamma}^2\phi = \lambda_{1,\gamma}\omega_{1,\gamma}^2\phi \tag{2.3.7}$$

in which we add another index to ω_1, ω_2 in order to highlight their dependence on γ, which now will be useful to remember.

Notice that the spectrum of this operator is well defined. Then observe that $\lambda_{1,\gamma} = -1$ is an eigenvalue with corresponding eigenfunction $\omega_{1,\gamma}$. Since we know that $\omega_{1,\gamma} > 0$ so -1 must be the first eigenvalue. The other eigenvalues are given by the min-max characterization

$$\lambda_{1,\gamma}^j(\Omega) = \min_{\substack{dim(E)=j \\ E \subseteq H_0^1(\Omega)}} \max_{u \in E} \frac{\int_\Omega |\nabla u|^2 - u^2 + \gamma\omega_{2,\gamma}^2 u^2}{\int_\Omega \omega_{1,\gamma}^2 u^2} \tag{2.3.8}$$

thus we have an increasing sequence of eigenvalues diverging to infinity for (2.3.4).

Analogously we have a sequence of eigenvalues for the equation (2.3.5). So we can proceed as in [14], Lemma 3.3 (focusing case), Lemma 3.4 (defocusing case) and Lemma 3.6 (mixed case) which finds solutions with respect to β_1, β_2 of the equations

$$-f_i(\beta_i) = \lambda_{i,\gamma}^k \ , \ i = 1, 2 \ , \ k = 1, 2, 3, \ldots. \tag{2.3.9}$$

where

$$f_i(\beta_i) = \frac{2}{g_i(\beta_i)} + 1 \text{ for } i = 1, 2.$$

Remark 2.3.5. *Note that when one family of equations is focusing or defocusing, $f_i'(\beta_i) \neq 0$. In fact, for example for the first n equations*

$$f_1'(\beta_1) = -\frac{2}{g_1(\beta_1)^2} \sum_{k=1}^n \frac{\mu_k}{(\mu_k - \beta_1)^2}$$

which is strictly negative in the focusing case and strictly positive in the defocusing case.

Simply trying to solve (2.3.9) for $k \in \mathbb{N}$, we can summarize the result in this way:

1. in the focusing case there are infinitely many candidates for bifurcation parameters;

2. in the defocusing case there are at most finitely many candidates for bifurcation parameters;

3. in all mixed cases there are at most finitely many candidates for bifurcation points.

The reader can find all the calculations in [14]. We will call $\beta_{1,k}, \beta_{2,k}$, $k = 1, 2, \ldots$ the eventual solutions of (2.3.9).

We will in prove the following proposition that some of these values are effectively bifurcation points.

Theorem 2.3.6. *Suppose that the system (2.3.1) is focusing or defocusing for u_i. Suppose that there is no solution of system (2.3.6) except the trivial one with the parameters $\beta_{1,k}, \beta_2, \gamma$. Then every couple of the form*

$$(\beta_{1,k}, \beta_2), \text{ for } k = 1, 2, \dots, \text{ with } \beta_2 \neq \beta_{2,l}, l = 1, 2, \dots$$

is a bifurcation couple of solutions of system (2.3.1) from the branch τ.

Analogously, if the system (2.3.1) is focusing or defocusing for v_j and there is no solution of system (2.3.6) except the trivial one with the parameters $\beta_1, \beta_{2,l}, \gamma$. Then every couple of the form

$$(\beta_1, \beta_{2,l}), \text{ for } l = 1, 2, \dots, \text{ with } \beta_1 \neq \beta_{1,k}, k = 1, 2, \dots$$

is a bifurcation couple of solutions of system (2.3.1) from the branch τ.

Lastly, if u_i and v_j are both focusing or both defocusing for system (2.3.1) and there is no solution of system (2.3.6) except the trivial one with the parameters $\beta_{1,k}, \beta_{2,l}, \gamma$. Then every couple of the form

$$(\beta_{1,k}, \beta_{2,l}), \text{ for } k, l = 1, 2, \dots$$

is a bifurcation couple of solutions of system (2.3.1) from the branch τ.

Remark 2.3.7. *One can naturally ask if these values are global bifurcation points. In our particular case $N = 1$, when $n = 2$ or $m = 2$ we can apply standard theorems and obtain locally a smooth curve of bifurcating solutions in the first two cases of Theorem 2.3.6.*

Thus we need to give some conditions under which system (2.3.6) has no solutions, except the trivial one. If we focus on the case $N = 1$, we can use tools from Sturm-Liouville Theory to investigate the nondegeneracy of the positive solution (ω_1, ω_2) of system (2.3.2). We will make a blow up analysis, standard in phase separation models, to understand the behaviour of the limiting profile, which will give us informations for γ sufficiently large.

Proposition 2.3.8. *If $N = 1$, for γ sufficiently large, system (2.3.6) has only the trivial solution.*

Moreover, in this chapter, we want to understand how the eigenvalues $\lambda^j_{i,\gamma}$ behave when γ goes to infinity. Let $D_1 = \{\omega_{1,\infty} > 0\}$ and $D_2 = \{\omega_{2,\infty} > 0\}$ and consider the two limit eigenvalue problems

$$-\Delta\phi - \phi = \lambda\omega^2_{1,\infty}\phi, \ \phi \in H^1_0(D_1)$$

$$-\Delta\phi - \phi = \lambda\omega^2_{2,\infty}\phi, \ \phi \in H^1_0(D_2)$$

which are the linearizations of the limit problem for $(\omega_{1,\gamma}, \omega_{2,\gamma})$, see [43].

Call $(\lambda^j(D_1))_{j=1,2,\dots}$ the sequence of eigenvalues for the first problem and $(\lambda^j(D_2))_{j=1,2,\dots}$ the sequence for the second problem.

Proposition 2.3.9. *It holds*

$$\lambda^j(D_1) = \lim_{\gamma\to\infty} \lambda^j_{1,\gamma}(\Omega),$$

$$\lambda^j(D_2) = \lim_{\gamma\to\infty} \lambda^j_{2,\gamma}(\Omega).$$

The rest of the chapter is divided into four sections. In the first one, Section 2.4 we give a very short introduction to Bifurcation Theory. The reader that already knows the basics of this theory can simply skip this part. For the interested one, we just state the main propositions, that are useful throughout the chapter. In Section 2.5 we calculate the linearized system, diagonalize and analize it. We will give prove results about (2.3.6) in dimension one in Section 2.6. Last Section 2.7 will be devoted to prove a sufficient condition to obtain bifurcation as stated in Theorem 2.3.6.

2.4 Bifurcation

Let us recall the definition of bifurcation value which, roughly speaking, appears when we have the rising or disappearance of solutions when varying an equation with respect to a parameter. As a reference one can see [2] for more details.

Definition 2.4.1. *Take an equation of the form*

$$F(\lambda, u) = 0, \tag{2.4.1}$$

where $\lambda \in \mathbb{R}, u \in X, F : \Lambda \times X \to Y$, X, Y *are Banach spaces. Suppose that* $F(\lambda, 0) = 0$ *for all* $\lambda \in \mathbb{R}$. *We say that* λ^* *is a bifurcation value or that* $(\lambda^*, 0)$ *is a bifurcation point if there is a sequence* $(\lambda_n, u_n) \in \mathbb{R} \times X$ *with* $u_n \neq 0$ *and* $F(\lambda_n, u_n) = 0$ *such that*

$$(\lambda_n, u_n) \to (\lambda^*, 0).$$

The first step in investigating bifurcation values is the following condition.

Proposition 2.4.2. *If* λ^* *is a bifurcation value for equation (2.4.1), then the partial derivative* $F_u(\lambda^*, 0)$ *is not invertible.*

The proof of this Proposition relies on the Implicit Function Theorem. Note that this condition is just necessary, so it is useful to find possible bifurcation values, but we need also a sufficient condition. We will use a result stated in [32], Theorem II.7.3. In the following X, Z will be Banach spaces and $U \subset X$. Given a linear operator L, we will indicate with $N(L)$ the kernel of the operator and with $R(L)$ its range.

