Estimating Parameters under Equality and Inequality Restrictions

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Estimating Parameters under Equality and Inequality Restrictions

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This is to certify that the work presented in the dissertation entitled *Estimating Parameters under Equality and Inequality Restrictions* submitted by *Adarsha Kumar Jena*, Roll Number 512MA1006, is a record of original research carried out by him under my supervision and guidance in partial fulfillment of the requirements of the degree of *Doctor of Philosophy* in *Mathematics*. Neither this dissertation nor any part of it has been submitted earlier for any degree or diploma to any institute or university in India or abroad.

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Dedication

Dedicated
to
my Parents

Adarsha Kumar Jena

Declaration of Originality

I, *Adarsha Kumar Jena*, Roll Number *512MA1006* hereby declare that this dissertation entitled *Estimating Parameters under Equality and Inequality Restrictions* presents my original work carried out as a doctoral student of NIT Rourkela and, to the best of my knowledge, contains no material previously published or written by another person, nor any material presented by me for the award of any degree or diploma of NIT Rourkela or any other institution. Any contribution made to this research by others, with whom I have worked at NIT Rourkela or elsewhere, is explicitly acknowledged in the dissertation. Works of other authors cited in this dissertation have been duly acknowledged under the sections “Reference” or “Bibliography”. I have also submitted my original research records to the scrutiny committee for evaluation of my dissertation.

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Abstract

The problem of estimating statistical parameters under equality or inequality (order) restrictions has received considerable attention by several researchers due to its vast applications in various physical, industrial and biological experiments. For example, the problem of estimating the common mean of two normal populations when the variances are unknown has a long history and is popularly known as "common mean problem". This problem is also referred as Meta-Analysis, where samples (data) from multiple sources are combined with a common objective. The "common mean problem" has its origin in the recovery of inter-block information when dealing with Balanced Incomplete Block Designs (BIBDs) problems. In this thesis, we study problem of estimating parameters and quantiles of two or more normal and exponential populations when the parameters are equal or ordered from decision theoretic point of view.

In **Chapter 1**, we give the motivation and do a detailed review of literature for the following problems. In **Chapter 2**, we discuss some basic definitions and decision theoretic results which are useful in developing the subsequent chapters. In **Chapter 3**, the problem of estimating the common mean of two normal populations has been considered when it is known a priori that the variances are ordered. Under order restriction on the variances, some new alternative estimators have been proposed including one that uses the maximum likelihood estimator (MLE). These new estimators beat some of the existing popular estimators in terms of stochastic domination as well as Pitman measure of closeness criterion. In **Chapter 4,** we have considered the problem of estimating quantiles for $k(\geq 2)$ normal populations with a common mean. A general result has been proved which helps in obtaining better estimators. Introducing the principle of invariance, sufficient conditions for improving estimators in certain equivariant classes have been derived. As a consequence some complete class results have been proved. A detailed simulation study has been carried out in order to numerically compare the performances of all the proposed estimators for the cases $k = 3$ and 4. A similar type of result has also been obtained for estimating the quantile vector. In **Chapter 5**, we deal with the problem of estimating quantiles and ordered scales of two exponential populations under equality assumption on the location parameters using type-II censored samples. First, we consider the estimation of quantiles of first population when type-II censored samples are available from two exponential populations. Sufficient conditions for improving equivariant estimators have been derived and as a consequence improved estimators have been obtained. A detailed simulation study has been carried out to compare the performances of improved estimators along with some of the existing ones. Further, we deal with the problem of estimating vector of ordered scale parameters. Under order restriction on the scale parameters, we derive the restricted maximum likelihood estimator for the vector parameter. We obtain classes of equivariant estimators and prove some inadmissibility results. Consequently, improved estimators have been derived. Finally a numerical comparison has been done among all the proposed estimators. In **Chapter 6**, the problem of estimating ordered quantiles of two exponential populations is considered assuming equality of location parameters. Under order restriction, we propose new estimators which are the isotonized version of some baseline estimators. A sufficient condition for improving equivariant estimators are derived under order restriction on quantiles. Consequently, estimators improving upon the baseline estimators are derived. Further, the problem of estimating ordered quantiles of two exponential populations is considered assuming equality of the scale parameters using type-II censored samples. Under order restrictions on the quantiles, isotonized version of some existing estimators have been proposed. Bayes estimators have been derived for the quantiles assuming order restriction on the quantiles. In **Chapter 7**, we consider the estimation of the common scale parameter of two exponential populations when the location parameters satisfy a simple ordering. Bayes estimators using uniform prior and a conditional inverse gamma prior have been obtained. Finally all the derived estimators have been numerically compared along with some of the existing estimators. In **Chapter 8**, we give an overall conclusion of the results obtained in the thesis and discuss some of our future research work.

***Keywords*:** *Admissibility*; *Bayes estimator*; *Common mean*; *Equivariant estimator*; *Inadmissibility*; *Isotonic regression*; *Maximum likelihood estimator* (MLE); *Ordered parameters*; *Quantiles*; *Quadratic loss* ; *Relative risk performance*; *Squared error loss*; *Type-II censored samples*; *Uniformly minimum variance unbiased estimator* (UMVUE).

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Chapter 1

Introduction

1.1 Introduction

The problem of estimating statistical parameters under equality or inequality (order) restrictions has received considerable attention by several researchers in the recent years. For example, the problem of estimating the common mean of two normal populations when the variances are unknown has a long history in the literature and is popularly known as "common mean problem". This problem is also referred as Meta-Analysis, where samples (data) from multiple sources are combined with a common objective. The "common mean problem" has its origin in the recovery of inter-block information when dealing with Balanced Incomplete Block Designs (BIBDs) problems. Here two independent unbiased estimators (intra-block and inter-block) for the treatment contrasts are available. The target is to develop an estimator by combining intra-block and inter-block, which may perform better than either of these. Similarly, the problem of estimating parameters under certain inequality (order) restrictions is of considerable interest and has been extensively studied by several researchers in the recent past. This type of statistical models arise in various physical, agricultural, industrial, biological and medical experiments. Below we discuss certain practical situations where modeling of the problem leads to the assumption of equality or/and inequality restrictions on the involved parameters.

1. Suppose there are n laboratories or operators evaluating a given product. It is quite possible to assume that the locations of the measured aspect of the product to be the same, where as the scales may differ due to laboratory techniques or facilities. The assumption on the distribution of the measured quantity may follow a particular location-scale family.

2. A particular type of products (electrical/mechanical) has been manufactured by different companies and to be lunched in the market. Because of market restrictions, the minimum guarantee periods (location) of the products may be same, whereas the average lives (scale) may be different. The life times of the products may follow certain life-time distributions. On the basis of prior information, one may be interested to estimate the parameters.

3. Suppose the farmers of a country use three types of treatments to grow the crops: treatment-I (using chemical fertilizers), treatment-II (using organic manures) and

treatment-III (without using any fertilizers). Let $\theta_1, \theta_2, \theta_3$ be denote the average yields by using the three types of treatments respectively. It is natural that $\theta_1 \geq \theta_2 \geq \theta_3$ and one would be interested in estimating one or all of $(\theta_1, \theta_2, \theta_3)$.

In this thesis, we have considered the problem of estimating equal or ordered parameters when the underlying distribution is either normal or exponential. Moreover, we have focused on estimating quantiles of these populations when the concerned or nuisance parameters are equal or ordered. We note that for these distributions, quantiles are linear function of location and scale parameters.

1.2 A Review of Literature

In this section we give a detailed review of literature on certain problems which are relevant and useful for developing the chapters of thesis.

1.2.1 Estimation of Common Mean of Two Normal Populations

The problem of estimating common mean of two normal populations is an age old problem and has a long history in the literature of statistical inference. The problem has received considerable attention by several authors in the last few decades due to its practical applications as well as theoretical challenges involve in it. Particularly, the problem has been well investigated from classical as well as decision theoretic point of view when there is no order restrictions on the variances. The problem is quite popular in the literature and is popularly known as "common mean problem". The problem has been originated from the study of recovery of inter-block information while dealing with balanced incomplete block design (BIBD) problems (see Shah (1964)). Probably, Yates (1940) was the first to consider the problem under normality assumption. Let $(X_{i1}, X_{i2}, \ldots, X_{in_i})$; $i = 1, 2$ be a random sample taken from the i^{th} normal population $N(\mu, \sigma_i^2)$. The problem is to estimate the common parameter μ when the variances are unknown and unequal with respect to the loss function

$$L_1(d, \mu) = (d - \mu)^2$$

or,

$$L_2(d, \mu) = \left(\frac{d - \mu}{\sigma_1}\right)^2,$$

where d is an estimator for μ. Let us define the random variables

$$\bar{X}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij}, \ S_i^2 = \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2; i = 1, 2.$$

The random variables $\bar{X}_i$ and S_i^2 are statistically independent and minimal sufficient for this model. We also note that these minimal sufficient statistics are not complete.

One of the first pioneering research work in this direction was done by Graybill and Deal (1959). They proposed a new combined estimator for the common mean μ by taking convex combination of $\bar{X}_1$ and $\bar{X}_2$ with weights as the functions of sample variances. Their combined estimator is given by

$$d_{GD} = \frac{n_1(n_1-1)S_2^2\bar{X}_1 + n_2(n_2-1)S_1^2\bar{X}_2}{n_1(n_1-1)S_2^2 + n_2(n_2-1)S_1^2}.$$

They have proved that the combined estimator d_{GD} performs better than both $\bar{X}_1$ and $\bar{X}_2$ in terms of variances (loss function L_1) when the sample sizes are at least 11. The estimator d_{GD} is also known as the best asymptotically normal and conditionally unbiased. After then a lot of research work has been done in this direction by several authors using classical as well as decision theoretic approaches. Their target has been to derive either some alternative estimators for μ which may compete with d_{GD} or proving some decision theoretic results like admissibility or minimaxity. Very surprisingly still now it remains an open problem whether d_{GD} is admissible or inadmissible.

For small sample sizes ($n_1, n_2 \leq 10$) Zacks (1966) proposed two classes of testimators using F-test. Let us denote $\tau = \sigma_2^2/\sigma_1^2$. Case-1: Consider testing the hypothesis $H_0 : \tau = 1$ against $H_1 : \tau \neq 1$. If H_0 is accepted then use the grand mean of two samples for estimating μ, otherwise use the Graybill-Deal estimator d_{GD}. Mathematically the testimator is written as

$$d_1(\tau^*) = \begin{cases} \frac{\bar{X}_1+\bar{X}_2}{2}, & \text{if } \frac{1}{\tau^*} \leq \frac{S_2^2}{S_1^2} \leq \tau^* \\ d_{GD}, & \text{otherwise,} \end{cases}$$

where τ^* is the critical value of the F-tests and $1 \leq \tau^* \leq \infty$. Case-2: Consider the testing procedure for the three alternatives, $H_0 : \tau = 1$, $H_1 : \tau > 1$ and $H_2 : \tau < 1$. If H_0 is true use the grand mean as an estimator for μ. If either H_1 or H_2 holds true then use the sample mean which has the smaller variance as the estimator for μ. This estimator can be written as

$$d_2(\tau^*) = \begin{cases} \bar{X}_2, & \text{if } \frac{S_2^2}{S_1^2} < \frac{1}{\tau^*} \\ \frac{\bar{X}_1+\bar{X}_2}{2}, & \text{if } \frac{1}{\tau^*} \leq \frac{S_2^2}{S_1^2} \leq \tau^* \\ \bar{X}_1, & \text{if} \frac{S_2^2}{S_1^2} > \tau^*. \end{cases}$$

Finally the author compared numerically the performances of all the estimators in these two classes.

Mehta and Gurland (1969) considered the estimation of μ under the assumption that the nuisance parameters follow a certain simple ordering say $\sigma_1^2 \leq \sigma_2^2$. They have proposed the following class of estimators.

$$d(\psi) = \psi(T)\bar{X}_1 + (1-\psi(T))\bar{X}_2,$$

where $T = S_2^2/S_1^2$. Depending upon the choices of ψ, the following three types of estimators can be proposed.

(i) $\delta_1 = d(\psi)$, where $\psi(T) = \frac{(c+T)}{c+a+T}$,

(ii) $\delta_2 = d(\psi)$, where

$$\psi(T) = \begin{cases} \frac{1}{2}, & \text{if } T < k \\ \frac{T}{T+a}, & \text{if } T \geq k. \end{cases}$$

(iii) $\delta_3 = d(\psi)$, where $\psi(T) = \frac{\sqrt{(c+T)}}{\sqrt{(c+T)}+a}$ and a, c, k are specific constants to be suitably chosen.

For the case $n_1 = n_2$, they proved that the estimator δ_1 performs better than d_{GD} for some choices of a, c and k. Further for some choices of k they have also proved that the estimator δ_2 performs better than d_{GD}. Finally, authors numerically compared the efficiencies of all the above three estimators.

Zacks (1970) considered the problem of estimation of common mean μ using the decision theoretic approach. The author proposed an equivariant class of estimators for μ which is given by

$$d_Z = \bar{X}_1 + (\bar{X}_2 - \bar{X}_1)\phi(T_1, T_2),$$

where $T_1 = \frac{S_1}{(\bar{X}_1 - \bar{X}_2)^2}$, and $T_2 = \frac{S_2}{(\bar{X}_2 - \bar{X}_1)^2}$. This class contains estimators that was previously proposed by Zacks (1966) and Mehta and Gurland (1969). He also proved that the estimator $\bar{X}_1$ is minimax with respect to the loss L_2. Further using a symmetric loss function, he proved that the grand mean is minimax. Zacks also derived the generalized Bayes estimator with respect to the Jeffrey's prior which is known as fiducial equivariant estimators. They also proved that these Bayes estimators are weakly admissible.

Khatri and Shah (1974) considered a general class of estimators for μ which is given by

$$d_{KS} = (1 - \phi(W))\bar{X}_1 + \phi(W)\bar{X}_2,$$

where $\phi(W) = \frac{n_2 S_1^2}{n_2 S_1^2 + n_1 c S_2^2}$ and $c = \frac{(n_1-3)(n_2-1)}{(n_2-3)(n_1-1)}$. The estimator d_{KS} improves on $\bar{X}_1$ in terms of variance if $n_2 \geq 2$. Further d_{KS} improves upon both $\bar{X}_1$ and $\bar{X}_2$ if $(n_1 - 7)(n_2 - 7) \geq 16$. Hence the estimator d_{KS} can be used in certain situations where d_{GD} fails to improve upon $\bar{X}_1$ and $\bar{X}_2$.

Cohen and Sackrowitz (1974) constructed the following class of estimators when $n_1 = n_2 = n(say)$.

$$d_{CS} = (1 - C_n H(z))\bar{X}_1 + C_n H(z)\bar{X}_2,$$

where

$$C_n = \begin{cases} \frac{(n-3)^2}{(n+1)(n-1)}, & \text{if n is odd,} \\ \frac{n-4}{n+2}, & \text{if n is even.} \end{cases}$$

and

$$H(z) = \begin{cases} F(1,(3-n)/2,(n-1)/2,z), & \text{for } 0 \le z \le 1, \\ -\frac{(n-3)}{(n-1)z}F(1,(5-n)/2,(n+1)/2,1/z), & \text{for } z \ge 1. \end{cases}$$

Here F is the hyper geometric function. For $n \geq 5$, they have shown that the estimator d_{CS} improves upon $\bar{X}_1$ using L_1. Further for the case $n \geq 10$, the estimator

$$d^*{}_{CS} = (1 - H(z))\bar{X}_1 + H(z)\bar{X}_2$$

performs better than both $\bar{X}_1$ and $\bar{X}_2$.

Brown and Cohen (1974) proposed the following class of estimators given by,

$$d_{BC1}(b) \;=\; \bar{X}_1 + \left\{\frac{(bS_1^2/n_1(n_2-1))(\bar{X}_2-\bar{X}_1)}{S_1^2/n_1(n_1+1)+S_2^2/n_2(n_2+2)+(\bar{X}_2-\bar{X}_1)^2/(n_2+2)}\right\},$$

where $0 < b \leq b_{max}(n_1, n_2) = 2(n_2+2)/n_2E(max(V^{-1}, V^{-2}))$. Here V follows F distribution with $(n_2{+}2)$ and $(n_1{-}1)$ degrees of freedom. The estimator $d_{BC1}(b)$ is unbiased for μ. When $n_2 \geq 3$, they have shown that $d_{BC1}(b)$ performs better than $\bar{X}_1$. They also established that for $n_2 = 2, n_1 \geq 2$, the estimator $d_{BC1}(b)$ is not better than $\bar{X}_1$ for any choices of b. Further, authors constructed a different class of unbiased estimators of the form

$$d_{BC2}(p,b) \;=\; \bar{X}_1 + \left\{\frac{(bS_1^2/n_1(n_1-1))(\bar{X}_2-\bar{X}_1)}{p(S_1^2/n_1(n_1-1)+S_2^2/n_2(n_2-1))+(1-p)(\bar{X}_2-\bar{X}_1)^2}\right\},$$

where $0 < p < 1$ and $0 < b < b_{max}(n_1, n_2-3)$. When $n_1 \geq 2$, $n_2 \geq 3$, they have proved that there exist values of b (> 0) for which $d_{BC2}(p, b)$ performs better than $\bar{X}_1$. They have also generalized some of their results to $k(\geq 2)$ normal populations.

Bhattacharya (1980) proposed a class of estimators that includes the estimators proposed by Brown and Cohen (1974) and Khatri and Shah (1974).

Sinha and Mouqadem (1982) proved the admissibility of the estimator d_{GD} in certain classes of estimators. They defined the class D and its members D_0, D_1, D_2 as follows.

$$\begin{aligned} D &= \{d = \bar{X}_1 + (\bar{X}_2-\bar{X}_1)\phi;\ 0 \le \phi(S_1^2, S_2^2, \bar{X}_2-\bar{X}_1) \le 1\}, \\ D_0 &= \{d = \bar{X}_1 + (\bar{X}_2-\bar{X}_1)\phi,\ 0 \le \phi(\frac{S_2^2}{S_1^2}) \le 1\}, \\ D_1 &= \{d = \bar{X}_1 + (\bar{X}_2-\bar{X}_1)\phi,\ 0 \le \phi(S_1^2, S_2^2) \le 1\}, \\ D_2 &= \{d = \bar{X}_1 + (\bar{X}_2-\bar{X}_1)\phi,\ 0 \le \phi(\frac{S_1^2}{(\bar{X}_2-\bar{X}_1)^2}, \frac{S_2^2}{(\bar{X}_2-\bar{X}_1)^2}) \le 1\}. \end{aligned}$$

The loss function is taken as L_1. The authors proved that the estimator d_{GD} is admissible in the class D_0 for $n_1 = n_2 \geq 2$ and extended admissible in D.

Bhattacharya (1986) observed that the conclusions of Cohen and Sackrowitz (1974) regarding improvements upon $\bar{X}_1$ and $\bar{X}_2$ are not correct. He proved that the estimator d_{CS} dominates $\bar{X}_1$ when $n \geq 7$ and, both $\bar{X}_1$ and $\bar{X}_2$ when $n \geq 15$.

Kubokawa (1987a) considered a general class of estimators for estimating μ which is given by

$$d_\phi(a, b, c) = \bar{X}_1 + \frac{a(\bar{X}_2 - \bar{X}_1)}{1 + R\phi(S_1^2, S_2^2, (\bar{X}_2 - \bar{X}_1)^2)},$$

where $R = \{bS_2^2 + c(\bar{X}_1 - \bar{X}_2)^2\}/S_1^2$ and ϕ is any positive real valued function. The estimator $d_\phi(a, b, c)$ improves upon $\bar{X}_1$ and is also minimax with respect to the loss L_2 when $0 \leq a \leq 2$, $b \geq c > 0$. Furthermore, for the choice of $\phi = 1 + d/\{bS_2^2 + c(\bar{X}_1 - \bar{X}_2)^2\}$, the above estimator is reduces to

$$d_1(a, b, c, d) = \bar{X}_1 + \frac{aS_1^2(\bar{X}_2 - \bar{X}_1)}{S_1^2 + bS_2^2 + c(\bar{X}_1 - \bar{X}_2) + d}.$$

For particular choices of a, b, c and d, the above class produces estimators which were proposed by Graybill and Deal (1959), Khatri and Shah (1974), Brown and Cohen (1974), Bhattacharya (1979) and Kubokawa (1987b).

Kubokawa (1989) proposed a class of estimators that dominate $\bar{X}_1$ in terms of Pitman Measure of Closeness (PMC). In particular he proved that the $\hat{\mu}_{GD}$ dominates $\bar{X}_1$ and $\bar{X}_2$ if the sample sizes are at least 5.

Nanayakkara and Cressie (1991) proposed a new class of estimators for the common mean μ which is given by

$$d_{NC}(r) = \Big(\frac{\alpha_1 \bar{X}_1}{S_1^r} + \frac{\alpha_2 \bar{X}_2}{S_2^r}\Big) / \Big(\frac{\alpha_1}{S_1^r} + \frac{\alpha_2}{S_2^r}\Big), \ r > 0.$$

For the case $r = 2$, they have obtained necessary and sufficient condition on α_1 and α_2 for which the estimator $d_{NC}(r)$ improves upon $\bar{X}_1$ and $\bar{X}_2$.

Kelleher (1996) considered the problem of estimating common mean for small and equal sample sizes. The author obtained a Bayes estimator by considering a prior for the ratio of variances $\sigma_1^2/\sigma_2^2 = \tau$.

$$d_B(R) = \frac{\int_0^\infty \tau f(R|\tau) dQ(\tau)}{\int_0^\infty (\tau + 1) f(R|\tau) dQ(\tau)}.$$

where $R = S_2^2/S_1^2$. He also proved numerically that $d_B(R)$ perform better than the estimator proposed by Zacks (1966).

Mitra and Sinha (2007) studied the common mean problem from Bayesian point of view. He has obtained the generalized Bayes estimator with respect to Jeffrey's prior. The prior is taken as

$$f(\mu, \sigma_1^2, \sigma_2^2) = (\sqrt{\sigma_1^2 + \sigma_2^2})/(\sigma_1^2\sigma_2^2)^{3/2}, \ -\infty < \mu < \infty, \ \sigma_1^2 > 0, \ \sigma_2^2 > 0.$$

The estimator is given by

$$d_{MS} = \frac{\int_0^\infty \tau^{n/2}(\tau+1)^n/(a\tau^2 + b\tau + c)^{n+1} d\tau}{\int_0^\infty \tau^{n/2}(\tau+1)^{n+1}/(a\tau^2 + b\tau + c)^{n+1} d\tau}.$$

The authors also numerically compared the risk performance of d_{MS} with the estimators proposed by Graybill and Deal (1959), Sinha (1979) and Sinha and Mouqadem (1982).

Pal et al. (2007) have obtained variance of the maximum likelihood estimator (MLE) of μ and the variance of d_{GD} numerically. They have shown by using simulation that, in most of the parameter ranges, the MLE has the smaller variance than d_{GD}.

Tripathy and Kumar (2010) revisited the problem of estimating common mean of two normal populations when the variances are unknown and unequal. Authors have established some decision theoretic results using a quadratic loss function. They have also obtained an alternative estimator for μ, by modifying the estimator proposed by Moore and Krishnamoorthy (1997). Their estimator is given by

$$d_{TK} = \frac{\bar{X}_1\sqrt{n_1}c_{n_2}S_2 + \bar{X}_2\sqrt{n_2}c_{n_1}S_1}{\sqrt{n_1}c_{n_2}S_2 + \sqrt{n_2}c_{n_1}S_1}$$

where $c_{n_i} = \frac{\Gamma(\frac{n_i-1}{2})}{\sqrt{2}\Gamma(\frac{n_i}{2})}$; $i = 1, 2$. Authors also obtained sufficient conditions for improving certain classes of equivariant estimators for the common mean. Through a simulation study, they have numerically compared the risk values of all the proposed estimators and recommended for their use. Their numerical comparison reveals that the estimator d_{TK} compete well with the estimator proposed by Tripathy and Kumar (2010).

1.2.2 Estimating Common Mean of Several Normal Populations (A Generalization to $k(\geq 3)$ Populations)

In this section we review the literature on the problem of estimating common mean of $k(\geq 3)$ normal populations when the variances are unknown and possibly unequal. Suppose $(X_{i1}, X_{i2}, \ldots, X_{in_i})$; $i = 1, 2, \ldots, k$, be a random sample taken from the i^{th} normal population $N(\mu, \sigma_i^2)$. Consider the problem of estimating μ with respect to the losses L_1 and L_2 as defined in previous section. Let us define the random variables

$$\bar{X}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} X_{ij}, \ S_i^2 = \sum_{j=1}^{n_i}(X_{ij} - \bar{X}_i)^2, i = 1, 2, \ldots, k.$$

Probably Norwood and Hinkelmann (1977) was the first to consider the problem for $(k \geq 2)$ normal populations. In fact, the authors have generalized the estimator given by Graybill and

Deal (1959) to $k(\geq 2)$ populations and the estimator is given by

$$d_{NH} = \frac{\sum_{i=1}^{k} n_i(n_i-1)\bar{X}_i/S_i^2}{\sum_{i=1}^{k} n_i(n_i-1)/S_i^2}.$$

It has been shown that the estimator d_{NH} performs better than each of $\bar{X}_i$ with respect to the loss L_1 if and only if each $n_i \geq 11$ or one $n_i = 10$, and all other $n_j \geq 18, i \neq j : j = 1, 2, \ldots, n_i$ and $i = 1, 2, \ldots, k$.

Shinozaki (1978) constructed a general class of estimators that contains d_{NH} and derived conditions for improving upon each $\bar{X}_i$. Their proposed estimator is given by

$$d_{SZ} = \frac{\sum_{i=1}^{k} c_i n_i(n_i-1)\bar{X}_i/S_i^2}{\sum_{i=1}^{k} c_i n_i(n_i-1)/S_i^2}.$$

The author established that the estimator d_{SZ} performs better than each $\bar{X}_i$ if and only if $n_i \geq 8$ and $(n_i - 7)(n_j - 7) \geq 16$ for $i \neq j$.

Sinha (1979) established that the estimator d_{NH} is inadmissible with respect to a general type of loss function, when for some i $\sigma_i^2 \leq \sigma_j^2$, $i \neq j$. In fact Sinha's result generalizes the inadmissibility result of Mehta and Gurland (1969).

Bhattacharya (1984) developed two general inequalities and used these to obtain a better estimator for μ. He also obtained improvements over shinozaki's (Shinozaki (1978)) result.

Kubokawa (1987c) considered the estimation of μ for $(k \geq 2)$ normal populations with respect to a symmetric loss function defined by

$$L(d, \mu) = \psi(|d-\mu|^r),\ 0 < r < \infty,$$

where ψ is a decreasing concave function of non negative real numbers and satisfies $\psi(0) = 0$. Author proposed a general class of estimators which is given by

$$d_K = \frac{\sum_{i=1}^{k} c_i(n_i-1)\bar{X}_i/S_i^2}{\sum_{i=1}^{k} c_i(n_i-1)/S_i^2},$$

where c_is are positive constants. Further he proved that the estimator d_K is better than each $\bar{X}_i$ if $n_i \geq 6$ and $c_j/c_i \leq 2(n_j - 5)/(n_i + 1)$ for $i \neq j; i, j = 1, 2, \ldots, k$.

Sarkar (1991) extended the results of kubokawa (Kubokawa (1989)) to k normal populations.

Moore and Krishnamoorthy (1997) constructed a new type of combined estimator for μ by taking convex combination of $\bar{X}_i$s with weights inversely proportinal to their standard errors which is given by

$$d_{MK} = \frac{\sum_{i=1}^{k} \sqrt{n_i(n_i-1)}\bar{X}_i/S_i}{\sum_{i=1}^{k} \sqrt{n_i(n_i-1)}/S_i}.$$

The authors numerically compared the estimator d_{MK} with that of d_{NH} in terms of the variances through a simulation study. Their numerical study reveals that the estimator d_{MK} performs better than d_{NH} when either the sample sizes are small or the population variances are close to each other.

Tripathy and Kumar (2015) have investigated the problem of estimating common mean μ of several normal populations using a decision theoretic approach with respect to a quadratic loss function. They have modified the estimator proposed by Moore and Krishnanmoorthy (1997) and is given by

$$d_{TK} = \frac{\sum_{i=1}^{k} \sqrt{n_i}\bar{X}_i/b_{n_i-1}S_i}{\sum_{i=1}^{k} \sqrt{n_i}/b_{n_i-1}S_i}.$$

The authors obtained classes of affine and location equivariant estimators and proved some inadmissibility results in these classes. As a consequence some complete class results have been proved. In addition to these, the authors also numerically compared the risk values of their proposed estimators with other well known estimators (including the MLE which has been obtained numerically) for the case $k = 3, 4$ using the Monte-Carlo simulation method. Finally recommendations have been made for the use of all these estimators for various choices of the parameters.

1.2.3 Estimating Common Mean (Variance) when the Nuisance Parameters are Ordered

The problem of estimating common mean or variance when the nuisance parameters (parameters other than our study of interest) satisfying certain ordering has received attentions by few researchers in the recent past. Suppose $(X_{i1}, X_{i2}, \ldots, X_{in_i}); i = 1, 2$ is a random sample taken from the i^{th} normal population $N(\mu, \sigma_i^2)$. The problem of interest is to estimate the common parameter μ under the assumption that $\sigma_1^2 \leq \sigma_2^2$.

Perhaps Sinha (1979) was the first to consider this model with equal sample sizes when the loss function is strictly increasing in $|d - \mu|$. He proposed a new estimator which dominates d_{GD} stochastically as well as universally. The proposed estimator is given by

$$d_S = \begin{cases} d_{GD}, & \text{if } \frac{S_1^2}{n_1-1} \leq \frac{S_2^2}{n_2-1} \\ \frac{\bar{X}_1+\bar{X}_2}{2}, & \text{if } \frac{S_1^2}{n_1-1} > \frac{S_2^2}{n_2-1}. \end{cases}$$

Elfessi and Pal (1992) considered the same model and proposed new estimators for both equal and unequal sample sizes. Their estimators are given by

$$d_{EP} = \begin{cases} d_{GD}, & \text{if } S_1^2 \leq S_2^2 \\ \delta \bar{X}_1 + (1-\delta)\bar{X}_2, & \text{if } S_1^2 > S_2^2, \end{cases}$$

and

$$d_{EP} = \begin{cases} d_{GD}, & \text{if } \frac{S_1^2}{n_1-1} \leq \frac{S_2^2}{n_2-1} \\ \frac{n_1\bar{X}_1+n_2\bar{X}_2}{n_1+n_2}, & \text{if } \frac{S_1^2}{n_1-1} > \frac{S_2^2}{n_2-1}, \end{cases}$$

for equal and unequal sample sizes respectively, where $\delta = S_1^2/(S_1^2 + S_2^2)$. The authors proved that the estimator d_{EP} dominates d_{GD} universally as well as stochastically when $\sigma_1^2 \leq \sigma_2^2$. They also obtained the percentage of risk improvement of d_{EP} over d_{GD} using both absolute error loss and squared error loss, numerically.

Misra and van der Meulen (1997) generalized the results obtained by Elfessi and Pal (1992) to $k(\geq 2)$ normal populations. Furthermore they have proved that the proposed new estimator dominates its old counter part (d_{NH}, extension of Graybill-Deal estimator to the case $k(\geq 2)$) in terms of Pitman measure of closeness criteria.

Chang et al. (2012) considered the problem of estimating common and ordered means of two normal populations assuming that the variances follow a simple ordering. They have proposed a class of estimators for the common mean μ of the form

$$\hat{\mu}(\gamma) = \gamma\bar{X}_1 + (1-\gamma)\bar{X}_2,$$

where γ is a function of $s_1^2, s_2^2, \bar{x}_1 - \bar{x}_2$ and $0 \leq \gamma \leq 1$. Here $\bar{x}_i$ and s_i^2 are the sample mean and variance of the i^{th} population respectively. They have proved that the estimator $\hat{\mu}(\gamma)$ dominates the Graybill-Deal estimator stochastically. Similarly they have chosen two classes of plug-in type estimators for the ordered means μ_1, and μ_2; $\mu_1 \leq \mu_2$ as

$$\hat{\mu}_1(\gamma) = \min(\bar{X}1, \gamma\bar{X}_1 + (1-\gamma)\bar{X}_2) \text{ and } \hat{\mu}_2(\gamma) = \max(\bar{X}_1, \gamma\bar{X}_1 + (1-\gamma)\bar{X}_2),$$

respectively. The estimator $\hat{\mu}_2(\gamma)$ dominates stochastically $\bar{X}_2$ where as a similar type of result does not hold true in the case of $\hat{\mu}_1(\gamma)$.

Gupta and Singh (1992) investigated the problem of estimating common variance of two normal populations when it is known a priori that the means follow a simple ordering say $\mu_1 \leq \mu_2$. Under order restrictions on the means, authors established that the restricted MLEs of the common variance and the ordered means dominate their old counter parts (the unrestricted MLEs, that is, estimators without taking account order restrictions on the means) in terms of Pitman measure of closeness criteria.

Tripathy et al. (2013) considered the problem of estimating common standard deviation (σ) of two normal populations under order restrictions on the means using a scale invariant loss function. A general minimaxity result has been proved and a class of minimax estimators is derived. An admissibility result is proved in this class. Further a class of equivariant estimators with respect to a subgroup of affine group is considered and dominating estimators in this class are obtained. The risk performance of all these proposed estimators is compared through a simulation study.

1.2.4 Estimating Common Parameter in Exponential Populations

It is quite clear from the literature that the estimation of common parameter (common mean) in the case of normal populations has been extensively studied by several authors in the recent past. Along the same direction, when the distribution is two-parameter exponential, some study has also been done by authors. In fact, the model has applications in the study of reliability, life testing and survival analysis, hence the problem is also known as the estimation of "common minimum guarantee time". To be very specific, let $(X_{i1}, X_{i2}, ..., X_{in_i})$ be a random sample taken from the i^{th} population Ex(μ, σ_i); $i = 1, 2, \ldots k$. Here μ is the location parameter which is common to all the populations, and σ_is are known as the scale parameters. The parameter μ is also referred as the minimum guarantee time and σ_i as the mean residual life times in the literature due to its application in reliability. The problem is to estimate the common parameter μ when the scale parameters are unknown. This model has been investigated from classical as well as decision theoretic point of view. Below we discuss certain related results in chronological order.

Probably, Ghosh and Razmpour (1984) were the first to consider this model and proposed various estimators such as the maximum likelihood estimator (MLE), a modification of the MLE (MMLE) and the uniformly minimum variance unbiased estimator (UMVUE) for μ when the scale parameters are unknown. They have also numerically compared the mean squared errors and the biases of these estimators for small and large sample sizes.

Pal and Sinha (1990) considered the same model as above and compared the performances of the MLE, MMLE and UMVUE in terms of mean squared error (MSE) and PMC criterion as well. Further they obtained a class of estimators that performs better than the MLE with respect to the squared error loss function and PMC criterion. Though they have observed that this class contains some variants of the MMLE and the UMVUE, however the variant of MMLE has been shown to be inadmissible with respect to the squared error loss function. Further, Jin and Pal (1992) suggested a wide class of estimators that dominates the MLE using a convex loss function.

Jin and Crouse (1998b) considered this problem of estimating common location parameter μ of several exponential populations when the scale parameters are unknown and unequal, using a general class of convex loss functions. Authors have derived a larger class of estimators that contains the MMLE and the UMVUE. Latter on Jin and Crouse (1998a) proved an identity and used it to compare the performances of the quantiles using squared error loss function. In addition to this they have used this identity to obtain a class of estimators for the common location parameter which dominates the MLE and the UMVUE.

The problem of estimation of common location parameter μ when the nuisance parameters (σ_is) are known to satisfy certain simple ordering (inequality restrictions) is of special interest. Tripathy et al. (2014) considered the estimation of the common location parameter μ when it is known a priori that the scale parameters follow the ordering $\sigma_1 \leq \sigma_2$. Using the affine invariant (quadratic) loss function, they have obtained certain new estimators which dominate the MLE, MMLE and the UMVUE under order restricted scale parameters. They have also obtained the percentage of risk improvements of these new estimators over the old ones numerically.

The problem of estimation of common scale parameter of several exponential populations when the location parameters are unknown and unequal has also received considerable attention in the literature. Specifically, let $(X_{i1}, X_{i2}, ..., X_{in_i})$ be a random sample taken from the i^{th} population Ex(μ_i, σ); $i = 1, 2, \dots k$. Here σ is the scale parameter which is common to all the populations, and μ_is are known as the location parameters. Rukhin and Zidek (1985) considered the problem of estimating the linear parametric function of the form $\theta = \sum_{i=1}^{k} \alpha_i \mu_i + \eta\sigma$ for k exponential populations. For $\eta > \{(nk+1) \sum_{i=1}^{k} \alpha_i\}/nk$ or $\theta = \alpha_1\mu_1 + \eta\sigma$ and $0 \leq \eta < \alpha_1/n$ they have constructed an estimator which improves upon the best affine equivariant estimator of θ for almost all parameter values. In this case if (i) $\alpha = e_j$ the basis vector then θ is a quantile of the j^{th} population, if (ii) $\alpha = 0$ and $\eta = 1$ then $\theta = \sigma$ is the common scale parameter and if (iii) $\alpha = (k^{-1}, \dots, k^{-1})$ then $\theta = k^{-1} \sum_{i_1}^{k} \mu_i + \eta\sigma$ which are very much statistical interest.

Pandey and Singh (1979) derived certain basic estimators for the common scale parameter σ namely the MLE, the unbiased estimator and further obtained new estimators which dominate these.

Madi and Tsui (1990) considered the estimation of common scale parameter σ with respect to a large class of bowl shaped loss functions. Authors proved the inadmissibility of the best affine equivariant estimator. Moreover the authors derived a class of improved estimators. Finally they used a simulation study to numerically obtain the percentage of risk reduction.

Madi and Leonard (1996) investigated the problem of estimating common scale parameter σ and the parametric function $\theta = \sum_{i=1}^{k} \alpha_i \mu_i + \eta\sigma$ for k exponential populations. Authors proposed some Bayes estimators and compared the risk of these estimators with that of the estimators previously proposed by Rukhin and Zidek (1985) and Madi and Tsui (1990).

1.2.5 Estimating Parameters under Order Restriction

The problem of estimation when it is known apriori that they follow certain ordering is quite interesting and has applications in industry, agriculture and medical experiments. For a detailed review and some applications, we refer to Barlow et al. (1972), Robertson et al. (1988) and van Eeden (2006). Below we discuss certain results which are relevant and useful for our study.

Most of the results on estimating ordered parameters deal with finding maximum likelihood estimator or its isotonic version and these results have been well addressed in Barlow et al. (1972), Robertson et al. (1988). Probably, Blumenthal and Cohen (1968) were the first to consider this problem using decision theoretic approach. Suppose X_{ij}; $i = 1, 2$ and $j = 1, 2, \ldots, n$ are independent random samples taken from distributions having density functions $f_i(x - \theta_i)$, $i = 1, 2$. Here θ_i are the location parameters. Blumenthal and Cohen (1968) considered the estimation of (θ_1, θ_2) when it is known a priori that $\theta_1 \leq \theta_2$. Using squared error loss function, they derived sufficient conditions for admissibility and minimaxity of Pitman estimator of the ordered parameters (θ_1, θ_2).

Kumar and Sharma (1988) considered the problem of estimating (θ_1, θ_2); $\theta_1 \leq \theta_2$ when the samples are taken from normal populations $N(\theta_i, \sigma_i^2)$ with respect to the sum of squared error loss functions. They obtained a class of minimax estimators for (θ_1, θ_2) and within this an admissible class of estimators was obtained. When $\sigma_1 \neq \sigma_2$, it is proved that some of these estimators are improved by the MLE itself. They also obtained a sufficient condition for the minimaxity of the analogue of Pitman estimator when the density function f_i belongs to a general location family.

Suppose there are k independent normal populations with means $\theta_1, \theta_2, \ldots, \theta_k$ and common variance unity. It is known a priori that $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_k$. Kumar and Sharma (1989) considered the problem of estimating the ordered normal means with respect to the sum of the squared error loss functions. They proved that the Pitman estimator $\underset{\sim}{\delta}_p$ (the generalized Bayes estimator with respect to the uniform prior in restricted parameter space) is minimax. For the case $k = 2$, it is also proved that the components of $\underset{\sim}{\delta}_p$ for estimating θ_1 and θ_2 are minimax (see Cohen and Sackrowitz (1970)). They also pointed out that a similar type of result does not hold for the case $k = 3$. They further proved the admissibility of the Pitman estimator $\underset{\sim}{\delta}_p$ in a subclass of estimators.

Kaur and Singh (1991) have considered the estimation of ordered means of two exponential populations when the sample sizes are equal. Using isotonic regression, they have obtained improved estimators over the usual MLE for the ordered means. The authors also obtained the asymptotic efficiency of improved estimators over the MLE.

Vijayasree and Singh (1991) investigated the problem of simultaneous estimation of ordered parameters from two exponential populations. They have derived a class of mixed estimators for the ordered means. In this class, an admissible class of estimators have been obtained. They have also studied the efficiencies of mixed estimators relative to sample means. Similar type of results have also been obtained by Vijayasree and Singh (1993) for component wise estimation of ordered means.

Pal and Kushary (1992) have studied the problem of estimating ordered location parameters of two exponential populations under squared error loss function. The authors have obtained some baseline estimators for the location parameters without assuming order restriction. Under order restrictions, authors obtained improved estimatorrs that dominate these baseline estimators.

Vijayasree et al. (1995) have considered the problem of estimating ordered location and scale parameters of k exponential populations. They derived sufficient conditions for inadmissibility of usual estimators for location and scale parameters under order restrictions with respect to mean squared error. As a consequence, they proposed improved estimators for the location as well as scale parameters.

Oono and Shinozaki (2005) considered the problem of estimating ordered means and linear function of it, for two normal populations with respect to a squared error loss function. The authors have proposed some plug-in type estimators for ordered means and obtained necessary and sufficient condition for this estimator to improve upon unrestricted MLE. For linear function of ordered means, the restricted MLE always improves upon the unrestricted MLE when the variance is known. However, when the variance is unknown the restricted MLE does not always improve upon unrestricted MLE.

Kumar et al. (2005a) studied the problem of estimating order means ($\theta_1 \leq \theta_2 \leq \ldots \leq \theta_k$) of k normal populations. They have assumed that the variances are known and unequal. For the case $k = 3$, they have shown that the components of Pitman estimator namely δ_{p1} and δ_{p3}, are failed to be minimax for θ_1 and θ_3 respectively. The authors also obtained the MLE for $(\theta_1, \theta_2, \theta_3)$ and compared the risk of it with the risk of Pitman estimator $\underset{\sim}{\delta}_p$ numerically.

Kumar et al. (2005b) considered k independent normal populations with means θ_1, θ_2,..., θ_k respectively and common variance unity. The authors proposed two well known estimators for estimating (θ_1, θ_2, ..., θ_k); $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_k$ namely the Pitman estimator ($\underset{\sim}{\delta}_p$) and the maximum likelihood estimator $\underset{\sim}{\delta}_{MLE}$. Applying an argument developed by Brown (1979), they obtained some James-Stein type estimators for ordered normal means. They observed through simulation study that many of these estimators dominate $\underset{\sim}{\delta}_{MLE}$ and $\underset{\sim}{\delta}_p$ substantially.

Nagatsuka et al. (2009) have considered the Bayesian estimation of ordered parameters of two exponential populations. Taking account of the prior information on ordering, they obtained Bayes estimators for the order parameters. Through a simulation study, they have shown that the proposed Bayes estimators perform better than the restricted MLEs.

Tripathy and Kumar (2011) considered the simultaneous estimation of quantiles of k normal populations when the variance is common and the means follow certain ordering for equal sample sizes. More specifically, let $(X_{i1}, X_{i2}, \ldots, X_{in})$; $i = 1, 2, \ldots, k$ be random samples taken from k normal populations with a common variance σ^2 and the means $\mu_1, \mu_2, \ldots, \mu_k$ such that $\mu_1 \leq \mu_2 \leq \cdots \mu_k$. Under this set up they have estimated the quantiles $\theta_i = \mu_i + \eta\sigma$. They proved the minimaxity of the best affine equivariant estimator. For the case $k = 2$, they have proposed a class of mixed estimators and proved a minimaxity results for this class. Certain admissible estimators have been derived within the class of minimax estimators. They also proposed certain generalized Bayes estimators and using these they obtained some heuristic type estimators for the quantiles. Finally they numerically compared all the proposed estimators using a simulation study.

Jana and Kumar (2015) have considered the simultaneous estimation of scale parameters (σ_1 and σ_2) from two exponential populations when the location parameter is common using a decision theoretic approach. They proposed some new estimators for ordered scale parameters which improve upon the usual estimators such as the MLE and the UMVUE. The percentage of risk improvement by the new estimators over the old ones have also been obtained numerically.

Chang and Shinozaki (2015) considered the estimation of ordered means from two normal populations when the variances are ordered and unordered using modified Pitman nearness criterion.

Recently, Pedram and Bazyari (2017) considered the estimation of ordered means of two normal populations when the variances are unknown and unequal. Using the squared error loss function, they derived a necessary and sufficient condition for the plug-in type estimators to improve upon the unrestricted MLE. They also derived improved estimators under modified Pitman measure of closeness criterion for improving the unbiased estimators. They also noticed that the uniform improvement is seen for unbiased estimator when the means are equal. They have illustrated the situation with certain practical examples.

It is worth mentioning that, apart from normal and exponential distribution some study also has been done on estimating ordered parameters in case of other distributions like gamma, Pareto, Lomax etc. We refer to Misra et al. (2002), Meghnatisi and Nematollahi (2009), Gunasekera (2017), Petropoulos (2017) and the refernces cited there in, for estimating ordered parameters in case of distributions other than normal and two-parameter exponential.

1.2.6 Estimation of Quantiles

The problem of estimation of quantiles is important and has received considerable attention by several researchers in the recent past. Needless to say, the estimators of quantiles are widely used in the study of reliability, life testing and survival analysis. We refer to Epstein and Sobel (1954) and Saleh (1981) for some applications of quantiles. In most of the literature the estimation of quantiles has been done in the case of exponential and normal distribution. Below we discuss certain results on estimation of quantiles which are relevant for developing the chapters in this thesis.

Probably Zidek (1969) was the first to consider the problem of estimating quantiles in the case of normal population. To be very specific, Let $\underset{\sim}{X} = (X_1, X_2, \ldots, X_n)$ and $\underset{\sim}{Y} = (Y_1, Y_2, \ldots, Y_k)$ be two independent random vectors from normal populations with $E(\underset{\sim}{X}) = 0$, $E(\underset{\sim}{Y}) = \mu$, $Cov(\underset{\sim}{X}) = \sigma^2 I$ and $Cov(\underset{\sim}{Y}) = \sigma^2 I$. Zidek (1969) investigated the estimation of a quantity $\theta = A\mu + \eta\sigma$ using a quadratic loss. For particular choice of $k = 1,$ and $A = 1,$ the quantity θ reduces to a quantile. He proved that the best equivariant estimator (BEE) of θ for a given matrix A and a given vector $\eta \neq 0$ is inadmissible whenever $|\eta|$ is sufficiently large. Latter Zidek (1971) proved that each member of a certain class of estimators of $\theta = \mu + \eta\sigma$ for a given vector $\eta(\neq 0)$ is inadmissible with respect to a quadratic loss function. The author also proved that the BEE is inadmissible with respect to the quadratic loss function.

Rukhin and Strawderman (1982) considered the problem of estimating the quantile $\theta = \mu + \eta\sigma$ $(\eta > 0)$ of an exponential population where μ and σ are the unknown location and scale parameters respectively. The authors established that the BEE is inadmissible whenever $0 \leq \eta < \frac{1}{n}$ or $\eta > 1 + \frac{1}{n}$ and the loss function is quadratic. Specifically, for $\eta > 1 + \frac{1}{n}$, they have obtained an improved estimator for the quantile.

Rukhin (1983) considered the problem of estimating the quantile, $\theta = \mu + \eta\sigma$ of a normal population $N(\mu, \sigma^2)$. Using a differential inequality approach, he constructed a class of minimax estimators which dominates the BEE of θ with respect to the quadratic loss function.

Rukhin (1986) considered the problem of estimating the quantile $\theta = \mu + \eta\sigma$ of an exponential population. Author proved that the BEE is admissible against the scale invariant squared error loss function if $\frac{1}{n} \leq \eta \leq 1 + \frac{1}{n}$. A class of minimax estimators was also derived for the case $\eta > 1 + \frac{1}{n}$. This class contains a generalized Bayes estimator which is shown to be admissible.

Sharma and Kumar (1994) considered the problem of estimating quantile of the first population when two exponential populations are available with unknown different scale parameters σ_1, σ_2 and a unknown common location parameter μ. The authors have shown that the UMVUE and the best affine equivariant estimator (BAEE), based on one population, of the quantile can be improved by using information available from both the samples. They obtained an inadmissibility condition for a class of affine equivariant estimators. Latter Kumar and Sharma (1996) generalized their inadmissibility results to k (≥ 2) exponential populations.

Kumar and Tripathy (2011) considered the estimation of quantiles of two normal populations when the mean is common and the variances are different. Authors proposed certain new estimators of the quantile using the estimators of the common mean. They also established that the estimators of the quantile can be improved if one can improve the estimators of either the common mean or the variance. They have derived sufficient conditions for improving estimators in the class of equivariant estimators. They have also numerically compared the risk values of various proposed estimators.

Tripathy and Kumar (2017) studied the problem of estimating quantile vector for $k(\geq 2)$ exponential populations with common location parameter μ and possibly different scale parameters $\sigma_1, \sigma_2, \ldots, \sigma_k$. The authors proposed some estimators based on the MLE, MMLE and the UMVUE. Furthermore they have also derived some inadmissibility results introducing affine and location group of transformations to the model. Finally, through a simulation study, authors have numerically compared all the proposed estimators with respect to the sum of quadratic losses.

1.2.7 Estimating Parameters Using Censored Samples

The problem of estimating parameters using censored samples is quite realistic and has received considerable attention by several researchers in the recent past. Particularly, the problem has been studied extensively when the distribution function is a two-parameter exponential due to the practical applications. Estimation using censored samples has wide range of applications in the fields of science, engineering, social sciences, public health and medicines in the study of reliability, life testing and survival analysis. Basically, the various types of censoring schemes available are of type-I (number of failures are random), type-II (censoring time is random) or some modification of these. For some detailed review on estimation of parameters from an exponential populations using these types of conventional censoring schemes we refer to Lawless (2003), and Johnson et al. (1994). We also note that the type-II censoring scheme is a special case of progressive type-II censoring scheme. Below we discuss some literature on estimation of parameters from an exponential population using censored sampling scheme from decision theoretic point of view.

Chandrasekar et al. (2002) have derived the minimum risk equivariant estimator for the location as well as the scale parameter of an exponential population under progressive type-II censored sampling scheme. The authors have established the results for location, scale and location-scale models separately.

Madi (2010) has investigated the problem of estimating scale parameter of an exponential population under progressive type-II censoring scheme. Applying Brewster-Zidek technique (Brewster and Zidek (1974)), author obtained a smooth estimator that improves over the minimum risk equivariant estimator. He has established the result under a large class of bowl-shaped loss functions. For a detailed review and some recent results on estimation of parameters under progressive type-II censored samples from a two parameter exponential distribution, using classical as well as decision theoretic approach, we refer to Balakrishnan and Cramer (2014) and the references cited there in. Recently, Tripathi et al. (2018) considered the estimation of a linear parametric function of location and scale from an exponential distribution using doubly censored sampling scheme. They have derived some decision theoretic results using a convex loss function. More importantly, in this thesis we have concentrated on estimation problems under type-II censored samples from exponential populations using decision theoretic approach.

It has been observed from the literature review that most of the research works on estimating parameters using censored samples has been studied using one population. We note that, a little attention has been paid in estimating parameters, using censored samples from two or more populations. Below we discuss certain results on estimation of parameters using censored samples from two or more exponential populations.

Probably, Chiou and Cohen (1984) were the first to consider the problem of estimating parameters using censored samples when more than one population is available. The authors have considered two exponential populations with a common location parameter and different scale parameters. Using type-II censored samples from these two populations, they have obtained the UMVUE and the MLE of the common location parameter with respect to the squared error loss function. They have also discussed certain results for $k(\geq 2)$ populations.

Elfessi and Pal (1991) have considered the problem of estimating common scale parameter of $k(\geq 2)$ exponential populations under squared error loss function using type-II censored samples. The authors have obtained a Stein type estimator that improves upon the best affine equivariant estimator. Further the authors estimated the vector of location parameters and constructed some improved estimators.

Yike and Heliang (1999) considered the estimation of ordered location parameters of two exponential populations using multiple type-II censoring scheme. Authors derived Bayes estimators using a non-informative prior.

Tripathy (2016) has revisited the statistical model considered by Chiou and Cohen (1984) and obtained certain decision theoretic results. Author proposed a class of affine equivariant estimators for the common location parameter and derived sufficient conditions for improving estimators in this class. Consequently, author proposed new estimators that improve upon the MLE and the UMVUE. Author also compared the risk values of all these estimators numerically using a simulation study.

Recently, Tripathy (2017) considered the same model that has been considered by Tripathy (2016), with the prior information that the scale parameters follow a simple ordering. Utilizing the prior information, the author has derived a sufficient condition for improving estimators in the class of equivariant estimators. Furthermore, applying integrated expression of risk difference (IERD) approach of Kubokawa (1994), an improved class of estimators has been derived. The risk values of all the proposed estimators have been compared through a simulation study using Monte-Carlo simulation method.

1.3 Objectives

From the above literature review one may see that the problem of estimation of common mean of two or more normal populations when there is no order restrictions on the variances has been studied extensively and various alternative estimators are available. In contrary to this, when it is known a priori that the variances follow certain ordering a less attention has been paid, for example, see Elfessi and Pal (1992), Misra and van der Meulen (1997) and Chang et al. (2012). An aim will be to construct alternative estimators for the common mean when the variances are ordered. The estimation of quantiles for two normal populations with a common mean has been studied by Kumar and Tripathy (2011). A generalization of their results to $k(\geq 2)$ normal populations is quite expected. The problem of estimating quantiles of two or more exponential populations has been considered by Sharma and Kumar (1994) and Kumar and Sharma (1996) using full samples, however in certain situations we may not able to observe all the samples. The problem may also be studied using censored samples. Further estimating ordered parameters using censored samples may be studied. In the literature, the problem of estimating ordered location or scale parameters has been studied. Further target will be to consider the problem of estimating function of ordered parameters.

In view of the above, in this thesis we have considered the problem of estimating equal or ordered parameters when the underlying distribution is either normal or exponential from a decision theoretic point of view. Moreover, we have also focused on estimating quantiles (which are linear function of location and scale parameters) of these populations under equality and/or inequality restrictions on the location/scale parameters.

1.4 A Summary of the Results Obtained in the Thesis

The thesis is organized as follows. In **Chapter 1**, we do a detailed review of literature for the following problems: (i) estimation of a common parameter, (ii) estimation of ordered parameters, (iii) estimation of quantiles and (iv) estimation of parameters using censored samples. In **Chapter 2**, some basic definitions and decision theoretic results have been discussed which will be useful in developing the subsequent chapters.

In **Chapter 3**, we have revisited the problem of estimating a common mean 'μ' of two normal populations $N(\mu, \sigma_1^2)$ and $N(\mu, \sigma_2^2)$ when it is known a priori that the nuisance parameters (variances) follow the simple ordering, that is, when $\sigma_1^2 \leq \sigma_2^2$. In order to evaluate the performance of an estimator, we use the loss functions

$$L_1(d, \underset{\sim}{\alpha}) = \left(\frac{d-\mu}{\sigma_1}\right)^2, \quad L_2(d, \underset{\sim}{\alpha}) = |d-\mu|, \quad \text{and} \quad L_3(d, \underset{\sim}{\alpha}) = (d-\mu)^2$$

where d is an estimator for estimating the common mean μ and $\underset{\sim}{\alpha} = (\mu, \sigma_1^2, \sigma_2^2)$. Let $\underset{\sim}{X} = (X_1, X_2, \ldots, X_m)$ and $\underset{\sim}{Y} = (Y_1, Y_2, \ldots, Y_n)$ be independent random samples taken from two normal populations $N(\mu, \sigma_1^2)$ and $N(\mu, \sigma_2^2)$ respectively. The minimal sufficient statistics (not complete) for this model exists and is given by $(\bar{X}, \bar{Y}, S_1^2, S_2^2)$, where we denote

$$\bar{X} = \frac{1}{m}\sum_{i=1}^{m} X_i,\ \bar{Y} = \frac{1}{n}\sum_{j=1}^{n} Y_j,\ S_1^2 = \sum_{i=1}^{m}(X_i - \bar{X})^2,\ S_2^2 = \sum_{j=1}^{n}(Y_j - \bar{Y})^2.$$

In Section 3.2, we discuss some existing results without considering ordering of the variances for the common mean. When there is order restriction on the variances, we propose the restricted MLE (Maximum Likelihood Estimator) for 'μ' which can be obtained numerically. Under the same set up, Elfessi and Pal (1992) proposed new estimators that dominate the well known Graybill-Deal (see Graybill and Deal (1959)) estimator stochastically (also universally), for equal and unequal sample sizes separately. Further these results have been extended to the case of $k(\geq 2)$ normal populations by Misra and van der Meulen (1997). Being motivated by these results, we in Section 3.3, construct some new estimators that dominate some other popular estimators for the common mean proposed by Khatri and Shah (1974), Moore and Krishnamoorthy (1997), Tripathy and Kumar (2010) and Brown and Cohen (1974), stochastically as well as universally for equal as well as unequal sample sizes. In Section 3.4, we have shown that these new estimators also dominate their old counter part in terms of Pitman measure of closeness. The concept of invariance has been introduced to our problem in Section 3.5. Sufficient conditions have been derived for improving estimators in the class of affine and location equivariant, under order restrictions on the variances. It has been observed that all the well known estimators (that have been considered) fall into these classes. As a consequence improved estimators have been obtained. Interestingly, these new estimators coincide with the estimators proposed in Section 3.3 and 3.4 for unequal sample sizes. Finally, in Section 3.6, we carried out a detailed simulation study in order to numerically compare the performances of all these proposed estimators including that proposed by Elfessi and Pal (1992). Specifically, we have calculated the percentage of risk improvements of new one over their respective old one and the percentage of relative risk performances of all the new estimators with respect to Graybill-Deal estimator. It has been observed that none of the estimators beat others uniformly using these three types of loss functions.

In **Chapter 4,** we have considered the problem of estimating quantiles for $k(\geq 2)$ normal populations under equality assumption on the mean μ. Let there be $k(\geq 2)$ normal populations, each having a common mean and possibly different variances. To be very specific, let $(X_{i1}, X_{i2}, \ldots, X_{in_i})$ be a random sample of size n_i available from the i^{th} normal population $N(\mu, \sigma_i^2)$; $i = 1, 2, \ldots, k$. Here, we assume that the parameters μ and σ_i^2; $i = 1, 2, \ldots, k$ are unknown. The problem is to estimate the quantile, $\theta = \mu + \eta\sigma_1$ of the first population, when the other $k-1$ populations are available, with respect to the quadratic loss function $L(d, \mu, \sigma_1^2) = \left(\frac{d-\theta}{\sigma_1}\right)^2$, where d is an estimate for estimating the quantile θ. Here $0 \neq \eta = \Phi^{-1}(p)$; $0 < p < 1$ and $\Phi(.)$ denotes the cumulative distribution function of a standard normal random variable. In Section 4.2, we have considered the estimation of quantiles of the first population when the other $k-1$, populations are available. We note that Kumar and Tripathy (2011), considered the estimation of quantiles of two normal populations when the mean is common, using decision theoretic approach. The main objective of this work (in Section 4.2) is to extend some of their decision theoretic results to the case of $k(\geq 2)$ populations. In Section 4.2.1, we prove a general result which helps in obtaining better estimators for the quantiles. Using these results, some improved estimators have been constructed. We introduce the concept of invariance and obtain the classes of affine and location equivariant estimators in Section 4.2.2. Using the orbit-by-orbit improvement technique of Brewster and Zidek (1974) for improving equivariant estimators, we derive sufficient conditions for improving estimators in these classes. As a consequence some complete class results have been proved. In Section 4.2.3, we carry out a detailed simulation study in order to numerically compare the performances of all the proposed estimators for the case $k = 3$ and 4. Finally we conclude our remarks in Section 4.2.4 also discuss two practical examples illustrating the use of estimators for quantiles.

Next, in Section 4.3 we consider the estimation of the vector quantile $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$, where $\theta_i = \mu + \eta\sigma_i$. The loss function is taken as sum of the quadratic losses,

$$L(\underset{\sim}{d}, \underset{\sim}{\theta}) = \sum_{i=1}^{k} \left(\frac{d_i - \theta_i}{\sigma_i}\right)^2,$$

where $\underset{\sim}{d} = (d_1, d_2, \ldots, d_k)$ is an estimator of $\underset{\sim}{\theta}$. In Section 4.3.1, we derive a basic result which helps in constructing improved estimators for quantile vector $\underset{\sim}{\theta}$. In Section 4.3.2, we derive affine and location equivariant estimators. Sufficient conditions for improving estimators in these classes have been obtained for the case $k = 2$. In the process, two complete class results have been proved. In Section 4.3.3, an extensive simulation study has been carried out in order to numerically compare the relative risk performances of various proposed estimators. We conclude in Section 4.3.4 with some examples.

In **Chapter 5**, we deal with the problem of estimating quantiles and ordered scales of two exponential populations under equality assumption on the location parameter using type-II censored samples. First, in Section 5.2, we consider the estimation of quantile $\theta = \mu + \eta\sigma_1$, when type-II censored samples are available from two exponential populations Ex(μ, σ_1) and Ex(μ, σ_2). More specifically, let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}$ $(2 \leq r \leq m)$ and $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(s)}$ $(2 \leq s \leq n)$ be the ordered observations taken from random samples of sizes m and n which follows exponential distributions with a common location parameter μ and possibly different scale parameters σ_1 and σ_2 respectively. We denote Ex(μ, σ_i) the exponential population with probability density function

$$f(t, \mu, \sigma_i) = \frac{1}{\sigma_i} \exp\{-(t-\mu)/\sigma_i\}, \quad t > \mu, \sigma_i > 0, -\infty < \mu < \infty;\ i = 1, 2.$$

The problem is to estimate the p^{th} quantile $\theta = \mu + \eta\sigma_1$ of the first population, where $0 < \eta = -\log(1-p)$; $0 < p < 1$. The loss function is taken as

$$L(d, \underset{\sim}{\alpha}) = \left(\frac{d-\theta}{\sigma_1}\right)^2$$

where d is an estimate for estimating the quantile θ and $\underset{\sim}{\alpha} = (\mu, \sigma_1, \sigma_2)$. We evaluate the performance of an estimator for quantile with the help of the risk function $R(d, \underset{\sim}{\alpha}) = E_{\mu,\sigma_1,\sigma_2}(L(d, \underset{\sim}{\alpha}))$. The complete and sufficient statistics for this model is $(U_1 - Z, U_2 - Z, Z)$, where $Z = \min(X_{(1)}, Y_{(1)})$, $U_1 = \frac{1}{m}[\sum_{i=1}^{r} X_{(i)} + (m-r)X_{(r)}]$, and $U_2 = \frac{1}{n}[\sum_{j=1}^{s} Y_{(j)} + (n-s)Y_{(s)}]$. In Section 5.2.1, we construct some basic estimators such as the MLE (d_{ML}), a modification to it call it the modified MLE (d_{MM}) and the uniformly minimum variance unbiased estimator (UMVUE) d_{MV}. These are given by

$$d_{ML} = Z + \eta\hat{\sigma}_{1ML}, \ \ d_{MM} = Z - \frac{1}{\hat{p}} + \eta\hat{\sigma}_{1ML}, \ \ d_{MV} = Z + \frac{V_1V_2(k-1)}{(r-1)V_2 + (s-1)V_1} + kV_1,$$

where $\hat{\sigma}_{1ML} = m(U_1 - Z)/r$, $\hat{\sigma}_{2ML} = n(U_2 - Z)/s$, $\hat{p} = m/\hat{\sigma}_{1ML} + n/\hat{\sigma}_{2ML}$, $V_1 = U_1 - Z$ and $V_2 = U_2 - Z$. In Section 5.2.2, we propose a class of estimators which contains the UMVUE of quantiles θ and obtain estimators dominating the UMVUE, d_{MV}. In Section 5.2.3, we derive sufficient conditions for improving equivariant estimators and as a consequence some complete class results have been obtained. Most importantly, in Section 5.2.4, we carry out a simulation study to numerically compare the risk values as well as the percentage of relative risk improvements of all the proposed estimators which may be useful for practical purposes. Finally we conclude with our remarks in Section 5.2.5. It is worth mentioning that, the theoretical results obtained in this section generalizes the results of Sharma and Kumar (1994) where they studied the problem for full and equal sample sizes.

Next, in Section 5.3, we deal with the problem of estimating vector $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$; such that $\sigma_1 \leq \sigma_2$. The loss function is taken as

$$L(\hat{\underset{\sim}{d}}, \underset{\sim}{\sigma}) = \sum_{i=1}^{2} \Big(\frac{d_i - \sigma_i}{\sigma_i}\Big)^2$$

where $\hat{\underset{\sim}{d}} = (d_1, d_2)$ is an estimator for $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$. Section 5.3.1 introduces the MLE and the UMVUE without considering order restrictions on the scale parameters. Then under order restriction on the scale parameters, we derive the restricted maximum likelihood estimator for $\underset{\sim}{\sigma}$. It has been proved that the restricted MLE performs better than the MLE using the quadratic loss function. Further we derive some complete class results in certain class of estimators. In Section 5.3.2, we obtain classes of equivariant estimators and prove some inadmissibility results in these classes. Using these results, we obtain improved estimators which dominate the MLE and the UMVUE with respect to the above loss function. In Section 5.3.3, a detailed simulation study has been carried out to numerically compare the relative risk performances of all the proposed estimators and recommendations have been made regarding their use. Section 5.3.4 concludes the remarks with some examples.

In **Chapter 6**, we consider the estimation of ordered quantiles of two exponential populations under equality restrictions on either the location or scale parameters. First, in Section 6.2, we consider two exponential populations with a common location parameter μ and possibly different scale parameters σ_1 and σ_2. Let θ_1 and θ_2 be the p^{th} quantiles of the two populations respectively. Here $\theta_i = \mu + \eta\sigma_i$; $i = 1, 2$, $\eta = -\ln(1-p)$; $0 < p < 1$. The problem is to estimate the quantiles θ_i, when it is known a priori that $\theta_1 \leq \theta_2$. The loss function is taken as,

$$L(d_i, \theta_i) = \Big(\frac{d_i - \theta_i}{\sigma_i}\Big)^2$$

where d_i is an estimator for θ_i; $i = 1, 2$. The performance of an estimator will be evaluated using the risk function $R(d_i, \theta_i) = E_{\mu,\sigma_i}\{L(d_i, \theta_i)\}$. In Section 6.2.1, we derive some baseline estimators without assuming ordering of quantiles. Further, using isotonic version of unrestricted MLEs, we propose some new estimators (call it restricted MLEs) for the quantiles under order restriction. Using the existing estimators for ordered scale and common location, we propose some new plug-in type of estimators for the ordered quantiles. In Section 6.2.2, we consider some classes of estimators for the quantiles. Sufficient conditions for improving estimators in this class have been proved. As a result, new estimators improving upon the MLE, the UMVUE, a modification to the MLE and the restricted MLE have been obtained. We note that an analytical comparison of the risk values of all these estimators is not possible.

Hence, a detailed simulation study has been done in Section 6.2.3, to compare the percentage of relative risk improvement of all these proposed estimators. We recommend using estimators for quantiles under order restrictions. Finally we conclude our remarks in Section 6.2.4.

In Section 6.3, we consider two exponential populations with a common scale parameter σ and different location parameters μ_1 and μ_2. We further assume that the location parameters are non-negative (which is important from application point of view). The problem of estimating ordered quantiles $\theta_i = \mu_i + \eta\sigma;\ i = 1, 2$ and $\theta_1 \leq \theta_2$ is considered using quadratic loss function when the samples drawn are type-II censored. In Section 6.3.1, we discuss some basic estimators such as the MLE, a modification to the MLE, the UMVUE, and the best affine equivariant, without considering ordering of the quantiles. Further, under order restrictions on the quantiles, isotonized version of all these estimators have been proposed. In Section 6.3.2, Bayes estimators have been derived for θ_1 and θ_2 assuming order restrictions on the quantiles. For this purpose we have considered two types of priors namely the non-informative prior and the conditional prior. In Section 6.3.3, a detailed simulation study has been carried out in order to numerically compare the risk values of all the proposed estimators.

In **Chapter 7**, we consider the estimation of the common scale parameter σ of two exponential populations when the location parameters satisfy the simple ordering $\mu_1 \leq \mu_2$ and $\mu_i \geq 0; i = 1, 2$. It is noted that this model was considered by Madi and Leonard (1996) without imposing order restriction on location parameters. The main goal of this chapter is to derive certain Bayes estimators for the common scale parameter σ, under the assumption that location parameters are ordered. In Section 7.1, we discuss some basic results and propose the restricted MLE for σ. In Section 7.2, we find Bayes estimators using uniform prior and a conditional inverse gamma prior, taking into account the order restrictions on the location parameters. Exact expressions for these two Bayes estimators have been obtained. It seems quite difficult to evaluate the risk values of these estimators analytically. In Section 7.3, taking the advantages of computational facilities, we compare the performance of our estimator with that of Madi and Leonard (1996) with respect to the quadratic loss function using Monte-Carlo simulation method numerically. It has been revealed that the proposed estimators perform quite satisfactorily in comparison to other estimators, when it is known apriori that the location parameters are ordered.

In **Chapter 8**, we give an overall conclusions of the thesis and discuss some of our future study.

Chapter 2

Some Definitions and Basic Results

In this chapter we discuss certain results and definitions from classical as well as decision theoretic point of view, and will be useful in developing chapters in the thesis. A thorough discussion on these topic can be found in Berger (1985), Lehmann and Casella (2006), Ferguson (1967), and Rohatgi and Saleh (2003).

Suppose X is a random variable defined on a sample space Ω having probability distribution function P_θ, where θ is an unknown parameter associated to it. This θ takes values in a set known as parameter space Θ. The goal of an estimation problem is to estimate either the parameter θ or a measurable function of it $h(\theta)$ using the observations $(X_1, X_2, \ldots, X_n)$ from X. A non-randomized decision rule d is defined as a function from the sample space Ω to the action space $\mathcal{A}$ which is defined as the convex closure of the set $h(\Theta) = \{h(\theta) : \theta \in \Theta\}$. Let us denote $\mathcal{D}$ as the class of all non-randomized decision rules. In this thesis we mostly concentrate on finding non-randomized decision rules as it is essentially complete. Based on the observed sample $X = x$, the parameter $h(\theta)$ is estimated by $d(x)$ and due to which a loss is incurred. Let us denote the loss by $L(d(x), h(\theta))$. In our thesis we have considered the loss function $L(d(x), h(\theta))$ as non-negative and real valued in both the arguments. The average loss is known as the risk and is defined by $R\left(h(\theta), d(x)\right) = E_\theta\left\{L\left(h(\theta), d\left(x\right)\right)\right\}$. The target is to obtain a good estimator as there may be more than one estimator for $h(\theta)$ available. The followings give criteria to choose a better estimator in terms of risk values.

Definition 2.0.1 *An estimator d_1 is said to be as good as an estimator d_2, if $R\left(\theta, d_1\right) \leq R\left(\theta, d_2\right)$ for all $\theta \in \Theta$. An estimator d_1 is said to perform better than another estimator d_2 if $R\left(\theta, d_1\right) \leq R\left(\theta, d_2\right)$ for all $\theta \in \Theta$ and strict inequality holds for some values of $\theta \in \Theta$.*

Definition 2.0.2 *Two estimators d_1 and d_2 are said to be equivalent if $R\left(\theta, d_1\right) = R\left(\theta, d_2\right)$, for all $\theta \in \Theta$.*

Definition 2.0.3 *An estimator d is said to be admissible if no other estimator d_0 dominates it in terms of risk. If an estimator is not admissible is known as inadmissible.*

It seems from the above definition that admissibility property of an estimator is a weak optimality criterion. Nevertheless, it is a desirable property that every estimator should enjoy. In practice, it is difficult to obtain an estimator satisfying the admissibility criteria in the class of all the estimators $\mathcal{D}$. Hence, one must concentrate on subclasses of the class $\mathcal{D}$ for better estimators. Considering this, we have the following definitions.

Definition 2.0.4 *A class of estimators C is said to be complete if for any other estimator d not belonging to C, there exists an estimator d_0 belonging to C such that it dominates d.*

Definition 2.0.5 *A class of estimators C is said to be essentially complete if for any other estimator d not belonging to C, there exists an estimator d_0 belonging to C such that it is as good as d.*

Definition 2.0.6 *A class of estimators C is said to be minimal complete if the class C is complete and no other proper subclass of C is complete.*

Definition 2.0.7 *A class of estimators C is said to be minimal essentially complete if the class C is essentially complete and no other proper subclass of C is essentially complete.*

We note that, to obtain good estimators one needs to concentrate only in the minimal(essentially) complete class, as it contains all the desirable estimators. Further, when the minimal (essentially) complete class exist, it contains all the admissible estimators. However, these classes may not exist always. We also noticed that admissibility is a weak optimality criteria. For example, constant estimators are admissible which of no use. Hence, one should look for some other optimality criteria that helps in obtaining good estimators. The following two optimality criteria are commonly used. (i) by ordering the estimators, and (ii) by restricting the class of estimators. First we will discuss certain results which are related to the method of ordering the estimators. Using this approach, estimators are ordered as per their worst performances in the sense of risk function. This concept leads to the criteria of minimaxity.

Definition 2.0.8 *An estimator d_0 satisfying the relation*

$$\sup_{\theta \in \Theta} R(\theta, d_0) = \inf_{d \in \mathcal{D}} \sup_{\theta \in \Theta} R(\theta, d)$$

is called a minimax estimator. The right hand side of the equality represents minimax risk value of the problem.

Alternatively, a minimax estimator minimizes the maximum risks of all the estimators in the class. Sometimes admissible estimators become minimax and vice-versa, which is given in the following theorem.

Theorem 2.0.1 (i) *A unique minimax estimator is admissible.*
(ii) *An admissible estimator having constant risk is minimax.*

Another method for obtaining optimal estimators is due to Bayesian principle. In this approach the parameter θ is treated as a random variable having certain probability. The distribution of θ is known as the prior distribution denoted by $\pi(\theta)$ and is defined on $(\Theta, \mathcal{B}(\Theta))$. Here $\mathcal{B}(\Theta)$ denotes the $\sigma-$field of subsets of Θ. The prior distribution π is called proper if $\pi(\Theta)$ is finite. It is called improper if $\pi(\Theta) = \infty$, and $\int_\Theta p_\theta(x) d\pi(\theta) < \infty$ for almost all x (a.e. (μ)). Here $p_\theta(x)$ is the density of P_θ with respect to a $\sigma-$finite measure μ on measurable space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$, where $\mathcal{B}(\mathcal{X})$ is a $\sigma-$field of subsets of $\mathcal{X}$. To find a Bayes estimator we proceed in the following manner. First we obtain posterior distribution of θ given $X = x$. It is obtained by dividing the joint density of θ and X with the marginal distribution of X. It is given by

$$\frac{p_\theta(x) d\pi(\theta)}{\int_\Theta p_\theta(x) d\pi(\theta)},$$

where the prior distribution $\pi(\theta)$ may be proper or improper. The posterior risk of an estimator is obtained as

$$\frac{\int_\Theta L(\theta, d(x)) p_\theta(x) d\pi(\theta)}{\int_\Theta p_\theta(x) d\pi(\theta)}.$$

Then we have the definition for Bayes estimator.

Definition 2.0.9 *An estimator* d *is said to be Bayes estimator if it minimizes the posterior risk with respect to a proper prior* π. *If the prior is improper, that is* $\int \pi(\theta)d\theta = \infty$, *then the estimator* d *minimizing the posterior risk is called a generalized Bayes estimator.*

It is also noted that the Bayes estimator minimizes the Bayes risk when the prior distribution $\pi(\theta)$ is proper. The Bayes risk is defined by

$$r(\pi, d) = \int_{\Theta} R(\theta, d) d\pi(\theta).$$

Hence, for a Bayes estimator d_0 with respect to a proper prior π, we have

$$r(\pi, d_0) = \inf_{d \in \mathcal{D}} r(\pi, d).$$

It is not difficult to observe that under a squared error loss function, the Bayes estimator becomes the mean of the posterior distribution of θ given X. Similarly for an absolute loss function, the Bayes estimator turns out to be the median of the posterior distribution of θ given X. Further for a weighted squared error loss function, say

$$L(\theta, d) = w(\theta)(\theta - d)^2,$$

where $w(\theta) > 0$ for all $\theta \in \Theta$, the form of Bayes estimator turns out to be

$$d_\pi = \frac{\int \theta w(\theta) d\pi(\theta|x)}{\int w(\theta) d\pi(\theta|x)}.$$

In many situations the minimum Bayes risk is not obtainable, hence one may find an estimator having risk close to it.

Definition 2.0.10 *Let* $\epsilon > 0$. *An estimator* d_0 *is said to be* $\epsilon-$*Bayes with respect to a prior distribution* π, *if*

$$r(\pi, d_0) \leq \inf_{d \in \mathcal{D}} r(\pi, d) + \epsilon.$$

Definition 2.0.11 *An estimator* d_0 *is said to be extended Bayes, if* d_0 *is* $\epsilon-$*Bayes with respect to some prior distribution of* θ.

Definition 2.0.12 *Let* d_n *be a Bayes estimator with respect to a sequence of prior distributions* $\{\pi_n\}$. *An estimator* d_0 *is said to be the limit of Bayes rule if* $d_n(x) \to d_0(x)$ *in the sense of distribution for almost all* x.

Definition 2.0.13 *Let* Θ^* *be the class of all prior distribution on* Θ. *A prior distribution* π^* *is said to be least favorable, if*

$$\inf_{d \in \mathcal{D}} r(\pi^*, d) = \sup_{\pi \in \Theta^*} \inf_{d \in \mathcal{D}} r(\pi, d).$$

Hence a least favorable prior distribution tends to maximize the minimum Bayes risk. The right hand side of the above expression is known as the lower value of an estimation problem.

The following theorems gives the condition under which a Bayes estimator is admissible or minimax.

Theorem 2.0.2 *If a Bayes estimator with respect to a prior distribution π is unique up to equivalence then it is admissible.*

The uniqueness condition can be relaxed if the risk function is continuous, which is stated below.

Theorem 2.0.3 *Let Θ be the one dimensional Euclidean space and assume that the risk function $R(\theta, d)$ is a continuous function of θ for all d. If the estimator d_0 is Bayes with respect to some prior π defined on Θ for which the risk is finite, and the support of π is the whole space Θ then d_0 is admissible.*

Theorem 2.0.4 *Let d_0 be a Bayes estimator with respect to a prior π for all $\theta \in \Theta$ and*

$$R(\theta, d_0) \leq r(\pi, d_0),$$

then δ_0 is minimax and π is least favorable.

Theorem 2.0.5 *Let $\{\pi_n\}$ be a sequence of proper prior distributions and d_n be the Bayes estimator corresponding to π_n. If*

$$\sup_{\theta \in \Theta} R(\theta, d) = \lim_{n \to \infty} r(\pi_n, d_n),$$

then d is a minimax estimator.

Still now we have discussed certain procedures by employing which, one can get optimal estimators. Particularly, we have discussed the method of ordering the estimators in a given problem. The next procedure is to restrict attention to certain classes and obtain optimal estimators. First we concentrate on the class of unbiased estimators.

Definition 2.0.14 *An estimator d for estimating $h(\theta)$ is said to be unbiased if $E(d) = h(\theta)$ for all $\theta \in \Theta$.*

It is easy to see that when the loss function is squared error, the risk of an unbiased estimator turns out to be its variance. It is desirable to have an estimator that enjoys the property of minimum risk in this class. If such an estimator exist in this class then we call it as the uniformly minimum variance unbiased estimator (UMVUE) of $h(\theta)$. In this regard the following result is very much useful.

Theorem 2.0.6 *Let $T(X)$ be a complete sufficient statistic for estimating $h(\theta)$, $\theta \in \Theta$. If we can find some function of T, say $\psi(T)$, which is unbiased estimator of $h(\theta)$, then $\psi(T)$ is the UMVUE of $h(\theta)$.*

Another approach to obtain optimal estimators is by introducing the concept of invariance to the estimation problem and we discuss it below in detail.

Suppose G is the group of measurable transformations whose elements are defined on the sample space Ω. The group operation is composition. If g_1 and g_2 are any arbitrary elements of G, then the composition $g_2 g_1$ is defined as the transformation $g_2 g_1(x) \to g_2(g_1(x))$. If for all $g \in G$ and for all $\theta \in \Theta$, there exists a unique $\bar{g}(\theta) \in \Theta$ such that the distribution of $g(X)$ is given by $P_{\bar{g}(\theta)}$, then the family of distributions $\{P_\theta : \theta \in \Theta\}$ is called invariant under the group G. It is noted that $\bar{G} = \{\bar{g} : g \in G\}$ is the induced transformation on Θ. Since G is a group, g^{-1} exists and hence $P_{\bar{g}(\theta)}(X \in S) = P_\theta(X \in g^{-1}(B))$ holds for all sets $B \in \mathcal{B}(\Omega)$.

Further, a loss function $L(\theta, d)$ is said to be invariant under the group G if for all $g \in G$ and $d \in \mathcal{D}$, there exists a unique rule $d_0 \in \mathcal{D}$ such that

$$L(\theta, d) = L(\bar{g}(\theta), d_0)$$

for all $\theta \in \Theta$.

The group G while acting on the sample space Ω, also induces automatically another group $\tilde{G} = \{\tilde{g} : g \in G\}$ defined on action space $\mathcal{A}$. We note that that the groups $\bar{G} = \{\bar{g} : g \in G\}$ and $\tilde{G} = \{\tilde{g} : g \in G\}$ are homomorphic images of the group G.

Definition 2.0.15 *An estimation problem under the group of transformations G is said to be invariant if the family of probability distributions P_θ and the loss function $L(\theta, d)$ are both invariant under the group G.*

For an invariant estimation problem, a non randomized estimator $d \in \mathcal{D}$ is said to be equivariant if

$$d(g(x)) = \tilde{g}(d(x)),$$

for all $x \in \Omega$ and $g \in G$.

Now the two elements θ_1 and θ_2 are said to be equivalent if for $\theta_1,\ \theta_2 \in \Theta$, the relation $\theta_1 = \bar{g}(\theta_2)$ holds true where $\bar{g}$ be an element of $\bar{G}$. This relation is an equivalence relation thus partitions the set Θ into different disjoint classes known as equivalent classes. These equivalent classes are known as the orbits of the parameter space Θ. A non-randomized equivariant decision rule satisfies the relation,

$$R(\bar{g}(\theta), d) = R(\theta, d),$$

for all $\bar{g} \in \bar{G}$ and $\theta \in \Theta$. Hence on the orbits of Θ, the risk of an equivariant estimator remains constant that is the risk is independent of parameter θ. Alternatively, one can say that the risk of an equivariant estimator is constant if the group G is transitive.

Theorem 2.0.7 *Suppose that a given decision problem is invariant under a finite group G. If an invariant decision rule d is admissible within the class of all invariant rules, it is admissible.*

Application of Brewster-Zidek Technique

It always remains an anxiety for a statistician to provide improved estimators in certain classes, or to obtain estimators that perform better than some of the baseline estimators like MLE and the UMVUE. Regarding that, (Brewster and Zidek (1974)) provided an useful technique which helps in obtaining improved estimators in the class of equivariant estimators. Though the method was developed looking into only the equivarinat class, it can be used in general. Suppose our target is to improve upon an estimator say d_0. Now consider a class C which contains the estimator d_0. Suppose d_θ is the minimizing choice in that class C for each $\theta \in \Theta$. A suitable choice of C may leave the estimator d_θ free of θ. For example, let $X = (X_1, X_2, \ldots, X_n)$ be a random sample taken from a population with uniform distribution $X \sim U[0, \sigma]$. One wishes to estimate the parameter σ. Let us consider the loss function as the scale invariant which is given by

$$L(\sigma, d) = \left(\frac{\sigma - d}{\sigma}\right)^2.$$

It can be seen that a complete and sufficient statistics for this problem is $X_{(n)} = \max\{X_1, X_2, \ldots, X_n\}$. The MLE and the UMVUE of σ are obtained as $X_{(n)}$ and $\frac{n+1}{n} X_{(n)}$ respectively. Looking at the form of these two estimators, one may choose a class of estimators $D = \{aX_{(n)}, a > 0\}$ for estimating σ. In this class the best estimator is given by $d_0 = \frac{n+2}{n+1} X_{(n)}$ hence improve upon both the MLE and the UMVUE.