Definition 2.4.3. *A continuous mapping* $F : U \to Z$ *is a nonlinear Fredholm operator if it is Fréchet differentiable on* U *and if* $DF(x)$ *satisfies for* $x \in U$*:*

- $\dim N(DF(x)) < \infty$;
- $\operatorname{codim} R(DF(x)) < \infty$;
- $R(DF(x))$ *is closed in* Z.

Moreover we call the difference $\dim N(DF(x)) - \operatorname{codim} R(DF(x))$ *the Fredholm index of* $DF(x)$.

This class of operators is a good one because it allows us to restrict to a finite dimensional problem, via the Liapunov-Schmidt reduction.

Now we need to introduce the notion of Morse index for operators.

Assume for a family of linear operators $A(\lambda) \in L(X, Z)$ for $\lambda \in \mathbb{R}$ to be Fredholm of index 0. Then the generalized eigenspace E_{λ_0} of the eigenvalue 0 of $A(\lambda_0)$ is finite-dimensional, and it is perturbed to an invariant space E_λ for $A(\lambda)$ of the same dimension for λ near λ_0. The eigenvalue 0 of $A(\lambda_0)$ perturbs to eigenvalues of $A(\lambda)$ near 0 (the so-called 0-group) that are the eigenvalues of $A(\lambda) \in L(E_\lambda, E_\lambda)$.

Definition 2.4.4. *Assume that zero is a locally hyperbolic equilibrium of $A(\lambda) \in L(X, Z)$ for all $\lambda \in (\lambda_0 - \delta, \lambda_0) \cup (\lambda_0, \lambda_0 + \delta)$, which means that there is no eigenvalue in the 0-group of $A(\lambda)$ on the imaginary axis. Let $n^{>}(\lambda)$ be the number of all eigenvalues in the 0-group of $A(\lambda)$ (counting multiplicities) in the positive complex half-plane. This number is constant for $\lambda \in (\lambda_0 - \delta, \lambda_0)$ and for $\lambda \in (\lambda_0, \lambda_0 + \delta)$, and it is called the local Morse index of $A(\lambda)$ at 0. The number*

$$\mathcal{X}(A(\lambda), \lambda_0) = n^{>}(\lambda_0 - \varepsilon) - n^{>}(\lambda_0 + \varepsilon) \ , \ 0 < \varepsilon < \delta$$

is the crossing number of the family $A(\lambda)$ at $\lambda = \lambda_0$ through 0.

Note that a nonzero crossing number means that there is a change in the local Morse index.

Now suppose moreover that X is continuously embedded in Z and that Z is endowed with a scalar product.

Definition 2.4.5. *A continuous mapping $G : U \to Z$ is called a potential operator (with respect to that scalar product) if there exists a continuously differentiable mapping $g : U \to \mathbb{R}$ such that*

$$Dg(x)h = (G(x), h) \text{ for all } x \in U, h \in X.$$

The function g is called the potential of G.

Suppose $F \in C(\Lambda \times X, Y)$ and $D_u F \in C(\Lambda \times X, L(X, Y))$ such that $A(\lambda) := D_u F(0, \lambda)$ is a family of Fredholm operators of index 0 with closed domain $X \subset Z$ for all $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$. Suppose moreover that $F(\cdot, \lambda)$ is a family of potential operators.

Theorem 2.4.6. *Let F satisfies the hypotheses summarized above. Let 0 be an isolated eigenvalue of $A(\lambda_0) = D_x F(0, \lambda_0)$. If zero is a locally hyperbolic equilibrium of $A(\lambda) = D_x F(0, \lambda)$ for $\lambda \in (\lambda_0 - \delta, \lambda_0) \cup (\lambda_0, \lambda_0 + \delta)$ and if the crossing number $\chi(A(\lambda), \lambda_0)$ of the family $A(\lambda)$ at $\lambda = \lambda_0$ through 0 is nonzero, then $(0, \lambda_0)$ is a bifurcation point of $F(x, \lambda) = 0$.*

This theorem means that any change of the local Morse index of $A(\lambda) = D_x F(0, \lambda)$ at $\lambda = \lambda_0$ implies bifurcation of $F(x, \lambda) = 0$ at $\lambda = \lambda_0$.

2.5 The linearized system

Thus our first step is linearizing system (2.3.1) in order to find possible bifurcation values via Proposition 2.4.2. So, for $i = 1, \dots, n, j = 1, \dots, m$ we get:

$$\begin{cases} -t\Delta\varphi_i - t\varphi_i - [3t\mu_i(\alpha_1^i)^2\omega_1^2 + t\beta_1 \sum_{k\neq i}(\alpha_1^k)^2\omega_1^2]\varphi_i - 2t\beta_1 \sum_{k\neq i} \alpha_1^k\alpha_1^i\omega_1^2\varphi_k \\ \qquad +\gamma' \sum_{k=1}^m (\alpha_2^k)^2\omega_2^2\varphi_i + 2\gamma' \sum_{k=1}^m \alpha_2^k\alpha_1^i\omega_1\omega_2\psi_k = 0 \text{ in } \Omega \\ \\ -\Delta\psi_j - \psi_j - [3\nu_j(\alpha_2^j)^2\omega_2^2 + \beta_2 \sum_{k\neq j}(\alpha_2^k)^2\omega_2^2]\psi_j - 2\beta_2 \sum_{k\neq j} \alpha_2^k\alpha_2^j\omega_2^2\psi_k \\ \qquad +\gamma' \sum_{k=1}^n (\alpha_1^k)^2\omega_1^2\psi_j + 2\gamma' \sum_{k=1}^n \alpha_1^k\alpha_2^j\omega_1\omega_2\varphi_k = 0 \text{ in } \Omega \end{cases}$$

for $i = 1, \dots, n, j = 1, \dots, m$ which, using vector notation $\varphi = (\varphi_1, \dots, \varphi_n)$, $\psi = (\psi_1, \dots, \psi_m)$, $\alpha_1 = (\alpha_1^1, \dots, \alpha_1^n)$ and $\alpha_2 = (\alpha_2^1, \dots, \alpha_2^m)$ and setting

$$f_1(\beta_1) = (\frac{2}{g_1(\beta_1)} + 1) \ , \ f_2(\beta_2) = (\frac{2}{g_2(\beta_2)} + 1)$$

$$C_1 = [tf_1(\beta_1)\omega_1^2 + \gamma\omega_2^2]I_n - 2t\beta_1\omega_1^2\alpha_1 \otimes \alpha_1 \in \mathcal{M}^{n\times n}$$

$$C_2 = [f_2(\beta_2)\omega_2^2 + \gamma\omega_1^2]I_m - 2\beta_2\omega_2^2\alpha_2 \otimes \alpha_2 \in \mathcal{M}^{m\times m}$$

$$K_{1,2} = 2\gamma'\omega_1\omega_2\alpha_1 \otimes \alpha_2 \in \mathcal{M}^{n\times m} \ , \ K_{2,1} = 2\gamma'\omega_1\omega_2\alpha_2 \otimes \alpha_1 \in \mathcal{M}^{m\times n}$$

$$M = \begin{pmatrix} C_1 & K_{1,2} \\ K_{2,1} & C_2 \end{pmatrix}$$

where I_n is the identity $n \times n$ matrix, $\otimes$ is the usual tensor product of two vectors, $\mathcal{M}^{m\times n}$ is the space of matrices with m rows and n columns, as calculated in [14], we can rewrite the system in a matricial form as

$$(2.5.1) \qquad -\Delta \begin{pmatrix} t\varphi \\ \psi \end{pmatrix} - \begin{pmatrix} t\varphi \\ \psi \end{pmatrix} + M \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = 0$$

Proof of Proposition 2.3.4. First note that

$$(w \otimes w)w = ||w||^2 w$$

so the tensor matrix $w \otimes w$ has eigenvalue $||w||^2$ and it has multiplicity 1, because all the other eigenvalues are zero. In fact, if you consider $\bar{w} \in \mathcal{L}(w)^\perp$, then

$$(w \otimes w)\bar{w} = 0.$$

Thus, about the tensor matrices $\alpha_1 \otimes \alpha_1$ and $\alpha_2 \otimes \alpha_2$, we have that there exists two basis of eigenvectors constructing two orthonormal matrices $M_1 \in \mathcal{O}(n)$ and $M_2 \in \mathcal{O}(m)$ such that

$$M_1^T \alpha_1 \otimes \alpha_1 M_1 = D_1 = \begin{pmatrix} ||\alpha_1||^2 & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 0 \end{pmatrix} \in \mathcal{M}^{n\times n},$$

$$M_2^T\alpha_2 \otimes \alpha_2 M_2 = D_2 = \begin{pmatrix} ||\alpha_2||^2 & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 0 \end{pmatrix} \in \mathcal{M}^{m\times m}.$$

M_1 is constructed in this way. The first eigenvector is $\frac{\alpha_1}{||\alpha_1||^2}$. Vectors in the kernel have this form

$$b_i^k = \begin{cases} -\alpha_1^i & \text{if } k = 1 \\ \alpha_1^1 & \text{if } k = i \\ 0 & \text{if } k = 0 \end{cases}, \ i = 2, \ldots, n$$

Notice that $b_i \cdot \alpha_1 = 0$ for $i = 2, \ldots, n$, thus we can extract an orthonormal basis from the set $\{b_2, \ldots, b_n\}$ (which we call again $b_i, i = 2, \ldots, n$).

Constructing analogously M_2 from the vectors

$$c_j^k = \begin{cases} -\alpha_2^j & \text{if } k = 1 \\ \alpha_2^1 & \text{if } k = j \\ 0 & \text{if } k = 0 \end{cases}, \ j = 2, \ldots, m$$

we have found the shape of the two orthonormal matrices

$$M_1 = \left(\left. \frac{\alpha_1}{||\alpha_1||} \right| b_2 \left| \cdots \right| b_n \right)$$

$$M_2 = \left(\left. \frac{\alpha_2}{||\alpha_2||} \right| c_2 \left| \cdots \right| c_m \right).$$

Note that the vectors in the matrix notation are column vector. Observe that

$$M_1^T\alpha_1 \otimes \alpha_2 M_2 = D = \begin{pmatrix} ||\alpha_1||||\alpha_2|| & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0 \end{pmatrix} \in \mathcal{M}^{n\times m}. \tag{2.5.2}$$

Analogously $M_2^T\alpha_2 \otimes \alpha_1 M_1 = D^T = D \in \mathcal{M}^{m\times n}$. These two identities will be useful to diagonalize the system. In fact, we can make the orthogonal transformation of the functions (φ, ψ)

$$\begin{cases} \varphi' = M_1^T\varphi \\ \psi' = M_2^T\psi \end{cases} \tag{2.5.3}$$

Thus, substituting (2.5.3), applying M_1^T from the left in the first n equations of the system (2.5.1) and M_2^T form the left in the remaining m equations and using the identities like (2.5.2), we have

$$-\Delta \begin{pmatrix} t\varphi' \\ \psi' \end{pmatrix} - \begin{pmatrix} t\varphi' \\ \psi' \end{pmatrix} + \begin{pmatrix} t[f_1(\beta_1)\omega_1^2 + \gamma\omega_2^2]I_n & 0 \\ 0 & [f_2(\beta_2)\omega_2^2 + \gamma\omega_1^2]I_n \end{pmatrix} \begin{pmatrix} \varphi' \\ \psi' \end{pmatrix} \tag{2.5.4}$$

$$+2 \begin{pmatrix} -2t\beta_1\omega_1^2 D_1 & \gamma'\omega_1\omega_2 D \\ \gamma'\omega_1\omega_2 D^T & -2\beta_2\omega_2^2 D_2 \end{pmatrix} \begin{pmatrix} \varphi' \\ \psi' \end{pmatrix} = 0 \text{ in } \Omega.$$

In system (2.5.4) we have the strong competition term just in the first component of φ', ψ'.

By straightforward calculations we have that

$$f_i(\beta_i) - 2\beta_i||\alpha_i||^2 = 3 \ , \ i = 1,2 \qquad \text{and} \qquad \gamma'||\alpha_1||||\alpha_2|| = \gamma\sqrt{t}.$$

□

The next step is to understand what is the behaviour of the eigenvalues as $\gamma \to \infty$. For this, we need to recall a summary of results stated in [43], [56].

Theorem 2.5.1. *Let $\omega_{1,\gamma}, \omega_{2,\gamma}$ be a positive solution of (2.3.2) uniformly bounded in $L^\infty(\Omega)$. Then there exists a pair $(\omega_{1,\infty}, \omega_{2,\infty})$ of Lipschitz continuous functions such that, up to a subsequence, there holds*

1. $\omega_{1,\gamma} \to \omega_{1,\infty}$, $\omega_{2,\gamma} \to \omega_{2,\infty}$ *in* $C^{0,\alpha}(\overline{\Omega}) \cap H^1(\Omega)$, *for all* $\alpha \in (0,1)$;
2. $\omega_{1,\infty}\omega_{2,\infty} \equiv 0$ *in* Ω *and* $\int_\Omega \gamma\omega_{1,\gamma}\omega_{2,\gamma} \to 0$ *as* $\gamma \to +\infty$;
3. *reflection law:* $\lim_{x \to x_0^+} |\nabla\omega_1(x)| = \lim_{x \to x_0^-} |\nabla\omega_2(x)|$, *where* $x \to x_0^\pm$ *means that the functions are approaching the interface from opposite sides.*

Moreover, if we set

$$D_i = \{x \in \Omega : \omega_{i,\infty} > 0\} \ , \ i = 1,2,$$

the couple $(\omega_{1,\infty}, \omega_{2,\infty})$ *satisfies the system*

$$\begin{cases} -\Delta\omega_{1,\infty} - \omega_{1,\infty} = -\omega_{1,\infty}^3 & \text{in } D_1 \\ -\Delta\omega_{2,\infty} - \omega_{2,\infty} = -\omega_{2,\infty}^3 & \text{in } D_2 \end{cases} \tag{2.5.5}$$

Because $\omega_{1,\infty}, \omega_{2,\infty}$ are positive and cannot exist in the same region by Point 2 in the Theorem, the two open sets D_1 and D_2 are disjoint. This is a consequence of the effect of the competition parameter γ: when this is very large, going to infinity, the two species cannot coexist, hence they separate in two domains. Moreover, by a standard approach in phase separation models, we have that $\omega_{1,\gamma}$ tends uniformly to 0 on compact sets of D_2 exponentially in γ. In the same way, $\omega_{2,\gamma}$ tends uniformly to 0 on compact sets of D_1 exponentially in γ.

Proposition 2.5.2. *In the same setting as Theorem 2.5.1, if we consider a point $x_0 \in D_1$ with the property that there exists $r, a > 0$ such that $\omega_{1,\infty}(x) \geq a$ in $B(x_0, r)$, then for all $\varepsilon > 0$ there exist two constants $C, c > 0$ such that*

$$\omega_{1,\gamma}(x) \leq Ce^{-c\gamma}, \text{ for all } x \in B(x_0, r - \varepsilon) \subset D_1.$$

In the same way, if we consider a point $x_0 \in D_2$ with the property that there exists $r, a > 0$ such that $\omega_{2,\infty}(x) \geq a$ in $B(x_0, r)$, then for all $\varepsilon > 0$ there exist two constants $C, c > 0$ such that

$$\omega_{2,\gamma}(x) \leq Ce^{-c\gamma}, \text{ for all } x \in B(x_0, r - \varepsilon) \subset D_2.$$

Now we prove Proposition 2.3.9; it is important to remember that the eigenvalues of the problem (2.3.7) are found via min-max characterization, as expressed in (2.3.8).