Sometimes it is not possible to obtain a class C such that d_θ is free from the parameter. When d_θ does not vary significantly with respect to θ, but is different from d_0, then it is possible to improve d_0. Let $d_0 \in \{d_a : a \in \mathbb{R}\}$ with $d_0 = d_{a_0}$. Let $\hat{a}_\theta$ be the choice of a that minimizes the risk $R(\theta, d_a)$ for each θ. Let

$$\underline{a} = \inf_{\theta \in \Theta} \hat{a}_\theta \text{ and } \bar{a} = \sup_{\theta \in \Theta} \hat{a}_\theta.$$

If $R(\theta, d_a)$ is a strictly convex function of a, then the estimator d_a is improved by $d_{\underline{a}}$ if $\underline{a} > a_0$ and by $d_{\bar{a}}$ if $\bar{a} < a_0$. Further the class of estimators $\{d_a : \underline{a} \leq a \leq \bar{a}\}$ form a minimal complete class.

The orbit-by-orbit improvement technique may be applied to reduce the risk of an equivariant estimator on the orbits of some invariant statistics T. Usually a maximal invariant statistic is preferred for the purpose. Let us consider estimators of the form $d_{\phi(T)}$ where $\phi = \phi_0$ gives d_0. The conditional risk function of $d_{\phi(T)}$ given $T = t$ is given by

$$R(\theta, d_{\phi(T)} | T = t) = E\{L[\theta, d_{\phi(T)}(X)] \mid T = t\}.$$

Let $R(\theta, d_{\phi(T)})$ be a strictly convex function of $\phi(T)$ and that $\phi^*_\theta(T)$ minimizing choice that minimizes the risk $R(\theta, d_{\phi(T)})$ for each θ and for a given $T = t$. Further, define the sets

$$A_1 = \left\{t : \underline{\phi}(t) = \inf_{\theta \in \Theta} \phi^*_\theta(t) > \phi_0(t)\right\} \text{ and } A_2 = \left\{t : \overline{\phi}(t) = \sup_{\theta \in \Theta} \phi^*_\theta(t) < \phi_0(t)\right\}.$$

For the equivariant estimator $d_{\phi(T)}$, define the function $\phi^*(t)$ as,

$$\phi^*(t) = \begin{cases} \underline{\phi}(t) & \text{if } t \in A_1, \\ \overline{\phi}(t) & \text{if } t \in A_2, \\ \phi_0(t) & \text{if elsewhere.} \end{cases}$$

Then the estimator $d_{\phi^*(T)}$ dominates d_0 provided $P_\theta(A_1 \cup A_2) > 0$ for some choices of $\theta \in \Theta$.

Chapter 3

Estimation of Common Mean of Two Normal Populations with Order Restricted Variances

3.1 Introduction

In this chapter, we have revisited the problem of estimating common mean of two normal populations, when the variances (nuisance parameters) follow a simple ordering, say $\sigma_1^2 \leq \sigma_2^2$.

Suppose we have two independent normal populations with a common mean μ and possibly different variances σ_1^2 and σ_2^2. More specifically, let $\underset{\sim}{X} = (X_1, X_2, \ldots, X_m)$ and $\underset{\sim}{Y} = (Y_1, Y_2, \ldots, Y_n)$ be independent random samples taken from two normal populations $N(\mu, \sigma_1^2)$ and $N(\mu, \sigma_2^2)$ respectively. The problem is to estimate the common mean μ under the assumption that the variances follow the ordering $\sigma_1^2 \leq \sigma_2^2$. In order to evaluate the performance of an estimator the loss functions

$$L_1(d, \underset{\sim}{\alpha}) = \left(\frac{d-\mu}{\sigma_1}\right)^2, \tag{3.1.1}$$

$$L_2(d, \underset{\sim}{\alpha}) = |d-\mu|, \tag{3.1.2}$$

and

$$L_3(d, \underset{\sim}{\alpha}) = (d-\mu)^2, \tag{3.1.3}$$

will be used, where d is an estimator for estimating the common mean μ and $\underset{\sim}{\alpha} = (\mu, \sigma_1^2, \sigma_2^2)$. Further the risk of an estimator d is defined by $R(d, \mu) = E_{\underset{\sim}{\alpha}}\{L_i(d, \mu)\}$; $i = 1, 2, 3$.

It is worth mentioning that, the problem of estimating the common mean of two or more normal populations, without taking account the order restrictions on the variances, is quite popular and has a long history in the literature of statistical inference. In fact, the origin of the problem has been in the recovery of inter-block information in the problems of balanced incomplete block designs, which probably was revealed by Yates (1940). Moreover, the problem has been attended by several pioneer researchers in the last few decades, due to its practical applications as well as the challenges involve in it. This well known problem arises in situations, where two or more measuring devices in a laboratory are used to measure certain quantity, several independent agencies are employed to test the effectiveness of certain new drugs produced by an industry, two or more different methods have been used to evaluate the performance of certain characters. Under these circumstances, if it is assumed that, the samples drawn follow normal distributions, then the task boils down to draw inference on the common mean when the variances are unknown and probably unknown and unequal. We refer to Chang and Pal (2008), Lin and Lee (2005) and Kelleher (1996) for applications as well as examples of such situations. We note that, when there is no order restrictions on the variances, one of the first attempts in estimating the common mean μ was made by Graybill and Deal (1959). They have obtained a combined estimator by taking convex combination of two sample means with weights as the functions of sample variances. They established that their proposed combined estimator performs better than the sample means in terms of mean squared error when the sample sizes are at least 11. Since then it has been a great interest for researchers to find some decision theoretic as well as classical results in this direction. In fact, their main goal has been to obtain either some competitors to Graybill-Deal estimator or some other alternative estimators which may perform better than both the sample means. Also few attempts have been made to prove the admissibility or inadmissibility of the Graybill-Deal estimator. For a detailed literature review with some applications and recent updates on estimating the common mean of two or more normal populations, we refer to Khatri and Shah(1974), Brown and Cohen (1974), Cohen and Sackrowitz (1974), Moore and Krishnamoorthy (1997), Pal et al. (2007), Tripathy and Kumar (2010), Tripathy and Kumar (2015) and the references cited therein.

On the other hand, a little attention has been paid in estimating the common mean μ when it is known a priori, that the variances follow certain ordering, say, $\sigma_1^2 \leq \sigma_2^2$. Probably Elfessi and Pal (1992) was the first to consider this model with some justification and propose an estimator which performs better than the Graybill-Deeal (Graybill and Deal (1959)) estimator. In fact, their proposed estimator performs better than the Graybill-Deal estimator in terms of stochastic domination as well as universally. Latter on their results have been extended to the case of a general $k(\geq 2)$ normal populations by Misra and van der Meulen (1997). Chang et al. (2012) considered the same model and obtained a class of improved estimators. In particular, their class contains the estimators previously proposed by Elfessi and Pal (1992). However, it is also essential to compare all the improved estimators and see their performances for practical purposes. It should be noted that, Tripathy and Kumar (2010) considered several well known estimators for the common mean without taking account the order restrictions on the variances. Their numerical study reveals that, none of the estimators including that of Graybill-Deal dominates others completely in the whole parameter space in terms of the risk values. In fact, all the existing estimators compete well with each others. This fact motivates us to study the performances of all the proposed estimators for the common mean μ under order restrictions on the variances. Being motivated from the above works, we in this paper study the problem which focuses on the following directions. Our first goal is to propose some specific estimators which may perform better than the estimators proposed by Moore and Krishnamoorthy (1997) , Khatri and Shah (1974), Brown and Cohen (1974), Tripathy and Kumar (2010) under order restrictions on the variances. Second is to obtain a class inadmissible estimators under order restrictions on the variances. Third is to compare the performances of all the proposed estimators which is very much essential from application point of view.

The rest of the work is organized as follows. In Section 3.2, we have discussed some basic results and propose a new plug-in type restricted MLE for the common mean μ, which has been obtained numerically. In Section 3.3, we have constructed some alternative estimators for the common mean when it is known a priori that the variances are ordered. It has also been shown that the proposed estimators dominate some of the existing well known estimators including that proposed by Moore and Krishnamoorthy (1997) , Khatri and Shah (1974), Brown and Cohen (1974), Tripathy and Kumar (2010) stochastically as well as universally. Moreover, in Section 3.4 we have proved that these new estimators also dominate their respective old counter parts in terms of Pitman (see Pitman (1937)) measure of closeness criteria. The concept of invariance has been introduced in Section 3.5, and derived some inadmissibility results under order restrictions on the variances. Particularly, the sufficient conditions for improving estimators which are invariant under affine and location group of transformations have been derived. Consequently, improved estimators have been derived. Interestingly these new improved estimators turns out to be the same as obtained in Section 3.2 as well as proposed by Elfessi and Pal (1992). We also note that it is difficult to compare all the proposed estimators analytically. Utilizing the computational facilities available now-a-days, we have compared the risk values of all the proposed estimators numerically with respect to the loss function (3.1.1) in Section 3.6 through the Monte-Carlo simulation method. The percentage of risk improvements have also been noted which is quite significant. Finally the percentage of relative risk performances of all the estimators have been evaluated with respect to the Graybill-Deal estimator and the conclusions have been made regarding their performances which is not been done so far in the literature. This is a major contribution to the chapter as well as to the current literature.

3.2 Some Basic Results

In this section, we consider the model and propose some alternative estimators for the common mean μ when it is known apriori that the variances follow the ordering $\sigma_1^2 \leq \sigma_2^2$.

Let $\underset{\sim}{X} = (X_1, X_2, \ldots, X_m)$ and $\underset{\sim}{Y} = (Y_1, Y_2, \ldots, Y_n)$ be independent random samples taken from two normal populations with a common mean μ and possibly different variances σ_1^2 and σ_2^2 respectively. Let $N(\mu, \sigma_i^2)$ be the normal population with mean μ and variance σ_i^2; $i = 1, 2$. The target is to derive certain estimators of the common mean μ when it is known a priori that the variances are ordered that is $\sigma_1^2 \leq \sigma_2^2$ or equivalently $\sigma_1 \leq \sigma_2$. We note that a minimal sufficient statistics (not complete) for this model exists and is given by $(\bar{X}, \bar{Y}, S_1^2, S_2^2)$ where,

$$\bar{X} = \frac{1}{m}\sum_{i=1}^{m} X_i,\ \bar{Y} = \frac{1}{n}\sum_{j=1}^{n} Y_j,\ S_1^2 = \sum_{i=1}^{m}(X_i - \bar{X})^2,\ S_2^2 = \sum_{j=1}^{n}(Y_j - \bar{Y})^2. \qquad (3.2.1)$$

We also note that, $\bar{X} \sim N(\mu, \sigma_1^2/m)$, $\bar{Y} \sim N(\mu, \sigma_2^2/n)$, $S_1^2/\sigma_1^2 \sim \chi^2_{m-1}$, and $S_2^2/\sigma_2^2 \sim \chi^2_{n-1}$. When there is no order restrictions on the variances, a number of estimators have been proposed by several researchers in the recent past. Let us consider the following well known estimators for the common mean μ when there is no order restriction on the variances.

$$d_{GD} = \frac{m(m-1)S_2^2\bar{X} + n(n-1)S_1^2\bar{Y}}{m(m-1)S_2^2 + n(n-1)S_1^2} \quad \text{(Graybill and Deal (1959))},$$

$$d_{KS} = \frac{m(m-3)S_2^2\bar{X} + n(n-3)S_1^2\bar{Y}}{m(m-3)S_2^2 + n(n-3)S_1^2} \quad \text{(Khatri and Shah (1974))},$$

$$d_{MK} = \frac{\bar{X}\sqrt{m(m-1)}S_2 + \bar{Y}\sqrt{n(n-1)}S_1}{\sqrt{m(m-1)}S_2 + \sqrt{n(n-1)}S_1} \quad \text{(Moore and Krishnamoorthy (1997))},$$

$$d_{TK} = \frac{\bar{X}\sqrt{m}c_nS_2 + \bar{Y}\sqrt{n}c_mS_1}{\sqrt{m}c_nS_2 + \sqrt{n}c_mS_1} \quad \text{(Tripathy and Kumar (2010))},$$

$$
\begin{aligned}
d_{BC1} &= \bar{X} + \Big\{ \frac{(\bar{Y}-\bar{X})b_1 S_1^2/m(m-1)}{S_1^2/m(m-1) + S_2^2/(n(n+2)) + (\bar{Y}-\bar{X})^2/(n+2)} \Big\}, \\
d_{BC2} &= \bar{X} + (\bar{Y}-\bar{X}) \Big\{ \frac{b_2 n(n-1)S_1^2}{n(n-1)S_1^2 + m(m-1)S_2^2} \Big\} \text{ (Brown and Cohen (1974))}, \\
d_{GM} &= \frac{m\bar{X} + n\bar{Y}}{m+n} \quad \text{(grand mean)},
\end{aligned}
$$

where $c_m = \frac{\Gamma(\frac{m-1}{2})}{\sqrt{2}\Gamma(\frac{m}{2})}$, $c_n = \frac{\Gamma(\frac{n-1}{2})}{\sqrt{2}\Gamma(\frac{n}{2})}$, $0 < b_1 < b_{\max}(m,n)$, $0 < b_2 < b_{\max}(m, n-3)$, and $b_{max}(m,n) = 2(n+2)/nE(\max(V^{-1}, V^{-2}))$. Here V is a random variable having F-distribution with $(n+2)$ and $(m-1)$ degrees of freedom.

Finally we consider the MLE of μ whose closed form does not exist. The MLEs of μ can be obtained numerically by solving the following system of three equations in three unknowns μ, σ_1^2, and σ_2^2 as,

$$
\mu = \frac{\frac{m}{\sigma_1^2}\bar{x} + \frac{n}{\sigma_2^2}\bar{y}}{\frac{m}{\sigma_1^2} + \frac{n}{\sigma_2^2}}, \tag{3.2.2}
$$

$$
\sigma_1^2 = \frac{s_1^2}{m} + \Big(\frac{n\sigma_1^2}{n\sigma_1^2 + m\sigma_2^2}\Big)^2 (\bar{x}-\bar{y})^2, \tag{3.2.3}
$$

$$
\sigma_2^2 = \frac{s_2^2}{n} + \Big(\frac{m\sigma_2^2}{n\sigma_1^2 + m\sigma_2^2}\Big)^2 (\bar{x}-\bar{y})^2. \tag{3.2.4}
$$

Here $(\bar{x}, \bar{y}, s_1^2, s_2^2)$ denotes the observed values of $(\bar{X}, \bar{Y}, S_1^2, S_2^2)$. Let the solution of the above system of equations be $\hat{\mu}_{ML}$, σ_{1ML}^2, and σ_{2ML}^2. These are the MLEs of μ, σ_1^2 and σ_2^2 respectively, when there is no order restrictions on the variances.

Next we discuss some basic results when it is known a priori that the variances are ordered, that is when $\sigma_1^2 \leq \sigma_2^2$. Let us define $\beta = \frac{n(n-1)S_1^2}{m(m-1)S_2^2 + n(n-1)S_1^2}$. Under order restrictions on the variances that is when $\sigma_1^2 \leq \sigma_2^2$, Elfessi and Pal (1992) proposed a new estimator, call it $\hat{d}_{EP}$, and is given by

$$
\hat{d}_{EP} = \begin{cases} (1-\beta)\bar{X} + \beta\bar{Y}, & \text{if } \frac{S_1^2}{m-1} \leq \frac{S_2^2}{n-1} \\ \beta^*\bar{X} + (1-\beta^*)\bar{Y}, & \text{if } \frac{S_1^2}{m-1} > \frac{S_2^2}{n-1}, \end{cases}
$$

where

$$
\beta^* = \begin{cases} \beta, & \text{if } m = n \\ \frac{m}{m+n}, & \text{if } m \neq n. \end{cases}
$$

In the above definition of $\hat{d}_{EP}$ for the case $m = n$, when $\beta^* = \beta$, we mean β as well as the conditions must be simplified for $m = n$.

It is well known that the estimator $\hat{d}_{EP}$ dominates d_{GD} stochastically as well as universally when $\sigma_1^2 \leq \sigma_2^2$. Further Misra and van der Meulen (1997) extended these dominance results to the case of $k(\geq 2)$ normal populations and also shown that the estimator $\hat{d}_{EP}$ performs better than d_{GD} in terms of Pitman measure of closeness criteria. The MLE of μ has been obtained by solving the system of equations numerically as shown above (see equations (3.2.2) to (3.2.4))). Under order restriction on the variances, that is, when $\sigma_1^2 \leq \sigma_2^2$, using the isotonic regression we obtain plug-in type restricted MLEs (numerically) of σ_1^2 and σ_2^2 respectively as,

$$\sigma_{1R}^2 = \begin{cases} \sigma_{1ML}^2, & \text{if } \sigma_{1ML}^2 \le \sigma_{2ML}^2 \\ \frac{1}{2}(\sigma_{1ML}^2 + \sigma_{2ML}^2), & \text{if } \sigma_{1ML}^2 > \sigma_{2ML}^2, \end{cases}$$

and

$$\sigma_{2R}^2 = \begin{cases} \sigma_{2ML}^2, & \text{if } \sigma_{1ML}^2 \le \sigma_{2ML}^2 \\ \frac{1}{2}(\sigma_{1ML}^2 + \sigma_{2ML}^2), & \text{if } \sigma_{1ML}^2 > \sigma_{2ML}^2. \end{cases}$$

Substituting these estimators in (3.2.2), we get a plug-in type restricted MLE, (call it d_{RM}) of μ as

$$d_{RM} = \frac{m\sigma_{2R}^2 \bar{X} + n\sigma_{1R}^2 \bar{Y}}{m\sigma_{2R}^2 + n\sigma_{1R}^2}.$$

Further using the grand mean of the two populations, one gets another plug-in type restricted MLE of μ call it $\hat{d}_{RM}$ and is given by,

$$\hat{d}_{RM} = \begin{cases} \hat{\mu}_{ML}, & \text{if } \sigma_{1ML}^2 \le \sigma_{2ML}^2 \\ \frac{m\bar{X}+n\bar{Y}}{m+n}, & \text{if } \sigma_{1ML}^2 > \sigma_{2ML}^2. \end{cases}$$

Tripathy and Kumar (2010) pointed out that a theoretical comparison of all the estimators is quite impossible and hence only numerical comparison is possible. They have also mentioned that the estimators d_{MK} and d_{TK} compete with each other and perform better than d_{GD} when the variances are not far away from each other. It has also been noticed that for small values of the ratios of the variances the estimator d_{KS} compete with d_{GD}. It is quite evident that one needs to find alternative estimators for μ that will compete with d_{MK}, d_{TK}, and d_{KS}, when $\sigma_1^2 \le \sigma_2^2$ or equivalently $\sigma_1 \le \sigma_2$. In the next section, we construct some estimators which dominate these estimators stochastically as well as universally. Now onwards for convenient we will denote $\hat{d}_{GD}$ in place of $\hat{d}_{EP}$.

Remark 3.2.1 *One can construct another plug-in type estimator for μ by replacing the estimators σ_{1R}^2 and σ_{2R}^2 in d_{RM} by $\sigma_{1R}^2 = \min(\sigma_{1ML}^2, \frac{m\sigma_{1ML}^2+n\sigma_{2ML}^2}{m+n})$ and $\sigma_{2R}^2 = \max(\sigma_{2ML}^2, \frac{m\sigma_{1ML}^2+n\sigma_{2ML}^2}{m+n})$ respectively when $m \neq n$. It has been revealed from our numerical study (Section 3.6) that it acts as a competitor of d_{RM}.*

Remark 3.2.2 *The estimators d_{RM} and $\hat{d}_{RM}$ are seen to perform equally good, which has been checked by using a numerical study in Section3.6. Hence we only include $\hat{d}_{RM}$ for numerical comparison purpose.*

3.3 Stochastic Domination under Order Restriction on the Variances

In this section we propose some alternative estimators for the common mean μ when there is order restriction on the variances that is when it is known apriori that $\sigma_1^2 \le \sigma_2^2$ or equivalently $\sigma_1 \le \sigma_2$. Further we will prove that these alternative estimators dominate stochastically some of the existing well known estimators proposed by Moore and Krishnamoorthy (1997) , Khatri and

Shah (1974), Brown and Cohen (1974) and Tripathy and Kumar (2010) under order restriction on the variances.

Let $\beta_1 = \frac{\sqrt{n(n-1)}S_1}{\sqrt{m(m-1)}S_2+\sqrt{n(n-1)}S_1}$, $\beta_2 = \frac{\sqrt{n}c_m S_1}{\sqrt{m}c_n S_2+\sqrt{n}c_m S_1}$, and $\beta_3 = \frac{n(n-3)S_1^2}{m(m-3)S_2^2+n(n-1)S_1^2}$, $\beta_4 = \frac{b_2 S_1^2}{S_1^2+S_2^2}$. We propose the following estimators for the common mean μ, when the variances known to follow the ordering $\sigma_1^2 \leq \sigma_2^2$.

$$\hat{d}_{MK} = \begin{cases} (1-\beta_1)\bar{X} + \beta_1\bar{Y}, & \text{if } \frac{\sqrt{n-1}S_1}{\sqrt{m-1}S_2} \leq \sqrt{\frac{n}{m}} \\ \beta_1^*\bar{X} + (1-\beta_1^*)\bar{Y}, & \text{if } \frac{\sqrt{n-1}S_1}{\sqrt{m-1}S_2} > \sqrt{\frac{n}{m}}, \end{cases}$$

where

$$\beta_1^* = \begin{cases} \beta_1, & \text{if } m = n, \\ \frac{m}{m+n}, & \text{if } m \neq n. \end{cases}$$

$$\hat{d}_{TK} = \begin{cases} (1-\beta_2)\bar{X} + \beta_2\bar{Y}, & \text{if } \frac{S_1}{S_2} \leq \sqrt{\frac{n}{m}}\frac{c_n}{c_m} \\ \beta_2^*\bar{X} + (1-\beta_2^*)\bar{Y}, & \text{if } \frac{S_1}{S_2} > \sqrt{\frac{n}{m}}\frac{c_n}{c_m}, \end{cases}$$

where

$$\beta_2^* = \begin{cases} \beta_2, & \text{if } m = n, \\ \frac{m}{m+n}, & \text{if } m \neq n. \end{cases}$$

$$\hat{d}_{KS} = \begin{cases} (1-\beta_3)\bar{X} + \beta_3\bar{Y}, & \text{if } \frac{S_1^2}{S_2^2} \leq \frac{m-3}{n-3} \\ \beta_3^*\bar{X} + (1-\beta_3^*)\bar{Y}, & \text{if } \frac{S_1^2}{S_2^2} > \frac{m-3}{n-3}, \end{cases}$$

where

$$\beta_3^* = \begin{cases} \beta_3, & \text{if } m = n, \\ \frac{m}{m+n}, & \text{if } m \neq n. \end{cases}$$

Finally we define only for equal sample sizes,

$$\hat{d}_{BC2} = \begin{cases} (1-\beta_4)\bar{X} + \beta_4\bar{Y}, & \text{if } S_2^2 \geq (2b_2-1)S_1^2 \\ \beta_4\bar{X} + (1-\beta_4)\bar{Y}, & \text{if } S_2^2 \geq (2b_2-1)S_1^2. \end{cases}$$

In the above definitions of the estimators $\hat{d}_{MK}$, $\hat{d}_{TK}$, $\hat{d}_{KS}$ for the case $m = n$, when $\beta_i^* = \beta_i$; $i = 1, 2, 3$ we also mean that both β_i and the corresponding conditions must be simplified by putting $m = n$.

To proceed further we need the following concepts and definitions which will be very much handy in developing the sections. Let d_1 and d_2 be any two estimators of the unknown parameter say θ.

Definition 3.3.1 *The estimator d_1 is said to dominate another estimator d_2 stochastically if*

$$P_\theta[(d_2-\mu)^2 \le c] \le P_\theta[(d_1-\mu)^2 \le c], \ \forall\, c>0.$$

Definition 3.3.2 *Let the loss function $L(d,\theta)$ in estimating θ by d be a non-decreasing function of the error $|d-\theta|$. An estimator d_1 is said to dominate another estimator d_2 universally if*

$$EL(|d_1-\theta|) \le EL(|d_2-\theta|),$$

over the parameter space for all $L(,)$ non-decreasing. Further it was shown by Hwang (1985) that d_1 dominates d_2 universally if and only if d_1 dominates d_2 stochastically.

Next, we prove the following results for estimating the common mean μ, under order restriction on the variances that is when it is known a priori that, $\sigma_1^2 \le \sigma_2^2$ or equivalently $\sigma_1 \le \sigma_2$.

Theorem 3.3.1 *Let the loss function $L(.)$ be a non-decreasing function of the error $|d-\mu|$. Further assume that the variances are known to follow the ordering $\sigma_1^2 \le \sigma_2^2$. Then for estimating the common mean μ we have the following dominance results.*

(i) The estimator $\hat{d}_{MK}$ dominates d_{MK} stochastically and hence universally.

(ii) The estimator $\hat{d}_{TK}$ dominates d_{TK} stochastically and hence universally.

(iii) The estimator $\hat{d}_{KS}$ dominates d_{KS} stochastically and hence universally.

(iv) The estimator $\hat{d}_{BC2}$ dominates d_{BC2} stochastically and hence universally.

Proof 3.3.1 (i) *First we will prove the result for the case of equal sample sizes that is for $m=n$. Consider the estimator $\hat{d}_{MK}$, which is given by*

$$\hat{d}_{MK} = \begin{cases} (1-\beta_1)\bar{X}+\beta_1\bar{Y}, & \text{if } S_1 \le S_2, \\ \beta_1\bar{X}+(1-\beta_1)\bar{Y}, & \text{if } S_1 > S_2. \end{cases}$$

Our target is to show that,

$$P[(\hat{d}_{MK}-\mu)^2 \le c] \le P[(d_{MK}-\mu)^2 \le c], \ \forall\, c>0. \tag{3.3.1}$$

We note that,

$$\begin{aligned} P[(\hat{d}_{MK}-\mu)^2 \le c] &= P[(d_{MK}-\mu)^2 \le c|S_1 \le S_2]P(S_1 \le S_2) \\ &\quad + P[(\hat{d}_{MK}-\mu)^2 \le c|S_1 > S_2]P(S_1 > S_2), \ \forall\, c>0. \end{aligned}$$

Thus the above inequality (3.3.1) is reduced to,

$$P[(d_{MK}-\mu)^2 \le c|S_1 > S_2] \le P[(\hat{d}_{MK}-\mu)^2 \le c|S_1 > S_2], \ \forall\, c>0.$$

Let us denote $X_1^* = (1-\beta_1)\bar{X}+\beta_1\bar{Y}$ and $X_2^* = \beta_1\bar{X}+(1-\beta_1)\bar{Y}$. With these notations, the above inequality is further equivalent to prove,

$$P[-\sqrt{c} \le X_1^*-\mu \le \sqrt{c}|S_1 > S_2] \le P[-\sqrt{c}X_2^*-\mu \le \sqrt{c}|S_1 > S_2], \ \forall\, c>0.$$

It is easy to observe that, $X_1^* - \mu \sim N(0, \sigma^2)$*, where* $\sigma^2 = (1-\beta_1)^2\frac{\sigma_1^2}{m} + \beta_1^2\frac{\sigma_2^2}{m}$*. Further* X_2^* *follows a normal distribution with mean* 0 *and variance* $\sigma_*^2 = \beta_1^2\frac{\sigma_1^2}{m} + (1-\beta_1)^2\frac{\sigma_2^2}{m}$*. Thus incorporating al these information the above inequality reduces to,*

$$\frac{\int\int_{s_1>s_2}[2\Phi\Big(\frac{\sqrt{c}}{\sigma}\Big)-1]g_1(s_1)g_2(s_2)ds_1ds_2}{P(S_1>S_2)} \le \frac{\int\int_{s_1>s_2}[2\Phi\Big(\frac{\sqrt{c}}{\sigma_*}\Big)-1]g_1(s_1)g_2(s_2)ds_1ds_2}{P(S_1>S_2)}, \ \forall\, c>0,$$

where $\Phi(.)$ *denotes the cumulative distribution function of a standard normal random variable and* g_1*,* g_2 *denote the probability density function of the random variables* S_1*,* S_2 *respectively. This is further equivalent to say that,*

$$\Phi\Big(\frac{\sqrt{c}}{\sigma}\Big) \le \Phi\Big(\frac{\sqrt{c}}{\sigma_*}\Big),\ \forall\, c>0 \text{ and } S_1 > S_2. \tag{3.3.2}$$

The inequality (3.3.2) is equivalent to show that, $\sigma^2 > \sigma_*^2$*, when* $S_1 > S_2$*. The inequality is true as,* $\sigma^2 - \sigma_*^2 = ((1-\beta_1^2)-\beta_1^2)(\sigma_1^2-\sigma_2^2) = \frac{(S_2^2-S_1^2)(\sigma_1^2-\sigma_2^2)}{(S_1+S_2)^2} > 0$*, when* $\sigma_1^2 \le \sigma_2^2$ *and* $S_1 > S_2$*. The proof is completed when the sample sizes are equal, that is for the case* $m = n$*.*

Next we will prove the result for the case of unequal sample sizes that is for $m \neq n$*. To prove the result let us denote* $V_1 = \sqrt{\frac{m}{m-1}}S_1$ *and* $V_2 = \sqrt{\frac{n}{n-1}}S_2$*. With these notations, the estimator* $\hat{d}_{MK}$ *is seen to be,*

$$\hat{d}_{MK} = \begin{cases} d_{MK}, & \text{if } V_1 \le V_2 \\ \frac{m\bar{X}+n\bar{Y}}{m+n}, & \text{if } V_1 > V_2. \end{cases}$$

Proceeding as before, one needs to show that,

$$P[(d_{MK}-\mu)^2 \le c|V_1 > V_2] \le P[(\hat{d}_{MK}-\mu)^2 \le c|V_1 > V_2],\ \ \forall\, c > 0,$$

which is equivalent to show that,

$$P[-\sqrt{c} \le d_{MK}-\mu \le \sqrt{c}|V_1 > V_2] \le P[-\sqrt{c} \le \hat{d}_{MK}-\mu \le \sqrt{c}|V_1 > V_2],\ \ \forall\, c > 0.$$

It is easy to observe that given S_1, S_2 *the random variable* $d_{MK}-\mu$ *follows a normal distribution with mean* 0 *and variance* $\nu^2 = \frac{(m-1)s_2^2\sigma_1^2+(n-1)s_1^2\sigma_2^2}{(\sqrt{m(m-1)}s_2+\sqrt{n(n-1)}s_1)^2}$*. Also given* S_1 *and* S_2*,* $\hat{d}_{MK}-\mu$ *follows normal distribution with mean* 0 *and variance* $\nu_*^2 = \frac{m\sigma_1^2+n\sigma_2^2}{(m+n)^2}$*. Utilizing these information, the above inequality after some simplification, reduces to*

$$\Phi\Big(\frac{\sqrt{c}}{\nu}\Big) \le \Phi\Big(\frac{\sqrt{c}}{\nu_*}\Big),\ \forall\, c > 0.$$

This is further equivalent to show that $\nu^2 > \nu_*^2, \forall\, c > 0$ *when* $V_1 > V_2$*. This is equivalent to show that,*

$$\Phi\Big(\frac{\sqrt{c}}{\nu}\Big) \le \Phi\Big(\frac{\sqrt{c}}{\nu_*}\Big),\ \forall\, c > 0.$$

This is equivalent to,

$$\frac{\sigma_1^2 + \sigma_2^2\lambda^2}{(\sqrt{m} + \sqrt{n}\lambda)^2} > \frac{m\sigma_1^2 + n\sigma_2^2}{(m+n)^2}, \ \forall\, c > 0, \sigma_1 \le \sigma_2,$$

where we denote $\lambda = \frac{\sqrt{n-1}S_1}{\sqrt{m-1}S_2}$. *Denote,* $h(\lambda) = \frac{\sigma_1^2+\sigma_2^2\lambda^2}{(\sqrt{m}+\sqrt{n}\lambda)^2}$, *and our target is to show that,* $h(\lambda) > h(\sqrt{\frac{n}{m}})$, *for* $\lambda > \sqrt{\frac{n}{m}}$. *It is easy to check that* $\frac{dh}{d\lambda} \le 0$, *if* $\lambda \le \sqrt{\frac{n}{m}}\frac{\sigma_1^2}{\sigma_2^2} \le \sqrt{\frac{n}{m}}$, *as* $\sigma_1^2/\sigma_2^2 \le 1$. *Also* $\frac{dh}{d\lambda} > 0$, *when* $\lambda > \sqrt{\frac{n}{m}}$. *Hence the function* $h(\lambda)$ *is increasing in the interval* $[\sqrt{\frac{n}{m}}, \infty)$. *Further universal dominance follows from Definition 3.3.2. This proves* (i) *of the theorem for both equal and unequal sample sizes. The proofs of* (ii), (iii) *and* (iv) *are very much similar to the proof of* (i) *and hence have been omitted for brevity. This completes the proof of the Theorem 3.3.1.*

Remark 3.3.1 *It should be noted that, for the case of unequal sample sizes, one can construct an estimator which may dominate the estimator* d_{BC2} *stochastically as well as universally, however the conditions will be more restrictive.*

3.4 Pitman Measure of Closeness

In this section, we prove that the new proposed estimators $\hat{d}_{MK}, \hat{d}_{TK}, \hat{d}_{KS}, \hat{d}_{BC2}$, perform better than their old counter parts in terms of Pitman measure of closeness when it is known a priori that the variances follow the ordering $\sigma_1^2 \le \sigma_2^2$ or equivalently $\sigma_1 \le \sigma_2$. To prove the main results of this section, we need the following concepts. Let δ_1 and δ_2 be any two estimators of a real parametric function say $\psi(\theta)$. Pitman (1937) proposed a measure of relative closeness to the parametric function $\psi(\theta)$ for comparing two estimators in the following fashions.

Definition 3.4.1 *The estimator* δ_1 *should be preferred to* δ_2 *if for every* θ,

$$PMC_\theta(\delta_1, \delta_2) = P_\theta(|\delta_1 - \psi(\theta)| < |\delta_2 - \psi(\theta)| | \delta_1 \ne \delta_2) \ge \frac{1}{2},$$

and with strict inequality for some θ.

The following lemma will be useful for proving the main results of this section, which was proposed by Peddada and Khattree (1986).

Lemma 3.4.1 *Suppose the random vector* (X, Y) *has a bivariate normal distribution with* $E(X) = E(Y) = 0$ *and* $E(X^2) < E(Y^2)$. *Then* $P(|X| < |Y|) > \frac{1}{2}$.

Let $\underset{\sim}{\alpha} = (\mu, \sigma_1^2, \sigma_2^2)$ and $\Omega_R = \{\underset{\sim}{\alpha} = (\mu, \sigma_1^2, \sigma_2^2) : -\infty < \mu < \infty, 0 < \sigma_1^2 \le \sigma_2^2 < \infty\}$. We prove the following theorem.

Theorem 3.4.1 *For estimating the common mean* μ *of two normal populations we have the following dominance results.*

(i) $PMC(\hat{d}_{MK}, d_{MK}) > \frac{1}{2}, \forall\, \underset{\sim}{\alpha} \in \Omega_R$.

(ii) $PMC(\hat{d}_{TK}, d_{TK}) > \frac{1}{2}, \forall\, \underset{\sim}{\alpha} \in \Omega_R$.

(iii) $PMC(\hat{d}_{KS}, d_{KS}) > \frac{1}{2}, \forall\, \underset{\sim}{\alpha} \in \Omega_R$.

(iv) $PMC(\hat{d}_{BC2}, d_{BC2}) > \frac{1}{2}, \forall\, \underset{\sim}{\alpha} \in \Omega_R.$

Proof 3.4.1 (i) *We prove the result for equal and unequal sample sizes separately. Let the sample sizes be equal that is* $m = n$. *We note that the conditional distributions of* $\hat{d}_{MK} - \mu$ *and* $\hat{\mu}_{MK} - \mu$ *given* S_1 *and* S_2 *follows normal distribution with a common mean* 0 *and variances* σ_*^2 *and* σ^2 *respectively as given in the proof of the Theorem 3.3.1. It has also been shown there that,* $\sigma_*^2 \le \sigma^2$ *whenever* $\sigma_1 \le \sigma_2$ *and* $S_1 > S_2$. *Thus we have,*

$$E[(\hat{d}_{MK} - \mu)^2 | (S_1, S_2)] \le E[(\hat{\mu}_{MK} - \mu)^2 | (S_1, S_2)],$$

whenever $\sigma_1 \le \sigma_2$ *and* $S_1 > S_2$. *Now using Lemma 3.4.1, and the above inequality, it follows that,*

$$PMC(\hat{d}_{MK}, \hat{\mu}_{MK}) > \frac{1}{2}, \;\; \forall\; \underset{\sim}{\alpha} \in \Omega_R.$$

To prove for unequal sample sizes, we observe that the conditional distributions of $\hat{d}_{MK} - \mu$ *and* $\hat{\mu}_{MK} - \mu$ *given* S_1 *and* S_2 *follows normal distribution with a common mean* 0 *and variances* ν_*^2 *and* ν^2 *respectively. Further using that,* $\nu_*^2 \le \nu^2$, *when* $\sigma_1 \le \sigma_2$, *and* $\lambda > \sqrt{\frac{n}{m}}$. *Thus we have also in the case of unequal sample sizes,*

$$PMC(\hat{d}_{MK}, \hat{\mu}_{MK}) > \frac{1}{2}, \;\; \forall\; \underset{\sim}{\alpha} \in \Omega_R.$$

The proofs of (ii), (iii) *and* (iv) *are similar to the proof of* (i) *and hence has been omitted. This completes the proof of the theorem.*

In the next section we will introduce the concept of invariance to our problem and prove some inadmissibility results in the classes of equivariant estimators for the common mean.

3.5 Inadmissibility Results under Order Restrictions on the Variances

In this section we introduce the concept of invariance to the model problem and derive some inadmissibility results for both affine and location equivariant estimators under order restriction on the variances. As a consequence, improved estimators dominating the popular estimators without order restriction on the variances have been derived.

Affine Class

Let us introduce the concept of invariance to our problem. More specifically, consider the affine group of transformations, $G_A = \{g_{a,b} : g_{a,b}(x) = ax + b, a > 0, b \in R\}$. Under the transformation $g_{a,b}$, $X_i \to aX_i + b$, $Y_j \to aY_j + b$, and consequently the sufficient statistics $\bar{X} \to a\bar{X} + b$, $\bar{Y} \to a\bar{Y} + b$, $S_i^2 \to a^2 S_i^2$, $\mu \to a\mu + b$, $\sigma_i^2 \to a^2\sigma_i^2$ and the family of distributions remains invariant. The problem remains invariant if we choose the loss function as (3.1.1). The form of an affine equivariant estimator for estimating μ, based on the sufficient statistics $(\bar{X}, \bar{Y}, S_1^2, S_2^2)$ is obtained as

$$d_\Psi = \bar{X} + S_1 \Psi(\underset{\sim}{T}), \tag{3.5.1}$$

where $\underset{\sim}{T} = (T_1, T_2)$, $T_1 = \frac{\bar{Y}-\bar{X}}{S_1}$, $T_2 = \frac{S_2^2}{S_1^2}$ and Ψ is any real valued function.

Let us define a new function Ψ_0 for the affine equivariant estimator d_Ψ as,

$$\Psi_0(\underset{\sim}{t}) = \begin{cases} \frac{n}{n+m}\min(t_1,0), & \text{if } \Psi(\underset{\sim}{t}) < \frac{n}{n+m}\min(t_1,0), \\ \Psi(\underset{\sim}{t}), & \text{if } \frac{n}{n+m}\min(t_1,0) \le \Psi(\underset{\sim}{t}) \le \frac{n}{n+m}\max(t_1,0), \\ \frac{n}{n+m}\max(t_1,0), & \text{if } \Psi(\underset{\sim}{t}) > \frac{n}{n+m}\max(t_1,0). \end{cases} \tag{3.5.2}$$

The following theorem gives a sufficient condition for improving estimators in the class of affine equivariant estimators of the form (3.5.1), under the order restrictions on the variances.

Theorem 3.5.1 *Let d_Ψ be an affine equivariant estimator of the form (3.5.1) for estimating the common mean μ and the loss function be the affine invariant loss (3.1.1). The estimator d_Ψ is inadmissible and is improved by d_{Ψ_0} if $P(\Psi(\underset{\sim}{T}) \neq \Psi_0(\underset{\sim}{T})) > 0$, for some choices of the parameters $\underset{\sim}{\alpha}$; $\sigma_1 \le \sigma_2$.*

Proof 3.5.1 *The theorem can be proved by using a well known technique for improving equivariant estimators proposed by Brewster and Zidek (1974). To proceed, let us consider the conditional risk function of d_Ψ given $\underset{\sim}{T} = \underset{\sim}{t}$:*

$$R(\underset{\sim}{\alpha}, d_\Psi|\underset{\sim}{t}) = \frac{1}{\sigma_1^2} E\{(\bar{X} + S_1\Psi(\underset{\sim}{T}) - \mu)^2 | \underset{\sim}{T} = \underset{\sim}{t}\}.$$

The above risk function is convex in $\Psi(\underset{\sim}{t})$ and attains its minimum value at

$$\Psi(\underset{\sim}{t}, \underset{\sim}{\alpha}) = \frac{E\{(\mu - \bar{X})S_1 | \underset{\sim}{T} = \underset{\sim}{t}\}}{E\{S_1^2 | \underset{\sim}{T} = \underset{\sim}{t}\}}. \tag{3.5.3}$$

To evaluate the conditional expectations involved in the above expression, we use the following transformations. Let us define $V_1 = \frac{\sqrt{m}(\bar{X}-\mu)}{\sigma_1}$, $V_2 = \frac{\sqrt{m}(\bar{Y}-\mu)}{\sigma_1}$, $W_1 = \frac{S_1^2}{\sigma_1^2}$ and $W_2 = \frac{S_2^2}{\sigma_2^2}$ and $\rho = \frac{\sigma_2^2}{\sigma_1^2}$. With this substitution the expression for $\Psi(\underset{\sim}{t}, \underset{\sim}{\alpha})$ then reduces to,

$$\Psi(\underset{\sim}{t}, \rho) = -\frac{E(V_1 W_1^{\frac{1}{2}} | \underset{\sim}{T} = \underset{\sim}{t})}{\sqrt{m} E(W_1 | \underset{\sim}{T} = \underset{\sim}{t})}. \tag{3.5.4}$$

These conditional expectations have been evaluated in Tripathy and Kumar (2010) and are given by,

$$E(W_1 | \underset{\sim}{T} = \underset{\sim}{t}) = \frac{m+n-1}{\lambda},$$

and

$$E(V_1 W_1^{\frac{1}{2}} | \underset{\sim}{T} = \underset{\sim}{t}) = -\frac{n\sqrt{m}(m+n-1)t_1}{(n+m\rho)\lambda},$$

where $\lambda = \frac{mnt_1^2}{n+m\rho} + \frac{t_2}{\rho} + 1$, and where $\rho = \frac{\sigma_2^2}{\sigma_1^2} \ge 1$, as $\sigma_1^2 \le \sigma_2^2$. Substituting these expressions *in (3.5.4), we get the minimimizing choice of* $\Psi(\underset{\sim}{t}, \rho)$ *as,*

$$\hat{\Psi}(\underset{\sim}{t}, \rho) = \frac{nt_1}{n + m\rho}.$$

In order to prove the inadmissibility result of the theorem, we need the supremum and infimum value of $\hat{\Psi}(\underset{\sim}{t}, \rho)$ *with respect to* ρ *for fixed values of* $\underset{\sim}{T} = \underset{\sim}{t}$*. We consider the following two cases to obtain the supremum and infimum of* $\hat{\Psi}(\underset{\sim}{t}, \rho)$*.*
Case-I: Let $t_1 \geq 0$*. Now the function* $\hat{\Psi}(\underset{\sim}{t}, \tau)$ *is decreasing with respect to* $\rho \geq 1$*. Hence, we obtain*

$$\inf_{\rho \geq 1} \hat{\Psi}(\underset{\sim}{t}, \rho) = \lim_{\rho \to \infty} \hat{\Psi}(\underset{\sim}{t}, \rho) = 0 \textit{ and } \sup_{\rho \geq 1} \hat{\Psi}(\underset{\sim}{t}, \rho) = \lim_{\rho \to 1} \hat{\Psi}(\underset{\sim}{t}, \rho) = \frac{nt_1}{n + m}.$$

Case-II: Let $t_1 < 0$*. The function* $\hat{\Psi}(\underset{\sim}{t}, \rho)$ *is an increasing function of* ρ*. So, in this case we obtain,*

$$\inf_{\rho \geq 1} \hat{\Psi}(\underset{\sim}{t}, \rho) = \lim_{\rho \to 1} \hat{\Psi}(\underset{\sim}{t}, \rho) = \frac{nt_1}{n + m} \textit{ and } \sup_{\rho \geq 1} \hat{\Psi}(\underset{\sim}{t}, \rho) = \lim_{\rho \to \infty} \hat{\Psi}(\underset{\sim}{t}, \rho) = 0.$$

Combining the Case-I and Case-II, it is easy to define the function $\Psi_0(\underset{\sim}{t})$ *as given in (3.5.2). Utilizing the function* $\Psi_0(\underset{\sim}{t})$ *and as an application of Theorem 3.3.1 (in Brewster and Zidek (1974)), we get* $R(\underset{\sim}{\alpha}, d_{\Psi_0}) \leq R(\underset{\sim}{\alpha}, d_{\Psi})$*, when* $\sigma_1 \leq \sigma_2$*. This completes the proof of the theorem.*

Next we will apply Theorem 3.5.1, to obtain some improved estimators for the common mean μ, under the assumption that $\sigma_1^2 \leq \sigma_2^2$. It is easy to observe that all the estimators discussed in Section 3.2, for the common mean μ without considering order restriction on the variances, except the MLE d_{ML} (whose closed form does not exist) fall in the class $d_\Psi = \bar{X} + S_1\Psi(\underset{\sim}{T})$. We apply Theorem 3.5.1 to get their corresponding improved estimators under the assumption that $\sigma_1^2 \leq \sigma_2^2$. Let us first consider the estimator $d_{GD} = \bar{X} + S_1\Psi(\underset{\sim}{T})$, where $\Psi(\underset{\sim}{T}) = \frac{n(n-1)T_1}{m(m-1)T_2+n(n-1)}$. We observe that $\Psi(\underset{\sim}{t}) > \frac{n}{m+n}\max(0, t_1)$, when $\frac{s_1^2}{m-1} > \frac{s_2^2}{n-1}$. Hence the estimator d_{GD} is imported and the improved estimator is obtained as,

$$d_{GD}^a = \begin{cases} \frac{m(m-1)S_2^2\bar{X}+n(n-1)S_1^2\bar{Y}}{m(m-1)S_2^2+n(n-1)S_1^2}, & \text{if } \frac{S_1^2}{m-1} \leq \frac{S_2^2}{n-1} \\ \\ \frac{m\bar{X}+n\bar{Y}}{m+n}, & \text{if } \frac{S_1^2}{m-1} > \frac{S_2^2}{n-1}. \end{cases}$$

Similarly one can get the estimators which improve upon d_{KS}, d_{MK}, d_{TK}, d_{BC1}, and d_{BC2}, respectively as,

$$d_{KS}^a = \begin{cases} \frac{m(m-3)S_2^2\bar{X}+n(n-3)S_1^2\bar{Y}}{m(m-3)S_2^2+n(n-3)S_1^2}, & \text{if } \frac{S_1^2}{m-3} \leq \frac{S_2^2}{n-3} \\ \\ \frac{m\bar{X}+n\bar{Y}}{m+n}, & \text{if } \frac{S_1^2}{m-3} > \frac{S_2^2}{n-3}, \end{cases}$$

$$d_{MK}^a = \begin{cases} \frac{\sqrt{m(m-1)}S_2\bar{X}+\sqrt{n(n-1)}S_1\bar{Y}}{\sqrt{m(m-1)}S_2+\sqrt{n(n-1)}S_1}, & \text{if } \frac{S_1}{\sqrt{m-1}} \leq \frac{S_2}{\sqrt{n-1}} \\ \\ \frac{m\bar{X}+n\bar{Y}}{m+n}, & \text{if } \frac{S_1}{\sqrt{m-1}} > \frac{S_2}{\sqrt{n-1}}, \end{cases}$$

$$d_{TK}^a = \begin{cases} \frac{\sqrt{m}c_n S_2 \bar{X} + \sqrt{n}c_m S_1 \bar{Y}}{\sqrt{m}c_n S_2 + \sqrt{n}c_m S_1}, & \text{if } \frac{S_1}{S_2} \leq \sqrt{\frac{n}{m}} \frac{c_n}{c_m} \\ \frac{m\bar{X}+n\bar{Y}}{m+n}, & \text{if } \frac{S_1}{S_2} \leq \sqrt{\frac{n}{m}} \frac{c_n}{c_m}, \end{cases}$$

$$d_{BC1}^a = \begin{cases} d_{BC1}, & \text{if } \frac{S_2^2}{S_1^2} + n(\frac{\bar{Y}-\bar{X}}{S_1})^2 > \frac{n+2}{m(m-1)}[b_1(m+n)-n] \\ \frac{m\bar{X}+n\bar{Y}}{m+n}, & \text{if } \frac{S_2^2}{S_1^2} + n(\frac{\bar{Y}-\bar{X}}{S_1})^2 \leq \frac{n+2}{m(m-1)}[b_1(m+n)-n], \end{cases}$$

$$d_{BC2}^a = \begin{cases} d_{BC2}, & \text{if } \frac{m(m-1)S_2^2}{n(n-1)S_1^2} \geq b_2(1+\frac{m}{n}) - 1 \\ \frac{m\bar{X}+n\bar{Y}}{m+n}, & \text{if } \frac{m(m-1)S_2^2}{n(n-1)S_1^2} < b_2(1+\frac{m}{n}) - 1. \end{cases}$$

Remark 3.5.1 *It is interesting to note that, for unequal sample sizes that is $m \neq n$, the estimators $d_{GD}^a = \hat{d}_{GD}$, $d_{KS}^a = \hat{d}_{KS}$, $d_{MK}^a = \hat{d}_{MK}$, $d_{TK}^a = \hat{d}_{TK}$, $d_{BC2}^a = \hat{d}_{BC2}$. However, for equal sample sizes, application of the Theorem 3.5.1 produces different estimators.*

Location Class

A larger class of estimators than the class considered above is the class of location equivariant estimators. Let $G_L = \{g_c : g_c(x) = x + c, -\infty < c < \infty\}$ be the location group of transformations. Under the transformation g_c, we observe that, $\bar{X} \to \bar{X} + c$, $\bar{Y} \to \bar{Y} + c$, $S_1^2 \to S_1^2$, $S_2^2 \to S_2^2$, and the parameters $\mu \to \mu + c$, $\sigma_1 \to \sigma_1$. The family of probability distributions is invariant and consequently the estimation problem is also invariant under the loss (3.1.1). Based on the minimal sufficient statistics $(\bar{X}, \bar{Y}, S_1^2, S_2^2)$ the form of a location equivariant estimator for estimating the common mean μ is thus obtained as,

$$d_\psi = \bar{X} + \psi(\underset{\sim}{U}), \tag{3.5.5}$$

where $\underset{\sim}{U} = (T, S_1^2, S_2^2)$, $T = \bar{Y} - \bar{X}$, and ψ is a real valued function. Let us define a function ψ_0 for the location equivariant estimator d_ψ as

$$\psi_0(\underset{\sim}{t}) = \begin{cases} \frac{n}{n+m}\min\{t,0\}, & \text{if } \psi(\underset{\sim}{u}) < \frac{n}{m+n}\min\{t,0\}, \\ \psi(\underset{\sim}{u}), & \text{if } \frac{n}{n+m}\min\{t,0\} \leq \psi(\underset{\sim}{u}) \leq \frac{n}{n+m}\max\{t,0\}, \\ \frac{n}{n+m}\max\{t,0\}, & \text{if } \psi(\underset{\sim}{u}) > \frac{n}{n+m}\max\{t,0\}. \end{cases} \tag{3.5.6}$$

The following theorem gives a sufficient condition for improving location equivariant estimators under the condition that the variances follow the ordering $\sigma_1^2 \leq \sigma_2^2$.

Theorem 3.5.2 *Let d_ψ be a location equivariant estimator of the common mean μ and the loss function be (3.1.1). Let the function $\psi_0(\underset{\sim}{u})$ be as defined in (3.5.6). The estimator d_ψ is inadmissible and is improved by d_{ψ_0} if $P_{\underset{\sim}{\alpha}}(\psi(\underset{\sim}{U}) \neq \psi_0(\underset{\sim}{U})) > 0$ for some choices of the parameters $\underset{\sim}{\alpha}$; $\sigma_1^2 \leq \sigma_2^2$.*

Proof 3.5.2 *The proof of the theorem is similar to the proof of the Theorem 3.5.1, and hence has been omitted for brevity.*

Remark 3.5.2 *We also observe that all the estimators discussed in Section 3.2, except the MLE d_{ML} (closed form does not exist) belong to the class $d_\psi(\underset{\sim}{U}) = \bar{X} + \psi(\underset{\sim}{U})$. Hence as an application of the Theorem 3.5.2, produces improved estimators. However, it has been seen that these improved estimators are same as that obtained by applying Theorem 3.5.1 under order restrictions on the variances.*

Remark 3.5.3 *The performances of all the improved estimators which has been proposed in Section 3.2 as well as in this section by applying Theorem 3.5.1, will be evaluated in Section 3.6, using the affine invariant loss function L_1. Further the percentage of risk improvements upon their respective old counter parts also has been noted.*

Remark 3.5.4 *We note that the estimator d_{GM}, also belongs to the classes given in (3.5.1) and (3.5.5). However, the conditions in Theorem 3.5.1 and Theorem 3.5.2 for improving it, do not satisfied. Hence the estimator could not be improved by applying either Theorem 3.5.1 or Theorem 3.5.2, under $\sigma_1^2 \le \sigma_2^2$.*

3.6 A Simulation Study

It should be noted that, in Section 3.2 we have constructed the plug-in type restricted MLE $\hat{d}_{RM}$ for the common mean μ, taking into account the order restriction on the variances. Moreover, in Sections 3.3 and 3.4 we have also constructed some alternative such as $\hat{d}_{MK}$, $\hat{d}_{TK}$, $\hat{d}_{KS}$, and $\hat{d}_{BC2}$ and proved theoretically that these estimators dominate their old counterparts in terms of stochastic domination as well as Pitman measure of closeness criterion. Further in Section 3.5, we have derived the estimators namely d^a_{GD}, d^a_{KS}, d^a_{MK}, d^a_{TK}, d^a_{BC1}, d^a_{BC2} as an application of Theorems 3.5.1 and 3.5.2. In addition to these estimators, we have also included the improved estimator proposed by Elfessi and Pal (1992) which we denote by $\hat{d}_{GD}$ for the convenience. However, from an application point of view, it is very much essential to see their performances among themselves as well as to see how much they improve upon their old counterparts. It seems quite impossible to compare the risk functions of all these improved estimators analytically. Taking advantages of the computational facilities available now-a-days, we in this section try to compare the performances of all these improved estimators numerically, which may be handy for practical purposes. For the purpose of numerical comparison, we have generated $20,000$ random samples of sizes m and n respectively from $N(\mu, \sigma_1^2)$ and $N(\mu, \sigma_2^2)$, with the condition that $\sigma_1^2 \le \sigma_2^2$. For comparing the performances of all the improved estimators we use the affine invariant loss function (3.1.1). However the percentage of risk improvements of an improved estimator with respect to its old counter part all the three loss functions have been used. In order to simulate the risk values the well known Monte-Carlo simulation method have been employed. The accuracy of the simulation has been checked and the error has been checked which is seen up to 10^{-3}. To proceed further, we define the percentage of risk improvement of all the improved estimators over their old counter parts as follows. $P1 = \left(1 - \frac{R(\hat{d}_{GD},\mu)}{R(d_{GD},\mu)}\right) \times 100, P2 = \left(1 - \frac{R(\hat{d}_{KS},\mu)}{R(d_{KS},\mu)}\right) \times 100, P3 = \left(1 - \frac{R(\hat{d}_{MK},\mu)}{R(d_{MK},\mu)}\right) \times 100, P4 = \left(1 - \frac{R(\hat{d}_{TK},\mu)}{R(d_{TK},\mu)}\right) \times 100, P5 = \left(1 - \frac{R(d^a_{GD},\mu)}{R(d_{GD},\mu)}\right) \times 100, P6 = \left(1 - \frac{R(d^a_{KS},\mu)}{R(d_{KS},\mu)}\right) \times 100, P7 = \left(1 - \frac{R(d^a_{MK},\mu)}{R(d_{MK},\mu)}\right) \times 100, P8 = \left(1 - \frac{R(d^a_{TK},\mu)}{R(d_{TK},\mu)}\right) \times 100, P9 = \left(1 - \frac{R(\hat{d}_{RM},\mu)}{R(d_{ML},\mu)}\right) \times 100.$

Next we define the percentage of relative risk performances of all the improved estimators with respect to the estimator d_{GD} as follows. $R1 = \left(1 - \frac{R(\hat{d}_{GD},\mu)}{R(d_{GD},\mu)}\right) \times 100$, $R2 = \left(1 - \frac{R(\hat{d}_{KS},\mu)}{R(d_{GD},\mu)}\right) \times 100$, $R3 = \left(1 - \frac{R(\hat{d}_{MK},\mu)}{R(d_{GD},\mu)}\right) \times 100$, $R4 = \left(1 - \frac{R(\hat{d}_{TK},\mu)}{R(d_{GD},\mu)}\right) \times 100$, $R5 = \left(1 - \frac{R(d^a_{GD},\mu)}{R(d_{GD},\mu)}\right) \times 100$, $R6 = \left(1 - \frac{R(d^a_{MK},\mu)}{R(d_{GD},\mu)}\right) \times 100$, $R7 = \left(1 - \frac{R(d^a_{BC1},\mu)}{R(d_{GD},\mu)}\right) \times 100$, $R8 = \left(1 - \frac{R(d^a_{BC2},\mu)}{R(d_{GD},\mu)}\right) \times 100$, $R9 = \left(1 - \frac{R(\hat{d}_{RM},\mu)}{R(d_{GD},\mu)}\right) \times 100$.

It is easy to observe that the risk values of all the estimators are functions of τ with respect to the loss function L_1 as given in (3.1.1), where we denote $\tau = \sigma_1^2/\sigma_2^2$. We note that, when the sample sizes are unequal the estimators $\hat{d}_{GD} = d^a_{GD}$, $\hat{d}_{KS} = d^a_{KS}$, $\hat{d}_{MK} = d^a_{MK}$, $\hat{d}_{TK} = d^a_{TK}$. Further we see that for equal sample sizes the estimators $\hat{d}_{GD} = \hat{d}_{KS}$ and $\hat{d}_{MK} = \hat{TK}$. In our simulation study we have taken $b_1 = \frac{1}{2}b_{\max}(m,n)$ and $b_2 = \frac{1}{2}b_{\max}(m,n-3)$, where the values of $b_{\max}(m,n)$ have been used from the tables given in Brown and Cohen (1974). Moreover we observe that for $b_2 = 1$, the estimator $d_{GD} = d_{BC2}$ and for $b_2 = 0$, it reduces to $\bar{X}$. The percentage of risk improvements of d^a_{BC1} over d_{BC1}, d^a_{BC2} over d_{BC2} and $\hat{d}_{BC2}$ upon d_{BC2} are seen to be very marginal and hence have not been tabulated. The simulation study has been done for various combinations of sample sizes and many ranges of the parameter space. For illustration purpose we have presented the percentage of risk improvements as well as the percentage of relative risk improvements of all the estimators for some choices of sample sizes in Tables 3.6.1 - 3.6.7. In Tables 3.6.1and 3.6.2 we have presented the percentage of risk improvements of all the estimators for equal and unequal sample sizes with respect to the loss (3.1.1). In Table 3.6.1, the percentage of risk improvements of all the estimators have been presented for the sample sizes $(5,5)$, $(12,12)$, $(20,20)$ and $(30,30)$. The first and seventh column represent the values of τ and the rest of the columns represents the percentage of risk improvements of all the estimators. In each cell corresponding to one value of τ there corresponds three values of percentage of risk improvements that gives for three different sample sizes $(5,5)$, $(12,12)$ and $(20,20)$ respectively. Table 3.6.2, is divided into two parts, specifically the first half (column second to sixth) represents the percentage of risk performances for all the estimators with sample sizes $(5,10)$, $(12,20)$ and the second part (column seventh to eleventh) represents for the sample sizes $(10,5)$ and $(20,12)$. In this table the first column also gives the values of τ and the columns second to eleven represents the percentage of risk improvements of all the estimators with respect to their old counter parts. In this table each cell contains two values of percentage of risk improvements which correspond to one value of τ in the cell of the first column. In a very similar fashion the percentage of risk improvements of all the estimators have been presented in Tables 3.6.3 to 3.6.5 for equal and unequal sample sizes with respect to the losses (3.1.2) and (3.1.3). The percentage of relative risk performances of all the improved estimators with respect to d_{GD} (denoted as $Ri; i = 1, 2, 7$) have been presented in Tables 3.6.6 and 3.6.7 for equal and unequal sample sizes respectively. Specifically in Table 3.6.6 we have presented the percentage of relative risk performances of all the improved estimators for the sample sizes $(5,5)$, $(12,12)$ and $(20,20)$. The Table 3.6.6 consists of eight columns and each column have several cells. Corresponding to each value of τ in the first column there corresponds three values of percentage of relative risk values from columns second to eight. These three values correspond to three sample sizes $(5,5)$, $(12,12)$ and $(20,20)$ respectively. In a very similar way we have presented the percentage of relative risk improvements of all the improved estimators for the unequal sample sizes $(5,10)$, $(12,20)$, $(10,5)$ and $(20,12)$ in Table 3.6.7. Moreover, we have also plotted the risk values of all the improved estimators with respect to the loss function $(3.1.1)$ against the choices of τ in the Figure 3.6.1. Specifically Figure 3.6.1 (a)-(b) gives the plot for equal sample sizes where as Figure 3.6.1 (c)-(f) gives the plot for the unequal sample sizes. We note that the estimators

$\hat{d}_{GD}$, $\hat{d}_{KS}$, $\hat{d}_{MK}$, $\hat{d}_{TK}$, $\hat{d}_{RM}$, d^a_{GD}, d^a_{MK}, d^a_{BC1}, d^a_{BC2} have been denoted by GDI, KSI, MKI, TKI, RML, GDA, MKA, BC1A and BC2A respectively in the Figure 3.6.1 (a)-(f).

The following observations have been made during our simulation study as well as from the tables, which we discuss separately for equal and unequal sample sizes.

Case I: $m = n$.

- (a) The percentage of risk improvements as well as the risk values of all the new estimators upon their respective old estimators decreases as the sample sizes increases for fixed values of the parameters, with respect to the loss functions L_1, L_2 and L_3.
- (b) Let the loss function be L_1. The percentage of risk improvement of $\hat{d}_{GD}$ over d_{GD} (see $P1$) is seen maximum up to 12%, d^a_{GD} over d_{GD} (see $P5$) is seen maximum up to 10%, $\hat{d}_{MK}$ over d_{MK} ($P3$) is seen maximum up to 7%, d^a_{MK} over d_{MK} ($P6$) is seen maximum up to 6%, where as for $\hat{d}_{RM}$ over d_{ML} ($P9$) is seen maximum up to 20%.
- (c) Let the loss function be L_2. The maximum percentage of risk improvement $\hat{d}_{GD}$, d^a_{GD}, $\hat{d}_{MK}$, d^a_{MK}, and $\hat{d}_{RM}$ over their respective old counter parts are seen near to 6%, 5%, 4%, 3% and 7% respectively. The maximum percentage of risk improvement is seen in the case of $\hat{d}_{RM}$ for small sample sizes and when σ_1^2 and σ_2^2 very very close to each other.
- (d) Let the loss function be L_3. The maximum percentage of risk improvement of $\hat{d}_{GD}$, d^a_{GD}, $\hat{d}_{MK}$, d^a_{MK}, and $\hat{d}_{RM}$ upon their respective old estimators are seen respectively as 11%, 10%, 7%, 5% and 20%. The maximum percentage of risk improvement has been seen for small sample sizes and when the variances are close to each other.
- (e) Here we note that, the percentage of risk improvements of all the new estimators upon their respective old estimators are approximated values only which have been obtained numerically and hence it may vary with sample sizes.
- (f) The above numerical results $(b) - (d)$ validates the theoretical findings in Sections 3.3, 3.4, and 3.5.
- (f) The simulated risk values of all the estimators such as $\hat{d}_{GD}$, d^a_{GD}, $\hat{d}_{MK}$, d^a_{MK}, d^a_{BC1}, d^a_{BC2}, and $\hat{d}_{RM}$,) decrease as the sample sizes increase. Further for the fixed sample sizes, as the values of τ varies from 0 to 1, the risk values of all the estimators decrease. It has been noticed that, for small values of τ (say approximately $0 < \tau < 0.25$), the percentage of relative risk improvement of ${d^a}_{BC1}$ is maximum seen up to 15%, that is when the variance of first population is much smaller than second population, ($\sigma_1^2 <<< \sigma_2^2$). For the values of τ near to 1, (say for the range $0.50 < \tau < 1$) the estimators $\hat{d}_{MK}$ and d^a_{MK} have almost same percentage of relative risk improvements. For moderate values of τ (say $0.50 < \tau < 0.75$), the estimators $\hat{d}_{MK}$ and d^a_{MK} perform equally well, however as the sample sizes increases from moderate to large, the performance of these two estimators decrease and compete well with $\hat{d}_{GD}$. In fact the dominance region of $\hat{d}_{MK}$ and d^a_{MK} over $\hat{d}_{GD}$ decreases. We also noticed that the estimators $\hat{d}_{GD}$ and d^a_{GD}, $\hat{d}_{MK}$ and d^a_{MK}, and d^a_{BC1} and d^a_{BC2} compete well with each other.

Case II: $m \neq n$.

- (a) The percentage of risk improvements of all the new estimators decreases as the sample sizes increases for fixed values of σ_1^2 and σ_2^2 with respect to the loss functions L_1, L_2 and L_3.

- (b) Let us first consider the loss function L_1. The percentage of risk improvement of $\hat{d}_{GD}$ upon d_{GD} (denoted as $P1$) is seen maximum up to 16%, the maximum percentage of risk improvement of $\hat{d}_{KS}$ over d_{KS} (denoted as $P2$) is seen near to 8%. The maximum percentage of risk improvement of $\hat{d}_{MK}$ and $\hat{d}_{TK}$ over their corresponding old estimators are seen near to 14% and 13% respectively. The maximum risk improvement of $\hat{d}_{RM}$ over d_{ML} is seen up to 15%. We also note that, these maximum risk improvements have been noticed when $m > n$ for all the estimators.

- (c) Let us consider the loss function L_2. The maximum percentage of risk improvement of $\hat{d}_{GD}$ over d_{GD} is seen up to 7%. The maximum percentage of risk improvement of $\hat{d}_{KS}$ over d_{KS} is seen near to 4%. The maximum percentage of risk improvement of $\hat{d}_{MK}$ over d_{MK} is seen near to 7%. The maximum percentage of risk improvement of $\hat{d}_{TK}$ over d_{TK} is seen near to 7%. The maximum percentage of risk improvement of $\hat{d}_{RM}$ over d_{ML} is seen near to 13%.

- (d) Consider the loss function L_3. The maximum percentage of risk improvement of $\hat{d}_{GD}$ over d_{GD} is seen up to 13%. The maximum percentage of risk improvement of $\hat{d}_{KS}$ over d_{KS} is seen near to 8%. The maximum percentage of risk improvement of $\hat{d}_{MK}$ over d_{MK} is seen near to 13%. The maximum percentage of risk improvement of $\hat{d}_{TK}$ over d_{TK} is seen near to 13%. The maximum percentage of risk improvement of $\hat{d}_{RM}$ over d_{ML} is seen near to 36%.

- (e) Here we note that, the percentage of risk improvements of all the new estimators upon their respective old estimators are approximated values only which have been obtained numerically and hence it may vary with sample sizes, however the trends remain the same.

- (f) The above numerical results $(b) - (d)$ also validates the theoretical findings in Sections 3.3, 3.4, and 3.5.

- (g) The simulated risk values of all the estimators such as $\hat{d}_{GD}$, $\hat{d}_{KS}$, $\hat{d}_{MK}$, $\hat{d}_{TK}$, d^a_{BC1}, d^a_{BC2}, and $\hat{d}_{RM}$, decrease as the sample sizes increase. It has been noticed that, for small values of τ (say approximately $0 < \tau < 0.15$), the percentage of relative risk improvements of d^a_{BC1} and d^a_{BC2} are maximum seen up to 12%, that is when the variance of first population is much smaller than the second population ($\sigma_1^2 <<< \sigma_2^2$). For the values of τ near to 1, (say $0.75 < \tau < 1$) the estimator $\hat{d}_{KS}$ (for $m < n$) and $\hat{d}_{MK}$, $\hat{d}_{TK}$ (when $m > n$) has maximum percentage of relative risk improvements. For moderate values of τ, the estimators $\hat{d}_{MK}$ and $\hat{d}_{TK}$ perform equally well, however as the sample sizes increase from moderate to large the performance of these two estimators decrease and the estimators $\hat{d}_{GD}$ and $\hat{d}_{KS}$ starts performing better.

From the above discussions and also from our simulation study the following conclusions can be drawn regarding the use of the proposed estimators in practice.

1. (a) First consider that the sample sizes are equal, that is $m = n$. When the variance of the first population is much smaller compare to the second, we recommend to use d^a_{BC1}. When the variance of both the populations are close to each other, we recommend to use either $\hat{d}_{MK}$ or d^a_{MK}, as they compete with each other. In other cases, that is neither the variances differ too much nor close enough, the estimators $\hat{d}_{MK}$ and d^a_{MK} can be used for small sample sizes (say $m, n \leq 10$), and $\hat{d}_{MK}$ or $\hat{d}_{GD}$ for moderate to large sample sizes.

2. (b) Next, consider that the sample sizes are unequal, that is, $m \neq n.$ When the variance of the first population is much smaller smaller than the second, we recommend to use either the estimator d^a_{BC1} or d^a_{BC2}. When the variances of both the populations are close to each other, the estimators $\hat{d}_{KS}$ or $\hat{d}_{TK}$ (for $m < n$) and $\hat{d}_{TK}$ or $\hat{d}_{MK}$ (for $m > n$) can be recommended for use. However for moderate ranges of τ, the estimators $\hat{d}_{MK}$ or $\hat{d}_{TK}$ (for $m < n$) and the estimators $\hat{d}_{KS}, \hat{d}_{GD}, \hat{d}_{RM}$ or $\hat{d}_{TK}$ (for $m > n$) can be recommended as they all perform equally well.

Table 3.6.1: Percentage of risk improvements of all the proposed estimators using the loss L_1 for the sample sizes $(m, n) = (5, 5), (12, 12), (20, 20), (30, 30)$

$\tau \downarrow$	$P1$	$P5$	$P3$	$P6$	$P9$	$\tau \downarrow$	$P1$	$P5$	$P3$	$P6$	$P9$
	2.49	1.53	0.95	0.53	17.90		9.67	8.76	5.84	4.33	12.03
0.05	0.00	0.00	0.00	0.00	0.00	0.55	3.96	2.82	2.11	1.29	3.56
	0.00	0.00	0.00	0.00	0.00		1.14	0.77	0.58	0.34	0.87
	0.00	0.00	0.00	0.00	0.00		0.73	0.45	0.37	0.21	0.50
	4.61	2.83	2.05	1.16	13.01		10.54	9.82	6.48	4.93	13.42
0.10	0.07	0.04	0.03	0.02	0.05	0.60	3.76	2.83	2.02	1.28	3.59
	0.00	0.00	0.00	0.00	0.00		1.80	1.26	0.93	0.56	1.43
	0.00	0.00	0.00	0.00	0.00		0.91	0.57	0.46	0.26	0.62
	6.20	4.03	3.06	1.80	10.36		8.85	9.51	5.48	4.58	12.71
0.15	0.13	0.07	0.06	0.03	0.19	0.65	4.83	3.68	2.62	1.68	4.48
	0.00	0.00	0.00	0.00	0.00		1.78	1.35	0.92	0.59	1.51
	0.00	0.00	0.00	0.00	0.00		0.76	0.56	0.39	0.24	0.60
	7.68	5.00	4.01	2.36	10.51		7.73	9.41	4.70	4.38	12.75
0.20	0.36	0.20	0.16	0.09	0.31	0.70	3.72	3.47	2.02	1.48	4.21
	0.01	0.00	0.00	0.00	0.00		2.40	1.87	1.25	0.81	2.13
	0.00	0.00	0.00	0.00	0.00		1.32	0.93	0.68	0.41	1.00
	8.56	5.74	4.56	2.74	11.03		6.14	8.75	3.81	4.03	11.13
0.25	0.43	0.25	0.20	0.11	0.36	0.75	2.88	3.17	1.55	1.29	3.93
	0.09	0.05	0.04	0.02	0.05		1.97	1.76	1.03	0.72	1.98
	0.00	0.00	0.00	0.00	0.00		1.30	1.07	0.67	0.44	1.17
	9.54	6.67	5.32	3.29	11.38		4.80	8.68	3.04	3.87	11.30
0.30	0.90	0.55	0.44	0.24	0.77	0.80	3.10	3.79	1.72	1.52	4.57
	0.09	0.05	0.04	0.02	0.07		2.61	2.40	1.37	0.99	2.69
	0.01	0.00	0.00	0.00	0.00		1.67	1.41	0.87	0.58	1.51
	9.63	7.16	5.49	3.52	10.81		2.69	8.04	1.76	3.45	10.91
0.35	1.44	0.92	0.72	0.42	1.17	0.85	2.51	3.79	1.38	1.45	4.69
	0.38	0.23	0.19	0.11	0.26		2.05	2.50	1.09	0.95	2.82
	0.04	0.02	0.02	0.01	0.02		1.62	1.56	0.84	0.62	1.70
	8.83	6.87	5.10	3.40	10.25		2.43	8.18	1.49	3.36	10.74
0.40	1.84	1.19	0.93	0.54	1.53	0.90	1.55	3.68	0.87	1.29	4.60
	0.53	0.32	0.26	0.15	0.38		0.69	1.95	0.35	0.62	2.25
	0.15	0.08	0.08	0.04	0.09		0.77	1.43	0.40	0.48	1.58
	9.94	7.93	5.80	3.94	11.81		1.36	7.91	0.74	3.12	10.39
0.45	2.33	1.62	1.23	0.74	2.11	0.95	1.86	4.57	1.03	1.61	5.68
	0.98	0.61	0.49	0.28	0.71		1.67	2.74	0.90	0.98	3.10
	0.30	0.16	0.15	0.08	0.17		0.15	1.37	0.08	0.39	1.51
	10.98	9.12	6.51	4.55	13.06		0.74	8.37	0.44	3.26	10.78
0.50	3.23	2.26	1.71	1.03	2.92	1.00	0.03	3.78	0.01	1.13	4.81
	1.10	0.71	0.56	0.32	0.85		0.78	2.73	0.44	0.87	3.11
	0.44	0.26	0.22	0.12	0.27		0.10	1.60	0.06	0.45	1.76

Table 3.6.2: Percentage of risk improvements of all the proposed estimators using the loss function L_1 for unequal sample sizes

$\tau \downarrow$	$(m,n)=(5,10),(12,20)$					$(m,n)=(10,5),(20,12)$				
	$P1$	$P2$	$P3$	$P4$	$P9$	$P1$	$P2$	$P3$	$P4$	$P9$
0.05	0.00	0.07	0.00	0.00	0.83	2.29	1.14	2.13	1.93	9.64
	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.00	0.00
0.10	0.05	0.28	0.00	0.00	1.62	3.99	1.92	4.26	3.88	8.94
	0.00	0.00	0.00	0.00	0.02	0.00	0.00	0.03	0.03	0.00
0.15	0.32	0.98	0.02	0.02	1.49	4.84	2.34	5.51	4.97	8.99
	0.01	0.01	0.00	0.00	0.01	0.03	0.01	0.14	0.13	0.01
0.20	0.44	1.61	0.04	0.04	2.16	5.60	2.44	6.66	6.03	6.79
	0.01	0.02	0.00	0.00	0.01	0.10	0.06	0.45	0.42	0.07
0.25	0.76	2.48	0.03	0.04	3.01	8.54	4.20	9.06	8.32	10.33
	0.06	0.09	0.00	0.00	0.1	0.30	0.20	0.94	0.88	0.20
0.30	1.12	3.12	0.07	0.10	3.53	8.28	3.92	9.56	8.73	9.12
	0.09	0.16	0.00	0.00	0.20	0.47	0.32	1.28	1.20	0.27
0.35	1.65	4.26	0.13	0.16	3.94	9.21	4.58	10.44	9.59	10.79
	0.27	0.41	0.00	0.00	0.55	1.25	0.91	2.33	2.22	0.75
0.40	1.63	4.54	0.10	0.13	3.81	10.27	5.25	11.24	10.37	10.52
	0.36	0.55	0.01	0.01	0.58	1.14	0.83	2.50	2.37	0.86
0.45	2.10	5.04	0.22	0.26	4.25	11.20	5.71	11.86	10.97	12.99
	0.51	0.79	0.02	0.02	0.94	1.60	1.13	3.31	3.15	1.21
0.50	2.62	6.31	0.18	0.22	5.22	11.02	5.56	11.68	10.80	10.47
	0.69	1.03	0.02	0.02	1.16	1.99	1.49	3.46	3.30	1.68
0.55	2.64	6.19	0.26	0.32	5.01	11.62	6.08	11.86	10.98	12.85
	1.07	1.49	0.05	0.06	1.59	2.51	1.94	4.09	3.91	2.25
0.60	3.14	7.23	0.27	0.32	6.27	12.50	6.46	12.25	11.35	11.94
	1.46	1.97	0.08	0.08	2.09	2.76	2.05	4.31	4.13	2.64
0.65	3.01	6.31	0.28	0.34	5.26	11.73	5.84	11.44	10.61	13.05
	1.40	1.92	0.07	0.08	2.08	2.85	2.13	4.26	4.09	2.51
0.70	3.04	6.68	0.31	0.36	5.00	12.36	6.27	11.74	10.90	13.57
	1.82	2.40	0.14	0.15	2.71	3.19	2.42	4.49	4.31	2.83
0.75	2.85	6.18	0.25	0.32	4.69	13.42	7.44	12.08	11.26	14.79
	1.63	2.20	0.09	0.09	2.51	3.99	3.11	4.88	4.71	3.77
0.80	3.48	7.06	0.37	0.43	5.59	11.77	5.94	10.35	9.57	13.6
	1.92	2.52	0.11	0.12	2.54	3.55	2.74	4.38	4.20	3.37
0.85	3.55	7.18	0.43	0.50	5.77	11.96	6.28	10.26	9.53	13.35
	2.17	2.82	0.13	0.14	2.94	4.62	3.72	4.74	4.58	4.44
0.90	3.87	7.43	0.48	0.56	5.73	12.15	6.63	9.61	8.90	13.8
	2.16	2.81	0.14	0.15	2.91	5.29	4.23	5.24	5.07	5.12
0.95	3.57	7.02	0.41	0.47	5.47	13.31	7.39	10.53	9.81	14.67
	1.95	2.58	0.10	0.11	2.83	3.44	2.65	3.28	3.14	3.44
1.00	3.35	6.97	0.19	0.24	5.10	11.82	6.37	8.82	8.19	14.16
	2.22	2.83	0.19	0.20	3.02	5.00	4.03	4.36	4.21	5.05

Table 3.6.3: Percentage of risk improvements of all the proposed estimators using the L_2 and L_3 loss functions

$(m,n)\downarrow$	$(\sigma_1^2,\sigma_2^2)\downarrow$	L_2- Loss					L_3- Loss				
		$P1$	$P5$	$P3$	$P6$	$P9$	$P1$	$P5$	$P3$	$P6$	$P9$
(5, 5)	(0.05, 0.10)	4.97	4.07	3.08	2.11	5.45	9.17	7.95	5.41	3.90	10.82
	(0.05, 0.30)	2.29	1.39	1.24	0.69	2.87	6.76	4.42	3.37	1.99	10.46
	(0.05, 0.50)	1.60	0.87	0.83	0.43	3.01	4.79	2.95	2.15	1.22	12.01
	(0.05, 0.70)	0.96	0.49	0.45	0.23	2.79	3.44	2.20	1.43	0.83	12.99
	(0.05, 1.00)	0.75	0.39	0.35	0.18	3.28	1.46	0.92	0.55	0.32	18.28
	(1.00, 1.10)	0.88	4.18	0.59	1.67	5.36	2.14	8.09	1.31	3.31	9.77
	(1.00, 1.50)	3.46	3.97	2.04	1.83	5.34	8.31	9.62	5.14	4.61	12.56
	(1.00, 2.00)	4.92	3.91	2.96	1.99	5.18	9.68	8.28	5.78	4.09	11.66
	(1.00, 2.50)	4.53	3.49	2.70	1.80	4.98	10.03	7.77	5.76	3.81	11.32
	(1.00, 3.00)	4.33	2.96	2.51	1.52	4.54	10.07	7.30	5.60	3.56	11.80
	(2.00, 2.10)	0.13	3.74	0.06	1.37	4.59	1.65	9.11	1.14	3.80	10.62
	(2.00, 2.30)	1.80	4.69	1.21	2.06	5.83	3.19	8.64	1.76	3.53	11.07
	(2.00, 2.50)	2.02	4.00	1.30	1.77	5.04	4.14	8.42	2.53	3.67	10.99
	(2.00, 2.70)	3.41	4.45	2.03	2.01	5.80	7.71	9.95	4.85	4.74	11.83
	(2.00, 3.00)	4.34	4.77	2.80	2.33	6.13	8.76	9.31	5.34	4.50	12.07
(12, 12)	(0.05, 0.10)	1.54	1.00	0.85	0.48	1.19	3.07	2.19	1.61	0.99	2.84
	(0.05, 0.30)	0.06	0.03	0.03	0.01	0.07	0.16	0.10	0.07	0.04	0.21
	(0.05, 0.50)	0.01	0.00	0.00	0.00	0.01	0.01	0.00	0.00	0.00	0.23
	(0.05, 0.70)	0.00	0.00	0.00	0.00	0.03	0.00	0.00	0.00	0.00	0.00
	(0.05, 1.00)	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	(1.00, 1.10)	0.60	1.76	0.31	0.57	2.15	0.45	3.35	0.23	1.06	4.23
	(1.00, 1.50)	1.56	1.42	0.84	0.59	1.73	3.87	3.21	2.10	1.42	3.95
	(1.00, 2.00)	1.15	0.90	0.59	0.39	1.09	2.78	1.97	1.46	0.89	2.58
	(1.00, 2.50)	0.76	0.52	0.40	0.24	0.67	1.91	1.24	0.96	0.56	1.54
	(1.00, 3.00)	0.62	0.35	0.32	0.17	0.44	1.04	0.68	0.52	0.30	0.91
	(2.00, 2.10)	0.50	1.95	0.33	0.67	2.39	0.73	3.80	0.41	1.23	4.74
	(2.00, 2.30)	1.19	2.02	0.67	0.76	2.48	1.10	3.28	0.60	1.10	4.17
	(2.00, 2.50)	1.55	1.90	0.86	0.77	2.29	2.66	3.56	1.45	1.38	4.41
	(2.00, 2.70)	1.52	1.58	0.77	0.64	1.90	3.21	3.34	1.79	1.41	4.14
	(2.00, 3.00)	1.46	1.34	0.79	0.57	1.55	2.99	2.82	1.60	1.18	3.49
(20, 20)	(0.05, 0.10)	0.55	0.33	0.29	0.16	0.38	0.83	0.55	0.42	0.24	0.62
	(0.05, 0.30)	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	(0.05, 0.50)	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	(0.05, 0.70)	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	(0.05, 1.00)	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	(1.00, 1.10)	0.50	1.10	0.29	0.37	1.24	1.67	2.51	0.90	0.92	2.85
	(1.00, 1.50)	0.96	0.79	0.50	0.34	0.89	2.16	1.59	1.12	0.70	1.81
	(1.00, 2.00)	0.53	0.30	0.27	0.15	0.33	1.17	0.73	0.59	0.34	0.82
	(1.00, 2.50)	0.24	0.14	0.13	0.07	0.16	0.50	0.30	0.25	0.14	0.33
	(1.00, 3.00)	0.10	0.06	0.04	0.02	0.07	0.24	0.14	0.11	0.06	0.16
	(2.00, 2.10)	0.05	1.18	0.02	0.35	1.39	1.62	2.86	0.87	1.01	3.30
	(2.00, 2.30)	0.90	1.21	0.49	0.46	1.37	1.66	2.24	0.89	0.83	2.55
	(2.00, 2.50)	0.91	0.97	0.50	0.39	1.09	1.43	1.77	0.75	0.66	1.98
	(2.00, 2.70)	1.09	0.92	0.56	0.39	1.02	2.79	2.20	1.47	0.96	2.48
	(2.00, 3.00)	1.08	0.79	0.57	0.35	0.87	1.91	1.43	0.99	0.62	1.59

Table 3.6.4: Percentage of risk improvements of all the proposed estimators using the L_2 and L_3 loss functions