Proof of Proposition 2.3.9. Take an open set $A_1 \subset D_1$ with compact closure in D_1. Clearly we have that $H_0^1(A_1) \subset H_0^1(D_1) \subset H_0^1(\Omega)$ and so, minimizing on a smaller set, we obtain

$$\lambda_{1,\gamma}^j(\Omega) \leq \lambda_{1,\gamma}^j(A_1).$$

Recall that, by Theorem 2.5.1, $\omega_{1,\gamma} \stackrel{\mathcal{C}^{0,\alpha}}{\to} \omega_{1,\infty}$, in particular $\omega_{1,\gamma} \stackrel{a.e.}{\to} \omega_{1,\infty}$. Now let u_γ be the function that gives $\lambda_{1,\gamma}^j(A_1)$. The sequence $\{u_\gamma\}$ is bounded in norm $H_0^1(A_1)$. By the compact embedding $H^1(\Omega) \subset\subset L^2(\Omega)$ given by Rellich Theorem, we obtain up to a subsequence that

$$u_\gamma \to u_\infty \text{ in } L^2(\Omega) \tag{2.5.6}$$

and by the uniform boundedness of $\{\omega_{1,\gamma}\}$ we get $\omega_{1,\gamma}u_\gamma \to \omega_{1,\infty}u_\infty$ in $L^2(\Omega)$. Moreover the operator $\nabla : H^1(\Omega) \to L^2(\Omega)$ is bounded and linear, thus we have $\nabla u_\gamma \rightharpoonup \nabla u_\infty$ in $L^2(A_1)$. Thus, testing the eigenvalue equation by $u_\gamma - u_\infty$ and integrating we get

$$\int_{A_1} \nabla u_\infty \cdot (\nabla u_\gamma - \nabla u_\infty) + |\nabla u_\gamma - \nabla u_\infty|^2 + Q(\gamma) = 0$$

with $Q(\gamma) \to 0$ as $\gamma \to +\infty$, because of (2.5.6). Thus, passing to the limit and using the weakly convergence of the gradient, we get

$$\int_{A_1} |\nabla u_\gamma - \nabla u_\infty|^2 = 0.$$

Remembering that $\omega_{2,\gamma}$ is exponentially decreasing on A_1 while γ goes to infinity, it's immediate to prove that

$$\lim_{\gamma \to +\infty} \lambda_{1,\gamma}^j(A_1) = \lambda^j(A_1)$$

which means that there is convergence of the eigenvalues and of the eigenfunctions of problem (2.3.7) in $H_0^1(A_1)$ to eigenvalues and eigenfunctions of the problem

$$-\Delta\phi - \phi = \lambda\omega_{1,\infty}^2\phi, \ \phi \in H_0^1(A_1).$$

Hence we obtain

$$\limsup_{\gamma \to \infty} \lambda_{1,\gamma}^j(\Omega) \leq \lambda^j(A_1). \tag{2.5.7}$$

By continuity of the eigenvalues with respect to the domain, for all $\varepsilon > 0$ we can find a set $A_\varepsilon \subset\subset D_1$ such that

$$\lambda^j(A_\varepsilon) \leq \lambda^j(D_1) + \varepsilon.$$

So, using (2.5.7), it follows

$$\lambda^j(D_1) \geq \lambda^j(A_\varepsilon) - \varepsilon \geq \limsup_{\gamma \to \infty} \lambda_{1,\gamma}^j(\Omega) - \varepsilon$$

for all $\varepsilon > 0$, thus

$$\lambda^j(D_1) \geq \limsup_{\gamma \to \infty} \lambda_{1,\gamma}^j(\Omega). \tag{2.5.8}$$

On the other side, by the identity

$$\int_\Omega |\nabla u_\gamma|^2 - u_\gamma^2 + \gamma\omega_{2,\gamma}^2 u_\gamma^2 = \lambda_{1,\gamma}(\Omega)\int_\Omega \omega_{1,\gamma}^2 u_\gamma^2,$$

one can easily obtain that $\int_\Omega \gamma\omega_{2,\gamma}^2 u_\gamma^2 \to 0$, which implies also that $u_\infty \equiv 0$ on D_2. From this fact, we can also infer that $u_\infty \in H_0^1(D_1)$, supposing $\partial D_1 \in \mathcal{C}^1$. Using the weak lower semicontinuity of the $H_0^1(\Omega)$-norm

$$\liminf_{\gamma\to+\infty}\int_\Omega |\nabla u_\gamma|^2 - u_\gamma^2 + \gamma\omega_{2,\gamma}^2 u_\gamma^2 \geq \int_\Omega |\nabla u_\infty|^2 - u_\infty^2,$$

which leads to

$$\lambda^j(D_1) \leq \liminf_{\gamma\to\infty}\lambda_{1,\gamma}^j(\Omega).$$

Thus, combining with (2.5.8)

$$\lambda^j(D_1) = \lim_{\gamma\to\infty}\lambda_{1,\gamma}^j(\Omega).$$

It works analogously with the equation (2.3.5). □

2.6 The competitive components

In order to work with the competitive system (2.3.6), we will state and prove our results in dimension 1. Without loss of generality, we can consider Ω as an interval containing a point x_0 such that $\omega_1(x_0) = \omega_2(x_0)$, which we can think as it is in the origin. There can be more than one point like x_0, but all the arguments are local, so we just focus on one.

We wil need some preliminary work. Suppose that there exists a sequence $\gamma_k \to \infty$ such that there is a nontrivial solution $(\varphi_1^{(k)}, \varphi_2^{(k)}) \in H_0^1(\Omega)$ of the system

$$\text{(2.6.1)} \qquad \begin{cases} (-\ddot{\varphi}_1 - \varphi_1 + 3\omega_1^2\varphi_1 + \gamma_k\omega_2^2\varphi_1)\, t + 2\gamma_k\sqrt{t}\omega_2\omega_1\varphi_2 = 0 \\ -\ddot{\varphi}_2 - \varphi_2 + 3\omega_2^2\varphi_2 + \gamma_k\omega_1^2\varphi_2 + 2\gamma_k\sqrt{t}\omega_1\omega_2\varphi_1 = 0 \end{cases}$$

We can get rid of the parameter t noting that $(\varphi_1, \varphi_2) = (\sqrt{t}\varphi_1^{(k)}, \varphi_2^{(k)})$ solves

$$\text{(2.6.2)} \qquad \begin{cases} -\ddot{\varphi}_1 - \varphi_1 + 3\omega_1^2\varphi_1 + \gamma_k\omega_2^2\varphi_1 + 2\gamma_k\omega_2\omega_1\varphi_2 = 0 \\ -\ddot{\varphi}_2 - \varphi_2 + 3\omega_2^2\varphi_2 + \gamma_k\omega_1^2\varphi_2 + 2\gamma_k\omega_1\omega_2\varphi_1 = 0 \end{cases}$$

which is actually the linearized system of (2.3.2). So we will just consider $(\varphi_1^{(k)}, \varphi_2^{(k)})$ as a solution of (2.6.2). The first remark is that $(\dot{\omega}_1, \dot{\omega}_2)$ solves system (2.6.2). Notice that also $c(\varphi_1, \varphi_2)$ is a solution for all $c \in \mathbb{R}$, thus we can normalize the solution. Because we have by the Sobolev embeddings that $H_0^1(\Omega) \hookrightarrow C^{0,\alpha}(\Omega)$, we can normalize taking the L^∞ norm

$$||\varphi_1^{(k)}||_\infty = 1 \ , \ ||\varphi_2^{(k)}||_\infty = 1.$$

We apply a blow up argument to see the behaviour of $(\varphi_1^{(k)}, \varphi_2^{(k)})$ close to the interface. This argument comes from [15], to which we refer for more details. Set

$$\begin{array}{rclcrcl} V_1^{(R)}(x) & = & \frac{1}{R}\omega_1^{(k)}(Rx) & , & V_2^{(R)}(x) & = & \frac{1}{R}\omega_2^{(k)}(Rx) \\ \Phi_1^{(R)}(x) & = & \varphi_1^{(k)}(Rx) & , & \Phi_2^{(R)}(x) & = & \varphi_2^{(k)}(Rx) \end{array}$$

where $R = \gamma_k^{-\frac{1}{4}} \overset{\gamma\to+\infty}{\longrightarrow} 0$. Note that all the families of functions $\{V_1^{(R)}\}$, $\{V_2^{(R)}\}$, $\{\Phi_1^{(R)}\}$, $\{\Phi_2^{(R)}\}$ are equicontinuous and uniformly bounded on compact sets of $\mathbb{R}$. Thus they converge in $C^2_{loc}(\mathbb{R})$ to solutions of the systems

$$\begin{cases} \ddot{V}_1 = V_2^2 V_1 \\ \ddot{V}_2 = V_1^2 V_2 \end{cases} \tag{2.6.3}$$

$$\begin{cases} -\ddot{\Phi}_1 = -V_2^2\Phi_1 - 2V_1V_2\Phi_2 \\ -\ddot{\Phi}_2 = -V_1^2\Phi_2 - 2V_1V_2\Phi_1 \end{cases} \tag{2.6.4}$$

with $(V_1, V_2), (\Phi_1, \Phi_2) \in H^1(\mathbb{R})^2$ and $||\Phi_1||_\infty = ||\Phi_2||_\infty = 1$.