$(m,n)\downarrow$	$(\sigma_1^2,\sigma_2^2)\downarrow$	L_2- Loss					L_3- Loss				
		$P1$	$P2$	$P3$	$P4$	$P9$	$P1$	$P2$	$P3$	$P4$	$P9$
(5, 10)	(0.05, 0.10)	1.24	2.77	0.12	0.14	2.32	2.60	5.90	0.24	0.29	5.36
	(0.05, 0.30)	0.11	0.44	0.00	0.01	0.57	0.30	1.15	0.01	0.01	2.23
	(0.05, 0.50)	0.04	0.13	0.00	0.00	0.28	0.11	0.35	0.01	0.01	2.07
	(0.05, 0.70)	0.01	0.05	0.00	0.00	0.31	0.01	0.12	0.00	0.00	1.33
	(0.05, 1.00)	0.00	0.02	0.00	0.00	0.25	0.00	0.12	0.00	0.00	2.06
	(1.00, 1.10)	1.90	3.70	0.16	0.19	3.02	3.24	6.78	0.27	0.32	5.33
	(1.00, 1.50)	1.66	3.51	0.16	0.20	2.51	3.00	6.63	0.26	0.33	4.77
	(1.00, 2.00)	1.19	2.91	0.10	0.13	2.06	2.28	5.62	0.19	0.23	3.96
	(1.00, 2.50)	0.84	2.11	0.07	0.09	1.42	1.80	4.54	0.12	0.16	2.99
	(1.00, 3.00)	0.68	1.76	0.05	0.06	1.22	1.35	3.88	0.11	0.14	2.80
	(2.00, 2.10)	1.52	3.27	0.10	0.12	2.04	3.03	6.51	0.25	0.29	4.12
	(2.00, 2.30)	1.55	3.42	0.10	0.13	2.17	3.46	7.04	0.29	0.35	4.62
	(2.00, 2.50)	1.92	3.66	0.27	0.31	2.65	3.24	6.95	0.27	0.33	4.31
	(2.00, 2.70)	1.93	3.94	0.20	0.24	2.51	3.40	6.94	0.38	0.46	4.41
	(2.00, 3.00)	1.31	2.96	0.10	0.12	1.92	2.94	6.62	0.26	0.32	4.41
(12,20)	(0.05, 0.10)	0.34	0.49	0.02	0.02	0.51	0.76	1.09	0.03	0.04	1.27
	(0.05, 0.30)	0.00	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.04
	(0.05, 0.50)	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	(0.05, 0.70)	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	(0.05, 1.00)	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	(1.00, 1.10)	1.12	1.47	0.07	0.07	1.49	2.26	2.90	0.20	0.20	3.02
	(1.00, 1.50)	0.73	1.01	0.02	0.03	1.04	1.75	2.30	0.13	0.14	2.46
	(1.00, 2.00)	0.41	0.55	0.02	0.02	0.59	0.77	1.10	0.03	0.03	1.19
	(1.00, 2.50)	0.14	0.22	0.00	0.00	0.21	0.32	0.51	0.01	0.01	0.66
	(1.00, 3.00)	0.04	0.07	0.00	0.01	0.08	0.15	0.25	0.01	0.01	0.28
	(2.00, 2.10)	1.21	1.54	0.08	0.08	1.65	2.00	2.51	0.17	0.18	2.62
	(2.00, 2.30)	0.93	1.23	0.05	0.06	1.23	2.20	2.83	0.12	0.13	3.14
	(2.00, 2.50)	0.99	1.28	0.06	0.06	1.39	2.05	2.72	0.08	0.09	2.79
	(2.00, 2.70)	0.92	1.20	0.05	0.06	1.26	2.11	2.74	0.19	0.20	2.88
	(2.00, 3.00)	0.65	0.87	0.05	0.06	0.97	1.62	2.15	0.11	0.12	2.23

Table 3.6.5: Percentage of risk improvements of all the proposed estimators using the L_2 and L_3 loss functions

$(m,n)\downarrow$	$(\sigma_1^2,\sigma_2^2)\downarrow$	L_2- Loss					L_3- Loss				
		$P1$	$P2$	$P3$	$P4$	$P9$	$P1$	$P2$	$P3$	$P4$	$P9$
(10, 5)	(0.05, 0.10)	4.43	2.19	5.05	4.62	3.62	11.47	6.06	11.74	10.85	8.67
	(0.05, 0.30)	1.78	0.82	2.53	2.27	0.62	6.29	3.18	6.74	6.17	1.95
	(0.05, 0.50)	0.99	0.43	1.39	1.24	0.30	3.91	1.98	4.20	3.81	0.84
	(0.05, 0.70)	0.68	0.30	0.97	0.87	0.28	2.35	0.85	2.62	2.34	2.83
	(0.05, 1.00)	0.19	0.06	0.42	0.36	1.39	3.07	1.58	2.14	1.98	24.67
	(1.00, 1.10)	5.93	3.12	5.04	4.67	6.44	10.66	5.34	8.59	7.93	12.20
	(1.00, 1.50)	6.06	3.23	6.12	5.68	7.53	12.23	6.60	11.40	10.59	16.63
	(1.00, 2.00)	4.98	2.50	5.69	5.25	8.86	11.51	6.25	11.62	10.77	22.51
	(1.00, 2.50)	4.14	1.96	5.12	4.68	10.14	10.22	5.15	10.96	10.11	27.65
	(1.00, 3.00)	3.70	1.82	4.76	4.34	12.64	8.63	4.17	9.61	8.81	36.34
	(2.00, 2.10)	5.55	2.85	4.56	4.21	6.90	12.12	6.39	9.53	8.86	15.36
	(2.00, 2.30)	5.99	3.18	5.20	4.82	7.89	11.81	6.17	9.83	9.10	17.69
	(2.00, 2.50)	6.17	3.23	5.57	5.18	8.36	13.98	7.94	12.20	11.40	19.99
	(2.00, 2.70)	5.71	3.00	5.40	4.98	8.43	12.80	6.70	11.76	10.95	19.95
	(2.00, 3.00)	5.19	2.55	5.15	4.74	8.41	12.32	6.60	11.49	10.67	21.59
(20,12)	(0.05, 0.10)	0.85	0.61	1.73	1.65	0.65	2.20	1.64	3.86	3.69	1.79
	(0.05, 0.30)	0.04	0.02	0.10	0.10	0.02	0.03	0.01	0.24	0.22	0.01
	(0.05, 0.50)	0.00	0.00	0.01	0.01	0.00	0.02	0.01	0.05	0.04	0.00
	(0.05, 0.70)	0.00	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	(0.05, 1.00)	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.02	0.00
	(1.00, 1.10)	2.25	1.81	2.30	2.22	2.16	4.09	3.16	4.12	3.97	3.96
	(1.00, 1.50)	1.28	0.96	2.00	1.91	1.10	2.89	2.15	4.31	4.14	2.76
	(1.00, 2.00)	0.95	0.72	1.85	1.76	0.81	2.22	1.69	3.76	3.60	2.22
	(1.00, 2.50)	0.64	0.47	1.30	1.24	0.56	1.17	0.84	2.48	2.35	1.16
	(1.00, 3.00)	0.33	0.24	0.81	0.76	0.35	0.80	0.60	1.77	1.67	2.54
	(2.00, 2.10)	2.38	1.90	2.30	2.22	2.29	4.32	3.40	3.91	3.77	4.27
	(2.00, 2.30)	2.34	1.83	2.53	2.45	2.29	4.89	3.94	4.82	4.66	4.80
	(2.00, 2.50)	1.90	1.47	2.34	2.25	1.80	3.75	2.91	4.37	4.21	3.62
	(2.00, 2.70)	1.95	1.51	2.40	2.31	1.82	4.09	3.19	4.89	4.73	3.72
	(2.00, 3.00)	1.35	1.02	1.98	1.89	1.24	3.41	2.59	4.78	4.60	3.30

Table 3.6.6: Percentage of relative risk improvements of all the proposed estimators using L_1 loss function for equal sample sizes

$\tau \downarrow$	$(m,n) = (5,5), (12,12), (20,20)$						
	$R1$	$R5$	$R3$	$R6$	$R7$	$R8$	$R9$
0.05	2.49 0.00 0.00	1.53 0.00 0.00	-38.62 -43.12 -42.78	-39.2 -43.12 -42.78	12.58 1.61 0.41	11.38 1.45 0.35	0.05 1.21 0.32
0.15	6.20 0.13 0.00	4.03 0.07 0.00	-6.72 -18.0 -20.08	-8.10 -18.03 -20.08	10.83 2.20 1.08	7.61 1.78 0.94	1.37 1.19 0.50
0.25	8.56 0.43 0.09	5.74 0.25 0.05	3.51 -7.41 -9.66	1.67 -7.51 -9.68	8.45 0.22 -0.06	3.27 -0.19 -0.06	3.97 0.49 0.20
0.35	9.63 1.44 0.38	7.16 0.92 0.23	9.89 -1.11 -3.84	8.01 -1.42 -3.92	0.60 0.04 -0.45	-7.41 -0.92 -0.34	5.28 0.65 0.05
0.45	9.94 2.33 0.98	7.93 1.62 0.61	12.84 2.63 -0.36	11.12 2.15 -0.58	-5.11 -3.45 -1.56	-15.28 -4.32 -1.29	5.37 0.78 0.27
0.55	9.67 3.96 1.14	8.76 2.82 0.77	13.66 5.01 1.06	12.28 4.21 0.82	-7.56 -4.41 -3.37	-18.95 -4.89 -2.98	6.32 1.73 0.42
0.65	8.85 4.83 1.78	9.51 3.68 1.35	15.23 6.89 2.84	14.42 5.99 2.51	-15.55 -5.67 -3.39	-29.57 -5.98 -2.95	6.79 2.54 0.90
0.75	6.14 2.88 1.97	8.75 3.17 1.76	14.04 6.36 4.00	14.23 6.11 3.71	-20.20 -10.87 -6.07	-35.89 -11.26 -5.22	6.05 1.79 1.27
0.85	2.69 2.51 2.05	8.04 3.79 2.50	12.00 6.63 4.15	13.52 6.69 4.02	-27.25 -12.13 -5.53	-45.30 -12.67 -4.49	4.80 2.65 2.07
0.95	1.36 1.86 1.67	7.91 4.57 2.74	11.57 6.78 4.23	13.70 7.33 4.30	-31.84 -13.05 -7.46	-50.84 -13.47 -6.29	4.84 3.38 2.35
1.00	0.74 0.03 0.78	8.37 3.78 2.73	11.08 5.46 4.32	13.60 6.53 4.74	-33.81 -14.72 -8.38	-53.79 -14.89 -6.99	5.15 2.70 2.24

Table 3.6.7: Percentage of relative risk improvements of all the proposed estimators using the L_1 loss function for unequal sample sizes

$\tau \downarrow$	$(m,n) = (5,10),(12,20),(10,5),(20,12)$						
	$R1$	$R2$	$R3$	$R4$	$R7$	$R8$	$R9$
	0.00	-8.40	-22.10	-24.25	3.97	3.47	-3.99
0.05	0.00	-0.45	-29.81	-30.50	0.35	0.27	0.33
	2.29	6.96	-65.90	-61.52	10.61	10.11	7.68
	0.00	0.30	-58.05	-56.91	0.81	0.78	0.66
	0.32	-6.71	-0.89	-1.90	0.28	-0.72	-3.97
0.15	0.00	-0.72	-8.51	-8.86	0.02	-0.02	-0.09
	4.84	8.73	-24.02	-22.02	10.93	8.69	9.80
	0.03	0.62	-30.95	-30.27	1.83	1.55	1.27
	0.76	-2.54	5.40	4.90	-8.43	-9.70	-3.33
0.25	0.06	-0.30	0.16	-0.03	-2.79	-2.68	-0.88
	8.54	11.22	-6.86	-5.77	12.74	9.56	11.50
	0.30	0.91	-17.54	-17.10	2.15	1.70	1.43
	1.65	1.04	8.13	7.93	-12.74	-14.36	-2.05
0.35	0.27	-0.06	2.79	2.66	-3.44	-3.19	-0.84
	9.21	10.33	2.58	3.15	9.34	4.57	10.56
	1.25	1.69	-9.74	-9.45	2.24	1.70	1.79
	2.10	3.97	9.30	9.37	-21.76	-23.55	-1.47
0.45	0.51	0.57	4.40	4.34	-7.00	-6.60	-0.70
	11.20	11.11	8.95	9.24	8.29	2.04	11.46
	1.60	1.86	-3.97	-3.80	1.07	0.67	1.65
	2.64	5.46	9.10	9.27	-27.05	-29.09	0.66
0.55	1.07	1.42	4.94	4.93	-8.75	-8.11	0.12
	11.62	10.64	11.50	11.65	5.30	-2.01	10.53
	2.51	2.46	0.97	1.06	-0.51	-1.02	2.08
	3.01	7.05	9.06	9.42	-36.55	-38.42	0.51
0.65	1.40	1.97	4.48	4.51	-12.67	-11.52	0.80
	11.73	10.04	13.25	13.31	3.21	-5.23	10.69
	2.85	2.57	3.13	3.16	-1.54	-2.32	2.34
	2.85	7.75	8.60	9.07	-40.76	-42.88	0.90
0.75	1.63	2.42	4.04	4.12	-15.96	-14.17	1.28
	13.42	11.18	15.97	15.96	2.32	-7.23	11.77
	3.99	3.54	5.17	5.19	-2.28	-3.05	3.29
	3.55	8.41	8.11	8.62	-48.20	-49.50	1.58
0.85	2.17	2.97	4.18	4.27	-17.86	-15.29	1.79
	11.96	9.08	14.53	14.51	-2.59	-14.01	10.09
	4.62	4.11	6.32	6.32	-2.58	-3.29	3.71
	3.57	9.03	7.13	7.77	-56.52	-58.42	2.38
0.95	1.95	2.86	2.98	3.11	-22.16	-19.13	1.78
	13.31	10.47	15.94	15.91	-2.05	-13.87	11.33
	3.44	2.89	5.46	5.45	-5.24	-5.98	2.74
	3.35	9.04	6.75	7.40	-61.01	-62.06	1.03
1.00	2.22	3.09	2.89	3.02	-23.97	-20.07	2.14
	11.82	8.46	14.89	14.85	-5.57	-18.70	9.86
	5.00	4.52	6.49	6.49	-3.18	-3.88	4.43

3.7 Conclusions

In this chapter we have reinvestigated the problem of estimating the common mean of two normal populations, when the variances are known to follow certain ordering say $\sigma_1^2 \leq \sigma_2^2$. It should be noted that, Elfessi and Pal (1992) considered this problem and obtained an estimator which dominates the well known Graybill-Deal (Graybill and Deal (1959)) estimator in terms of stochastic domination as well as Pitman measure of closeness criterion. In a very similar fashion, in Sections 3.2, 3.3, and 3.4 we have proposed some new estimators which beats some of the well known estimators proposed by Khatri and Shah (1974), Moore and Krishnamoorthy (1997), Tripathy and Kumar (2010), Brown and Cohen (1974) and the MLE (closed form does not exist) respectively, in terms of stochastic domination as well as Pitman measure of closeness criteria under the assumption that the variances are ordered. In addition to this, we have also derived sufficient conditions for improving estimators in the classes of equivariant estimators. As a consequence improved estimators have been derived using the affine invariant loss (3.1.1). More interestingly it has been seen that, the estimators obtained are turning out to be the same as proposed in Section 3.2 including the estimator proposed by Elfessi and Pal (1992). In order to evaluate the performances of all the improved estimators, we have compared numerically the risk values of all these estimators through the simulation study using the Monte-Carlo simulation method. It has been seen that the percentage of risk improvements for all the improved estimators are quite significant which further strengthens the findings in Sections 3.2, 3.3, and 3.4. It has also been concluded that, like in the case of without restrictions on the variances, none of the estimators completely dominates others in terms of the risk values. Finally we have recommended for the use of these improved well structured estimators in practice under order restriction on the variances which lacks in the literature. We hope the present study in the chapter certainly add more value to the current literature on "common mean problem" as well and fills the gap.

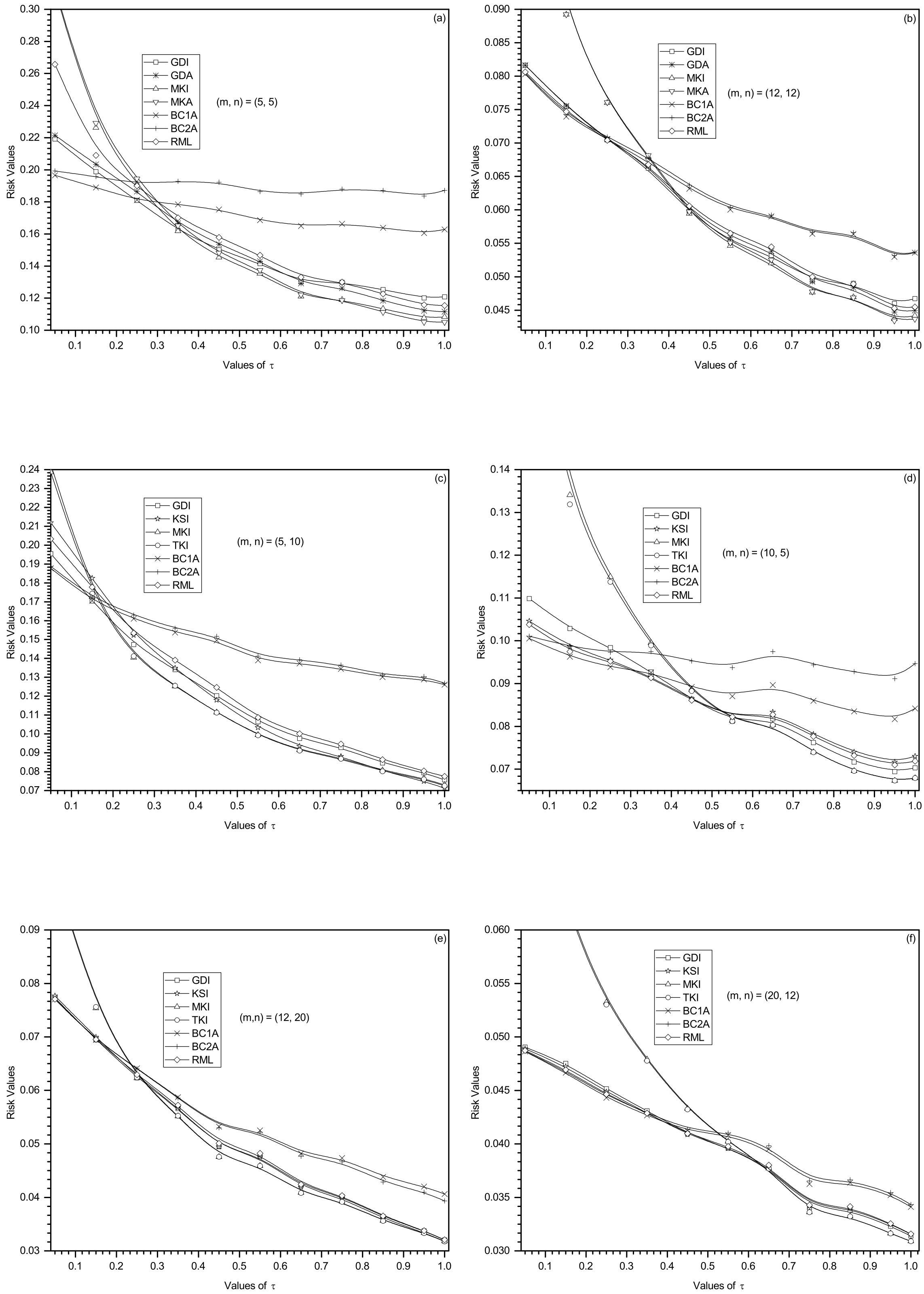

Figure 3.6.1: $(a)-(f)$ Comparison of risk values of several estimators for common mean μ using the loss L_1 for sample sizes $(5,5)$, $(12,12)$, $(5,10)$, $(10,5)$, $(12,20)$ and $(20,12)$ respectively.

Chapter 4

Estimating Quantiles of Several Normal Populations with a Common Mean

4.1 Introduction

In the previous chapter (Chapter 3), we have considered two normal populations with a common mean 'μ' and different variances σ_1^2, and σ_2^2. In fact, we have re-investigated the problem of estimating the common mean under the assumption that, the variances follow certain simple ordering say $\sigma_1^2 \leq \sigma_2^2$. In the present chapter we intend to investigate the problem of estimating quantiles in several normal populations with a common mean, however without assuming any restrictions on the variances.

We note that, the problem of estimating quantiles for any distribution function is certainly important due to its real life applications and also the challenges involve in it. Particularly, the application of quantiles for exponential populations are seen in the study of reliability, life testing, survival analysis and related fields. We refer to Epstein and Sobel (1954) and Saleh (1981) for some practical applications of quantiles. The problem of estimating quantiles of normal population was probably first considered by Zidek (1969, 1971) from a decision theoretic point of view. In fact, he estimated a general function of mean and standard deviation, which in particular case reduces to a quantile $\theta = \mu + \eta\sigma$. He established some interesting results including inadmissibility of the best affine equivariant estimator under certain conditions on $|\eta|$ using a quadratic loss function. Further Rukhin (1983) derived a class of minimax estimators for the quantile θ, each of which improves upon the best equivariant estimator when only one population is available. Lately, Kumar and Tripathy (2011) considered the estimation of the quantiles of the first population with a common mean when two normal populations are available using a quadratic loss from a decision theoretic point of view. For a detailed review and recent updates on estimation of quantiles on normal populations we refer to Kumar and Tripathy (2011) and the references cited there in.

The model under consideration is quite popular in the literature and has many applications in real life situations. For validity and some application of our model (assuming equality of mean/location parameter), we refer to Hahn and Nelson (1970), Vazquez et al. (2007) and has been well addressed by Chang and Pal (2008) and Tripathy and Kumar (2015). Under such situations, one may wishes to draw inference on either common mean or quantiles, assuming that the data follow a normal distribution.

The problem considered in this chapter is of great interest and also quite challenging, as it uses the information for common mean to draw inference on the quantiles. To be very specific,

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for the case $k = 2$, Kumar and Tripathy (2011) established that, improving the estimators of a common mean, one can certainly improve estimators of the quantiles. They also derived some inadmissibility conditions for estimators belonging to some equivariant classes. For some recent results on estimating common mean of two normal populations we refer to Tripathy and Kumar (2010), where as for the case $(k \geq 3)$ we refer to Tripathy and Kumar (2015) and the references cited there in. Our main target in this chapter is to generalize some of the results of Kumar and Tripathy (2011) to $k(\geq 3)$ normal populations. The rest of the chapter is organized as follows. In Section 4.2, we consider several $(k \geq 3)$ normal populations with a common mean and the variances are different. We have considered the estimation of the p^{th} quantile of the first population using the quadratic loss. In Section 4.2.1, we prove a general result which helps in obtaining better estimators for the quantiles. We introduce the concept of invariance to the model in Section 4.2.2 and derive sufficient conditions for improving estimators which are equivariant under affine and location group of transformations. Consequently, two complete class results are obtained for estimating the quantiles. More importantly, in Section 4.2.3, we carry out a detailed simulation study (for the cases $k = 2$ and $k = 3$) in order to numerically compare the risk performances of all the proposed estimators. Finally, we recommend using estimators for quantiles in certain situations, which may be of great interest for practical purposes. In Section 4.2.4, we give our conclusions and also discuss two practical examples illustrating the use of estimators for quantiles.

It is also important to note that, in the literature most of the results on quantile estimation are for a single parameter, $\theta = \mu + \eta\sigma$. In Section 4.3, we consider the problem of simultaneous estimation of a vector $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$ of $k(\geq 2)$ quantiles. This model is certainly important from theoretical as well as application point of view. For some results on simultaneous estimation of location and scale parameters with application, we refer to Bai and Durairajan (1998), Alexander and Chandrasekar (2005) and Tsukuma (2012). In Section 4.3.1, we derive a basic result which helps in constructing certain improved estimators for the quantile vector $\underset{\sim}{\theta}$. In Section 4.3.2, we derive affine and location equivariant estimators. Sufficient conditions for improving estimators in the class of affine and location equivariant class have been derived for the case $k = 2$. In the process, two complete class results have been proved there. In Section 4.3.3, a detailed simulation study has been done in order to numerically compare the relative risk performances of some of our proposed estimators. We conclude with some practical examples in Section 4.3.4.

4.2 Estimating Quantiles of Normal Population with a Common Mean

Let there be $k(\geq 2)$ independent normal populations, each having a common mean and possibly different variances. To be very specific, let $(X_{i1}, X_{i2}, \ldots, X_{in_i})$ be a random sample of size n_i available from the i^{th} normal population $N(\mu, \sigma_i^2)$; $i = 1, 2, \ldots, k$. Here, we assume that the parameters μ and σ_i^2; $i = 1, 2, \ldots, k$ are unknown. The problem is to estimate the quantile, $\theta = \mu + \eta\sigma_1$ of the first population with respect to a quadratic loss function,

$$L(d, \mu, \sigma_1^2) = \left(\frac{d - \theta}{\sigma_1}\right)^2, \tag{4.2.1}$$

where d is an estimate for estimating the quantile θ. Here $0 \neq \eta = \Phi^{-1}(p)$; $0 < p < 1$ and $\Phi(.)$ denotes the cumulative distribution function of a standard normal random variable. For the case $\eta = 0$, the problem boils down to estimating the common mean of $k(\geq 2)$ normal populations and it has been recently dealt by Tripathy and Kumar (2015). It should be noted that, for the case $k = 2$, the problem of estimation for quantile θ, has been well investigated by Kumar and Tripathy (2011). The main objective of this section is two fold. First, to generalize some of their decision theoretic results, to a general $k(\geq 2)$ normal populations. Second, an attempt is to compare the relative risk performances of some of our proposed estimators for quantiles numerically, specifically for the case $k = 3$ and $k = 4$, which may be handy for practical purposes.

4.2.1 Some Improved Estimators for Quantiles

Suppose that there are $k(\geq 2)$ independent normal populations with a common unknown mean μ and possibly unknown different variances σ_i^2; $i = 1, 2, \ldots, k$ respectively are available. Specifically, let $(X_{i1}, X_{i2}, \ldots, X_{in_i})$ be a random sample of size n_i available from the ith normal population $N(\mu, \sigma_i^2)$; $i = 1, 2, \ldots, k$. We are interested to estimate the quantiles, $\theta = \mu + \eta\sigma_1$ of the first population when other $k - 1$ normal populations are available with respect to the loss function (4.2.1). A minimal sufficient statistic for our model is $(\bar{X}_1, \ldots, \bar{X}_k, S_1^2, \ldots, S_k^2)$, where we define the random variables,

$$\bar{X}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} X_{ij},\ S_i^2 = \sum_{j=1}^{n_i}(X_{ij} - \bar{X}_i)^2,\ i = 1, 2, \ldots, k.$$

It is easy to observe that, $\bar{X}_i \sim N(\mu, \sigma_i^2/n_i)$ and $S_i^2 \sim \sigma_i^2\chi^2_{n_i-1}$; $i = 1, 2, \ldots, k$. All these random variables are mutually independent.

The model permits to take advantage of μ being common and utilize all the results available for common mean to estimate quantiles θ. On the other hand, as the sufficient statistic is not complete, it is hard to derive the UMVUE (uniformly minimum variance unbiased estimator). Also the MLE (maximum likelihood estimator) is not obtainable in closed form (see Pal et al. (2007)). Further, to proceed we follow the arguments of Kumar and Tripathy (2011), and consider the baseline estimator for quantile θ as $d_1 = \bar{X}_1 + \eta b_{n_1} S_1$, where

$$b_\xi = \frac{\Gamma(\frac{\xi}{2})}{\sqrt{2}\Gamma(\frac{\xi+1}{2})},\ \ \xi = 2, 3, \ldots. \tag{4.2.2}$$

It should be noted that the estimator d_1 is the best affine equivariant and also minimax based on the sufficient statistic $(\bar{X}_1, S_1^2)$.

To continue with, below (Theorem 4.2.1, 4.2.2), we prove some results which directly generalize the results obtained in Section 2 of Kumar and Tripathy (2011) to $k(\geq 2)$ normal populations. First we borrow their results (Theorem 2.1, Remark 2.1, 2.2) which also remains valid for $k(\geq 2)$ populations.

Theorem 4.2.1 *Let d_M be an estimator of the common mean μ, and d_S be an estimator of σ_1. Consider the estimator of $\theta = \mu + \eta\sigma_1$ of the form d $= d_M + \eta d_S$. Also, assume that, given d_S, the estimator d_M is conditionally unbiased for μ, that is,*

$$E(d_M|d_S) = \mu. \tag{4.2.3}$$

Then we have,

$$E(d-\theta)^2 = E(d_M-\mu)^2 + \eta^2 E(d_S-\sigma_1)^2. \qquad (4.2.4)$$

Proof 4.2.1 *The left hand side of (4.2.4) is given by*

$$E(d-\theta)^2 = E(d_M-\mu)^2 + \eta^2(d_S-\sigma_1)^2 + 2\eta E(d_M-\mu)(d_S-\sigma_1). \qquad (4.2.5)$$

Now the term $E(d_M-\mu)(d_S-\sigma_1) = 0$ by the condition $E(d_M|d_S) = \mu$. Hence we have the result.

Remark 4.2.1 *The condition (4.2.3) holds if, in particular we choose d_M as unbiased estimator for common mean μ and d_S to be independent of d_M. For example, one can choose $d_M = \bar{X}_1$ and $d_S = cS_1$, for some suitable constant c.*

Remark 4.2.2 *It is clear from above Theorem 4.2.1, that to construct a good estimator for the quantile θ, it is sufficient to have a good estimator for the common mean μ and/or a good estimator for σ_1.*

To proceed further, we define, $\underset{\sim}{S} = (S_1, \ldots, S_k)$ and $d_\psi = \psi_1(\underset{\sim}{S})\bar{X}_1 + \psi_2(\underset{\sim}{S})\bar{X}_2 + \ldots + \psi_k(\underset{\sim}{S})\bar{X}_k$, such that, $\sum_{i=1}^k \psi_i(\underset{\sim}{S}) = 1$. We have the following remark.

Remark 4.2.3 *Taking $d_M = d_\psi$, and $d_S = bS_1/\eta \ \ (\eta \neq 0)$, in Theorem 4.2.1, we have proved the following theorem.*

Theorem 4.2.2 *Let d_ψ be an estimator of the common mean μ. Consider an estimator of θ as $d_\psi(b) = d_\psi + bS_1$. Then $d_\psi(b)$ has smaller risk than $d_1(b) = \bar{X}_1 + bS_1$ with respect to the quadratic loss function (4.2.1) if and only if d_ψ has smaller risk than $\bar{X}_1$ for estimating common mean μ.*

Proof 4.2.2 *To prove the result, let us consider the risk difference of $d_\psi(b)$ and $d_1(b)$. Let*

$$\begin{aligned}
\triangle &= R(d_\psi(b),\theta) - R(d_1(b),\theta) \\
&= \frac{1}{\sigma_1^2}\{E(d_\psi(b)-\theta)^2 - E(d_1(b)-\theta)^2\} \\
&= \frac{1}{\sigma_1^2}\{E(d_\psi-\mu)^2 - E(\bar{X}_1-\mu)^2\} \\
&= R(d_\psi,\mu) - R(\bar{X}_1,\mu).
\end{aligned}$$

Hence the risk difference $\triangle = R(d_\psi(b),\theta) - R(d_1(b),\theta) \leq 0$, is equivalent to say that $R(d_\psi,\mu) - R(\bar{X}_1,\mu) \leq 0$. This proves the theorem.

Remark 4.2.4 *Take $d_M = d_\psi$, and d_S to be the best affine equivariant estimator of σ_1, that is $d_S = b_{n_1}S_1$. Then using Theorem 4.2.1, we prove the following theorem.*

Theorem 4.2.3 *For estimating quantile $\theta = \mu + \eta\sigma_1$ with respect to the scale invariant loss function (4.2.1), the estimator $d_\psi(b)$ has minimum risk if we choose $b = \eta b_{n_1}$.*

Proof 4.2.3 *Let us consider the risk function of $d_\psi(b)$:*

$$R(d_\psi(b), \theta) \;=\; \frac{1}{\sigma_1^2}\{E(d_\psi - \mu)^2 + E(bS_1 - \eta\sigma_1)^2 + 2E(d_\psi - \mu)(bS_1 - \eta\sigma_1)\}.$$

Now using Theorem 4.2.1, the term $E(d_\psi - \mu)(bS_1 - \eta\sigma_1) = 0$. Further it is easy to observe that the above risk is a convex function in b, hence its minimizing choice is obtained by differentiating it with respect to b and equating to zero. Thus the minimizing choice is given by $b = \frac{\eta\sigma_1 E(S_1)}{E(S_1^2)}$. Now utilizing the fact that $E(S_1) = \frac{\sigma_1\sqrt{2}\Gamma(\frac{n_1}{2})}{\Gamma(\frac{n_1-1}{2})}$ and $E(S_1^2) = (n_1-1)\sigma_1^2$, and substituting in the above we get the choice of b for which the risk will be minimum as $\frac{\eta\Gamma(\frac{n_1}{2})}{\sqrt{2}\Gamma(\frac{n_1+1}{2})}$. This proves the theorem.

From the above discussion it is clear that, in order to construct a better estimator than $d_1 = \bar{X}_1 + \eta b_{n_1} S_1$, for quantile $\theta = \mu + \eta\sigma_1$, one needs to replace $\bar{X}_1$ in d_1 by improved estimators of common mean of the form d_ψ. Below, we construct some estimators for quantiles θ, which are better than d_1, in which $\bar{X}_1$ is being replaced by estimators of common mean μ proposed by Norwood and Hinkelmann (1977), Shinozaki (1978), Moore and Krishnamoorthy (1997) and Tripathy and Kumar (2015) and finally using the grand mean. It should be noted that the well known popular estimator proposed Graybill and Deal (1959) for $k = 2$, has been extended to $k \geq 3$ by Norwood and Hinkelmann (1977).

$$d_{NH} = \hat{\mu}_{NH} + \eta b_{n_1} S_1,$$

where

$$\hat{\mu}_{NH} = \frac{\sum_{i=1}^k n_i(n_i-1)\bar{X}_i/S_i^2}{\sum_{i=1}^k n_i(n_i-1)/S_i^2}, \quad \text{Norwood and Hinkelmann (1977),}$$

$$d_{SZ} = \hat{\mu}_{SZ} + \eta b_{n_1} S_1,$$

where

$$\hat{\mu}_{SZ} = \frac{\sum_{i=1}^k n_i(n_i-3)\bar{X}_i/S_i^2}{\sum_{i=1}^k n_i(n_i-3)/S_i^2}, \quad \text{Shinozaki (1978),}$$

$$d_{MK} = \hat{\mu}_{MK} + \eta b_{n_1} S_1,$$

where

$$\hat{\mu}_{MK} = \frac{\sum_{i=1}^k \sqrt{n_i(n_i-1)}\bar{X}_i/S_i}{\sum_{i=1}^k \sqrt{n_i(n_i-1)}/S_i}, \quad \text{Moore and Krishnamoorthy (1997),}$$

$$d_{TK} = \hat{\mu}_{TK} + \eta b_{n_1} S_1,$$

where

$$\hat{\mu}_{TK} = \frac{\sum_{i=1}^{k} \sqrt{n_i}\bar{X}_i/(b_{n_i-1}S_i)}{\sum_{i=1}^{k} \sqrt{n_i}/(b_{n_i-1}S_i)}, \quad \text{Tripathy and Kumar (2015),}$$

and finally the estimator based on the grand sample mean,

$$d_{GM} = \hat{\mu}_{GM} + \eta b_{n_1} S_1,$$

where

$$\hat{\mu}_{GM} = \frac{\sum_{i=1}^{k} n_i \bar{X}_i}{\sum_{i=1}^{k} n_i}.$$

The following two theoretical comparisons are immediate, which follows directly from the results given in Norwood and Hinkelmann (1977) and Shinozaki (1978) where they have obtained the results for common mean.

Theorem 4.2.4 *For estimating the quantiles, $\theta = \mu + \eta\sigma_1$ with respect to the loss function (4.2.1), the estimator*

(i) d_{NH} has smaller risk than d_1 if and only if, the sample sizes $n_i \geq 11$ or one of the $n_i = 10$ and all other $n_j \geq 18$ where i is different from j.

(ii) d_{SZ} has smaller risk than d_1 if and only if $(n_1 - 1) \geq 7$ and $(n_1 - 7)(n_j - 7) \geq 16$ for any $j \neq 1$.

Proof 4.2.4 *The proof is trivial after using Theorem 4.2.1 and the results of Norwood and Hinkelmann (1977) and Shinozaki (1978) for estimating a common mean.*

Remark 4.2.5 *In Section 4.2.3, we carry out a detailed simulation study to numerically compare all these estimators for the case $k = 3$ and $k = 4$ populations, which validate the theoretical results.*

4.2.2 Inadmissibility Results for Equivariant Estimators

In this section we introduce the concept of invariance to our problem of estimating quantiles and prove some inadmissibility results for estimators which are equivariant under affine and location group of transformations.

Let us introduce the affine group of transformations, $G_A = \{g_{a,b}(x) = ax + b, a > 0, b \in \mathbb{R}\}$ to our problem. Under this group of transformations, $\bar{X}_i \to a\bar{X}_i + b$, $S_i^2 \to a^2 S_i^2, i = 1, 2, \ldots, k$. The parameters $\mu \to a\mu + b$, $\sigma_i \to a\sigma_i, i = 1, 2, \ldots, k$, and $\theta \to a\theta + b$, where θ is the quantile. The problem becomes invariant if we choose the loss function (4.2.1). The decision rule d must satisfy the relation,

$$d(a\bar{X}_1 + b, \cdots, a\bar{X}_k + b, a^2 S_1^2, , \cdots, a^2 S_k^2) = ad(\bar{X}_1, \cdots, \bar{X}_k, S_1^2, \cdots, S_k^2) + b.$$

Now choosing $b = -a\bar{X}_1$ where $a = 1/S_1$, and simplifying we obtain the form of an affine equivariant estimator as,

$$d_\Psi = \bar{X}_1 + S_1 \Psi(\underset{\sim}{T}, \underset{\sim}{R}), \tag{4.2.6}$$

where Ψ is any real valued function, $\underset{\sim}{T} = (T_2, T_3, \ldots, T_k)$, $\underset{\sim}{R} = (R_2, R_3, \ldots, R_k)$, $T_i = (\bar{X}_i - \bar{X}_1)/S_1$, and $R_i = S_i^2/S_1^2$; $i = 2, 3, \ldots, k$.

Let us denote,

$$\Psi_1(\underset{\sim}{t}, \underset{\sim}{r}) = \begin{cases} \eta b_{2n+3}, & \text{if } t_i \geq 0; i = 2, 3, \ldots, k \\ \eta b_{2n+3} + \sum_{i=2}^{k} t_i, & \text{if } t_i < 0; i = 2, 3, \ldots, k \\ \eta b_{2n+3} + \sum_{j=2}^{p} t_{l_j}, & \text{if Case 2 (given in the proof) holds} \\ \eta b_{2n+3} + \sum_{j=p+1}^{k} t_{l_j}, & \text{if Case 3 (given in the proof) holds} \end{cases} \tag{4.2.7}$$

and

$$\Psi_2(\underset{\sim}{t}, \underset{\sim}{r}) = \begin{cases} \eta b_{2n+3}, & \text{if } t_i < 0; i = 2, 3, \ldots, k \\ \eta b_{2n+3} + \sum_{i=2}^{k} t_i, & \text{if } t_i \geq 0; i = 2, 3, \ldots, k \\ \eta b_{2n+3} + \sum_{j=2}^{p} t_{l_j}, & \text{if Case 5 (given in the proof) holds} \\ \eta b_{2n+3} + \sum_{j=p+1}^{k} t_{l_j}, & \text{if Case 6 (given in the proof) holds,} \end{cases} \tag{4.2.8}$$

where $(l_2, l_3, \ldots, l_k)$ is a permutation of numbers $(2, 3, \ldots, k)$ and $2 \leq p \leq k$.

Now for an affine equivariant estimator d_Ψ of the quantile θ, as given in (4.2.6), we define functions Ψ_{10} and Ψ_{20} as follows.

$$\Psi_{10}(\underset{\sim}{t}, \underset{\sim}{r}) = \max(\Psi(\underset{\sim}{t}, \underset{\sim}{r}), \Psi_1(\underset{\sim}{t}, \underset{\sim}{r})), \tag{4.2.9}$$

and

$$\Psi_{20}(\underset{\sim}{t}, \underset{\sim}{r}) = \min(\Psi(\underset{\sim}{t}, \underset{\sim}{r}), \Psi_2(\underset{\sim}{t}, \underset{\sim}{r})). \tag{4.2.10}$$

Let us denote $\underset{\sim}{\alpha} = (\mu, \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2)$. Next we prove a theorem regarding inadmissibility of estimators which are equivariant under affine group of transformations.

Theorem 4.2.5 *Let d_Ψ be an affine equivariant estimator of the form (4.2.6) for estimating the quantile θ, and the loss function be (4.2.1) or the squared error. Let the functions $\Psi_{10}(\underset{\sim}{t}, \underset{\sim}{r})$ and $\Psi_{20}(\underset{\sim}{t}, \underset{\sim}{r})$ be defined as in (4.2.9) and (4.2.10) respectively.*

(i) When $\eta > 0$, the estimator d_Ψ is improved by $d_{\Psi_{10}}$, if there exist some values of parameters $\underset{\sim}{\alpha}$ such that, $P_{\underset{\sim}{\alpha}}(\Psi(\underset{\sim}{T}, \underset{\sim}{R}) \neq \Psi_{10}(\underset{\sim}{T}, \underset{\sim}{R})) > 0$.

(ii) When $\eta < 0$, the estimator d_Ψ is improved by $d_{\Psi_{20}}$, if there exist some values of parameters $\underset{\sim}{\alpha}$ such that, $P_{\underset{\sim}{\alpha}}(\Psi(\underset{\sim}{T}, \underset{\sim}{R}) \neq \Psi_{20}(\underset{\sim}{T}, \underset{\sim}{R})) > 0$.

Proof 4.2.5 *The proof of the theorem can be done by applying the orbit-by-orbit improvement technique for improving equivariant estimators proposed by Brewster and Zidek (1974). The proof can be done easily by putting θ in place of μ in the proof of Theorem 2.1 in Tripathy and Kumar (2015).*

Consider the conditional risk function of d_Ψ given $(\underset{\sim}{T}, \underset{\sim}{R})$:

$$R(d_\Psi, \underset{\sim}{\alpha} | (\underset{\sim}{T}, \underset{\sim}{R})) = \frac{1}{\sigma_1^2} E\{(\bar{X}_1 + S_1 \Psi(\underset{\sim}{T}, \underset{\sim}{R}) - \theta)^2 | (\underset{\sim}{T}, \underset{\sim}{R}) = (\underset{\sim}{t}, \underset{\sim}{r})\}. \tag{4.2.11}$$

It can be easily seen that the above risk function (4.2.11) is a convex function in Ψ, hence the *minimizing choice of Ψ is obtained as,*

$$\Psi(\underset{\sim}{t},\underset{\sim}{r}) \quad = \quad -\frac{E((\bar{X}_1-\theta)S_1|(\underset{\sim}{T},\underset{\sim}{R})=(\underset{\sim}{t},\underset{\sim}{r}))}{E(S_1^2|\underset{\sim}{T},\underset{\sim}{R})=(\underset{\sim}{t},\underset{\sim}{r}))}. \tag{4.2.12}$$

Let us introduce the new variables $V_i = \frac{\sqrt{n_1}(\bar{X}_i-\mu)}{\sigma_1}$ *and* $W_i = \frac{S_i^2}{\sigma_i^2}$; $i = 2,\ldots,k$. *Also denote* $\tau_i = \frac{n_1\sigma_i^2}{n_i\sigma_1^2}$, $i = 2,3,\ldots,k$. *The above expression (4.2.12), reduces to*

$$\Psi(\underset{\sim}{t},\underset{\sim}{r},\underset{\sim}{\tau}) = -\frac{E\{V_1\sqrt{W_1}|(\underset{\sim}{T},\underset{\sim}{R})=(\underset{\sim}{t},\underset{\sim}{r})\}}{\sqrt{n_1}E(W_1|(\underset{\sim}{T},\underset{\sim}{R})=(\underset{\sim}{t},\underset{\sim}{r}))}+\eta\sigma_1\frac{E\{\sqrt{W_1}|(\underset{\sim}{T},\underset{\sim}{R})=(\underset{\sim}{t},\underset{\sim}{r})\}}{E(W_1|(\underset{\sim}{T},\underset{\sim}{R})=(\underset{\sim}{t},\underset{\sim}{r}))}, \tag{4.2.13}$$

where $\underset{\sim}{\tau} = (\tau_2,\tau_3,\ldots,\tau_k)$. *The conditional expectations in the right hand side of (4.2.13) have been computed in Tripathy and Kumar (2015) and are given by*

$$E\{W_1|(\underset{\sim}{T},\underset{\sim}{R})\} = \frac{2(n+1)}{\lambda}, \tag{4.2.14}$$

$$E\{\sqrt{W_1}|(\underset{\sim}{T},\underset{\sim}{R})\} = \frac{2(n+1)b_{2n+3}}{\sqrt{\lambda}}, \tag{4.2.15}$$

and

$$E\{V_1\sqrt{W_1}|(\underset{\sim}{T},\underset{\sim}{R})\} = -\frac{2\sqrt{n_1}(n+1)}{\tau\lambda}\sum_{i=2}^{k}\frac{t_i}{\tau_i}, \tag{4.2.16}$$

where $n = \frac{3k+n_1-6}{2}+\sum_{i=2}^{k}\frac{n_i-3}{2}$, $\lambda = 1+\sum_{i=2}^{k}\frac{n_1t_i^2}{\tau_i}+\sum_{i=2}^{k}\frac{n_1r_i}{n_i\tau_i}-\frac{n_1}{\tau}(\sum_{i=2}^{k}\frac{t_i}{\tau_i})^2$, *and* $\tau = 1+\sum_{i=2}^{k}\frac{1}{\tau_i}$. *Substituting all these values and simplifying we get the minimizing choice as,*

$$\Psi(\underset{\sim}{t},\underset{\sim}{r}) = \frac{1}{\tau}\sum_{i=2}^{k}\frac{t_i}{\tau_i}+\eta\sqrt{\lambda}b_{2n+3}. \tag{4.2.17}$$

To apply orbit-by-orbit improvement technique of Brewster and Zidek (1974), we need to find the upper and lower bounds of $\Psi(\underset{\sim}{t},\underset{\sim}{r})$ *for fixed choices of* $(\underset{\sim}{t},\underset{\sim}{r})$ *and* η.

Case 1: *Let* $\eta > 0$, *and* $t_i \geq 0$, $i = 2,3,\ldots,k$. *In this case we obtain,*

$$\inf\Psi(\underset{\sim}{t},\underset{\sim}{r},\underset{\sim}{\tau}) = \eta b_{2n+3}, \quad \sup\Psi(\underset{\sim}{t},\underset{\sim}{r},\underset{\sim}{\tau}) = +\infty. \tag{4.2.18}$$

Case 2: $\eta > 0$, *and let* $(l_2,l_3,\ldots,l_k)$ *be a permutation of* $(2,3,\ldots,k)$ *such that* $t_{l_2} < 0$, $t_{l_3} < 0$, $\ldots$, $t_{l_p} < 0$, *and* $t_{l_{p+1}} \geq 0$, $t_{l_{p+2}} \geq 0$, $\ldots$, $t_{l_k} \geq 0$, $2 \leq p \leq k$.

$$\inf\Psi(\underset{\sim}{t},\underset{\sim}{r},\underset{\sim}{\tau}) \geq \eta b_{2n+3}+\sum_{j=2}^{p}t_{l_j}, \quad \sup\Psi(\underset{\sim}{t},\underset{\sim}{r},\underset{\sim}{\tau}) = +\infty. \tag{4.2.19}$$

Case 3: $\eta > 0$, *and let* $(l_2,l_3,\ldots,l_k)$ *be a permutation of* $(2,3,\ldots,k)$ *such that* $t_{l_2} \geq 0$, $t_{l_3} \geq 0$,

$\ldots, t_{l_p} \geq 0$, *and* $t_{l_{p+1}} < 0, t_{l_{p+2}} < 0, \ldots, t_{l_k} < 0, 2 \leq p \leq k$.

$$\inf \Psi(\underset{\sim}{t}, \underset{\sim}{r}, \underset{\sim}{\tau}) \geq \eta b_{2n+3} + \sum_{j=p+1}^{k} t_{l_j}, \quad \sup \Psi(\underset{\sim}{t}, \underset{\sim}{r}, \underset{\sim}{\tau}) = +\infty. \tag{4.2.20}$$

Case 4: $\eta < 0$, *and let* $t_i < 0, i = 2, 3, \ldots, k$. *In this case we obtain,*

$$\inf \Psi(\underset{\sim}{t}, \underset{\sim}{r}, \underset{\sim}{\tau}) = -\infty, \quad \sup \Psi(\underset{\sim}{t}, \underset{\sim}{r}, \underset{\sim}{\tau}) = \eta b_{2n+3}. \tag{4.2.21}$$

Case 5: $\eta < 0$, *and let* $(l_2, l_3, \ldots, l_k)$ *be a permutation of* $(2, 3, \ldots, k)$ *such that* $t_{l_2} < 0, t_{l_3} < 0, \ldots, t_{l_p} < 0$, *and* $t_{l_{p+1}} \geq 0, t_{l_{p+2}} \geq 0, \ldots, t_{l_k} \geq 0, 2 \leq p \leq k$.

$$\inf \Psi(\underset{\sim}{t}, \underset{\sim}{r}, \underset{\sim}{\tau}) = -\infty, \quad \sup \Psi(\underset{\sim}{t}, \underset{\sim}{r}, \underset{\sim}{\tau}) \leq \eta b_{2n+3} + \sum_{j=p+1}^{k} t_{l_j}. \tag{4.2.22}$$

Case 6: $\eta < 0$, *and let* $(l_2, l_3, \ldots, l_k)$ *be a permutation of* $(2, 3, \ldots, k)$ *such that* $t_{l_2} \geq 0, t_{l_3} \geq 0, \ldots, t_{l_p} \geq 0$, *and* $t_{l_{p+1}} < 0, t_{l_{p+2}} < 0, \ldots, t_{l_k} < 0, 2 \leq p \leq k$.

$$\inf \Psi(\underset{\sim}{t}, \underset{\sim}{r}, \underset{\sim}{\tau}) = -\infty, \quad \sup \Psi(\underset{\sim}{t}, \underset{\sim}{r}, \underset{\sim}{\tau}) \leq \eta b_{2n+3} + \sum_{j=2}^{p} t_{l_j}. \tag{4.2.23}$$

Now combining all the Cases 1-6, we can define functions $\Psi_1(\underset{\sim}{t}, \underset{\sim}{r}), \Psi_2(\underset{\sim}{t}, \underset{\sim}{r})$ *as defined in (4.2.7) and (4.2.8) respectively. Using these functions, it is easy to define* $\Psi_{10}(\underset{\sim}{t}, \underset{\sim}{r})$ *and* $\Psi_{20}(\underset{\sim}{t}, \underset{\sim}{r})$ *as given in (4.2.9) and (4.2.10) respectively. An application of Theorem 3.1 (in Brewster and Zidek (1974)) we have,*

$$R(d_{\Psi_{10}}, \underset{\sim}{\alpha}) \leq R(d_{\Psi}, \underset{\sim}{\alpha}),$$

provided $P_{\underset{\sim}{\alpha}}(\Psi_{10} \neq \Psi) > 0$ *for some choices of* $\underset{\sim}{\alpha}$, *when* $\eta > 0$. *Further,*

$$R(d_{\Psi_{20}}, \underset{\sim}{\alpha}) \leq R(d_{\Psi}, \underset{\sim}{\alpha}),$$

provided $P_{\underset{\sim}{\alpha}}(\Psi_{20} \neq \Psi) > 0$ *for some choices of* $\underset{\sim}{\alpha}$, *when* $\eta < 0$. *This completes the proof of the theorem.*

All the estimators for quantiles $\theta = \mu + \eta\sigma_1$, constructed in Section 4.2.1, can be expressed in the form (4.2.6). We write these estimators in the form (4.2.6) as below.

$$d_{NH} = \bar{X}_1 + S_1 \Psi_{NH}(\underset{\sim}{T}, \underset{\sim}{R})$$

where

$$\Psi_{NH}(\underset{\sim}{T}, \underset{\sim}{R}) = \eta b_{n_1} + \frac{\sum_{i=2}^{k} n_i(n_i - 1)T_i/R_i}{n_1(n_1 - 1) + \sum_{i=2}^{k} n_i(n_i - 1)/R_i}.$$

$$d_{SZ} = \bar{X}_1 + S_1 \Psi_{SZ}(\underset{\sim}{T}, \underset{\sim}{R}),$$

where

$$\Psi_{SZ}(\underset{\sim}{T}, \underset{\sim}{R}) = \eta b_{n_1} + \frac{\sum_{i=2}^{k} n_i(n_i - 3)T_i/R_i}{n_1(n_1 - 3) + \sum_{i=2}^{k} n_i(n_i - 3)/R_i}.$$

$$d_{MK} = \bar{X}_1 + S_1 \Psi_{MK}(\underset{\sim}{T}, \underset{\sim}{R}),$$

where

$$\Psi_{MK}(\underset{\sim}{T}, \underset{\sim}{R}) = \eta b_{n_1} + \frac{\sum_{i=2}^{k} \sqrt{n_i(n_i - 1)}T_i/\sqrt{R_i}}{\sqrt{n_1(n_1 - 1)} + \sum_{i=2}^{k} \sqrt{n_i(n_i - 1)}/\sqrt{R_i}}.$$

$$d_{TK} = \bar{X}_1 + S_1 \Psi_{TK}(\underset{\sim}{T}, \underset{\sim}{R}),$$

where

$$\Psi_{TK}(\underset{\sim}{T}, \underset{\sim}{R}) = \eta b_{n_1} + \frac{\sum_{i=2}^{k} \sqrt{n_i}T_i/(b_{n_i-1}\sqrt{R_i})}{\sqrt{n_1}/b_{n_1-1} + \sum_{i=2}^{k} \sqrt{n_i}/(b_{n_i-1}\sqrt{R_i})}.$$

Finally, we have the estimator based on the grand mean,

$$d_{GM} = \bar{X}_1 + S_1 \Psi_{GM}(\underset{\sim}{T}, \underset{\sim}{R}),$$

where

$$\Psi_{GM}(\underset{\sim}{T}, \underset{\sim}{R}) = \eta b_{n_1} + \frac{\sum_{i=2}^{k} n_i T_i}{\sum_{i=1}^{k} n_i}.$$

Next we propose an estimator for quantiles $\theta = \mu + \eta\sigma_1$ based on the maximum likelihood estimator for common mean. It should be noted that the closed form of the MLE of common mean is not obtainable. To get the MLE of the common mean μ for $k(\geq 2)$ populations, one needs to solve numerically the system of $k+1$ equations in $k+1$ variables $(\mu, \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2)$. Specifically we obtain the MLE for $k = 3$ and $k = 4$ populations numerically. Let $\hat{\mu}_{ML}$ be the MLE for μ obtained by solving the system of equations. Using this estimator for common mean we propose an estimator for the quantile $\theta = \mu + \eta\sigma_1$ as

$$d_{ML} = \hat{\mu}_{ML} + \eta b_{n_1} S_1.$$

Remark 4.2.6 *It is interesting to see that, all the proposed estimators for the quantiles which are based on some popular estimators for common mean fall into the class (4.2.6). However, it has been seen using a simulation study, none of these estimators could be improved by using Theorem 4.2.5. It is also interesting to note that, the choices of Ψ for all these estimators lie inside the interval $[\Psi_1, +\infty)$ (when $\eta > 0$) and $(-\infty, \Psi_2]$ (when $\eta < 0$) with probability 1. It can be also noted that, all these estimators form a complete class. We carry out a detailed simulation study in Section 4.2.3 , to numerically compare all these well structured estimators.*

Next we consider a smaller group (that is location group) of transformations which will give rise to a larger class of estimators. Also we derive a sufficient condition for improving estimators which are equivariant under this group of transformations.

Let us introduce the location group of transformations $G_L = \{g_c(x) = x + c, c \in \mathbb{R}\}$ to our problem. Under this transformation, $\bar{X}_i \to \bar{X}_i + c, S_i^2 \to S_i^2$, $i = 1, 2, \ldots, k$. The parameters $\mu \to \mu + c, \sigma_i \to \sigma_i$ and the quantile $\theta \to \theta + c$. The problem remains invariant if we choose the loss function (5.2.1), and the form of a location equivariant estimator d must satisfy the relation,

$$d(\bar{X}_1 + c, \cdots, \bar{X}_k + c, S_1^2, \cdots, S_k^2) = d(\bar{X}_1, \cdots, \bar{X}_k, S_1^2, \cdots, S_k^2) + c.$$

Now choosing $c = -\bar{X}_1$, and simplifying, we obtain the form of a location equivariant estimator as,

$$d_\phi = \bar{X}_1 + \phi(\underset{\sim}{U}, \underset{\sim}{S}), \tag{4.2.24}$$

where $\underset{\sim}{U} = (U_2, U_3, \ldots, U_k)$, $U_i = \bar{X}_i - \bar{X}_1$, $i = 2, 3, \ldots, k$, and $\underset{\sim}{S} = (S_1, \ldots, S_k)$.

To proceed further, let us denote

$$\phi_1(\underset{\sim}{u}, \underset{\sim}{s}) = \begin{cases} 0, & \text{if } u_j \geq 0; j = 2, 3, \ldots, k \\ \sum_{j=2}^{k} u_j, & \text{if } u_j < 0; j = 2, 3, \ldots, k \\ \sum_{j=2}^{p} u_{l_j}, & \text{if Case 2 (given in the proof) holds} \\ \sum_{j=p+1}^{k} u_{l_j}, & \text{if Case 3 (given in the proof) holds} \end{cases} \tag{4.2.25}$$

and

$$\phi_2(\underset{\sim}{u}, \underset{\sim}{s}) = \begin{cases} 0, & \text{if } u_j < 0; j = 2, 3, \ldots, k \\ \sum_{j=2}^{k} u_j, & \text{if } u_j \geq 0; j = 2, 3, \ldots, k \\ \sum_{j=p+1}^{k} u_{l_j}, & \text{if Case 5 (given in the proof) holds} \\ \sum_{j=2}^{p} u_{l_j}, & \text{if Case 6 (given in the proof) holds,} \end{cases} \tag{4.2.26}$$

where $(l_2, l_3, \ldots, l_k)$ is a permutation of numbers $(2, 3, \ldots, k)$ and $2 \leq p \leq k$.

To prove the next result (Theorem 4.2.6), we define two functions $\phi_{10}(\underset{\sim}{u}, \underset{\sim}{s})$ and $\phi_{20}(\underset{\sim}{u}, \underset{\sim}{s})$ for location equivariant estimator d_ϕ as,

$$\phi_{10}(\underset{\sim}{u}, \underset{\sim}{s}) = \max(\phi(\underset{\sim}{u}, \underset{\sim}{s}), \phi_1(\underset{\sim}{u}, \underset{\sim}{s})) \tag{4.2.27}$$

and

$$\phi_{20}(\underset{\sim}{u}, \underset{\sim}{s}) = \min(\phi(\underset{\sim}{u}, \underset{\sim}{s}), \phi_2(\underset{\sim}{u}, \underset{\sim}{s})). \tag{4.2.28}$$

Next we have the result, which helps in proving inadmissibility of estimators which are equivariant under location group of transformations.

Theorem 4.2.6 *Let d_ϕ be a location equivariant estimator as given in (4.2.24) and the loss function be the quadratic loss as given in (4.2.1) or any squared error loss. Let the functions ϕ_{10} and ϕ_{20} be defined as in (4.2.27) and (4.2.28) respectively.*

(i) When $\eta > 0$, the location equivariant estimator d_ϕ is inadmissible and is improved by $d_{\phi_{10}}$, if there exist some values of parameters $\underset{\sim}{\alpha}$, such that $P_{\underset{\sim}{\alpha}}(\phi(\underset{\sim}{U}, \underset{\sim}{S}) \neq \phi_{10}(\underset{\sim}{U}, \underset{\sim}{S})) > 0$.

(i) When $\eta < 0$, the location equivariant estimator d_ϕ is inadmissible and is improved by $d_{\phi_{20}}$, if there exist some values of parameters $\underset{\sim}{\alpha}$, such that $P_{\underset{\sim}{\alpha}}(\phi(\underset{\sim}{U}, \underset{\sim}{S}) \neq \phi_{20}(\underset{\sim}{U}, \underset{\sim}{S})) > 0$.

Proof 4.2.6 *The proof is similar to the proof of the Theorem 3.1 of Tripathy and Kumar (2015), where one needs to replace μ by θ. However, for the sake of completeness, we give the details.*

Consider the conditional risk function of d_ϕ *given* $(\underset{\sim}{U}, \underset{\sim}{S})$:

$$R(d_\phi, \underset{\sim}{\alpha}|(\underset{\sim}{U}, \underset{\sim}{S})) = \frac{1}{\sigma_1^2} E\{(\bar{X}_1 + \phi(\underset{\sim}{U}, \underset{\sim}{S}) - \theta)^2 |(\underset{\sim}{U}, \underset{\sim}{S})\}.$$

It is easy to see that the above risk function is a convex function of ϕ*. Therefore minimizing choice of* ϕ *is obtained as,*

$$\phi(\underset{\sim}{u}, \underset{\sim}{s}) = \theta - E\{\bar{X}_1|(\underset{\sim}{U}, \underset{\sim}{S}) = (\underset{\sim}{u}, \underset{\sim}{s})\}.$$

The conditional expectation in the right hand side has been evaluated in Tripathy and Kumar (2015). Utilizing their expression and substituting above the minimizing choice reduces to

$$\phi(\underset{\sim}{u}, \underset{\sim}{s}) = \eta\sigma_1 + \frac{1}{\tau} \sum_{j=2}^{k} \frac{u_j}{\tau_j}, \tag{4.2.29}$$

where $\tau = 1 + \sum_{i=2}^{k} \frac{1}{\tau_i}$*, and* $\tau_j = \frac{n_1 \sigma_j^2}{n_j \sigma_1^2}$.

In order to apply orbit-by-orbit improvement technique of Brewster and Zidek (1974), we need to obtain the supremum and infimum of $\phi(\underset{\sim}{u}, \underset{\sim}{s})$ *with respect to* $\underset{\sim}{\tau} = (\tau_2, \tau_3, \ldots, \tau_k)$*, for fixed values of* $\underset{\sim}{u}$, $\underset{\sim}{s}$ *and* η*. Analyzing the right hand side term in (4.2.29), we obtain the following results.*

Case 1: *Let* $\eta > 0$*, and* $u_j \geq 0$, $j = 2, 3, \ldots, k$*. In this case, we obtain,*

$$\inf \phi(\underset{\sim}{u}, \underset{\sim}{s}) = 0, \quad \sup \phi(\underset{\sim}{u}, \underset{\sim}{s}) = +\infty.$$

Case 2: *Let* $\eta > 0$*, and* $(l_2, \ldots, l_k)$ *be a permutation of* $(2, \ldots, k)$ *such that,* $u_{l_2} < 0$, $u_{l_3} < 0$, $\ldots$, $u_{l_p} < 0$, $u_{l_{p+1}} \geq 0$, $u_{l_{p+2}} \geq 0$, $\ldots$, $u_{l_k} \geq 0$; $2 \leq p \leq k$*. Then the supremum and infimum of* $\phi(\underset{\sim}{u}, \underset{\sim}{s})$ *is obtained as,*

$$\inf \phi(\underset{\sim}{u}, \underset{\sim}{s}) \geq \sum_{j=2}^{p} u_{l_j}, \quad \sup \phi(\underset{\sim}{u}, \underset{\sim}{s}) = +\infty.$$

Case 3: *Let* $\eta > 0$*, and* $(l_2, \ldots, l_k)$ *be a permutation of* $(2, \ldots, k)$ *such that,* $u_{l_2} \geq 0$, $u_{l_3} \geq 0$, $\ldots$, $u_{l_p} \geq 0$, $u_{l_{p+1}} < 0$, $u_{l_{p+2}} < 0$, $\ldots$, $u_{l_k} < 0$; $2 \leq p \leq k$*. Then the supremum and infimum of* $\phi(\underset{\sim}{u}, \underset{\sim}{s})$ *is obtained as,*

$$\inf \phi(\underset{\sim}{u}, \underset{\sim}{s}) \geq \sum_{j=p+1}^{k} u_{l_j}, \quad \sup \phi(\underset{\sim}{u}, \underset{\sim}{s}) = +\infty.$$

Case 4: *Let* $\eta < 0$*, and* $u_j \geq 0$, $j = 2, 3, \ldots, k$*. In this case, we obtain,*

$$\inf \phi(\underset{\sim}{u}, \underset{\sim}{s}) = -\infty, \quad \sup \phi(\underset{\sim}{u}, \underset{\sim}{s}) = 0.$$

Case 5: *Let* $\eta < 0$*, and* $(l_2, \ldots, l_k)$ *be a permutation of* $(2, \ldots, k)$ *such that,* $u_{l_2} < 0$, $u_{l_3} < 0$, $\ldots$, $u_{l_p} < 0$, $u_{l_{p+1}} \geq 0$, $u_{l_{p+2}} \geq 0$, $\ldots$, $u_{l_k} \geq 0$; $2 \leq p \leq k$*. Then the supremum and infimum* *of* $\phi(\underset{\sim}{u}, \underset{\sim}{s})$ *is obtained as,*

$$\inf \phi(\underset{\sim}{u}, \underset{\sim}{s}) = -\infty, \quad \sup \phi(\underset{\sim}{u}, \underset{\sim}{s}) \leq \sum_{j=p+1}^{k} u_{l_j}.$$

Case 6: *Let* $\eta < 0$, *and* $(l_2, \ldots, l_k)$ *be a permutation of* $(2, \ldots, k)$ *such that,* $u_{l_2} \geq 0$, $u_{l_3} \geq 0$, $\ldots$, $u_{l_p} \geq 0$, $u_{l_{p+1}} < 0$, $u_{l_{p+2}} < 0$, $\ldots$, $u_{l_k} < 0$; $2 \leq p \leq k$. *Then the supremum and infimum of* $\phi(\underset{\sim}{u}, \underset{\sim}{s})$ *is obtained as,*

$$\inf \phi(\underset{\sim}{u}, \underset{\sim}{s}) = -\infty, \quad \sup \phi(\underset{\sim}{u}, \underset{\sim}{s}) \leq \sum_{j=2}^{p} u_{l_j}.$$

Combining all the above Cases 1-6, we define the functions,

$$\phi_1(\underset{\sim}{u}, \underset{\sim}{s}) = \begin{cases} 0, & \text{if } u_j \geq 0; j = 2, 3, \ldots, k \\ \sum_{j=2}^{k} u_j, & \text{if } u_j < 0; j = 2, 3, \ldots, k \\ \sum_{j=2}^{p} u_{l_j}, & \text{if Case 2 holds} \\ \sum_{j=p+1}^{k} u_{l_j}, & \text{if Case 3 holds} \end{cases} \tag{4.2.30}$$

and

$$\phi_2(\underset{\sim}{u}, \underset{\sim}{s}) = \begin{cases} 0, & \text{if } u_j < 0; j = 2, 3, \ldots, k \\ \sum_{j=2}^{k} u_j, & \text{if } u_j \geq 0; j = 2, 3, \ldots, k \\ \sum_{j=p+1}^{k} u_{l_j}, & \text{if Case 5 holds} \\ \sum_{j=2}^{p} u_{l_j}, & \text{if Case 6 holds.} \end{cases} \tag{4.2.31}$$

as given in (4.2.25) and (4.2.26). Utilizing these functions, it is easy to define the functions $\phi_{10}(\underset{\sim}{u}, \underset{\sim}{s})$ *and* $\phi_{20}(\underset{\sim}{u}, \underset{\sim}{s})$ *as given in (4.2.27) and (4.2.28) respectively. An application of Theorem 3.1 (in Brewster and Zidek (1974)) we have,*

$$R(d_{\phi_{10}}, \underset{\sim}{\alpha}) \leq R(d_{\phi}, \underset{\sim}{\alpha}),$$

provided $P_{\underset{\sim}{\alpha}}(\phi_{10} \neq \phi) > 0$ *for some choices of* $\underset{\sim}{\alpha}$, *when* $\eta > 0$. *Further,*

$$R(d_{\phi_{20}}, \underset{\sim}{\alpha}) \leq R(d_{\phi}, \underset{\sim}{\alpha}),$$

provided $P_{\underset{\sim}{\alpha}}(\phi_{20} \neq \phi) > 0$ *for some choices of* $\underset{\sim}{\alpha}$, *when* $\eta < 0$. *This completes the proof of the theorem.*

Remark 4.2.7 *It can be also noted that, all the estimators discussed in Section 4.2.1, belong to the class (4.2.24). Unfortunately, none of these could be improved by using Theorem 4.2.6. In fact, the choices of* ϕ *for all these estimators lie inside the interval* $[\phi_1, +\infty)$ *(when* $\eta > 0$*) and* $(-\infty, \phi_2]$ *(when* $\eta < 0$*) with probability 1. However, all these estimators form a complete class.*

4.2.3 A Simulation Study

In Section 4.2.1 , we have constructed various estimators for the quantiles $\theta = \mu + \eta\sigma_1$, which are better than the baseline estimator $d_1 = \bar{X}_1 + \eta b_{n_1} S_1$, under some conditions on sample sizes. Further it has been shown in Section 4.2.2, that these well structured estimators form a complete class except d_{ML} whose closed form does not exist. We have also given an analytical comparison of the risk values of estimators d_1 with d_{NH} and d_{SZ}. However, for practical point of view, we need estimators that can be used in practice. It seems impossible to analytically compare all these popular estimators. Taking advantage of computational resources, in this section, we carry out a detailed simulation study to numerically compare all these estimators. To be more specific, we consider the estimators d_1, d_{NH}, d_{SZ}, d_{MK}, d_{TK}, d_{GM} and d_{ML} for comparison purpose, which are based on the estimators for common mean as given in Section 4.2.1, for the case $k(>2)$. Numerically comparing all these estimators for a general $k(\geq 2)$ is quite impossible. So for simplicity, we consider only the case $k = 3$ and $k = 4$. It should be noted that for $k = 2$, Kumar and Tripathy (2011) carried out a detailed simulation study and numerically compared some of their proposed estimators for quantile θ.

In order to numerically compare all these estimators for quantiles θ, we have generated 20,000 random samples of different sizes each from k (3 or 4) normal populations, with a common mean μ and different variances σ_i^2; $i = 1, 2, 3, 4$ respectively. The loss function is taken as (4.2.1). It is easy to see that, with respect to the loss function (4.2.1), the risk values of all the estimators for quantiles θ are functions of ρ_2, ρ_3 (for the case $k = 3$) and ρ_2, ρ_3, ρ_4 (for the case $k = 4$), where $\rho_i = \sigma_i/\sigma_1$; $i = 2, 3, \ldots, k$ only, when the sample sizes and η being fixed. The simulation study has been carried out for wide range of the values of ρ_i, but results for selected values are reported here. The risk value of d_1 is constant and is obtained as $1/n_1 + \eta^2(1 - (n_1 - 1)b_{n_1}^2)$. A high level of accuracy has been achieved for our simulation in the sense that the standard error of simulation is small. It has been checked that the error is of the order 10^{-3}. In fact for the case $k = 3$ populations the standard error of simulation is seen to vary between 0.0001 and 0.0018 whereas for the case $k = 4$, it is seen to vary between 0.001 and 0.008. It is also noticed that as the number of populations increase the simulation error increases. To proceed further, we define the percentage of relative risk performances of all the estimators with respect to d_1 as following.

$$\begin{aligned}
PR1 &= \frac{Risk(d_1) - Risk(d_{NH})}{Risk(d_1)} \times 100, \; PR2 = \frac{Risk(d_1) - Risk(d_{SZ})}{Risk(d_1)} \times 100, \\
PR3 &= \frac{Risk(d_1) - Risk(d_{MK})}{Risk(d_1)} \times 100, \; PR4 = \frac{Risk(d_1) - Risk(d_{TK})}{Risk(d_1)} \times 100, \\
PR5 &= \frac{Risk(d_1) - Risk(d_{GM})}{Risk(d_1)} \times 100, \; PR6 = \frac{Risk(d_1) - Risk(d_{ML})}{Risk(d_1)} \times 100.
\end{aligned}$$

For illustration purpose, we have presented the percentage of relative risk values of all the estimators in Tables 4.2.1-4.2.3 (for $k = 3$). The risk value of all estimators is also a function of $|\eta|$. For presentation purpose, we choose $\eta = 1.960$. To make the simulation study more impact, we study the case $k = 3$ and $k = 4$ separately in detail. In Table 4.2.1 we present the percentage of relative risk values, for equal sample sizes $(10, 10, 10)$, and $(30, 30, 30)$. It can be also observed that, for equal sample sizes and for fixed η, the estimators $d_{MK} = d_{TK}$ and $d_{SZ} = d_{NH}$. In each tables, the first column gives the values of ρ_2 and the second column gives the values of ρ_3. In each cell corresponding to one value of ρ_2 there corresponds seven (7) values of ρ_3. A pair of values (ρ_2, ρ_3) corresponds four values of percentage of relative risk improvements of estimators. In Tables 4.2.2-4.2.3, we present the relative risk performances of estimators for unequal sample sizes $(10, 15, 20)$, and $(40, 30, 20)$ respectively. A detailed numerical comparison for $k = 4$ population has also been done, however we only present the comments and omit the tables for brevity.

The simulated risk values of d_{NH} and d_{SZ} decrease and converge to the risk value of d_1

as ρ_is increase, whereas the risk values of d_{MK}, d_{TK} and d_{GM} diverge from the risk value of d_1 when ρ_is increase. It is also noticed that the estimator d_{ML} seems to converge to the risk value of d_1 as the sample sizes and ρ_is increase. As the sample sizes increase the the risk values of all the estimators decrease. Also it can be noted that as $|\eta|$ increases, the risk values of all the estimators increase for fixed sample sizes. The following observations can be made from our simulation study and the Tables 4.2.1-4.2.3. We discuss separately, the case for $k = 3$ and $k = 4$.

1(a) Consider the case of equal sample sizes ($k = 3$).

For small sample sizes and for small values of ρ_i $(0 < \rho_i < 0.5)$, the estimator d_{ML} performs the best, and the percentage of relative risk improvement is noticed as high as 33.50%. For $\rho_i = 1$, the estimator d_{ML} has the highest percentage of relative risk improvement and it is around 21.74%. When the values of ρ_i are in the neighborhood of 1, the estimator d_{ML} seems to compete with either d_{MK} or d_{GM}. For large values of ρ_i none of the estimators perform well, even worse than d_1.

For moderate to large sample sizes, and for small values of ρ_i, the estimator d_{ML} has the best percentage of relative risk performance. It is noticed as high as 34.75%. For values of $\rho_i = 1$, the estimator d_{ML} performs the best. The percentage of relative risk performance is noticed as high as 23.35%. For values of ρ_i in the neighborhood of 1, the estimator d_{MK} and d_{ML} compete each other. For large values of ρ_i, the estimator d_{NH} seems to compete with d_{ML}. In this case, other estimators such as d_{GM} and d_{MK} does not even perform better than d_1.

2(a) Consider the case of unequal sample sizes ($k = 3$).

For small sample sizes and for small values of ρ_i $(0 < \rho_i < 0.5)$, the estimators d_{NH} and d_{SZ} compete with d_{ML}. For $\rho_i = 1$, the estimator d_{GM} has the highest percentage of relative risk performance and it is noticed around 24.63%. For values of ρ_i in the neighborhood of 1, the estimator d_{ML} compete with either d_{MK} or d_{TK}. For large values of ρ_i, the estimators d_{NH}, d_{ML} and d_{SZ} perform equally well.

For moderate to large sample sizes, and for small values of ρ_i $(0 < \rho_i < 0.5)$, the estimator d_{ML} seems to compete with either d_{NH} or d_{SZ}. For $\rho_i = 1$, the estimator d_{GM} has the best performance and the percentage of relative risk improvement is noticed around 19.27%. For values of ρ_i in the neighborhood of 1, the estimator d_{ML} compete with either d_{MK} or d_{TK}. For large values of ρ_i the estimators d_{NH}, d_{SZ} and d_{ML} are good competitors of each other. In this case other estimators do not even perform better than d_1.

1(b) Consider that the sample sizes are equal ($k = 4$).

For small sample sizes and for small values of ρ_i $(0 < \rho_i < 0.5)$, the estimator d_{ML} compete well with d_{NH}. When the values of ρ_i, are in the neighborhood of 1, the estimator d_{ML} seems to compete with d_{TK}. For large values of ρ_i, the estimators d_{NH} and d_{ML} perform equally well. However as the ρ_is increase further none of the estimators perform well, even does not compete with d_1.

For moderate to large sample sizes, and for small values of ρ_i $(0 < \rho_i < 0.5)$, the estimator d_{ML} seems to compete with d_{NH}. For values of ρ_i in the neighborhood of 1, the estimator d_{ML} seems to compete with either d_{MK} or d_{NH}. For large values of ρ_i, the estimator d_{NH} and d_{ML} perform equally well, whereas other estimators do not perform better than d_1.

2(b) Consider that the sample sizes are unequal ($k = 4$).

For small sample sizes, and for small values of ρ_i ($0 < \rho_i < 0.5$), the estimator d_{ML} seems to compete with either d_{NH} or d_{SZ}. For moderate values of ρ_i, that is in the neighborhood of 1, the estimators d_{MK} and d_{TK} compete with d_{ML}. For large values of ρ_i, none of the estimators perform better than d_1.

For moderate to large sample sizes, and for small values of ρ_i ($0 < \rho_i < 0.5$), the estimator d_{ML} performs the best. For $\rho_i = 1$, the estimator d_{TK} performs the best and the percentage of relative risk performances is noticed as high as 24.45%. For values of ρ_i in the neighborhood of 1, the estimators d_{SZ} and d_{TK} compete with each other. For large values of ρ_i, the estimators d_{SZ}, d_{ML} and d_{NH} compete each other.

From the above discussion, we conclude the following regarding the use of estimators for practical purposes.

(i) It is interesting to note that, none of the estimators for estimating quantiles θ, out perform in terms of risk values for all values of parameters with respect to the loss (4.2.1). It has been observed that as the values of $|\eta|$ increases the percentage of relative risk performances of all the estimators decreases.

(ii) Consider the case of equal sample sizes. For small sample sizes and for small values of ρ_i ($0 < \rho_i < 0.5$), we recommend to use either d_{ML} or d_{NH}. For moderate values of ρ_i ($0.5 < \rho_i < 1.5$), the estimator d_{MK} or d_{ML} may be used. For particular cases $k = 2, 3$ and $\rho_i = 1$, the estimator d_{GM} or d_{ML} may be used preferably. When the number of populations increase that is when $k > 3$, the performance of the estimator d_{GM} decreases and even perform badly for the case $\rho_i = 1$. For large values of ρ_i (> 3), none of the estimators perform better than d_1.

For moderate to large sample sizes and when the values of ρ_is are either small ($0 < \rho_i < 0.5$) or large ($\rho_i > 3$) we recommend using either d_{ML} or d_{NH}, whereas for moderate values of ρ_i, the estimator d_{MK} or d_{ML} may be used.

(iii) Consider the case of unequal sample sizes. For small sample sizes and for small values of ρ_i ($0 < \rho_i < 0.5$), we recommend using either d_{NH} or d_{SZ} (for $k = 3$). Also it has been noticed that, for the case $k > 3$ the estimator d_{ML} compete well with either d_{SZ} or d_{NH}. So any one of the estimator can be used. For moderate values of ρ_i ($0.5 < \rho_i < 1.5$), we recommend using either d_{MK}, d_{TK} or d_{ML} except for the case $k = 2, 3$ and $\rho_i = 1$, where d_{GM} can be preferred. For large values of ρ_i, none of the estimators perform better than d_1.

For moderate to large sample sizes, and for small values of ρ_i one may use the estimator d_{ML}. For large values of ρ_i we recommend to use any one of the three estimators d_{SZ}, d_{ML} or d_{NH} as they perform equally well. For moderate values of ρ_i ($0.5 < \rho_i < 1.5$), we recommend using either d_{ML} or d_{SZ}. For particular cases, $k = 2, 3$ and for $\rho_i = 1$, one may use d_{GM}.

(iv) A similar type of observations were made for other combinations of sample sizes and η. It has also been noticed that, as the value of $|\eta|$ increases, the risk values increase and the percentage of relative risk performances of all the estimators with respect to d_1 decrease. The results of simulation study also validate the theoretical findings obtained in Section 4.2.1 .

4.2.4 Conclusions

In this section, we have considered the problem of estimating quantiles $\theta = \mu + \eta\sigma_1$, of the first normal population, when other $k - 1$, $(k \geq 2)$ normal populations are available with a common mean μ and different variances σ_i^2; $i = 1, 2, \ldots, k$. It should be noted that, for the case $k = 2$, the problem has been well investigated by Kumar and Tripathy (2011). We have generalized some of their decision theoretic results (Theorems 2.1, 3.1 and 4.1 in Kumar and Tripathy (2011)) to a general $k(\geq 2)$ normal populations. Many of their results is a particular case of our results, which is a major contribution to the literature. We have made an attempt to numerically compare the risk values of some of our proposed estimators for the case $k = 3$ and $k = 4$. We also recommend using these estimators in practice in certain situations. Next we consider two examples below to illustrate the use of estimators for quantiles θ.

Example 4.2.1 *We have taken the data reported in Meier (1953) and analyzed by Jordan and Krishnamoorthy (1996). The data are about the percentage of albumin in plasma protein in human subjects. From their paper, we have* $n_1 = 12$, $n_2 = 15$, $n_3 = 7$, $n_4 = 16$, $\bar{x}_1 = 62.3$, $\bar{x}_2 = 60.3$, $\bar{x}_3 = 59.5$, $\bar{x}_4 = 61.5$ *and* $s_1^2 = 142.846$, $s_2^2 = 109.76$, $s_3^2 = 200.598$, *and* $s_4^2 = 277.695$. *It seems that the variances are not significantly different from each other. Let us choose* $\eta = 3.0$ *The various estimates for quantiles are computed as* $d_1 = 72.86823$, $d_{NH} = 71.56313$, $d_{SZ} = 71.56824$, $d_{MK} = 71.54019$, $d_{TK} = 71.53802$, $d_{GM} = 71.62023$, $d_{ML} = 71.54409$. *In this situation we recommend to use* d_{ML}.

Example 4.2.2 *The second data set are taken from Eberhardt et al. (1989), who reported the data on selenium in non-fat milk power by combining the results of four independent measurement methods. From their paper we have,* $n_1 = 8$, $n_2 = 12$, $n_3 = 14$, $n_4 = 8$, $\bar{x}_1 = 105.00$, $\bar{x}_2 = 109.75$, $\bar{x}_3 = 109.50$, $\bar{x}_4 = 113.25$ *and* $s_1^2 = 599.977$, $s_2^2 = 228.228$, $s_3^2 = 35.477$, *and* $s_4^2 = 235.48$. *Let us choose* $\eta = 3.0$ *The various estimates for quantiles are computed as* $d_1 = 131.8028$, $d_{NH} = 136.4049$, $d_{SZ} = 136.393$, $d_{MK} = 136.4621$, $d_{TK} = 136.4641$, $d_{GM} = 136.2314$, $d_{ML} = 136.3778$. *This is the case where the variances differ significantly and the sample sizes are small. It is clearly seen that none of the estimators are good in comparison to* d_1.

Table 4.2.1: Relative risk performances of various estimators for quantiles with $\eta = 1.960$

$(n_1, n_2, n_3) \rightarrow$		(10,10,10)				(30,30,30)			
$\rho_2 \downarrow$	$\rho_3 \downarrow$	$PR1$	$PR3$	$PR5$	$PR6$	$PR1$	$PR3$	$PR5$	$PR6$
0.25	0.25	30.98	30.84	28.19	31.76	31.81	31.61	28.63	33.86
	0.75	29.41	28.81	25.69	31.64	31.82	31.15	27.82	31.38
	1.00	30.75	29.79	25.14	30.76	31.83	30.96	26.26	31.22
	1.25	30.61	29.37	22.75	30.99	31.38	30.19	23.25	33.37
	2.00	30.27	28.65	13.65	30.61	32.89	31.54	16.10	31.28
	3.00	29.36	27.56	-5.09	31.30	31.50	29.55	-4.99	31.78
	4.00	30.63	28.56	-29.64	31.03	31.94	30.14	-30.33	34.16
0.75	0.25	30.85	30.15	26.93	32.03	31.99	31.10	27.41	31.81
	0.75	22.67	23.31	23.22	24.61	26.75	26.88	26.53	26.87
	1.00	22.11	22.63	22.28	21.34	25.43	25.57	25.21	24.70
	1.25	23.24	23.75	22.81	21.63	23.17	22.89	21.49	26.76
	2.00	20.06	19.34	11.54	21.28	23.51	22.42	14.69	22.97
	3.00	20.07	18.37	-5.23	20.65	21.41	18.88	-6.64	22.58
	4.00	19.79	17.29	-30.76	20.64	21.31	18.15	-31.82	20.39
1.00	0.25	30.55	29.43	24.69	29.49	31.53	30.54	25.59	31.33
	0.75	22.74	23.45	23.34	22.99	25.03	25.13	24.74	23.82
	1.00	18.82	20.16	20.74	21.02	20.93	21.39	21.59	23.35
	1.25	18.38	19.74	19.99	17.20	21.47	21.85	21.63	20.23
	2.00	15.97	16.23	11.13	15.78	17.95	17.12	10.98	19.00
	3.00	15.44	13.86	-7.29	14.83	17.80	15.48	-6.53	17.98
	4.00	14.53	11.45	-34.26	15.90	17.16	13.48	-34.54	20.24
1.25	0.25	30.66	29.56	23.31	30.21	31.30	30.01	23.02	31.35
	0.75	20.81	21.32	20.46	20.73	24.55	24.36	23.071	24.43
	1.00	18.89	20.07	20.11	16.84	21.11	21.32	20.94	19.03
	1.25	16.79	18.15	18.31	15.86	17.24	17.66	17.47	16.12
	2.00	14.09	14.41	10.06	11.93	14.36	13.55	08.36	14.12
	3.00	11.30	09.33	-10.40	12.55	14.31	11.88	-8.70	13.09
	4.00	10.54	07.99	-33.22	11.54	14.09	10.04	-35.59	13.82
2.00	0.25	31.01	29.59	15.34	29.55	31.73	30.24	14.46	30.90
	0.75	20.15	19.65	12.66	19.10	22.27	21.13	13.22	21.10
	1.00	16.06	15.99	10.29	15.00	18.76	18.15	12.38	17.60
	1.25	13.21	13.47	08.96	12.20	15.07	14.48	09.59	14.52
	2.00	08.42	07.48	00.62	08.70	11.69	09.41	01.45	09.50
	3.00	04.81	01.27	-18.56	07.50	08.82	03.72	-18.27	06.75
	4.00	05.66	-1.11	-43.86	05.21	07.43	00.24	-44.79	04.73
3.00	0.25	31.11	29.40	-3.55	31.91	31.43	29.65	-4.90	31.18
	0.75	21.21	18.94	-5.57	19.82	21.83	19.11	-6.71	21.89
	1.00	17.13	15.68	-4.90	14.82	17.20	15.00	-7.65	18.35
	1.25	11.77	10.26	-8.37	12.46	13.51	11.16	-9.17	13.40
	2.00	04.30	00.98	-18.44	05.26	09.19	04.47	-17.12	09.27
	3.00	01.72	-6.14	-37.39	03.88	05.44	-2.80	-36.62	06.86
	4.00	03.19	-7.98	-60.18	03.91	05.25	-6.57	-63.18	03.20
4.00	0.25	29.81	27.81	-30.98	30.27	32.84	30.88	-29.45	30.87
	0.75	19.36	16.86	-30.44	19.89	22.21	18.89	-33.16	20.77
	1.00	15.13	12.17	-33.17	16.15	17.30	13.83	-33.47	16.67
	1.25	10.64	07.43	-36.25	10.33	12.98	08.92	-35.91	12.65
	2.00	04.19	-2.26	-45.33	05.19	06.19	-0.48	-43.94	07.98
	3.00	03.17	-7.40	-57.77	05.01	03.21	-8.25	-63.68	04.36
	4.00	-0.65	-16.46	-90.18	02.66	04.70	-10.73	-88.25	04.53

Table 4.2.2: Relative risk performances of various estimators for quantiles with $\eta = 1.960$

$(n_1, n_2, n_3) \rightarrow$		(10,15,20)					
$\rho_2 \downarrow$	$\rho_3 \downarrow$	$PR1$	$PR2$	$PR3$	$PR4$	$PR5$	$PR6$
0.25	0.25	31.51	31.52	31.42	31.42	30.30	31.47
	0.75	29.83	29.84	29.36	29.36	27.62	31.17
	1.00	32.31	32.31	31.70	31.71	28.81	31.71
	1.25	32.59	32.59	31.75	31.75	27.05	31.57
	2.00	31.27	31.27	30.08	30.08	17.72	30.60
	3.00	32.67	32.68	31.49	31.49	04.41	30.28
	4.00	32.12	32.12	31.03	31.04	-18.60	30.54
0.75	0.25	31.58	31.58	31.14	31.15	29.63	31.29
	0.75	27.12	27.17	27.08	27.10	27.33	28.75
	1.00	26.75	26.79	26.91	26.92	26.85	27.13
	1.25	25.03	25.07	25.28	25.29	24.57	25.39
	2.00	23.74	23.77	23.36	23.36	16.61	24.32
	3.00	23.70	23.76	22.41	22.40	01.28	22.74
	4.00	22.74	22.80	20.75	20.73	-22.09	22.64
1.00	0.25	31.03	31.04	30.33	30.34	27.80	32.06
	0.75	25.44	25.51	25.33	25.35	25.57	25.51
	1.00	24.08	24.10	24.43	24.44	24.63	24.13
	1.25	23.72	23.75	24.34	24.35	24.33	22.15
	2.00	21.14	21.16	21.43	21.42	16.11	19.91
	3.00	19.54	19.58	18.85	18.82	-0.09	18.92
	4.00	18.86	18.88	17.72	17.68	-21.41	19.08
1.25	0.25	31.98	31.98	31.30	31.31	27.77	31.23
	0.75	27.21	27.26	26.84	26.86	26.36	24.96
	1.00	22.58	22.63	23.04	23.05	23.26	23.02
	1.25	20.24	20.27	21.11	21.12	21.13	21.61
	2.00	17.34	17.32	18.27	18.25	14.17	17.46
	3.00	16.62	16.61	16.59	16.55	-0.20	16.30
	4.00	16.54	16.49	15.28	15.22	-23.12	14.93
2.00	0.25	30.00	30.01	28.94	28.96	19.42	30.41
	0.75	24.06	24.13	23.22	23.24	18.84	24.95
	1.00	21.60	21.66	20.96	20.97	17.29	21.17
	1.25	17.81	17.91	18.31	18.32	15.97	18.25
	2.00	13.75	13.59	14.13	14.08	08.40	13.94
	3.00	10.43	10.19	09.52	09.43	-7.63	10.15
	4.00	08.36	08.16	06.15	06.03	-31.03	09.31
3.00	0.25	32.59	32.60	31.32	31.34	09.78	30.53
	0.75	24.39	24.47	22.34	22.37	06.37	24.95
	1.00	21.41	21.47	19.81	19.82	05.99	20.94
	1.25	18.35	18.38	17.17	17.17	04.19	17.01
	2.00	10.92	10.76	09.81	09.76	-3.84	10.73
	3.00	07.95	07.53	04.08	03.94	-21.16	06.57
	4.00	06.58	06.27	01.56	01.38	-40.77	04.56
4.00	0.25	31.55	31.55	30.31	30.33	-8.76	30.65
	0.75	25.87	25.93	23.34	23.37	-8.89	24.40
	1.00	22.18	22.25	19.79	19.81	-10.39	20.17
	1.25	17.38	17.35	15.46	15.45	-12.30	16.16
	2.00	10.91	10.71	08.24	08.17	-20.11	08.95
	3.00	05.61	05.29	00.64	00.48	-35.83	05.31
	4.00	02.65	02.24	-5.46	-5.70	-61.06	03.28

Table 4.2.3: Relative risk performances of various estimators for quantiles with $\eta = 1.960$

$(n_1, n_2, n_3) \rightarrow$		(40,30,20)					
$\rho_2 \downarrow$	$\rho_3 \downarrow$	$PR1$	$PR2$	$PR3$	$PR4$	$PR5$	$PR6$
0.25	0.25	31.57	31.56	31.31	31.31	25.99	32.05
	0.75	31.59	31.59	30.48	30.49	24.93	31.04
	1.00	32.05	32.05	30.89	30.90	24.65	29.75
	1.25	31.38	31.38	30.04	30.05	22.25	32.17
	2.00	31.05	31.04	28.85	28.86	12.65	31.33
	3.00	31.74	31.74	29.63	29.64	-2.75	31.86
	4.00	30.47	30.47	28.44	28.45	-26.09	31.94
0.75	0.25	30.80	30.80	30.07	30.06	24.53	32.25
	0.75	22.52	22.53	22.82	22.83	22.11	20.52
	1.00	21.65	21.67	21.59	21.60	21.37	20.48
	1.25	21.75	21.77	21.06	21.08	20.26	20.43
	2.00	20.54	20.54	18.53	18.56	12.43	17.72
	3.00	18.19	18.17	14.56	14.61	-7.26	20.04
	4.00	19.39	19.38	15.24	15.30	-27.92	18.14
1.00	0.25	29.86	29.85	28.86	28.84	21.65	29.39
	0.75	20.41	20.42	20.86	20.86	20.35	20.21
	1.00	18.57	18.61	18.64	18.66	19.27	18.27
	1.25	16.94	16.96	16.48	16.51	16.97	16.95
	2.00	15.68	15.70	13.55	13.59	09.73	14.34
	3.00	15.08	15.10	10.54	10.60	-8.12	14.23
	4.00	14.66	14.67	09.44	09.52	-30.19	12.69
1.25	0.25	30.80	30.80	29.66	29.64	20.40	31.06
	0.75	20.02	20.02	20.29	20.29	18.73	19.11
	1.00	16.80	16.83	16.69	16.70	16.75	16.47
	1.25	14.35	14.37	13.91	13.93	14.42	15.04
	2.00	12.16	12.22	08.86	08.92	05.61	12.31
	3.00	11.18	11.21	05.66	05.74	-11.04	11.67
	4.00	10.28	10.29	03.73	03.82	-34.69	09.16
2.00	0.25	29.14	29.13	27.15	27.13	05.39	30.05
	0.75	16.08	16.08	14.69	14.69	04.06	17.60
	1.00	12.93	12.94	11.37	11.38	03.37	14.03
	1.25	09.52	09.58	07.06	07.09	01.24	09.29
	2.00	07.80	07.85	01.96	02.03	-5.55	08.69
	3.00	05.39	05.43	-3.71	-3.60	-23.08	06.14
	4.00	07.27	07.32	-5.39	-5.24	-46.02	05.40
3.00	0.25	31.42	31.40	29.05	29.02	-16.60	29.81
	0.75	16.34	16.34	13.70	13.69	-19.31	14.38
	1.00	12.19	12.23	08.22	08.23	-21.63	12.34
	1.25	08.30	08.34	03.16	03.19	-24.54	10.53
	2.00	04.32	04.39	-5.49	-5.40	-31.68	03.88
	3.00	03.28	03.33	-11.15	-11.01	-47.51	04.42
	4.00	04.35	04.37	-13.39	-13.22	-69.69	02.18
4.00	0.25	30.10	30.09	27.47	27.45	-52.39	30.05
	0.75	15.24	15.25	11.38	11.38	-54.65	15.03
	1.00	11.74	11.75	06.76	06.77	-55.75	11.24
	1.25	07.76	07.81	01.08	01.12	-58.05	09.97
	2.00	03.68	03.79	-9.54	-9.44	-66.89	04.17
	3.00	02.85	02.90	-15.15	-14.99	-81.77	04.76
	4.00	02.81	02.84	-19.69	-19.49	-106.28	04.63

4.3 Estimating Quantile Vector in Several Normal Populations with a Common Mean

In the previous section (Section 4.2), we have considered the estimation of quantiles of the first population when the other $k-1$ normal populations with a common mean are available. In this section we consider the same model and estimate the quantile vector $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$; $\theta_i = \mu + \eta\sigma_i$, where $\eta = \Phi^{-1}(p)$; $0 < p < 1$. Here $\Phi(.)$ denotes the cumulative distribution function of a standard normal random variable. The loss function is taken as the sum of quadratic losses given by,

$$L(\underset{\sim}{d}, \underset{\sim}{\theta}) = \sum_{i=1}^{k} \Big(\frac{d_i - \theta_i}{\sigma_i}\Big)^2, \tag{4.3.1}$$

where $\underset{\sim}{d} = (d_1, d_2, \ldots, d_k)$ is an estimator of $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$.

4.3.1 A General Result and Some Improved Estimators

Suppose $(X_{i1}, X_{i2}, \ldots, X_{in_i})$; $i = 1, 2, \ldots, k$ be independent random samples taken from $k(\geq 2)$ normal populations $N(\mu, \sigma_i^2)$. We observe that a minimal sufficient statistic for this model exists and is given by $(\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_k, S_1^2, S_2^2, \ldots, S_k^2)$ where

$$\bar{X}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} X_{ij},\ S_i^2 = \sum_{j=1}^{n_i}(X_{ij} - \bar{X}_i)^2;\ i = 1, 2, \ldots, k.$$

We note that the maximum likelihood estimator (MLE) for μ, is not obtainable in closed form (see Pal et al. (2007) for the case $k = 2$). Also the minimal sufficient statistics for this problem are not complete, hence the usual approaches to find uniformly minimum variance unbiased estimator (UMVUE) for individual quantile do not work as ancillary statistics may carry relevant information for the parameter of interest. Therefore, it is not known if a UMVUE exists or not, and it is difficult to find even if one exists. Further, it is known that when we have only one population (say $\underset{\sim}{X}$) the best affine equivariant estimator for estimating quantile $\theta_1 = \mu + \eta\sigma_1$ is minimax (see Kiefer (1957)). When we have two populations the problem of estimating the first component θ_1 has been considered by Kumar and Tripathy (2011). Also in the Section 4.2, we have estimated the first component θ_1, when $k(\geq 2)$ populations are available, which generalizes the results of Kumar and Tripathy (2011). Following their arguments, a natural way to construct improved estimators for $\underset{\sim}{\theta}$ is to combine the improved estimators for the common mean and the improved estimators for the respective standard deviations. We first propose a basic estimator for $\underset{\sim}{\theta}$ as,

$$\underset{\sim}{d} = (d_1, d_2, \ldots, d_k),$$

where $d_i = \bar{X}_1 + cS_i$; $i = 1, 2, \ldots$ and $c \in \mathbb{R}$.

Let us define

$$C_n = \frac{\eta\sqrt{2}}{n-k}\sum_{i=1}^{k}\left[\frac{\Gamma(\frac{n_i}{2})}{\Gamma(\frac{n_i-1}{2})}\right]. \tag{4.3.2}$$

where we denote $n = \sum_{i=1}^{k} n_i$.

Theorem 4.3.1 *If we estimate the quantile vector $\underset{\sim}{\theta}$ by $\underset{\sim}{d} = (\bar{X}_1 + cS_1, \bar{X}_1 + cS_2, \ldots, \bar{X}_1 + cS_k)$ with respect to the loss function (4.3.1), then the value of c that minimizes the risk is obtained as C_n.*

Proof 4.3.1 *To prove the theorem, let us consider the risk function of $\underset{\sim}{\theta}$, with respect to the loss (4.3.1):*

$$R(\underset{\sim}{d}, \underset{\sim}{\theta}) = \sum_{i=1}^{k} E\left(\frac{d_i - \theta_i}{\sigma_i}\right)^2.$$

The above risk is a convex function of c and hence its minimizing value is obtained by differentiating with respect to c and equating to zero, and is given by

$$c = \frac{\eta \sum_{i=1}^{k} E(S_i/\sigma_i)}{\sum_{i=1}^{k} E(S_i^2/\sigma_i^2)}.$$

Further we note that, $S_i^2/\sigma_i^2 \sim \chi^2_{n_i-1}; i = 1, 2, \ldots, k$ hence $E(S_i^2/\sigma_i^2) = n_i - 1$ and $E(S_i/\sigma_i) = \frac{\sqrt{2}\Gamma(n_i/2)}{\Gamma((n_i-1)/2)}$. Substituting all these values we obtain the minimizing choice after simplification as,

$$c = \frac{\eta\sqrt{2}}{n-k} \sum_{i=1}^{k} \left[\frac{\Gamma(\frac{n_i}{2})}{\Gamma(\frac{n_i-1}{2})}\right] = C_n \ (say).$$

This completes the proof of the theorem.

Remark 4.3.1 *The above result in Theorem 4.3.1 will remain unchanged if $\bar{X}_1$ is replaced by any one of the $\bar{X}_i$; $i = 1, 2, \ldots, k$.*

Let us denote $\underset{\sim}{d}^{\bar{X}_1} = (\bar{X}_1 + C_n S_1, \bar{X}_1 + C_n S_2, \ldots, \bar{X}_1 + C_n S_k)$. Next, we give a general result which in parallel to Theorem 2.1 of Kumar and Tripathy (2011) that valid for estimating only θ_1.

Theorem 4.3.2 *Suppose $\underset{\sim}{d}_M = (d_M, d_M, \ldots, d_M)_k$ be an estimator for $\underset{\sim}{\mu} = (\mu, \mu, \ldots, \mu)_k$, and $\underset{\sim}{d}_S = (d_{S_1}, d_{S_2}, \ldots, d_{S_k})$ be an estimator for $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_k)$. Consider $\underset{\sim}{d}_Q = (d_{Q1}, d_{Q2}, \ldots, d_{Qk}) = \underset{\sim}{d}_M + \eta \underset{\sim}{d}_S$ as an estimator for $\underset{\sim}{\theta}$. Further, assume that given $\underset{\sim}{d}_S = (d_{S_1}, d_{S_2}, \ldots, d_{S_k})$, the estimator d_M is conditionally unbiased for μ, that is*

$$E(d_M | d_{S_i}) = \mu, \tag{4.3.3}$$

for $i = 1, 2 \ldots, k$, then,

$$E\sum_{i=1}^{k}(d_{Qi} - \theta_i)^2 = kE(d_M - \mu)^2 + \eta^2 \sum_{i=1}^{k} E(d_{S_i} - \sigma_i)^2. \tag{4.3.4}$$

Proof 4.3.2 *The proof is similar to the arguments used in proving Theorem 2.1 of Kumar and Tripathy (2011), hence omitted.*

Remark 4.3.2 *It is easy to observe that, condition (4.3.4) will satisfy if we choose d_M to be an unbiased estimator for μ and $d_{S_1}, d_{S_2}, \ldots, d_{S_k}$ are independent of d_M. For example we may take $d_M = \bar{X}_1$ and $d_{S_i} = S_i$, $i = 1, 2, \ldots, k$.*

Remark 4.3.3 *As a consequence of Theorem 4.3.2, to construct a good estimator for $\underset{\sim}{\theta}$, it is sufficient to have a good estimator for μ and/or a good estimator for σ_i, $i = 1, 2, \ldots, k$.*

Remark 4.3.4 *Let $d_M = d_\phi$, where $d_\phi = \sum_{i=1}^{k} \phi_i(S_1, S_2, \ldots, S_k)\bar{X}_i$ be any unbiased estimator for μ, such that $\sum_{i=1}^{k} \phi_i(S_1, S_2, \ldots, S_k) = 1$ and $d_{S_i} = cS_i/\eta$, ($\eta \neq 0$),$i = 1, 2, \ldots, k$. It is easy to see that, the condition of Theorem 4.3.2 satisfies and we prove the following result.*

Theorem 4.3.3 *Let $d_\phi = \sum_{i=1}^{k} \phi_i(S_1, S_2, \ldots, S_k)\bar{X}_i$ such that $\sum_{i=1}^{k} \phi_i(S_1, S_2, \ldots, S_k) = 1$ be an estimator for the common mean μ. Consider the estimator $\underset{\sim}{d}_\phi(c) = (d_\phi + cS_1, d_\phi + cS_2, \ldots, d_\phi + cS_k)$ for estimating quantile vector $\underset{\sim}{\theta}$. Then $\underset{\sim}{d}_\phi(c)$ has smaller risk than $\underset{\sim}{d}$ with respect to the sum of quadratic loss (4.3.1) if and only if d_ϕ has smaller risk than $\bar{X}_1$. Further, $\underset{\sim}{d}_\phi(c)$ has minimum risk with respect to the loss (4.3.1) when $c = C_n$.*

Proof 4.3.3 *The proof of first part of the result is very much similar to the proof of Theorem 4.2.2. The second part follows from Theorem 4.3.1 by replacing $\bar{X}_1$ with d_ϕ and the fact that d_ϕ is conditionally unbiased.*

We note that the minimizing choice of c is C_n which is symmetric in n_i $i = 1, 2, \ldots, k$. One may construct an estimator for the quantile $\underset{\sim}{\theta}$ using any one of the $\bar{X}_i$ for the common mean. Let us denote $\underset{\sim}{d}^* = (\bar{X}_i + cS_1, \bar{X}_i + cS_2, \ldots, \bar{X}_i + cS_k)$, $i = 2, 3, \ldots, k$. The results of the Theorem 4.3.3 will remain true if we replace $\underset{\sim}{d}$ by $\underset{\sim}{d}^*$. Hence we have the following remark.

Remark 4.3.5 *Let $d_\phi = \sum_{i=1}^{k} \phi_i(S_1, S_2, \ldots, S_k)\bar{X}_i$; $\sum_{i=1}^{k} \phi_i = 1$ be an estimator for the common mean μ. Consider the estimator $\underset{\sim}{d}_\phi(c) = (d_\phi + cS_1, d_\phi + cS_2, \ldots, d_\phi + cS_k)$ for estimating quantile vector $\underset{\sim}{\theta}$. Then $\underset{\sim}{d}_\phi(c)$ has smaller risk than $\underset{\sim}{d}^*$ with respect to the sum of quadratic loss (4.3.1) if and only if d_ϕ has smaller risk than $\bar{X}_i$. Further, $\underset{\sim}{d}_\phi(c)$ has minimum risk with respect to the loss (4.3.1) when $c = C_n$. Let us denote $\underset{\sim}{d}^{\bar{X}_i} = (\bar{X}_i + C_nS_1, \bar{X}_i + C_nS_2, \ldots, \bar{X}_i + C_nS_k)$; $i = 2, 3, \ldots, k$.*

Remark 4.3.6 *Following the Theorem 4.3.3, one can easily construct good estimators for $\underset{\sim}{\theta}$ by replacing $\bar{X}_1$ in $\underset{\sim}{d}^{\bar{X}_1}$ or $\bar{X}_i$ in $\underset{\sim}{d}^{\bar{X}_i}$ by any improved estimator of the form d_ϕ for the common mean μ.*

To emphasize the case $k = 2$ and $k(\geq 3)$ we construct the following estimators for $k = 2$ and $k(\geq 3)$ separately. Following the above remarks and Theorem 4.3.2, we first propose the following estimators for $\underset{\sim}{\theta}$ taking $k = 2$, which have smaller risk than $\underset{\sim}{d}^{\bar{X}_1}$ or/and $\underset{\sim}{d}^{\bar{X}_2}$ under certain conditions on the sample sizes. For $k = 2$, we have $n = n_1 + n_2$, so that $C_n = C_{n_1+n_2}$.

$$
\begin{aligned}
\underset{\sim}{d}^{GM} &= (\hat{\mu}_{GM} + C_nS_1, \hat{\mu}_{GM} + C_nS_2), \\
\underset{\sim}{d}^{GD} &= (\hat{\mu}_{GD} + C_nS_1, \hat{\mu}_{GD} + C_nS_2), \\
\underset{\sim}{d}^{KS} &= (\hat{\mu}_{KS} + C_nS_1, \hat{\mu}_{KS} + C_nS_2), \\
\underset{\sim}{d}^{CS} &= (\hat{\mu}_{CS} + C_nS_1, \hat{\mu}_{CS} + C_nS_2), \\
\underset{\sim}{d}^{MK} &= (\hat{\mu}_{MK} + C_nS_1, \hat{\mu}_{MK} + C_nS_2), \\
\underset{\sim}{d}^{TK} &= (\hat{\mu}_{TK} + C_nS_1, \hat{\mu}_{TK} + C_nS_2), \\
\underset{\sim}{d}^{BC1} &= (\hat{\mu}_{BC1} + C_nS_1, \hat{\mu}_{BC1} + C_nS_2), \\
\underset{\sim}{d}^{BC2} &= (\hat{\mu}_{BC2} + C_nS_1, \hat{\mu}_{BC2} + C_nS_2).
\end{aligned}
$$

Here we denote $\hat{\mu}_{GM} = \frac{n_1\bar{X}_1+n_2\bar{X}_2}{n_1+n_2}$, $\hat{\mu}_{TK} = \frac{\sqrt{n_1}\ b_{n_2-1}S_2\bar{X}_1+\sqrt{n_2}\ b_{n_1-1}S_1\bar{X}_2}{\sqrt{n_1}\ b_{n_2-1}S_2+\sqrt{n_2}\ b_{n_1-1}S_1}$, and $\hat{\mu}_{GD}$, $\hat{\mu}_{KS}$, $\hat{\mu}_{BC1}$, $\hat{\mu}_{BC2}$, $\hat{\mu}_{CS}$, $\hat{\mu}_{MK}$, are estimators for the common mean μ, as defined in Tripathy and Kumar (2010), with $\bar{X} = \bar{X}_1$, $\bar{Y} = \bar{X}_2$, $m = n_1$, and $n = n_2$. Although the closed form of the MLE of μ is not available, one can obtain it numerically by solving a system of three equations in three unknowns. Let us denote $\hat{\mu}_{ML}$ as the MLE of the common mean. Using this estimator for the common mean we propose an estimator for the quantile vector $\underset{\sim}{\theta}$ as,

$$\underset{\sim}{d}^{ML} \quad = \quad (\hat{\mu}_{ML} + C_n S_1, \hat{\mu}_{ML} + C_n S_2).$$

All these estimators belong to the class $\underset{\sim}{d}_{\phi}(C_n)$ and will be compared numerically in Section 4.3.3.

Theorem 4.3.4 *Let the estimators* $\underset{\sim}{d}^{\bar{X}_1}$, $\underset{\sim}{d}^{\bar{X}_2}$, $\underset{\sim}{d}^{GD}$, $\underset{\sim}{d}^{KS}$, $\underset{\sim}{d}^{BC1}$, $\underset{\sim}{d}^{BC2}$, *and* $\underset{\sim}{d}^{CS}$ *as defined above for estimating* $\underset{\sim}{\theta}$. *The loss function be taken as the sum of the quadratic losses (4.3.1).*

(i) The estimator $\underset{\sim}{d}^{GD}$ *performs better than both* $\underset{\sim}{d}^{\bar{X}_1}$ *and* $\underset{\sim}{d}^{\bar{X}_2}$ *if and only if* $n_1, n_2 \geq 11$.

(ii) The estimator $\underset{\sim}{d}^{KS}$ *performs better than both* $\underset{\sim}{d}^{\bar{X}_1}$ *and* $\underset{\sim}{d}^{\bar{X}_2}$ *if and only if* $(n_1-7)(n_2-7) \geq 16$.

(iii) The estimator $\underset{\sim}{d}^{BC1}$ *performs better than* $\underset{\sim}{d}^{\bar{X}_1}$ *if and only if* $n_1 \geq 2$, $n_2 \geq 3$ *and for* $0 < b_1 < b_{\max}(n_1, n_2)$.

(iv) The estimator $\underset{\sim}{d}^{BC2}$ *performs better than* $\underset{\sim}{d}^{\bar{X}_1}$ *if and only if* $n_1 \geq 2$, $n_2 \geq 6$ *and for* $0 < b_2 < b_{\max}(n_1, n_2 - 3)$.

(v) The estimator $\underset{\sim}{d}^{CS}$ *performs better than* $\underset{\sim}{d}^{\bar{X}_1}$ *if* $n_1 = n_2 \geq 7$.

Here b_1, b_2 *and* $b_{\max}(n_1, n_2)$ *are as defined in Kumar and Tripathy (2011).*

Proof 4.3.4 *The proof of* (i)-(v) *can be done by using Theorem 4.3.3 and the arguments given in the proof of Theorem 2.4 in Kumar and Tripathy (2011).*

Remark 4.3.7 *The estimator* $\underset{\sim}{d}^{MK}$ *uses the estimator proposed by Moore and Krishnamoorthy (1997) that uses the estimates of standard deviation instead of variance. Their estimator does not improve upon* $\bar{X}_1$ *uniformly. The estimator* $\underset{\sim}{d}^{TK}$ *proposed by Tripathy and Kumar (2010), also does not improve upon* $\bar{X}_1$ *uniformly. As our numerical results shows (in Section 4.3.3), these two estimators perform quite well for moderate values of* $\sigma_2/\sigma_1 > 0$ *and also they are good competitor of each other.*

Next we propose the estimators for the general case of $k(\geq 3)$. The well known popular estimator for the common mean μ, proposed by Graybill and Deal (1959) for the case $k = 2$, has been extended to the case $k \geq 3$ by Norwood and Hinkelmann (1977). Using this estimator we propose

$$\underset{\sim}{d}^{NH} \quad = \quad (\hat{\mu}_{NH} + C_n S_1, \hat{\mu}_{NH} + C_n S_2, \ldots, \hat{\mu}_{NH} + C_n S_k),$$

where

$$\hat{\mu}_{NH} = \frac{\sum_{i=1}^{k} n_i(n_i - 1)\bar{X}_i/S_i^2}{\sum_{i=1}^{k} n_i(n_i - 1)/S_i^2}, \quad \text{Norwood and Hinkelmann (1977).}$$

$$\underset{\sim}{d}^{SZ} = (\hat{\mu}_{SZ} + C_n S_1, \hat{\mu}_{SZ} + C_n S_2, \ldots, \hat{\mu}_{SZ} + C_n S_k),$$

where

$$\hat{\mu}_{SZ} = \frac{\sum_{i=1}^{k} n_i(n_i - 3)\bar{X}_i/S_i^2}{\sum_{i=1}^{k} n_i(n_i - 3)/S_i^2}, \quad \text{Shinozaki (1978).}$$

$$\underset{\sim}{d}^{MK} = (\hat{\mu}_{MK} + C_n S_1, \hat{\mu}_{MK} + C_n S_2, \ldots, \hat{\mu}_{MK} + C_n S_k),$$

where

$$\hat{\mu}_{MK} = \frac{\sum_{i=1}^{k} \sqrt{n_i(n_i - 1)}\bar{X}_i/S_i}{\sum_{i=1}^{k} \sqrt{n_i(n_i - 1)}/S_i}, \quad \text{Moore and Krishnamoorthy (1997).}$$

$$\underset{\sim}{d}^{TK} = (\hat{\mu}_{TK} + C_n S_1, \hat{\mu}_{TK} + C_n S_2, \ldots, \hat{\mu}_{TK} + C_n S_k),$$

where

$$\hat{\mu}_{TK} = \frac{\sum_{i=1}^{k} \sqrt{n_i}\bar{X}_i/(b_{n_i-1}S_i)}{\sum_{i=1}^{k} \sqrt{n_i}/(b_{n_i-1}S_i)}, \quad \text{Tripathy and Kumar (2015).}$$

We propose the estimator based on the grand sample mean,

$$\underset{\sim}{d}^{GM} = (\hat{\mu}_{GM} + C_n S_1, \hat{\mu}_{GM} + C_n S_2, \ldots, \hat{\mu}_{GM} + C_n S_k),$$

where

$$\hat{\mu}_{GM} = \frac{\sum_{i=1}^{k} n_i \bar{X}_i}{\sum_{i=1}^{k} n_i}.$$

Finally we propose the estimator based on the MLE of μ which can be obtained by solving a system of $k+1$ non linear equations in $k+1$ variables $(\mu, \sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2)$. Let $\hat{\mu}_{ML}$ be the MLE for μ obtained by solving the system of equations. Using the MLE $\hat{\mu}_{ML}$ for the common mean μ, we propose an estimator for the quantile vector $\underset{\sim}{\theta}$ as,

$$\underset{\sim}{d}^{ML} = (\hat{\mu}_{ML} + C_n S_1, \hat{\mu}_{ML} + C_n S_2, \ldots, \hat{\mu}_{ML} + C_n S_k).$$

The following two theoretical comparisons are immediate, which follows directly from the results given in Norwood and Hinkelmann (1977) and Shinozaki (1978) where they have obtained the results for the common mean.

Theorem 4.3.5 *For estimating the quantile vector, $\underset{\sim}{\theta}$ with respect to the loss function (4.2.1), the estimator*

(i) $\underset{\sim}{d}^{NH}$ has smaller risk than $\underset{\sim}{d}^{\bar{X}_1}$ if and only if, the sample sizes $n_i \geq 11$ or one of the $n_i = 10$ and all other $n_j \geq 18$ where i is different from j.

(ii) $\underset{\sim}{d}^{SZ}$ *has smaller risk than* $\underset{\sim}{d}^{\bar{X}_1}$ *if and only if* $(n_1-1) \geq 7$ *and* $(n_1-7)(n_j-7) \geq 16$ *for any* $j \neq 1$.

Proof 4.3.5 *The proof is trivial after using Theorem 4.3.3 and the results of Norwood and Hinkelmann (1977) and Shinozaki (1978) for the common mean.*

Remark 4.3.8 *In Section 4.3.3, we carry out a detailed simulation study to numerically compare all these estimators for the case* $k = 2$ *and* $k = 3$, *which validate the theoretical results.*

4.3.2 Inadmissibility Results for Equivariant Estimators

In this subsection, we introduce the concept of invariance to the problem of estimating quantile vector and derive classes of affine and location equivariant estimators. Further sufficient conditions for improving estimators in these classes have been derived. Consequently some complete class results are also proved.

Consider the group $G_A = \{g_{a,b} : g_{a,b}(x) = ax + b, a > 0, b \in \mathbb{R}\}$ of affine transformations. Under the transformation, $\bar{X}_i \to a\bar{X}_i+b$, $S_i^2 \to a^2 S_i^2$, $\mu \to a\mu+b$, $\sigma_i^2 \to a^2\sigma_i^2$, and $\underset{\sim}{\theta} \to a\underset{\sim}{\theta}+b\underset{\sim}{e}$, where $\underset{\sim}{e} = (1, 1, \ldots, 1)_k$ and $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$, $\theta_i = \mu + \eta\sigma_i, i = 1, 2, \ldots, k$. The problem considered is invariant if we choose the loss function as sum of the affine invariant loss functions (4.3.1). Based on the sufficient statistics $(\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_k, S_1^2, S_2^2, \ldots, S_k^2)$, the form of an affine equivariant estimator for estimating the vector $\underset{\sim}{\theta}$ is obtained as,

$$\underset{\sim}{d}(a\bar{X}_1 + b, \cdots, a\bar{X}_k + b, a^2S_1^2, , \cdots, a^2S_k^2) = a\underset{\sim}{d}(\bar{X}_1, \cdots, \bar{X}_k, S_1^2, \cdots, S_k^2) + b\underset{\sim}{e}.$$

Now choosing $b = -a\bar{X}_1$ where $a = 1/S_1$, and simplifying we obtain the form of an affine equivariant estimator as,

$$\begin{aligned} \underset{\sim}{d}_{\underset{\sim}{\Psi}} &= (\bar{X}_1 + S_1\Psi_1(\underset{\sim}{T}, \underset{\sim}{R}), \bar{X}_1 + S_1\Psi_2(\underset{\sim}{T}, \underset{\sim}{R}), \ldots, \bar{X}_1 + S_1\Psi_k(\underset{\sim}{T}, \underset{\sim}{R})) \\ &= (d_{\psi_1}, d_{\psi_1}, \ldots, d_{\psi_k}) \end{aligned} \tag{4.3.5}$$

where $\underset{\sim}{T} = (T_2, T_3, \ldots, T_k)$, $\underset{\sim}{R} = (R_2, R_3, \ldots, R_k)$, $T_i = (\bar{X}_i - \bar{X}_1)/S_1$, and $R_i = S_i^2/S_1^2$; $i = 2, 3, \ldots, k$.

Let us consider the case $k = 2$ and derive the following inadmissibility results for equivariant estimators. Hence, as per the notation used above, for $k = 2$, we have $\underset{\sim}{T} = T_2$ and $\underset{\sim}{R} = R_2$. Specifically we obtain the form of an equivariant estimator for the case $k = 2$ as

$$\begin{aligned} (d_1(\bar{X}_1, \bar{X}_2, S_1^2, S_2^2), d_2(\bar{X}_1, \bar{X}_2, S_1^2, S_2^2)) &= (\bar{X}_1 + S_1\Psi_1(T_2, R_2), \bar{X}_1 + S_1\Psi_2(T_2, R_2)) \\ &= (d_{\Psi_1}, d_{\Psi_2}) \\ &= \underset{\sim}{d}_{\underset{\sim}{\Psi}} \text{ say,} \end{aligned} \tag{4.3.6}$$

where $T_2 = \frac{\bar{X}_2 - \bar{X}_1}{S_1}$ and $R_2 = \frac{S_2^2}{S_1^2}$. To proceed further let us denote, $M_1 = \min(t_2, 0)$, and $M_2 = \max(t_2, 0)$. For the affine equivariant estimator $\underset{\sim}{d}_{\underset{\sim}{\Psi}}$ we define the following functions.

$$\underset{\sim}{\Psi}^0 = (\min(\max(\Psi_1, M_1), M_2), \min(\max(\Psi_2, M_1), M_2)) \tag{4.3.7}$$

$$\underset{\sim}{\Psi}^1 = (\max\{M_1 + \eta C_{n_1+n_2}, \Psi_1\}, \max\{M_1 + \eta C_{n_1+n_2}\sqrt{r_2}, \Psi_2\}), \tag{4.3.8}$$

$$\underset{\sim}{\Psi}^2 = (\min\{M_2 + \eta C_{n_1+n_2}, \Psi_1\}, \min\{M_2 + \eta C_{n_1+n_2}\sqrt{r_2}, \Psi_2\}). \tag{4.3.9}$$

Here t_2 and r_2 denote the observed values of T_2 and R_2.

The following theorem gives a sufficient condition for improving estimators in the class of $\underset{\sim}{d}_{\underset{\sim}{\Psi}}$.

Theorem 4.3.6 *Let $\underset{\sim}{d}_{\underset{\sim}{\Psi}}$ be an affine equivariant estimator of the form (4.3.6) of the quantile vector $\underset{\sim}{\theta}$, and the loss function be sum of the quadratic loss (4.3.1) or the sum of squared errors. Let the functions $\underset{\sim}{\Psi}^0$, $\underset{\sim}{\Psi}^1$ and $\underset{\sim}{\Psi}^2$ be defined as in (4.3.7), (4.3.8) and (4.3.9) respectively. Let $\underset{\sim}{\alpha} = (\mu, \sigma_1^2, \sigma_2^2)$.*

(i) When $\eta = 0$, the estimator $\underset{\sim}{d}_{\underset{\sim}{\Psi}}$ is improved by $\underset{\sim}{d}_{\underset{\sim}{\Psi}^0}$ if $P_{\underset{\sim}{\alpha}}(\underset{\sim}{\Psi}^0 \neq \underset{\sim}{\Psi}) > 0$ for some choices of $\underset{\sim}{\alpha}$.

(ii) When $\eta > 0$, the estimator $\underset{\sim}{d}_{\underset{\sim}{\Psi}}$ is improved by $\underset{\sim}{d}_{\underset{\sim}{\Psi}^1}$ if $P_{\underset{\sim}{\alpha}}(\underset{\sim}{\Psi}^1 \neq \underset{\sim}{\Psi}) > 0$ for some choices of $\underset{\sim}{\alpha}$.

(iii) When $\eta < 0$, the estimator $\underset{\sim}{d}_{\underset{\sim}{\Psi}}$ is improved by $\underset{\sim}{d}_{\underset{\sim}{\Psi}^2}$ if $(\underset{\sim}{\Psi}^2 \neq \underset{\sim}{\Psi}) > 0$ for some choices of $\underset{\sim}{\alpha}$.

Proof 4.3.6 *To prove the theorem we use a result due to Brewster and Zidek (Brewster and Zidek (1974)). Consider the conditional risk function of $\underset{\sim}{d}_{\underset{\sim}{\Psi}}$ given (T_2, R_2) :*

$$\begin{aligned} R((\underset{\sim}{d}_{\underset{\sim}{\Psi}}, \underset{\sim}{\theta})|(T_2, R_2)) &= E\{L(\underset{\sim}{d}_{\underset{\sim}{\Psi}}, \underset{\sim}{\theta})|(T_2, R_2)\} \\ &= \frac{1}{\sigma_1^2} E\{(\bar{X} + S_1\Psi_1(T_2, R_2) - \mu - \eta\sigma_1)^2|(T_2, R_2) = (t_2, r_2)\} \\ &\quad + \frac{1}{\sigma_2^2} E\{(\bar{X} + S_1\Psi_2(T_2, R_2) - \mu - \eta\sigma_2)^2|(T_2, R_2) = (t_2, r_2)\}. \end{aligned} \tag{4.3.10}$$

The above risk function (4.3.10) is a sum of two convex functions in Ψ_1 and Ψ_2, which is a convex function. The minimizing choices of $\Psi_1(T_2, R_2)$ and $\Psi_2(T_2, R_2)$, are obtained respectively as,

$$\Psi_1(t_2, r_2) = -\frac{E\{(\bar{X} - \mu)S_1|(T_2, R_2)\}}{E(S_1^2|(T_2, R_2))} + \eta\sigma_1 \frac{E(S_1|(T_2, R_2))}{E(S_1^2|(T_2, R_2))}$$

and

$$\Psi_2(t_2, r_2) = -\frac{E\{(\bar{X} - \mu)S_1|(T_2, R_2)\}}{E(S_1^2|(T_2, R_2))} + \eta\sigma_2 \frac{E(S_1|(T_2, R_2))}{E(S_1^2|(T_2, R_2))}.$$

Using the conditional expectations derived in Kumar and Tripathy (2011), the minimizing choices for $\Psi_1(t_2, r_2)$ and $\Psi_2(t_2, r_2)$ are simplified and are given by

$$\Psi_1((t_2, r_2), \rho) = \frac{t_2}{1+\rho} + \eta C_{n_1+n_2}\sqrt{\lambda} \tag{4.3.11}$$

and

$$\Psi_2((t_2, r_2), \rho) = \frac{t_2}{1+\rho} + \eta C_{n_1+n_2}\sqrt{\frac{n_2\rho}{n_1}}\sqrt{\lambda}. \tag{4.3.12}$$

Here $\lambda = \frac{n_1 t_2^2}{1+\rho} + \frac{n_1 r_2}{n_2 \rho} + 1$, *and* $\rho = \frac{n_1 \sigma_2^2}{n_2 \sigma_1^2}$.

In order to prove the theorem, we need to find the infimum and supremum values of $\Psi_1(t_2, r_2, \rho)$ *and* $\Psi_2(t_2, r_2, \rho)$ *with respect to* $\rho > 0$, *for all values of* η *and* (t_2, r_2). *After analyzing the terms* $\Psi_1(t_2, r_2, \rho)$ *and* $\Psi_2(t_2, r_2, \rho)$, *for separate values of* η, *we have the following cases:*

(i) When $\eta = 0$, *and* $t_2 \in \mathbb{R}$,

$$\inf_{\rho} \Psi_1(t_2, r_2, \rho) = M_1 \ \textit{and} \ \sup_{\rho} \Psi_1(t_2, r_2, \rho) = M_2$$
$$\inf_{\rho} \Psi_2(t_2, r_2, \rho) = M_1 \ \textit{and} \ \sup_{\rho} \Psi_2(t_2, r_2, \rho) = M_2. \tag{4.3.13}$$

(ii) When $\eta > 0$, *and* $t_2 \in \mathbb{R}$, *we have*

$$\inf_{\rho} \Psi_1(t_2, r_2, \rho) \geq M_1 + \eta C_{n_1+n_2} \ (\textit{equality holds if } t_2 > 0)$$
$$\textit{and} \ \sup_{\rho} \Psi_1(t_2, r_2, \rho) = +\infty$$
$$\inf_{\rho} \Psi_2(t_2, r_2, \rho) \geq M_1 + \eta C_{n_1+n_2} \sqrt{r_2} (\textit{equality holds if } t_2 < 0)$$
$$\textit{and} \ \sup_{\rho} \Psi_2(t_2, r_2, \rho) = +\infty. \tag{4.3.14}$$

(iii) When $\eta < 0$, $t_2 \in \mathbb{R}$, *we have*

$$\sup_{\rho} \Psi_1(t_2, r_2, \rho) \leq M_2 + \eta C_{n_1+n_2} \ (\textit{equality holds if } t_2 < 0)$$
$$\textit{and} \ \inf_{\rho} \Psi_1(t_2, r_2, \rho) = -\infty$$
$$\sup_{\rho} \Psi_2(t_2, r_2, \rho) \leq M_2 + \eta C_{n_1+n_2} \sqrt{r_2} \ (\textit{equality holds if } t_2 > 0)$$
$$\textit{and} \ \inf_{\rho} \Psi_2(t_2, r_2, \rho) = -\infty. \tag{4.3.15}$$

Utilizing the expressions (4.3.13)-(4.3.15), for $\eta = 0$, $\eta > 0$ *and* $\eta < 0$, *respectively, for an affine equivariant estimator* $\underset{\sim}{d}_{\underset{\sim}{\Psi}} = (d_{\Psi_1}, d_{\Psi_2})$, *we can easily define the functions* $\underset{\sim}{\Psi}^0$, $\underset{\sim}{\Psi}^1$, $\underset{\sim}{\Psi}^2$ *as in (4.3.7)-(4.3.9) respectively. An application of orbit-by-orbit improvement technique for improving equivariant estimators of Brewster and Zidek (1974), proves the theorem.*

Remark 4.3.9 *The above theorem is basically a complete class result. It tells that for an equivariant estimator of the form (4.3.6),*

(i) if $P_{\underset{\sim}{\alpha}}(\{\Psi_1 \in [\min(T_2, 0), \max(T_2, 0)]^c\} \bigcup \{\Psi_2 \in [\min(T_2, 0), \max(T_2, 0)]^c\}) > 0$, *then the estimator* $\underset{\sim}{d}_{\underset{\sim}{\Psi}}$ *is improved by* $\underset{\sim}{d}_{\underset{\sim}{\Psi}^0}$, *when* $\eta = 0$.

(ii) if $P(\{\Psi_1 < \min(T_2, 0) + \eta C_{n_1+n_2}\} \bigcup \{\Psi_2 < \min(T_2, 0) + \eta C_{n_1+n_2} \sqrt{R_2}\}) > 0$, *then the estimator* $\underset{\sim}{d}_{\underset{\sim}{\Psi}^1}$ *will improve upon* $\underset{\sim}{d}_{\underset{\sim}{\Psi}}$, *when* $\eta > 0$,

(iii) if $P(\{\Psi_1 > \max(T_2, 0) + \eta C_{n_1+n_2}\} \bigcup \{\Psi_2 > \max(T_2, 0) + \eta C_{n_1+n_2} \sqrt{R_2}\}) > 0$, *then the estimator* $\underset{\sim}{d}_{\underset{\sim}{\Psi}^2}$ *will improve upon* $\underset{\sim}{d}_{\underset{\sim}{\Psi}}$ *when* $\eta < 0$.

Here $[a, b]^c$ stands for complement of the interval $[a, b]$ in $\mathbb{R}$.

Remark 4.3.10 *All the estimators discussed in Section 4.3.1 (except $\underset{\sim}{d}^{ML}$ whose closed form does not exist), belong to the class (4.3.6). But it has been seen that for none of these estimators, the choices of Ψ_1 and Ψ_2 satisfy the above conditions in Remark 4.3.9. So the estimators considered can not be improved by using Theorem 4.3.6, but they form a complete class. The result we write as a theorem below.*

Theorem 4.3.7 *Let the loss function be (4.3.1).*

(i) The class of estimators $\{\underset{\sim}{d}_{\underset{\sim}{\Psi}} : \Psi_1 \in [\min(T_2, 0), \max(T_2, 0)]$ and $\Psi_2 \in [\min(T_2, 0), \max(T_2, 0)]\}$ is complete for $\eta = 0$.

(ii) The class of estimators $\{\underset{\sim}{d}_{\underset{\sim}{\Psi}} : \Psi_1 > \min(T_2, 0) + \eta C_{n_1+n_2}$ and $\Psi_2 > \min(T_2, 0) + \eta C_{n_1+n_2}\sqrt{R_2}\}$ is complete for $\eta > 0$.

(iii) The class of estimators $\{\underset{\sim}{d}_{\underset{\sim}{\Psi}} : \Psi_1 < \max(T_2, 0) + \eta C_{n_1+n_2}$ and $\Psi_2 < \max(T_2, 0) + \eta C_{n_1+n_2}\sqrt{R_2}\}$ is complete for $\eta < 0$.

Proof 4.3.7 *The proof is immediate from Theorem 4.3.6.*

Next, we consider a smaller group of transformations and hence a larger class of estimators for estimating the quantile vector $\underset{\sim}{\theta}$. Consider the group $G_L = \{g_c : g_c(x) = c + x, c \in \mathbb{R}\}$ of location transformations. Under the transformation, $\bar{X}_i \to \bar{X}_i + c$, $S_i^2 \to S_i^2$, $\mu \to \mu + c$, $\sigma_i \to \sigma_i$, $\theta_i = \mu + \eta\sigma_i \to \theta_i + c$ where $i = 1, 2, \ldots, k$.

The estimation problem is invariant if we take the loss function as the sum of squared error losses (4.3.1), and the form of a location equivariant estimator for estimating the vector $\underset{\sim}{\theta}$ based on the sufficient statistics $(\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_k, S_1^2, S_2^2, \ldots, S_k^2)$ is obtained as

$$\underset{\sim}{d}_{\underset{\sim}{\psi}} = (\bar{X}_1 + \psi_1(\underset{\sim}{U}, \underset{\sim}{S}), \bar{X}_2 + \psi_2(\underset{\sim}{U}, \underset{\sim}{S}), \ldots, \bar{X}_2 + \psi_2(\underset{\sim}{U}, \underset{\sim}{S}), \tag{4.3.16}$$

where $\underset{\sim}{U} = (U_2, U_3, \ldots, U_k)$ and $\underset{\sim}{S} = (S_1, S_2, \ldots, S_k)$ and $U_i = \bar{X}_i - \bar{X}_1$; $i = 2, 3, \ldots, k$. To obtain the inadmissibility result let us consider the case $k = 2$. Let us denote $N_1 = \min(u_2, 0)$ and $N_2 = \max(u_2, 0)$. For a location equivariant estimator $\underset{\sim}{d}_{\underset{\sim}{\psi}}$, define the functions $\underset{\sim}{\psi}^0$, $\underset{\sim}{\psi}^1$ and $\underset{\sim}{\psi}^2$ as,

$$\underset{\sim}{\psi}^0(\underset{\sim}{u}) = (\min(\max(\psi_1, N_1), N_2), \min(\max(\psi_2, N_1), N_2)) \tag{4.3.17}$$

$$\underset{\sim}{\psi}^1(\underset{\sim}{u}) = (\max\{N_1, \psi_1\}, \max\{N_1, \psi_2\}), \tag{4.3.18}$$

$$\underset{\sim}{\psi}^2(\underset{\sim}{u}) = (\min\{N_2, \psi_1\}, \min\{N_2, \psi_2\}). \tag{4.3.19}$$

The following theorem gives a sufficient condition for improving estimators in the class $\underset{\sim}{d}_{\underset{\sim}{\psi}}$.

Theorem 4.3.8 *Let $\underset{\sim}{d}_{\underset{\sim}{\psi}}$ be a location equivariant estimator of the quantile vector $\underset{\sim}{\theta}$ and the loss function be sum of the quadratic losses (4.3.1) or the sum of squared errors. Let the functions $\underset{\sim}{\psi}^0$, $\underset{\sim}{\psi}^1$ and $\underset{\sim}{\psi}^2$ be defined as in (4.3.17), (4.3.18) and (4.3.19) respectively.*

(i) When $\eta = 0$, the estimator $\underset{\sim}{d}_{\underset{\sim}{\psi}}$ is improved by $\underset{\sim}{d}_{\underset{\sim}{\psi}^0}$ if $P_{\underset{\sim}{\alpha}}(\underset{\sim}{\psi}_0 \neq \underset{\sim}{\psi}) > 0$ for some choices of $\underset{\sim}{\alpha}$.

(ii) When $\eta > 0$, the estimator $\underset{\sim}{d}_{\psi}$ is improved by $\underset{\sim}{d}_{\psi^1}$ if $P_{\underset{\sim}{\alpha}}(\underset{\sim}{\psi}^1 \neq \underset{\sim}{\psi}) > 0$ for some choices of $\underset{\sim}{\alpha}$.

(iii) When $\eta < 0$, the estimator $\underset{\sim}{d}_{\psi}$ is improved by $\underset{\sim}{d}_{\psi^2}$ if $P_{\underset{\sim}{\alpha}}(\underset{\sim}{\psi}^2 \neq \underset{\sim}{\psi}) > 0$ for some choices of $\underset{\sim}{\alpha}$.

Proof 4.3.8 *The proof is similar to the arguments used in proving Theorem 4.3.6. The details of the proof is omitted.*

Remark 4.3.11 *Similar to Theorem 4.3.6 above, Theorem 4.3.8 is also a complete class result. It tells that for an estimator of the form (4.3.16),*

(i) if $P(\{\psi_1 \in [\min(U_2, 0), \max(U_2, 0)]^c\} \bigcup \{\psi_2 \in [\min(U_2, 0), \max(U_2, 0)]^c\}) > 0$ then the estimator $\underset{\sim}{d}_{\psi}$ is improved by $\underset{\sim}{d}_{\psi^0}$, when $\eta = 0$,

(ii) if $P(\{\psi_1 < \min(U_2, 0)\} \bigcup \{\psi_2 < \min(U_2, 0)\}) > 0$, then the estimator $\underset{\sim}{d}_{\psi^1}$ will improve upon $\underset{\sim}{d}_{\psi}$, for $\eta > 0$, and

(iii) if $P(\{\psi_1 > \max(U_2, 0)\} \bigcup \{\psi_2 > \max(U_2, 0)\}) > 0$, then the estimator $\underset{\sim}{d}_{\psi^2}$ will improve upon $\underset{\sim}{d}_{\psi}$ when $\eta < 0$.

Remark 4.3.12 *All the estimators discussed in Section 4.3.1 (except $\underset{\sim}{d}^{ML}$ whose closed form does not exist), belong to the class (4.3.16). But it has also been seen that for none of these estimators the choices of ψ_1 and ψ_2 satisfy the above conditions in Remark 4.3.11. So the estimators considered can not be improved by using Theorem 4.3.8, but they form a complete class. This we write as a theorem.*

Theorem 4.3.9 *Let the loss function be (4.3.1).*

(i) The class of estimators $\{\underset{\sim}{d}_{\psi} : \psi_1 \in [\min(U_2, 0), \max(U_2, 0)]$ and $\psi_2 \in [\min(U_2, 0), \max(U_2, 0)]\}$ is complete for $\eta = 0$.

(ii) The class of estimators $\{\underset{\sim}{d}_{\psi} : \psi_1 > \min(U_2, 0)$ and $\psi_2 > \min(U_2, 0)\}$ is complete for $\eta > 0$.

(ii) The class of estimators $\{\underset{\sim}{d}_{\psi} : \psi_1 < \max(U_2, 0)$ and $\psi_2 < \max(U_2, 0)\}$ is complete for $\eta < 0$.

Proof 4.3.9 *The proof is immediate from Theorem 4.3.8.*

Remark 4.3.13 *We note that, the inadmissibility results have been derived only for the case $k = 2$ populations. The case of general $k(\geq 3)$ remains unresolved. However, we feel that all the proposed estimators in Section 4.3.1 for the case $k(\geq 3)$ will form a complete class.*

4.3.3 Numerical Comparisons

In the previous subsections we have derived several estimators for the quantile vector $\underset{\sim}{\theta}$ such as $\underset{\sim}{d}^{\bar{X}_1}$, $\underset{\sim}{d}^{\bar{X}_2}$, $\underset{\sim}{d}^{GD}$, $\underset{\sim}{d}^{GM}$, $\underset{\sim}{d}^{KS}$, $\underset{\sim}{d}^{BC1}$, $\underset{\sim}{d}^{BC2}$, $\underset{\sim}{d}^{CS}$, $\underset{\sim}{d}^{MK}$, $\underset{\sim}{d}^{TK}$, and $\underset{\sim}{d}^{ML}$ when $k = 2$. Further for the case $k(\geq 3)$ we have also constructed various estimators for $\underset{\sim}{\theta}$ such as, $\underset{\sim}{d}^{NH}$, $\underset{\sim}{d}^{SZ}$, $\underset{\sim}{d}^{MK}$, $\underset{\sim}{d}^{TK}$, $\underset{\sim}{d}^{GM}$, and $\underset{\sim}{d}^{ML}$. We have also shown that these well structured estimators, except $\underset{\sim}{d}^{ML}$, belong to the class (4.3.6) and (4.3.16). It seems quite difficult to compare the risk values of all these estimators analytically. But for practical purposes, one needs the estimator to be used. Taking the advantages of computational resources, we in this section compare numerically the simulated risk values of all these estimators which may be handy for practical purposes. For evaluating the risk function, we use the loss function (4.3.1).

We first discuss the numerical results for the case $k = 2$ normal populations. For numerical comparison purpose, we have generated 20,000 random samples $\underset{\sim}{X}_1$ of sizes n_1 and 20,000 random samples $\underset{\sim}{X}_2$ of sizes n_2 from normal populations with equal mean and different variances. It can be easily checked that all the risks values are functions of $\tau = \frac{\sigma_2}{\sigma_1} > 0$, for fixed values of n_1, n_2 and $|\eta|$. The approximate value of π is taken to be 3.1416. We have computed the risk values of all the estimators taking various choices of τ and the sample sizes. However, for illustration purpose we present the risk values for some selected choices of τ and n_1, n_2. We also observe that when the values of τ increase from 0 to ∞ the risk values converge for all the estimators except $\underset{\sim}{d}^{GM}$ and $\underset{\sim}{d}^{\bar{X}_2}$. As the sample sizes increases the risk values of all the estimators decrease for fixed $|\eta|$. Further, the risk values increase as η increases for fixed values of τ and sample sizes. If we choose the value of b_1 and b_2 near 0 the estimators $\underset{\sim}{d}^{BC1}$ and $\underset{\sim}{d}^{BC2}$ tends to $\underset{\sim}{d}^{\bar{X}_1}$. Also if we choose the value of b_2 near 1 the estimator $\underset{\sim}{d}^{BC2}$ tends to $\underset{\sim}{d}^{GD}$. So for numerical comparison a convenient choice would be an intermediate value which we take as $\frac{1}{2}b_{max}$. The value of $b_{max}(n_1, n_2)$ have been taken from the tabulated values given in Brown and Cohen (1974). We also note that, when the sample sizes are equal the estimator $\underset{\sim}{d}^{GD}$ becomes same as $\underset{\sim}{d}^{KS}$ and $\underset{\sim}{d}^{MK}$ becomes same as $\underset{\sim}{d}^{TK}$. When the sample sizes are unequal the estimator $\underset{\sim}{d}^{CS}$ is not defined, so for unequal sample sizes we do not include it for numerical comparison purpose. A massive simulation study has been conducted separately for the cases $n_1 = n_2$, $n_1 > n_2$ and $n_1 < n_2$. The simulated risk values have been plotted against τ for all the estimators in Fig. 4.3.1 and Fig. 4.3.2. In Fig. 4.3.1 the sample sizes have been taken as equal, whereas in Fig. 4.3.2, the simulated risk values have been plotted for unequal sample sizes. In Figures 1, and 2 we label $X, Y, GM, GD, KS, BC1, BC2, CS, MK, TK$ and ML for the estimators $\underset{\sim}{d}^{\bar{X}_1}$, $\underset{\sim}{d}^{\bar{X}_2}$, $\underset{\sim}{d}^{GM}$, $\underset{\sim}{d}^{GD}$, $\underset{\sim}{d}^{KS}$, $\underset{\sim}{d}^{BC1}$, $\underset{\sim}{d}^{BC2}$, $\underset{\sim}{d}^{CS}$, $\underset{\sim}{d}^{MK}$, $\underset{\sim}{d}^{TK}$ $\underset{\sim}{d}^{ML}$ respectively. In Tables 4.3.1-4.3.3, we have presented the simulated values of the percentage of relative risk improvement of all the estimators with respect to $\underset{\sim}{d}^{\bar{X}_1}$, which are defined as

$$
\begin{aligned}
PR1 &= \left(1 - \frac{Risk(\underset{\sim}{d}^{\bar{X}_2})}{Risk(\underset{\sim}{d}^{\bar{X}_1})}\right) \times 100, \; PR2 = \left(1 - \frac{Risk(\underset{\sim}{d}^{GM})}{Risk(\underset{\sim}{d}^{\bar{X}_1})}\right) \times 100, \\
PR3 &= \left(1 - \frac{Risk(\underset{\sim}{d}^{GD})}{Risk(\underset{\sim}{d}^{\bar{X}_1})}\right) \times 100, \; PR4 = \left(1 - \frac{Risk(\underset{\sim}{d}^{KS})}{Risk(\underset{\sim}{d}^{\bar{X}_1})}\right) \times 100, \\
PR5 &= \left(1 - \frac{Risk(\underset{\sim}{d}^{BC1})}{Risk(\underset{\sim}{d}^{\bar{X}_1})}\right) \times 100, \; PR6 = \left(1 - \frac{Risk(\underset{\sim}{d}^{BC2})}{Risk(\underset{\sim}{d}^{\bar{X}_1})}\right) \times 100, \\
PR7 &= \left(1 - \frac{Risk(\underset{\sim}{d}^{CS})}{Risk(\underset{\sim}{d}^{\bar{X}_1})}\right) \times 100, \; PR8 = \left(1 - \frac{Risk(\underset{\sim}{d}^{MK})}{Risk(\underset{\sim}{d}^{\bar{X}_1})}\right) \times 100,
\end{aligned}
$$

$$PR9 = \left(1 - \frac{Risk(\underset{\sim}{d}^{TK})}{Risk(\underset{\sim}{d}^{\bar{X}_1})}\right) \times 100, \; PR10 = \left(1 - \frac{Risk(\underset{\sim}{d}^{ML})}{Risk(\underset{\sim}{d}^{\bar{X}_1})}\right) \times 100.$$

The following observations can be made from the Tables 4.3.1-4.3.3 and the Figures 4.3.1-4.3.2 as well as from our simulation study. For illustration purpose, we have presented the risk functions only for the case $\eta = 1.960$.

Case 1: $\mathbf{n_1 = n_2}$

(i) Figure 4.3.1 represents the risk values of all the estimators for the equal sample sizes and $\eta = 1.960$. In Figure 4.3.1, (a)-(c) it represents the risk values for sample sizes small to moderate that is (6,6), (8,8) and (12,12) whereas (d)-(f) the sample sizes are taken as moderate to large (20,20), (30,30) and (40,40). It has been noticed that the risk values of the estimators $\underset{\sim}{d}^{\bar{X}_1}$, $\underset{\sim}{d}^{BC1}$, $\underset{\sim}{d}^{BC2}$ and $\underset{\sim}{d}^{CS}$ are decreasing as τ increases from 0 to ∞. The estimator $\underset{\sim}{d}^{GD}$ first increases and attains maximum value then decreases. The estimators $\underset{\sim}{d}^{GM}$, and $\underset{\sim}{d}^{MK}$ first decrease attains minimum (in the neighborhood of $\tau = 1$) then increases. The estimator $\underset{\sim}{d}^{\bar{X}_2}$ increases as τ varies from 0 to ∞. It has also been noticed that all the estimators (except $\underset{\sim}{d}^{GM}$ and $\underset{\sim}{d}^{\bar{X}_2}$) converge to the estimator $\underset{\sim}{d}^{\bar{X}_1}$ which is true as these estimators are consistent.

(ii) The percentage of relative risk performances of all the estimators with respect to $\underset{\sim}{d}^{\bar{X}_1}$ decrease as τ varies from 0 to ∞. Let us first consider the case of small sample sizes ($m, n \leq 10$). For small values of τ ($\tau < 0.25$) the estimators $\underset{\sim}{d}^{\bar{X}_2}$ and $\underset{\sim}{d}^{ML}$ has the maximum percentage of relative risk improvement and it is seen near to 98.88%. For moderate values of τ ($0.75 < \tau < 2.5$) the estimators $\underset{\sim}{d}^{GM}$ and $\underset{\sim}{d}^{MK}$ compete each other however when $\tau = 1$, the estimator $\underset{\sim}{d}^{GM}$ has the maximum percentage of relative risk improvement and it is seen near to 15.68%. For large values of τ, the estimator $\underset{\sim}{d}^{BC1}$ has the maximum percentage of relative risk improvement.

Consider the case of moderate sample sizes ($12 \leq n_1, n_2 \leq 20$). For small values of τ, the estimator $\underset{\sim}{d}^{ML}$ has the best performance and the percentage of relative risk improvement is seen near to 89.78%. For moderate values of τ ($0.75 < \tau < 2.5$) the estimators $\underset{\sim}{d}^{MK}$ and $\underset{\sim}{d}^{GD}$ perform equally well, however for $\tau = 1$, the estimator $\underset{\sim}{d}^{GM}$ has the maximum percentage of relative risk performances. For large values of τ, ($\tau > 3.5$) the estimators $\underset{\sim}{d}^{BC1}$ and $\underset{\sim}{d}^{ML}$ compete with each other.

Consider the case of large sample sizes ($n_1, n_2 \geq 30$). For small values of τ the estimators $\underset{\sim}{d}^{ML}$ and $\underset{\sim}{d}^{GD}$ compete with each other and the percentage of relative risk performance has been noticed near to 90.40%. For moderate values of τ ($0.75 < \tau < 2.5$,) the estimators $\underset{\sim}{d}^{GD}$, $\underset{\sim}{d}^{ML}$ and $\underset{\sim}{d}^{MK}$ compete with each other, however for $\tau = 1$, the estimator $\underset{\sim}{d}^{GM}$ has the best performance. For large values of τ, the estimators $\underset{\sim}{d}^{BC1}$ and $\underset{\sim}{d}^{BC2}$ compete with $\underset{\sim}{d}^{ML}$.

Case 2: $(\mathbf{n_1 < n_2})$

(i) Fig. 4.3.2, ((a), (c) and (e)) represents the risk values of all the estimators for $\eta = 1.960$ and the sample sizes (4,10), (12,20) and (30,40). The risk values of the estimators $\underset{\sim}{d}^{\bar{X}_1}$, is decreasing as τ increases. The risk values of $\underset{\sim}{d}^{GD}$, $\underset{\sim}{d}^{KS}$ increase and attains maximum then decrease as τ increases. The risk values of all the estimators converge to the risk of $\underset{\sim}{d}^{\bar{X}_1}$ except $\underset{\sim}{d}^{\bar{X}_2}$ and $\underset{\sim}{d}^{GM}$.

(ii) Consider the small sample sizes ($n_1, n_2 \leq 10$). For small values of $\tau < 0.25$, the estimator $\underset{\sim}{d}^{\bar{X}_2}$ and $\underset{\sim}{d}^{ML}$ compete with each other and the percentage of relative risk improvement is seen near to 98.88%. For moderate values of τ $(0.75 < \tau < 3,)$ the estimators $\underset{\sim}{d}^{TK}$ and $\underset{\sim}{d}^{GM}$ compete each other, however for $\tau = 1$, the estimator $\underset{\sim}{d}^{GM}$ has the best performance. For large values of τ $(\tau > 3.0,)$ the estimator $\underset{\sim}{d}^{BC1}$ performs the best and the percentage of relative risk performance.

Consider the case of moderate sample sizes $(12 \leq n_1, n_2 \leq 20)$. For small values of τ the estimator $\underset{\sim}{d}^{ML}$ has the maximum percentage of relative risk performance and it is seen near to 98.88%. For moderate values of τ $(0.75 < \tau < 3)$ the estimators $\underset{\sim}{d}^{TK}$, $\underset{\sim}{d}^{MK}$ and $\underset{\sim}{d}^{KS}$ compete each other, however for $\tau = 1$, $\underset{\sim}{d}^{GM}$ has the best performance. For large values of τ $(\tau > 3)$ the estimator $\underset{\sim}{d}^{BC1}$ has the maximum percentage of relative risk improvement.

Consider the case of large sample sizes $(n_1, n_2 \geq 30)$. For small values of τ $(\tau \leq 0.25)$, the estimators $\underset{\sim}{d}^{KS}$, $\underset{\sim}{d}^{GD}$ and $\underset{\sim}{d}^{ML}$ compete each other. For moderate values of τ $(0.25 < \tau < 3.)$ the estimators $\underset{\sim}{d}^{GD}$, $\underset{\sim}{d}^{KS}$, $\underset{\sim}{d}^{TK}$, $\underset{\sim}{d}^{MK}$ and $\underset{\sim}{d}^{ML}$ compete each other. For large values of τ the estimators $\underset{\sim}{d}^{ML}$ and $\underset{\sim}{d}^{BC1}$ compete each other.

Case-3:$\mathbf{n_1 > n_2}$

(i) Fig. 4.3.2, ((b), (d) and (f)) represent the risk values of all the estimators for $\eta = 1.960$ and for the sample sizes (10,4), (20,12) and (40,30). The risk values of $\underset{\sim}{d}^{\bar{X}_1}$ is decreasing as τ increases. The risk values of $\underset{\sim}{d}^{GD}$, $\underset{\sim}{d}^{KS}$, $\underset{\sim}{d}^{BC1}$ and $\underset{\sim}{d}^{BC2}$ decrease as τ increases. The risk values of estimators $\underset{\sim}{d}^{GM}$, and $\underset{\sim}{d}^{\bar{X}_2}$ first decrease attains minimum then increase with respect to τ.

(ii) Consider the case of small sample sizes $(n_1, n_2 \leq 10)$. For small values of τ $(\tau \leq 0.25)$ the estimator $\underset{\sim}{d}^{ML}$ has maximum percentage of relative risk performance and it is noticed near to 97.7%, for moderate values of τ $(0.75 < \tau < 2.0)$ the estimators $\underset{\sim}{d}^{TK}$ and $\underset{\sim}{d}^{GM}$ compete each other, however for $\tau = 1$, the estimator $\underset{\sim}{d}^{GM}$ has the best performance. For large values of τ, $(\tau > 3)$ the estimator $\underset{\sim}{d}^{BC1}$ has the best performance.

Consider the case of moderate sample sizes $(12 \leq n_1, n_2 \leq 20)$. For small values of τ $(\tau < 0.25)$ the estimator $\underset{\sim}{d}^{ML}$ has the best performance, for moderate values of τ $(0.75 \leq \tau < 2.0)$, the estimator $\underset{\sim}{d}^{KS}$ and $\underset{\sim}{d}^{GD}$ compete each other. For $\tau = 1$ the estimator $\underset{\sim}{d}^{GM}$ performs the best. For large values of τ the estimator $\underset{\sim}{d}^{BC1}$ and $\underset{\sim}{d}^{ML}$ compete each other.

Consider the case of large sample sizes $(n_1, n_2 \geq 30)$. For small values of τ the estimators $\underset{\sim}{d}^{ML}$ has the maximum percentage of risk improvement, for moderate values of τ the estimators $\underset{\sim}{d}^{ML}$, $\underset{\sim}{d}^{GD}$, $\underset{\sim}{d}^{KS}$, $\underset{\sim}{d}^{TK}$, and $\underset{\sim}{d}^{MK}$ compete each other. However for $\tau = 1$ the estimator $\underset{\sim}{d}^{GM}$ has the best performance. For large values of τ the estimators $\underset{\sim}{d}^{ML}$, $\underset{\sim}{d}^{GD}$, $\underset{\sim}{d}^{BC1}$, $\underset{\sim}{d}^{BC2}$ and $\underset{\sim}{d}^{KS}$ perform equally well.

On the basis of the above discussion and observations the following recommendations may be done for the use of the estimators.

(i) We conclude from the above discussion that, none of the estimators completely dominate others in terms of the risk function for the full range of the parameters.

(ii) When the sample sizes are small that is $n_1, n_2 \leq 10$, the estimators $\underset{\sim}{d}^{ML}$ and $\underset{\sim}{d}^{\bar{X}_2}$ can be used if τ is near to 0. For values of τ in the neighborhood of 1, the estimators $\underset{\sim}{d}^{MK}$ and $\underset{\sim}{d}^{TK}$ may be used, however for $\tau = 1$ that is when the variances are of the two populations are same, the estimator $\underset{\sim}{d}^{GM}$ should be used. For large values of τ we recommend to use $\underset{\sim}{d}^{BC1}$.

(iii) When the sample sizes are from moderate to large the estimators $\underset{\sim}{d}^{ML}$, $\underset{\sim}{d}^{GD}$, or $\underset{\sim}{d}^{KS}$ may be used if τ is near to 0, however for moderate values of τ we recommend to use either of the estimators $\underset{\sim}{d}^{GD}$, $\underset{\sim}{d}^{KS}$, $\underset{\sim}{d}^{MK}$, $\underset{\sim}{d}^{TK}$, or $\underset{\sim}{d}^{ML}$. For values of $\tau = 1$, the estimator $\underset{\sim}{d}^{GM}$ is strongly recommended to use. For large values of τ, the estimators $\underset{\sim}{d}^{ML}$, $\underset{\sim}{d}^{BC1}$, or $\underset{\sim}{d}^{BC2}$ may be used.

(iv) A similar type of observations have been made for other combinations of sample sizes and η.

<u>Numerical Results for the Case $k = 3$</u>

As discussed before, we have constructed several estimators for the quantile vector $\underset{\sim}{\theta}$, when $k \geq 3$. Specifically, we have constructed $\underset{\sim}{d}^{NH}$, $\underset{\sim}{d}^{SZ}$, $\underset{\sim}{d}^{MK}$, $\underset{\sim}{d}^{TK}$, $\underset{\sim}{d}^{GM}$, and $\underset{\sim}{d}^{ML}$. Also, we note that it is impossible to evaluate the risk functions analytically. Hence, for the particular case $k = 3$, we carry out a simulation study to compare the risk functions of all these estimators numerically. In a similar way we have generated 20,000 random samples each from the normal populations $N(\mu, \sigma_i^2)$; $i = 1, 2, 3$ for various sample sizes n_1, n_2 and n_3. The risk will be evaluated using the loss function (4.3.1). To compare the performances of all the proposed estimators we calculate the percentage of relative risk improvements of an estimator (say $\underset{\sim}{d}$) with respect to $\underset{\sim}{d}^{\bar{X}_1}$, as

$$P(\underset{\sim}{d}) = \Big(1 - \frac{Risk(\underset{\sim}{d})}{Risk(\underset{\sim}{d}^{\bar{X}_1})}\Big) \times 100.$$

It is easy to observe that the risk values of all the estimators are functions of $\tau_2 = \sigma_2/\sigma_1$ and $\tau_3 = \sigma_3/\sigma_1$ with respect to the loss function (4.3.1). The simulation study has been done by taking various combinations of the sample sizes, and many choices of the parameters. However, for illustrative purpose we have presented the percentage of relative risk values of all the estimators for some selected choices of the sample sizes, parameters and η. In Tables 4.3.4 to 4.3.6 the percentage of relative risk values of all the estimators have been presented for the case of equal sample sizes $(10, 10, 10)$, $(20, 20, 20)$ and $(30, 30, 30)$, whereas Tables 4.3.7 to 4.3.9 represent for unequal sample sizes. We also note that when the sample sizes are equal the estimators $\underset{\sim}{d}^{NH} = \underset{\sim}{d}^{SZ}$ and $\underset{\sim}{d}^{MK} = \underset{\sim}{d}^{TK}$. Tables 4.3.4 to 4.3.6 have ten columns and each column is again divided into several cells. The first two columns represents the values of τ_2 and τ_3. The columns 3rd to 8th represent the percentage of relative risk values of all the estimators. Further in each cell, one value of τ_2 (column 1) corresponds to seven values of τ_3 (column 2). In a similar way the percentage of relative risk values of all the estimators in Tables 4.3.7 to 4.3.9 have been presented for unequal sample sizes. The following observations have been made from the Tables as well as from our simulation study.

(i) Like the case of $k = 2$, none of the estimators completely dominate others in terms of the risk function for the full range of the parameters.

(ii) Consider that the sample sizes are small and equal. For the values of τ_2 and τ_3 close to 0, the estimator $\underset{\sim}{d}^{NH}$ and $\underset{\sim}{d}^{ML}$ has the maximum percentage of relative risk performances. Also, a similar type of behavior has been noticed when the values of τ_2 and τ_3 are large. For $\tau_2 = \tau_3 = 1$, the estimator $\underset{\sim}{d}^{GM}$ has the best performance. When the values of τ_2 and τ_3 are close to each other and small the estimator $\underset{\sim}{d}^{MK}$ has the best percentage of relative risk performances, however as the τ_2 and τ_3 become large this estimator does not perform well and is being dominated by $\underset{\sim}{d}^{NH}$ or $\underset{\sim}{d}^{ML}$. Consider that the sample sizes are moderate to large. For τ_2 and τ_3 close to 0, the estimators $\underset{\sim}{d}^{ML}$ and $\underset{\sim}{d}^{NH}$ perform equally well. For large values of τ_2 and τ_3, the estimator $\underset{\sim}{d}^{ML}$ has the best performance. When $\tau_2 = \tau_3 = 1$, the estimator $\underset{\sim}{d}^{GM}$ has the best performance. Further, when the values of τ_2 and τ_3 close to each other but are small the estimator $\underset{\sim}{d}^{MK}$ has the best performance. However as the values of τ_2 and τ_3 become large this estimator is being dominated by either $\underset{\sim}{d}^{NH}$ or $\underset{\sim}{d}^{ML}$.

(iii) Consider that the sample sizes are unequal. For the values of τ_2 and τ_3 close to 0, the estimators $\underset{\sim}{d}^{NH}$ and $\underset{\sim}{d}^{SZ}$ compete with $\underset{\sim}{d}^{ML}$. For the values of $\tau_2 = \tau_3 = 1$, the estimator $\underset{\sim}{d}^{GM}$ has the best percentage of relative risk performance. For large values of τ_2 and τ_3 the estimators $\underset{\sim}{d}^{NH}$, $\underset{\sim}{d}^{SZ}$ and $\underset{\sim}{d}^{ML}$ compete with each other. We have also noticed that, for the values of τ_2 and τ_3 in the neighborhood of 1, the estimators $\underset{\sim}{d}^{MK}$ and $\underset{\sim}{d}^{TK}$ performs better compare to other estimators.

(iv) A similar type of observations have been made for other combinations of sample sizes and η.

The following conclusions can be drawn from the above discussions regarding the use of the estimators.

(i) None of the estimators completely dominate others in terms of the risk values for the full range of the parameters.

(ii) For the values of τ_2 and τ_3 are close to 0, we recommend to use either $\underset{\sim}{d}^{NH}$ or $\underset{\sim}{d}^{SZ}$ (small sample sizes) and $\underset{\sim}{d}^{ML}$ for moderate to large sample sizes. For all the sample sizes, when $\tau_2 = \tau_3 = 1$, that is when the populations have same variances the estimator $\underset{\sim}{d}^{GM}$ is recommended for use. For large values of τ_2 and τ_3, we recommend to use either $\underset{\sim}{d}^{NH}$ or $\underset{\sim}{d}^{SZ}$ (small sample sizes) and $\underset{\sim}{d}^{ML}$ (for large sample sizes). When the sample sizes are small, and the values of τ_2 and τ_3 are in the neighborhood of 1, the estimators $\underset{\sim}{d}^{MK}$ and $\underset{\sim}{d}^{TK}$ can be used. For other cases, we recommend to use $\underset{\sim}{d}^{NH}$ or $\underset{\sim}{d}^{SZ}$.