Thus we can apply a result about the nondegeneracy of the blow up system, see [15], Theorem 1.3. We state here what we need.

Theorem 2.6.1. *If V_1, V_2 is a nonnegative solution of system (2.6.3) and $\Phi_1, \Phi_2 \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfy system (2.6.4), then*

1. *there exists $x_0 \in \mathbb{R}$ such that $V_2(y - x_0) = V_1(x_0 - y)$ for $y \in \mathbb{R}$;*
2. *Either*
$$\begin{cases} V_1(-\infty) = 0, \dot{V}_1(-\infty) = 0, \dot{V}_1 > 0, \dot{V}_1(\infty) = \sqrt{T_\infty}, \\ V_2(\infty) = 0, \dot{V}_2(\infty) = 0, \dot{V}_2 < 0, \dot{V}_2(-\infty) = -\sqrt{T_\infty}, \end{cases}$$
or likewise with V_1 and V_2 interchanged, where $T_\infty > 0$ is the constant that satisfies
$$\dot{V}_1^2 + \dot{V}_2^2 - V_1^2V_2^2 = T_\infty \ , \text{ in } \mathbb{R};$$
3. *$(\Phi_1, \Phi_2) \equiv C(\dot{V}_1, \dot{V}_2)$, for some $C \in \mathbb{R}$.*

Remark 2.6.2. *For simlpicity, we suppose $C = 1$. Moreover, as we already said, without loss of generality we can suppose $x_0 = 0$. So the fact that $V_1(-x) = V_2(x)$ implies $\dot{V}_1(-x) = -\dot{V}_2(x)$ tell us that the only element in the kernel of the linearized operator has the simmetry*
$$\Phi_1(-x) = -\Phi_2(x).$$

Next step is to control the behaviour of $\varphi_1^{(k)}, \varphi_2^{(k)}$ near the interface.

Lemma 2.6.3. *If we consider the difference $\phi^{(k)} = \varphi_1^{(k)} - \varphi_2^{(k)}$, then for all $\varepsilon > 0$ there exist K, K' positive such that if $z \in [-\gamma^{-\frac{1}{4}}K', -\gamma^{-\frac{1}{4}}K]$ we have*

$$\lim_{\gamma\to+\infty} |\phi^{(k)}(z) - \sqrt{T_\infty}| < 2\varepsilon, \tag{2.6.5}$$

and if $z \in [\gamma^{-\frac{1}{4}}K, \gamma^{-\frac{1}{4}}K']$ we have again

$$\lim_{\gamma\to+\infty} |\phi^{(k)}(z) - \sqrt{T_\infty}| < 2\varepsilon. \tag{2.6.6}$$

Moreover for $z \in [-\gamma^{-\frac{1}{4}}K', -\gamma^{-\frac{1}{4}}K] \cup [\gamma^{-\frac{1}{4}}K, \gamma^{-\frac{1}{4}}K']$ it holds

$$\lim_{\gamma\to+\infty} |\dot{\phi}^{(k)}(z)| < \varepsilon. \tag{2.6.7}$$

Proof. We know, as was studied in [15], the main features of the profile of Φ_1, Φ_2. For example $\Phi_1(x) = \dot{V}_1(x)$ goes to 0 as $x \to -\infty$ and to $\sqrt{T_\infty} > 0$ as $x \to +\infty$ and it is positive by point 3 of Theorem 2.6.1. So in particular for all $\varepsilon > 0$ there exist $K > 0$ such that if $x > K$ then $|\Phi_1(x) - \sqrt{T_\infty}| < \varepsilon$. Then consider $K' > K$ and $z \in [\gamma^{-\frac{1}{4}}K, \gamma^{-\frac{1}{4}}K']$ and setting $z = \gamma^{-\frac{1}{4}}x$

$$\lim_{\gamma\to+\infty} |\varphi_1^{(k)}(z) - \sqrt{T_\infty}| = \lim_{\gamma\to+\infty} |\varphi_1^{(k)}(\gamma^{-\frac{1}{4}}x) - \sqrt{T_\infty}| = |\Phi_1(x) - \sqrt{T_\infty}| < \varepsilon.$$

On the same side for $\varphi_2^{(k)}$ holds

$$\lim_{\gamma\to+\infty} |\varphi_2^{(k)}(z)| < \varepsilon.$$

While on the other side of the interface $z \in [-\gamma^{-\frac{1}{4}}K', -\gamma^{-\frac{1}{4}}K]$ we have

$$\lim_{\gamma\to+\infty} |\varphi_2^{(k)}(z) + \sqrt{T_\infty}| < \varepsilon \ , \ \lim_{\gamma\to+\infty} |\varphi_1^{(k)}(z)| < \varepsilon.$$

These inequalities imply that $\varphi_1^{(k)}, \varphi_2^{(k)}$ do not oscillate while getting closer to the interface and moreover they approach some precise values.

Now considering $\phi^{(k)}$, we immediately get (2.6.5) and (2.6.6). Making analogous calculations with the fact that

$$\lim_{x\to\pm\infty} \ddot{\Phi}_i(x) = 0 \ , \ i = 1, 2,$$

one immediately gets (2.6.7). □

These calculations where valid when close to the interface, where the two components interact together. Now we will need some estimates to understand the behaviour of φ_1, φ_2 related to ω_1, ω_2 in the domains D_1, D_2.

If we restrict our attention to the domain $D_1 = \{x \in \Omega : \omega_{1,\infty}(x) > 0\}$ and consider a compact set $K_1 \subset D_1$ then we have, as we already said before, that $\omega_{2,\gamma}$ tends to 0 uniformly in K_1 and exponentially in γ as $\gamma \to +\infty$.

Moreover $(\varphi_1^{(k)}, \varphi_2^{(k)}) \to (\phi_1, \phi_2)$ uniformly in K_1 by Ascoli-Arzelà Theorem.

The first equation of the competitive system (2.6.2) tends to a solution of

$$-\ddot{\phi}_1 - \phi_1 + 3\omega_{1,\infty}^2 \phi_1 = 0 \text{ on } K_1.$$

For the second equation we can test against φ_2 and integrate over K_1 (leaving the index k in order not to make heavy the notation)

$$\int_{K_1} \dot{\varphi}_2^2 - \varphi_2^2 + 3\omega_2^2\varphi_2^2 + \gamma\omega_1^2\varphi_2^2 + 2\gamma\omega_1\omega_2\varphi_1\varphi_2 = 0$$

in which the terms with ω_2 tend to 0. Because all the other terms are bounded, we get that

$$\int_{K_1} \gamma\omega_1^2\varphi_2^2 \to 0,$$

which means that also ω_1 and φ_2 tend to have disjoint supports. Thus $\phi_2 \equiv 0$ on K_1.