Table 4.3.1: Percentage of relative risk improvements of various estimators of normal quantiles with $\eta = 1.960$, $(n_1, n_2) = (8, 8), (12, 12), (20, 20), (40, 40)$

$\tau \downarrow$	$PR1$	$PR2$	$PR3$	$PR5$	$PR6$	$PR7$	$PR8$	$PR10$
	98.72	74.20	98.72	55.40	53.86	30.17	98.46	98.73
0.05	98.76	74.20	98.76	70.66	75.75	63.20	98.52	98.76
	98.76	74.19	98.76	83.70	88.49	84.65	98.54	98.76
	98.80	74.22	98.80	92.81	94.90	95.05	98.58	98.80
	89.50	68.21	89.45	50.46	48.60	27.28	88.06	89.50
0.15	89.73	68.29	89.74	64.02	68.62	57.31	88.45	89.76
	89.75	68.39	89.79	76.20	80.40	76.96	88.59	89.79
	90.16	68.66	90.20	84.69	86.62	86.77	89.05	90.20
	75.29	59.03	75.28	42.83	40.68	22.89	73.34	75.43
0.25	75.77	59.25	75.89	54.65	57.81	48.33	73.97	75.93
	76.07	59.45	76.28	64.74	68.12	65.22	74.43	76.29
	76.32	59.75	76.57	71.96	73.48	73.62	74.77	76.58
	40.53	36.89	41.99	24.76	22.66	12.60	41.28	41.92
0.50	41.25	37.75	43.22	31.97	33.03	27.52	42.35	43.21
	41.60	38.23	44.02	37.94	39.44	37.74	43.01	44.03
	41.79	38.54	44.60	42.16	42.88	42.95	43.48	44.60
	17.86	24.31	23.92	15.07	13.37	07.07	24.71	23.45
0.75	18.06	24.87	25.04	19.29	19.57	15.87	25.52	24.77
	17.79	24.86	25.47	22.42	22.99	21.74	25.63	25.40
	17.45	25.02	25.81	24.66	24.94	24.90	25.81	25.79
	-0.86	15.68	13.55	09.61	08.38	04.16	15.01	12.70
1.00	-0.18	16.58	15.10	12.37	12.39	09.65	16.14	14.64
	01.15	17.20	16.38	14.65	14.93	13.72	16.98	16.24
	00.56	16.99	16.51	15.83	15.97	15.76	16.85	16.48
	-15.31	10.10	08.78	06.77	05.86	02.75	10.05	07.96
1.25	-17.90	09.50	09.08	08.06	08.01	06.04	09.81	08.69
	-16.36	10.49	10.32	09.62	09.72	08.80	10.81	10.18
	-16.82	10.67	11.02	10.71	10.78	10.60	11.17	10.97
	-31.46	04.87	05.88	05.07	04.33	01.97	06.51	05.19
1.50	-32.18	04.77	06.44	06.00	05.87	04.38	06.62	06.29
	-34.66	04.40	06.96	06.76	06.80	06.21	06.68	06.88
	-34.23	05.07	08.09	07.90	07.92	07.80	07.54	08.08
	-67.82	-5.69	02.45	02.94	02.54	01.13	01.45	02.30
2.00	-70.73	-6.03	03.48	03.54	03.47	02.61	01.95	03.46
	-69.55	-4.90	04.53	04.35	04.37	03.99	03.00	04.51
	-72.31	-5.85	04.57	04.52	04.53	04.48	02.63	04.57
	-116.05	-18.14	01.11	01.89	01.68	00.77	-1.87	01.24
2.50	-115.39	-17.45	02.27	02.46	02.40	01.82	-0.84	02.38
	-120.07	-18.71	02.59	02.68	02.66	02.47	-1.00	02.68
	-119.29	-18.49	02.74	02.78	02.77	02.75	-0.81	02.76
	-169.15	-31.75	00.49	01.45	01.21	00.57	-4.27	01.09
3.00	-170.01	-31.47	01.27	01.56	01.57	01.22	-3.49	01.42
	-172.46	-31.42	02.11	02.12	02.09	01.93	-2.55	02.17
	-176.39	-32.49	02.20	02.17	02.17	02.14	-2.59	02.20
	-293.23	-61.96	00.38	00.88	00.78	00.36	-6.55	00.80
4.00	-304.09	-65.67	00.48	00.81	00.82	00.66	-7.06	00.68
	-311.09	-66.74	01.19	01.20	01.19	01.10	-6.02	01.23
	-319.53	-69.42	01.03	01.06	01.06	01.05	-6.51	01.05

Table 4.3.2: Percentage of relative risk improvements of various estimators of normal quantiles with $\eta = 1.960$, $(n_1, n_2) = (4, 10), (12, 20), (30, 40)$

τ	$PR1$	$PR2$	$PR3$	$PR4$	$PR5$	$PR6$	$PR8$	$PR9$	$PR10$
	98.74	90.72	98.72	98.74	48.01	53.65	98.51	98.53	98.74
0.05	98.69	84.86	98.69	98.69	79.81	83.67	98.53	98.54	98.69
	98.67	80.62	98.67	98.67	91.94	93.71	98.50	98.50	98.67
	89.99	82.96	89.77	89.92	43.60	48.41	88.72	88.83	89.98
0.15	89.34	77.33	89.35	89.35	72.34	75.63	88.45	88.46	89.36
	89.21	73.63	89.24	89.24	83.16	84.76	88.31	88.31	89.24
	76.50	71.03	75.95	76.33	36.96	40.55	74.39	74.57	76.48
0.25	75.22	66.08	75.27	75.28	61.08	63.67	73.81	73.84	75.30
	74.94	62.96	75.07	75.07	69.93	71.23	73.52	73.53	75.08
	46.57	44.83	45.59	46.46	22.71	23.98	44.43	44.65	46.53
0.50	43.61	41.04	44.23	44.28	36.01	37.19	42.89	42.93	44.31
	42.16	39.03	43.48	43.49	40.56	41.16	42.17	42.17	43.50
	29.00	29.96	28.53	29.48	14.74	15.03	28.38	28.58	28.88
0.75	24.71	27.02	27.10	27.17	22.40	22.85	26.75	26.77	27.11
	20.91	25.07	25.53	25.53	24.10	24.37	25.31	25.32	25.52
	18.38	21.72	19.56	20.22	10.59	10.54	20.24	20.38	19.29
1.00	11.35	18.16	17.45	17.48	14.86	15.01	17.78	17.79	17.36
	07.2	17.09	16.77	16.77	15.93	16.05	16.96	16.96	16.74
	09.62	15.57	13.89	14.02	07.97	07.77	15.03	15.12	13.32
1.25	02.77	13.30	12.91	12.93	11.06	11.09	13.50	13.51	12.74
	-4.45	11.25	11.45	11.45	11.02	11.02	11.77	11.77	11.41
	03.39	11.56	10.57	10.33	06.29	06.10	12.03	12.07	09.84
1.50	-8.37	07.64	09.18	09.15	08.16	08.13	09.66	09.65	08.99
	-15.42	06.25	08.30	08.29	07.98	07.99	08.34	08.33	08.27
	-13.23	01.65	05.59	03.91	04.05	03.82	06.86	06.78	04.87
2.00	-29.23	-1.70	05.12	05.03	04.86	04.83	05.06	05.03	05.00
	-44.68	-4.36	04.66	04.64	04.65	04.64	03.58	03.57	04.65
	-29.19	-6.74	03.43	00.82	02.96	02.77	04.27	04.11	02.86
2.50	-54.92	-12.23	03.36	03.25	03.32	03.29	02.20	02.15	03.32
	-73.59	-14.60	02.94	02.93	02.98	02.97	00.74	00.73	02.95
	-49.24	-17.00	02.39	-0.89	02.31	02.15	02.43	02.18	01.92
3.00	-87.01	-25.22	02.26	02.13	02.37	02.34	-0.28	-0.36	02.31
	-110.99	-26.79	02.34	02.33	02.34	02.31	-0.94	-0.96	02.36
	-76.25	-31.59	00.89	-3.42	01.63	01.51	-0.37	-0.76	00.76
3.50	-120.65	-38.15	01.77	01.66	01.85	01.83	-1.67	-1.76	01.82
	-159.48	-43.07	01.61	01.59	01.63	01.64	-3.08	-3.10	01.62
	-83.63	-34.56	01.41	-2.41	01.64	01.53	00.17	-0.21	01.22
3.75	-138.71	-45.64	01.33	01.21	01.49	01.47	-2.67	-2.77	01.41
	-181.46	-49.60	01.60	01.59	01.59	01.58	-3.22	-3.25	01.61
	-99.21	-42.8	00.90	-3.31	01.42	01.31	-0.97	-1.42	00.81
4.00	-157.38	-53.73	00.97	00.86	01.18	01.16	-3.7	-3.80	01.08
	-199.50	-55.59	01.39	01.39	01.38	01.37	-3.54	-3.57	01.40

Table 4.3.3: Percentage of relative risk improvements of various estimators of normal quantiles with $\eta = 1.960$, $\eta = 1.960$, $(n_1, n_2) = (10, 4), (20, 12), (40, 30)$

τ	$PR1$	$PR2$	$PR3$	$PR4$	$PR5$	$PR6$	$PR8$	$PR9$	$PR10$
	96.66	47.61	96.65	96.61	31.74	06.15	96.10	96.04	96.66
0.05	97.65	59.62	97.65	97.65	73.60	76.84	97.28	97.28	97.65
	98.16	66.31	98.16	98.16	90.21	93.39	97.90	97.89	98.16
	74.93	38.57	74.89	73.82	25.03	04.75	73.28	73.07	75.01
0.15	81.95	51.36	82.00	81.98	62.19	64.41	80.44	80.41	82.01
	85.51	58.82	85.58	85.57	78.61	81.31	84.22	84.21	85.58
	51.02	28.76	51.30	49.01	17.79	03.27	50.24	50.00	51.33
0.25	60.73	40.31	61.25	61.21	46.87	48.12	59.58	59.54	61.27
	66.65	47.91	67.08	67.08	61.81	63.78	65.37	65.36	67.09
	11.92	13.43	17.02	15.51	07.06	01.24	18.18	18.11	16.37
0.50	23.63	21.38	27.08	26.99	21.28	21.32	26.88	26.85	27.00
	30.09	27.26	33.27	33.26	30.78	31.58	32.56	32.55	33.27
	-7.17	07.28	05.65	05.53	03.32	00.57	07.09	07.16	04.80
0.75	01.51	12.33	12.52	12.51	10.49	10.37	13.11	13.11	12.31
	07.33	16.57	17.34	17.34	16.42	16.67	17.52	17.52	17.31
	-22.64	04.15	01.39	02.30	01.87	00.34	02.23	02.41	01.16
1.00	-13.11	07.75	06.84	06.88	06.14	05.98	07.26	07.27	06.68
	-9.14	10.03	09.55	09.55	09.27	09.34	09.76	09.77	09.52
	-37.14	01.91	-0.45	00.84	01.11	00.21	-0.44	-0.18	-0.40
1.25	-27.24	04.17	04.00	04.06	03.82	03.70	03.75	03.78	03.96
	-21.59	06.40	06.57	06.58	06.36	06.38	06.46	06.46	06.55
	-50.31	00.63	-0.75	00.55	00.87	00.17	-1.31	-1.01	-0.17
1.50	-42.87	00.85	02.20	02.30	02.41	02.32	01.18	01.23	02.27
	-38.20	02.26	04.45	04.46	04.37	04.39	03.55	03.56	04.45
	-89.75	-3.66	-2.23	-0.43	00.41	00.09	-4.51	-4.06	-0.24
2.00	-77.46	-4.60	01.17	01.26	01.41	01.33	-1.18	-1.11	01.33
	-72.94	-5.01	02.44	02.45	02.46	02.45	00.30	00.31	02.46
	-130.42	-7.14	-2.33	-0.55	00.26	00.06	-5.74	-5.21	-0.18
2.50	-125.44	-11.86	00.60	00.68	00.85	00.81	-3.25	-3.16	00.78
	-115.91	-13.22	1.64	01.65	01.64	01.63	-1.63	-1.61	01.66
	-190.10	-12.31	-2.55	-0.66	00.13	00.03	-7.50	-6.85	-0.24
3.00	-175.49	-19.09	00.36	00.42	00.56	00.53	-4.33	-4.23	00.50
	-162.38	-21.35	01.32	01.32	01.29	01.28	-2.49	-2.47	01.32
	-250.22	-17.03	-2.10	-0.49	00.10	00.03	-8.03	-7.31	-0.17
3.50	-234.56	-27.38	00.29	00.34	00.43	00.42	-5.11	-5.00	00.40
	-225.80	-34.25	00.75	00.76	00.78	00.77	-4.54	-4.51	00.78
	-285.75	-19.42	-1.88	-0.42	00.16	00.04	-7.97	-7.23	00.06
3.75	-264.67	-31.69	00.20	00.25	00.36	00.33	-5.41	-5.30	00.32
	-264.47	-41.68	00.55	00.56	00.59	00.60	-5.42	-5.39	00.57
	-328.00	-23.98	-2.36	-0.77	00.08	00.02	-9.20	-8.41	-0.09
4.00	-296.86	-35.41	00.37	00.40	00.44	00.40	-5.16	-5.04	00.45
	-296.24	-47.50	00.50	00.51	00.53	00.53	-5.68	-5.65	00.52

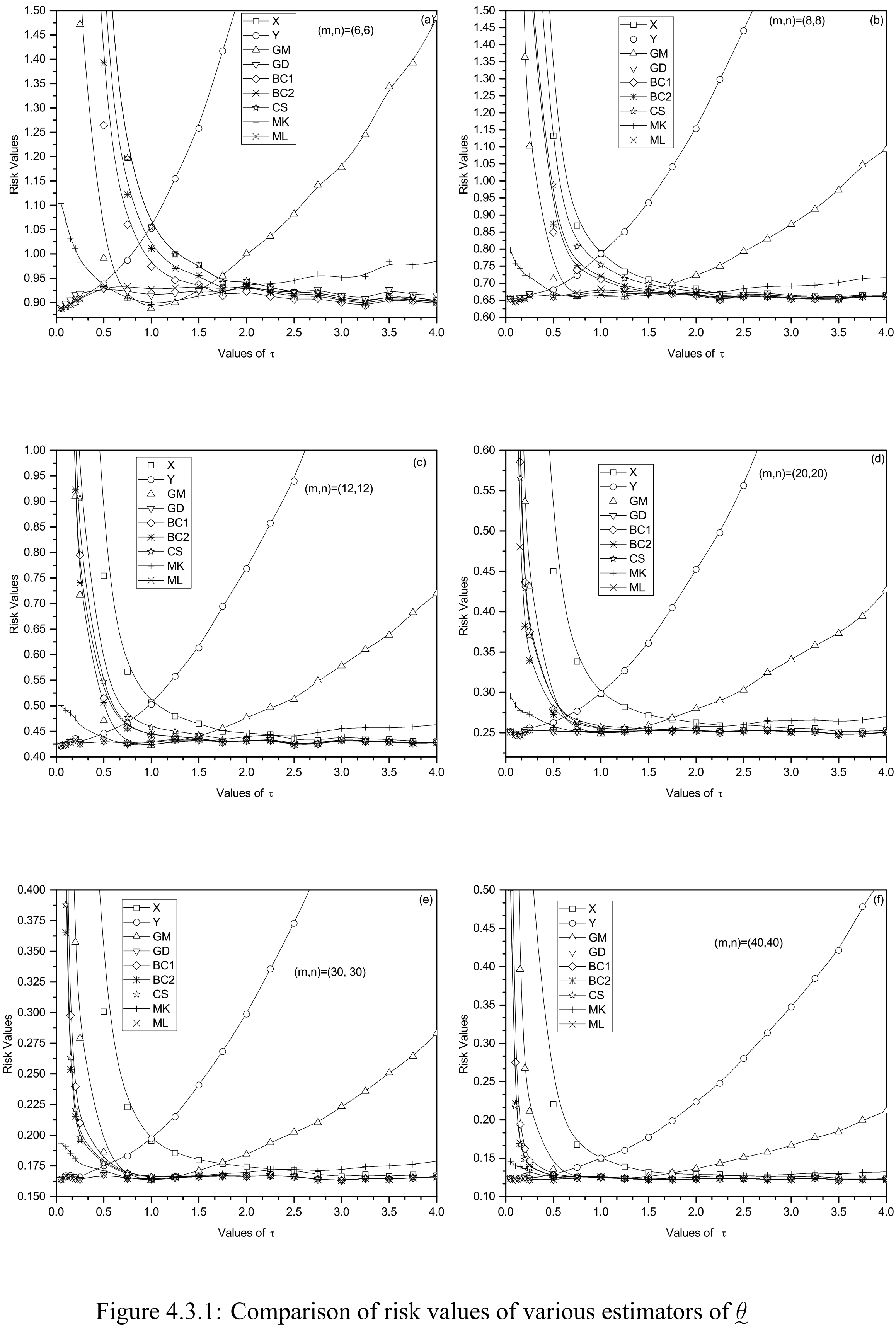

Figure 4.3.1: Comparison of risk values of various estimators of $\underset{\sim}{\theta}$

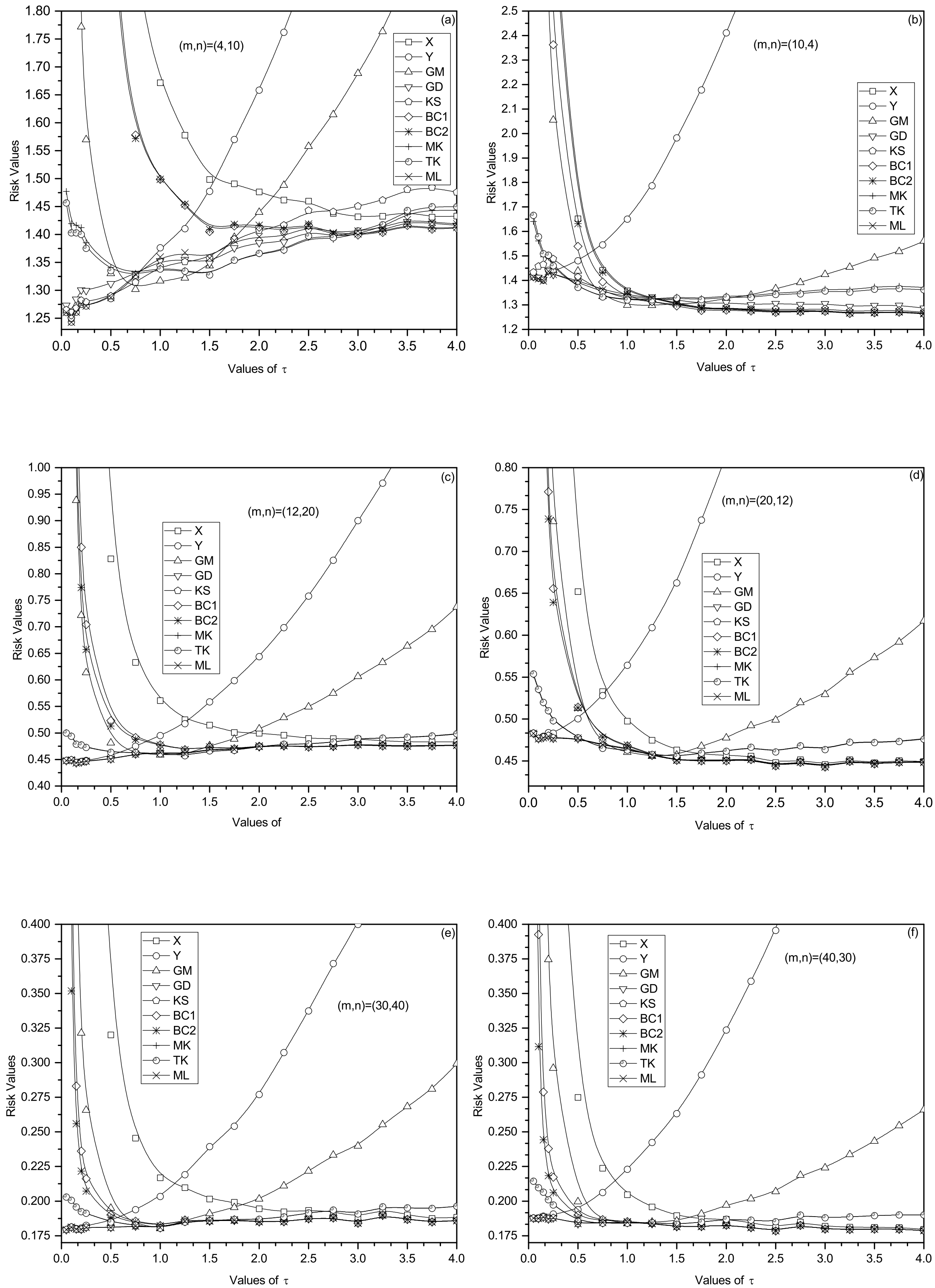

Figure 4.3.2: Comparison of risk values of various estimators of $\underset{\sim}{\theta}$

Table 4.3.4: Relative risk performances of various estimators for quantile vector when k=3

$(n_1, n_2, n_3) \rightarrow$		(10,10,10)							
$\tau_2 \downarrow$	$\tau_3 \downarrow$	$P(\underset{\sim}{d}^{X_2})$	$P(\underset{\sim}{d}^{X_3})$	$P(\underset{\sim}{d}^{NH})$	$P(\underset{\sim}{d}^{SZ})$	$P(\underset{\sim}{d}^{MK})$	$P(\underset{\sim}{d}^{TK})$	$P(\underset{\sim}{d}^{GM})$	$P(\underset{\sim}{d}^{ML})$
0.25	0.25	79.09	79.14	81.50	81.50	81.01	81.01	73.71	81.25
	0.50	72.40	57.98	73.08	73.08	72.18	72.18	65.95	72.86
	0.75	70.59	33.24	70.95	70.95	69.36	69.36	61.82	70.81
	1.00	69.38	-1.00	69.57	69.57	67.38	67.38	56.92	69.48
	1.25	69.35	-40.08	69.50	69.50	66.91	66.91	52.54	69.37
	2.00	69.33	-215.55	69.35	69.35	65.76	65.76	32.21	69.08
	3.00	68.32	-589.33	68.38	68.38	64.36	64.36	-9.75	68.23
0.75	0.25	33.00	70.37	70.75	70.75	69.23	69.23	61.82	70.61
	0.50	22.61	39.46	43.60	43.60	43.65	43.65	41.88	43.32
	0.75	19.08	19.16	31.61	31.61	32.46	32.46	32.35	31.37
	1.00	16.57	0.23	26.25	26.25	27.17	27.17	27.12	26.00
	1.25	15.49	-19.65	23.50	23.50	24.04	24.04	23.00	23.19
	2.00	15.97	-97.25	21.62	21.62	20.88	20.88	13.45	21.14
	3.00	13.94	-256.29	19.42	19.42	17.55	17.55	-5.74	18.71
1.00	0.25	-0.83	69.93	70.05	70.05	67.79	67.79	57.36	69.96
	0.50	-0.55	35.71	38.91	38.91	38.58	38.58	35.97	38.72
	0.75	1.45	16.89	26.38	26.38	27.29	27.29	27.20	26.14
	1.00	0.56	0.50	20.01	20.01	21.27	21.27	21.79	19.65
	1.25	-0.09	-16.76	16.74	16.74	17.82	17.82	17.83	16.46
	2.00	0.30	-80.27	13.26	13.26	13.19	13.19	8.55	12.78
	3.00	1.12	-199.59	12.52	12.52	11.38	11.38	-4.71	11.72
1.25	0.25	-38.24	69.46	69.60	69.60	67.07	67.07	53.08	69.36
	0.50	-27.94	35.59	37.95	37.95	37.00	37.00	32.46	37.85
	0.75	-18.73	16.33	24.10	24.10	24.68	24.68	23.71	23.68
	1.00	-17.02	0.23	16.78	16.78	17.75	17.75	17.68	16.56
	1.25	-14.44	-13.37	13.79	13.79	14.91	14.91	15.02	13.49
	2.00	-11.50	-67.92	9.69	9.69	10.06	10.06	7.07	9.30
	3.00	-11.00	-174.03	8.31	8.31	7.21	7.21	-5.59	7.74
2.00	0.25	-218.00	69.25	69.29	69.29	65.85	65.85	32.52	68.90
	0.50	-139.01	34.91	36.64	36.64	34.28	34.28	19.05	36.37
	0.75	-96.38	14.37	20.46	20.46	19.87	19.87	12.61	19.95
	1 .00	-80.58	0.58	13.43	13.43	13.40	13.40	8.86	12.95
	1.25	-68.09	-13.22	9.20	9.20	9.57	9.57	6.63	8.77
	2.00	-58.65	-57.06	4.55	4.55	4.14	4.14	0.28	4.29
	3.00	-52.43	-141.53	3.03	3.03	1.00	1.00	-9.87	2.86
3.00	0.25	-571.32	68.94	69.00	69.00	65.11	65.11	-6.49	68.88
	0.50	-355.44	34.00	35.64	35.64	32.56	32.56	-4.88	35.37
	0.75	-252.37	14.03	19.46	19.46	17.41	17.41	-5.70	18.73
	1.00	-210.48	-0.48	11.45	11.45	10.23	10.23	-6.63	10.78
	1.25	-177.78	-11.83	8.05	8.05	6.92	6.92	-6.36	7.23
	2.00	-142.02	-52.69	3.23	3.23	1.19	1.19	-9.74	3.18
	3.00	-131.67	-130.10	1.81	1.81	-1.96	-1.96	-17.52	1.14

Table 4.3.5: Relative risk performances of various estimators for quantile vector when k=3

$(n_1, n_2, n_3) \rightarrow$		(30,30,30)							
$\tau_2 \downarrow$	$\tau_3 \downarrow$	$P(\underset{\sim}{d}^{X_2})$	$P(\underset{\sim}{d}^{X_3})$	$P(\underset{\sim}{d}^{NH})$	$P(\underset{\sim}{d}^{SZ})$	$P(\underset{\sim}{d}^{MK})$	$P(\underset{\sim}{d}^{TK})$	$P(\underset{\sim}{d}^{GM})$	$P(\underset{\sim}{d}^{ML})$
0.25	0.25	79.52	79.43	82.13	82.13	81.59	81.59	74.18	82.13
	0.50	73.10	58.04	74.13	74.13	73.09	73.09	66.58	74.14
	0.75	71.29	31.86	71.83	71.83	70.03	70.03	61.85	71.85
	1.00	70.88	1.97	71.32	71.32	69.21	69.21	58.55	71.34
	1.25	70.11	-41.89	70.51	70.51	68.05	68.05	53.16	70.52
	2.00	69.71	-225.20	70.04	70.04	66.90	66.90	32.42	70.04
	3.00	69.82	-591.29	70.06	70.06	66.36	66.36	-8.31	70.07
0.75	0.25	33.43	71.35	71.98	71.98	70.33	70.33	62.41	71.99
	0.50	22.96	39.41	44.51	44.51	44.09	44.09	42.05	44.51
	0.75	19.16	20.01	33.78	33.78	33.91	33.91	33.43	33.77
	1.00	17.96	-0.74	28.35	28.35	28.44	28.44	27.97	28.36
	1.25	15.40	-20.86	24.92	24.92	24.83	24.83	23.53	24.92
	2.00	14.96	-105.12	22.17	22.17	20.81	20.81	12.42	22.17
	3.00	14.46	-263.07	21.38	21.38	19.15	19.15	-5.41	21.36
1.00	0.25	1.45	71.10	71.55	71.55	69.39	69.39	58.66	71.56
	0.50	0.71	37.48	41.45	41.45	40.69	40.69	37.8	41.46
	0.75	0.79	17.21	28.48	28.48	28.68	28.68	28.31	28.46
	1.00	-1.38	0.57	21.67	21.67	22.10	22.10	22.28	21.66
	1.25	-0.72	-18.59	18.29	18.29	18.46	18.46	18.09	18.26
	2.00	0.05	-83.65	14.93	14.93	14.29	14.29	9.31	14.92
	3.00	-0.53	-214.01	12.89	12.89	10.97	10.97	-6.89	12.86
1.25	0.25	-43.68	70.73	71.04	71.04	68.40	68.40	53.17	71.05
	0.50	-25.66	36.80	40.13	40.13	38.85	38.85	34.03	40.13
	0.75	-20.12	16.53	25.91	25.91	25.78	25.78	24.42	25.90
	1.00	-17.36	1.31	19.40	19.40	19.69	19.69	19.42	19.38
	1.25	-15.83	-14.62	15.25	15.25	15.50	15.50	15.25	15.23
	2.00	-12.95	-73.05	10.97	10.97	10.33	10.33	6.51	10.97
	3.00	-13.22	-188.65	9.06	9.06	6.90	6.90	-8.13	9.05
2.00	0.25	-225.57	70.05	70.33	70.33	67.19	67.19	33.06	70.34
	0.50	-136.71	35.97	38.34	38.34	35.75	35.75	20.25	38.36
	0.75	-98.70	14.03	21.77	21.77	20.61	20.61	12.71	21.76
	1.00	-82.51	0.38	14.94	14.94	14.27	14.27	9.27	14.92
	1.25	-72.60	-13.46	10.92	10.92	10.45	10.45	6.77	10.90
	2.00	-59.35	-61.22	6.39	6.39	5.06	5.06	0.31	6.39
	3.00	-56.03	-150.7	4.52	4.52	1.70	1.70	-10.51	4.56
3.00	0.25	-592.52	69.73	69.98	69.98	66.37	66.37	-8.15	69.98
	0.50	-375.10	33.83	36.40	36.40	33.03	33.03	-7.30	36.39
	0.75	-263.62	14.73	21.35	21.35	19.03	19.03	-5.47	21.33
	1.00	-214.55	0.06	13.19	13.19	11.15	11.15	-6.86	13.16
	1.25	-181.92	-12.24	9.60	9.60	7.93	7.93	-6.01	9.59
	2.00	-147.67	-57.16	4.89	4.89	2.40	2.40	-9.31	4.88
	3.00	-136.69	-136.04	2.86	2.86	-1.48	-1.48	-18.72	2.88

Table 4.3.6: Relative risk performances of various estimators for quantile vector when k=3

$(n_1, n_2, n_3) \rightarrow$		(20,20,20)							
$\tau_2 \downarrow$	$\tau_3 \downarrow$	$P(\underset{\sim}{d}^{X_2})$	$P(\underset{\sim}{d}^{X_3})$	$P(\underset{\sim}{d}^{NH})$	$P(\underset{\sim}{d}^{SZ})$	$P(\underset{\sim}{d}^{MK})$	$P(\underset{\sim}{d}^{TK})$	$P(\underset{\sim}{d}^{GM})$	$P(\underset{\sim}{d}^{ML})$
0.25	0.25	79.40	79.54	82.02	82.02	81.49	81.49	74.06	82.01
	0.50	73.06	58.77	74.08	74.08	73.06	73.06	66.60	74.09
	0.75	70.65	31.86	71.27	71.27	69.73	69.73	62.08	71.30
	1.00	69.71	-1.13	70.15	70.15	68.03	68.03	57.33	70.18
	1.25	69.65	-46.82	70.01	70.01	67.41	67.41	51.92	70.03
	2.00	70.05	-222.40	70.27	70.27	66.94	66.94	32.87	70.29
	3.00	69.42	-576.58	69.67	69.67	66.01	66.01	-6.77	69.69
0.75	0.25	33.18	71.45	71.92	71.92	70.14	70.14	62.17	71.96
	0.50	23.94	40.17	45.12	45.12	44.77	44.77	42.71	45.11
	0.75	17.55	18.12	32.44	32.44	32.81	32.81	32.47	32.40
	1.00	17.38	0.63	27.89	27.89	28.17	28.17	27.78	27.87
	1.25	16.16	-20.82	25.01	25.01	24.96	24.96	23.58	25.01
	2.00	15.42	-99.94	22.19	22.19	21.00	21.00	12.91	22.14
	3.00	14.04	-259.90	20.60	20.60	18.50	18.50	-5.36	20.57
1.00	0.25	-2.08	70.45	70.83	70.83	68.62	68.62	57.65	70.85
	0.50	-0.21	37.28	41.01	41.01	40.29	40.29	37.42	41.02
	0.75	0.23	17.02	28.06	28.06	28.41	28.41	28.08	28.02
	1.00	-0.56	-0.42	21.19	21.19	21.76	21.76	21.94	21.10
	1.25	-0.50	-18.19	17.98	17.98	18.47	18.47	18.27	17.94
	2.00	0.83	-80.48	15.39	15.39	14.96	14.96	10.21	15.33
	3.00	0.67	-209.21	13.51	13.51	11.75	11.75	-5.69	13.48
1.25	0.25	-38.19	70.64	70.91	70.91	68.25	68.25	53.28	70.91
	0.50	-28.96	35.83	39.07	39.07	37.84	37.84	32.93	39.08
	0.75	-23.42	15.30	24.16	24.16	24.09	24.09	22.65	24.14
	1.00	-18.23	-1.34	17.70	17.70	18.21	18.21	18.02	17.66
	1.25	-15.76	-15.20	14.89	14.89	15.37	15.37	15.17	14.84
	2.00	-12.26	-73.46	10.75	10.75	10.34	10.34	6.64	10.66
	3.00	-12.57	-183.13	8.95	8.95	7.21	7.21	-6.84	8.89
2.00	0.25	-230.94	69.08	69.35	69.35	66.00	66.00	30.95	69.38
	0.50	-142.69	34.39	36.86	36.86	34.44	34.44	18.77	36.87
	0.75	-101.97	14.68	21.69	21.69	20.59	20.59	12.59	21.67
	1.00	-83.97	-0.84	13.92	13.92	13.38	13.38	8.42	13.86
	1.25	-71.47	-12.95	10.92	10.92	10.61	10.61	7.08	10.88
	2.00	-61.58	-60.31	5.44	5.44	3.98	3.98	-0.95	5.46
	3.00	-56.47	-148.25	3.86	3.86	1.07	1.07	-10.81	3.96
3.00	0.25	-588.31	69.44	69.65	69.65	65.87	65.87	-8.40	69.67
	0.50	-376.73	34.09	36.23	36.23	32.75	32.75	-7.67	36.24
	0.75	-255.25	15.20	21.61	21.61	19.56	19.56	-3.98	21.55
	1.00	-209.11	-0.02	12.92	12.92	11.28	11.28	-5.93	12.85
	1.25	-180.65	-12.81	8.74	8.74	7.21	7.21	-6.51	8.69
	2.00	-150.18	-55.91	4.48	4.48	1.89	1.89	-10.27	4.52
	3.00	-139.76	-138.82	2.06	2.06	-2.93	-2.93	-20.84	2.21

Table 4.3.7: Relative risk performances of various estimators for quantile vector when k=3

$(n_1, n_2, n_3) \rightarrow$		(20,30,40)							
$\tau_2 \downarrow$	$\tau_3 \downarrow$	$P(\underset{\sim}{d}^{X_2})$	$P(\underset{\sim}{d}^{X_3})$	$P(\underset{\sim}{d}^{NH})$	$P(\underset{\sim}{d}^{SZ})$	$P(\underset{\sim}{d}^{MK})$	$P(\underset{\sim}{d}^{TK})$	$P(\underset{\sim}{d}^{GM})$	$P(\underset{\sim}{d}^{ML})$
0.25	0.25	75.68	76.35	77.47	77.47	77.10	77.11	74.22	77.46
	0.50	67.35	61.36	68.08	68.08	67.45	67.46	64.75	68.09
	0.75	65.85	49.82	66.28	66.28	65.30	65.3	61.35	66.28
	1.00	65.10	34.95	65.35	65.35	64.02	64.03	57.86	65.36
	1.25	64.28	14.60	64.46	64.46	62.83	62.83	52.89	64.47
	2.00	63.72	-67.27	63.89	63.89	61.89	61.89	36.62	63.89
	3.00	63.76	-228.21	63.86	63.86	61.50	61.51	4.92	63.87
0.75	0.25	42.21	65.92	66.12	66.12	64.99	65.00	61.40	66.12
	0.50	27.20	38.25	39.86	39.86	39.24	39.25	38.68	39.86
	0.75	21.47	24.76	29.44	29.45	29.27	29.28	29.38	29.45
	1.00	18.68	15.32	24.73	24.74	24.73	24.73	24.66	24.73
	1.25	16.64	5.68	21.67	21.66	21.65	21.65	20.79	21.65
	2.00	16.25	-25.66	19.48	19.48	18.89	18.88	13.31	19.49
	3.00	16.55	-84.42	18.91	18.92	17.69	17.68	1.59	18.93
1.00	0.25	22.40	65.08	65.21	65.21	63.84	63.85	58.50	65.21
	0.50	13.21	35.58	36.73	36.74	35.81	35.82	34.59	36.74
	0.75	9.91	21.41	24.68	24.69	24.46	24.46	24.52	24.68
	1.00	7.86	12.37	19.25	19.26	19.34	19.35	19.52	19.26
	1.25	7.75	5.55	16.66	16.67	16.84	16.85	16.74	16.67
	2.00	6.47	-20.01	13.16	13.16	13.08	13.08	9.73	13.16
	3.00	6.08	-67.61	11.54	11.55	10.94	10.93	-0.48	11.52
1.25	0.25	-5.30	64.66	64.77	64.77	63.16	63.17	54.98	64.77
	0.50	-1.86	34.33	35.18	35.19	34.02	34.03	31.76	35.19
	0.75	-1.42	19.40	22.27	22.28	21.90	21.91	21.45	22.27
	1.00	-1.62	11.15	16.66	16.67	16.68	16.68	16.67	16.66
	1.25	-0.50	4.42	14.08	14.09	14.31	14.31	14.20	14.08
	2.00	-1.47	-17.62	9.92	9.91	10.04	10.04	7.59	9.89
	3.00	-0.29	-57.39	8.93	8.93	8.58	8.57	-0.36	8.90
2.00	0.25	-106.88	64.79	64.85	64.85	62.90	62.91	43.67	64.85
	0.50	-61.11	33.05	33.57	33.58	31.77	31.78	23.82	33.59
	0.75	-43.14	18.04	19.96	19.96	18.99	18.99	15.08	19.97
	1.00	-33.91	10.72	14.41	14.43	13.90	13.91	11.66	14.42
	1.25	-28.94	4.10	10.94	10.94	10.75	10.75	9.03	10.93
	2.00	-23.90	-13.67	6.64	6.64	6.59	6.59	4.10	6.61
	3.00	-22.83	-47.63	4.62	4.60	3.95	3.93	-3.40	4.60
3.00	0.25	-331.36	63.52	63.58	63.58	61.45	61.47	17.97	63.58
	0.50	-182.69	32.30	32.79	32.80	30.55	30.57	9.73	32.80
	0.75	-126.00	17.84	19.35	19.37	17.60	17.61	5.03	19.37
	1.00	-95.71	9.94	13.40	13.41	12.37	12.37	4.06	13.39
	1.25	-84.01	3.86	9.79	9.80	9.13	9.13	2.54	9.77
	2.00	-69.08	-13.83	4.74	4.72	3.90	3.89	-2.04	4.74
	3.00	-61.30	-42.31	3.18	3.16	1.83	1.81	-7.14	3.16

Table 4.3.8: Relative risk performances of various estimators for quantile vector when k=3

$(n_1, n_2, n_3) \rightarrow$		(40,30,20)							
$\tau_2 \downarrow$	$\tau_3 \downarrow$	$P(\underset{\sim}{d}^{X_2})$	$P(\underset{\sim}{d}^{X_3})$	$P(\underset{\sim}{d}^{NH})$	$P(\underset{\sim}{d}^{SZ})$	$P(\underset{\sim}{d}^{MK})$	$P(\underset{\sim}{d}^{TK})$	$P(\underset{\sim}{d}^{GM})$	$P(\underset{\sim}{d}^{ML})$
0.25	0.25	60.15	57.58	62.48	62.48	62.02	62.01	51.80	62.46
	0.50	49.92	27.61	50.76	50.76	49.73	49.73	41.93	50.77
	0.75	47.44	-6.22	47.96	47.96	46.35	46.35	38.23	47.98
	1.00	46.90	-51.69	47.24	47.24	45.12	45.13	35.39	47.26
	1.25	46.07	-106.18	46.43	46.43	44.14	44.15	32.26	46.44
	2.00	45.38	-342.95	45.68	45.68	42.87	42.88	19.42	45.69
	3.00	45.13	-844.29	45.40	45.40	42.30	42.31	-4.81	45.41
0.75	0.25	11.80	44.84	46.26	46.25	45.21	45.19	36.94	46.26
	0.50	6.59	13.49	20.89	20.89	21.01	21.01	19.17	20.88
	0.75	4.57	-2.78	13.67	13.67	13.86	13.87	13.47	13.67
	1.00	4.05	-18.26	11.16	11.17	11.10	11.11	10.96	11.16
	1.25	4.33	-33.48	9.94	9.95	9.66	9.67	9.32	9.95
	2.00	3.31	-103.10	8.28	8.29	7.24	7.25	4.41	8.28
	3.00	3.52	-246.89	8.12	8.12	6.57	6.59	-2.59	8.12
1.00	0.25	-16.60	44.30	45.33	45.32	43.71	43.69	33.03	45.34
	0.50	-7.76	12.84	18.70	18.69	18.67	18.66	16.46	18.70
	0.75	-6.24	-2.70	10.46	10.47	10.65	10.65	10.39	10.44
	1.00	-5.37	-15.19	7.70	7.72	7.71	7.72	7.95	7.71
	1.25	-5.14	-27.76	6.37	6.38	6.20	6.21	6.39	6.36
	2.00	-3.71	-79.06	5.17	5.17	4.38	4.40	3.05	5.17
	3.00	-3.31	-184.55	4.68	4.69	3.19	3.21	-2.81	4.68
1.25	0.25	-54.94	44.23	45.02	45.01	42.96	42.93	28.51	45.04
	0.50	-25.85	11.88	17.01	17.01	16.74	16.73	13.46	16.99
	0.75	-17.89	-2.42	9.01	9.01	9.06	9.06	8.28	9.01
	1.00	-14.49	-13.23	6.16	6.17	6.15	6.16	6.14	6.15
	1.25	-11.71	-25.17	5.09	5.11	4.84	4.85	5.00	5.09
	2.00	-10.27	-68.02	3.56	3.57	2.73	2.75	1.85	3.56
	3.00	-9.05	-154.36	3.09	3.09	1.70	1.72	-2.77	3.08
2.00	0.25	-211.33	43.68	44.36	44.35	41.54	41.51	10.63	44.37
	0.50	-99.23	10.73	15.19	15.19	14.10	14.09	4.05	15.17
	0.75	-64.39	-1.43	7.42	7.41	6.79	6.79	2.07	7.40
	1.00	-48.84	-11.67	4.41	4.42	3.94	3.95	1.49	4.41
	1.25	-43.08	-21.84	2.72	2.75	1.88	1.89	0.13	2.77
	2.00	-33.94	-53.82	1.76	1.77	0.48	0.50	-1.22	1.77
	3.00	-31.55	-122.95	1.27	1.28	-0.85	-0.82	-5.05	1.29
3.00	0.25	-546.88	43.19	43.90	43.89	40.64	40.61	-26.41	43.90
	0.50	-246.71	11.35	15.09	15.08	13.42	13.40	-11.93	15.07
	0.75	-155.50	-1.64	6.60	6.61	5.43	5.42	-8.28	6.59
	1.00	-118.31	-10.68	3.72	3.74	2.53	2.53	-6.82	3.72
	1.25	-101.43	-19.94	2.15	2.17	0.63	0.64	-6.78	2.18
	2.00	-79.46	-48.78	1.10	1.11	-0.96	-0.94	-6.68	1.09
	3.00	-71.92	-110.70	0.68	0.69	-2.14	-2.11	-9.27	0.71

Table 4.3.9: Relative risk performances of various estimators for quantile vector when k=3

$(n_1, n_2, n_3) \rightarrow$		(10,20,30)							
$\tau_2 \downarrow$	$\tau_3 \downarrow$	$P(\underset{\sim}{d}^{X_2})$	$P(\underset{\sim}{d}^{X_3})$	$P(\underset{\sim}{d}^{NH})$	$P(\underset{\sim}{d}^{SZ})$	$P(\underset{\sim}{d}^{MK})$	$P(\underset{\sim}{d}^{TK})$	$P(\underset{\sim}{d}^{GM})$	$P(\underset{\sim}{d}^{ML})$
0.25	0.25	76.35	77.15	77.76	77.76	77.40	77.41	75.88	77.74
	0.50	67.88	64.30	68.45	68.45	67.92	67.94	66.46	68.43
	0.75	65.58	55.26	65.84	65.85	64.98	65.00	62.38	65.78
	1.00	65.25	45.23	65.41	65.42	64.29	64.31	59.68	65.37
	1.25	64.38	32.25	64.49	64.50	63.17	63.19	55.82	64.42
	2.00	64.09	-21.32	64.12	64.14	62.39	62.41	42.25	64.08
	3.00	63.76	-131.02	63.80	63.81	61.86	61.88	14.80	63.76
0.75	0.25	48.98	66.45	66.53	66.54	65.58	65.60	63.51	66.50
	0.50	30.61	39.38	40.17	40.20	39.53	39.56	39.45	40.12
	0.75	23.92	27.23	29.81	29.85	29.56	29.59	29.90	29.70
	1.00	21.68	19.78	25.54	25.58	25.48	25.50	25.59	25.46
	1.25	20.76	13.91	23.68	23.72	23.55	23.56	23.02	23.69
	2.00	17.99	-9.66	19.90	19.94	19.48	19.49	14.88	19.89
	3.00	17.21	-49.69	18.77	18.81	17.82	17.82	4.14	18.80
1.00	0.25	32.57	65.44	65.49	65.50	64.38	64.41	60.87	65.49
	0.50	19.13	36.55	37.02	37.05	36.17	36.20	35.64	36.91
	0.75	14.67	24.13	25.88	25.92	25.53	25.55	25.78	25.80
	1.00	12.35	16.58	20.34	20.39	20.34	20.36	20.69	20.33
	1.25	10.71	10.70	17.46	17.49	17.68	17.69	17.69	17.37
	2.00	9.93	-6.50	14.28	14.32	14.43	14.44	11.91	14.30
	3.00	10.01	-37.14	13.21	13.26	12.84	12.83	3.64	13.18
1.25	0.25	14.57	64.90	64.92	64.93	63.62	63.65	58.29	64.92
	0.50	8.01	35.37	35.77	35.80	34.72	34.75	33.36	35.69
	0.75	5.01	21.99	23.22	23.26	22.78	22.80	22.66	23.10
	1.00	4.93	15.33	18.11	18.17	18.04	18.06	18.32	18.06
	1.25	4.71	9.71	14.87	14.92	15.10	15.12	15.28	14.87
	2.00	3.92	-5.46	11.29	11.32	11.64	11.64	9.85	11.22
	3.00	4.52	-32.92	10.04	10.06	9.96	9.95	2.45	9.94
2.00	0.25	-63.57	64.41	64.43	64.44	62.92	62.95	49.16	64.40
	0.50	-36.75	33.91	34.14	34.17	32.64	32.67	27.05	34.10
	0.75	-24.47	20.34	21.14	21.19	20.21	20.23	17.69	21.13
	1.00	-19.63	13.43	15.41	15.46	15.02	15.04	13.66	15.42
	1.25	-18.57	7.90	11.40	11.43	11.32	11.33	10.23	11.37
	2.00	-14.94	-5.04	7.28	7.26	7.59	7.58	5.78	7.22
	3.00	-13.10	-25.87	5.49	5.44	5.63	5.61	0.34	5.41
3.00	0.25	-226.81	64.46	64.48	64.48	62.81	62.85	31.05	64.48
	0.50	-124.15	33.03	33.22	33.24	31.36	31.40	16.42	33.20
	0.75	-86.72	19.87	20.42	20.48	19.03	19.06	10.28	20.48
	1.00	-67.56	12.60	14.18	14.23	13.24	13.25	7.25	14.21
	1.25	-55.55	8.00	10.84	10.88	10.34	10.35	5.98	10.84
	2.00	-46.86	-4.25	5.84	5.81	5.67	5.65	1.69	5.85
	3.00	-42.46	-25.22	3.70	3.60	3.16	3.11	-3.71	3.68

4.3.4 Conclusions

We note here that, in the literature most of the results on estimation of quantiles are for a single parameter $\theta = \mu + \eta\sigma$ either using one or more populations. In this section, we consider the simultaneous estimation of the quantile vector $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$ which is important from an application point of view. The loss function is taken as the sum of the quadratic loss functions. It should be noted that, Kumar and Tripathy (2011) considered this model with $k = 2$ and estimated the first component θ_1 with respect to a quadratic loss function. We have implemented the Brewster and Zidek (1974) technique to the case of estimating a vector parameter, which is interesting. Further we have proposed some new estimators such as the $\underset{\sim}{d}^{\bar{X}_2}$, $\underset{\sim}{d}^{GM}$, and $\underset{\sim}{d}^{ML}$ which were not considered previously. First, we derived sufficient conditions for improving equivariant estimators and in the process some complete class results obtained for the case $k = 2$. We have constructed some improved estimators using one of our result obtained in Section 4.3.1. However, the analytical comparison of these estimators is not possible. We have conducted a detailed simulation study to numerically compare these estimators which can be used in practice. Specifically, we have done the numerical comparison for the case $k = 2$ and $k = 3$. Our conclusions regarding the use of the estimators are completely based on the simulation study as no analytical comparison is possible among all the estimators. It will be interesting to generalize the results to case of $k \geq 3$ normal populations, where proving inadmissibility of these estimators will be challenging. However, we feel that, for the case $k \geq 3$, the well structured estimators will form a complete class.

Below we present some examples where our model fits well and also compute the estimates for practical purposes. In the examples below we have taken the value of $\eta = 1.960$ for convenient.

Example 4.3.1 *We consider the example discussed in Hines et al. (2008), (p. 290). Suppose a manufacturer of video display units produces two micro circuit designs design A and design B. He wants to test whether the two design produce same current flow. The summarized data for design A are given by* $n_1 = 15$, $\underset{\sim}{d}^{\bar{x}_1} = 24.2$, $s_1^2 = 10$ *where as the data for design B are given by* $n_2 = 10$, $\underset{\sim}{d}^{\bar{x}_2} = 23.9$, $s_2^2 = 20$. *It is also given that both the data follow normal distributions with a common mean. The experimental conditions ensures that the variances are unequal. This is a situation where our model will be very much useful. The several estimators for quantiles are calculated as* $\underset{\sim}{d}^{\bar{X}_1} = (25.97, 26.71)$, $\underset{\sim}{d}^{\bar{X}_2} = (25.67, 26.41)$, $\underset{\sim}{d}^{GM} = (25.85, 26.59)$, $\underset{\sim}{d}^{GD} = (25.92, 26.65)$, $\underset{\sim}{d}^{KS} = (25.92, 26.66)$, $\underset{\sim}{d}^{BC1} = (25.97, 26.71)$, $\underset{\sim}{d}^{BC2} = (25.94, 26.68)$, $\underset{\sim}{d}^{MK} = (25.88, 26.61)$, $\underset{\sim}{d}^{TK} = (25.88, 26.61)$ *and* $\underset{\sim}{d}^{ML} = (25.92, 26.65)$. *If the variances of both the data set differ significantly we may use either the estimator* $\underset{\sim}{d}^{GD}$, $\underset{\sim}{d}^{ML}$, *or* $\underset{\sim}{d}^{BC1}$. *If the variances differ marginally we may use either* $\underset{\sim}{d}^{KS}$, *or* $\underset{\sim}{d}^{MK}$.

Example 4.3.2 *Rohatgi and Saleh (2003), (p.515) discussed one example regarding the mean life time (in hours) of light bulbs. Suppose a random sample of 9 bulbs has sample mean 1309 hours with standard deviation of 420 hours. A second sample of 16 bulbs chosen from a different batch has sample mean 1205 hours and standard deviation 390 hours. A two sample t-test fails to reject the hypothesis that the means are equal. This is a situation where our model will be useful. Suppose we want to know the life time of both the bulbs at any instant of time then we can use our estimators. The various estimators are calculated as* $\underset{\sim}{d}^{\bar{X}_1} = (1543.45, 1526.70)$, $\underset{\sim}{d}^{\bar{X}_2} = (1439.45, 1422.70)$, $\underset{\sim}{d}^{GM} = (1476.89, 1460.14)$, $\underset{\sim}{d}^{GD} = (1460.82, 1444.08)$, $\underset{\sim}{d}^{KS} = (1458.47, 1441.73)$, $\underset{\sim}{d}^{BC1} = (1501.44, 1484.69)$, $\underset{\sim}{d}^{BC2} = (1498.38, 1481.64)$, $\underset{\sim}{d}^{MK} = (1474.51, 1457.77)$, $\underset{\sim}{d}^{TK} = (1474.18, 1457.43)$ *and* $\underset{\sim}{d}^{ML} = (1457.08, 1440.33)$. *Also a F-test fails to reject the hypothesis that the population variances are equal. In this situation we recommend to use either* $\underset{\sim}{d}^{TK}$, *or* $\underset{\sim}{d}^{MK}$.

Chapter 5

Estimating Quantiles and Ordered Scales of Two Exponential Populations with a Common Location Using Censored Samples

5.1 Introduction

In previous chapters, we have considered estimation of common mean and quantiles of two or more normal populations when all the samples are available. However, in this chapter, we consider the estimation of quantiles and ordered scale parameters using type-II censored samples from two exponential populations, assuming equality restrictions on the location parameter.

In practice it is not always possible to observe all the sample values because of some constraints like time and cost in certain life testing experiments. Under such circumstances type-II censored samples are very much useful for inference purposes. The problem of estimating parameters of exponential distribution using censored samples has received considerable attention and has been studied by several authors in the recent past. The applications of this type of models are seen in industry, public health, business, social sciences and related fields which arise naturally in the study of reliability, life testing and survival analysis. Let us consider a situation where the assumption of equality on the location parameters is justified. Suppose two brands of electrical products have been newly lunched in the market. The life times of the products being random follow exponential distribution. It is also expected that the minimum guarantee time (or equivalently the location parameter μ) of both the products are same due to market competition whereas the residual life times(or equivalently the scale parameters) may be different. To carry out a life testing procedure, say m and n units from each of the two brands have been put for life testing. The experimenter could able to observe only $r(\leq m)$ and $s(\leq n)$ failure times. On the basis of these sample values one needs to draw the inference on the mean life times or the quantiles of the products. Most of the commonly used censoring schemes available in the literature are type-I (when number of observations are random and time is fixed), type-II (when number of observations are fixed and time is random), random censoring (both time and number of observations are random) or a mixture of these. For a quick review on estimation of parameters of exponential population using such types of

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[0]The content of this chapter (Section 5.3) has been published in *Chilean Journal of Statistics*, Vol. 8, No. 1, Pages 87 - 101.

conventional censoring schemes, we refer to Lawless (2003) and Johnson et al. (1994). Some practical examples also have been discussed in Lawless (2003) where these types of censoring schemes are useful.

It should be noted that, type-II censoring is a special case of progressive type-II censoring scheme. A lot of attention has been paid in estimating the parameters of an exponential population using progressive type-II censored samples by several authors in the recent past. For some classical as well as decision theoretic results in this direction, we refer to Balakrishnan and Sandhu (1996), Chandrasekar et al. (2002), and Madi (2010). For some recent updates and detailed review on estimation of parameters of an exponential population using progressive type-II censored samples, one may refer to Balakrishnan and Cramer (2014) and the references cited therein.

A lot of attention has been paid on estimation of parameters using censored samples when single population is available using a decision theoretic approach. However, a less attention has been paid to estimating the parameters when more than one exponential population is available. For example, Chiou and Cohen (1984) considered estimation of the common location parameter of two exponential populations using type-II right censored data when the scale parameters are unknown. Elfessi and Pal (1991) considered the estimation of common scale and the location parameters of $k(\geq 2)$ exponential populations using type-II right censored data. Yike and Heliang (1999) considered the Bayesian estimation of ordered location parameters of two exponential populations under a multiple type-II censoring scheme. Tripathy (2016) obtained classes of equivariant estimators and derived some inadmissibility results for estimating the common location parameter of two exponential populations using type-II right censored data.

The main objective of this chapter is to estimate the quantiles and ordered scales of two exponential populations assuming location parameters to be equal and the samples are type-II censored, using decision theoretic approach. First (in Section 5.2), we take up the problem of estimating quantiles $\theta = \mu + \eta\sigma_1$, of the first population, when the parameter μ is common. Exponential quantiles are very much useful in the study of reliability, life testing and survival analysis and some related areas. For some practical application of exponential quantiles we refer to Epstein (1962), Epstein and Sobel (1954) and Saleh (1981). We refer to Ghosh and Razmpour (1984), Rukhin (1986), Jin and Crouse (1998b), Sharma and Kumar (1994) and Jin and Crouse (1998a) for some excellent results and review on estimation of common location or/and quantiles of two or more exponential populations when full sample is available. In Section 5.2.1 we discuss the model and present some basic results. In Section 5.2.2, we propose a class of estimators which contain the UMVUE of quantiles θ and obtain estimators dominating the UMVUE. In Section 5.2.3, we derive sufficient conditions for improving equivariant estimators and as a consequence some complete class results have been obtained. Most importantly, in Section 5.2.4, we carry out a simulation study to numerically compare the risk values as well as the percentage of relative risk improvements of all the proposed estimators which may be useful for practical purposes. Finally we conclude with our remarks in Section 5.2.5.

Next (in Section 5.3) we consider the same model with inequality restrictions (ordered restriction) on the scale parameters, that is $\sigma_1 \leq \sigma_2$, and estimate the vector $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$. In the above example, if one of the brands (say first brand) uses the traditional technology and the other (second brand) uses the modern technology, then it is natural to assume that $\sigma_1 \leq \sigma_2$. Under this situation one wishes to draw inference on the vector parameter $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$. The problem of estimating the ordered parameters of various distribution functions has been studied by several researchers in the recent past, when full samples are available. For some results on estimation of ordered parameters of two or more exponential populations we refer to Misra and Singh (1994), Jin and Pal (1991), Vijayasree et al. (1995), and Jana and Kumar (2015). Some work has been done in estimating the ordered parameters (means or variances) when the underlying distribution is normal. We refer to Chang et al. (2012) and Tripathy and Kumar (2011) for some results on estimating ordered parameters of normal populations. In Section 5.3, we consider the simultaneous estimation of ordered scale parameters, that is, the vector $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2) : \sigma_1 \leq \sigma_2$ using type-II right censored samples from two exponential populations. Section 5.3.1 introduces the MLE and the UMVUE without considering order restriction on the scale parameters. Then under order restriction on the scale parameters, we derive the restricted maximum likelihood estimator for $\underset{\sim}{\sigma}$. In Section 5.3.2, we obtain classes of equivariant estimators and prove some inadmissibility results in these classes. Using these results, we obtain improved estimators which dominate the MLE and the UMVUE with respect to the risk function. In Section 5.3.3, a detailed simulation study has been carried out in order to numerically compare the relative risk performances of all the proposed estimators and recommendations have been made regarding their use. Finally, Section 5.3.4 concludes the remarks.

5.2 Estimating Quantiles of Exponential Populations with Common Location Using Censored Samples

Suppose we have type-II right censored random samples from two exponential populations with a common location parameter and possibly different scale parameters. More specifically, let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}$ $(2 \leq r \leq m)$ and $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(s)}$ $(2 \leq s \leq n)$ be the ordered observations taken from random samples of sizes m and n which follow exponential distributions with a common location parameter μ and possibly different scale parameters σ_1 and σ_2 respectively. We denote $Ex(\mu, \sigma_i)$ the exponential population with probability density function

$$f(t, \mu, \sigma_i) = \frac{1}{\sigma_i} \exp\{-(t-\mu)/\sigma_i\}, \quad t > \mu, \sigma_i > 0, -\infty < \mu < \infty;\ i = 1, 2. \qquad (5.2.1)$$

The problem is to estimate the p^{th} quantile $\theta = \mu + \eta\sigma_1$ of the first population, where $0 < \eta = -\log(1-p)$; $0 < p < 1$. The loss function is taken as

$$L(d, \underset{\sim}{\alpha}) = \left(\frac{d-\theta}{\sigma_1}\right)^2, \qquad (5.2.2)$$

where d is an estimate for estimating the quantile θ and $\underset{\sim}{\alpha} = (\mu, \sigma_1, \sigma_2)$. We evaluate the performance of an estimator for quantile with the help of the risk function

$$R(d, \underset{\sim}{\alpha}) = E_{\underset{\sim}{\alpha}}(L(d, \underset{\sim}{\alpha})).$$

We note that, for $\eta = 0$, the problem reduces to the problem of estimating common location parameter μ of two exponential populations using type-II censored samples and has been well investigated by Chiou and Cohen (1984) and Tripathy (2016). However, we extend some of their decision theoretic results to the case of estimating quantiles, that is, when $\eta \neq 0$. Moreover, the results of Sharma and Kumar (1994) can be derived as a particular case of our results by choosing $m = r$, $n = s$ and $m = n$. Basically they have obtained some inadmissibility results for estimating quantiles θ assuming the sample sizes are equal. They also obtained estimators which dominate the UMVUE in terms of risk values. However, in practice one would be interested to know the percentage of risk improvements approximately. Taking advantages of computational facilities we compare all the proposed estimators numerically. Hence it fills the gap in the literature which is not available.

5.2.1 Construction of Some Basic Estimators for Quantiles

In this section, we discuss the model and derive some baseline estimators for the quantile $\theta = \mu + \eta\sigma_1$. Specifically we obtain the MLE, a modification to the MLE, and the UMVUE for θ.

Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}$, $(2 \leq r \leq m)$ be the r smallest ordered observations taken from a random sample of size m having probability density function $Ex(\mu, \sigma_1)$ as given in (5.2.1). Similarly, let $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(s)}$, $(2 \leq s \leq n)$ be the s smallest ordered observations taken from a random sample of size n having probability density function $Ex(\mu, \sigma_2)$ as given in (5.2.1). The samples drawn from two populations are assumed to be statistically independent.

For this particular model a sufficient statistic is (U_1, U_2, Z), where $Z = \min(X_{(1)}, Y_{(1)})$, $U_1 = \frac{1}{m}[\sum_{i=1}^{r} X_{(i)} + (m-r)X_{(r)}]$, and $U_2 = \frac{1}{n}[\sum_{j=1}^{s} Y_{(j)} + (n-s)Y_{(s)}]$. The joint probability density function of $\underset{\sim}{U} = (U_1, U_2, Z)$ is given by,

$$f_{\underset{\sim}{U}}(\underset{\sim}{u}) = K(u_1 - z)^{r-1}(u_2 - z)^{s-1}\Big(\frac{r-1}{u_1 - z} + \frac{s-1}{u_2 - z}\Big) \exp\left\{ -\frac{m(u_1 - \mu)}{\sigma_1} - \frac{n(u_2 - \mu)}{\sigma_2} \right\},$$
$$u_1 > x_{(1)}, u_2 > y_{(1)}, z > \mu \tag{5.2.3}$$

where $K = \frac{m^r n^s}{\Gamma s \Gamma r \sigma_1^r \sigma_2^s}$, (see Chiou and Cohen (1984) and Tripathy (2016)). We also note that the random variable Z follows an exponential distribution with location parameter μ and scale parameter $1/p$, where $p = m/\sigma_1 + n/\sigma_2$. The MLEs of μ, σ_1 and σ_2 are obtained by Tripathy (2016) and are given by Z, $m(U_1 - Z)/r$ (say $\hat{\sigma}_{1ML}$), and $n(U_2 - Z)/s$ (say $\hat{\sigma}_{2ML}$) respectively. Using the MLEs of μ and σ_1, we obtain the MLE of the quantile $\theta = \mu + \eta\sigma_1$ as

$$d_{ML} = Z + \eta\hat{\sigma}_{1ML}. \tag{5.2.4}$$

Further using the modified MLE of the common location parameter μ (motivated by Ghosh and Razmpour (1984)), we propose a modification to the MLE of the quantile θ as

$$d_{MM} = Z - \frac{1}{\hat{p}} + \eta\hat{\sigma}_{1ML}, \tag{5.2.5}$$

where $\hat{p} = m/\hat{\sigma}_{1ML} + n/\hat{\sigma}_{2ML}$. It is also noted that the sufficient statistics $(U_1 - Z, U_2 - Z)$ and Z are independent and also complete (see Chiou and Cohen (1984)). Using the complete and sufficient statistics $(U_1 - Z, U_2 - Z, Z)$, one can easily obtain the UMVUE of the common location parameter μ as given in Tripathy (2016) and derived by Chiou and Cohen (1984). Let us denote $V_1 = U_1 - Z$, $V_2 = U_2 - Z$. We note that $E(V_1) = \frac{r}{m}\sigma_1 - p^{-1}$ and $E[(\frac{V_2}{s-1})^{-1} + (\frac{V_1}{r-1})^{-1}]^{-1} = p^{-1}$. Using these results one can easily derive the UMVUE of the quantile θ as,

$$d_{MV} = Z + \frac{V_1 V_2 (k-1)}{(r-1)V_2 + (s-1)V_1} + kV_1, \tag{5.2.6}$$

where $k = \eta m/r$.

5.2.2 Improving Upon the UMVUE

In this section, we consider a class of estimators which contain the UMVUE for $\theta = \mu + \eta\sigma_1$. Using a technique of Brewster and Zidek (Brewster and Zidek (1974)), we obtain an estimator which dominates the UMVUE with respect to the loss function (5.2.2). Let us consider the class of estimators for estimating the quantile $\theta = \mu + \eta\sigma_1$ as $D = \{d_c : c \in R\}$ where

$$d_c = Z + \frac{V_1V_2(k-1)}{(r-1)V_2 + (s-1)V_1} + kcV_1. \tag{5.2.7}$$

It should be noted that this class contains the UMVUE d_{MV} for $c = 1$.

Let us denote $c_1 = \frac{r}{r+1}$, $c_2 = \frac{\eta m(r-2)+1}{\eta m(r-1)}$, $c_{12} = \max\{c_1, c_2\}$, $c_3 = \hat{c}(\beta^-)$, where $\beta^- = \frac{r+1}{2m} - \frac{\sqrt{(r+1)^2 - 4\eta m}}{2m}$. Further define the constants

$$c_* = \begin{cases} c_1, & \text{if } c > c_1 \\ c_2, & \text{if } c < c_2 \\ c, & \text{otherwise} \end{cases} \tag{5.2.8}$$

$$c^* = \begin{cases} c_{12}, & \text{if } c > c_{12} \\ c_3, & \text{if } c < c_3 \\ c, & \text{otherwise} \end{cases} \tag{5.2.9}$$

Theorem 5.2.1 *The class of estimators d_c is inadmissible and is improved by d_{c_*} if $c_* \neq c$ when $\eta > r/m$ and by d_{c^*} if $c^* \neq c$ when $0 < \eta < r/m$.*

Proof 5.2.1 *Consider the risk function of d_c with respect to the quadratic loss function (5.2.2).*

$$R(d_c, \underline{\alpha}) = \frac{1}{\sigma_1^2} E\{Z + \frac{V_1V_2}{(r-1)V_2 + (s-1)V_1}\left(\eta\frac{m}{r} - 1\right) + \eta\frac{m}{r}cV_1 - \theta\}^2. \tag{5.2.10}$$

It is easy to see that the above risk function is a convex function with respect to c, hence the minimizing choice is obtained as,

$$\hat{c}(v_1, v_2, \underline{\alpha}) = \frac{\theta EV_1 - EZEV_1 - (\eta\frac{m}{r} - 1)E\{\frac{V_1^2V_2}{(r-1)V_2+(s-1)V_1}\}}{\eta\frac{m}{r}EV_1^2}. \tag{5.2.11}$$

We also note that $EV_1 = \frac{r}{m}\sigma_1 - p^{-1}$, $EZ = \mu + p^{-1}$, $EV_1^2 = \frac{\sigma_1 np^{-1}}{\sigma_2 m}[\frac{\sigma_1}{m}(r+1)r + \frac{\sigma_2}{n}r(r-1)]$, and $E[\frac{V_1^2V_2}{(r-1)V_2+(s-1)V_1}] = \frac{r}{m}\sigma_1 p^{-1}$. Substituting all these values in (3.2) and simplifying we get,

$$\hat{c}(\lambda) = \frac{\eta r - 2\eta\lambda m + m\lambda^2}{\eta(r+1-2m\lambda)}, \tag{5.2.12}$$

where $\lambda = (\sigma_1 p)^{-1}$ and $0 < \lambda < \frac{1}{m}$.

To apply Brewster and Zidek technique (Brewster and Zidek (1974)) for improving estimators, we need to find the supremum and infimum of $\hat{c}(\lambda)$ for fixed V_1, V_2 and η. Consider the derivative of $\hat{c}(\lambda)$. We have $\hat{c}'(\lambda) = g(\lambda)/\eta(r+1-2m\lambda)^2$, where $g(\lambda) = -2m(m\lambda^2 - (r+1)\lambda + \eta)$. The derivative is simply $g(\lambda)$ multiplied by a positive factor. It is easy to see that $g(\lambda)$

is a concave function of λ. *The maximum value of* $g(\lambda)$ *is* $\frac{(r+1)^2}{4m} - \eta$ *attained at* $\lambda = (r+1)/2m$. *Next we consider two separate cases as* $\eta > \frac{(r+1)^2}{4m}$ *or* $\eta \leq \frac{(r+1)^2}{4m}$.

Case I: $\eta > (r+1)^2/4m$. *In this case the maximum of* $g(\lambda)$ *is negative, hence* $g(\lambda) < 0$ *which leads to* $\hat{c}^{'}(\lambda) < 0$. *The function* $\hat{c}(\lambda)$ *is decreasing with respect to* λ. *Hence we obtain*

$$\sup_{0<\lambda<\frac{1}{m}} \hat{c}(\lambda) = \frac{r}{r+1}, \textit{ and } \inf_{0<\lambda<\frac{1}{m}} \hat{c}(\lambda) = \frac{\eta m(r-2)+1}{\eta m(r-1)}. \tag{5.2.13}$$

Case II: $\eta < (r+1)^2/4m$. *In this case the maximum value of* $g(\lambda)$ *is positive and hence it will have two roots* $\lambda^+ = \frac{r+1}{2m} + \frac{\sqrt{(r+1)^2-4\eta m}}{2m}$, *and* $\lambda^- = \frac{r+1}{2m} - \frac{\sqrt{(r+1)^2-4\eta m}}{2m}$. *Also it is can be seen easily that* $0 < \lambda^- < \lambda^+$. *But* λ^+ *is always greater than* $1/m$. *Further* λ^- *also lies outside our concerned interval if* $\eta > r/m$. *Hence for this case the function* $\hat{c}(\lambda)$ *is decreasing and we obtain*

$$\sup_{0<\lambda<\frac{1}{m}} \hat{c}(\lambda) = \frac{r}{r+1} \textit{ and } \inf_{0<\lambda<\frac{1}{m}} \hat{c}(\lambda) = \frac{\eta m(r-2)+1}{\eta m(r-1)}. \tag{5.2.14}$$

If $\eta < r/m$, *then* $\lambda^- < 1/m$. *For this case the function* $g(\lambda)$ *decreasing in the interval* $(0, \lambda^-]$ *and increasing in the interval* $(\lambda^-, 1/m]$. *Hence we obtain,*

$$\sup_{0<\lambda<\frac{1}{m}} \hat{c}(\lambda) = \max\{\hat{c}(0), \hat{c}(1/m)\} \textit{ and } \inf_{0<\lambda<\frac{1}{m}} \hat{c}(\lambda) = \hat{c}(\lambda^-). \tag{5.2.15}$$

Let us denote $c_1 = \frac{r}{r+1}$, $c_2 = \frac{\eta m(r-2)+1}{\eta m(r-1)}$. *Utilizing the results from case I and II, define the constants* c_* *and* c^* *as above. That is,*

$$c_* = \begin{cases} c_1, & \textit{if } c > c_1 \\ c_2, & \textit{if } c < c_2 \\ c, & \textit{otherwise} \end{cases}$$

$$c^* = \begin{cases} c_{12}, & \textit{if } c > c_{12} \\ c_3, & \textit{if } c < c_3 \\ c, & \textit{otherwise} \end{cases}$$

Now applying the orbit-by-orbit improvement technique of Brewster-Zidek technique (Brewster and Zidek (1974)), we have the theorem. Next we obtain improved estimators for the UMVUE of the quantile θ by an application of the Theorem (5.2.1).

Theorem 5.2.2 *Let the loss function be quadratic loss as given in (5.2.2). The uniformly minimum variance unbiased estimator(UMVUE)* $d_{MV} = d_1$ *for the quantile* $\theta = \mu + \eta\sigma_1$ *is inadmissible and is improved by* d_{c_1} *when* $\eta > (r+1)/2m$. *For* $1/m < \eta < (r+1)/2m$ *the UMVUE is improved by* d_{c_2}. *For* $0 < \eta < 1/m$ *the estimator* d_{MV} *is admissible and can not be improved.*

Remark 5.2.1 *The class of estimators* $\{d_c : c_2 \leq c \leq c_1\}$ *form an essentially complete class when* $\eta > r/m$. *The class of estimators* $\{d_c : c_3 \leq c \leq c_1\}$ *form an essentially complete class when* $(r+1)/2m < \eta < r/m$. *The class* $\{d_c : c_3 \leq c \leq c_2\}$ *form an essentially complete class when* $1/m < \eta < (r+1)/2m$. *Finally the class* $\{d_c : c_3 < c < c_2\}$ *is the essentially complete class in the class* D *when* $0 < \eta < 1/m$, *and can not be improved on by any* d_c.

Remark 5.2.2 *Using the above results it is easy to write the improved estimator which improves upon the UMVUE of the quantile θ. Let $\eta > r/m$ then the estimator which improves upon the UMVUE is obtained as $d_{MVI} = Z + \frac{V_1 V_2 (k-1)}{(r-1)V_2 + (s-1)V_1} + (\frac{\eta m}{r+1})V_1$. It is easy to write the improved estimator for the case $1/m < \eta < (r+1)/2m$. The estimator $d_1 = d_{MV}$ can not be improved by any d_c when $0 < \eta < 1/m$. In Section 5.2.4, we numerically evaluate the risk functions of these improved estimators and show the percentage of improvement over the UMVUE d_{MV}.*

5.2.3 An Inadmissibility Result for Affine Equivariant Estimators

In this section, we introduce the concept of invariance to our problem and obtain a broad class of estimators for quantiles $\theta = \mu + \eta\sigma_1$, which are invariant under an affine group of transformations. Further sufficient conditions for improving these estimators are obtained.

Let $G_A = \{g_{a,b} : g_{a,b}(x) = ax + b, a \in \mathbb{R}^+, b \in \mathbb{R}\}$ be an affine group of transformations. Under this transformation the problem remain invariant and the form of an affine equivariant estimator for estimating the quantile θ, based on the sufficient statistics (V_1, V_2, Z) is obtained as

$$d(Z, V_1, V_2) = Z + V_1\phi(V) = d_\phi, \text{ (say)}, \tag{5.2.16}$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}$ and $V = V_2/V_1$. To proceed further, let us define the functions ϕ_1 and ϕ_2 as follows.

$$\phi_1(v) = \begin{cases} \frac{m}{r+s}\Big(\eta - \frac{1}{m}\Big), & \text{if } 0 < v \leq \frac{1}{1-\eta m} \\ \frac{m+n\tau^+ v}{r+s}\Big(\eta - \frac{1}{m+n\tau^+}\Big), & \text{if } v > \frac{1}{1-\eta m} \end{cases}$$

$$\phi_2(v) = \frac{m}{r+s}\Big(\eta - \frac{1}{m}\Big),$$

where $\tau^+ = -\frac{m}{n} + \frac{1}{n}\sqrt{\frac{m(v-1)}{\eta v}}$. For the affine equivariant estimator d_ϕ, we define the functions ϕ_1^* and ϕ_2^* as below.

$$\phi_1^*(v) = \begin{cases} \phi_1, & \text{if } \phi < \phi_1 \\ \phi, & \text{otherwise} \end{cases} \tag{5.2.17}$$

$$\phi_2^*(v) = \begin{cases} \phi_2, & \text{if } \phi < \phi_2 \\ \phi, & \text{otherwise} \end{cases} \tag{5.2.18}$$

Now it is immediate to propose the main result of this section which will help in deriving improved estimators for the quantiles θ with respect to the quadratic loss function (5.2.2).

Theorem 5.2.3 *For the affine equivariant estimator d_ϕ (as given in (5.2.16)), define the functions ϕ_1^* and ϕ_2^* as given in (5.2.17) and (5.2.18) respectively. Let the loss function be the affine invariant loss (5.2.2).*

- *The estimator d_ϕ is inadmissible and is improved by $d_{\phi_1^*}$, if there exist some values of the parameters $\underline{\alpha} = (\mu, \sigma_1, \sigma_2)$ such that, $P(d_\phi \neq d_{\phi_1^*}) > 0$ when $\eta < 1/m$.*

- *The estimator d_ϕ is inadmissible and is improved by $d_{\phi_2^*}$, if there exist some values of the parameters $\underline{\alpha} = (\mu, \sigma_1, \sigma_2)$ such that, $P(d_\phi \neq d_{\phi_2^*}) > 0$ when $\eta > 1/m$.*

Proof 5.2.2 *Consider the conditional risk function of d_ϕ given $V = v$.*

$$R((d_\phi, \underline{\alpha})|V = v) = \frac{1}{\sigma_1^2} E[(d_\phi - \theta)^2 | V = v],$$
$$= \frac{1}{\sigma_1^2} E[(Z + V_1\phi(V) - \theta)^2 | V = v]. \quad (5.2.19)$$

It can be easily seen that the above risk function (5.2.19) is a convex function in ϕ. Therefore the minimizing value of ϕ for fixed values of V is obtained as,

$$\hat{\phi}(v, \sigma_1, \sigma_2) = \eta\sigma_1 \frac{E(V_1|V = v)}{E(V_1^2|V = v)} - \frac{1}{p}\frac{E(V_1|V = v)}{E(V_1^2|V = v)}. \quad (5.2.20)$$

To evaluate the above expression in (5.2.20), we have the joint probability density function of (U_1, U_2, Z) as given in (5.2.3). Let us use the transformation $V_1 = U_1 - Z$, $V_2 = U_2 - Z$ and $Z = Z$. The inverse transformation is given by $U_1 = V_1 + Z$, $U_2 = V_2 + Z$, and $Z = Z$. The Jacobian is obtained as $J = 1$. Hence, the joint probability density function of (Z, V_1, V_2) is obtained as,

$$f_{V_1,V_2,Z}(v_1, v_2, z) = \frac{m^r n^s}{\sigma_1^r \sigma_2^s}\Big[\frac{v_1^{r-1} v_2^{s-2}}{\Gamma r\Gamma(s-1)} + \frac{v_1^{r-2} v_2^{s-1}}{\Gamma s\Gamma(r-1)}\Big] \exp\{-\frac{m}{\sigma_1}(v_1 + z - \mu) - \frac{n}{\sigma_2}(v_2 + z - \mu)\},$$
$$v_1 > 0, v_2 > 0, z > \mu.$$

Using the independence of (V_1, V_2) and Z one can easily write the joint probability density function of (V_1, V_2) and is given by,

$$f_{V_1,V_2}(v_1, v_2) = \frac{m^r n^s p^{-1}}{\sigma_1^r \sigma_2^s}\Big[\frac{v_1^{r-1} v_2^{s-2}}{\Gamma r\Gamma(s-1)} + \frac{v_1^{r-2} v_2^{s-1}}{\Gamma s\Gamma(r-1)}\Big] \exp\Big\{-\frac{m}{\sigma_1}v_1 - \frac{n}{\sigma_2}v_2\Big\}, \quad v_1 > 0, v_2 > 0.$$

We need to calculate the conditional density of V_1 given V. Let us use the transformation, $V = \frac{V_2}{V_1}$, $V_1 = V_1$. The inverse transformation is given by $V_2 = VV_1$, $V_1 = V_1$. The Jacobian of this transformation is obtained as V_1. Hence the joint probability density function of (V_1, V) is obtained as,

$$f_{V_1,V}(v_1, v) = \frac{m^r n^s p^{-1}}{\sigma_1^r \sigma_2^s}\Big[\frac{v_1^{r+s-2} v^{s-2}}{\Gamma r\Gamma(s-1)} + \frac{v_1^{r+s-2} v^{s-1}}{\Gamma s\Gamma(r-1)}\Big] \exp\Big\{-\frac{m}{\sigma_1}v_1 - \frac{n}{\sigma_2}vv_1\Big\}, \quad v_1 > 0, v > 0.$$

The marginal density function of V is given by

$$f_V(v) = \frac{m^r n^s p^{-1}\Gamma(r+s-1)}{\sigma_1^r \sigma_2^s}\Big(\frac{m}{\sigma_1} + \frac{n}{\sigma_2}v\Big)^{1-r-s}\Big[\frac{v^{s-2}}{\Gamma r\Gamma(s-1)} + \frac{v^{s-1}}{\Gamma s\Gamma(r-1)}\Big], \quad v > 0.$$

It is easy to observe that, the conditional probability density function of V_1 given $V = v$, is a gamma distribution with shape parameter $r + s - 1$ and scale parameter $\frac{\sigma_1\sigma_2}{m\sigma_2 + n\sigma_1 v}$. Here the gamma probability density function with a shape parameter α and a scale parameter β is defined as,

$$g(x, \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}, \quad x > 0, \alpha > 0, \beta > 0.$$

So, the conditional expectations are calculated and obtained as

$$E(V_1|V=v)=\frac{(r+s-1)\sigma_1\sigma_2}{m\sigma_2+n\sigma_1 v}, \tag{5.2.21}$$

and

$$E(V_1^2|V=v)=(r+s-1)(r+s)\Big(\frac{\sigma_1\sigma_2}{m\sigma_2+n\sigma_1 v}\Big)^2. \tag{5.2.22}$$

Substituting these conditional expectations from (5.2.21) and (5.2.22) in (5.2.20), and simplifying, we have the minimizing choice of $\hat{\phi}(\tau,v)$ *for fixed* v *as,*

$$\hat{\phi}(\tau,v)=\frac{m+n\tau v}{r+s}\Big(\eta-\frac{1}{m+n\tau}\Big), \tag{5.2.23}$$

where $\tau=\frac{\sigma_1}{\sigma_2}>0$ *and* $v>0$.

In order to apply the orbit-by-orbit improvement technique of Brewster and Zidek (Brewster and Zidek (1974)), we need to obtain the supremum and infimum of $\hat{\phi}(\tau,v)$ *with respect to* τ *for fixed values of* v *and* η. *Consider the derivative of* $\hat{\phi}(\tau)$. *We have* $\hat{\phi}'(\tau)=\frac{1}{(r+s)(m+n\tau)^2}h(\tau)$, *where* $h(\tau)=\eta n^3 v\tau^2+2mn^2v\eta\tau+mnv(\eta m-1)+mn$. *It can be noticed that the* $\hat{\phi}'(\tau)$ *is simply* $h(\tau)$ *multiplied by a positive factor. To analyze* $h(\tau)$, *it is easy to observe that* $h(\tau)$ *is a convex function of* τ. *The minimum value is obtained at* $\tau=-\frac{m}{n}$, *which is negative. Hence in the concerned region* $(0,\infty)$, *the minimum is attained at* $\tau=0$. *The minimum value being* $\eta m^2nv-mnv+mn$. *Below we discuss two separate cases* $\eta<1/m$ *and* $\eta>1/m$ *for calculating supremum and infimum of* ϕ.

Case I: $\eta<1/m$. *For this case there are two possibilities that is either the minimum value of* $h(\tau)$ *is positive or negative. Suppose it is positive that is* $v<1/(1-\eta m)$. *If the minimum is positive then* $h(\tau)\geq 0$ *for all* $\tau>0$. *Hence the function* $\hat{\phi}(\tau,v)$ *is increasing in* τ *for all* $\tau>0$. *In this case the supremum and infimum of* $\hat{\phi}(\tau,v)$ *is obtained as,*

$$\sup_{\tau>0}\hat{\phi}(\tau,v)=\infty,\ \ \textit{and}\ \ \inf_{\tau>0}\hat{\phi}(\tau,v)=\frac{m}{r+s}\Big(\eta-\frac{1}{m}\Big). \tag{5.2.24}$$

If the minimum value is negative that is when $v\geq 1/(1-\eta m)$, *then the function* $h(\tau)$ *crosses the* τ *axis. The function* $h(\tau)$ *has two real roots say* τ^- *(smaller root) and* τ^+ *(larger root). It can be easily checked that* $\tau^-<0<\tau^+$. *Here* $\tau^+=-\frac{m}{n}+\frac{1}{n}\sqrt{\frac{m(v-1)}{\eta v}}$. *It can be seen that for* $\tau\in(0,\tau^+]$ *the function* $h(\tau)<0$ *and for* $\tau\in[\tau^+,\infty)$ *the function* $h(\tau)\geq 0$. *That is* $\hat{\phi}(\tau,v)$ *is decreasing in* $(0,\tau^+]$ *and increasing in* $[\tau^+,\infty)$. *Hence we obtain,*

$$\sup_{\tau>0}\hat{\phi}(\tau,v)=\max\{\phi(0,v),\phi(\infty,v)\}=\infty \tag{5.2.25}$$

and

$$\inf_{\tau>0}\hat{\phi}(\tau,v)=\frac{m+nv\tau^+}{r+s}\Big(\eta-\frac{1}{m+n\tau^+}\Big). \tag{5.2.26}$$

Case-II: $\eta>1/m$. *In this case, the minimum value of* $h(\tau)$ *is always positive. Hence the* *function $\hat{\phi}(\tau, v)$ is increasing with respect to τ. Hence we obtain the supremum and infimum of $\hat{\phi}(\tau, v)$ as,*

$$\sup_{\tau>0} \hat{\phi}(v, \tau) = \infty, \quad and \quad \inf_{\tau>0} \hat{\phi}(v, \tau) = \frac{m}{r+s}\Big(\eta - \frac{1}{m}\Big).$$

Utilizing the results from case I and II it is easy to define the functions ϕ_1 and ϕ_2. Further using the functions ϕ_1 and ϕ_2, we define the functions ϕ_1^* and ϕ_2^* as given in (5.2.17) and (5.2.18) respectively. Applying the orbit-by-orbit improvement technique of (Brewster and Zidek (1974)) (see Theorem 3.1.1 in Brewster and Zidek (1974)), the result follows.

Remark 5.2.3 *The above theorem basically gives a complete class result. It simply tells that any affine equivariant estimator d_ϕ of the form (5.2.16) can be improved if $P(\phi < \phi_1) > 0$ (when $\eta < 1/m$) or $P(\phi < \phi_2) > 0$ (when $\eta > 1/m$).*

Remark 5.2.4 *The class of estimators $\{d_\phi : \phi \geq \phi_1\}$ for estimating the quantiles θ form a complete class with respect to the loss function (5.2.2) when $\eta < 1/m$. The class of estimators $\{d_\phi : \phi \geq \phi_2\}$ for estimating the quantiles θ form a complete class with respect to the loss function (5.2.2) when $\eta > 1/m$.*

It is easy to note that, all the estimators such as the MLE d_{ML}, a modification to the MLE d_{MM} (MM) and the UMVUE d_{MV} considered for the quantiles θ belong to the class d_ϕ as given in (5.2.16).

Remark 5.2.5 *Though the estimators d_{ML} and d_{MM} belong to the class d_ϕ in (5.2.16), the condition for improving these estimators does not satisfy which has been observed from our simulation study. Hence we are not able to get improved estimator for d_{ML} and d_{MM}. However, the UMVUE d_{MV} has been improved by using Theorem (5.2.3) , when $\eta < 1/m$ and denote the improved estimator as d_{MVA}. A numerical comparison of this estimator with other estimators has been done using Monte-Carlo simulation method in Section 5.2.4.*

5.2.4 Simulation Study

In the previous sections, we have proposed various estimators for the quantiles θ such as the MLE d_{ML}, a modification to the MLE (MM) d_{MM} and the UMVUE d_{MV}. Further improved estimators d_{MVI} and d_{MVA} dominating the UMVUE have also been derived. However, it should be noted that the analytical comparison of risk values for all these estimators is not possible. Taking the advantages of computational facilities, we in this section numerically evaluate the risk values of all these estimators. For this purpose, we have generated 20,000 type-II censored random samples each from two exponential populations having probability density function (5.2.1) with a common location parameter μ and different scale parameters σ_1, σ_2. The loss function is taken as (5.2.2). We use Monte-Carlo simulation method to compute the simulated risk values of each estimator. The accuracy of simulation has been checked and the standard error is of the order of 10^{-4}. It can be easily seen that with respect to the loss function (5.2.2), the risk values of all the estimators are function of $\tau = \sigma_2/\sigma_1 > 0$, for fixed sample sizes and fixed η. The simulation study has been conducted for wide range of the parameters, however for illustrative purpose we report the simulated risk values for some selected choices of parameters. Let us define the percentage of relative risk improvements (RRI) of all the estimators with respect to the MLE as,

$$R1 = \Big(1 - \frac{Risk(d_{MM})}{Risk(d_{ML})}\Big) \times 100,\ R2 = \Big(1 - \frac{Risk(d_{MV})}{Risk(d_{ML})}\Big) \times 100,$$
$$R3 = \Big(1 - \frac{Risk(d_{MVI})}{Risk(d_{ML})}\Big) \times 100,\ R4 = \Big(1 - \frac{Risk(d_{MVA})}{Risk(d_{ML})}\Big) \times 100.$$

Also we define the percentage of risk improvement of improved estimators over their old counterparts,

$$P1 = \Big(1 - \frac{Risk(d_{MVI})}{Risk(d_{MV})}\Big) \times 100,\ P2 = \Big(1 - \frac{Risk(d_{MVA})}{Risk(d_{MV})}\Big) \times 100.$$

Further we define the censoring factors ($k1$ and $k2$) for both the populations as the ratio of number of observed samples to the total number of samples. That is for the first population $k1 = r/m$ and for the second population $k2 = s/n$. It can be noticed that the censoring factors $k1$ and $k2$ always lie between 0 and 1. A massive simulation study has been carried out by considering various combinations of sample sizes and η. However, for illustration purpose, we present (in Table 5.2.1) the percentage of relative risk performances as well as percentage of risk improvements for sample sizes $(m, n) = (8, 8)$ and for $\eta = 1.5$, $\eta = 0.01$. The first column gives the values of τ. Corresponding to one value of τ, there corresponds four values of relative risk performances for an estimator. These four values correspond to $k1 = k2 = 0.25, 0.50, 0.75, 1.00$ respectively.

The following conclusions can be drawn from our simulation study as well as the Tables 5.2.1-5.2.4, and Figures 5.2.1-5.2.4.

1. Let $\eta > r/m$ or $(r+1)/2m$. The percentage of relative risk values $R2$ increases with respect to both τ and $k1$, $k2$, whereas the relative risk value $R3$ increases for small values of τ and then starts decreasing after attending maximum somewhere near 1.0. The behavior of $R1$ is not clear.

2. Let $\eta < 1/m$. The relative risk improvement ($R1$, $R2$, $R4$) of all the estimators with respect to the MLE d_{ML} increases as the censoring factors $k1$ and $k2$ increase for fixed sample sizes. Also $R1$, $R2$ and $R4$ increases with respect to τ and attains its maximum somewhere near $\tau = 1$, then slowly decrease. Further, as τ becomes large the risk values of all the estimators converge to some constant value.

3. The percentage of improvements of d_{MVI} over d_{MV} ($P1$) is maximum around 39% and the percentage of improvements of d_{MVA} over d_{MV} ($P2$) is near to 15%. As the censoring factors $k1$ and $k2$ increase the percentage of improvement becomes negligible. The maximum improvement is obtained near $\tau = 1$.