By simmetry, the same happens when we consider a compact set $K_2 \subset D_2$. $\phi_1 \equiv 0$ on K_2 while ϕ_2 satisfies

$$-\ddot{\phi}_2 - \phi_2 + 3\omega_{2,\infty}^2 \phi_2 = 0 \text{ on } K_2.$$

Proof of Proposition 2.3.8. If we set $w = \omega_{1,\infty} - \omega_{2,\infty}$ we have that

$$-\ddot{w} - w + w^3 = 0 \text{ in } \Omega.$$

Set also the operators $L_1 = -\Delta - I + w^2$, $L_2 = L_1 + 2w^2$. So if we consider the eigenvalue problem

$$\left\{ \begin{array}{ll} L_1\phi = \lambda\phi & \text{in } \Omega \\ \phi \in H_0^1(\Omega) \end{array} \right. \tag{2.6.8}$$

we have that $\lambda = 0$ is an eigenvalue with corresponding eigenfunction w. Moreover, because $w \in H_0^1(\Omega)$, is of class C^1 by a reflection law stated in [56] and it changes sign at least one time. So 0 is not the first eigenvalue, at least the second one: $\lambda_j(L_1) = 0$ for some $j \in \mathbb{Z}, j > 1$.

Now set $\phi^{(k)} = \varphi_1^{(k)} - \varphi_2^{(k)}$. Then, gluing together the limit equations in D_1 and D_2 and the estimates (2.6.5),(2.6.6) close to the interface, $\phi^{(k)}$ in the limit as $\gamma \to +\infty$ solves a Sturm-Liouville problem

$$\left\{ \begin{array}{ll} -\ddot{\phi}_\infty - \phi_\infty + 3w^2\phi_\infty = 0 & \text{in } \Omega \\ \phi_\infty = 0 & \text{on } \partial\Omega \end{array} \right. \tag{2.6.9}$$

for which we have another solution $\dot{w} = \dot{\omega}_{1,\infty} - \dot{\omega}_{2,\infty}$ of (2.6.9) that doesn't satisfies the boundary condition, but has at least two zeroes inside the interval Ω. By Sturm Oscillation Lemma, we have that between two zeroes of $\dot{w}$ there is one zero of ϕ_∞. So 0 is not the principal eigenvalue of the associated eigenvalue problem to the system (2.6.9), moreover we have $\lambda_i(L_2) = 0$ for some $i \in \mathbb{Z}, i \geq j$.

But if we compare the two operators L_1, L_2, we have that the eigenvalues of L_2 are all greater than the eigenvalues of L_1. Thus

$$0 = \lambda_i(L_2) \geq \lambda_j(L_2) > \lambda_j(L_1) = 0,$$

which is a contradiction, coming from the assumption of the existence of the solution $(\varphi_1^{(k)}, \varphi_2^{(k)})$. $\square$

2.7 Sufficient condition for bifurcation

This section will be devoted to the proof of Theorem 2.3.6. In order to check if the values $\beta_{i,k}, i = 1, 2, k = 1, 2, \dots$ are bifurcation values, we want to apply Theorem 2.4.6 and see if there are changes in the morse index of the energy functional J when passing through these values. Part of the calcultations are repeated in the same way as in [14], Lemma 4.1.

Proof of Theorem 2.3.6. The energy functional J has the form

$$\begin{aligned} J_{\beta_1,\beta_2}(u, v) &= t\left(\tfrac{1}{2}\textstyle\sum_{i=1}^n \int_\Omega |\nabla u_i|^2 - u_i^2 - \tfrac{1}{4}\sum_{i=1}^n \int_\Omega \mu_i u_i^4 - \tfrac{\beta_1}{2}\sum_{k<i} \int_\Omega u_i^2 u_k^2\right) \\ &\quad + \tfrac{1}{2}\textstyle\sum_{j=1}^m \int_\Omega |\nabla v_j|^2 - v_j^2 - \tfrac{1}{4}\sum_{j=1}^m \int_\Omega \nu_j v_j^4 - \tfrac{\beta_2}{2}\sum_{k<j} \int_\Omega v_j^2 v_k^2 \\ &\quad + \tfrac{\gamma'}{2}\textstyle\sum_{i=1}^n \sum_{j=1}^m \int_\Omega u_i^2 v_j^2. \end{aligned}$$

It is well defined for $N \leq 3$ on the space $\mathcal{H} = (H_0^1(\Omega))^{n+m}$ and it is of class C^2. The hessian of J at a point (β_1, β_2, u, v) in the branch τ has this form

$$\begin{aligned} Q_{\beta_1,\beta_2}(\phi, \psi) &= \langle J''_{\beta_1,\beta_2}(u,v)\begin{pmatrix}\phi\\ \psi\end{pmatrix}, \begin{pmatrix}\phi\\ \psi\end{pmatrix}\rangle \\ &= t\left(\int_\Omega |\nabla\phi|^2 - |\phi|^2\right) + \int_\Omega |\nabla\psi|^2 - |\psi|^2 + \int_\Omega \langle M\begin{pmatrix}\phi\\ \psi\end{pmatrix}, \begin{pmatrix}\phi\\ \psi\end{pmatrix}\rangle \end{aligned}$$

As in [14] if we have only the trivial solution of (2.3.6) the kernel of Q_{β_1,β_2} with respect to the value $\beta_{1,k}$ is

$$V_{1,k} = \{\phi \in H_0^1(\Omega)^n : -\Delta\phi_i - \phi_i + \gamma\omega_{2,\gamma}^2\phi_i = \lambda_{1,\gamma}^k\omega_{1,\gamma}^2\phi_i,\ i = 1, \ldots, n \text{ and } \sum_{i=1}^n \alpha_1^i\phi_i = 0\},$$

while with respect to the value $\beta_{2,l}$ is

$$V_{2,l} = \{\psi \in H_0^1(\Omega)^m : -\Delta\psi_j - \psi_j + \gamma\omega_{1,\gamma}^2\psi_j = \lambda_{2,\gamma}^l\omega_{2,\gamma}^2\psi_j,\ j = 1, \ldots, m \text{ and } \sum_{j=1}^m \alpha_2^j\psi_j = 0\}$$

for $k, l = 1, 2, \ldots$. We have different cases, in which we touch one possible bifurcation value only for β_1, or else only for β_2 or else more for both of them, as long as we don't come across any other possible bifurcation value with respect to the parameter γ. Respectively, in these cases, the kernel $\mathcal{K}$ is of this form

$$(1)\ \mathcal{K} = V_{1,k} \times \{0\}$$

$$(2)\ \mathcal{K} = \{0\} \times V_{2,l}$$

$$(3)\ \mathcal{K} = V_{1,k} \times V_{2,l}$$

Thus, at one of the values $(\bar\beta_1, \bar\beta_2) \in \{(\beta_1, \beta_2) : \beta_1 = \beta_{1,k} \text{ or } \beta_2 = \beta_{2,l})\}$, we can decompose $\mathcal{H}$ in the direct sum of its positive eigenspace V^+, the kernel $\mathcal{K}$ and the negative eigenspace V^-. Obviously $Q_{\bar\beta_1,\bar\beta_2} > 0$ on V^+ and $Q_{\bar\beta_1,\bar\beta_2} < 0$ on V^-. Moreover we have the expansion

$$Q_{\beta_1,\beta_2} = Q_{\bar\beta_1,\bar\beta_2} + (\beta_1 - \bar\beta_1)\frac{\partial}{\partial\beta_1}Q(\bar\beta_1, \bar\beta_2) + (\beta_2 - \bar\beta_2)\frac{\partial}{\partial\beta_2}Q(\bar\beta_1, \bar\beta_2) + o(|\beta_1 - \bar\beta_1| + |\beta_2 - \bar\beta_2|)$$

as $(\beta_1, \beta_2) \to (\bar\beta_1, \bar\beta_2)$. This formula implies that $Q_{\beta_1,\beta_2} > 0$ on V^+ and $Q_{\beta_1,\beta_2} < 0$ on V^- if (β_1, β_2) is sufficiently close to $(\bar\beta_1, \bar\beta_2)$. Thus we have to study the derivative of the hessian with respect to β_1 in case (1), to β_2 in case (2), to both of them in case (3).