4. Consider for small values of η that is $\eta < 1/m$. When the values of τ are close to 0, the estimator d_{MM} has the maximum percentage of relative risk performance. For moderate values of τ $(0.25 < \tau < 3.00)$, the estimator d_{MVA} has the maximum percentage of relative risk improvement and is seen to vary from 30% to 47%. However, for large values of τ (≥ 3.0) the estimator d_{MM} performs the best and the percentage of relative risk improvement is seen near to 45%.

5. Consider that $\eta > 1.0$ or $\eta > (r+1)/2m$. When the values of τ are close to 0, and $k1$ and $k2$ also close to 0, the estimator d_{MM} has the maximum percentage of relative risk

performance and is seen near to 1%. For moderate to large values of τ, the estimator d_{MVI} has the maximum percentage of relative risk improvement and it is seen near to 36%.

6. From our simulation study we notice, that the amount of improvement of d_{MVA} over d_{MV} decreases as the values of τ increases. The improvement is not significant as the values of k_1 and k_2 increases for $\eta < 1/m$. We also observe that the estimator d_{MVI} gives maximum percentage of improvement over d_{MV} for the case $\eta > r/m$.

7. On the basis of our computational results, we recommend the following. When η is small and the values of τ are close to 0, we recommend to use d_{MM}. For moderate values of τ we recommend to use d_{MVA} whereas for large values of τ the estimator d_{MM} is recommended. When $\eta > (r+1)/2m$, and for small values of τ, we recommend to use d_{MM} whereas for moderate to large values of τ we recommend using the estimator d_{MVI}.

8. A similar type of observations have been made for other combinations of $k1$, $k2$ and the sample sizes.

5.2.5 Conclusions

We have considered the estimation of quantiles of two exponential populations assuming that the location parameters are equal using type-II censored samples from a decision theoretic point of view. We have derived some baseline estimators such as the MLE, the modified MLE and the UMVUE for the quantile θ. We also obtained estimators which dominate the UMVUE for $\eta > 1/m$. Further inadmissibility results have been proved for affine equivariant estimators. It should be noted that when the censoring factors $k1$ and $k2$ become 1, the problem reduces to the full sample problem which was earlier studied by several authors including Sharma and Kumar (1994) . Though they have obtained improved estimators analytically, it is essential to know the percentage of risk improvement approximately. In this regard our results add one more dimension to their results and may be handy for practical purposes for $k1 = k2 = 1$. Also we have obtained the results when the sample sizes are not equal and $k1 = k2 = 1$. The present work also extends the results of Tripathy (2016) to the case of $\eta \neq 0$ which is new.

Next, we present an example where our model fits well and compute the estimates for the quantile $\theta = \mu + \eta\sigma_1$.

Example 5.2.1 *(Simulated Data) Suppose two brands of electronic devices each having* 30 *units are placed for a life testing experiment. It is known that, the lifetimes (in hours) of each unit follows an exponential distribution with same minimum guarantee time. The experimenter could able to observe only* 10 *units of failures (in hours) from each brands of devices because of some constraints. The data for both the brands are obtained as Brand 1: 59.69, 60.18, 68.33, 113.78, 155.78, 203.83, 237.86, 243.67, 251.62, 301.49; Brand 2: 37.62, 73.03, 100.54, 103.61, 106.37, 110.72, 119.26, 135.59, 169.75, 177.03.*

On the basis of above data, we have computed the statistic values as $Z = 37.62$, $V_1 = 219.91$, *and* $V_2 = 118.18$. *Let* $\eta = 2.0$, *then the various estimates for the quantile* $\theta = \mu + \eta\sigma_1$ *have been computed as* $d_{ML} = 1357.13$, $d_{MM} = 1349.448$, $d_{MV} = 1399.84$, $d_{MVI} = 1279.88$. *In this situation, we recommend to use* d_{MVI}.

Table 5.2.1: Relative risk performances of different estimators for quantile θ when $\eta = 1.5$ with $k1 = k2 = 0.25, 0.50, 0.75, 1.00$

$\tau \downarrow$	(m,n)=(8,8)				(m,n)=(16,16)			
	$R1$	$R2$	$R3$	$P1$	$R1$	$R2$	$R3$	$P1$
	0.71	-81.30	-25.16	30.96	0.46	-27.64	-0.42	21.32
0.05	2.30	-25.68	1.73	21.81	1.10	-11.10	1.94	11.74
	2.57	-11.89	3.83	14.05	2.47	-6.90	3.09	9.35
	4.38	-6.71	5.34	11.29	2.43	-3.23	2.98	6.03
	0.34	-48.47	2.60	34.40	0.59	-20.47	7.78	23.45
0.25	1.60	-17.28	9.07	22.47	1.12	-8.32	5.57	12.83
	2.76	-9.13	8.51	16.17	1.19	-4.11	4.12	7.90
	3.91	-4.96	8.31	12.65	1.82	-2.34	4.09	6.29
	0.18	-22.85	21.96	36.48	0.27	-9.89	15.60	23.20
0.75	1.02	-8.88	16.86	23.65	0.17	-3.31	7.95	10.90
	1.20	-4.34	11.87	15.54	0.51	-1.96	5.73	7.54
	1.65	-2.30	9.86	11.89	0.90	-1.16	4.93	6.02
	0.13	-17.03	26.53	37.22	0.24	-7.54	17.68	23.45
1.00	0.53	-6.06	17.61	22.31	0.14	-2.57	8.85	11.14
	1.11	-3.57	13.11	16.11	0.57	-1.67	6.72	8.26
	1.32	-1.80	10.10	11.69	0.75	-0.92	5.42	6.29
	0.19	-14.31	29.15	38.02	0.19	-6.15	18.09	22.84
1.25	0.19	-4.27	17.18	20.57	0.19	-2.15	9.29	11.21
	0.95	-2.85	14.30	16.68	0.30	-1.17	6.53	7.62
	1.04	-1.42	10.64	11.90	0.51	-0.68	5.44	6.08
	0.01	-6.92	32.14	36.54	0.00	-2.66	18.37	20.48
2.00	0.11	-2.38	18.66	20.56	0.07	-1.13	10.04	11.04
	0.50	-1.56	14.41	15.73	0.20	-0.68	7.31	7.93
	0.29	-0.64	9.60	10.18	0.31	-0.39	5.73	6.10
	0.03	-5.36	33.31	36.71	0.05	-2.32	19.73	21.55
2.50	0.14	-1.90	19.78	21.27	0.19	-1.14	11.73	12.73
	0.12	-0.86	13.15	13.90	0.13	-0.48	7.30	7.75
	0.28	-0.49	10.91	11.35	0.13	-0.24	5.35	5.59
	0.06	-4.59	34.50	37.38	0.03	-1.75	20.51	21.88
3.00	0.07	-1.40	19.40	20.52	0.04	-0.64	10.68	11.25
	0.16	-0.77	13.47	14.13	0.16	-0.44	8.04	8.45
	0.35	-0.46	11.33	11.75	0.23	-0.25	6.40	6.63
	0.04	-3.53	34.62	36.85	0.00	-0.94	19.65	20.40
3.50	0.17	-1.38	20.94	22.02	0.06	-0.58	10.92	11.44
	0.28	-0.76	15.01	15.66	0.06	-0.28	7.31	7.58
	0.16	-0.30	10.47	10.74	0.12	-0.16	5.72	5.88
	0.08	-3.33	35.63	37.70	0.00	-1.04	20.49	21.31
4.00	0.09	-0.99	20.04	20.83	0.13	-0.64	12.44	13.01
	0.18	-0.57	15.11	15.59	0.02	-0.19	7.45	7.63
	0.30	-0.34	11.68	11.98	0.10	-0.13	5.76	5.89
	0.00	-2.27	34.00	35.47	0.01	-0.90	20.51	21.22
4.50	0.05	-0.75	20.25	20.85	0.09	-0.50	11.62	12.06
	0.11	-0.42	14.85	15.21	0.12	-0.26	8.36	8.60
	0.29	-0.31	11.93	12.20	0.12	-0.13	6.28	6.41
	0.01	-2.02	33.30	34.62	0.00	-0.66	19.43	19.96
5.00	0.07	-0.72	19.60	20.18	0.05	-0.35	11.64	11.95
	0.00	-0.25	13.44	13.66	0.02	-0.14	7.46	7.60
	0.03	-0.13	10.35	10.47	-0.03	-0.04	4.76	4.80

Table 5.2.2: Relative risk performances of different estimators for quantile θ when $\eta = 1.5$ with $k1 = k2 = 0.25, 0.50, 0.75, 1.00$

$\tau\downarrow$	(m,n)=(8,12)				(m,n)=(12,8)			
	$R1$	$R2$	$R3$	$P1$	$R1$	$R2$	$R3$	$P1$
0.05	0.96	-74.69	-18.35	32.25	0.57	-44.07	-6.27	26.24
	2.06	-23.81	2.89	21.56	1.00	-14.95	1.53	14.34
	3.04	-11.81	4.72	14.79	2.62	-8.69	3.09	10.84
	4.10	-6.18	5.55	11.05	3.48	-4.59	3.93	8.15
0.25	0.66	-36.55	12.28	35.76	0.32	-33.59	3.04	27.42
	1.68	-14.10	11.97	22.84	1.32	-13.02	5.41	16.31
	1.75	-6.70	9.18	14.89	1.90	-6.67	4.73	10.70
	3.86	-4.30	9.76	13.49	2.78	-3.66	5.08	8.43
0.75	0.42	-13.66	30.09	38.49	-0.02	-18.70	13.92	27.48
	0.87	-5.87	18.54	23.06	0.64	-7.40	9.62	15.85
	1.28	-3.27	14.07	16.79	1.29	-4.19	7.78	11.49
	1.24	-1.55	10.49	11.86	1.99	-2.38	6.75	8.92
1.00	0.40	-10.26	32.37	38.66	-0.06	-14.85	17.75	28.39
	0.36	-3.42	18.90	21.59	0.82	-6.71	11.41	16.98
	0.74	-2.14	13.82	15.63	0.80	-3.03	7.80	10.52
	0.38	-0.85	9.15	9.92	0.93	-1.57	5.90	7.36
1.25	0.45	-9.04	33.31	38.84	0.12	-14.16	19.57	29.55
	0.43	-3.00	19.89	22.23	0.31	-4.78	11.11	15.17
	0.67	-1.71	14.19	15.64	0.78	-2.70	8.44	10.85
	0.74	-0.87	11.13	11.89	0.84	-1.33	6.43	7.66
2.00	0.07	-3.07	34.16	36.12	0.01	-8.42	22.89	28.89
	0.23	-1.52	21.03	22.22	0.18	-2.90	12.98	15.43
	0.20	-0.72	13.63	14.25	0.46	-1.61	9.70	11.13
	0.34	-0.43	11.15	11.53	0.81	-0.99	7.54	8.45
2.50	0.16	-2.95	35.83	37.68	-0.04	-6.21	23.61	28.08
	0.11	-0.95	20.18	20.94	0.01	-1.95	12.38	14.06
	0.21	-0.58	14.86	15.35	0.36	-1.26	9.75	10.88
	0.25	-0.31	10.87	11.15	0.13	-0.48	6.47	6.92
3.00	0.14	-2.30	36.46	37.89	-0.03	-5.13	23.72	27.45
	0.06	-0.67	19.93	20.47	-0.01	-1.48	12.96	14.23
	0.24	-0.53	15.75	16.20	0.10	-0.80	8.90	9.63
	0.20	-0.24	11.47	11.69	0.35	-0.52	7.54	8.02
3.50	0.13	-1.99	36.12	37.37	-0.08	-3.62	24.94	27.56
	0.07	-0.56	21.17	21.61	0.05	-1.40	13.40	14.60
	0.21	-0.44	15.79	16.15	0.19	-0.77	9.84	10.53
	0.13	-0.17	10.96	11.12	0.26	-0.41	7.69	8.07
4.00	0.07	-1.29	34.91	35.74	-0.06	-3.31	23.97	26.41
	0.04	-0.40	20.45	20.77	0.09	-1.29	14.10	15.20
	0.00	-0.16	13.36	13.50	0.13	-0.63	9.73	10.29
	0.16	-0.16	11.69	11.83	0.16	-0.31	7.44	7.73
4.50	0.04	-0.90	33.84	34.43	-0.01	-3.18	26.06	28.34
	0.03	-0.32	20.53	20.79	0.03	-0.96	14.17	14.99
	0.05	-0.18	14.30	14.46	0.07	-0.47	10.04	10.46
	0.10	-0.11	11.61	11.71	0.03	-0.20	7.01	7.20
5.00	0.07	-1.02	35.49	36.15	-0.07	-2.22	24.20	25.85
	0.02	-0.27	20.05	20.27	0.07	-0.94	14.61	15.41
	0.09	-0.20	15.06	15.23	0.13	-0.50	10.04	10.49
	0.09	-0.10	11.37	11.47	0.09	-0.21	7.52	7.71

Table 5.2.3: Relative risk performances of different estimators for quantile θ when $\eta = 0.01$ with $k1 = k2 = 0.25, 0.50, 0.75, 1.00$

$\tau \downarrow$	(m,n)=(8,8)				(m,n)=(16,16)			
	$R1$	$R2$	$R4$	$P2$	$R1$	$R2$	$R4$	$P2$
0.05	27.00	9.91	13.03	3.46	40.55	38.37	38.43	0.09
	39.18	35.78	35.85	0.11	45.39	44.57	44.57	0.00
	42.54	41.43	41.43	0.01	46.79	46.52	46.52	0.00
	45.60	44.97	44.97	0.00	47.66	47.43	47.43	0.00
0.25	30.01	19.93	30.70	13.45	42.42	41.88	43.10	2.09
	40.72	39.08	40.51	2.35	46.37	46.34	46.54	0.37
	43.63	42.81	43.25	0.76	47.95	47.94	47.92	0.03
	46.21	45.78	45.93	0.26	48.62	48.57	48.59	0.04
0.75	30.63	22.39	33.70	14.57	43.04	42.94	44.20	2.19
	41.95	40.92	43.00	3.52	47.17	47.16	47.44	0.52
	44.83	44.66	45.40	1.33	47.98	47.99	48.10	0.22
	46.66	46.58	46.88	0.56	48.56	48.52	48.64	0.22
1.00	31.03	23.76	34.46	14.03	42.29	41.93	43.20	2.18
	42.72	41.64	43.31	2.86	46.96	47.13	47.36	0.43
	46.57	46.59	47.19	1.11	48.31	48.22	48.33	0.21
	47.02	47.20	47.34	0.27	48.31	48.35	48.38	0.06
1.25	31.34	23.79	33.63	12.90	43.06	42.69	43.97	2.22
	41.72	41.27	42.61	2.28	47.17	47.05	47.30	0.47
	45.93	46.00	46.51	0.94	47.73	47.94	47.99	0.11
	46.04	46.02	46.20	0.34	48.52	48.64	48.65	0.02
2.00	30.62	22.18	31.21	11.60	42.73	41.95	42.98	1.78
	41.62	40.94	41.84	1.52	46.10	46.35	46.43	0.13
	44.85	44.65	44.90	0.45	47.70	47.76	47.78	0.03
	46.08	45.92	46.06	0.26	49.00	48.97	48.98	0.00
2.50	29.23	20.06	28.40	10.42	41.28	40.82	41.52	1.17
	40.87	39.80	40.51	1.18	45.71	45.76	45.81	0.08
	44.15	43.98	44.11	0.23	47.73	47.81	47.82	0.01
	45.98	46.12	46.15	0.05	47.34	47.30	47.30	0.00
3.00	29.44	20.13	28.29	10.22	39.62	39.13	39.66	0.87
	41.07	40.09	40.64	0.91	45.32	45.20	45.22	0.03
	44.29	43.72	43.88	0.27	48.69	48.80	48.80	0.00
	46.14	46.13	46.16	0.05	47.19	47.21	47.21	0.00
3.50	29.12	21.00	27.45	8.16	39.59	38.56	38.95	0.64
	40.39	39.02	39.51	0.79	44.45	44.35	44.37	0.03
	43.77	43.25	43.35	0.17	46.60	46.53	46.53	0.00
	46.04	45.89	45.91	0.02	46.77	46.72	46.72	0.00
4.00	28.68	17.91	24.47	7.99	39.46	38.51	38.90	0.63
	39.96	38.87	39.24	0.61	45.10	44.85	44.86	0.01
	43.27	42.84	42.88	0.07	46.23	46.10	46.10	0.00
	45.06	44.76	44.77	0.02	47.33	47.19	47.19	0.00
4.50	28.99	17.85	24.42	7.99	38.24	37.19	37.48	0.47
	39.56	37.88	38.19	0.49	43.70	43.61	43.62	0.01
	44.91	44.41	44.46	0.08	46.00	45.91	45.91	0.00
	44.80	44.48	44.47	0.01	47.04	46.89	46.89	0.00
5.00	27.82	16.13	22.36	7.42	38.32	37.75	37.97	0.36
	39.62	38.22	38.47	0.39	43.32	42.74	42.75	0.01
	43.76	42.95	42.98	0.04	45.61	45.53	45.53	0.00
	44.58	44.27	44.28	0.02	46.72	46.76	46.76	0.00

Table 5.2.4: Relative risk performances of different estimators for quantile θ when $\eta = 0.01$ with $k1 = k2 = 0.25, 0.50, 0.75, 1.00$

$\tau \downarrow$	(m,n)=(8,12)				(m,n)=(12,8)			
	$R1$	$R2$	$R4$	$P2$	$R1$	$R2$	$R4$	$P2$
0.05	28.60	15.31	17.24	2.28	35.75	29.88	30.58	1.00
	39.36	36.21	36.24	0.03	42.83	41.65	41.67	0.03
	42.84	41.53	41.53	0.00	45.01	44.40	44.40	0.00
	43.82	43.02	43.02	0.00	46.17	45.85	45.85	0.00
0.25	33.76	26.27	34.71	11.45	35.95	30.94	35.83	7.07
	42.18	41.10	42.35	2.13	43.41	42.57	43.16	1.02
	44.42	43.95	44.43	0.85	46.00	45.56	45.80	0.42
	46.37	46.38	46.44	0.12	46.65	46.47	46.50	0.06
0.75	36.07	32.66	38.76	9.06	35.28	31.91	37.63	8.40
	43.16	42.64	43.94	2.26	43.82	43.33	44.46	1.99
	45.62	45.34	45.97	1.14	45.81	45.64	46.06	0.78
	47.86	47.80	48.09	0.55	47.10	46.97	47.20	0.43
1.00	35.27	32.17	37.56	7.95	35.51	32.45	37.95	8.14
	44.17	43.76	44.66	1.59	44.37	44.42	45.02	1.07
	46.07	46.11	46.41	0.55	46.61	46.53	46.78	0.46
	47.70	47.70	47.91	0.39	47.49	47.60	47.76	0.29
1.25	36.31	32.98	37.71	7.05	34.38	30.51	36.39	8.45
	44.09	43.69	44.57	1.54	43.96	43.62	44.53	1.61
	45.96	45.69	46.15	0.85	46.05	46.12	46.37	0.46
	47.09	47.03	47.19	0.29	47.54	47.53	47.67	0.28
2.00	37.27	33.76	37.85	6.17	33.99	30.70	35.43	6.82
	43.74	43.17	43.74	0.99	44.15	43.97	44.41	0.77
	46.13	45.90	46.05	0.27	46.15	46.17	46.31	0.26
	47.85	47.82	47.91	0.15	47.78	47.81	47.84	0.06
2.50	36.54	32.06	35.96	5.73	32.51	28.63	33.45	6.75
	43.34	42.83	43.21	0.66	42.21	41.43	41.88	0.75
	45.25	44.70	44.83	0.22	45.75	45.88	45.95	0.12
	46.75	46.50	46.53	0.05	46.43	46.22	46.25	0.03
3.00	35.65	31.73	34.76	4.43	32.90	28.67	32.56	5.44
	43.58	43.08	43.38	0.53	41.56	40.41	40.67	0.44
	44.39	43.75	43.83	0.14	45.45	45.41	45.43	0.04
	46.84	46.75	46.76	0.02	45.90	45.60	45.60	0.00
3.50	36.01	31.77	34.96	4.67	31.16	25.05	29.17	5.50
	43.05	42.22	42.48	0.45	41.60	41.22	41.44	0.36
	44.37	43.94	43.98	0.06	44.32	44.10	44.12	0.03
	47.53	47.51	47.52	0.01	45.92	45.92	45.92	0.00
4.00	34.99	31.17	33.92	3.98	30.47	25.14	28.46	4.44
	42.20	41.49	41.72	0.39	41.70	40.91	41.04	0.20
	44.18	43.73	43.77	0.06	44.10	44.06	44.07	0.02
	46.48	46.29	46.30	0.01	46.00	45.86	45.86	0.00
4.50	33.97	28.41	31.66	4.54	29.74	23.26	26.44	4.14
	41.32	40.45	40.60	0.25	40.89	40.18	40.28	0.15
	43.89	43.49	43.50	0.02	43.61	43.38	43.39	0.01
	45.72	45.51	45.52	0.00	45.59	45.47	45.46	0.00
5.00	33.80	28.91	31.24	3.28	30.32	25.48	27.81	3.13
	42.32	41.88	41.98	0.16	41.05	40.34	40.38	0.06
	43.44	42.93	42.93	0.00	44.28	44.43	44.44	0.00
	45.97	45.78	45.78	0.00	46.13	46.08	46.09	0.00

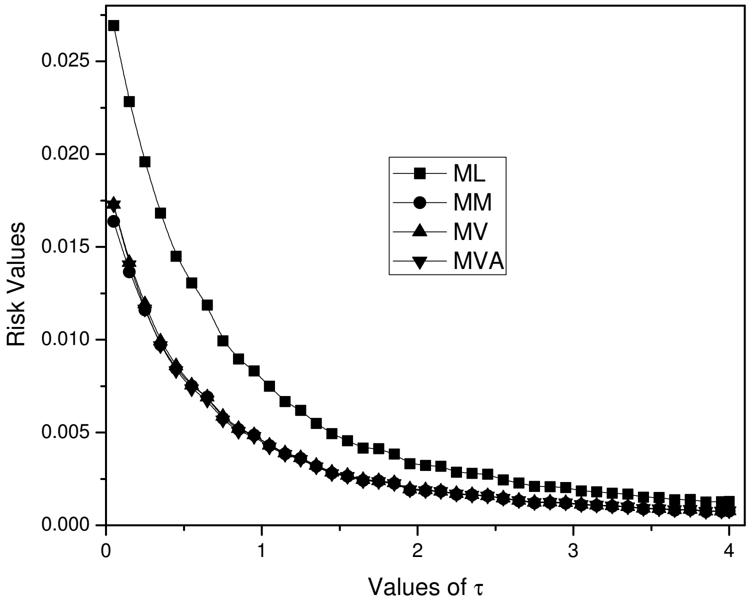

Figure 5.2.1: Comparison of risk values of improved estimators for quantile θ when $k1 = k2 = 0.5$, $m = n = 8$ and $\eta = 0.01$

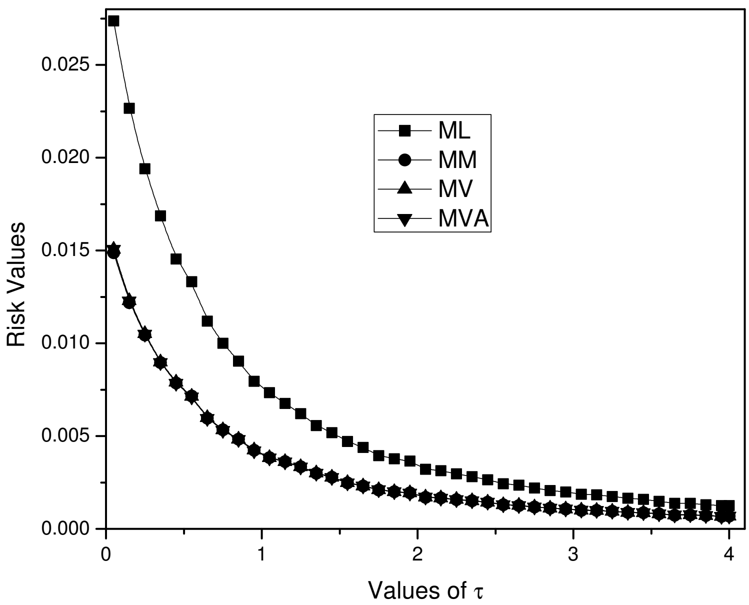

Figure 5.2.2: Comparison of risk values of improved estimators for quantile θ when $k1 = k2 = 1.00$, $m = n = 8$ and $\eta = 0.01$

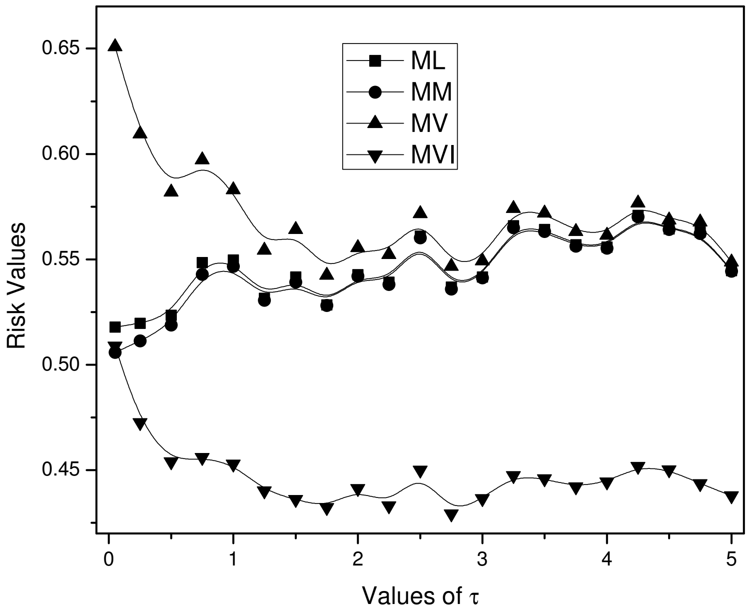

Figure 5.2.3: Comparison of risk values of improved estimators for quantile θ when $k1 = k2 = 0.5$, $m = n = 8$ and $\eta = 1.50$

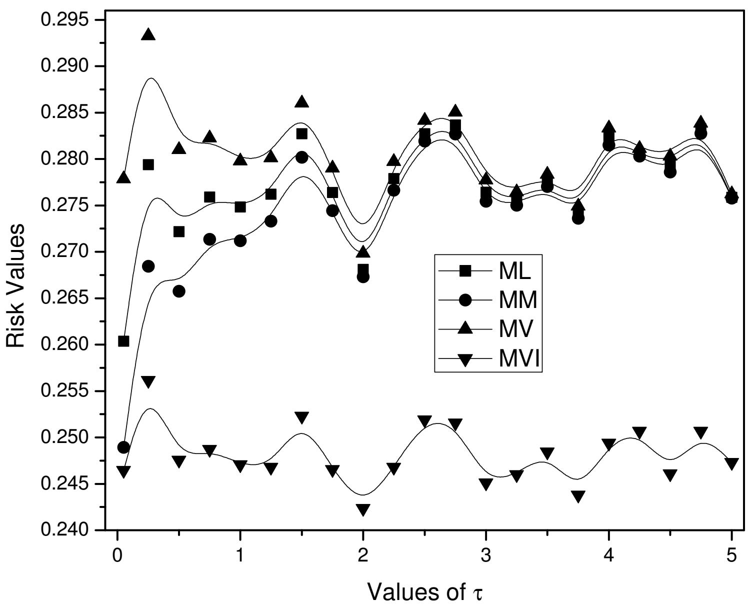

Figure 5.2.4: Comparison of risk values of improved estimators for quantile θ when $k1 = k2 = 1.0$, $m = n = 8$ and $\eta = 1.50$

5.3 Estimating Ordered Scale of Two Exponential Populations with a Common Location under Type-II Censoring

In this section we consider the simultaneous estimation of ordered scale parameters of two exponential populations under equality restriction on the location parameter and the samples are type-II censored.

Suppose type-II censored samples are available from two exponential populations with a common location parameter μ and possibly different scale parameters σ_1 and σ_2 respectively. More specifically, let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}$ $(2 \leq r \leq m)$ and $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(s)}$ $(2 \leq s \leq n)$ be ordered observations taken from two random samples of sizes m and n which follow $Ex(\mu, \sigma_1)$ and $Ex(\mu, \sigma_2)$ respectively. This type of data are known as type-II right censored data. Here $Ex(\mu, \sigma_i)$ denotes the exponential population with location parameter 'μ' and scale parameter σ_i, $i = 1, 2$. The probability density function of $Ex(\mu, \sigma_i)$ is given by

$$f(t, \mu, \sigma_i) = \frac{1}{\sigma_i} \exp\Big\{ - \Big(\frac{t-\mu}{\sigma_i}\Big)\Big\},\ t > \mu, \sigma_i > 0, -\infty < \mu < \infty, i = 1, 2. \tag{5.3.1}$$

The parameter 'μ' which is common to both populations is known as the location parameter (equivalently minimum guarantee time) and the σ_i's are known as the scale parameters (equivalently residual life times). The problem is to estimate the vector parameter $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$ under the assumption that $\sigma_1 \leq \sigma_2$ using a decision theoretic approach. The loss function is taken as

$$L(\hat{\underset{\sim}{d}}, \underset{\sim}{\sigma}) = \sum_{i=1}^{2} \Big(\frac{d_i - \sigma_i}{\sigma_i}\Big)^2, \tag{5.3.2}$$

where $\hat{\underset{\sim}{d}} = (d_1, d_2)$ is an estimator for $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$. The performance of an estimator will be evaluated using the risk function defined as

$$R(\hat{\underset{\sim}{d}}, \underset{\sim}{\sigma}) = E_{\underset{\sim}{\sigma}}\{L(\hat{\underset{\sim}{d}}, \underset{\sim}{\sigma})\}. \tag{5.3.3}$$

The model we consider in this section has applications in industry, business, medical research in the study of reliability, life testing and survival analysis. For example, two new brands of electronic devices say brand A (which uses traditional technology) and brand B (which uses modern technology), having $m(\geq 2)$ and $n(\geq 2)$ units each are placed for life testing. The experimenter could observe only r $(\leq m)$ and s $(\leq n)$ number of failures from brand A and brand B respectively, due to some constraints like time and cost. It may be noted that, the lifetime of each unit from the two brands is random and follows exponential distribution. It is also expected that the minimum guarantee time (μ) for both brands are the same due to market competition, whereas the residual life time (σ_1) of brand A can not exceed the residual life time (σ_2) of brand B. Under this situation one may be interested in drawing inference on the vector parameter $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$. For some more examples of similar nature we refer to Jana and Kumar (2015), and Barlow et al. (1972).

It should be noted that, for full sample case (that is $r = m$, and $s = n$) Jana and

[0]The content of this chapter (Section 5.3) has been published in *Chilean Journal of Statistics*, Vol. 8, No. 1, Pages 87 - 101.

Kumar (2015) considered the componentwise estimation of ordered scale parameters of two exponential populations when the location parameter is common.

5.3.1 Some Basic Results

In this section, we consider the model (5.3.1) and obtain some basic estimators for the vector parameter $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$ assuming that $\sigma_1 \leq \sigma_2$. To be very specific, let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}$, $(2 \leq r \leq m)$ be the r smallest ordered observations taken from a random sample of size $m(\geq 2)$ which follows $Ex(\mu, \sigma_1)$. Likewise let $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(s)}$, $(2 \leq s \leq n)$ be the s smallest ordered observations taken from a random sample of size $n(\geq 2)$ following $Ex(\mu, \sigma_2)$. We assume that these two samples have been drawn independently from two populations. Let us denote $Z = \min(X_{(1)}, Y_{(1)})$, $V_x = U_x - Z$, $V_y = U_y - Z$, where $U_x = [\sum_{i=1}^{r} X_{(i)} + (m - r)X_{(r)}]/m$ and $U_y = [\sum_{i=1}^{s} Y_{(i)} + (n-s)Y_{(s)}]/n$. The complete and sufficient statistics for this problem is given by (Z, V_x, V_y). The joint probability density function of (Z, V_x, V_y) is given by

$$
\begin{aligned}
f_{V_x,V_y,Z}(v_x, v_y, z) = \frac{m^r n^s}{\sigma_1^r \sigma_2^s} &\Big[\frac{v_x^{r-1} v_y^{s-2}}{\Gamma r \Gamma(s-1)} + \frac{v_x^{r-2} v_y^{s-1}}{\Gamma s \Gamma(r-1)}\Big] \\
&\exp\{-\frac{m}{\sigma_1}(v_x + z - \mu) - \frac{n}{\sigma_2}(v_y + z - \mu)\}, v_x > 0, v_y > 0, z > \mu.
\end{aligned}
\tag{5.3.4}
$$

The statistic Z is independent of (V_x, V_y). In the next lines to follow, when we say the MLE (the UMVUE) of the vector parameter $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$ we mean "the collection of the MLEs (the UMVUEs) for each component σ_i and put together to form the vector".

When there is no order restriction among the scale parameters σ_1 and σ_2 the MLE of $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$ is given by

$$
\begin{aligned}
\hat{\underset{\sim}{\sigma}}_{ml} &= \Big(\frac{m}{r}V_x, \frac{n}{s}V_y\Big) \\
&= (\hat{\sigma}_{1ml}, \hat{\sigma}_{2ml}), \text{ say,}
\end{aligned}
\tag{5.3.5}
$$

(see Tripathy (2016) and Chiou and Cohen (1984)). The uniformly minimum variance unbiased estimator for the vector $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$ is given by

$$
\begin{aligned}
\hat{\underset{\sim}{\sigma}}_{mv} &= \Big(\frac{m}{r}\Big(V_x + V_*^{-1}\Big), \frac{n}{s}\Big(V_y + V_*^{-1}\Big)\Big) \\
&= (\hat{\sigma}_{1mv}, \hat{\sigma}_{2mv}), \text{ say,}
\end{aligned}
\tag{5.3.6}
$$

where $V_* = (\frac{V_x}{r-1})^{-1} + (\frac{V_y}{s-1})^{-1}$ (see Tripathy (2016) and Chiou and Cohen (1984)).

When it is known a priori that the scale parameters follow certain ordering that is $\sigma_1 \leq \sigma_2$, these estimators need not be good enough to estimate the vector $\underset{\sim}{\sigma}$. Hence improved estimators can be obtained by using its isotonic regression with proper weights. Using the mini-max formula (see Barlow et al. (1972)), one can easily write the restricted MLEs of both σ_1 and σ_2 as

$$
\hat{\sigma}_{ir} = \min_{i \leq t_1 \leq 2} \max_{1 \leq s_1 \leq i} Av(s_1, t_1), \;\; i = 1, 2,
$$

where

$$Av(s_1,t_1) = \frac{\sum_{j=s_1}^{t_1} n_j\hat{\sigma}_j}{\sum_{j=s_1}^{t_1} n_j}, s_1 \le t_1, s_1, t_1 \in \{1,2\}.$$

Here we denote $n_1 = r$ and $n_2 = s$. Explicitly we obtain the estimators for σ_1 and σ_2 as

$$\hat{\sigma}_{1r} = \min\Big(\frac{m}{r}V_x, \frac{mV_x + nV_y}{r+s}\Big) \text{ and } \hat{\sigma}_{2r} = \max\Big(\frac{n}{s}V_y, \frac{mV_x + nV_y}{r+s}\Big).$$

Using these estimators for σ_1 and σ_2 we construct the restricted MLE (call it $\hat{\underset{\sim}{\sigma}}_{rm}$) of $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$ as

$$\hat{\underset{\sim}{\sigma}}_{rm} = (\hat{\sigma}_{1r}, \hat{\sigma}_{2r}). \tag{5.3.7}$$

It is easy to observe that the risk of the MLE $\hat{\underset{\sim}{\sigma}}_{ml}$ and the UMVUE $\hat{\underset{\sim}{\sigma}}_{mv}$ are respectively given by

$$R(\hat{\underset{\sim}{\sigma}}_{ml}, \underset{\sim}{\sigma}) = \frac{1}{r} + \frac{1}{s},$$

and

$$R(\hat{\underset{\sim}{\sigma}}_{mv}, \underset{\sim}{\sigma}) = \frac{1}{r} + \frac{1}{s} + \Big\{\Big(\frac{m}{r\sigma_1}\Big)^2 + \Big(\frac{n}{s\sigma_2}\Big)^2\Big\}E(V_*^{-2}).$$

Theorem 5.3.1 *Let $\hat{\underset{\sim}{\sigma}}_{ml}$ and $\hat{\underset{\sim}{\sigma}}_{rm}$ be the MLE and the restricted MLE of $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2) : \sigma_1 \le \sigma_2$ respectively. Let the loss function be the sum of the quadratic losses as given in (5.3.2). Then we have $R(\hat{\underset{\sim}{\sigma}}_{ml}, \underset{\sim}{\sigma}) \ge R(\hat{\underset{\sim}{\sigma}}_{rm}, \underset{\sim}{\sigma})$.*

Proof 5.3.1 *Consider the risk difference of $\hat{\underset{\sim}{\sigma}}_{rm}$ and $\hat{\underset{\sim}{\sigma}}_{ml}$:*

$$\begin{aligned}
\Delta &= R(\hat{\underset{\sim}{\sigma}}_{rm}, \underset{\sim}{\sigma}) - R(\hat{\underset{\sim}{\sigma}}_{ml}, \underset{\sim}{\sigma}) \\
&= K_1 \int_1^\infty \frac{(1-z)z^{r-2}\{(1+z) - 2\rho\}\{z(s-1)nr + (r-1)ms\}}{(rz + s\rho)^{r+s+1}} dz \\
&\quad + K_2 \int_1^\infty \frac{(z-1)z^{r-2}\{(1+z) - 2z/\rho\}\{z(s-1)nr + (r-1)ms\}}{(rz + s\rho)^{r+s+1}} dz \\
&= \Delta_1 + \Delta_2, \ (say),
\end{aligned}$$

where $K_1 = \frac{s^{s+1}r^{r-1}\Gamma(r+s+1)\rho^s}{(r+s)^2(m+n\rho)}$, $K_2 = \frac{s^{s-1}r^{r+1}\Gamma(r+s+1)\rho^{s+2}}{(r+s)^2(m+n\rho)}$ and $0 < \rho = \sigma_1/\sigma_2 \le 1$. It is easy to observe that, both terms Δ_1 and Δ_2 are non-positive when $0 < \rho \le 1$. This completes the proof of the theorem.

Next, we consider a general class of estimators for estimating the vector $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$ and derive a sufficient condition for improving estimators in this class under the assumption, that the scale parameters are ordered, that is, $\sigma_1 \le \sigma_2$. Consider the class of estimators

$$D_{\mathbf{c}} = \{\hat{\underset{\sim}{d}}_{\mathbf{c}} = (\hat{d}_{c_1}, \hat{d}_{c_2}) : \mathbf{c} = (c_1, c_2), c_1, c_2 \in \mathbb{R}\}, \tag{5.3.8}$$

where $\hat{d}_{c_1} = c_1 V_x$, and $\hat{d}_{c_2} = c_2 V_y$. This class contains the MLE with choices of $c_1 = m/r$ and $c_2 = n/s$.

To proceed further we define a vector $\mathbf{c}^*$ for the class of estimators $D_{\mathbf{c}}$ as,

$$\mathbf{c}^* = (\min(\max(c_1, c_{1*}), c_1^*), \min(\max(c_2, c_{2*}), c_2^*)), \tag{5.3.9}$$

where

$$c_{1*} = \frac{m(r(m+n)) - m}{mr(r-1) + nr(r+1)},\ c_1^* = \frac{m}{r},\ c_{2*} = \frac{n}{s+1},\ \text{and}\ c_2^* = \frac{n(s(m+n)) - n}{ns(s-1) + ms(s+1)}.$$

Next, we prove a general inadmissibility result for the class of estimators $D_{\mathbf{c}}$.

Theorem 5.3.2 *Let $\hat{\underset{\sim}{d}}_{\mathbf{c}}$ be the class of estimators for estimating the vector parameter $\underset{\sim}{\sigma}$ as given in (5.3.8) and the loss function be taken as in (5.3.2). Define a vector $\mathbf{c}^*$ as in (5.3.9). Then the class of estimators $\hat{\underset{\sim}{d}}_{\mathbf{c}}$ is inadmissible and is improved by $\hat{\underset{\sim}{d}}_{\mathbf{c}^*}$ if $\mathbf{c} \neq \mathbf{c}^*$.*

Proof 5.3.2 *Let us consider the risk of the estimator $\hat{\underset{\sim}{d}}_{\mathbf{c}}$ with respect to the loss function (5.3.2).*

$$R(\hat{\underset{\sim}{d}}_{\mathbf{c}}, \hat{\underset{\sim}{\sigma}}) = E\Big(\frac{\hat{d}_{c_1} - \sigma_1}{\sigma_1}\Big)^2 + E\Big(\frac{\hat{d}_{c_2} - \sigma_2}{\sigma_2}\Big)^2.$$

The above risk is a convex function in both c_1 and c_2, hence the minimizing choices of c_1 and c_2 have been obtained as

$$\hat{c}_1 = \frac{\sigma_1 EV_x}{EV_x^2}\ \textit{and}\ \hat{c}_2 = \frac{\sigma_2 EV_y}{EV_y^2}.$$

We note that $EV_x = \frac{r}{m}\sigma_1 - p^{-1}$, $EV_y = \frac{s}{n}\sigma_2 - p^{-1}$, $EV_x^2 = \frac{n\sigma_1}{m\sigma_2}p^{-1}\{\frac{r(r-1)\sigma_2}{n} + \frac{r(r+1)\sigma_1}{m}\}$, $EV_y^2 = \frac{m\sigma_2}{n\sigma_1}p^{-1}\{\frac{s(s-1)\sigma_1}{m} + \frac{s(s+1)\sigma_2}{n}\}$ where we denote $p = \frac{m}{\sigma_1} + \frac{n}{\sigma_2}$. Substituting all these values and after some simplification we get

$$\hat{c}_1(\rho) = \frac{m(r(m+n\rho) - m)}{mr(r-1) + nr(r+1)\rho},\ \textit{and}\ \hat{c}_2(\rho) = \frac{n(s(m+n\rho) - n\rho)}{s(s-1)n\rho + s(s+1)m},$$

where we denote $\rho = \sigma_1/\sigma_2$; $0 < \rho \leq 1$.

In order to obtain the result we need to obtain the supremum and infimum of $\hat{c}_1(\rho)$ and $\hat{c}_2(\rho)$ with respect to ρ for fixed sample sizes. It is easy to observe that the function $\hat{c}_1(\rho)$ is a decreasing function in ρ $(0 < \rho \leq 1)$. Hence its infimum is attained as $\rho \to 1$ and supremum is attained as $\rho \to 0$. We have

$$\inf \hat{c}_1(\rho) = \frac{m(r(m+n)) - m}{mr(r-1) + nr(r+1)} = c_{1*}\ \textit{and}\ \sup \hat{c}_1(\rho) = \frac{m}{r} = c_1^*.$$

Similarly the infimum and supremum of $\hat{c}_2(\rho)$ are obtained as

$$\inf \hat{c}_2(\rho) = \frac{n}{s+1} = c_{2*}\ \textit{and}\ \sup \hat{c}_2(\rho) = \frac{n(s(m+n)) - n}{ns(s-1) + ms(s+1)} = c_2^*.$$

Utilizing these results one can easily define the vector $\mathbf{c}^*$ as given in (5.3.9). Now using the orbit-by-orbit improvement technique of Brewster and Zidek (1974), we have proved the theorem.

Remark 5.3.1 *The class of estimators $D_{\mathbf{c}} = \{\hat{\underset{\sim}{d}}_{\mathbf{c}} : \mathbf{c} = (c_1, c_2), c_{1*} \leq c_1 \leq c_1^*, c_{2*} \leq c_2 \leq c_2^*\}$ is complete.*

Remark 5.3.2 *Consider the restricted parameter space $\sigma_1 \le \sigma_2$. The estimator $\hat{\underset{\sim}{d}}_{c^*}$ dominates $\hat{\underset{\sim}{d}}_c$ if either $c_1 \in [c_{1*}, c_1^*]^c$ or $c_2 \in [c_{2*}, c_2^*]^c$. The MLE $\hat{\underset{\sim}{\sigma}}_{ml}$ can not be improved by using Theorem 5.3.2 as for the MLE, $c_1 \in [c_{1*}, c_1^*]$ and $c_2 \in [c_{2*}, c_2^*]$. Here $[a,b]^c$ denotes the compliment of the interval $[a,b]$ for any real numbers a and b.*

In the next section, we prove some general inadmissibility results for the classes of affine and scale equivariant estimators. As a consequence estimators dominating the MLE $\hat{\underset{\sim}{\sigma}}_{ml}$, the UMVUE $\hat{\underset{\sim}{\sigma}}_{mv}$ and the restricted MLE $\hat{\underset{\sim}{\sigma}}_{rm}$ have been obtained.

5.3.2 Improving Equivariant Estimators under Order Restriction

In this section, we introduce the concept of invariance to our problem and derive a sufficient condition for improving estimators which are equivariant under affine group of transformations.

Let $G_A = \{g_{a,b} : g_{a,b}(x) = ax + b, a > 0, -\infty < b < \infty\}$ be a group of affine transformations. Let us define, $V_x = U_x - Z$, $V_y = U_y - Z$. Under the transformation $g_{a,b}$, the sufficient statistics being transformed as $V_x \to aV_x$, $V_y \to aV_y$ and $Z \to aZ + b$. The parameters $\mu \to a\mu + b$, and $\underset{\sim}{\sigma} \to a\underset{\sim}{\sigma}$ as $\sigma_i \to a\sigma_i, i = 1, 2$ such that the ordering remains intact. In order that the loss function (5.3.2) to be invariant, the estimator $\underset{\sim}{d} = (d_1, d_2)$ satisfies the relation

$$\underset{\sim}{d}(aZ + b, aV_x, aV_y) = a\underset{\sim}{d}(Z, V_x, V_y).$$

Substituting $b = -aZ$ where $a = 1/V_x$, and simplifying, we obtain the form of an affine equivariant estimator for estimating the vector prameter $\underset{\sim}{\sigma}$ based on (V_x, V_y, Z) as,

$$\begin{aligned} \underset{\sim}{d}(Z, V_x, V_y) &= V_x(\xi_1(V), \xi_2(V)), \\ &= \underset{\sim}{d}_\xi, \text{ (say)}, \end{aligned} \tag{5.3.10}$$

where $\xi = (\xi_1, \xi_2)$, $V = \frac{V_y}{V_x}$ and $\xi_i : [0, \infty) \to \mathbb{R}$, $i = 1, 2$ are real valued functions of V.

To prove the main result of this section let us define a vector valued function ξ^* for the class of estimators $\underset{\sim}{d}_\xi$ as

$$\xi^*(v) = (\min(\max(\xi_1, \xi_{1*}), \xi_1^*), \max(\xi_2, \xi_{2*})), \tag{5.3.11}$$

where $\xi_{1*} = \frac{m}{r+s}$, $\xi_1^* = \frac{m+nv}{r+s}$, and $\xi_{2*} = \frac{m+nv}{r+s}$.

Theorem 5.3.3 *Let $\underset{\sim}{d}_\xi$ be the affine class of estimators as given in (5.3.10) for estimating the vector parameter $\underset{\sim}{\sigma}$. Let the loss function be taken as (5.3.2). Then the estimator $\underset{\sim}{d}_\xi$ is inadmissible and is improved by $\underset{\sim}{d}_{\xi^*}$ if there exist some values of the parameters $\sigma_1, \sigma_2; \sigma_1 \le \sigma_2$, such that $P(\underset{\sim}{d}_\xi \ne \underset{\sim}{d}_{\xi^*}) > 0$.*

Proof 5.3.3 *The proof of the theorem can be done by using a result of Brewster and Zidek (Brewster and Zidek (1974)). To complete the proof, let us consider the conditional risk function of $\underset{\sim}{d}_\xi$ given V.*

$$R(\underset{\sim}{d}_\xi, \underset{\sim}{\sigma}|V = v) = E\Big\{\Big(\frac{d_{\xi_1} - \sigma_1}{\sigma_1}\Big)^2|V = v\Big\} + E\Big\{\Big(\frac{d_{\xi_2} - \sigma_2}{\sigma_2}\Big)^2|V = v\Big\}.$$

The above risk function is a convex function of both ξ_1 and ξ_2. The minimizing choices of these *functions are obtained as*

$$\hat{\xi}_1(v) = \frac{\sigma_1 E(V_x|V=v)}{E(V_x^2|V=v)} \text{ and } \hat{\xi}_2(v) = \frac{\sigma_2 E(V_x|V=v)}{E(V_x^2|V=v)}. \tag{5.3.12}$$

It is easy to observe that, the conditional probability density function of V_x given $V=v$, is a gamma distribution with shape parameter '$r+s-1$' and scale parameter '$\frac{\sigma_1\sigma_2}{m\sigma_2+n\sigma_1 v}$'. Here the gamma probability density function with a shape parameter 'α' and a scale parameter 'β' is defined as

$$g(x,\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}, \ \ x>0, \alpha>0, \beta>0.$$

So, the conditional expectations are calculated and are obtained as

$$E(V_x|V=v) = \frac{(r+s-1)\sigma_1\sigma_2}{m\sigma_2+n\sigma_1 v}, \tag{5.3.13}$$

and

$$E(V_x^2|V=v) = (r+s-1)(r+s)\Big(\frac{\sigma_1\sigma_2}{m\sigma_2+n\sigma_1 v}\Big)^2. \tag{5.3.14}$$

Substituting the conditional expectations from (5.3.13) and (5.3.14) in (5.3.12) and simplifying we get the minimizing choices as,

$$\hat{\xi}_1(v) = \frac{m+n\rho v}{r+s} \text{ and } \hat{\xi}_2(v) = \frac{m+n\rho v}{\rho(r+s)}.$$

In order to apply the Brewster-Zidek technique (Brewster and Zidek (1974)), it is necessary to obtain the supremum and infimum of both $\hat{\xi}_1(v)$ and $\hat{\xi}_2(v)$ for fixed values of v and fixed values of sample sizes. It is easy to note that $\hat{\xi}_1(v)$ is an increasing function of $0<\rho\le 1$ for fixed v and m,n,r,s. Hence the infimum and supremum of $\hat{\xi}_1(v)$ are obtained as

$$\inf_{0<\rho\le 1} \hat{\xi}_1(v) = \frac{m}{r+s} = \xi_{1*}, \text{say and } \sup_{0<\rho\le 1} \hat{\xi}_1(v) = \frac{m+nv}{r+s} = \xi_1^*, \text{say}.$$

Similarly it is easy to obtain the spremum and infimum of $\hat{\xi}_2$ and are given by

$$\inf_{0<\rho\le 1} \hat{\xi}_2(v) = \frac{m+nv}{r+s} = \xi_{2*}, \text{say and } \sup_{0<\rho\le 1} \hat{\xi}_2(v) = +\infty.$$

Now using the above results it is easy to define the vector valued function ξ^* as given in (5.3.11). Using Theorem 5.3.3 of Brewster and Zidek (see Brewster and Zidek (1974)) for improving equivariant estimators we get $R(\underset{\sim}{d}_\xi, \underset{\sim}{\sigma}) \ge R(\underset{\sim}{d}_{\xi^*}, \underset{\sim}{\sigma})$ when $0<\rho\le 1$. Hence the proof is completed.

Remark 5.3.3 *The class of estimators $\underset{\sim}{d}_\xi$ such that $\xi_{1*} \le \xi_1 \le \xi_1^*$ and $\xi_2 \ge \xi_{2*}$ form an admissible class of estimators within the class of all estimators of the form $\underset{\sim}{d}_\xi$.*

Next we apply the Theorem 5.3.3 to obtain improved estimators which dominate the MLE $\hat{\underset{\sim}{\sigma}}_{ml}$, the UMVUE $\hat{\underset{\sim}{\sigma}}_{mv}$ and the restricted MLE $\hat{\underset{\sim}{\sigma}}_{rm}$ when $\sigma_1 \le \sigma_2$. We note that, the estimators

$\hat{\underset{\sim}{\sigma}}_{ml}$, $\hat{\underset{\sim}{\sigma}}_{mv}$ and $\hat{\underset{\sim}{\sigma}}_{rm}$ belong to the class given in (5.3.10). Applying Theorem 5.3.3, we obtain the improved estimators dominating $\hat{\underset{\sim}{\sigma}}_{ml}$, $\hat{\underset{\sim}{\sigma}}_{mv}$ and $\hat{\underset{\sim}{\sigma}}_{rm}$ respectively as

$$\hat{\underset{\sim}{\sigma}}_{am} = V_x[\min(\max(\xi_{1m}(V), \xi_{1*}(V)), \xi_1^*(V)), \max(\xi_{2m}(V), \xi_{2*}(V)))], \tag{5.3.15}$$

where $\xi_{1m}(V) = \frac{m}{r}, \xi_{2m}(V) = \frac{n}{s}V$,

$$\hat{\underset{\sim}{\sigma}}_{av} = V_x[\min(\max(\xi_{1v}(V), \xi_{1*}(V)), \xi_1^*(V)), \max(\xi_{2v}(V), \xi_{2*}(V)))], \tag{5.3.16}$$

where $\xi_{1v}(V) = \frac{m}{r}(1 + \frac{V}{(r-1)V+(s-1)}), \xi_{2v}(V) = \frac{n}{s}(V + \frac{V}{(r-1)V+(s-1)})$, and

$$\hat{\underset{\sim}{\sigma}}_{ar} = V_x[\min(\max(\xi_{1r}(V), \xi_{1*}(V)), \xi_1^*(V)), \max(\xi_{2r}(V), \xi_{2*}(V)))], \tag{5.3.17}$$

where

$$\xi_{1r}(V) = \begin{cases} \frac{m}{r}, & \text{if } \frac{m}{r}V_x \leq \frac{n}{s}V_y, \\ \frac{m+nV}{r+s}, & \text{if } \frac{m}{r}V_x > \frac{n}{s}V_y, \end{cases}$$

$$\xi_{2r}(V) = \begin{cases} \frac{n}{s}V, & \text{if } \frac{m}{r}V_x \leq \frac{n}{s}V_y, \\ \frac{m+nV}{r+s}, & \text{if } \frac{m}{r}V_x > \frac{n}{s}V_y. \end{cases}$$

Remark 5.3.4 *We note that the risk values of the above improved estimators could not be obtained in closed form. Hence a simulation study has been done in Section 5.3.3, to evaluate numerically the risk performances of all these estimators.*

Next, we consider a smaller group of transformations which will lead to form a larger class of estimators. Consider the smaller scale group of transformations $G_S = \{g_a : g_a(x) = ax, a > 0\}$. With the help of this group structure, the sufficient statistics being transformed as $Z \to aZ$, $V_x \to aV_x$ and $V_y \to aV_y$. Also the parameters $\mu \to a\mu$, $\sigma_i \to a\sigma_i$; $i = 1, 2$, so that the vector $\underset{\sim}{\sigma} \to a\underset{\sim}{\sigma}$. The loss function (5.3.2) will be invariant if the estimator $\underset{\sim}{\delta}$ satisfies the relation

$$\underset{\sim}{\delta}(aZ, aV_x, aV_y) = a\underset{\sim}{\delta}(Z, V_x, V_y).$$

Choosing $a = 1/V_x$, and simplifying we get the form of a scale equivariant estimator for estimating $\underset{\sim}{\sigma}$, based on (Z, V_x, V_y) as

$$\begin{aligned} \underset{\sim}{\delta}(Z, V_x, V_y) &= V_x(\psi_1(U, V), \psi_2(U, V)) \\ &= \underset{\sim}{\delta}_\psi, \text{ say} \end{aligned} \tag{5.3.18}$$

where $U = Z/V_x$, $V = V_y/V_x$ and ψ_1 and ψ_2 are real valued functions of U and V.

Let us define the following functions

$$\psi_1^0 = \frac{m(1+u) + n(u+v)}{r+s+1},\ \psi_{11}^0 = \frac{m(1+u)}{r+s+1}, \psi_2^0 = \psi_1^0. \tag{5.3.19}$$

For the scale equivariant estimator $\underset{\sim}{\delta}_\psi$ define the vector valued function ψ^* as,

$$\psi^* = (\psi_1^*, \psi_2^*) \tag{5.3.20}$$

where the functions ψ_1^* and ψ_2^* are defined as

$$\psi_1^* = \begin{cases} \psi_1^0, & \text{if } u > 0, \psi_1 > \psi_1^0 \text{ or } u < 0, \psi_1 < \psi_1^0, u + v < 0, \\ \psi_{11}^0, & \text{if } u < 0, \psi_1 < \psi_{11}^0, u + v > 0, \\ \psi_1, & \text{otherwise.} \end{cases}$$

and

$$\psi_2^* = \begin{cases} \psi_2^0, & \text{if } u < 0, \psi_2 < \psi_2^0, \\ \psi_2, & \text{otherwise.} \end{cases}$$

Theorem 5.3.4 *Let $\underset{\sim}{\delta}_\psi$ be the class of scale equivariant estimators for estimating the vector parameter $\underset{\sim}{\sigma}$ as given in (5.3.18). Let the loss function be as given in (5.3.2). Define the vector valued function ψ^* as in (5.3.20). Then the estimator $\underset{\sim}{\delta}_\psi$ is inadmissible and is improved by $\underset{\sim}{\delta}_{\psi^*}$ if there exist some values of parameters μ, σ_1, $\sigma_2 : \sigma_1 \le \sigma_2$, such that $P(\underset{\sim}{\delta}_{\psi^*} \neq \underset{\sim}{\delta}_\psi) > 0$.*

Proof 5.3.4 *Let us consider the conditional risk function of $\underset{\sim}{\delta}_\psi$ given $(U, V) = (u, v)$.*

$$R(\underset{\sim}{\delta}_\psi, \mu, \underset{\sim}{\sigma} | (U, V)) = \frac{1}{\sigma_1^2} E\{(V_x \psi_1(U, V) - \sigma_1)^2 | (U, V)\} + \frac{1}{\sigma_2^2} E\{(V_x \psi_2(U, V) - \sigma_2)^2 | (U, V)\}.$$

It is easy to observe that the above risk function is a sum of two convex functions in ψ_1 and ψ_2, hence their minimizing choices have been obtained as

$$\begin{aligned} \hat{\psi}_1(u, v) &= \frac{\sigma_1 E(V_x | (U, V) = (u, v))}{E(V_x^2 | (U, V)) = (u, v)} \\ &= \frac{\int_D t^{r+s} \exp\{-(m(1+u) + n\rho(u+v))t\} dt}{\int_D t^{r+s+1} \exp\{-(m(1+u) + n\rho(u+v))t\} dt}, \end{aligned}$$

where $D = \{t : 0 < t < \infty, \sigma_1 u t > \mu\}$.

$$\begin{aligned} \hat{\psi}_2(u, v) &= \frac{\sigma_2 E(V_x | (U, V) = (u, v))}{E(V_x^2 | (U, V)) = (u, v)} \\ &= \frac{\int_{D^*} t^{r+s} \exp\{-(m(1+u)\rho^* + n(u+v))t\} dt}{\int_{D^*} t^{r+s+1} \exp\{-(m(1+u)\rho^* + n(u+v))t\} dt}, \end{aligned}$$

where $D^* = \{t : 0 < t < \infty, \sigma_2 u t > \mu\}$, and $\rho^* = 1/\rho$.

To apply Brewster and Zidek (1974) technique, we first obtain the bounds for the first component $\hat{\psi}_1(u, v)$. Let

$$G_1(c) = \int_c^\infty t^{r+s} \exp\{-(m(1+u) + n\rho(u+v))t\} dt$$

and

$$G_2(c) = \int_c^\infty t^{r+s+1} \exp\{-(m(1+u) + n\rho(u+v))t\} dt,$$

where $c = \max(0, \mu/\sigma_1 u)$. Next we consider three separate cases to analyze the terms and obtain the bounds.

Case I: *Let $\mu > 0, u > 0$. In this case $c = \mu/\sigma_1 u$. It is easy to see that $\hat{\psi}_1(u, v)$ is a decreasing function of $\mu \in (0, \infty)$ as $G_2(c) \geq cG_1(c)$. Hence we obtained*

$$\inf \hat{\psi}_1(u, v) = 0, \;\; \textit{and} \;\; \sup \hat{\psi}_1(u, v) = \frac{m(1 + u) + n(u + v)}{r + s + 1}.$$

Case II: *Let $\mu < 0$, $u < 0$. In this case we have $c = \mu/\sigma_1 u$ so that the range of integration becomes $(0, c)$. Hence*

$$G_1(c) = \int_0^c t^{r+s} \exp\{-(m(1 + u) + n\rho(u + v))t\}dt$$

and

$$G_2(c) = \int_0^c t^{r+s+1} \exp\{-(m(1 + u) + n\rho(u + v))t\}dt.$$

In this case we can easily see that, $G_2(c) \leq cG_1(c)$. The function $\hat{\psi}_1(u, v)$ is an increasing function of μ as $G_2(c) \leq cG_1(c)$. We have $\lim_{\mu\to-\infty} \hat{\psi}_1(u, v) = \lim_{\mu\to-\infty} \frac{G_1(c)}{G_2(c)} = \frac{m(1+u)+n\rho(u+v)}{r+s+1}$. Hence

$$\begin{aligned} \inf_{\rho,\mu} \hat{\psi}_1(u, v) &= \frac{m(1 + u)}{r + s + 1}, \; if \; u + v > 0, \\ &= \frac{m(1 + u) + n(u + v)}{r + s + 1} \; if \; u + v < 0. \end{aligned}$$

In this case the supremum is obtained as

$$\sup_{\mu\to 0} \hat{\psi}_1(u, v) = \infty.$$

Case III: *Let $\mu < 0$, $u > 0$. In this case $c = 0$. Hence the function $\hat{\psi}_1(u, v)$ simplifies to*

$$\hat{\psi}_1(u, v) = \frac{m(1 + u) + n\rho(u + v)}{r + s + 1}.$$

We obtain the supremum and infimum as

$$\inf_{\rho\to 0} \hat{\psi}_1(u, v) = \frac{m(1 + u)}{r + s + 1},$$

and

$$\sup_{\rho\to 1} \hat{\psi}_1(u, v) = \frac{m(1 + u) + n(u + v)}{r + s + 1}.$$

To obtain the supremum and infimum of the second component we proceed as follows. Let us denote,

$$H_1(c) = \int_c^\infty t^{r+s} \exp\{-(m(1 + u)\rho^* + n(u + v))t\},$$

and

$$H_2(c) = \int_c^\infty t^{r+s+1} \exp\{-(m(1+u)\rho^* + n(u+v))t\},$$

where $c = \max(0, \mu/\sigma_2 u)$. *As in the case of the first component we also consider three separate cases for the second component as below.*

Case I: $\mu > 0, u > 0$. *In this case* $c = \mu/\sigma_2 u$. *It is easy to observe that* $\psi_2(u,v)$ *is a decreasing function of* μ. *So* $\inf_{\mu,\rho} \psi_2(u,v) = \lim_{\mu\to\infty} \psi_2(u,v) = 0$ *and* $\sup_{\mu,\rho} \psi_2(u,v) = \infty$.

Case II: $\mu < 0, u < 0$. *In this case* $c = \mu/\sigma_2 u$. *Also we have observed that* $\psi_2(u,v)$ *is an increasing function of* μ. *Hence* $\inf \psi_2(u,v) = \lim_{\mu\to-\infty} \psi_2(u,v) = \frac{m(1+u)\rho^*+n(u+v)}{r+s+1}$. *Hence* $\inf \psi_2(u,v) = \frac{m(1+u)+n(u+v)}{r+s+1}$. *Also we obtain* $\sup \psi_2(u,v) = \infty$.

Case III: $\mu < 0, u > 0$. *In this case* $c = 0$ *and hence* $\psi_2(u,v) = \frac{m(1+u)\rho^*+n(u+v)}{r+s+1}$. *Hence* $\sup \psi_2(u,v) = \infty$ *and* $\inf \psi_2(u,v) = \frac{m(1+u)+n(u+v)}{r+s+1}$.

Next define the following functions.

$$\psi_1^0 = \frac{m(1+u)+n(u+v)}{r+s+1},\ \psi_{11}^0 = \frac{m(1+u)}{r+s+1}, \psi_2^0 = \psi_1^0.$$

For the scale equivariant estimator $\underline{\delta}_\psi$ *define the functions* ψ_1^* *and* ψ_2^* *as follows.*

$$\begin{aligned} \psi_1^* &= \psi_1^0,\ \textit{if } u > 0, \psi_1 > \psi_1^0 \textit{ or } u < 0, \psi_1 < \psi_1^0, u+v < 0, \\ &= \psi_{11}^0,\ \textit{if } u < 0, \psi_1 < \psi_{11}^0, u+v > 0, \\ &= \psi_1,\ \textit{otherwise.} \end{aligned}$$

$$\begin{aligned} \psi_2^* &= \psi_2^0,\ \textit{if } u < 0, \psi_2 < \psi_2^0, \\ &= \psi_2,\ \textit{otherwise.} \end{aligned}$$

Using the above results it is easy to define a vector ψ^* *as*

$$\psi^* = (\psi_1^*, \psi_2^*).$$

Using the orbit-by-orbit improvement technique of Brewster and Zidek (1974), the theorem is proved.

Next, we use the above result to obtain estimators improving upon the MLE $\hat{\underset{\sim}{\sigma}}_{ml}$, the UMVUE $\hat{\underset{\sim}{\sigma}}_{mv}$, and the restricted MLE $\hat{\underset{\sim}{\sigma}}_{rm}$. We note that the estimators $\hat{\underset{\sim}{\sigma}}_{ml}$, $\hat{\underset{\sim}{\sigma}}_{mv}$, and $\hat{\underset{\sim}{\sigma}}_{rm}$, also belong to the class given in (5.3.18). As an application of Theorem 5.3.4, the following improved estimators have been obtained. The estimator which improves upon $\hat{\underset{\sim}{\sigma}}_{ml}$ is given by

$$\hat{\underset{\sim}{\sigma}}_{sm} = V_x(\psi_{1m}^*, \psi_{2m}^*) \tag{5.3.21}$$

where

$$\psi_{1m}^* = \begin{cases} \psi_1^0, & \text{if } u > 0, \psi_{1m} > \psi_1^0 \text{ or } u < 0, \psi_{1m} < \psi_1^0, u+v < 0, \\ \psi_{11}^0, & \text{if } u < 0, \psi_{1m} < \psi_{11}^0, u+v > 0, \\ \psi_{1m}, & \text{otherwise,} \end{cases}$$

$$\psi_{2m}^* = \begin{cases} \psi_2^0, & \text{if } u < 0, \psi_{2m} < \psi_2^0, \\ \psi_{2m}, & \text{otherwise,} \end{cases}$$

and

$$\psi_{1m} = \frac{m}{r}, \;\; \psi_{2m} = \frac{n}{s} V_y.$$

The estimator which improves upon $\hat{\underset{\sim}{\sigma}}_{mv}$ is given by

$$\hat{\underset{\sim}{\sigma}}_{sv} = V_x(\psi_{1v}^*, \psi_{2v}^*) \tag{5.3.22}$$

where

$$\psi_{1v}^* = \begin{cases} \psi_1^0, & \text{if } u > 0, \psi_{1v} > \psi_1^0 \text{ or } u < 0, \psi_{1v} < \psi_1^0, u + v < 0, \\ \psi_{11}^0, & \text{if } u < 0, \psi_{1v} < \psi_{11}^0, u + v > 0, \\ \psi_{1v}, & \text{otherwise.} \end{cases}$$

$$\psi_{2v}^* = \begin{cases} \psi_2^0, & \text{if } u < 0, \psi_{2v} < \psi_2^0, \\ \psi_{2v}, & \text{otherwise,} \end{cases}$$

and

$$\psi_{1v} = \frac{m}{r}\Big(1 + \frac{V}{(r-1)V + (s-1)}\Big), \;\; \psi_{2v} = \frac{n}{s}\Big(V + \frac{V}{(r-1)V + (s-1)}\Big).$$

Similarly the estimator which improves upon $\hat{\underset{\sim}{\sigma}}_{rm}$ is given by

$$\hat{\underset{\sim}{\sigma}}_{sr} = V_x(\psi_{1r}^*, \psi_{2r}^*) \tag{5.3.23}$$

where

$$\psi_{1r}^* = \begin{cases} \psi_1^0, & \text{if } u > 0, \psi_{1r} > \psi_1^0 \text{ or } u < 0, \psi_{1r} < \psi_1^0, u + v < 0, \\ \psi_{11}^0, & \text{if } u < 0, \psi_{1r} < \psi_{11}^0, u + v > 0, \\ \psi_{1r}, & \text{otherwise.} \end{cases}$$

$$\psi_{2r}^* = \begin{cases} \psi_2^0, & \text{if } u < 0, \psi_{2r} < \psi_2^0, \\ \psi_{2r}, & \text{otherwise,} \end{cases}$$

and

$$\psi_{1r} = \begin{cases} \frac{m}{r}, & \text{if } \frac{m}{r} V_x \le \frac{n}{s} V_y, \\ \frac{m+nV}{r+s}, & \text{if } \frac{m}{r} V_x > \frac{n}{s} V_y, \end{cases}$$

$$\psi_{2r} = \begin{cases} \frac{n}{s} V, & \text{if } \frac{m}{r} V_x \le \frac{n}{s} V_y, \\ \frac{m+nV}{r+s}, & \text{if } \frac{m}{r} V_x > \frac{n}{s} V_y. \end{cases}$$

Remark 5.3.5 *The improved estimators $\hat{\underset{\sim}{\sigma}}_{sm}$, $\hat{\underset{\sim}{\sigma}}_{sv}$ and $\hat{\underset{\sim}{\sigma}}_{sr}$ obtained by using Theorem 5.3.4 have been numerically compared in Section 5.3.3.*

5.3.3 Numerical Comparisons

In Section 5.3.2, we have proposed improved estimators namely $\hat{\underset{\sim}{\sigma}}_{am}$, $\hat{\underset{\sim}{\sigma}}_{av}$, $\hat{\underset{\sim}{\sigma}}_{ar}$, $\hat{\underset{\sim}{\sigma}}_{sm}$, $\hat{\underset{\sim}{\sigma}}_{sv}$, and $\hat{\underset{\sim}{\sigma}}_{sr}$, for $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$ using Theorem 5.3.3 and 5.3.4 when there is order restriction on σ_is that is, $\sigma_1 \leq \sigma_2$. These estimators have been improved upon $\hat{\underset{\sim}{\sigma}}_{ml}$, $\hat{\underset{\sim}{\sigma}}_{mv}$, $\hat{\underset{\sim}{\sigma}}_{rm}$. In Section 5.3.1, we have also shown that the estimator $\hat{\underset{\sim}{\sigma}}_{rm}$ improves upon $\hat{\underset{\sim}{\sigma}}_{ml}$. It seems impossible to compare the risk performances of all the estimators analytically. The performance of each estimator was evaluated numerically using simulations. In order to numerically compare the performances of all the estimators, we have generated 20,000 random type-II censored samples each from two exponential populations with a common location parameter μ and different scale parameters σ_1 and σ_2 such that $\sigma_1 \leq \sigma_2$. It is easy to observe that the risk function of all these estimators with respect to the loss (5.3.2) is only a function of τ, where $0 < \tau = \sigma_1/\sigma_2 \leq 1$ for fixed sample sizes m, n, r and s. We note that the risk of the estimator $\hat{\underset{\sim}{\sigma}}_{ml}$ is constant $1/r + 1/s$, however the simulated risk values have been used for comparison purpose in our simulation. To proceed further we define the following percentage of relative risk for all the estimators with respect to the estimator $\hat{\underset{\sim}{\sigma}}_{mv}$ as

$$PR1 = \left(1 - \frac{Risk(\hat{\underset{\sim}{\sigma}}_{ml})}{Risk(\hat{\underset{\sim}{\sigma}}_{mv})}\right) \times 100, \quad PR2 = \left(1 - \frac{Risk(\hat{\underset{\sim}{\sigma}}_{rm})}{Risk(\hat{\underset{\sim}{\sigma}}_{mv})}\right) \times 100,$$

$$PR3 = \left(1 - \frac{Risk(\hat{\underset{\sim}{\sigma}}_{am})}{Risk(\hat{\underset{\sim}{\sigma}}_{mv})}\right) \times 100, \quad PR4 = \left(1 - \frac{Risk(\hat{\underset{\sim}{\sigma}}_{av})}{Risk(\hat{\underset{\sim}{\sigma}}_{mv})}\right) \times 100,$$

$$PR5 = \left(1 - \frac{Risk(\hat{\underset{\sim}{\sigma}}_{ar})}{Risk(\hat{\underset{\sim}{\sigma}}_{mv})}\right) \times 100, \quad PR6 = \left(1 - \frac{Risk(\hat{\underset{\sim}{\sigma}}_{sm})}{Risk(\hat{\underset{\sim}{\sigma}}_{mv})}\right) \times 100,$$

$$PR7 = \left(1 - \frac{Risk(\hat{\underset{\sim}{\sigma}}_{sv})}{Risk(\hat{\underset{\sim}{\sigma}}_{mv})}\right) \times 100, \quad PR8 = \left(1 - \frac{Risk(\hat{\underset{\sim}{\sigma}}_{sr})}{Risk(\hat{\underset{\sim}{\sigma}}_{mv})}\right) \times 100.$$

The simulated risk values are checked correct up to 4 decimal places. It has been observed from our simulation study that the risk values of the estimators $\hat{\underset{\sim}{\sigma}}_{rm}$, $\hat{\underset{\sim}{\sigma}}_{am}$ and $\hat{\underset{\sim}{\sigma}}_{ar}$ are very similar, hence for presentation purpose we have excluded the estimators $\hat{\underset{\sim}{\sigma}}_{ar}$ and $\hat{\underset{\sim}{\sigma}}_{am}$.

The censoring factors for the first and second populations are $k_1 = r/m$ and $k_2 = s/n$ respectively. We note that the values of k_1 and k_2 are always lie between 0 and 1. The simulation study has been done for various combinations of sample sizes and τ ranging from 0 to 1. The simulated risk values as well as the percentage of relative risk have been computed for the choices $m = n$, $m \neq n$, $k_1 = k_2$ and $k_1 \neq k_2$. The risk values of $\hat{\underset{\sim}{\sigma}}_{ml}$ (labeled as MLE) $\hat{\underset{\sim}{\sigma}}_{mv}$ (labeled as UMV), $\hat{\underset{\sim}{\sigma}}_{rm}$ (labeled as RML), $\hat{\underset{\sim}{\sigma}}_{av}$ (labeled as AMV), $\hat{\underset{\sim}{\sigma}}_{sm}$ (labeled as SML), $\hat{\underset{\sim}{\sigma}}_{sv}$ (labeled as SMV) and $\hat{\underset{\sim}{\sigma}}_{sr}$ (labeled as SRM) have been presented in the Figures 5.3.1 and 5.3.2. Specifically, we have presented the risk values of all the estimators for the choices $m = n = 8$, $k_1 = k_2 = 0.25$ (Figure 5.3.1(a)), $k_1 = k_2 = 0.75$ (Figure 5.3.1(b)), $k_1 = 0.25$, $k_2 = 0.75$ (Figure 5.3.1(c)), $k_1 = 0.75$, $k_2 = 0.25$ (Figure 5.3.1(d)). The graphs for the unequal sample sizes $m = 12$, $n = 20$, $k_1 = k_2 = 0.25$ (Figure 5.3.1(e)) and $k_1 = k_2 = 0.75$ (Figure 5.3.1(f)) are also presented. Similarly, in the Figure 5.3.2(a)-5.3.2(f) the risk values have been presented for the sample sizes $m = 12, n = 20$ and $m = 20, n = 12$ with various combinations of k_1 and k_2 (mentioned in the graphs).

The following observations have been made from our simulation study, see Tables 5.3.1-5.3.4 and Figures 5.3.1, 5.3.2.

1. The percentage of relative risk performances of each estimator with respect to $\hat{\underset{\sim}{\sigma}}_{mv}$ decreases as the censoring factors for first and second populations k_1 and k_2 increase

from 0 to 1 for fixed values of m, n. However, as the sample sizes increase for fixed censoring factors (k_1 and k_2) the percentage of relative risk decreases.

2. The percentage of relative risk improvement for $\hat{\underset{\sim}{\sigma}}_{ml}$ varies between 3% and 41%. The percentage of relative risk improvement for $\hat{\underset{\sim}{\sigma}}_{rm}$ varies between 5% and 46%. The percentage of relative risk improvement for $\hat{\underset{\sim}{\sigma}}_{av}$ varies between 0% and 51%. The percentage of relative risk improvement for $\hat{\underset{\sim}{\sigma}}_{sm}$ varies between 2% and 41%. The percentage of relative risk improvement for $\hat{\underset{\sim}{\sigma}}_{sv}$ varies between 0% and 51% whereas for $\hat{\underset{\sim}{\sigma}}_{sr}$ it is varying between 2% and 46%.

3. The percentage of risk improvement for $\hat{\underset{\sim}{\sigma}}_{rm}$ over $\hat{\underset{\sim}{\sigma}}_{ml}$ varies between 0% and 34%. The percentage of risk improvement of $\hat{\underset{\sim}{\sigma}}_{av}$ over $\hat{\underset{\sim}{\sigma}}_{mv}$ has been quite significant and is varying between 1% and 51%. The percentage of risk improvement of $\hat{\underset{\sim}{\sigma}}_{sm}$ over $\hat{\underset{\sim}{\sigma}}_{ml}$ varies between 2% and 27%. The percentage of risk improvement for $\hat{\underset{\sim}{\sigma}}_{sv}$ over $\hat{\underset{\sim}{\sigma}}_{mv}$ varies between 0% and 51% however, for $\hat{\underset{\sim}{\sigma}}_{sr}$ over $\hat{\underset{\sim}{\sigma}}_{rm}$ is very small and is noticed between 0% and 2%.

4. The maximum percentage of relative risk improvement has been seen for each estimator when $\tau \to 1$, and k_1 and k_2 tending to 0.

5. For small values of τ (~0), the percentage of relative risk performance of $\hat{\underset{\sim}{\sigma}}_{sr}$ has the highest percentage of relative risk improvement ~46%. For moderate values of τ the estimator $\hat{\underset{\sim}{\sigma}}_{sr}$ also has the best performance. However, as τ approaches 1, it competes with $\hat{\underset{\sim}{\sigma}}_{av}$.

6. Similar observations hold for other combinations of r, m and s, n.

7. Based on above discussion and our simulation study, we recommend using the estimator $\hat{\underset{\sim}{\sigma}}_{sr}$ when the values of τ is moderate or very small (~0). For large values of $\tau \leq 1$ either of the estimators $\hat{\underset{\sim}{\sigma}}_{sr}$ or $\hat{\underset{\sim}{\sigma}}_{av}$ can be used.

Example 5.3.1 *Suppose two brands (brand A and B) of electronic devices have been introduced in the market. It is known that the brand A uses traditional methodology where as brand B uses modern technology. The lifetimes are assumed to follow exponential distribution. It is also expected that the minimum guarantee time for both the products remain same due to market competition where as the residual life times of brand A never exceeds the residual life times of brand B. Say* 20 *units from each brand A and B put for a life test. Then the following failure times (in hours) from brand A and B have been observed . Brand A:* 760.60, 768.34, 1159.43, 1179.04, 1224.18, 1966.99, 4125.64, 4216.05, 7554.39, 8415.60, *Brand B:* 259.29, 698.10, 857.57, 1471.89, 1987.32, 3486.55, 4922.22, 4941.09, 5333.26, 5869.24. *Here* $m = n = 20$ *and* $r = s = 10$. *On the basis of these type-II censored samples we can easily compute* $Z = 259.29$*,* $V_x = 5517.01$ *and* $V_y = 4166.65$. *The various estimators for the vector parameter* $\hat{\underset{\sim}{\sigma}} = (\sigma_1, \sigma_2)$ *are computed as,* $\hat{\underset{\sim}{\sigma}}_{ml} = (11034.04, 8333.30)$, $\hat{\underset{\sim}{\sigma}}_{mv} = (11561.56, 8860.82)$, $\hat{\underset{\sim}{\sigma}}_{rm} = (9683.67, 9683.67)$, $\hat{\underset{\sim}{\sigma}}_{av} = (9683.67, 9683.67)$, $\hat{\underset{\sim}{\sigma}}_{sm} = (9716.44, 8333.30)$, $\hat{\underset{\sim}{\sigma}}_{sv} = (9716.44, 8860.82)$, $\hat{\underset{\sim}{\sigma}}_{sr} = (9683.67, 9683.67)$. *In this situation we recommend to use the estimator* $\hat{\underset{\sim}{\sigma}}_{sr}$.

5.3.4 Conclusions

We have considered the simultaneous estimation of ordered scale parameters σ_is using type-II right censored samples from two exponential populations with common location parameter in a decision theoretic approach. We note that Jana and Kumar (2015) considered the componentwise estimation of ordered scale parameters when full samples (that is $r = m, s = n$) are available from two exponential populations. We have succeeded in applying Brewster and Zidek (1974) technique for simultaneous estimation of parameters. We have derived a sufficient condition for improving estimators belonging to a broad class of equivariant estimators. This class contains the MLE, and the UMVUE for estimating $\underset{\sim}{\sigma}$. As a consequence, estimators dominating the MLE, and the UMVUE in terms of risk values are obtained using the prior information $\sigma_1 \leq \sigma_2$. In fact the results obtained in this section generalizes some of their results for simultaneous estimation of ordered scale parameters σ_is using samples from two exponential populations with a common location. Also we discuss an example where our model fits well and compute estimates for the ordered scale parameters σ_is.