For case (1), with $\psi = 0$, we obtain

$$\frac{\partial}{\partial\beta_1}Q = \frac{\partial t}{\partial\beta_1}\left(\int_\Omega |\nabla\phi|^2 - |\phi|^2 + \langle C_1\phi, \phi\rangle\right) + t\int_\Omega \langle\frac{\partial}{\partial\beta_1}C_1\phi, \phi\rangle$$

but $\phi \in V_{1,k}$, so the first term is 0. By just a simple calculation as in [14], Lemma 4.1, one can show that

$$\langle\frac{\partial}{\partial\beta_1}C_1\phi, \phi\rangle = \omega_1^2 f_1'(\beta_{1,k})|\phi|^2,$$

so we obtain that

$$\frac{\partial}{\partial\beta_1}Q = t\int_\Omega \omega_1^2 f_1'(\beta_{1,k})|\phi|^2$$

Thus we have bifurcation if $f_1'(\beta_{1,k})$ has fixed sign, which surely happens in the focusing and defocusing case for parameters $\mu_i, i = 1, \dots, n$. It may fail in some mixed case.

For case (2), with $\phi = 0$, we obtain

$$\frac{\partial}{\partial\beta_2}Q = \int_\Omega \langle \frac{\partial}{\partial\beta_2}C_2\psi, \psi\rangle$$

but $\psi \in V_{2,l}$ and as before we get

$$\frac{\partial}{\partial\beta_2}Q = \int_\Omega \omega_2^2 f_2'(\beta_{2,l})|\psi|^2$$

Thus we have bifurcation if $f_2'(\beta_{2,l})$ has fixed sign, which surely happens in the focusing and defocusing case for parameters $\nu_j, j = 1, \dots, m$. It may fail in some mixed case.

For case (3), in a similar way, we have

$$\begin{aligned}
\tfrac{\partial}{\partial\beta_1}Q &= \tfrac{\partial t}{\partial\beta_1}\left(\textstyle\int_\Omega |\nabla\phi|^2 - |\phi|^2 + \langle C_1\phi, \phi\rangle\right) \\
&\quad + t\textstyle\int_\Omega \langle \tfrac{\partial}{\partial\beta_1}C_1\phi, \phi\rangle + 2\gamma\int_\Omega \omega_1\omega_2\langle \alpha_2 \otimes \tfrac{\partial}{\partial\beta_1}(\tfrac{\alpha_1}{||\alpha_1||^2})\phi, \psi\rangle \\
&= t\textstyle\int_\Omega \omega_1^2 f_1'(\beta_{1,k})|\phi|^2
\end{aligned}$$

and

$$\begin{aligned}
\tfrac{\partial}{\partial\beta_2}Q &= \tfrac{\partial t}{\partial\beta_2}\left(\textstyle\int_\Omega |\nabla\phi|^2 - |\phi|^2 + \langle C_1\phi, \phi\rangle\right) \\
&\quad + 2\gamma\textstyle\int_\Omega \omega_1\omega_2\langle (\tfrac{\partial}{\partial\beta_2}\alpha_2) \otimes \tfrac{\alpha_1}{||\alpha_1||^2}\phi, \psi\rangle + \int_\Omega \langle \tfrac{\partial}{\partial\beta_2}C_2\psi, \psi\rangle \\
&= \textstyle\int_\Omega \omega_2^2 f_2'(\beta_{2,l})|\psi|^2,
\end{aligned}$$

that we got always applying the fact that $\phi \in V_{\beta_{1,k}}, \psi \in V_{\beta_{2,l}}$ and [14]. Thus in this case we have bifurcation for $(\beta_{1,k}, \beta_{2,l})$ in one of the following cases:

(i) $f_1'(\beta_{1,k})$ and $f_2'(\beta_{2,l})$ are both different from 0 and have the same sign;

(ii) $f_1'(\beta_{1,k}) = 0$ and $f_2'(\beta_{2,l}) \neq 0$;

(iii) $f_1'(\beta_{1,k}) \neq 0$ and $f_2'(\beta_{2,l}) = 0$.

Case *(i)* for example can happen if the two families of equations are both in the same case, focusing or defocusing. Case *(ii)* and *(iii)* instead can happen in some mixed case for one family of equations and focusing or defocusing for the other family. □

Remark 2.7.1. *According to bifurcation theory, it is a standard question whether we have a global bifurcation point. By Crandall-Rabinowitz Theorem, this happens if the crossing number is odd. The value of the crossing number is given by the kernel $\mathcal{K}$ as in previous proof.*

In case (1), this is equal to $n-1$ times the dimension of the eigenspace

$$\{\phi \in H_0^1(\Omega) : -\Delta\phi - \phi + \gamma\omega_{2,\gamma}^2\phi = \lambda_{1,\gamma}^k\omega_{1,\gamma}^2\phi\}.$$

We can set it to be the number n_k. Thus the question is whether $(n-1)n_k$ is odd or not. If $n = 2$ and $n_k = 1$, which holds for example when $N = 1$, then Crandall-Rabinowitz Theorem applies and yields locally a smooth curve of bifurcating solutions.

In case (2) one should look at the dimension m_l of the eigenspace

$$\{\phi \in H_0^1(\Omega) : -\Delta\phi - \phi + \gamma\omega_{1,\gamma}^2\phi = \lambda_{2,\gamma}^l \omega_{2,\gamma}^2\phi\},$$

and in the same way as before one can apply global bifurcation Theorem when $m = 2$, $m_l = 1$ and $N = 1$.

In case (3) we have that the crossing number is equal to $(n-1)n_k + (m-1)m_l$ which is given by a sum of two integers and it is greater than one, so we cannot obtain further information.

Bibliography

[1] A. Aftalion, C. Sourdis, *Interface Layer of a Two-Component Bose-Einstein Condensate*, ArXiv,1509.08328

[2] A. Ambrosetti, G. Prodi, *A Primer of Nonlinear Analysis*, Cambridge University Press, 1995

[3] H. Aref: *Point vortex dynamics: A classical mathematical playground.* J. Math. Phys 48 (2007), 1-22.

[4] H. Aref: *Stirring by chaotic advection.* J. Fluid. Mech. 143 (1984) 1–21.

[5] C. Bandle, M. Flucher: *Harmonic radius and concentration of energy; hyperbolic radius and Liouville's equations* $\Delta u = e^U$ *and* $\Delta U = U^{\frac{n+2}{n-2}}$. SIAM Review 38 (1996), 191–238.

[6] T. Bartsch: *Periodic solutions of singular first-order Hamiltonian systems of* N*-vortex type.* Arch. Math. (2016), DOI: 10.1007/s00013-016-0928-9.

[7] T. Bartsch, Q. Dai: *Periodic solutions of the N-vortex Hamiltonian system in planar domains.* J. Diff. Eq. 260 (3) (2016), 2275–2295.

[8] T. Bartsch, Q. Dai, B. Gebhard: *Periodic solutions of the N-vortex Hamiltonian system near the domain boundary.* Preprint.

[9] T. Bartsch, N. Dancer, and Z.-Q. Wang, *A Liouville theorem, a-priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system*, Calc. Var. 37 (2010), 345-361.

[10] T. Bartsch, B. Gebhard: *Global continua of periodic solutions of singular first-order Hamiltonian systems of N-vortex type.* arXiv:1604.01576.

[11] T. Bartsch, A. Pistoia: *Critical points of the* N*-vortex Hamiltonian in bounded planar domains and steady state solutions of the incompressible Euler equations*, SIAM J. Appl. Math. **75 (2)** (2015), 726–744.

[12] T. Bartsch, A. Pistoia, and T. Weth: N*-vortex equilibria for ideal fluids in bounded planar domains and new nodal solutions of the sinh-Poisson and the Lane-Emden-Fowler equations.* Comm. Math. Phys. **297 (3)** (2010), 653–686.

[13] T. Bartsch, M. Sacchet, *Periodic solutions with prescribed minimal period of the 2-vortex problem in domains*, arXiv:1608.06775.