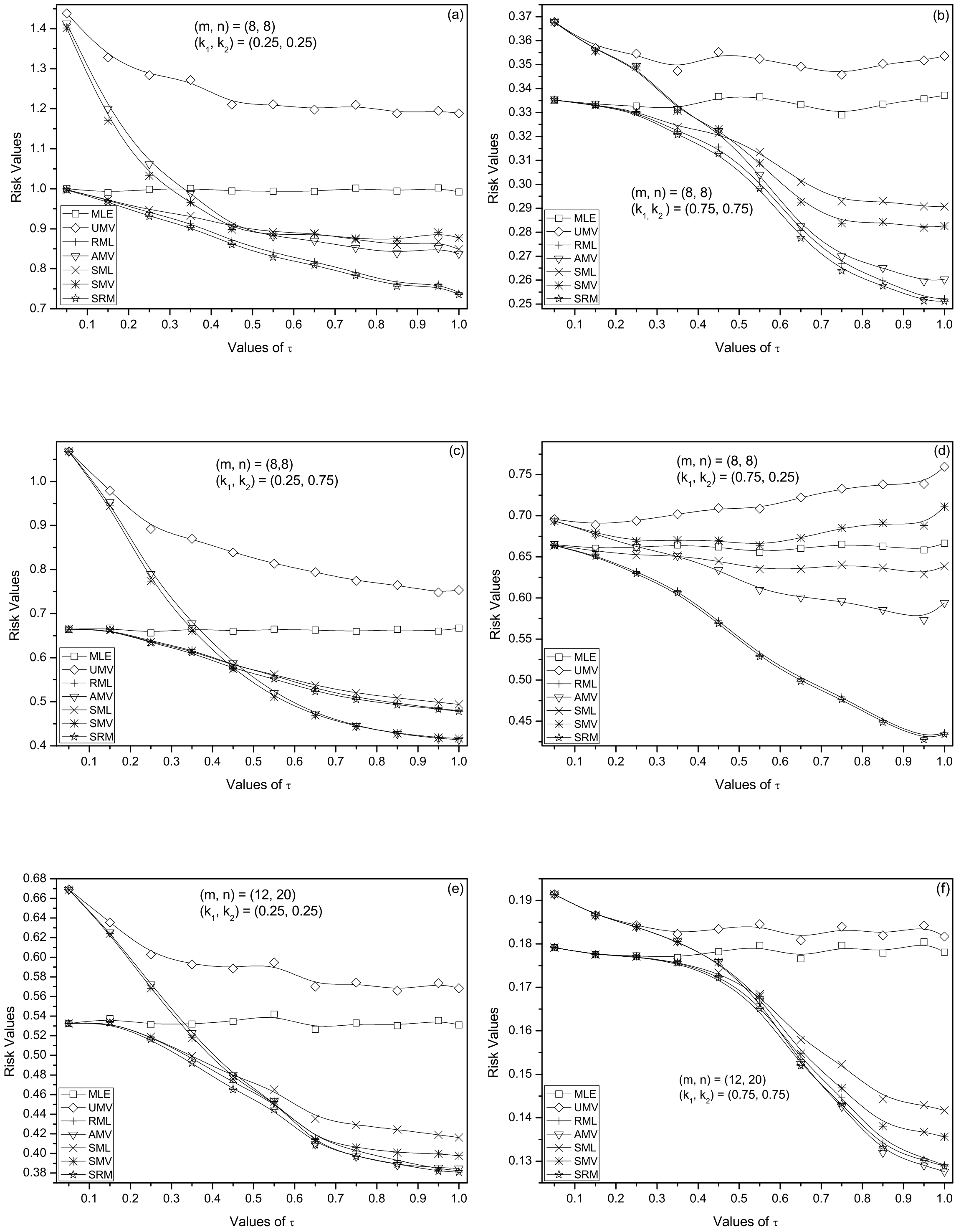

Figure 5.3.1: Comparison of risk values of various estimators of σ

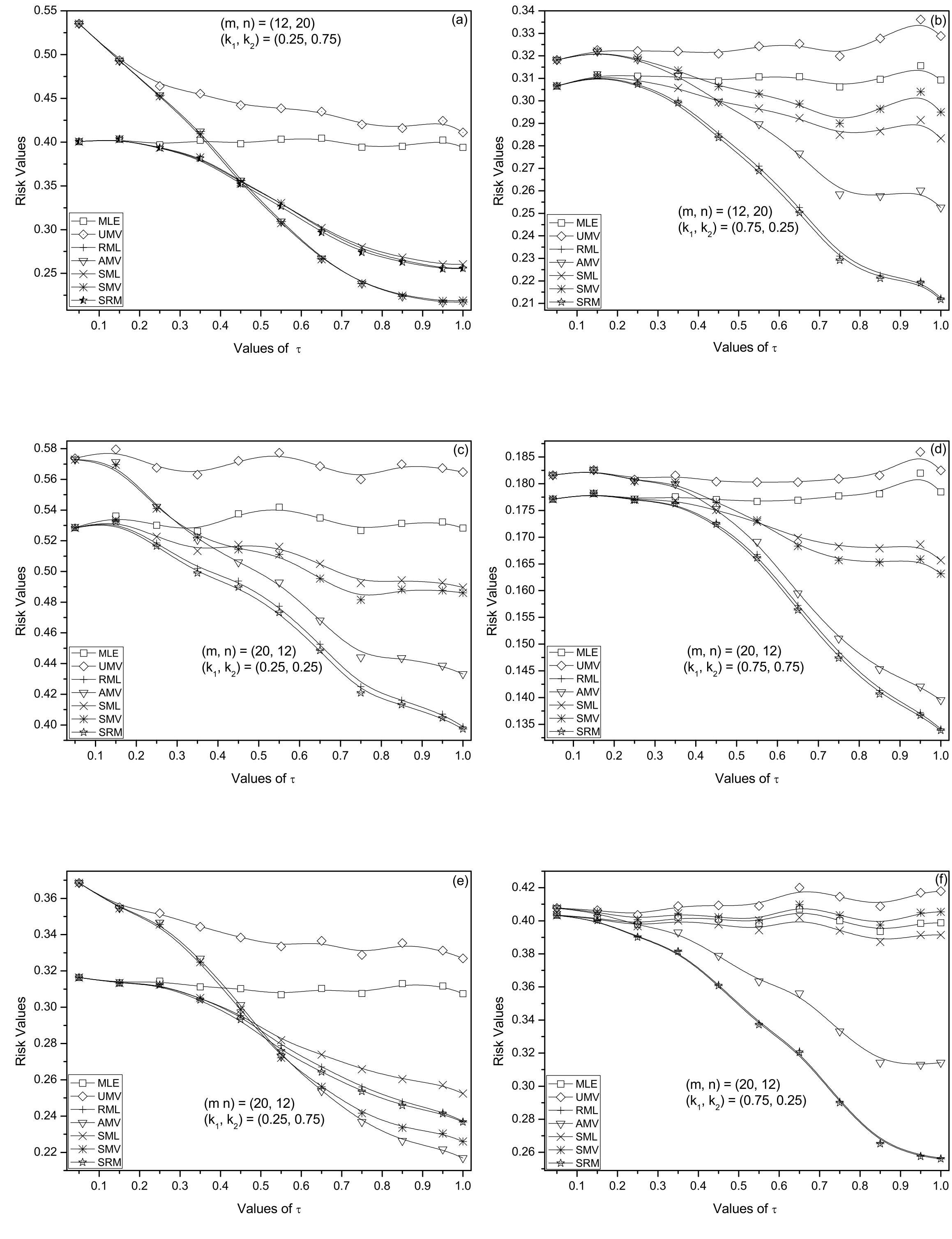

Figure 5.3.2: Comparison of risk values of various estimators of $\underset{\sim}{\sigma}$

Table 5.3.1: Comparison of relative risk performances of different estimators of $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$ when $(m, n) = (8, 8)$ and $k_1 = k_2 = 0.25, 0.50, 0.75, 1.00$

τ	$PR1$	$PR2$	$PR3$	$PR4$	$PR5$	$PR6$	$PR7$	$PR8$
	29.14	29.34	29.34	02.05	29.34	29.39	02.75	29.42
0.05	13.55	13.55	13.55	00.01	13.55	13.55	00.01	13.55
	08.56	08.56	08.56	00.00	08.56	08.56	00.00	08.56
	06.41	06.41	06.41	00.00	06.41	06.41	00.00	06.41
	27.00	27.91	27.91	05.49	27.91	27.97	07.47	28.11
0.10	11.54	11.65	11.65	00.39	11.65	11.67	00.50	11.68
	07.52	07.54	07.54	00.04	07.54	07.55	00.05	07.55
	05.24	05.24	05.24	00.00	05.24	05.24	00.01	05.24
	25.33	26.96	26.96	09.58	26.96	26.88	12.13	27.30
0.15	10.62	11.15	11.15	01.47	11.15	11.14	01.79	11.27
	06.63	06.73	06.73	00.24	06.73	06.75	00.32	06.77
	04.94	04.98	04.98	00.07	04.98	04.98	00.09	04.99
	21.72	26.21	26.21	17.53	26.21	25.68	19.89	26.89
0.25	09.89	11.75	11.75	04.39	11.75	11.55	04.91	12.11
	06.21	06.83	06.83	01.41	06.83	06.83	01.72	07.00
	03.95	04.21	04.21	00.61	04.21	04.22	00.71	04.30
	20.52	27.56	27.56	21.67	27.56	26.03	23.43	28.40
0.35	08.29	12.88	12.88	09.06	12.88	12.03	09.81	13.55
	04.72	07.88	07.88	05.24	07.88	07.24	05.22	08.35
	04.45	06.04	06.04	02.64	06.04	05.71	02.68	06.28
	20.75	30.14	30.14	25.40	30.14	27.47	26.18	31.06
0.45	08.45	17.07	17.07	14.78	17.07	14.66	14.58	18.06
	04.93	10.53	10.53	08.60	10.53	08.91	08.23	11.19
	03.80	07.82	07.82	05.98	07.82	06.78	05.83	08.37
	18.16	31.03	31.03	27.13	31.03	26.61	26.83	31.84
0.55	07.34	19.14	19.14	17.74	19.14	14.98	16.64	20.10
	04.72	14.63	14.63	13.68	14.63	11.20	12.34	15.55
	03.16	12.06	12.06	11.55	12.06	08.95	10.04	12.77
	18.38	33.16	33.16	28.18	33.16	27.45	27.00	33.89
0.65	08.16	24.42	24.42	22.80	24.42	18.48	20.58	25.53
	04.07	17.76	17.76	17.46	17.76	12.06	14.60	18.62
	03.32	15.98	15.98	15.50	15.98	10.88	12.79	16.80
	16.55	34.16	34.16	29.53	34.16	27.22	27.40	34.89
0.75	07.89	26.97	26.97	24.80	26.97	19.02	21.10	27.88
	04.72	22.51	22.51	21.26	22.51	14.67	16.95	23.29
	03.69	21.32	21.32	20.40	21.32	13.51	15.60	22.04
	17.19	36.46	36.46	30.17	36.46	28.68	27.44	37.06
0.85	07.36	28.27	28.27	25.41	28.27	18.84	20.78	28.85
	04.32	25.95	25.95	24.78	25.95	15.98	18.99	26.52
	02.61	23.86	23.86	23.32	23.86	13.60	16.89	24.39
	15.94	36.16	36.16	29.04	36.16	27.81	26.13	36.68
0.95	06.66	29.98	29.98	27.14	29.98	19.94	22.12	30.50
	04.74	27.66	27.66	25.65	27.66	16.96	19.35	28.07
	02.91	27.05	27.05	25.84	27.05	15.27	18.22	27.32
	16.97	37.88	37.88	29.52	37.88	29.82	26.74	38.41
1.00	06.16	29.36	29.36	26.39	29.36	18.36	20.73	29.62
	04.56	29.87	29.87	27.76	29.87	18.97	21.44	30.10
	03.10	28.05	28.05	26.36	28.05	16.07	18.64	28.19

Table 5.3.2: Comparison of relative risk performances of different estimators of $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$ when $(m, n) = (12, 12)$ and $k_1 = k_2 = 0.25, 0.50, 0.75, 1.00$

τ	$PR1$	$PR2$	$PR3$	$PR4$	$PR5$	$PR6$	$PR7$	$PR8$
	17.60	17.62	17.62	00.19	17.62	17.63	00.30	17.64
0.05	08.25	08.25	08.25	00.00	08.25	08.25	00.00	08.25
	05.76	05.76	05.76	00.00	05.76	05.76	00.00	05.76
	03.79	03.79	03.79	00.00	03.79	03.79	00.00	03.79
	16.70	17.05	17.05	01.24	17.05	17.06	01.64	17.15
0.10	06.93	06.94	06.94	00.02	06.94	06.94	00.02	06.95
	04.46	04.46	04.46	00.00	04.46	04.46	00.00	04.46
	03.97	03.97	03.97	00.00	03.97	03.97	00.00	03.97
	15.42	16.75	16.75	04.16	16.75	16.78	04.91	17.06
0.15	06.21	06.23	06.23	00.07	06.23	06.24	00.10	06.24
	04.01	04.02	04.02	00.02	04.02	04.03	00.02	04.03
	03.10	03.10	03.10	00.00	03.10	03.10	00.00	03.10
	13.22	16.02	16.02	08.24	16.02	15.85	09.53	16.64
0.25	05.71	06.38	06.38	01.37	06.38	06.29	01.50	06.51
	03.64	03.74	03.74	00.26	03.74	03.76	00.31	03.79
	02.83	02.87	02.87	00.08	02.87	02.87	00.11	02.88
	11.57	17.16	17.16	13.26	17.16	15.93	14.16	18.03
0.35	06.35	09.47	09.47	05.21	09.47	08.78	05.31	09.90
	04.07	05.35	05.35	02.06	05.35	05.08	02.04	05.56
	02.95	03.33	03.33	00.65	03.33	03.29	00.72	03.43
	11.76	21.24	21.24	18.88	21.24	18.82	19.24	22.38
0.45	05.48	11.37	11.37	09.17	11.37	09.81	08.84	12.16
	03.66	07.06	07.06	04.91	07.06	06.09	04.70	07.52
	02.76	04.99	04.99	03.06	04.99	04.33	02.84	05.26
	10.65	23.60	23.60	21.70	23.60	19.38	20.77	24.60
0.55	04.51	14.81	14.81	14.28	14.81	11.47	12.91	15.78
	02.64	10.47	10.47	09.96	10.47	07.71	08.62	11.15
	02.40	08.22	08.22	07.30	08.22	06.25	06.22	08.70
	10.33	25.52	25.52	23.42	25.52	19.89	21.86	26.49
0.65	05.04	19.41	19.41	18.50	19.41	13.81	15.79	20.36
	03.08	15.23	15.23	14.66	15.23	10.21	11.79	15.95
	02.07	13.24	13.24	13.04	13.24	08.63	10.14	13.83
	09.91	28.96	28.96	26.22	28.96	21.13	23.32	29.73
0.75	04.27	22.47	22.47	21.56	22.47	14.47	17.28	23.28
	02.47	20.02	20.02	19.69	20.02	12.06	14.81	20.74
	02.01	17.97	17.97	17.90	17.97	11.24	13.55	18.71
	09.06	29.83	29.83	26.42	29.83	20.85	22.81	30.49
0.85	04.56	26.22	26.22	25.01	26.22	16.62	19.53	26.85
	03.36	24.27	24.27	23.40	24.27	15.21	17.62	24.91
	02.04	22.87	22.87	22.53	22.87	13.36	16.00	23.42
	09.40	31.48	31.48	27.09	31.48	21.31	22.56	31.89
0.95	04.52	28.58	28.58	26.74	28.58	17.68	20.48	28.95
	02.58	26.64	26.64	25.57	26.64	14.68	17.66	26.90
	02.16	25.98	25.98	25.23	25.98	14.93	17.51	26.32
	09.82	32.72	32.72	27.89	32.72	22.66	23.22	33.08
1.00	04.51	29.13	29.13	26.61	29.13	17.23	19.53	29.31
	02.61	28.25	28.25	26.91	28.25	15.67	18.53	28.38
	02.16	26.32	26.32	25.21	26.32	14.47	16.93	26.43

Table 5.3.3: Comparison of relative risk performances of different estimators of $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$ when $(m, n) = (12, 20)$ and $k_1 = k_2 = 0.25, 0.50, 0.75, 1.00$

τ	$PR1$	$PR2$	$PR3$	$PR4$	$PR5$	$PR6$	$PR7$	$PR8$
	20.85	20.85	20.85	00.00	20.85	20.85	00.01	20.85
0.05	09.67	09.67	09.67	00.00	09.67	09.67	00.00	09.67
	06.95	06.95	06.95	00.00	06.95	06.95	00.00	06.95
	04.67	04.67	04.67	00.00	04.67	04.67	00.00	04.67
	16.94	17.01	17.01	00.25	17.01	17.02	00.29	17.04
0.10	08.03	08.03	08.03	00.00	08.03	08.03	00.00	08.03
	05.30	05.30	05.30	00.00	05.30	05.30	00.00	05.30
	03.91	03.91	03.91	00.00	03.91	03.91	00.00	03.91
	14.26	04.78	14.78	01.42	14.78	14.83	01.60	14.91
0.15	07.25	07.32	07.32	00.09	07.32	07.30	00.08	07.32
	05.20	05.20	05.20	00.00	05.20	05.20	00.01	05.20
	03.25	03.25	03.25	00.00	03.25	03.25	00.00	03.25
	12.11	14.76	14.76	06.30	14.76	14.68	06.87	15.29
0.25	05.57	05.82	05.82	00.52	05.82	05.81	00.57	05.87
	03.76	03.81	03.81	00.09	03.81	03.81	00.10	03.82
	03.07	03.09	03.09	00.03	03.09	03.09	00.03	03.09
	10.42	16.51	16.51	11.84	16.51	16.10	12.53	17.38
0.35	05.30	07.54	07.54	03.64	07.54	07.29	03.68	07.85
	03.48	04.30	04.30	01.23	04.30	04.28	01.30	04.44
	02.04	02.17	02.17	00.23	02.17	02.17	00.26	02.20
	08.97	20.95	20.95	19.75	20.95	19.50	20.02	22.10
0.45	04.11	09.48	09.48	08.18	09.48	08.87	08.21	10.15
	02.44	04.86	04.86	03.46	04.86	04.57	03.49	05.18
	02.35	03.57	03.57	01.73	03.57	03.42	01.70	03.73
	08.07	24.62	24.62	25.34	24.62	22.04	25.07	25.88
0.55	04.14	14.80	14.80	14.74	14.80	13.00	14.16	15.64
	02.64	09.34	09.34	08.88	09.34	08.31	08.57	09.92
	01.93	06.05	06.05	05.46	06.05	05.43	05.25	06.48
	07.46	27.14	27.14	28.35	27.14	23.55	27.40	28.25
0.65	03.72	19.98	19.98	21.19	19.98	17.00	20.01	21.05
	02.30	15.34	15.34	16.15	15.34	12.72	14.75	16.12
	01.55	12.02	12.02	12.60	12.02	10.02	11.50	12.64
	07.20	30.36	30.36	31.70	30.36	26.14	30.34	31.43
0.75	03.33	25.13	25.13	26.50	25.13	20.33	23.93	25.96
	02.62	20.99	20.99	22.00	20.99	17.08	19.80	21.89
	01.60	18.46	18.46	19.57	18.46	14.87	17.37	19.21
	05.88	30.67	30.67	31.75	30.67	24.99	29.53	31.32
0.85	03.96	28.17	28.17	28.79	28.17	22.35	25.56	28.87
	02.48	26.24	26.24	27.46	26.24	20.98	24.25	26.99
	01.95	25.36	25.36	26.28	25.36	19.99	22.74	25.96
	06.48	33.71	33.71	34.04	33.71	27.87	31.58	34.12
0.95	03.02	30.77	30.77	31.80	30.77	24.50	28.34	31.15
	02.26	29.56	29.56	30.72	29.56	23.37	26.78	29.93
	01.51	29.72	29.72	30.88	29.72	23.10	26.40	30.00
	05.91	31.40	31.40	31.24	31.40	25.04	28.54	31.65
1.00	02.78	30.26	30.26	31.02	30.26	23.26	26.98	30.38
	01.52	28.53	28.53	29.59	28.53	21.16	24.81	28.60
	01.35	28.53	28.53	29.50	28.53	21.41	24.61	28.63

Table 5.3.4: Comparison of relative risk performances of different estimators of $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$ when $(m, n) = (20, 12)$ and $k_1 = k_2 = 0.25, 0.50, 0.75, 1.00$

τ	$PR1$	$PR2$	$PR3$	$PR4$	$PR5$	$PR6$	$PR7$	$PR8$
	07.72	07.72	07.72	00.04	07.72	07.72	00.06	07.73
0.05	04.07	04.07	04.07	00.00	04.07	04.07	00.00	04.07
	02.84	02.84	02.84	00.00	02.84	02.84	00.00	02.84
	01.81	01.81	01.81	00.00	01.81	01.81	00.00	01.81
	07.95	08.06	08.06	00.41	08.06	08.05	00.52	08.09
0.10	03.50	03.50	03.50	00.00	03.50	03.50	00.01	03.50
	02.23	02.23	02.23	00.00	02.23	02.23	00.00	02.23
	01.64	01.64	01.64	00.00	01.64	01.64	00.00	01.64
	06.95	07.56	07.56	01.60	07.56	07.40	01.81	07.68
0.15	03.35	03.36	03.36	00.04	03.36	03.36	00.06	03.36
	01.91	01.91	01.91	00.00	01.91	01.91	00.00	01.91
	01.43	01.43	01.43	00.00	01.43	01.43	00.00	01.43
	06.51	08.58	08.58	04.18	08.58	07.79	04.36	08.86
0.25	02.88	03.21	03.21	00.57	03.21	03.07	00.57	03.26
	02.47	02.55	02.55	00.12	02.55	02.50	00.09	02.56
	01.52	01.54	01.54	00.06	01.54	01.55	00.07	01.56
	06.50	10.73	10.73	07.34	10.73	08.77	06.99	11.23
0.35	02.63	04.36	04.36	02.46	04.36	03.46	01.99	04.55
	02.14	02.69	02.69	00.76	02.69	02.44	00.68	02.77
	01.38	01.70	01.70	00.41	01.70	01.53	00.32	01.72
	05.22	12.97	12.97	10.96	12.97	08.50	09.12	13.50
0.45	02.31	06.69	06.69	05.70	06.69	04.42	04.54	07.12
	01.91	04.25	04.25	02.87	04.25	02.96	02.14	04.46
	01.41	02.49	02.49	01.29	02.49	01.86	00.94	02.60
	05.77	16.94	16.94	14.42	16.94	09.98	11.08	17.56
0.55	02.93	11.29	11.29	09.89	11.29	06.47	07.08	11.87
	01.86	07.14	07.14	05.98	07.14	04.04	04.06	07.49
	01.67	05.84	05.84	04.57	05.84	03.37	02.85	06.10
	06.16	20.52	20.52	17.46	20.52	11.45	12.73	21.18
0.65	03.06	15.75	15.75	13.95	15.75	07.71	08.73	16.38
	02.14	11.96	11.96	10.54	11.96	05.53	06.09	12.45
	01.37	09.57	09.57	08.56	09.57	04.16	04.65	09.94
	05.85	24.08	24.08	20.48	24.08	12.27	13.94	24.72
0.75	02.49	19.45	19.45	17.51	19.45	07.46	09.28	19.95
	01.95	18.16	18.16	16.48	18.16	06.95	08.26	18.66
	01.26	16.25	16.25	15.14	16.25	05.76	07.09	16.69
	05.70	25.51	25.51	21.07	25.51	11.49	12.94	25.96
0.85	02.87	23.77	23.77	20.93	23.77	08.90	10.65	24.20
	02.04	22.28	22.28	19.87	22.28	07.80	09.12	22.64
	01.58	21.18	21.18	19.35	21.18	07.24	08.48	21.54
	06.37	28.61	28.61	22.85	28.61	12.77	13.46	28.96
0.95	03.38	26.53	26.53	23.03	26.53	10.36	11.66	26.86
	02.00	25.76	25.76	23.09	25.76	08.56	10.02	25.97
	01.54	26.44	26.44	24.21	26.44	08.45	09.98	26.66
	06.40	28.35	28.35	22.39	28.35	12.99	13.51	28.62
1.00	02.79	27.48	27.48	23.98	27.48	09.84	11.62	27.61
	02.00	26.95	26.95	24.10	26.95	08.90	10.45	27.11
	01.36	26.54	26.54	24.34	26.54	08.22	09.89	26.64

Chapter 6

Estimating Ordered Quantiles of Two Exponential Populations with a Common Location or Scale

.

6.1 Introduction

In Chapter 5, we have considered the problem of estimating quantiles and ordered scale of two exponential populations, under equality assumption on the location parameters when the samples are type-II censored. In this chapter we consider the estimation of ordered quantiles of two exponential populations under equality restriction on either the location or scale parameter using decision theoretic approach. First we take up the problem of estimating ordered quantiles of two exponential populations when the location parameter is common.

The problem of estimation of parameters when they known to follow certain ordering is quite popular and has a rich literature in statistical inference. Particularly when the underlying distribution is either exponential or normal, the problem has got considerable attention by several researchers in the recent past due to its real world applications. Estimation of parameters under order restrictions, has its origin in the study of isotonic regression, bio-assays, reliability and arises naturally in various agricultural, industrial, and biomedical experiments. Suppose two brands (say A and B) of mechanical products has been produced where brand A uses traditional methodologies and brand B uses modern technology. The life times of these two products being random follow exponential distributions. Also it is quite expected that the minimum guarantee period or warranty period (μ) will remain same for both the brands due to market competition, whereas the mean residual life times of A will never exceed that of B. For another example, suppose an electronic item contains k important components and has been put for a life testing experiment. The life times of these components may follow exponential distribution. It is natural to assume that the minimum guarantee time to failure remains same (because of the warranty period of the system), whereas the residual life times of the individual components are ordered. For some more examples of this nature when the data follow exponential distribution, we refer to Jana and Kumar (2015), Misra and Singh (1994), and Kaur and Singh (1991). Under these circumstances it is customary to draw inference on the associated parameters or some function of it, say, mean or quantiles.

The problem of estimating parameters under order restriction has been well addressed

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in Barlow et al. (1972) and Robertson et al. (1988). There most of the results are dealt with deriving the maximum likelihood estimators (MLEs) and the isotonic versions of the unrestricted MLEs. In the recent past, the problem has got attention by several authors, particularly when the underlying distribution is exponential or normal. Vijayasree et al. (1995) considered the component-wise estimation of ordered scale (location) parameters when the location(scale) parameters are known as well as unknown. They proposed some new estimators which improve upon the MLEs. Kaur and Singh (1991) considered the estimation of ordered means of two exponential populations and showed that the isotonic version of the MLEs of the means dominate the unrestricted MLEs in terms of the mean squared error (MSE). Misra and Singh (1994) considered the componentwise estimation of ordered location parameters of two exponential populations when the scale parameters are known. They have shown that the minimum risk equivariant estimators (MREs) are inadmissible. Further they derived the class of mixed estimators and show their efficiency with respect to the MREs. Vijayasree and Singh (1993) considered the estimation of ordered means of two exponential populations and obtained some inadmissibility results for the class of mixed estimators. We note that, along the same direction, some study also has been done when the underlying distribution is normal. For some results on estimation of ordered location (mean) and scale (standard deviation) parameters of normal populations, we refer to Kumar and Tripathy (2011), Oono and Shinozaki (2005), Kumar and Sharma (1988), Kumar and Sharma (1989) and the references cited therein.

The problem under consideration has its importance in the sense that, the estimators for the common location parameter have been utilized to estimate the ordered quantiles. For the same statistical model, Jana and Kumar (2015) considered the estimation of scale parameters and proved some decision theoretic results. They proposed some new estimators for ordered scale parameters which improve upon the usual estimators with respect to a quadratic loss. Also Tripathy et al. (2014) considered the estimation of common location parameter when the scale parameters follow some known ordering. In this study, we consider the estimation of quantiles when it is known a priori that they satisfy the ordering, $\theta_1 \leq \theta_2$, where θ_i; $i = 1, 2$ denotes the p^{th} quantile of the i^{th} population. In fact, the basic purpose of the study is to extend some of the results obtained by Jana and Kumar (2015) to the estimation of ordered quantiles. The problem of estimation of quantiles is quite important for its practical applications as well as theoretical challenges involve in it. Applications of quantiles are seen in the study of reliability, life testing and survival analysis. We refer to Li et al. (2012) and Guo and Krishnamoorthy (2005) for some applications and related results on quantiles. The estimation of quantiles of exponential populations has been considered by several researchers lately in the literature. From a decision theoretic view point, the problem has got attention by several researchers in the recent past. We refer to Kumar and Sharma (1996) and Rukhin (1986) and the references cited therein for some decision theoretic results on estimating quantiles of exponential populations. In Section 6.2.1, we derive some baseline estimators without assuming ordering of quantiles. Further, using isotonic version of unrestricted MLEs, we propose some new estimators (call it restricted MLEs) for the quantiles under order restriction. Using the existing estimators for ordered scale and common location, we propose some new plug-in type of estimators for the ordered quantiles. In Section 6.2.2, we consider some classes of estimators for the quantiles. Sufficient conditions for improving estimators in this class have been proved. As a result, new estimators improving upon the MLE, the UMVUE, a modification to the MLE and the restricted MLE have been obtained. We note that an analytical comparison of the risk values of all these estimators is not possible. Hence, a detailed simulation study has been done in Section 6.2.3, to compare the percentage of relative risk improvement of all these proposed estimators. We recommend using estimators for quantiles under order restrictions. Finally we conclude our remarks in Section

6.2.4.

Next (in Section 6.3), we take up the problem of estimating ordered quantiles of two exponential populations, under equality assumption on the scale parameters when the samples drawn are type-II censored. The definition of type-II censoring scheme and its importance have been discussed in Chapter 5. Let us discuss a situation where assumption of equality on the scale parameter is justified. Suppose a particular type of product or item is produced by $k(\geq 2)$ different manufacturers or companies. Let $X_i; i = 1, 2$ denote the lifetimes of the products from the i^{th} manufacturer. Assume that all the manufacturers use modern statistical methods (for example, process control technique), and quality standards (like ISO 9000 series) during their production. This guarantees that the variations of the processes are minimum and are under control. Hence, we may assume that the scale parameters (σ_i) are close to each other. Furthermore, we may assume that the minimum guarantee periods of the products follow certain ordering (may be due to their target level and technology development). Under such scenario, it is quite reasonable to estimate the mean life times or the quantiles. We refer to Tripathy and Kumar (2011), Rukhin and Zidek (1985), Elfessi and Pal (1991) and the references cited therein for some related results and justification of our model. In Section 6.3.1, we propose certain basic estimators such as the MLE, a modification to the MLE, the UMVUE, the BAEE without taking into account the ordering of the quantiles. Further, we obtain the isotonic version of the MLE when there is order restrictions. In Section 6.3.2, Bayes estimators have been derived for the quantiles. In Section 6.3.3, a detailed simulation study has been carried out for numerically comparing the risk function of all the proposed estimators.

6.2 Estimating Ordered Quantiles of Two Exponential Populations with a Common Minimum Guarantee Time

In this section we consider the estimation of ordered quantiles from two exponential populations when the location parameter is common. More specifically, let $\underset{\sim}{X} = (X_1, X_2, \ldots, X_m)$ and $\underset{\sim}{Y} = (Y_1, Y_2, \ldots, Y_n)$ be independent random samples taken from two exponential populations Ex(μ, σ_1) and Ex(μ, σ_2) respectively. Here μ is the location parameter which is common to both the populations and σ_1, σ_2 are the scale parameters. In reliability and life-testing experiments the location parameter μ is also known as the minimum guarantee period to failure of an equipment and σ_is are known as the residual life times after the survival period. The quantile of the i^{th} population is $\theta_i = \mu + \eta\sigma_i$; $i = 1, 2$, where $\eta = -\log(1-p)$, $0 < p < 1$. The problem is to estimate the quantile θ_i when it is known a priori, that, they follow the ordering, $\theta_1 \leq \theta_2$. The loss function is taken as the quadratic loss given by

$$L(d_i, \theta_i) = \Big(\frac{d_i - \theta_i}{\sigma_i}\Big)^2, \tag{6.2.1}$$

where d_i is an estimator for θ_i; $i = 1, 2$. The performance of an estimator will be evaluated using the risk function

$$R(d_i, \theta_i) = E_{\mu,\sigma_i}\{L(d_i, \theta_i)\}. \tag{6.2.2}$$

6.2.1 Some Basic Results

In this section, we derive some basic results as well as construct some plug-in type estimators for the quantiles θ_is when $\theta_1 \leq \theta_2$, using some of the existing results in the literature.

Suppose $\underset{\sim}{X} = (X_1, X_2, \ldots, X_m)$ and $\underset{\sim}{Y} = (Y_1, Y_2, \ldots, Y_n)$ are independent random samples taken from two exponential populations $\text{Ex}(\mu, \sigma_1)$ and $\text{Ex}(\mu, \sigma_2)$ respectively. Here $\text{Ex}(\mu, \sigma_i)$ denotes the i^{th} exponential population having probability density function,

$$f(x) = \frac{1}{\sigma_i} \exp\Big\{ -\Big(\frac{x-\mu}{\sigma_i}\Big)\Big\},\ \ x > \mu,\ \ -\infty < \mu < \infty,\ \sigma_i > 0.$$

We assume that all the parameters are unknown however a prior information regarding the ordering is known in advance that is $\theta_1 \le \theta_2$. We are interested in estimating the quantiles θ_is when $\theta_1 \le \theta_2$ or equivalently $\sigma_1 \le \sigma_2$ using a decision theoretic approach.

We note that, when there is no order restriction on the quantiles, one can derive the basic estimators. Let us denote $Z = \min(X_{(1)}, Y_{(1)})$, where $X_{(1)}$ and $Y_{(1)}$ are minimum of the first and second samples respectively. Further define $V_1 = \frac{1}{m}\sum_{i=1}^{m}(X_i - Z)$ and $V_2 = \frac{1}{n}\sum_{j=1}^{n}(Y_j - Z)$. A complete and sufficient statistic for this model is (Z, V_1, V_2). The statistics Z and (V_1, V_2) are independent. Also the probability density function of Z and (V_1, V_2) are given by respectively,

$$f_Z(z) = p\exp(-p(z-\mu)),\ \ z > \mu, -\infty < \mu < \infty,$$

and

$$f_{\underset{\sim}{V}}(\underset{\sim}{v}) = \frac{m^m n^n}{\sigma_1^m \sigma_2^n p}\Big[\frac{v_1^{m-1}v_2^{n-2}}{\Gamma m \Gamma n-1} + \frac{v_1^{m-2}v_2^{n-1}}{\Gamma n \Gamma m-1}\Big]\exp(-mv_1/\sigma_1 - nv_2/\sigma_2), v_1, v_2 > 0,$$

where $p = \frac{m}{\sigma_1} + \frac{n}{\sigma_2}$. The MLE of μ is given by Z and the MLEs of σ_is are given by V_i; $i = 1, 2$. Using these, we can write the MLEs of θ_i as

$$d_i^L = Z + \eta V_i,\ i = 1, 2. \tag{6.2.3}$$

Also a modification to the MLE of μ can be used to obtain the modified MLE of θ_i as,

$$d_i^M = Z - \frac{V_1 V_2}{mV_2 + nV_1} + \eta V_i,\ i = 1, 2. \tag{6.2.4}$$

In a similar way one can obtain the UMVUE of θ_i as,

$$d_i^U = Z + \eta V_i + (\eta - 1)V^*,\ \ \text{where}\ \ V^* = \frac{V_1 V_2}{(m-1)V_2 + (n-1)V_1}. \tag{6.2.5}$$

When there is order restriction on the parameters that is under the prior information $\theta_1 \le \theta_2$, all the above estimators may not perform well. Hence using the isotonic regression on the MMLEs of θ_is with suitable weights one can get improved estimators dominating the MMLEs as,

$$d_i^R = Z - \frac{\hat{\sigma}_{1R}\hat{\sigma}_{2R}}{m\hat{\sigma}_{2R} + n\hat{\sigma}_{1R}} + \eta\hat{\sigma}_{iR},\ i = 1, 2, \tag{6.2.6}$$

where

$$\hat{\sigma}_{1R} = \min\Big(V_1, \frac{mV_1 + nV_2}{m+n}\Big),\ \hat{\sigma}_{2R} = \max\Big(V_2, \frac{mV_1 + nV_2}{m+n}\Big).$$

Next we prove a basic result regarding improving any estimator of the quantiles θ_i.

Theorem 6.2.1 *Let $\delta(\mu, \sigma_1) = \delta(\mu) + \eta\delta(\sigma_i)$ be an estimator of $\theta_i = \mu + \eta\sigma_i$; $i = 1, 2$. Let the loss function be (6.2.1). Let us consider a new estimator of the form $\delta^*(\mu, \sigma_i) = \delta^*(\mu) + \eta\delta^*(\sigma_i)$, where $\delta^*(\mu)$ and $\delta^*(\sigma_i)$ are improved estimators over $\delta(\mu)$ and $\delta(\sigma_i)$ respectively. Then $R(\delta^*(\mu, \sigma_i), \theta_i) \leq R(\delta(\mu, \sigma_i), \theta_i)$, provided the following conditions hold:*

1. *The estimators $\delta(\mu)$ and $\delta^*(\mu)$ are unbiased for μ and must be free from V_1 and V_2.*
2. *The estimators $\delta(\sigma_i)$ and $\delta^*(\sigma_i)$ are unbiased for σ_i and must be free from Z.*

Proof 6.2.1 *Let us consider the risk difference $\triangle = R(\delta^*(\mu, \sigma_i), \theta_i) - R(\delta(\mu, \sigma_i), \theta_i)$. After substituting the expressions for $\delta^*(\mu, \sigma_i)$, and $\delta(\mu, \sigma_i)$, then simplifying, the risk difference reduces to, $\triangle = R(\delta^*(\mu), \mu) - R(\delta(\mu), \mu) + \eta^2 R(\delta^*(\sigma_i), \sigma_i) - R(\delta(\sigma_i), \sigma_i) + 2\eta\{E\{(\delta^*(\mu) - \mu)(\delta^*(\sigma_i) - \sigma_i)\} - E\{(\delta(\mu) - \mu)(\delta(\sigma_i) - \sigma_i)\}\}$. It is given that, $R(\delta^*(\mu), \mu) - R(\delta(\mu), \mu) < 0$, and $R(\delta^*(\sigma_i), \sigma_i) - R(\delta(\sigma_i), \sigma_i) < 0$. Hence, the risk difference $\triangle \leq 0$, provided the conditions (1) and (2) hold. This proves the theorem..*

Remark 6.2.1 *We note that, Tripathy et al. (2014) obtained an improved estimator for the modified MLE of common location μ by using Brewster and Zidek (1974) technique. This improved estimator is same as the restricted MLE of μ. Hence, the restricted MLE of μ can be further improved by using the same method. This improved estimator for the common location μ has been used to construct plug-in type estimator for the quantiles under order restriction. For details see Remark 6.2.3 below.*

Remark 6.2.2 *It should be noted that, some plug-in type of estimators may be constructed which may or may not improve upon the original estimators. The plug-in type of estimators may be constructed by improving either the common location parameter μ or scale parameters σ_i. However, below we propose some plug-in type of estimators by replacing both the estimators of μ and σ_i with their improved versions.*

Remark 6.2.3 *Under order restrictions on σ_is Tripathy et al. (2014), obtained improved estimators for the common location parameter μ. Also Jana and Kumar (2015) obtained improved estimators for σ_is when $\sigma_1 \leq \sigma_2$. Using those improved estimators for the MLEs of μ and σ_is, we propose an estimator for the quantile θ_i as,*

$$d_i^l = \mu^l + \eta\sigma_i^l,$$

where μ^l is the improved estimator over the MLE of μ (equation (30) of (Tripathy et al. (2014)) and σ_i^l is the improved estimator for the MLE of σ_i as given in Corollary 1 or 3 of Jana and Kumar (2015). Further using the improvement over the UMVUE of μ (equation (31) of (Tripathy et al. (2014)) and improvement over the UMVUE of σ_i (see either Corollary 1 or 3 of Jana and Kumar (2015)) one can get another plug-in type estimator for θ_i as,

$$d_i^u = \mu^u + \eta\sigma_i^u.$$

Similarly using the improved estimator over the restricted MLE of μ (as given in equation (6.2.6)) and improvement over the restricted MLE of σ_i (see Corollary 1 Jana and Kumar (2015)) one can propose the estimator of the quantile θ_i as,

$$d_i^r = \mu^r + \eta\sigma_i^r.$$

These estimators have been numerically compared using a Monte-Carlo simulation procedure in Section 6.2.3. It has been observed that none of these improve uniformly upon their respective old estimators. This also follows from Theorem 6.2.1.

In the next section, we prove a general inadmissibility result for affine equivariant class of estimators and as a consequence, estimators dominating MLEs, and the UMVUEs in terms of risk values have been obtained.

6.2.2 Sufficient Conditions for Improving Equivariant Estimators under Order Restrictions

In this section, we derive the form of an affine class of estimator for the quantiles $\theta_i;\ i = 1, 2$. Utilizing a technique of Brewster and Zidek (1974) we derive sufficient conditions for improving estimators in this class under the assumption that $\theta_1 \le \theta_2$. First we consider the estimation of θ_1 when $\theta_1 \le \theta_2$.

Estimation of θ_1

Let us introduce the affine group of transformations, $G_A = \{g_{a,b} : g_{a,b}(x) = ax + b, a > 0, b \in \mathbb{R}\}$ to our problem. Under the transformation $g_{a,b}$, $Z \to aZ + b$, $V_i \to aV_i$, $\sigma_1 \to a\sigma_1$, $\mu \to a\mu + b$, and $\theta_1 = \mu + \eta\sigma_1 \to a\theta_1 + b$. The problem remains invariant with respect to the loss function (6.2.1), (with $i = 1$) and the form of an affine equivariant estimator for estimating θ_1, based on the sufficient statistics (Z, V_1, V_2) is thus obtained as,

$$\begin{aligned} d(Z, V_1, V_2) &= Z + V_1\phi(V) \\ &= d_\phi, \text{ say}, \end{aligned} \tag{6.2.7}$$

where $V = V_2/V_1$ and $\phi : (0, \infty) \to \mathbb{R}$ a real valued function. Let $\eta > 0$, be fixed. For the class of estimators d_ϕ, let us define the functions $\phi_*(v)$ and $\phi^*(v)$ as follows:
Case-I: Let $\eta < \frac{m}{(m+n)^2}$. For this values of η define

$$\phi_*(v) = \begin{cases} \frac{\eta m-1}{m+n}, & \text{if } v < \frac{1}{1-\eta m} \\ \frac{\eta m(1-v)+2\sqrt{\eta m v(v-1)}-v}{m+n}, & \text{if } \frac{1}{1-\eta m} < v < \frac{m}{m-\eta(m+n)^2} \\ \frac{m+nv}{m+n}(\eta - \frac{1}{m+n}), & \text{if } v > \frac{m}{n-\eta(m+n)^2}, \end{cases} \tag{6.2.8}$$

and

$$\phi^*(v) = \begin{cases} \frac{m+nv}{m+n}(\eta - \frac{1}{m+n}), & \text{if } v < \frac{1}{1-\eta m} \\ \max\{\frac{\eta m-1}{m+n}, \frac{m+nv}{m+n}(\eta - \frac{1}{m+n})\}, & \text{if } \frac{1}{1-\eta m} < v < \frac{m}{m-\eta(m+n)^2} \\ \frac{\eta m-1}{m+n}, & \text{if } v > \frac{m}{m-\eta(m+n)^2}. \end{cases} \tag{6.2.9}$$

Case-II: Let $\frac{m}{(m+n)^2} < \eta < \frac{1}{m}$. Define

$$\phi_*(v) = \begin{cases} \frac{\eta m-1}{m+n}, & \text{if } v < \frac{1}{1-\eta m} \\ \frac{m+nv}{m+n}(\eta - \frac{1}{m+n}), & \text{if } v > \frac{1}{1-\eta m}. \end{cases} \tag{6.2.10}$$

$$\phi^*(v) = \begin{cases} \frac{m+nv}{m+n}(\eta - \frac{1}{m+n}), & \text{if } v < \frac{1}{1-\eta m} \\ \frac{\eta m-1}{m+n}, & \text{if } v > \frac{1}{1-\eta m}. \end{cases} \tag{6.2.11}$$

Case-III: Let $\eta > \frac{1}{m}$. For this choice of η, define

$$\phi_*(v) = \frac{\eta m - 1}{m+n}, \;\; \phi^*(v) = \frac{m+nv}{m+n}(\eta - \frac{1}{m+n}). \tag{6.2.12}$$

Utilizing the functions $\phi_*(v)$ and $\phi^*(v)$ and for fixed values of η, we define the function, $\phi_0(v)$ as follows:

$$\phi_0(v) = \min\{\max\{\phi_*(v), \phi(v)\}, \phi^*(v)\}. \tag{6.2.13}$$

The following theorem is immediate.

Theorem 6.2.2 *Let d_ϕ be the class of affine equivariant estimators for estimating θ_1. The loss function is taken as (6.2.1). Let us define a function $\phi_0(v)$ as given in (6.2.13). Then the estimator d_ϕ is inadmissible and is improved by d_{ϕ_0} if there exist some values of the parameters $(\mu, \sigma_1, \sigma_2)$; $\theta_1 \le \theta_2$ such that $P(\phi_0(V) \neq \phi(V)) > 0$.*

Proof 6.2.2 *The proof of the theorem can be done by using a technique of Brewster and Zidek (1974). Let us consider the conditional risk function of d_ϕ given $V = v$.*

$$R(d_\phi, \theta_1) \;\; = \;\; \frac{1}{\sigma_1^2}E\{(Z + V_1\phi(V) - \mu - \eta\sigma_1)^2|V\}. \tag{6.2.14}$$

The above risk function (6.2.14) is a convex function in ϕ and the minimizing value is obtained as,

$$\phi(v, \sigma_1, \sigma_2) = \frac{\eta\sigma_1 E(V_1|V) - a^{-1}E(V_1|V)}{E(V_1^2|V)}.$$

The conditional expectations are evaluated as

$$E(V_1|V) = \frac{m+n-1}{K}, \; E(V_1^2|V) = \frac{(m+n-1)(m+n)}{K^2},$$

where $K = \frac{m}{\sigma_1} + \frac{n}{\sigma_2}v$. After substituting these values and simplifying one would get the minimizing choice as,

$$\phi(\tau, v) = \frac{[\eta - (m+n\tau)^{-1}][m + \tau n v]}{m+n}, \tag{6.2.15}$$

where $\tau = \sigma_1/\sigma_2 : 0 < \tau \le 1$.

We consider the following three separate cases in order to obtain the bounds of $\phi(\tau, v)$ for given $V = v$.

Case I: $\eta < \frac{m}{(m+n)^2}$. *The derivative of $\phi(\tau, v)$ is simply $g(\tau) = \eta n^3 v\tau^2 + 2\eta m n^2 v\tau + mn(\eta m v - v + 1)$ multiplied by a positive factor. This is a convex function of τ attaining minimum at $\tau = 0$ in the concerned region $(0, 1)$. The minimum value is $h(0) = mn(\eta m v - v + 1)$. If the minimum value is positive that is $0 < v < \frac{1}{1-\eta m}$, then $h(\tau) > 0$. Hence $\phi(\tau, v)$ is an increasing function in τ. Hence,*

$$\inf \phi(\tau, v) = \frac{m}{m+n}(\eta - \frac{1}{m}) = \phi(0, v), \;\; \sup \phi(\tau, v) = \frac{1}{m+n}(\eta - \frac{1}{m+n})(m + vn) = \phi(1, v).$$

If $\frac{1}{1-\eta m} < v < \frac{m}{m-\eta(m+n)^2}$, then the function $g(\tau)$ will have two real roots say τ^- and τ^+. The smaller root say τ^- is negative and the larger one $\tau^+ < 1$. The function $\phi(\tau, v)$ is *decreasing in* $(0,\tau^+)$ *and increasing in* $(\tau^+,1)$. *Hence we have*

$$\inf \phi(\tau,v)=\phi(\tau^+,v),\ \sup\phi(\tau,v)=\max\{\phi(0,v),\phi(1,v)\},$$

where $\tau^+=-\frac{m}{n}+\frac{1}{n}\sqrt{\frac{m(v-1)}{\eta v}}$.

If $v>\frac{m}{m-\eta(m+n)^2}$, *then* $\tau^+>1$. *Hence in the concerned region* $(0,1)$ *the function* $\phi(\tau,v)$ *is decreasing in* τ. *We obtain*

$$\inf \phi(\tau,v)=\phi(1,v),\ \sup\phi(\tau,v)=\phi(0,v).$$

Case II: $\frac{m}{(m+n)^2}<\eta<\frac{1}{m}$. *In this case* $\frac{m}{m-\eta(m+n)^2}<\frac{1}{1-\eta m}$. *The minimum value of* $g(\tau,v)$ *is positive for* $0<v<\frac{1}{1-\eta m}$. *Hence*

$$\inf \phi(\tau,v)=\phi(0,v),\ \sup\phi(\tau,v)=\phi(1,v).$$

If $v>\frac{1}{1-\eta m}$, *the minimum value of* $g(\tau,v)$ *is negative and crosses the* τ *axis. However the larger root* $\tau^+>1$. *Hence in the concerned region* $(0,1)$ *the function* $\phi(\tau,v)$ *is decreasing and we obtain,*

$$\inf \phi(\tau,v)=\phi(1,v),\ \sup\phi(\tau,v)=\phi(0,v).$$

Case III: $\eta>\frac{1}{m}$. *In this case the minimum value of* $g(\tau,v)$ *is always positive for all* v. *Hence we obtain,*

$$\inf \phi(\tau,v)=\phi(0,v),\ \sup\phi(\tau,v)=\phi(1,v).$$

Utilizing the above cases from I to III, one can easily define the functions ϕ_* *and* ϕ^* *for different regions of* η. *Consequently,* ϕ_0 *is defined as in (6.2.13). Using a result of Brewster and Zidek (see Theorem 3.1.1 of Brewster and Zidek (1974)) for improving equivariant estimators, it is immediate that* $R(d_{\phi_0},\theta_1)\le R(d_\phi,\theta_1)$ *when* $\theta_1\le\theta_2$. *This completes the proof of the theorem.*

Next our target is to get improved estimators in the class d_ϕ. It is easy to observe that all the basic estimators such as, the MLE, MMLE, UMVUE and the restricted MLE for θ_1 belong to the class d_ϕ for some choices of ϕ. To get improved estimators, one needs to fix the value of η. For simplicity, let us choose the value of $\eta>\frac{1}{m}$ (Case-III above). Let us consider the MLE $d_1^L=Z+V_1\phi_l(V)$, where $\phi_l(V)=\phi(V)=\eta$. In order to obtain the improved estimator, the value of $\phi(v)=\eta$, must lie outside the interval $[\frac{\eta m-1}{m+n},\frac{m+nv}{m+n}(\eta-\frac{1}{m+n})]$ with non zero probability for some choices of parameters. We observe that, the condition $\eta<\frac{\eta m-1}{m+n}$ is not satisfied. Further, the condition $\eta>\frac{m+nv}{m+n}(\eta-\frac{1}{m+n})$ is equivalent to $v<\frac{\eta n^2+\eta mn+n}{n(\eta(m+n)-1)}=\eta_0,(say)$. Thus for $\eta>\frac{1}{m}$, the improved estimator for θ_1, which will dominate the MLE $d_1^L=Z+V_1\phi_l(V)$, call it d_1^{LI} and is obtained as,

$$d_1^{LI}=\begin{cases} Z+(\frac{mV_1+nV_2}{m+n})(\eta-\frac{1}{m+n}), & \text{if } V<\eta_0,\\ Z+\eta V_1, & \text{otherwise.}\end{cases} \tag{6.2.16}$$

Let us consider the MMLE $d_1^M=Z+V_1\phi_m(V)$, where $\phi_m(V)=\phi(V)=\eta-\frac{V}{mV+n}$. To get improved estimator for d_1^M, we notice that, the condition $\eta-\frac{v}{mv+n}<\frac{\eta m-1}{m+n}$ is not true. Further, the condition $\eta - \frac{v}{mv+n} > (\frac{\eta(m+n)-1}{m+n})$ to hold, v must belong to the interval $(0, v^+)$, where v^+ is the larger root of the equation $g(v) = n^2v^2(1-\eta(m+n)) - v(m^2+\eta mn^2+n^2) + m^2 + \eta m^3 = 0$. Thus the improved estimator which dominate the MMLE $d_1^M = Z + V_1\phi_m(V)$, is obtained as,

$$d_1^{MI} = \begin{cases} Z + (\frac{mV_1+nV_2}{m+n})(\frac{\eta(m+n)-1}{m+n}), & \text{if } V \in (0, v^+) \\ Z + V_1(\eta - \frac{V}{mV+n}), & \text{otherwise.} \end{cases} \tag{6.2.17}$$

Next, to get improved estimator for the UMVUE $d_1^U = Z + T_1\phi_u(V)$, where $\phi_u(V) = \phi(V) = \eta + \frac{V(\eta-1)}{(m-1)V+(n-1)}$. For this case, it is not possible to check the conditions analytically. However, it has been observed numerically, using simulation study, that the condition for improving the UMVUE does satisfy for many values of the parameters. Thus the improved estimator for the UMVUE is obtained as,

$$d_1^{UI}(V) = \begin{cases} Z + V_1(\frac{\eta m-1}{m+n}), & \text{if } \phi_u(V) < \frac{\eta m-1}{m+n} \\ Z + V_1(\frac{m+nV}{m+n})(\frac{\eta(m+n)-1}{m+n}), & \text{if } \phi_u(V) > (\frac{m+nV}{m+n})(\frac{\eta(m+n)-1}{m+n}), \\ Z + V_1\phi_u(V), & \text{otherwise.} \end{cases} \tag{6.2.18}$$

Let us denote $A(v) = \min(1, \frac{m+nv}{m+n})$, $B(v) = \max(v, \frac{m+nv}{m+n})$. Using these notations, the restricted MLE of θ_1 is given by

$$d_1^R = Z + V_1\phi_r(V),$$

where

$$\phi_r(V) = \eta A(v) - \frac{A(v)B(v)}{nA(v) + mB(v)}.$$

It is also not possible to check analytically, the condition for improving the RML d_1^R. However, using a simulation study we have checked numerically, that the condition for improving the RML hardly satisfy. In fact, $\phi_r(v)$ belongs to the interval $[\frac{\eta m-1}{m+n}, \frac{m+nv}{m+n}(\eta - \frac{1}{m+n})]$ with probability 1. Thus we are not able to get improved estimator for the RML using Theorem (6.2.2).

Remark 6.2.4 *The expressions of improved estimators for other choices of η can be obtained in a very similar manner. We note that, the conditions are not straight forward to derive analytically. However, the conditions for improving estimators, can be checked numerically. In Section 6.2.3, we have computed the risk values as well as the relative risk performances of all these estimators through a simulation study, numerically.*

Estimation of θ_2

In this section, we consider the estimation of θ_2 when $\theta_1 \leq \theta_2$. Let us consider the affine group of transformations, $G_A = \{g_{a,b} : g_{a,b}(x) = ax + b, a > 0, b \in \mathbb{R}\}$. Under the transformation $g_{a,b}$, the sufficient statistics get transformed and thus $Z \to aZ + b$, $V_i \to aV_i$, $\sigma_2 \to a\sigma_2$, $\mu \to a\mu + b$, and $\theta_2 = \mu + \eta\sigma_2 \to a\theta_2 + b$. The estimation problem remains invariant if we choose the loss function (6.2.1), (with $i = 2$) and the form of an affine equivariant estimator for estimating θ_2, based on the sufficient statistics (Z, V_1, V_2) is obtained as,

$$\begin{aligned} d(Z, V_1, V_2) &= Z + V_2\psi(W) \\ &= d_\psi, \text{ say,} \end{aligned} \tag{6.2.19}$$

where $W = V_1/V_2$ and $\psi : (0, \infty) \to \mathbb{R}$ a real valued function. Let η be fixed. Now for the affine equivariant estimator d_ψ, define the function ψ_* as give below.
Case-I: Let $\eta < \frac{n}{(m+n)^2}$. Define

$$\psi_*(w) = \begin{cases} \frac{mw+n}{m+n}(\eta - \frac{1}{m+n}), & \text{if } w < \frac{n}{n-\eta(m+n)^2} \\ \psi(\rho^+, w), & \text{if } w > \frac{n}{n-\eta(m+n)^2}, \end{cases} \tag{6.2.20}$$

where $\psi(\rho^+, w) = \frac{2\sqrt{\eta n w(w-1)} - \eta n(w-1) - w}{m+n}$.
Case-II: Let $\eta > \frac{n}{(m+n)^2}$. Define

$$\psi_*(w) = \frac{mw+n}{m+n}\Big(\eta - \frac{1}{m+n}\Big). \tag{6.2.21}$$

Utilizing the function $\psi_*(w)$, and for a fixed value of η, we define a new function $\psi_0(w)$ for the affine equivariant estimator d_ψ as,

$$\psi_0(w) = \max\{\psi_*(w), \psi(w)\}. \tag{6.2.22}$$

The following result is immediate.

Theorem 6.2.3 *Let d_ψ be the class of affine equivariant estimators for estimating θ_2 as given in (6.2.19). The loss function is taken as (6.2.1). Let us define a function $\psi_0(w)$ as in (6.2.22). Then the estimator d_ψ is inadmissible and is improved by d_{ψ_0} if there exist some values of the parameters $(\mu, \sigma_1, \sigma_2)$; $\theta_1 \leq \theta_2$ such that $P(\psi_0(W) \neq \psi(W)) > 0$.*

Proof 6.2.3 *The proof of the theorem can be done by using a technique of Brewster and Zidek (1974). Let us consider the conditional risk function of d_ψ given $W = w$.*

$$R(d_\psi, \theta_2) = \frac{1}{\sigma_2^2} E\{(Z + V_2\psi(W) - \mu - \eta\sigma_2)^2 | W\}. \tag{6.2.23}$$

The above risk function (6.2.23) is a convex function in ψ and the minimizing value is obtained as,

$$\psi(w, \sigma_1, \sigma_2) = \frac{\eta\sigma_2 E(V_2|W) - a^{-1}E(V_2|W)}{E(V_2^2|W)}.$$

The conditional expectations are evaluated as

$$E(V_2|W) = \frac{m+n-1}{M},\ E(V_2^2|W) = \frac{(m+n-1)(m+n)}{M^2},$$

where $M = \frac{n}{\sigma_2} + \frac{m}{\sigma_1}w$. After substituting these values and simplifying one would get the minimizing choice as,

$$\psi(\rho, w) = \frac{[\eta - (n + m\rho)^{-1}][n + \rho m w]}{m+n},$$

where $\rho = \sigma_2/\sigma_1$.

In this case also we consider the following three separate cases in order to obtain the bounds of $\psi(\rho, w)$ for fixed w.

Case I: $\eta < \frac{n}{(m+n)^2}$. *The derivative of* $\psi(\rho, w)$ *is simply* $h(\rho) = \eta m^3 w\rho^2 + 2\eta n m^2 w\rho + mn(\eta n w - w + 1)$ *multiplied by a positive factor. This is a convex function of* ρ *attaining minimum at* $\rho = 1$ *in the concerned region* $[1, \infty)$. *The minimum value is* $h(1) = \eta m^3 w + 2\eta n m^2 w + mn(\eta n w - w + 1)$. *If the minimum value is positive that is* $0 < w < \frac{1}{1-\eta n}$, *then* $h(\rho) > 0$. *Hence* $\psi(\rho, w)$ *is an increasing function in* $\rho \in [1, \infty)$. *Hence,*

$$\inf \psi(\rho, w) = \frac{1}{m+n}(\eta - \frac{1}{m+n})(mw + n), \ \sup \psi(\rho, w) = \infty.$$

If $\frac{1}{1-\eta n} < w < \frac{n}{n-\eta(m+n)^2}$, *then the function* $h(\rho) > 0 \ \forall \rho \in [1, \infty)$. *Hence* $\psi(\rho, w)$ *is increasing in the concerned region. We obtain*

$$\inf \psi(\rho, w) = \frac{1}{m+n}(\eta - \frac{1}{m+n})(mw + n), \ \sup \psi(\rho, w) = \infty.$$

If $w > \frac{n}{n-\eta(m+n)^2}$, *then* $h(\rho) < 0$. *Hence the function* $h(\rho)$ *will have two real roots say* ρ^- *and* ρ^+. *The smaller one* $\rho^- < 0$ *and the larger one* $\rho^+ > 1$ *if* $w > \frac{n}{n-\eta(m+n)^2}$. *Then* ρ^+ *is inside our concerned region* $[1, \infty)$. *Hence the function* $\psi(\rho, w)$ *is decreasing in the region* $[1, \rho^+]$ *and increasing in the region* $[\rho^+, \infty)$. *We obtain*

$$\inf \psi(\rho, w) = \psi(\rho^+, w), \ \sup \psi(\tau, w) = \max\{\psi(1, w), \psi(\infty, w)\} = \infty,$$

where $\rho^+ = -\frac{n}{m} + \frac{1}{m}\sqrt{\frac{n(w-1)}{\eta w}}$.

Case II: $\frac{n}{(m+n)^2} < \eta < \frac{1}{n}$. *In this case* $\frac{n}{n-\eta(m+n)^2} < \frac{1}{1-\eta n}$ *and let* $0 < w \leq \frac{1}{1-\eta n}$. *The minimum value of* $h(\rho)$ *is positive for* $0 < w < \frac{1}{1-\eta n}$. *Hence*

$$\inf \psi(\rho, w) = \psi(1, w), \ \sup \psi(\rho, w) = \infty.$$

If $w > \frac{1}{1-\eta n}$, *then* $w > \frac{n}{n-\eta(m+n)^2}$ *and* $h(1) > 0$. *Hence* $\psi(\rho, w)$ *is increasing in* $\rho \in [1, \infty)$. *We obtain,*

$$\inf \psi(\rho, w) = \psi(1, w), \ \sup \psi(\rho, w) = \infty.$$

Case III: $\eta > \frac{1}{n}$. *In this case the minimum value of* $h(\rho)$ *is always positive for all* ρ. *Hence we obtain,*

$$\inf \psi(\rho, w) = \psi(1, w), \ \sup \psi(\tau, w) = \infty.$$

Utilizing the above cases from I to III, one can easily define the function ψ_* *and consequently* ψ_0 *for different regions of* η. *Using a result of Brewster and Zidek (1974)(see Theorem 3.1.1 in Brewster and Zidek (1974)) for improving equivariant estimators, it is immediate that* $R(d_{\psi_0}, \theta_2) \leq R(d_\psi, \theta_2)$ *when* $\theta_1 \leq \theta_2$.

Next, using the Theorem 6.2.3, we will derive improved estimators which dominate the usual estimators such as, the MLE, the MMLE, the UMVUE and the restricted MLE RML for θ_2. We note that, all the basic estimators belong to the class d_ψ. To get the improved estimators we need to fix the value of η. For simplicity, let $\eta > \frac{1}{n} > \frac{n}{(m+n)^2}$. Let us first consider the MLE $d_2^L = Z + V_2\psi_l(W)$, where $\psi_l(W) = \psi(W) = \eta$. In order to get the improved estimator, the condition $\eta < \frac{(mw+n)(\eta m+\eta n-1)}{(m+n)^2}$ must satisfy. The condition is equivalent to say that $w > \frac{\eta mn+\eta m^2+n}{m(\eta(m+n)-1)}$.

Thus the improved estimator for MLE $d_2^L = Z + V_2\psi_l(W)$, can be written as,

$$d_2^{LI} = \begin{cases} Z + (\frac{mV_1+nV_2}{m+n})(\eta - \frac{1}{m+n}), & \text{if } w > \frac{\eta mn+\eta m^2+n}{m(\eta(m+n)-1)} \\ Z + \eta V_2, & \text{otherwise.} \end{cases} \tag{6.2.24}$$

Next, consider the modified MLE $d_2^M = Z+V_2\psi_m(W)$, where $\psi_m(W) = \eta - \frac{W}{m+nW}$. In order to get improved estimator the condition $\eta - \frac{w}{m+nw} < \frac{(mw+n)(\eta m+\eta n-1)}{(m+n)^2}$ must hold. Analytically, we are unable to prove the inequality. Using simulation study, numerically it has been checked that, the condition does not satisfy for many replications of sample sizes. However, for $\eta < \frac{n}{(m+n)^2}$, the condition does hold for some ranges of w, which have been seen numerically. Thus one can write the expression for the improved estimator for the modified MLE $d_2^M = Z + V_2\psi_m(W)$, call it d_2^{MI} and is given by

$$d_2^{MI} = \begin{cases} Z + V_2(\frac{mW+n}{n+m})(\eta - \frac{1}{m+n}), & \text{if } w < \frac{n}{n-\eta(m+n)^2} \\ Z + V_2\psi(\rho^+, w), & \text{if } w > \frac{n}{n-\eta(m+n)^2} \\ d_2^M, & \text{otherwise.} \end{cases} \tag{6.2.25}$$

Next, consider the UMVUE $d_2^U = Z + V_2\psi_u(W)$, where $\psi_u(W) = \eta + \frac{W(\eta-1)}{(m-1)+(n-1)W}$. In order to get the improved estimator, the condition $\eta + \frac{W(\eta-1)}{(m-1)+(n-1)W} < (\frac{mW+n}{n+m})(\eta - \frac{1}{m+n})$ must hold true for some ranges of w. We are unable to prove the inequality analytically. However, for any fixed choice of η, using a simulation study, it has been checked numerically that the condition satisfies for many values of w. Thus one can write, the expression for improved estimator for the UMVUE $d_2^U = Z + V_2\psi_u(W)$, as

$$d_2^{UI} = \begin{cases} d_{\psi_*}(W), & \text{if } \psi_u(W) < \psi_*(W) \\ Z + V_2\psi_u(W), & \text{otherwise.} \end{cases} \tag{6.2.26}$$

Remark 6.2.5 *We note that, though the estimator restricted MLE belong to the class of estimators d_ψ, the condition for improving it, is not satisfied. Hence we are unable to improve the restricted MLE using Theorem 6.2.3. In Section 6.2.3, we carry out a detailed simulation study to numerically compare the risk values of all these improved estimators.*

6.2.3 Simulation Study

In Section 6.2.1 we have proposed some baseline estimators such as the MLE d_i^L, a modification to the MLE d_i^M and the UMVUE d_i^U for the quantiles θ_i, $i = 1, 2$, without assuming order restriction on the quantiles as given in (6.2.3), (6.2.4) and (6.2.5) respectively. Further under order restrictions on the quantiles that is, when $\theta_1 \le \theta_2$, we have proposed the restricted MLEs d_i^R; $i = 1, 2$ for the quantiles. In Section 6.2.2, we have also obtained improved estimators for some of these estimators. Specifically, for estimating θ_1, we have proposed improved estimators d_1^{LI} and d_1^{UI} for the MLE and the UMVUE respectively. The expressions for these estimators has been obtained in (6.2.16) and (6.2.18) for the choice of $\eta > \frac{1}{m}$. We also note that the estimators d_1^R, d_1^{MI} perform equally well. Thus for estimating θ_1, we consider d_1^L, d_1^M, d_1^U, d_1^R, d_1^{LI} and d_1^{UI} for numerical comparison purpose. For estimating θ_2, we have proposed improved estimators d_2^{MI} (for $\eta < \frac{n}{(m+n)^2}$), d_2^{LI} (for $\eta > \frac{n}{(m+n)^2}$) and d_2^{UI}. It has also been noticed that the estimators d_2^{LI} (for $\eta < \frac{n}{(m+n)^2}$) and d_2^{MI} (for $\eta > \frac{n}{(m+n)^2}$) have not been included for numerical comparison purpose, as these estimators give either no improvements or very marginal improvements. Hence, in our numerical study, we consider only d_2^L, d_2^M, d_2^U, d_2^R, d_2^{MI} (for small $\eta < \frac{1}{n}$) d_2^{LI} (for moderate to large $\eta > \frac{1}{n}$) and d_2^{UI} when estimating θ_2.

Further we note that, the proposed plug-in type of estimators for the quantiles under the assumption, $\theta_1 \leq \theta_2$ (see Remark 6.2.3) using some of the existing results in the literature, does not improve upon the old as well as improved estimators obtained by using Theorem 6.2.2 and 6.2.3 uniformly. This has been noticed in our simulation study. Hence we exclude those plug-in type of estimators from our numerical study. For the numerical comparison purposes, we have generated 20,000 random samples each from two exponential populations with a common location parameter μ and different scale parameters σ_1, σ_2 such that, $\theta_1 \leq \theta_2$. It should be noted that, with respect to the loss function (6.2.1), the risk values of all the estimators are functions of $0 < \tau = \sigma_2/\sigma_1 \leq 1$ for fixed sample sizes and η. We have used Monte-Carlo simulation procedure to evaluate the risk functions for each estimator. Next, we define the percentage of relative risk improvement for each estimator say, d of θ_i with respect to the MLE d_i^L as,

$$R(d) = \Big(1 - \frac{R(d, \theta_i)}{R(d_i^L, \theta_i)}\Big) \times 100, \ i = 1, 2,$$

where d is any estimator for estimating θ_i.

The accuracy of simulation has been checked and the error of simulation is of the order of 10^{-3}. The risk values of the estimators has been evaluated by considering various choices of sample sizes as well as η. In Table 6.2.1, we present the percentage of relative risk performances of all the estimators for sample sizes $(5, 5)$, $(10, 10)$, $(5, 10)$, $(10, 5)$ for estimating θ_1 with choices of $\eta = 0.01$, and 1.5. The table consists of 3 columns and several rows. The first column represents the values of τ. Second and third column again divided into 5 sub columns each. The second column gives the percentage of relative risk values of various estimators for the choice of $\eta = 0.01$ whereas column 3 gives the values for $\eta = 1.5$. Corresponding to each value of the parameter τ, there corresponds four values in each cell of the sub columns. These four values correspond to four different combinations of sample sizes. For example in Table 6.2.1, the percentage of relative risk performances has been tabulated for the sample sizes $(5, 5)$, $(10, 10)$, $(5, 10)$ and $(10, 5)$ in each cell. In a very similar manner, the percentage of relative risk performances of various estimators for the quantile θ_2 have been tabulated in Table 6.2.2, with the choice of $\eta = 0.01$ and 1.5.

The following conclusions can be drawn from our simulation study as well as from the Tables 6.2.1-6.2.2. For convenient, we discuss the observations for estimating θ_1 and θ_2 separately.

Comments for θ_1

1. The risk values of all the estimators decrease as τ increases from 0 to 1 for fixed sample sizes and η. Also as the sample sizes increase the risk values of all the estimators decrease for fixed values of η and τ.

2. The percentage of relative risk improvement of all the estimators increase when $\eta < m/(m+n)^2$. The percentage of relative risk improvement of all the estimators decrease when $\eta > 1/m$ except d_1^U and d_1^{UI} as sample sizes increase for fixed values of τ and η.

3. Consider the case $\eta < m/(m+n)^2$. The percentage of relative risk performance of d_1^R is maximum (near to 47%) for small values of τ (< 0.35). For moderate to large values of τ ($0.35 < \tau \leq 1$) the estimator d_1^{UI} has the maximum percentage of relative risk values

(near to 50%). A similar type of pattern of improvements for the case $m/(m+n)^2 \leq \eta \leq 1/m$ has been noticed. Further for the case $\eta > 1/m$, the percentage of relative risk performance of d_1^M and d_1^R are quite comparable for small ranges of τ. For moderate to large values of τ $(0.35 < \tau \leq 1)$ the estimator d_1^{LI} has the maximum percentage of relative risk values near to 45%.

4. It has been observed that, by using the Theorem 6.2.2, the percentage of risk improvements for the new estimators d_1^{LI}, and d_1^{UI}, upon their respective old estimators d_1^L, and d_1^U are seen maximum up to 50%, and 2%, for the case $\eta < 1/m$. For the case $\eta > 1/m$, the percentage of risk improvements of d_1^{UI} over d_1^U is seen maximum up to 45%, and for d_1^{LI} it is seen up to 44%. This validates the theoretical results obtained in Theorem 6.2.2.

5. A similar type of observations have been made for other combinations of sample sizes (m, n) and fixed η.

Comments for θ_2

1. The percentage of relative risk performance of each estimator increases as the sample sizes increase for fixed values of τ when $\eta < n/(m+n)^2$. The percentage of relative risk performance of all the estimators decrease as sample sizes increase except d_2^U and d_2^{UI} (for $\eta > n/(m+n)^2$). The percentage of relative risk performance of d_2^U and d_2^{UI} decrease as the sample sizes increase.

2. Consider the case $\eta < n/(m+n)^2$. The percentage of relative risk performance of d_2^{MI} and d_2^R are quite good (seen near to 45%) in comparison to other estimators for small values of τ (< 0.35). For moderate to large values of τ $(0.35 < \tau \leq 1)$ the estimator d_2^{UI} has the maximum percentage of relative risk values and it is seen near to 48%. Next, consider the case of $\eta \geq n/(m+n)^2$. The percentage of relative risk improvements of d_2^R is maximum for all the values of τ.

3. The percentage of risk improvement of the estimator d_2^{MI} is very negligible, where as for d_2^{UI} it is seen between 1% and 3% when $\eta < n/(m+n)^2$. When $\eta > n/(m+n)^2$, the percentage of risk improvement of d_2^{LI} is seen maximum up to 25%. The percentage of risk improvement for d_2^{MI} is seen maximum up to 28% where as for d_2^{UI} it is seen near to 23%. This validates the theoretical results obtained in Theorem 6.2.3.

4. A similar type of observations have been made for other combinations of sample sizes (m, n) and η.

On the basis of our theoretical as well as computational studies, the following conclusions can be made regarding the use of the estimators for the quantiles θ_is when a priori condition $\theta_1 \leq \theta_2$ is available.

1. Consider the estimators for estimating θ_1 with respect to the loss function (6.2.1). For small values of τ $(\tau < 0.35)$ we recommend to use d_1^R. For moderate to large values of τ we recommend to use d_1^{UI} (when $\eta < 1/m$) and d_1^{LI} (when $\eta > 1/m$).

2. Consider the estimators for estimating θ_2 with respect to the loss function (6.2.1). For small values of τ, $(\tau < 0.35)$ the estimator d_2^R is recommended for use. For moderate to large values of τ the estimator d_2^{UI} (when $\eta < \frac{n}{(m+n)^2}$) and d_2^{UI} (when $\eta > \frac{n}{(m+n)^2}$) are recommended.

6.2.4 Conclusions

We have considered the estimation of quantiles $\theta_i = \mu + \eta\sigma_i$, $i = 1, 2$ of the two exponential populations under the restriction that $\theta_1 \leq \theta_2$. The loss function is taken as the quadratic loss. First we derive some baseline estimators for the quantiles when there is no ordering, such as the MLE, a modification to the MLE and the UMVUE. Under order restriction on the quantiles, we have obtained the restricted modified MLEs using isotonic regression which we call it RML. We derive sufficient conditions for improving these baseline estimators. Consequently, estimators dominating the MLE, the modification to the MLE, the UMVUE have been derived. A detailed simulation study has been carried out in order to evaluate the performance of these improved estimators under order restrictions. It has been noticed that in case of the MLE and the UMVUE the percentage of risk improvement is quite significant. This validates the theoretical findings in Section 6.2.2. The present work also extends some of the theoretical results of Jana and Kumar (2015) to the estimation of ordered quantiles. Finally we have recommended for the use of the estimators under order restrictions on the quantiles which is useful in practice. To the best of our knowledge, the problem of estimation of ordered quantiles has not been considered in the literature and hence the problem is new and has importance from a practical point of view. Below we discuss an example to compute the estimates of quantiles and recommend to use.

Example 6.2.1 *(Simulated Data) We have generated the following simulated data from two exponential populations with a common location parameter* $\mu = 100$ *and different scale parameters* $\sigma_1 = 15$ *and* $\sigma_2 = 25$. *Here we note that the condition* $\theta_1 \leq \theta_2$ *holds. Here we have taken* $m = 10$, *and* $n = 15$.
Sample 1: 128.61, 115.73, 105.63, 111.46, 100.14, 100.64, 106.13, 111.01, 103.14, 111.66.
Sample 2: 165.26, 109.50, 108.88, 112.57, 110.52, 154.40, 109.87, 118.37, 114.79, 116.08, 101.96, 136.84, 102.62, 115.31, 135.89.

On the basis of the above samples one can obtain the statistics $Z = 100.14$, $V_1 = 9.27$, *and* $V_2 = 20.71$. *Let* $\eta = 0.01$. *In this case the various estimators have been computed for* θ_1 *as* $d_1^L = 100.23$, $d_1^M = 99.68$, $d_1^U = 99.63$, $d_1^R = 99.68$, $d_1^{LI} = 99.81$, *and* $d_1^{UI} = 99.66$. *Here we recommend to use* d_1^{UI} *for* θ_1. *Using these data one can also compute the estimates of* θ_2 *as* $d_2^L = 100.35$, $d_2^M = 99.07$, $d_2^U = 98.76$, $d_2^R = 99.07$, $d_2^{MI} = 99.07$, *and* $d_2^{UI} = 98.79$. *In this case we recommend to use* d_2^{UI}.

Table 6.2.1: Relative risk performance of various estimators for quantile θ_1 for $(m, n) = (5, 5), (10, 10), (5, 10), (10, 5)$.

$\tau \downarrow$	$\eta = 0.01$					$\eta = 1.5$				
	$R(d_1^M)$	$R(d_1^U)$	$R(d_1^R)$	$R(d_1^{LI})$	$R(d_1^{UI})$	$R(d_1^M)$	$R(d_1^U)$	$R(d_1^R)$	$R(d_1^{LI})$	$R(d_1^{UI})$
	41.48	39.33	41.48	33.00	39.33	7.62	-11.78	7.62	0.01	-11.76
0.05	45.57	45.23	45.57	37.20	45.23	3.45	-5.13	3.45	0.00	-5.13
	41.54	40.25	41.54	26.22	40.25	6.59	-10.16	6.59	0.00	-10.16
	45.70	45.27	45.70	41.92	45.27	3.75	-5.57	3.75	0.00	-5.57
	41.80	40.31	41.80	33.62	40.41	6.28	-10.16	6.32	0.07	-10.02
0.10	46.49	46.19	46.49	38.91	46.19	3.07	-4.61	3.07	0.00	-4.61
	42.27	41.42	42.27	28.27	41.42	6.05	-8.59	6.05	0.00	-8.59
	45.89	45.57	45.89	42.29	45.59	3.72	-5.32	3.74	0.04	-5.24
	41.87	40.55	41.87	34.57	40.74	6.19	-9.55	6.63	0.80	-8.39
0.15	46.58	46.31	46.58	39.83	46.32	3.75	-4.70	3.75	0.00	-4.69
	43.33	42.67	43.33	30.49	42.76	5.31	-7.23	5.34	0.09	-7.09
	45.50	45.19	45.50	42.28	45.23	3.68	-5.14	3.85	0.34	-4.65
	42.71	41.82	42.72	36.48	42.53	5.24	-7.87	7.55	3.84	-2.60
0.25	47.48	47.34	47.48	42.12	47.41	2.51	-3.60	2.58	0.16	3.35
	45.32	45.15	45.33	34.71	45.46	4.20	-5.40	4.97	1.20	-3.84
	46.11	45.86	46.13	43.49	46.02	2.93	-4.49	4.13	2.06	-1.66
	43.97	43.26	44.03	38.83	44.31	4.18	-6.53	9.70	9.00	5.61
0.35	47.40	47.42	47.41	43.30	47.53	2.35	-3.18	3.42	1.72	-0.95
	45.22	45.35	45.24	36.78	45.87	3.06	-4.00	7.03	5.55	2.72
	46.39	45.97	46.42	44.45	46.32	2.51	-4.00	5.18	4.60	2.21
	43.79	43.30	43.95	39.73	44.48	4.05	-5.80	14.97	16.31	15.11
0.45	47.41	47.22	47.43	44.70	47.49	2.05	-2.75	6.44	6.34	4.97
	46.07	46.11	46.12	39.71	46.66	2.73	-3.29	12.95	13.55	12.61
	46.73	46.59	46.85	45.33	46.89	2.66	-3.91	7.61	8.19	6.89
	42.65	42.28	42.97	39.71	43.53	3.65	-5.16	19.20	22.67	23.30
0.55	47.52	47.53	47.58	45.50	47.76	1.56	-2.29	10.35	12.12	12.13
	45.67	45.58	45.77	41.02	46.22	2.06	-2.55	20.15	22.89	23.66
	46.86	46.91	47.02	45.72	47.23	2.63	-3.66	10.72	12.79	12.81
	43.80	43.33	44.13	41.52	44.58	3.02	-4.34	23.4	28.11	29.94
0.65	47.75	47.66	47.88	46.44	48.00	1.42	-2.02	15.42	18.65	19.76
	45.76	45.71	45.90	42.21	46.23	1.80	-2.13	26.67	30.79	32.64
	46.45	46.36	46.67	45.76	46.81	2.45	-3.44	12.89	16.24	17.24
	44.17	43.96	44.71	42.38	45.15	2.40	-3.73	24.96	30.87	33.64
0.75	47.81	47.82	48.02	47.08	48.08	1.24	-1.78	20.12	24.43	26.31
	45.80	45.84	46.17	43.38	46.44	1.68	-1.90	32.70	37.54	39.92
	46.57	46.48	46.82	46.11	46.93	2.10	-3.08	15.48	19.86	21.68
	44.19	43.94	44.89	43.00	45.23	2.37	-3.47	26.95	32.88	35.75
0.85	48.14	48.12	48.45	47.85	48.47	1.28	-1.67	24.96	29.80	32.03
	46.47	46.55	46.98	44.91	47.22	1.33	-1.56	36.49	41.63	44.25
	47.00	46.90	47.43	46.91	47.43	1.49	-2.63	15.74	20.79	23.00
	43.84	43.49	44.72	43.48	44.83	1.75	-2.88	27.75	33.85	36.92
0.95	47.59	47.58	47.93	47.48	47.88	0.68	-1.25	25.35	30.49	32.93
	46.55	46.47	47.05	45.61	47.10	0.84	-1.17	36.92	42.22	44.96
	46.92	46.98	47.61	47.18	47.56	1.29	-2.40	16.46	21.54	23.80
	44.18	43.79	45.17	43.97	45.14	2.15	-3.02	30.29	35.95	38.80
1.00	47.64	47.69	48.15	47.77	48.10	1.18	-1.49	27.66	32.44	34.73
	46.08	45.95	46.67	45.25	46.64	1.10	-1.25	39.43	44.43	47.02
	47.04	47.03	47.75	47.37	47.70	1.79	-2.65	18.37	23.24	25.41

Table 6.2.2: Relative risk performance of various estimators for quantile θ_2 for $(m,n) = (5,5),(10,10),(5,10),(10,5)$.

$\tau\downarrow$	$\eta=0.01$					$\eta=1.5$				
	$R(d_2^M)$	$R(d_2^U)$	$R(d_2^R)$	$R(d_2^{MI})$	$R(d_2^{UI})$	$R(d_2^M)$	$R(d_2^U)$	$R(d_2^R)$	$R(d_2^{LI})$	$R(d_2^{UI})$
	37.04	35.46	37.04	37.04	35.47	0.02	-0.02	0.02	0.00	-0.02
0.05	37.32	36.96	37.32	37.32	36.96	0.01	-0.01	0.01	0.00	-0.01
	39.33	38.08	39.33	39.33	38.08	0.00	-0.03	0.00	0.00	-0.03
	32.14	31.76	32.14	32.14	31.76	0.00	0.00	0.00	0.00	0.00
	41.09	39.72	41.09	41.09	39.80	0.14	-0.14	0.15	0.00	-0.14
0.10	43.63	43.37	43.63	43.63	43.37	0.00	-0.02	0.00	0.00	-0.02
	41.68	41.19	41.68	41.68	41.20	0.06	-0.15	0.06	0.00	-0.15
	42.14	41.96	42.14	42.14	41.99	0.00	-0.01	0.00	0.00	0.00
	41.55	40.26	41.55	41.55	40.47	0.13	-0.20	0.19	0.03	-0.17
0.15	45.05	44.84	45.05	45.05	44.84	0.07	-0.10	0.07	0.00	-0.10
	42.87	42.50	42.87	42.87	42.53	0.07	-0.26	0.07	0.00	-0.26
	44.16	43.94	44.16	44.16	44.01	0.05	-0.05	0.07	0.01	-0.04
	42.63	42.00	42.64	42.63	42.65	0.28	-0.44	0.91	0.39	-0.14
0.25	47.06	46.90	47.06	47.06	46.93	0.03	-0.15	0.07	0.02	-0.13
	44.00	44.01	44.00	44.00	44.19	0.53	-0.74	0.64	0.04	-0.71
	44.99	44.51	44.99	45.00	44.74	0.16	-0.15	0.66	0.38	0.17
	42.55	42.12	42.59	42.55	43.10	0.57	-0.81	2.69	1.38	0.24
0.35	47.49	47.40	47.49	47.49	47.50	0.37	-0.43	0.79	0.27	-0.22
	44.68	44.93	44.68	44.68	45.28	0.57	-0.98	1.36	0.43	-0.67
	46.17	45.92	46.18	46.17	46.35	0.24	-0.26	2.31	1.62	1.14
	43.94	43.30	43.98	43.94	44.59	0.67	-1.05	5.79	3.55	1.75
0.45	47.57	47.46	47.57	47.57	47.65	0.38	-0.53	2.45	1.47	0.68
	44.91	44.96	44.91	44.91	45.45	0.84	-1.33	3.32	1.46	-0.25
	46.20	45.90	46.22	46.21	46.47	0.36	-0.38	5.37	3.99	3.10
	43.58	43.19	43.65	43.58	44.47	1.00	-1.47	10.07	6.46	3.74
0.55	48.21	48.14	48.21	48.21	48.42	0.60	-0.74	6.02	4.08	2.72
	45.33	45.41	45.33	45.33	46.00	1.00	-1.58	6.39	3.48	1.13
	46.61	46.16	46.61	46.61	46.90	0.42	-0.50	10.44	8.11	6.64
	44.07	43.80	44.23	44.07	45.08	1.34	-1.86	14.62	9.72	6.10
0.65	47.94	48.03	47.95	47.95	48.34	0.66	-0.89	10.88	7.88	5.91
	46.63	46.59	46.64	46.63	47.24	1.19	-1.84	10.21	6.12	3.04
	46.15	45.82	46.18	46.16	46.55	0.71	-0.74	15.75	12.29	10.14
	44.16	43.60	44.43	44.16	45.07	1.53	-2.19	19.33	13.38	8.95
0.75	47.53	47.67	47.54	47.53	47.92	0.82	-1.06	16.53	12.35	9.69
	45.54	45.74	45.61	45.54	46.19	1.31	-2.08	13.92	8.87	5.15
	46.35	46.24	46.43	46.36	46.85	0.74	-0.83	21.84	17.49	14.78
	44.03	43.95	44.36	44.03	45.09	1.53	-2.38	23.48	16.69	11.65
0.85	47.10	47.17	47.11	47.10	47.45	0.59	-1.04	22.28	17.43	14.35
	46.76	46.75	46.85	46.76	47.34	1.35	-2.22	16.84	11.16	7.01
	46.62	46.65	46.73	46.62	47.11	0.79	-0.95	26.92	21.81	18.62
	44.44	44.18	44.82	44.44	45.29	1.78	-2.74	25.14	17.68	12.08
0.95	47.64	47.58	47.68	47.65	47.95	0.94	-1.30	25.58	19.88	16.29
	6.57	46.62	46.73	46.57	47.09	1.65	-2.51	19.22	12.78	8.11
	45.84	45.90	46.06	45.85	46.37	1.22	-1.28	29.51	23.56	19.82
	44.15	44.00	44.63	44.15	45.08	2.16	-3.04	25.69	17.76	11.82
1.00	47.17	47.19	47.21	47.17	47.46	0.99	-1.39	25.92	20.03	16.29
	46.09	46.17	46.22	46.09	46.57	1.98	-2.75	19.68	12.87	7.95
	46.47	46.40	46.71	46.47	46.89	1.11	-1.26	30.35	24.30	20.48

6.3 Estimating Ordered Quantiles of Two Exponential Populations with a Common Scale under Type-II Censoring

In this section, we consider the problem of estimating ordered quantiles from two exponential populations under equality assumption on the scale parameters using type-II censored samples. Specifically, suppose type-II censored samples are available from two exponential populations Ex(μ_1, σ) and Ex(μ_2, σ). Here μ_is are the location parameter and σ is the scale parameter. The quantile of the i^{th} population is given by $\theta_i = \mu_i + \eta\sigma$; $i = 1, 2$, where $\eta = -\ln(1-p)$; $0 < p < 1$. Our target is to estimate the quantiles θ_i, when it is known a priori that $\theta_1 \le \theta_2$; and $\mu_i \ge 0$. The loss function is taken as

$$L(d_i, \theta_i) = \Big(\frac{d_i - \theta_i}{\sigma}\Big)^2 \tag{6.3.1}$$

where d_i is an estimator for the quantile θ_i; $i = 1, 2$. The risk function of an estimator d_i is defined as

$$R(d_i, \theta_i) = EL(d_i, \theta_i).$$

6.3.1 Preliminaries and Some Basic Results

Suppose type-II censored random samples are available from two exponential populations with a common scale parameter σ and possibly different location parameters μ_1 and μ_2. Specifically, let $X_{(1)} \le X_{(2)} \le \cdots \le X_{(r)}$, $(2 \le r \le m)$ be the r smallest ordered observations taken from a random sample of size $m(\ge 2)$ which follows Ex(μ_1, σ). Likewise let $Y_{(1)} \le Y_{(2)} \le \cdots \le Y_{(s)}$, $(2 \le s \le n)$ be the s smallest ordered observations taken from a random sample of size $n(\ge 2)$ that follows Ex(μ_2, σ). We assume that these two samples have been drawn independently. Here μ_i is the location parameter and also known as the minimum guarantee time in the study of reliability, life testing and survival analysis. The scale parameter σ which is common to both the populations is known as the residual life time. Though the location parameter can take the value in the interval $(-\infty, \infty)$, from application point of view we assume that $\mu_i \ge 0$. It is worth mentioning that the same model has also been considered previously by Elfessi and Pal (1991) without considering ordering of the location parameters and investigated the problem of estimating the location and scale parameters from a decision theoretic point of view. Here Ex(μ_i, σ) denotes the exponential population having probability density function

$$f(x_i) = \frac{1}{\sigma}\exp\Big\{-\Big(\frac{x_i - \mu_i}{\sigma}\Big)\Big\}, \;\; x_i > \mu_i \ge 0, \;\; \sigma > 0. \tag{6.3.2}$$

It is easy to see that the joint statistics $(X_{(1)}, Y_{(1)}, T)$, is sufficient and also complete. These three random variables are independent. Here we denote,

$$X_{(1)} = \min_{1\le i\le m} X_i, Y_{(1)} = \min_{1\le j\le n} Y_j$$

and

$$T = \sum_{i=1}^{r}(X_i - X_{(1)}) + (m-r)(X_{(r)} - X_{(1)}) + \sum_{j=1}^{s}(Y_j - Y_{(1)}) + (n-s)(Y_{(s)} - Y_{(1)}).$$

Note that $X_{(1)} \sim \text{Ex}(\mu_1, \sigma/m)$, $Y_{(1)} \sim \text{Ex}(\mu_2, \sigma/n)$ and $T \sim Gamma(m+n-2, \sigma)$, where $Gamma(\alpha, \beta)$ denotes the gamma distribution with shape parameter α and scale parameter β. The joint probability density function of $(X_{(1)}, Y_{(1)}, T)$ is given by

$$f(x_{(1)}, y_{(1)}, t) = \frac{mnt^{r+s-3}}{\Gamma(r+s-2)}(1/\sigma)^{r+s} e^{-\frac{1}{\sigma}(mx_{(1)}+ny_{(1)}+t-m\mu_1-n\mu_2)}, \tag{6.3.3}$$

where $x_{(1)} \geq \mu_1$, $y_{(1)} \geq \mu_2$, $\sigma > 0$.