[14] T. Bartsch, R. Tian, Z. Q. Wang, *Bifurcations for a coupled Schrödinger system with multiple components*, Angew. Math. Phys. (2015) 66: 2109. doi:10.1007/s00033-015-0498-x

[15] H. Berestycki, T. C. Lin, J. Wei, C. Zhao, *On phase-separation models: asymptotics and qualitative properties*, Arch. Ration. Mech. Anal. 208 (2013), 163200.

[16] A. Boscaggin, P. J. Torres: *Periodic motions of fluid particles induced by a prescribed vortex path in a circular.* Phys. D 261 (2013), 81-84.

[17] L. A. Caffarelli, A. Friedman: *Convexity of solutions of semilinear elliptic equations.* Duke Math. J. 52 (1985), 431-456.

[18] L. A. Caffarelli, F. H. Lin, *An optimal partition problem for eigenvalues*, J. Sci. Comp. 31 (2007) 5-18.

[19] S.M. Chang, C.S. Lin, T.C. Lin, and W.W. Lin, *Segregated nodal domains of two-dimensional multispecies Bose-Einstein condensates*, Phys. D 196, 341-361 (2004)

[20] M. Conti, S. Terracini, G. Verzini, *An optimal partition problem related to non linear eigenvalues*, J. Funct. Anal. 198 (2003), no. 1, 160-196;

[21] M. Conti, S. Terracini, and G. Verzini, *Nehari's problem and competing species systems*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 19(6):871-888, 2002.

[22] M. Conti, S. Terracini, G. Verzini, *On a class of optimal partition problems related to the Fučík spectrum and to the monoticity formulae*, Calc. Var. 22 (2005), 45-72.

[23] Q. Dai: Periodic solutions of the N point-vortex problem in planar domains. Ph. D. Dissertation, University of Giessen 2014.

[24] E. N. Dancer, K. Wang, Z. Zhang, *Uniform Hölder estimate for singularly perturbed parabolic systems of Bose-Einstein condensates and competing species*, J. Diff. Eq. 251, 2737-2769 (2011).

[25] E. N. Dancer, K. Wang, Z. Zhang, *Dynamics of strongly competing systems with many species*, Trans. Am. Math. Soc. 364, 961-1005 (2012).

[26] E. N. Dancer, K. Wang, Z. Zhang, *The limit equation for the Gross-Pitaevskii equations and S. Terracini's conjecture*, J. Funct. Anal. 262(2), 1087-1131 (2012).

[27] W. Y. Ding: *A generalization of the Poincaré-Birkhoff theorem.* Proc. Amer. Math. Soc. 88 (1983), 341–346.

[28] M. Flucher: *Variational Problems with Concentration*, Birkhuser, 1999

[29] A. Fonda, M. Sabatini, F. Zanolin: *Periodic solutions of perturbed Hamiltonian systems in the plane by the use of the Poincaré-Birkhoff theorem.* Topol. Methods Nonlinear Anal. 40 (2012), 29–52.

[30] A. Fonda, A. J. Ureña: *A higher dimensional Poincaré-Birkhoff theorem for Hamiltonian flows.* Ann Inst. H. Poincaré, Non Lin. Anal. (2016), DOI:10.1016/j.anihpc.2016.04.002.

[31] D. S. Hall, M. R. Matthews, J. R. Ensher, C. E. Wieman, and E. A. Cornell, *Dynamics of component separation in a binary mixture of Bose-Einstein condensates*, Phys. Rev. Lett., 81(8):15391542, 1998.

[32] H. Kielhöfer, *Bifurcation Theory. An Introduction with Applications to PDEs*, Springer-Verlag New York, Applied Mathematical Sciences 156 (2004), DOI: 10.1007/b97365

[33] G. R. Kirchhoff: Vorlesungen über mathematische Physik. Teubner, Leipzig 1876.

[34] P. Le Calvez, J. Wang: *Some remarks on the Poincaré-Birkhoff theorem.* Proc. Amer. Math. Soc. 138 (2010), 703–715.

[35] E. Lieb, R. Seiringer, J.P. Solovej and J. Yngvason, *The Mathematics of the Bose Gas and its Condensation*, vol.34, Oberwolfach Seminars Series, Birkhaeuser (2005)

[36] C. C. Lin: *On the motion of vortices in two dimensions. I. Existence of the Kirchhoff-Routh function.* Proc. Nat. Acad. Sci. USA 27 (1941), 570-575.

[37] C. C. Lin: *On the motion of vortices in two dimensions. II. Some further investigations on the Kirchhoff-Routh function.* Proc. Nat. Acad. Sci. USA 27 (1941), 575-577.

[38] A. J. Majda, A. L. Bertozzi: Vorticity and Incompressible Flow. Cambridge University Press, Cambridge 2001.

[39] C. Marchioro, M. Pulvirenti: Mathematical Theory of Incompressible Nonviscous Fluids. Applied mathematical sciences 96, Springer, New York 1994.

[40] R. Martins, A. J. Ureña: *The star-shaped condition on Ding's version of the Poincaré-Birkhoff theorem.* Bull. Lond. Math. Soc. 39 (2007), 803–810.

[41] A. M. Micheletti, A. Pistoia: *Non degeneracy of critical points of the Robin function with respect to deformations of the domain.* Potential Anal. 40 (2) (2014), 103-116.

[42] P. K. Newton: The N-vortex problem. Springer-Verlag, Berlin 2001.

[43] B. Noris, H. Tavares, S. Terracini, G. Verzini, *Uniform Hölder Bounds for Nonlinear Schrödinger Systems with Strong Competition*, Comm. Pure Appl. Math. 63 (2010), 267302.

[44] S. B. Papp, J. M. Pino, and C. E. Wieman, *Tunable miscibility in a dual-species Bose-Einstein condensate*, Phys. Rev. Lett., 101:040402, 2008.

[45] L. Pitaevskii, S. Stringari, *Bose-Einstein condensation*, Oxford University Press, USA (2003)

[46] E. J. Routh: *Some applications of conjugate functions.* Proc. London Math. Soc. (S1) 12 (1) (1881), 73–89.

[47] Ch. Rüegg et al., *Bose-Einstein condensation of the triple states in the magnetic insulator tlcucl3*, Nature 423, 62-65 (2003)

[48] P. G. Saffman: Vortex Dynamics. Cambridge Univ. Press, Cambridge 1995.

[49] B. Sirakov, *Least energy solitary waves for a system of Nonlinear Schrödinger equations in* $\mathbb{R}^n$, Commun. Math. Phys. 271, 199-221 (2007).

[50] N. Soave, *On existence and phase separation of solitary waves for nonlinear Schrödinger systems modelling simultaneous cooperation and competition*, Calc. Var. Partial Differential Equations 53 (2015), 3-4, 689–718, DOI 10.1007/s00526-014-0764-3.

[51] N. Soave and A. Zilio, *Entire solutions with exponential growth for a nonlinear elliptic system modeling phase-separation*, Nonlinearity, 27 (2) (2014), 305-342.

[52] N. Soave and A. Zilio, *Multidimensional entire solutions for an elliptic system modelling phase separation.* Analysis and Partial Differential Equations, 9(5) (2016), 1019-1041

[53] N. Soave and A. Zilio, *Uniform bounds for strongly competing systems: the optimal Lipschitz case*. Archive Ration. Mech. Anal., 218 (2015), 647-697.

[54] M. Squassina, S. Zuccher, *Numerical computations for the spatial segregation limit of some 2D competition-diffusion systems*, Adv. Math. Sci. Appl. 18 (2008), 83104.

[55] M. Struwe: *Existence of periodic solutions of Hamiltonian systems on almost every energy surface*. Bol. Soc. Bras. Mat. 20 (1990), 49-58.

[56] H. Tavares, S. Terracini, *Regularity of the nodal set of segregated critical configurations under a weak reflection law*, Calculus of Variations and Partial Differential Equations 45 (3-4), 273-317.

[57] J.C. Wei and T. Weth, *Asymptotic behavior of solutions of planar systems with strong competition*, Nonlinearity 21, 305-317 (2008)