When there is no order restriction on the location parameters or equivalently on the quantiles, the MLEs of μ_1, μ_2, and σ are obtained as $X_{(1)}$, $Y_{(1)}$ and $\frac{T}{r+s}$ respectively. Consequently, the MLEs of θ_1 and θ_2 are given by

$$\delta_1^L = X_{(1)} + \eta\frac{T}{r+s} \text{ and } \delta_2^L = Y_{(1)} + \eta\frac{T}{r+s},$$

respectively. Furthermore, we observe that, $E\Big(X_{(1)} - \frac{T}{m(r+s-2)}\Big) = \mu_1$ and $E\Big(Y_{(1)} - \frac{T}{n(r+s-2)}\Big) = \mu_2$. This motivates us to propose a modification to the MLEs of θ_1 and θ_2 as

$$\delta_1^M = X_{(1)} - \frac{T}{m(r+s-2)} + \eta\frac{T}{r+s} \text{ and } \delta_2^M = Y_{(1)} - \frac{T}{n(r+s-2)} + \eta\frac{T}{r+s},$$

respectively. We also note that, the uniformly minimum variance unbiased estimators (UMVUEs) for θ_1 and θ_2 are given by

$$\delta_1^U = X_{(1)} + (\eta - \frac{1}{m})\frac{T}{r+s-2} \text{ and } \delta_2^U = Y_{(1)} + (\eta - \frac{1}{n})\frac{T}{r+s-2},$$

respectively.

Next, we find classes of equivariant estimators for both θ_1 and θ_2. Consider the group $G_A = \{g_{a,b_i} : g_{a,b_i}(x) = ax + b_i,\ a > 0,\ b_i \in R;\ i = 1, 2\}$ of affine transformations. Under this transformation, $X_i \to aX_i + b_1$, $X_{(1)} \to aX_{(1)} + b_1$,$Y_j \to aY_j + b_2$, $Y_{(1)} \to aY_{(1)} + b_2$, $T \to aT$, $\theta_1 \to a\theta_1 + b_1$ and $\theta_2 \to a\theta_2 + b_2$. The form of the equivariant estimators for θ_1 and θ_2 based on the sufficient statistics $(X_{(1)}, Y_{(1)}, T)$ are obtained as

$$\delta_{c_1} = X_{(1)} + c_1 T$$

and

$$\delta_{c_2} = Y_{(1)} + c_2 T,$$

respectively. Here c_1 and c_2 are suitably chosen constants. It is easy to observe that, the choice of c_1 that minimizes the risk function of the estimator δ_{c_1} is given by $c_1^* = \frac{\eta m-1}{m(r+s-1)}$, when the loss is (6.3.1). Hence the best equivariant estimator in the class δ_{c_1} is obtained as $\delta_{c_1^*} = X_{(1)} + \frac{\eta m-1}{m(r+s-1)}T$. Let us call this estimator as δ_1^E. Similarly for estimating θ_2, the best equivariant estimator is obtained as $\delta_2^E = Y_{(1)} + \frac{\eta n-1}{n(r+s-1)}T$.

When there is order restriction on the quantiles that is when it is known a priori that, $\theta_1 \leq \theta_2$ (equivalently $0 \leq \mu_1 \leq \mu_2$), the above estimators no longer perform better and improved estimators can be constructed (see Barlow et. al (1972)). Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be any estimators for θ_1 and θ_2 respectively when there is no order restriction. Using the method of isotonic regression on $\hat{\theta}_1$ and $\hat{\theta}_2$, one can get better estimators under order restrictions on the parameters. Now using min-max formula (see Barlow et al. (1972)), the isotonic regression of $\hat{\theta}_1$, and $\hat{\theta}_2$ with weights w_i; $i = 1, 2$ is given by

$$\hat{\theta}_{iR} = \min_{i \leq t_1 \leq 2} \max_{1 \leq s_1 \leq i} Av(s_1, t_1), \quad i = 1, 2,$$

where

$$Av(s_1, t_1) = \frac{\sum_{r=s_1}^{t_1} w_r \hat{\theta}_r}{\sum_{r=s_1}^{t_1} w_r}, \ s_1 \leq t_1, s_1, t_1 \in \{1, 2\}.$$

As $\theta_i \geq 0$, and taking $w_1 = r$ and $w_2 = s$, we get the isotonic estimators for θ_1 and θ_2 as $\hat{\theta}_{1R}$ and $\hat{\theta}_{2R}$. That is, one gets

$$\hat{\theta}_{1R} = \max\left(0, \min\left(\hat{\theta}_1, \frac{r\hat{\theta}_1 + s\hat{\theta}_2}{r+s}\right)\right), \text{ and } \hat{\theta}_{2R} = \max\left(0, \max\left(\hat{\theta}_2, \frac{r\hat{\theta}_1 + s\hat{\theta}_2}{r+s}\right)\right).$$

Using this estimator, one may easily construct improved estimators for the MLE, the modified MLE, the UMVUE and the best equivariant estimators, when there is order restrictions on the parameters. Replacing $\hat{\theta}_i$s by the MLEs, MMLEs, the UMVUEs and the best equivariant estimators one gets plug-in type restricted estimators for θ_is. Let us denote these plug-in type estimators by δ_{iR}^L, δ_{iR}^M, δ_{iR}^U and δ_{iR}^E respectively, where $i = 1, 2$.

Remark 6.3.1 *A detailed simulation study has been carried out to numerically compare the performances of all the above restricted plug-in type estimators with respect to their old counter parts in Section 6.3.3. It has been seen from our simulation study that the percentage or risk improvements of all the above estimators over their old counter parts give marginal improvements, when the values of* $\rho = (\mu_2 - \mu_1)/\sigma$ *close to zero. In other cases no improvements has been seen. Hence, for convenient we have chosen the plug-in type restricted estimators using the best equivariant estimators for* θ_1 *and* θ_2*, in order to compare with other proposed estimators (Bayes estimators obtained in the next section).*

Remark 6.3.2 *It is also observed from our simulation study while using type-II censored samples (with various choices of* r *and* s *such that* $r < m$ *and* $s < n$*), the plug-in type restricted estimators do not improve uniformly over their old counter parts. Hence, to obtain better estimators under order restrictions we use Bayesian approach in the next section.*

6.3.2 Bayesian Estimation of Ordered Quantiles

In this section, we derive Bayes estimators for the quantiles θ_1 and θ_2 incorporating the order restriction, that is assuming $\theta_1 \leq \theta_2$. For this, we have chosen two types of priors for the parameters (μ_1, μ_2, σ) namely the non-informative prior and the inverse gamma prior.

Bayesian Estimation with Uniform Prior

First we consider the joint prior distribution for the ordered location parameter as,

$$\pi_1(\mu_1, \mu_2) = 1, \ 0 \leq \mu_1 \leq \mu_2,$$

where c is a constant. For the scale parameter, we choose the prior as,

$$\pi_2(\sigma) = \frac{1}{\sigma}, \ \sigma > 0.$$

Also it is assumed that the priors are independent. In order to proceed further, let us denote the sufficient statistics $\underset{\sim}{Z} = (X_{(1)}, Y_{(1)}, T)$ as $\underset{\sim}{Z} = (X, Y, T)$. Using the notations for the sufficient statistics, the likelihood function is given by

$$L(x, y, t) = \frac{mnct^{r+s-3}}{\Gamma(r+s-2)\sigma^{r+s}} e^{-\frac{1}{\sigma}\{mx+ny+t-m\mu_1-n\mu_2\}}, \tag{6.3.4}$$

where $0 < \mu_1 < \min(x, y)$ and $\mu_1 < \mu_2 < y$, as $x > \mu_1$, $y > \mu_2$. Let us denote $t^* = \min(x, y)$. Hence the joint posterior density of (μ_1, μ_2, σ) is obtained by

$$g(\mu_1, \mu_2, \sigma|\underset{\sim}{Z}) = \frac{mnct^{r+s-3}}{A\Gamma(r+s-2)\sigma^{r+s+1}} e^{-\frac{1}{\sigma}\{mx+ny+t-m\mu_1-n\mu_2\}}, \tag{6.3.5}$$

where

$$A = \int_0^{t^*}\int_{\mu_1}^{y}\int_0^{\infty} g(\mu_1, \mu_2, \sigma|\underset{\sim}{Z}) d\sigma d\mu_2 d\mu_1.$$

Now under the weighted squared error loss function (6.3.1), for $i = 1$, with weight $\frac{1}{\sigma^2}$, the Bayes estimator of θ_1 is given by the expression

$$\begin{aligned} \delta_1^{B1} &= \frac{E(\frac{\theta_1}{\sigma^2}|\underset{\sim}{Z})}{E(\frac{1}{\sigma^2}|\underset{\sim}{Z})}, \\ &= \frac{E(\frac{\mu_1}{\sigma^2}|\underset{\sim}{Z})}{E(\frac{1}{\sigma^2}|\underset{\sim}{Z})} + \eta\frac{E(\frac{1}{\sigma}|\underset{\sim}{Z})}{E(\frac{1}{\sigma^2}|\underset{\sim}{Z})}, \end{aligned} \tag{6.3.6}$$

where $E(\frac{\theta_1}{\sigma^2}|\underset{\sim}{Z})$ and $E(\frac{1}{\sigma^2}|\underset{\sim}{Z})$ are the posterior means of $\frac{\theta_1}{\sigma^2}$ and $\frac{1}{\sigma^2}$ respectively. Hence the Bayes estimator of θ_1 under the loss function (6.3.1), is given by

$$\delta_1^{B1} = \frac{\int_0^{t^*}\int_{\mu_1}^{y}\int_0^{\infty}(\frac{\mu_1}{\sigma^2})g(\underset{\sim}{\mu*}|\underset{\sim}{Z})d\sigma d\mu_2 d\mu_1}{\int_0^{t^*}\int_{\mu_1}^{y}\int_0^{\infty}(\frac{1}{\sigma^2})g(\underset{\sim}{\mu*}|\underset{\sim}{Z})d\sigma d\mu_2 d\mu_1} + \eta\frac{\int_0^{t^*}\int_{\mu_1}^{y}\int_0^{\infty}(\frac{1}{\sigma})g(\underset{\sim}{\mu*}|\underset{\sim}{Z})d\sigma d\mu_2 d\mu_1}{\int_0^{t^*}\int_{\mu_1}^{y}\int_0^{\infty}(\frac{1}{\sigma^2})g(\underset{\sim}{\mu*}|\underset{\sim}{Z})d\sigma d\mu_2 d\mu_1}. \tag{6.3.7}$$

where $\underset{\sim}{\mu*} = (\mu_1, \mu_2, \sigma)$. Denote $\xi = mx + t$ and $w = mx + ny + t$. After a lot of mathematical calculations the integrals have been evaluated and after some simplification we obtain,

$$E\Big(\frac{\mu_1}{\sigma^2}|\underset{\sim}{Z}\Big) = \frac{mt^{r+s-3}\Gamma(r+s+1)}{A\Gamma(r+s-2)(r+s)}(B_1 - B_2), \tag{6.3.8}$$

and

$$E\Big(\frac{1}{\sigma}|\underset{\sim}{Z}\Big) = \frac{mt^{r+s-3}\Gamma(r+s)}{(r+s-1)A\Gamma(r+s-2)}(D_1 - D_2), \tag{6.3.9}$$

where we denote

$$\begin{aligned} B_1 &= \frac{t^*(w-(m+n)t^*)^{-(r+s)}}{(m+n)} + \frac{w^{1-(r+s)} - (w-(m+n)t^*)^{1-(r+s)}}{(m+n)^2(r+s-1)}, \\ B_2 &= \frac{t^*(\xi - mt^*)^{-(r+s)}}{m} + \frac{\xi^{1-(r+s)} - (\xi - mt^*)^{1-(r+s)}}{m^2(r+s-1)}, \end{aligned}$$

$$D_1 = \frac{1}{(m+n)}\{w^{1-(r+s)} - (w-(m+n)t^*)^{1-(r+s)}\},$$
$$D_2 = \frac{1}{m}\{\xi^{1-(r+s)} - (\xi - mt^*)^{1-(r+s)}\}.$$

Similarly we obtain,

$$E\Big(\frac{1}{\sigma^2}|\underset{\sim}{Z}\Big) = \frac{mt^{r+s-3}\Gamma(r+s)}{A\Gamma(r+s-2)}(E_1 - E_2),$$

where

$$E_1 = \frac{1}{(m+n)}\{w^{-(r+s)} - (w-(m+n)t^*)^{-(r+s)}\},$$
$$E_2 = \frac{1}{m}\{\xi^{-(r+s)} - (\xi - mt^*)^{-(r+s)}\}.$$

Substituting all these expressions in (6.3.6), we obtain the Bayes estimator for θ_1 as

$$\delta_1^{B1} = \frac{B_1 - B_2}{n(E_1 - E_2)} + \eta\frac{D_1 - D_2}{(r+s-1)(E_1 - E_2)}. \tag{6.3.10}$$

Next we derive Bayes estimator of θ_2 using the same vague prior (as above). The Bayes estimator of θ_2 under the loss function (6.3.1), is obtained as

$$\delta_2^{B1} = \frac{E(\frac{\theta_2}{\sigma^2}|\underset{\sim}{Z})}{E(\frac{1}{\sigma^2}|\underset{\sim}{Z})},$$
$$= \frac{E(\frac{\mu_2}{\sigma^2}|\underset{\sim}{Z})}{E(\frac{1}{\sigma^2}|\underset{\sim}{Z})} + \eta\frac{E(\frac{1}{\sigma}|\underset{\sim}{Z})}{E(\frac{1}{\sigma^2}|\underset{\sim}{Z})}, \tag{6.3.11}$$

where $E(\frac{\theta_2}{\sigma^2}|\underset{\sim}{Z})$ and $E(\frac{1}{\sigma^2}|\underset{\sim}{Z})$ are the posterior means of $\frac{\theta_2}{\sigma^2}$ and $\frac{1}{\sigma^2}$ respectively. Hence the Bayes estimator of θ_2 under the loss function (6.3.1), is given by

$$\delta_2^{B1} = \frac{\int_0^{t^*}\int_{\mu_1}^{y}\int_0^{\infty}(\frac{\mu_2}{\sigma^2})g(\underset{\sim}{\mu}^*)d\sigma d\mu_2 d\mu_1}{\int_0^{t^*}\int_{\mu_1}^{y}\int_0^{\infty}(\frac{1}{\sigma^2})g(\underset{\sim}{\mu}^*)d\sigma d\mu_2 d\mu_1} + \eta\frac{\int_0^{t^*}\int_{\mu_1}^{y}\int_0^{\infty}(\frac{1}{\sigma})g(\underset{\sim}{\mu}^*)d\sigma d\mu_2 d\mu_1}{\int_0^{t^*}\int_{\mu_1}^{y}\int_0^{\infty}(\frac{1}{\sigma^2})g(\underset{\sim}{\mu}^*)d\sigma d\mu_2 d\mu_1}. \tag{6.3.12}$$

In a very similar way we obtain the expression

$$E\Big(\frac{\mu_2}{\sigma^2}|\underset{\sim}{Z}\Big) = \frac{mt^{r+s-3}\Gamma(r+s)}{nA\Gamma(r+s-2)}(B_1^* - B_2^*), \tag{6.3.13}$$

where

$$B_1^* = \frac{y}{m}\{\xi^{-(r+s)} - (\xi - mt^*)^{-(r+s)}\} + \frac{\{\xi^{1-(r+s)} - (\xi - mt^*)^{1-(r+s)}\}}{mn(1-(r+s))}.$$

and

$$B_2^* = \frac{t^*(w-(m+n)t^*)^{-(r+s)}}{(m+n)} + \frac{m\{w^{1-(r+s)} - (w-(m+n)t^*)^{1-(r+s)}\}}{n(m+n)^2(1-(r+s))},$$

Hence the Bayes estimator of θ_2 with respect to the non informative prior is given by

$$\delta_2^{B1} = \frac{(B_1^* - B_2^*)}{n(E_1 - E_2)} + \eta \frac{(D_1 - D_2)}{(r+s-1)(E_1 - E_2)}. \tag{6.3.14}$$

Bayesian Estimation Using Inverse Gamma Prior

In this section we obtain the Bayes estimator of θ_is using a conditional prior for the ordered quantiles. We observe that, the scale parameter σ of an exponential population has a conjugate prior as inverse gamma. We note that the location parameters μ_is are also refereed as the minimum guarantee time in the study of reliability and life testing experiment. Hence, a reasonable prior density for the location parameter may be considered as an exponential type. Further we have also assumed $0 \le \mu_1 \le \mu_2$ which satisfies the boundedness criteria. A similar type of argument has been used to choose prior distribution for the ordered parameters in Yike and Heliang (1999) and Nagatsuka et al. (2009). In view of the above arguments we have considered the joint prior density of (μ_1, μ_2, σ) as follows (using conditional prior),

$$\pi(\mu_1, \mu_2, \sigma) = \pi_1(\mu_1|\mu_2, \sigma)\pi_2(\mu_2|\sigma)\pi_3(\sigma),$$

where

$$\pi_1(\mu_1|\mu_2, \sigma) = \frac{1}{\sigma} e^{-(\mu_2 - \mu_1)/\sigma}, \;\; \pi_2(\mu_2|\sigma) = \frac{1}{\sigma} e^{-\mu_2/\sigma} \tag{6.3.15}$$

and

$$\pi_3(\sigma) = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{e^{-\beta/\sigma}}{\sigma^{\alpha+1}}, \; \alpha > 0, \beta > 0. \tag{6.3.16}$$

The joint posterior density of μ_1, μ_2 and σ is given by

$$g^*((\mu_1, \mu_2, \sigma)|\underset{\sim}{Z}) = \frac{\beta^\alpha mnt^{r+s-3}}{A^*\Gamma(\alpha)\Gamma(r+s-2)} \frac{e^{-\frac{1}{\sigma}\{mx+ny+t+\beta+(2-n)\mu_2-(m+1)\mu_1\}}}{\sigma^{r+s+\alpha+3}}, \tag{6.3.17}$$

where

$$A^* = \int_0^{t^*} \int_{\mu_1}^{y} \int_0^\infty g^*((\mu_1, \mu_2, \sigma)|\underset{\sim}{Z}) d\sigma d\mu_2 d\mu_1.$$

Denoting $v = mx + ny + t + \beta$, $u = mx + 2y + t + \beta$, and after lot of mathematical calculations, we obtain

$$E\Big(\frac{\mu_1}{\sigma^2}|\underset{\sim}{Z}\Big) = \frac{mnt^{r+s-3}\beta^\alpha\Gamma(r+s+\alpha+2)}{A^*\Gamma(r+s-2)\Gamma(\alpha)(n-2)}(b_1 - b_2), \tag{6.3.18}$$

where

$$b_1 = \frac{t^*(v-(m+n-1)t^*)^{-(\alpha^*+2)}}{(m+n-1)} + \frac{\{v^{-(\alpha^*+1)} - (v-(m+n-1)t^*)^{-(\alpha^*+1)}\}}{(m+n-1)^2(\alpha^*+1)},$$

$$b_2 = \frac{t^*(u-(m+1)t^*)^{-(\alpha^*+2)}}{(m+1)} + \frac{\{u^{-(\alpha^*+1)} - (u-(m+1)t^*)^{-(\alpha^*+1)}\}}{(m+1)^2(\alpha^*+1)},$$

where $\alpha^* = r+s+\alpha$. Similarly we also obtain,

$$E\Big(\frac{1}{\sigma}|\underset{\sim}{Z}\Big) = \frac{mnt^{r+s-3}\beta^{\alpha}\Gamma(r+s+\alpha+1)}{A^*\Gamma(r+s-2)\Gamma(\alpha)(n-2)}(d_1-d_2), \tag{6.3.19}$$

where

$$d_1 = \frac{1}{(m+n-1)}\{v^{-(\alpha^*+1)} - (v-(m+n-1)t^*)^{-(\alpha^*+1)}\},$$

$$d_2 = \frac{1}{(m+1)}\{u^{-(\alpha^*+1)} - (u-(m+1)t^*)^{-(\alpha^*+1)}\}.$$

Similarly we obtain the conditional expectation,

$$E\Big(\frac{1}{\sigma^2}|\underset{\sim}{Z}\Big) = \frac{mnt^{r+s-3}\beta^{\alpha}\Gamma(r+s+\alpha+2)}{A^*\Gamma(r+s-2)\Gamma(\alpha)(n-2)}(e_1-e_2), \tag{6.3.20}$$

where

$$e_1 = \frac{1}{(m+n-1)}\{v^{-(\alpha^*+2)} - (v-(m+n-1)t^*)^{-(\alpha^*+2)}\},$$

$$e_2 = \frac{1}{(m+1)}\{u^{-(\alpha^*+2)} - (u-(m+1)t^*)^{-(\alpha^*+2)}\}.$$

Substituting all the above expressions in (6.3.6), we obtain the Bayes estimator of θ_1 as,

$$\delta_1^{B2} = \frac{(b_1-b_2)}{(e_1-e_2)} + \frac{\eta(d_1-d_2)}{(\alpha^*+1)(e_1-e_2)}. \tag{6.3.21}$$

To obtain the Bayes estimator of θ_2, we need to compute $E(\frac{\mu_2}{\sigma^2}|\underset{\sim}{Z})$. The conditional expectation has been calculated and obtained as,

$$E\Big(\frac{\mu_2}{\sigma^2}|\underset{\sim}{Z}\Big) = \frac{mnt^{r+s-3}\beta^{\alpha}\Gamma(\alpha^*+2)}{A^*\Gamma(r+s-2)\Gamma(\alpha)(n-2)}(b_1^*-b_2^*), \tag{6.3.22}$$

where

$$b_1^* = \frac{y}{m+1}\{u^{-(\alpha^*+2)} - (u-(m+1)t^*)^{-(\alpha^*+2)}\} - \frac{\{u^{-(\alpha^*+1)} - (u-(m+1)t^*)^{-(\alpha^*+1)}\}}{(m+1)(n-2)(\alpha^*+1)},$$

$$b_2^* = \frac{t^*(v-(m+n-1)t^*)^{-(\alpha^*+2)}}{(m+n-1)} - \frac{(m+1)\{v^{-(\alpha^*+1)} - (v-(m+n-1)t^*)^{-(\alpha^*+1)}\}}{(m+n-1)^2(\alpha^*+1)(n-2)}.$$

Hence the Bayes estimator of θ_2 is obtained as

$$\delta_2^{B2} = \frac{b_1^* - b_2^*}{e_1 - e_2} + \eta \frac{d_1 - d_2}{(\alpha^* + 1)(e_1 - e_2)}. \tag{6.3.23}$$

6.3.3 Numerical Comparisons

In Section 6.3.1, we have obtained various estimators such as $\delta_1^L, \delta_1^L, \delta_1^M, \delta_1^U, \delta_1^E$ for θ_1 when there is no order restrictions on the location parameters. Further when there is order restriction we have obtained δ_1^R, (plug-in type restricted estimator over the analogue of best equivariant estimator) δ_1^{B1} and δ_1^{B2} as given in Section 6.3.2. Similarly we have proposed $\delta_2^L, \delta_2^L, \delta_2^M, \delta_2^U, \delta_2^E$, δ_2^R, δ_2^{B1} and δ_2^{B2} for θ_2. It seems difficult to compare all these estimators analytically. However for the purpose of application, one needs the estimator that can be used. Further an intention is to know how these estimators behave as the choices of η and the censoring factors ($k1 = r/m$, $k2 = s/n$) varies. Taking the advantages of computational facilities, we have numerically compared the risk values of all these estimators using Monte-Carlo simulation method. In order to do so, we have generated 20,000 type-II censored random samples each from exponential distributions having the same scale parameter 1 and different location parameters such that $0 \leq \mu_1 \leq \mu_2$. To proceed further, we define the percentage of relative risk improvements of an estimator d with respect to the MLE δ_i^L as,

$$R(d) = \left(1 - \frac{R(d, \theta_i)}{R(\delta_i^L, \theta_i)}\right) \times 100.$$

The simulation results have been checked by taking various combinations of (α, β) and η. We have taken conveniently the choices of $c_i \geq 0$. However, one may choose $c_i < 0$, in that case the MLEs and the UMVUEs will not belong to the class d_{c_i}. In our simulation we have taken the choice of $\eta = 1.5$ conveniently. The choice of the hyper parameters have been taken as $\alpha = 3.5$ and $\beta = 3$ as a reasonable choice. An extensive simulation study has been conducted by taking various combinations of sample sizes with censoring factors $k1 = k2 = 0.25, 0.50, 0.75, 1$. For illustration purpose we have tabulated the percentage of relative risk performances of various estimators of θ_1 (Tables 6.3.1-6.3.3) for sample sizes $(12, 8)$, $(8, 12)$ and $(12, 12)$. Similarly the percentage of relative risk values of all the estimators for θ_2 have been tabulated in Tables 6.3.4-6.3.6. Table 6.3.1 contains 10 columns and each column divided into some cells. The first and the 6th column represent the choices of $\rho = (\mu_2 - \mu_1)/\sigma > 0$. The columns from second to fifth and 7th to 10th represent the percentage of relative risk values of estimators δ_1^E, δ_1^R, δ_1^{B1}, δ_1^{B2}. Further in each cell corresponding to one value of ρ there correspond four values of relative risk improvements. These four values correspond to the four choices of censoring factors $k1 = k2 = 0.25, 0.50, 0.75, 1$. The accuracy of the simulation has been checked and the error of the simulation has been checked which is seen maximum up to 10^{-2}.

The following observations can be drawn from our simulation study as well as the Tables 6.3.1-6.3.6.

Comments on θ_1

1. For $0 < \rho < 1.0$, the estimator δ_1^{B2} perform better than all the estimators. The percentage of relative risk improvement of δ_1^{B2} varies from 44% to 86%. The percentage of relative risk improvement of δ_1^{B1} varies from 42% to 82%. The percentage of relative risk improvement of δ_1^E varies from 28% to 46%. Similarly, the percentage of relative risk improvement of δ_1^R varies from $28\% - 63\%$. For $1.0 < \rho < 2.5$, the δ_1^{B1} perform

better among all the estimators. For $\rho > 2.5$, the percentage of relative risk improvement of all the estimators are nearly same except the estimator δ_1^{B2}.

2. The percentage of relative risk improvements of all the estimators are highly dependent on the hyper parameters α and β and η. This improvement is not uniform for all values of ρ. It may be positive or negative. However, when the parameters α and β are close to each other ($\alpha \approx \beta$), the percentage of relative risk improvements are noticeable.

3. When values of the ρ increases from zero to some large value, and $\alpha \approx \beta$, the percentage of relative risk improvement of δ_1^{B2} is positive and reaches its maximum value then goes down to zero and, finally enter into negative values. However the percentage of relative risk improvement of δ_1^{B1} is better for initial-values of ρ and then its performance becomes negligible.

4. We also observe that for most of the values of ρ, as the censoring factor increases the percentage of relative risk values of all the estimators decreases.

Comments on θ_2

1. For $0 < \rho < 3.5$, the estimator δ_2^{B2} perform better than all the estimators. The percentage of relative risk improvement of δ_2^{B2} varies from 22% to 77%. However the percentage of relative risk improvements of all other estimators are very less. For $\rho > 3.5$, the percentage of relative risk improvement of the δ_2^{B2} is nearly zero and as ρ further increases it becomes negative. But in this region of ρ, the percentage of relative risk improvements all other estimators, such as δ_2^E, δ_2^R and δ_2^{B1} are almost same.

2. As in the case of θ_1, the percentage of relative risk improvements of all the estimators for θ_2 are also highly dependent on the hyper parameters α and β and η. This improvement is not uniform for all values of ρ. It may be positive or negative. However, when the parameters α and β are close to each other and medium ($\alpha \approx \beta$), the percentage of relative risk improvements are noticeable.

On the basis of our computational results, the following conclusions can be drawn for estimating the quantiles θ_i when $\theta_1 \leq \theta_2$.

1. For the estimation of θ_1, we recommend to use $\hat{\theta}_{2bs}$ for small values of ρ $(0 < \rho < 1)$. For $1 < \rho < 2.5$, we recommend to use $\hat{\theta}_{1bs}$. For $\rho > 2.5$, we recommend to use $\hat{\theta}_{1R}$.

2. For the estimation of θ_2, we recommend to use $\hat{\theta}_{2bs}$ for small values of ρ $(0 < \rho < 3.5)$. For $\rho > 3.5$, we recommend to use $\hat{\theta}_{1R}$.

Table 6.3.1: Percentage of relative risk improvements of proposed estimators of θ_1 for sample sizes $(m, n) = (12, 8)$ with censoring factors $k1 = k2 = (0.25.0.5, 0.75, 1)$ and for $\eta = 1.5; \alpha = 3.5; \beta = 3.0$

$\rho \downarrow$	$R(\delta_1^E)$	$R(\delta_1^R)$	$R(\delta_1^{B1})$	$R(\delta_1^{B2})$	$\rho \downarrow$	$R(\delta_1^E)$	$R(\delta_1^R)$	$R(\delta_1^{B1})$	$R(\delta_1^{B2})$
0.05	5.93	5.41	6.42	35.34	2.25	6.70	6.70	6.51	84.12
0.05	1.83	1.79	1.34	12.50	2.25	1.71	1.71	1.65	60.21
0.05	0.09	0.62	0.88	7.86	2.25	0.37	0.37	0.25	45.37
0.05	0.10	1.10	1.87	6.67	2.25	0.00	0.00	0.03	33.53
0.25	6.19	6.19	7.98	48.63	2.50	6.52	6.52	6.62	79.60
0.25	2.04	2.18	4.26	28.16	2.50	1.89	1.89	1.70	54.46
0.25	0.22	0.52	3.10	20.99	2.50	0.20	0.20	0.18	39.14
0.25	0.05	0.46	3.30	18.43	2.50	0.00	0.00	0.03	29.14
0.50	6.06	6.06	8.38	61.45	2.75	6.45	6.45	6.41	73.36
0.50	1.41	1.42	3.97	41.84	2.75	1.77	1.77	1.69	48.08
0.50	0.18	0.21	2.32	32.46	2.75	0.17	0.17	0.16	32.37
0.50	0.10	0.16	2.07	27.95	2.75	0.08	0.08	0.06	20.84
0.75	6.40	6.40	7.71	70.38	3.00	6.43	6.43	6.42	65.56
0.75	1.54	1.54	2.90	50.80	3.00	2.54	2.54	2.01	40.09
0.75	0.33	0.33	1.17	39.37	3.00	0.13	0.13	0.13	21.73
0.75	0.11	0.11	0.92	33.44	3.00	0.02	0.02	0.05	14.13
1.00	6.80	6.80	7.47	77.54	3.25	6.12	6.12	6.46	54.75
1.00	1.22	1.22	2.27	57.24	3.25	1.64	1.64	1.67	26.67
1.00	0.04	0.04	0.55	44.84	3.25	0.26	0.26	0.19	12.68
1.00	0.11	0.11	0.39	36.91	3.25	0.05	0.05	0.04	2.64
1.25	6.15	6.15	6.99	82.50	3.50	6.60	6.60	6.60	43.21
1.25	1.34	1.34	1.92	61.78	3.50	1.48	1.48	1.56	11.89
1.25	0.08	0.08	0.37	47.91	3.50	0.21	0.21	0.18	-0.54
1.25	0.09	0.09	0.20	39.10	3.50	0.02	0.02	0.04	-7.30
1.50	6.46	6.46	6.63	85.68	3.75	6.54	6.54	6.45	28.78
1.50	1.68	1.68	1.95	64.47	3.75	1.61	1.61	1.61	-2.70
1.50	0.30	0.30	0.31	50.15	3.75	0.16	0.16	0.16	-15.24
1.50	0.00	0.00	0.08	40.21	3.75	0.05	0.05	0.02	-17.54
1.75	6.41	6.41	6.87	86.93	4.00	5.52	5.52	6.28	12.00
1.75	2.32	2.32	2.04	65.58	4.00	2.54	2.54	1.90	-16.42
1.75	0.20	0.20	0.21	49.48	4.00	0.27	0.27	0.18	-31.63
1.75	0.01	0.01	0.06	39.48	4.00	0.07	0.07	0.07	-35.96
2.00	6.89	6.89	6.84	86.64	4.25	5.36	5.36	6.14	-5.41
2.00	1.89	1.89	1.76	63.89	4.25	1.85	1.85	1.60	-37.52
2.00	0.02	0.02	0.14	46.53	4.25	0.05	0.05	0.13	-49.33
2.00	0.02	0.02	0.05	37.25	4.25	0.09	0.09	0.07	-53.71

Table 6.3.2: Percentage of relative risk improvements of proposed estimators of θ_1 for sample sizes $(m, n) = (8, 12)$ with censoring factors $k1 = k2 = (0.25.0.5, 0.75, 1)$ and for $\eta = 1.5; \alpha = 3.5; \beta = 3.0$

$\rho\downarrow$	$R(\delta_1^E)$	$R(\delta_1^R)$	$R(\delta_1^{B1})$	$R(\delta_1^{B2})$	$\rho\downarrow$	$R(\delta_1^E)$	$R(\delta_1^R)$	$R(\delta_1^{B1})$	$R(\delta_1^{B2})$
0.05	4.73	2.40	6.23	33.06	2.25	3.97	3.97	4.47	80.85
0.05	0.28	0.56	1.26	12.02	2.25	0.23	0.23	0.21	54.47
0.05	0.39	2.44	3.54	9.35	2.25	0.48	0.48	0.39	37.82
0.05	2.39	6.69	7.92	13.53	2.25	1.75	1.75	1.86	28.67
0.25	5.29	5.32	6.55	47.22	2.50	5.08	5.08	4.67	76.67
0.25	0.27	1.39	3.90	29.86	2.50	0.03	0.03	0.17	47.59
0.25	0.47	2.63	5.42	26.22	2.50	0.32	0.32	0.30	32.27
0.25	2.18	4.95	8.24	26.39	2.50	2.60	2.60	2.24	21.50
0.50	4.60	4.69	6.01	59.97	2.75	4.31	4.31	4.38	69.68
0.50	0.34	0.66	2.23	41.35	2.75	0.23	0.23	0.23	40.12
0.50	0.57	1.05	2.57	34.93	2.75	0.45	0.45	0.39	23.58
0.50	2.42	3.08	4.43	31.87	2.75	1.30	1.30	1.74	15.27
0.75	4.87	4.90	5.42	69.12	3.00	5.02	5.02	4.58	61.13
0.75	0.23	0.27	1.05	49.58	3.00	0.29	0.29	0.29	30.88
0.75	0.31	0.38	1.07	39.62	3.00	0.24	0.24	0.36	14.88
0.75	2.34	2.50	2.72	34.85	3.00	2.07	2.07	1.98	5.67
1.00	4.68	4.68	5.00	76.31	3.25	4.35	4.35	4.45	49.29
1.00	0.26	0.29	0.59	55.66	3.25	0.13	0.13	0.18	16.17
1.00	0.44	0.46	0.62	43.78	3.25	0.37	0.37	0.36	3.05
1.00	2.03	2.05	2.05	36.84	3.25	1.76	1.76	1.84	-4.68
1.25	4.41	4.41	4.81	81.28	3.50	4.45	4.45	4.48	36.08
1.25	0.23	0.23	0.35	59.42	3.50	0.17	0.17	0.19	1.69
1.25	0.36	0.36	0.44	45.91	3.50	0.40	0.40	0.39	-11.07
1.25	1.85	1.85	1.82	38.00	3.50	2.11	2.11	2.13	-16.31
1.50	5.00	5.00	4.90	84.49	3.75	4.23	4.23	4.38	22.12
1.50	0.32	0.32	0.35	61.57	3.75	0.10	0.10	0.17	-17.22
1.50	0.64	0.64	0.49	46.16	3.75	0.38	0.38	0.41	-26.26
1.50	1.74	1.74	1.95	37.82	3.75	2.06	2.06	2.12	-30.63
1.75	5.69	5.69	4.94	85.87	4.00	5.37	5.37	4.63	4.70
1.75	0.25	0.25	0.26	61.04	4.00	0.29	0.29	0.24	-32.21
1.75	0.44	0.44	0.40	45.48	4.00	0.56	0.56	0.42	-47.47
1.75	2.11	2.11	2.10	35.54	4.00	2.12	2.12	2.10	-44.74
2.00	4.86	4.86	4.61	84.30	4.25	4.71	4.71	4.29	-14.75
2.00	0.27	0.27	0.25	58.88	4.25	0.31	0.31	0.29	-51.30
2.00	0.51	0.51	0.41	42.31	4.25	0.28	0.28	0.41	-58.03
2.00	1.77	1.77	1.83	33.25	4.25	2.15	2.15	2.11	-65.62

Table 6.3.3: Percentage of relative risk improvements of proposed estimators of θ_1 for sample sizes $(m, n) = (12, 12)$ with censoring factors $k1 = k2 = (0.25.0.5, 0.75, 1)$ and for $\eta = 1.5; \alpha = 3.5; \beta = 3.0$

$\rho\downarrow$	$R(\delta_1^E)$	$R(\delta_1^R)$	$R(\delta_1^{B1})$	$R(\delta_1^{B2})$	$\rho\downarrow$	$R(\delta_1^E)$	$R(\delta_1^R)$	$R(\delta_1^{B1})$	$R(\delta_1^{B2})$
0.05	4.39	3.58	6.30	28.18	2.25	5.39	5.39	5.16	79.48
0.05	0.82	0.71	1.50	8.86	2.25	0.52	0.52	0.67	52.47
0.05	0.00	0.92	1.40	5.28	2.25	0.00	0.00	0.00	38.11
0.05	0.18	1.39	2.32	4.03	2.25	0.22	0.22	0.34	28.99
0.25	4.91	4.94	7.49	42.13	2.50	4.89	4.89	5.21	74.53
0.25	0.54	0.86	3.56	25.30	2.50	0.67	0.67	0.78	48.56
0.25	0.00	0.52	3.04	19.64	2.50	0.00	0.00	0.00	32.42
0.25	0.38	0.93	3.52	17.93	2.50	0.39	0.39	0.43	23.25
0.50	4.64	4.65	6.52	54.74	2.75	5.18	5.18	5.18	68.23
0.50	0.48	0.49	2.16	36.74	2.75	0.43	0.43	0.73	39.39
0.50	0.00	0.01	1.05	27.80	2.75	0.00	0.00	0.00	25.18
0.50	0.39	0.42	1.37	24.44	2.75	0.45	0.45	0.49	16.98
0.75	4.34	4.34	5.67	64.24	3.00	4.97	4.97	5.15	60.94
0.75	0.60	0.60	1.14	44.36	3.00	0.94	0.94	0.90	31.95
0.75	0.00	0.00	0.28	34.15	3.00	0.00	0.00	0.00	18.00
0.75	0.25	0.25	0.57	28.49	3.00	0.40	0.40	0.48	9.82
1.00	4.94	4.94	5.48	71.82	3.25	5.09	5.09	5.17	48.52
1.00	0.96	0.96	0.93	50.47	3.25	0.92	0.92	0.76	21.26
1.00	0.00	0.00	0.07	38.88	3.25	0.00	0.00	0.00	8.30
1.00	0.35	0.35	0.39	31.98	3.25	0.41	0.41	0.40	0.96
1.25	5.34	5.34	5.39	77.29	3.50	4.75	4.75	5.17	35.73
1.25	0.96	0.96	0.88	54.94	3.50	0.72	0.72	0.79	7.97
1.25	0.00	0.00	0.02	41.85	3.50	0.00	0.00	0.00	-3.64
1.25	0.57	0.57	0.51	33.85	3.50	0.39	0.39	0.45	-7.45
1.50	5.28	5.28	5.33	80.83	3.75	4.60	4.60	4.94	22.88
1.50	0.84	0.84	0.89	57.61	3.75	0.49	0.49	0.69	-11.25
1.50	0.00	0.00	0.00	43.15	3.75	0.00	0.00	0.00	-17.79
1.50	0.55	0.55	0.43	34.21	3.75	0.61	0.61	0.53	-24.59
1.75	5.26	5.26	5.21	82.32	4.00	5.05	5.05	5.00	4.84
1.75	0.85	0.85	0.78	58.22	4.00	0.81	0.81	0.82	-20.37
1.75	0.00	0.00	0.00	43.24	4.00	0.00	0.00	0.00	-32.87
1.75	0.51	0.51	0.45	33.35	4.00	0.33	0.33	0.31	-33.58
2.00	6.06	6.06	5.46	82.50	4.25	4.85	4.85	5.03	-12.07
2.00	0.95	0.95	0.90	57.25	4.25	0.96	0.96	0.94	-41.01
2.00	0.00	0.00	0.00	42.23	4.25	0.00	0.00	0.00	-50.38
2.00	0.34	0.34	0.33	31.84	4.25	0.38	0.38	0.42	-49.00

Table 6.3.4: Percentage of relative risk improvements of proposed estimators of θ_2 for sample sizes $(m, n) = (12, 8)$ with censoring factors $k1 = k2 = (0.25.0.5, 0.75, 1)$ and for $\eta = 1.5; \alpha = 3.5; \beta = 3.0$

$\rho\downarrow$	$R(\delta_2^E)$	$R(\delta_2^R)$	$R(\delta_2^{B1})$	$R(\delta_2^{B2})$	$\rho\downarrow$	$R(\delta_2^E)$	$R(\delta_2^R)$	$R(\delta_2^{B1})$	$R(\delta_2^{B2})$
0.05	4.87	6.42	9.33	48.10	2.25	4.45	4.45	4.56	82.93
0.05	0.29	1.91	4.01	31.52	2.25	0.14	0.14	0.71	59.10
0.05	0.43	2.11	4.24	23.65	2.25	0.66	0.66	1.27	43.57
0.05	2.20	3.48	5.11	15.31	2.25	2.32	2.32	2.52	34.81
0.25	5.64	5.79	4.83	46.37	2.50	4.70	4.70	4.51	79.85
0.25	0.24	0.47	1.03	27.29	2.50	0.20	0.20	0.43	55.99
0.25	0.56	0.70	1.16	21.20	2.50	0.44	0.44	0.58	41.12
0.25	2.31	2.49	2.57	18.76	2.50	1.64	1.64	1.41	33.81
0.50	3.68	3.69	5.31	56.33	2.75	4.23	4.23	4.46	74.56
0.50	0.30	0.32	0.15	33.71	2.75	0.25	0.25	0.41	50.97
0.50	0.17	0.19	-0.23	23.63	2.75	0.36	0.36	0.35	37.65
0.50	2.12	2.14	1.86	20.73	2.75	1.53	1.53	0.91	29.14
0.75	4.62	4.62	4.85	64.74	3.00	4.91	4.91	4.26	68.21
0.75	0.53	0.53	-0.40	41.45	3.00	0.18	0.18	0.39	42.43
0.75	0.20	0.20	-0.42	30.62	3.00	0.30	0.30	0.09	31.21
0.75	1.92	1.92	1.38	25.04	3.00	1.94	1.94	1.89	23.26
1.00	4.96	4.96	4.69	72.26	3.25	3.88	3.88	4.70	59.04
1.00	0.27	0.27	0.12	49.32	3.25	0.28	0.28	0.16	34.33
1.00	0.34	0.34	0.10	36.71	3.25	0.36	0.36	0.26	21.96
1.00	2.58	2.58	2.76	30.64	3.25	1.90	1.90	1.61	17.53
1.25	4.38	4.38	4.69	77.94	3.50	4.01	4.01	4.83	49.51
1.25	0.26	0.26	0.12	55.25	3.50	0.26	0.26	0.31	23.30
1.25	0.59	0.59	0.87	41.60	3.50	0.43	0.43	0.47	12.81
1.25	2.20	2.20	2.07	34.06	3.50	2.07	2.07	2.08	7.46
1.50	3.77	3.77	4.77	81.79	3.75	4.65	4.65	4.50	37.56
1.50	0.24	0.24	0.12	58.73	3.75	0.28	0.28	0.17	11.59
1.50	0.42	0.42	0.60	44.62	3.75	0.42	0.42	0.78	2.85
1.50	2.47	2.47	2.96	35.79	3.75	2.74	2.74	3.37	-2.59
1.75	4.68	4.68	4.48	83.95	4.00	4.70	4.70	4.24	24.20
1.75	0.18	0.18	0.66	61.00	4.00	0.13	0.13	0.69	-3.08
1.75	0.55	0.55	0.84	46.28	4.00	0.55	0.55	0.92	-10.65
1.75	1.72	1.72	1.39	37.42	4.00	1.94	1.94	1.63	-12.40
2.00	4.44	4.44	4.56	84.34	4.25	4.53	4.53	4.62	7.84
2.00	0.16	0.16	0.51	61.02	4.25	0.21	0.21	0.13	-18.72
2.00	0.59	0.59	0.97	45.49	4.25	0.14	0.14	-0.46	-23.26
2.00	2.15	2.15	2.37	37.36	4.25	1.79	1.79	1.63	-23.97

Table 6.3.5: Percentage of relative risk improvements of proposed estimators of θ_2 for sample sizes $(m, n) = (8, 12)$ with censoring factors $k1 = k2 = (0.25.0.5, 0.75, 1)$ and for $\eta = 1.5; \alpha = 3.5; \beta = 3.0$

$\rho\downarrow$	$R(\delta_2^E)$	$R(\delta_2^R)$	$R(\delta_2^{B1})$	$R(\delta_2^{B2})$	$\rho\downarrow$	$R(\delta_2^E)$	$R(\delta_2^R)$	$R(\delta_2^{B1})$	$R(\delta_2^{B2})$
0.05	5.83	8.18	15.31	48.44	2.25	6.08	6.08	6.50	85.36
0.05	1.66	4.23	7.53	31.78	2.25	1.54	1.54	1.50	62.37
0.05	0.07	2.17	4.12	24.46	2.25	0.04	0.04	0.34	46.32
0.05	0.02	2.31	3.58	19.72	2.25	0.07	0.07	0.17	36.26
0.25	6.78	7.32	9.59	48.93	2.50	6.32	6.32	6.55	82.11
0.25	1.28	1.72	3.42	29.52	2.50	1.40	1.40	1.86	58.18
0.25	0.18	0.56	0.72	21.08	2.50	0.13	0.13	0.34	42.30
0.25	0.04	0.36	0.44	17.56	2.50	0.16	0.16	0.81	31.47
0.50	6.29	6.43	7.77	58.28	2.75	6.17	6.17	6.58	76.27
0.50	1.84	1.93	2.08	37.59	2.75	1.57	1.57	1.63	51.51
0.50	0.16	0.22	0.25	28.07	2.75	0.01	0.01	0.52	36.12
0.50	0.08	0.19	0.33	23.16	2.75	0.09	0.09	0.22	26.74
0.75	7.01	7.03	7.09	67.41	3.00	6.63	6.63	6.52	70.37
0.75	2.08	2.09	1.60	46.55	3.00	2.11	2.11	1.50	46.04
0.75	0.13	0.12	0.10	35.61	3.00	0.19	0.19	0.01	30.08
0.75	0.12	0.14	0.47	29.18	3.00	0.05	0.05	0.09	19.84
1.00	6.55	6.55	6.78	74.96	3.25	6.37	6.37	6.40	60.75
1.00	1.75	1.75	1.87	54.12	3.25	2.05	2.05	1.42	35.05
1.00	0.18	0.19	0.06	41.77	3.25	0.13	0.13	0.43	19.58
1.00	0.13	0.13	0.59	33.73	3.25	0.06	0.06	0.10	11.59
1.25	7.27	7.27	6.57	80.66	3.50	6.01	6.01	6.41	50.54
1.25	2.00	2.00	1.57	59.74	3.50	1.62	1.62	1.81	23.13
1.25	0.20	0.20	-0.04	45.85	3.50	0.24	0.24	-0.24	9.69
1.25	0.03	0.03	-0.31	36.91	3.50	0.00	0.00	-0.12	3.88
1.50	6.75	6.75	6.54	84.56	3.75	6.73	6.73	6.38	37.95
1.50	1.27	1.27	2.01	63.14	3.75	1.71	1.71	1.73	9.03
1.50	0.24	0.24	0.16	48.92	3.75	0.23	0.23	0.08	-2.39
1.50	0.11	0.11	0.49	38.89	3.75	0.07	0.07	0.14	-11.11
1.75	7.47	7.47	6.65	87.01	4.00	7.05	7.05	6.41	23.44
1.75	2.07	2.07	1.41	65.37	4.00	1.76	1.76	1.64	-4.77
1.75	0.08	0.08	0.63	49.78	4.00	0.10	0.10	0.34	-17.80
1.75	0.06	0.06	0.00	39.58	4.00	0.06	0.06	0.18	-22.27
2.00	6.37	6.37	6.64	86.71	4.25	5.69	5.69	6.42	6.21
2.00	1.59	1.59	1.72	64.73	4.25	1.48	1.48	1.90	-22.56
2.00	0.20	0.20	0.06	49.24	4.25	0.24	0.24	0.08	-32.98
2.00	0.07	0.07	0.22	38.38	4.25	0.02	0.02	-0.28	-35.21

Table 6.3.6: Percentage of relative risk improvements of proposed estimators of θ_2 for sample sizes $(m, n) = (12, 12)$ with censoring factors $k1 = k2 = (0.25.0.5, 0.75, 1)$ and for $\eta = 1.5; \alpha = 3.5; \beta = 3.0$

$\rho\downarrow$	$R(\delta_2^E)$	$R(\delta_2^R)$	$R(\delta_2^{B1})$	$R(\delta_2^{B2})$	$\rho\downarrow$	$R(\delta_2^E)$	$R(\delta_2^R)$	$R(\delta_2^{B1})$	$R(\delta_2^{B2})$
0.05	4.99	6.47	10.44	39.75	2.25	4.92	4.92	5.32	80.43
0.05	0.56	2.17	5.01	25.68	2.25	1.27	1.27	0.54	57.33
0.05	0.00	1.33	2.41	18.99	2.25	0.00	0.00	0.05	40.92
0.05	0.52	2.12	3.26	17.41	2.25	0.30	0.30	-0.04	32.19
0.25	4.78	4.91	6.52	41.23	2.50	5.61	5.61	5.02	77.35
0.25	0.63	0.76	1.75	23.60	2.50	0.95	0.95	0.84	53.58
0.25	0.00	0.11	-0.08	16.22	2.50	0.00	0.00	-0.10	37.91
0.25	0.41	0.60	0.39	14.12	2.50	0.44	0.44	0.47	28.96
0.50	5.33	5.34	5.63	52.03	2.75	5.36	5.36	5.24	71.85
0.50	0.39	0.40	1.21	32.68	2.75	0.82	0.82	0.68	46.66
0.50	0.00	0.00	0.05	23.66	2.75	0.00	0.00	0.13	33.32
0.50	0.37	0.36	-0.04	18.85	2.75	0.60	0.60	0.89	24.00
0.75	5.67	5.67	5.69	61.95	3.00	5.78	5.78	5.23	65.04
0.75	0.82	0.82	0.96	40.65	3.00	0.95	0.95	0.80	40.12
0.75	0.00	0.00	-0.08	30.32	3.00	0.00	0.00	-0.05	26.15
0.75	0.62	0.62	0.94	25.41	3.00	0.41	0.41	0.43	17.98
1.00	4.53	4.53	5.22	69.78	3.25	4.51	4.51	5.09	54.77
1.00	0.85	0.85	1.08	48.18	3.25	0.81	0.81	0.96	29.69
1.00	0.00	0.00	-0.14	36.20	3.25	0.00	0.00	-0.34	17.88
1.00	0.47	0.47	0.67	29.23	3.25	0.41	0.41	0.20	11.64
1.25	4.79	4.79	5.07	75.66	3.50	5.22	5.22	5.17	45.17
1.25	0.98	0.98	0.83	53.36	3.50	0.85	0.85	0.69	18.94
1.25	0.00	0.00	0.04	40.17	3.50	0.00	0.00	-0.29	7.47
1.25	0.59	0.59	1.03	32.45	3.50	0.42	0.42	0.34	1.73
1.50	5.08	5.08	5.11	79.77	3.75	5.33	5.33	5.15	31.71
1.50	0.80	0.80	0.94	56.90	3.75	0.72	0.72	0.91	7.33
1.50	0.00	0.00	0.10	42.62	3.75	0.00	0.00	0.29	-5.26
1.50	0.26	0.26	0.01	34.05	3.75	0.72	0.72	1.29	-9.07
1.75	4.25	4.25	5.36	81.66	4.00	6.20	6.20	5.09	18.87
1.75	0.69	0.69	0.88	58.42	4.00	0.70	0.70	0.91	-8.95
1.75	0.00	0.00	0.20	43.65	4.00	0.00	0.00	0.18	-16.44
1.75	0.27	0.27	0.15	35.09	4.00	0.44	0.44	0.44	-18.47
2.00	5.89	5.89	5.16	82.83	4.25	5.79	5.79	5.06	0.05
2.00	0.88	0.88	0.72	58.45	4.25	0.80	0.80	0.87	-25.47
2.00	0.00	0.00	-0.25	43.67	4.25	0.00	0.00	-0.32	-31.52
2.00	0.25	0.25	0.06	34.57	4.25	0.38	0.38	0.24	-31.11

6.3.4 Conclusions

In this section we have considered the problem of estimating ordered quantiles from two exponential populations with a common scale when the samples drawn are type-II censored. First, utilizing the order restrictions on the location parameters, we propose certain plug-in type restricted estimators based on some baseline estimators using the principle of isotonic regression. It has been observed from our simulation study, that these plug-in type restricted estimators do not improve uniformly upon their old counter parts when censored samples are available. Furthermore, we have used two types of priors to get Bayes estimators for the ordered quantiles. These Bayes estimators have been obtained analytically. However, the Bayes estimators too do not improve upon the restricted plug-in type estimator for all values of the parameters. Finally, we have tabulated the percentage of relative risk improvements of all these proposed estimators and recommendations have been made for their use. It will be interesting to get Bayes estimators using some other useful priors for the parameters in order to get better dominance results.

Chapter 7

Bayesian Estimation of Common Scale Parameter of Two Exponential Populations with Order Restricted Locations

7.1 Introduction

Let $\underset{\sim}{X} = (X_1, X_2, \ldots, X_m)$ and $\underset{\sim}{Y} = (Y_1, Y_2, \ldots, Y_n)$ be random samples taken from two shifted exponential populations with a common scale parameter σ and different location parameters μ_1 and μ_2 respectively. Here μ_is are also known as the minimum guarantee times or survival periods of certain products, whereas σ is known as the mean residual life time after the minimum survival period. From our practical experience, it is natural to assume that the minimum guarantee times (minimum survival periods) are non negative, hence throughout we assume, that $\mu_i > 0$ (see Elfessi and Pal (1991) for justification). The problem is to estimate the common scale parameter σ when it is known a priori that the location parameters follow the ordering $\mu_1 \leq \mu_2$. The loss function is taken as the quadratic,

$$L(\delta, \sigma) = \left(\frac{\delta}{\sigma} - 1\right)^2, \tag{7.1.1}$$

where δ is an estimator for the common scale parameter σ. Further the risk of an estimator is defined as

$$E\{L(\delta, \sigma)\}.$$

It is natural to expect that, the estimators obtained without any restriction on the parameters can be improved quite significantly, if one imposes order restrictions on the parameters. We expect that, the existing estimators for σ (without any restrictions on the location parameters) can be improved under the assumption of order restrictions on the location parameters. In the literature, the problem of estimation of parameters under order restrictions has been well investigated by several authors in the recent past, from a classical as well as decision theoretic point of view. Particularly, the model under consideration has got its importance due to the practical applications in real world problems. For example, suppose a product/equipment is produced from two different manufacturers say M1 and M2 and let the life times of these products follow exponential distributions. Assume that both the manufacturers employ modern statistical techniques so that their variations will be minimized. Depending upon their technology development and the target level the manufactures want that the minimum guarantee period or the mean life times of one manufacture will be less or more than the other. Under such a scenario, it is quite practical to assume that the scale parameters are equal and the location parameters follow certain ordering. For some review on estimation of ordered parameters under various statistical model assumptions, we refer to Barlow et al. (1972) and Robertson et al. (1988). Some further related results on estimation of parameters under order restrictions in the case of exponential populations have been studied in Tripathy et al. (2014), Misra and Singh (1994), Kushary and Cohen (1989) and the references cited therein.

The model under consideration, has also been studied previously by Madi and Tsui (1990), and Madi and Leonard (1996), without assuming the order restrictions on the location parameters. Madi and Tsui (1990) considered the estimation of σ under a large class of bowl-shaped loss function and proved the inadmissibility of the best affine equivariant estimator. They derived a class of improved estimators for σ. Further Madi and Leonard (1996) derived a Bayes estimator for σ and compared its simulated risk values with that of the best affine equivariant estimator using Monte-Carlo simulation method. Their numerical study reveals that, the amount of risk reduction of the Bayes estimator over the best affine equivariant estimator is much higher than compared to the estimator proposed by Madi and Tsui (1990). They have also tabulated the approximate values of the risk reduction in their paper. For some more results on Bayesian estimation of ordered parameters in the case of exponential populations, we refer to Yike and Heliang (1999), and Nagatsuka et al. (2009). Elfessi and Pal (1991) considered the estimation of σ using type-II censored samples where as Pandey and Singh (1979) studied the problem using the loss function $L(\delta, \sigma) = \max\{\delta/\sigma - 1, \sigma/\delta - 1\}$. Due to the difficulties in deriving the analytical expressions of a Bayes estimator under order restrictions, some authors have shown their interest in developing sampling techniques that can be used numerically. Treating the order restriction on the parameters, Gelfand et al. (1992) proposed a Gibbs sampling procedure for finding approximate Bayes estimator. Some related results on estimating ordered parameters using certain numerical methods, one may refer to Molitor and Sun (2002) and the references cited there in. The problem of estimation of quantiles of exponential populations with common scale parameter has been considered by Vellaisamy (2003). Recently Nagamani and Tripathy (2017) considered the Bayesian estimation of common scale parameter of two gamma populations using some numerical methods. Further results on estimation of common parameter from exponential distribution using full sample and record data have been considered by Jana et al. (2016) and Arshad and Baklizi (2018).

In this chapter, we consider the statistical model of two exponential populations with a common scale, that has been earlier considered by Madi and Tsui (1990), and Madi and Leonard (1996), with the additional information that the location parameters are ordered and non-negative. The main target is to derive certain Bayes estimators for the common scale parameter σ, under the assumption that location parameters are ordered, and which is practically very much useful. The rest of work is organized as follows. In Section 7.2, we discuss some basic results and propose the restricted MLE of σ. In Section 7.3, we find Bayes estimators using uniform prior and a conditional inverse gamma prior, taking into account the order restrictions on the location parameters. Exact expressions for these two Bayes estimators have been obtained. It seems quite difficult to evaluate the risk values of these estimators analytically. In Section 7.4, taking the advantages of computational facilities, we compare the performance of our estimator with that of Madi and Leonard (1996) with respect to the quadratic loss function (7.1.1) using Monte-Carlo simulation method. It has been revealed that the proposed estimator perform quite satisfactorily in comparison to other estimators, when it is known a priori that the location parameters are ordered.

7.2 Ceratin Basic Results

Let $\underset{\sim}{X} = (X_1, X_2, \ldots, X_m)$ and $\underset{\sim}{Y} = (Y_1, Y_2, \ldots, Y_n)$ be random samples taken from two exponential populations with a common scale parameter σ and different location parameters μ_1 and μ_2 respectively. Let us define the followings random variable.

$$X_{(1)} = \min_{1 \leq j \leq m} X_j, \; Y_{(1)} = \min_{1 \leq j \leq n} Y_j,$$

and

$$T = \sum_{j=1}^{m} (X_i - X_{(1)}) + \sum_{j=1}^{n} (Y_i - Y_{(1)}).$$

It is easy to see that $(X_{(1)}, Y_{(1)}, T)$ is complete and sufficient for (μ_1, μ_2, σ). We also note that, $X_{(1)} \sim Exp(\mu_1, \frac{\sigma}{m})$, $Y_{(1)} \sim Exp(\mu_2, \frac{\sigma}{n})$ and $T \sim Gamma(m+n-2, \sigma)$ and they are all independent. Here $Gamma(m+n-2, \sigma)$ denotes gamma distribution with shape parameter $m+n-2$ and scale parameter σ.

When there is no order restriction on the μ_is, the MLEs for μ_1, μ_2 and σ are obtained as $X_{(1)}, Y_{(1)}$ and $\frac{T}{m+n}$ respectively. Further, the uniformly minimum variance unbiased estimator (UMVUE) of σ is obtained as $\frac{T}{m+n-2}$. The analogous of the best affine equivariant estimator (BAEE) based on the sufficient statistics $(X_{(1)}, Y_{(1)}, T)$ is obtained as $\frac{T}{m+n-1}$. Further we also note that, in the class cT, where c is any positive constant, the choice of c which minimizes the risk with respect to the loss (7.1.1) or any weighted squared error loss, is obtained as $c_0 = \frac{1}{m+n-1}$. Let us denote this estimator as $d_0 = c_0 T$. Madi and Leonard (1996) obtained the following Bayes estimator when there is no restrictions on the location parameters.

$$d_B = \frac{\sum_{j=1}^{3} \frac{1}{\zeta_j} \{(t + \beta + \xi_j - z_{(j)}\zeta_j)^{-(k-1)} - (t + \beta + \xi_j - z_{(j-1)}\zeta_{(j)})^{-(k-1)}\}}{(n + \alpha - 1) \sum_{j=1}^{3} \frac{1}{\zeta_j} \{(t + \beta + \xi_j - z_{(j)}\zeta_j)^{-k} - (t + \beta + \xi_j - z_{(j-1)}\zeta_{(j)})^{-k}\}},$$

where $k = m + n + \alpha$, $Z_1 = X_{(1)}$, $Z_2 = Y_{(1)}$, and $Z_{(1)} \leq Z_{(2)}$ the order statistics of Z_i, and the symbols ξ_j, ζ_j are as defined in (2.7), (2.8) of Madi and Leonard (1996). It has been shown that the estimator d_B completely dominates d_0 with respect to a quadratic loss. They have also obtained the amount of risk reduction over the best equivariant estimator d_0.

We note that when there is order restriction on the location parameters that is $\mu_1 \leq \mu_2$, the usual estimator for the scale parameter may not perform well and hence better estimators can be derived. When there is no order restriction on location parameters, the MLEs of μ_1 and μ_2 are given by $X_{(1)}$ and $Y_{(1)}$ respectively. Using the isotonic regression on the estimators of μ_i we obtain the following estimators for μ_1 and μ_2 when $\mu_1 \leq \mu_2$.

$$\hat{\mu}_{1R} = \max\{0, \min(X_{(1)}, \frac{mX_{(1)} + nY_{(1)}}{m+n})\}, \qquad \hat{\mu}_{2R} = \max\{0, \max(Y_{(1)}, \frac{mX_{(1)} + nY_{(1)}}{m+n})\}.$$

Using these estimators of μ_is we propose the following estimator of σ and named it as restricted MLE.

$$\hat{\sigma}_R = \frac{1}{m+n-1} \left[\sum_{j=1}^{m} (X_i - \hat{\mu}_{1R}) + \sum_{j=1}^{n} (Y_i - \hat{\mu}_{2R}) \right].$$

Now taking convex combination of c_0T and $\hat{\sigma}_R$ we get another estimator say

$$d_R = pc_0T + (1-p)\hat{\sigma}_R, \text{ where } 0 < p < 1.$$

Remark 7.2.1 *In the estimator d_R, it seems difficult to find an optimal choice of p for which the risk will be minimum analytically. However, using the computational resources, we have checked the risk values d_R by taking many choices of p $(0.01(0.01)1)$ numerically. It has been revealed that the risk values of d_R changes marginally with respect to p. Hence for convenience we have taken $p = 1/2$ in our numerical study.*

Remark 7.2.2 *A numerical comparison of the risk values of the estimator d_R (with $p = 1/2$) and $\hat{\sigma}_R$ with respect to the loss function (7.1.1), has been done using Monte-Carlo simulation method. The restricted MLE $\hat{\sigma}_R$ does not improve upon the MLE. However, it has been observed that the amount of risk improvement of d_R over $\hat{\sigma}_R$ is very negligible. Hence we have not presented the risk of d_R in tables.*

7.3 Bayesian Estimation under Order Restriction

In this Section, we derive Bayes estimators for the common scale parameter σ assuming that, the location parameters follow a certain ordering that is $\mu_1 \le \mu_2$. When some prior information is available about parameters of certain distribution of an random experiment, the Bayesian estimators are more useful for the estimation of that parameters. It is a usual practice to find conjugate prior for the parameters in the Bayesian estimation. For our model it seems difficult to get a conjugate prior for the parameters under the condition that, $\mu_1 \le \mu_2$, $\sigma > 0$. For convenience, we have chosen two different priors which satisfy the order restrictions. First we consider uniform prior on the parameter space that is $\sigma > 0$ and $\mu_1 \le \mu_2$.

Bayes Estimator Using Uniform Prior

We assume that the parameters (μ_1, μ_2) and σ have been observed independently. Let us consider the joint prior density of $\underset{\sim}{\mu} = (\mu_1, \mu_2)$ as

$$\pi_1(\mu_1, \mu_2) = 1, \quad \text{for} \quad 0 < \mu_1 \le \mu_2.$$

$$\text{and} \quad \pi_2(\sigma) = \frac{1}{\sigma} \quad \text{for} \quad \sigma > 0.$$

So the joint prior density is obtained as

$$\pi(\mu_1, \mu_2, \sigma) = 1/\sigma, \quad \text{for} \quad 0 < \mu_1 \le \mu_2, \sigma > 0. \tag{7.3.1}$$

To derive the Bayes estimator of σ, we denote $X_{(1)}, Y_{(1)}, T$ as X, Y, T respectively. Denote $w = mx + ny + t$. The joint density of (X, Y, T) is given by

$$f_{X,Y,T}(x, y, t) = \frac{mnt^{m+n-3}}{\Gamma(m+n-2)\sigma^{m+n}} e^{-\frac{1}{\sigma}(w - m\mu_1 - n\mu_2)}, t > 0, x > \mu_1, y > \mu_2.$$

Further the joint posterior density function of (μ_1, μ_2, σ) is given (X, Y, T) by

$$g((\mu_1, \mu_2, \sigma)|(x, y, t)) \propto \frac{mnt^{m+n-3}}{\Gamma(m+n-2)\sigma^{m+n+1}} e^{-\frac{1}{\sigma}\{w-m\mu_1-n\mu_2\}},$$

where $t > 0, x > \mu_1, y > \mu_2, \mu_1 \le \mu_2$. The marginal posterior density function of σ is seen to be proportional to

$$g(\sigma|(x, y, t)) \propto \int_0^{t^*}\int_{\mu_1}^{y} \frac{mnt^{m+n-3}}{\Gamma(m+n-2)\sigma^{m+n+1}} e^{-\frac{1}{\sigma}\{w-m\mu_1-n\mu_2\}} d\mu_2 d\mu_1, \qquad (7.3.2)$$

$t > 0, x > \mu_1, y > \mu_2, \mu_1 \le \mu_2$. Here we denote $t^* = \min(x, y)$ and the values $0 < \mu_1 < t^*$, $\mu_1 < \mu_2 < y$. It can be seen that with respect to the loss function (7.1.1), the Bayes estimator of σ is of the form

$$d_{UB} = \frac{E(\frac{1}{\sigma}|(x, y, t))}{E(\frac{1}{\sigma^2}|(x, y, t))}. \qquad (7.3.3)$$

After some tedious calculations, the expected values have been obtained as,

$$E(\frac{1}{\sigma}|(x, y, t)) = c_1\Gamma(m+n-1)\Bigg[\frac{1}{m+n}\{w^{1-(m+n)} - (w-(m+n)t^*)^{1-(m+n)}\} - \frac{1}{m}\{\xi^{1-(m+n)} - (\xi - mt^*)^{1-(m+n)}\}\Bigg] \qquad (7.3.4)$$

and

$$E(\frac{1}{\sigma^2}|(x, y, t)) = c_1\Gamma(m+n)\Bigg[\frac{1}{m+n}\{w^{-(m+n)} - (w-(m+n)t^*)^{-(m+n)}\} - \frac{1}{m}\{\xi^{-(m+n)} - (\xi - mt^*)^{-(m+n)}\}\Bigg] \qquad (7.3.5)$$

where $\xi = mx + t$, $c_1 = \frac{mt^{m+n-3}}{A\Gamma(m+n-2)}$ and $A = \int_0^{t^*}\int_{\mu_1}^{y}\int_0^{\infty} \frac{1}{\sigma} f_{X,Y,T}(x, y, t) d\sigma d\mu_2 d\mu_1$.
Substituting these expressions in (7.3.3), one gets the generalized Bayes estimator of σ as

$$d_{UB} = \frac{\frac{1}{M}\{w^{(1-M)} - (w - Mt^*)^{(1-M)}\} - \frac{1}{m}\{\xi^{(1-M)} - (\xi - mt^*)^{(1-M)}\}}{(M-1)\frac{1}{M}\{w^{-M} - (w - Mt^*)^{-M}\} - \frac{1}{m}\{\xi^{-M} - (\xi - mt^*)^{-M}\}},$$

where $M = m + n$. The following result is immediate.

Theorem 7.3.1 *The generalized Bayes estimator of σ using the uniform prior (7.3.1), with respect to the loss function (7.1.1) is given by d_{UB}.*

Remark 7.3.1 *In section 7.4, we have conducted a simulation study and numerically compared the simulated risk values of the estimator d_{UB} with other estimators.*

Bayes Estimator Using Conditional Prior

In this section, we obtain the Bayes estimator of σ considering the fact that, the parameters may not be independent. We observe that, the scale parameter σ of an exponential population has a conjugate prior as inverse gamma. We note that the location parameters μ_is are also refereed as the minimum guarantee time in the study of reliability and life testing experiment. Hence, a reasonable prior density for the location parameter may be considered as an exponential type. Further we have also assumed $0 \le \mu_1 \le \mu_2$ which satisfies the boundedness criteria. A similar type of argument has been used to choose prior distribution for the ordered parameters in Yike and Heliang (1999) and Nagatsuka et al. (2009). In view of the above arguments, we have considered the joint prior density of (μ_1, μ_2, σ) as follows (using conditional prior). First assume that the prior density function of σ be an inverse gamma which is given by

$$p(\sigma) = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{e^{-\beta/\sigma}}{\sigma^{\alpha+1}}, \ \alpha > 0, \beta > 0.$$

Further consider the conditional prior density functions of μ_2 given σ and μ_1 given (μ_2, σ) respectively as,

$$q(\mu_2|\sigma) = \frac{1}{\sigma} e^{-\mu_2/\sigma}, \mu_2 > 0$$

and

$$r(\mu_1|(\mu_2, \sigma)) = \frac{1}{\sigma} e^{-(\mu_2 - \mu_1)/\sigma}, \ \mu_1 \le \mu_2, \sigma > 0.$$

Hence the joint prior density function of (μ_1, μ_2, σ) is given by

$$\pi(\mu_1, \mu_2, \sigma) = r(\mu_1|\mu_2, \sigma) q(\mu_2|\sigma) p(\sigma), \mu_1 \le \mu_2, \sigma > 0. \tag{7.3.6}$$

The joint posterior density function of μ_1, μ_2 and σ is given by

$$g((\mu_1, \mu_2, \sigma)|(x, y, t)) \propto \frac{\beta^\alpha mnt^{m+n-3}}{\Gamma(\alpha)\Gamma(m+n-2)} \frac{e^{-\frac{1}{\sigma}\{mx+ny+t+\beta+(2-n)\mu_2-(m+1)\mu_1\}}}{\sigma^{m+n+\alpha+3}}, \quad \sigma > 0, \mu_1 \le \mu_2. \tag{7.3.7}$$

Hence the marginal posterior density function of σ, can be obtained as in the previous subsection and is given by

$$g(\sigma|(x, y, t)) \propto \int_0^{t^*} \int_{\mu_1}^{y} \frac{\beta^\alpha mnt^{m+n-3}}{\Gamma(\alpha)\Gamma(m+n-2)} \frac{e^{-\frac{1}{\sigma}\{mx+ny+t+\beta+(2-n)\mu_2-(m+1)\mu_1\}}}{\sigma^{m+n+\alpha+3}} d\mu_2 d\mu_1, \quad \sigma > 0, \mu_1 \le \mu_2. \tag{7.3.8}$$

The above integral (7.3.8) has been evaluated and after some simplification is obtained as,

$$g(\sigma|(x, y, t)) = c_2(b_1 - b_2 - b_3 + b_4), \tag{7.3.9}$$

where

$$b_1 = \frac{e^{-\frac{1}{\sigma}\{\xi^* - (m+1)t^*\}}}{(m+n)\sigma^{m+n+\alpha+1}}, \quad b_2 = \frac{e^{-\frac{\xi^*}{\sigma}}}{(m+n)\sigma^{m+n+\alpha+1}},$$
$$b_3 = \frac{e^{-\frac{1}{\sigma}\{w^* - (m+n-1)t^*\}}}{(m+n-1)\sigma^{m+n+\alpha+1}}, \quad b_4 = \frac{e^{-\frac{w^*}{\sigma}}}{(m+n-1)\sigma^{m+n+\alpha+1}}.$$

and $c_2 = \frac{\beta^\alpha mnt^{m+n-3}}{B(n-2)\Gamma(\alpha)\Gamma(m+n-2)}$, $\xi^* = mx + 2y + \beta + t$, $w^* = mx + ny + \beta + t$, where

$$B = \int_0^{t^*}\int_{\mu_1}^{y}\int_0^\infty \frac{\beta^\alpha mnt^{m+n-3}}{\Gamma(\alpha)\Gamma(m+n-2)} \frac{e^{-\frac{1}{\sigma}\{mx+ny+t+\beta+(2-n)\mu_2-(m+1)\mu_1\}}}{\sigma^{m+n+\alpha+3}} d\sigma d\mu_2 d\mu_1.$$

Similar to the previous case, we need to find $E(\frac{1}{\sigma}|(x,y,t))$ and $E(\frac{1}{\sigma^2}|(x,y,t))$. The expressions have been simplified and are obtained as,

$$E(\frac{1}{\sigma}|(x,y,t)) = c_2\Gamma(m+n+\alpha+1)[\hat{b}_1 - \hat{b}_2 - \hat{b}_3 + \hat{b}_4], \tag{7.3.10}$$

where

$$\hat{b}_1 = \frac{1}{m+1}(\xi^* - (m+1)t^*)^{-(m+n+\alpha+1)}; \qquad \hat{b}_2 = \frac{1}{m+1}\xi^{*-(m+n+\alpha+1)};$$
$$\hat{b}_3 = \frac{1}{m+n-1}(w^* - (m+n-1)t^*)^{-(m+n+\alpha+1)}; \; \hat{b}_4 = \frac{1}{m+n-1}w^{*-(m+n+\alpha+1)}.$$

Also we have,

$$E(\frac{1}{\sigma^2}|\underline{z}) = c_2\Gamma(m+n+\alpha+2)[b_1^* - b_2^* - b_3^* + b_4^*], \tag{7.3.11}$$

where

$$b_1^* = \frac{1}{m+1}(\xi^* - (m+1)t^*)^{-(m+n+\alpha+2)}; \qquad b_2^* = \frac{1}{m+1}\xi^{*-(m+n+\alpha+2)};$$
$$b_3^* = \frac{1}{m+n-1}(w^* - (m+n-1)t^*)^{-(m+n+\alpha+2)}; \; b_4^* = \frac{1}{m+n-1}w^{*-(m+n+\alpha+2)}.$$

Substituting all these expressions we obtain the Bayes estimator of σ as,

$$d_{CB} = \frac{1}{(m+n+\alpha+1)}\left(\frac{\hat{b}_1 - \hat{b}_2 - \hat{b}_3 + \hat{b}_4}{b_1^* - b_2^* - b_3^* + b_4^*}\right). \tag{7.3.12}$$

The following theorem is immediate to follow.

Theorem 7.3.2 *Let the loss function be (7.1.1) and consider the conditional prior as given in (7.3.6), for estimating σ. The Bayes estimator of σ with respect to the prior (7.3.6) and the loss function (7.1.1), is given by*

$$\hat{\sigma}_{B2} = \frac{1}{(m+n+\alpha+1)}\left(\frac{\hat{b}_1 - \hat{b}_2 - \hat{b}_3 + \hat{b}_4}{b_1^* - b_2^* - b_3^* + b_4^*}\right).$$

where b_i^* and $\hat{b}_i$, for $i = 1, 2, 3, 4$ are given above.

Remark 7.3.2 *In Section 7.4, we have numerically evaluated the risk values of d_{CB} using Monte-Carlo simulation. For various values of the sample sizes and the hyper parameters α and β the risk values have been computed and compared with the existing estimators.*

7.4 Simulation Study

In Section 7.2, we have proposed some basic estimators such as the MLE, UMVUE, BAEE, (without order restrictions on the location parameters) the restricted MLE d_R (with order restrictions on the location parameters) for the common scale parameter σ. Moreover, in Section 7.3, we have analytically derived two Bayes estimators (one with respect to the uniform prior and the other with respect to the conditional prior) assuming order restriction on the location parameters, that is $\mu_1 \le \mu_2$. In this section, we will evaluate the performance of all these estimators with respect to the BAEE d_0. It seems difficult to compare the risk function analytically. Hence, we numerically evaluate the risk functions using Monte-Carlo simulation procedure. For the purpose of numerical comparison, we have generated 20000 random samples each from two exponential populations with a common scale parameter and different location parameters such that, $\mu_1 \le \mu_2$. The loss function is taken as the quadratic loss (7.1.1). It is easy to observe that, with respect to the loss function (7.1.1), the risk values of all the estimators are function of μ_1/σ and μ_2/σ. The error of the simulation has been checked and it is seen of the order of 10^{-3}.

We note that, when there is no order restrictions on the location parameters, the estimator d_B proposed by Madi and Leonard (1996) has the maximum percentage of relative risk performance. The amount of risk reduction over the BAEE d_0 has been calculated by them and the values have been tabulated there. So we have taken both d_B and d_0 as the baseline estimators when there is no order restrictions on the location parameters. When prior information regarding the ordering of the location parameters is available, one may be interested to know the amount of risk reduction of new estimators. Hence to evaluate the performance of the proposed estimators, we have computed the percentage of relative risk improvement of the estimator d_B, d_R, d_{UB} and d_{CB} with respect to the BAEE d_0. The percentage of relative risk improvement of any estimator δ with respect to the the BAEE d_0 is given by

$$R(\delta) = \left(1 - \frac{R(\delta, \sigma)}{R(d_0, \sigma)}\right) \times 100.$$

Though the risk of d_0 is $1/(m + n - 1)$, we have used the simulated risk values for comparison purpose. We have not tabulated the relative risk values of d_R as the improvement is very negligible. The simulation study has been done by taking various combinations of (α, β). However, we have presented the risk values for $(\alpha, \beta) = (3, 3)$ and $(5.5, 5)$ as for these choices we have noticed maximum risk reduction. Also from our simulation study it has been observed that the choices of α and β should be taken close to each other to get maximum percentage of risk improvements. Let us denote d_{CB1} the estimator corresponding to the values of $(\alpha, \beta) = (5.5, 5)$ and d_{CB2} corresponding to the values of $(\alpha, \beta) = (3, 3)$. All the estimators have been compared taking various combinations of of μ_1/σ and μ_2/σ under the condition that the location parameters are ordered that is, $\mu_1 \le \mu_2$. The risk values have been computed for various choices of the sample sizes and various combinations of μ_1/σ_1 and μ_2/σ_2. However, for illustration purpose, we have presented the percentage of relative risk values for some specific choices of the sample sizes. In Table 7.4.1, we have presented the percentage of relative risk improvements of various estimators for sample sizes $(m, n) = (4, 5)$. The first two columns give the values of μ_1/σ and μ_2/σ. The columns from 3rd to 6th represent the percentage of relative risk values of d_{UB}, d_B, d_{CB1} and d_{CB2} respectively. In a similar way the percentage of relative risk values of various estimators for the sample sizes $(m, n) = (5, 4)$, $(10, 5)$, $(5, 10)$, $(12, 16)$, $(16, 12)$, $(15, 15)$ and $(25, 25)$ have been presented in Tables 7.4.2-7.4.8 respectively. The following conclusions can be drawn from our simulation study as well as the Tables 7.4.1-7.4.8.

1. The percentage of relative risk improvements of d_{UB} varies between 0.10% and 7%. The percentage of relative risk values of d_B varies between 13% and 44%. The percentage of relative risk values of d_{CB1} varies between 15% and 80%, where as that of d_{CB2} varies between 10% and 70%.

2. The percentage of relative risk improvements of all the estimators certainly depends on the hyper parameters α and β. As the sample sizes increases the amount of risk reduction over the estimator d_0 decreases which is true. Further we note that for some combinations of μ_1/σ and μ_2/σ the percentage of relative risk improver of d_{UB} becomes negative, showing no improvements. It also has been noticed that, the percentage of relative risk improvements of all estimators first increase and attains its maximum then starts decreasing as the difference of μ_1/σ and μ_2/σ increases. We also noticed that, when the parameters α and β are nearer to each other ($\alpha \approx \beta$), the percentage of relative risk improvements is quite satisfactory.

3. The percentage of risk improvements of d_{CB1} and d_{CB2} over the estimator d_B is quite noticeable and it is seen to be maximum up to 25% and 35% respectively, which is quite significant and satisfactory. This is the main contribution of the current work.

4. A very similar type of observations were made for other choices and combinations of sample sizes and the hyper parameters.

On the basis of above observations, and from our simulation study we recommend to use the estimator d_{CB1} or d_{CB2} for the common scale parameter σ, when it is known a priori that the location parameters μ_1 and μ_2 follow the ordering, $0 < \mu_1 \leq \mu_2$.

Table 7.4.1: Percentage of relative risk improvements of Bayes estimators for sample sizes $(m, n) = (4, 5)$

μ_1/σ	μ_2/σ	$R(d_{UB})$	$R(d_B)$	$R(d_{CB1})$	$R(d_{CB2})$
0.00	0.25	3.481	40.056	42.226	32.574
0.00	0.50	3.537	42.142	51.748	43.260
0.00	0.75	2.068	43.643	59.606	53.517
0.00	1.00	1.379	44.253	66.698	61.375
0.05	0.25	4.817	39.868	42.142	34.203
0.05	0.50	3.993	41.832	51.019	44.023
0.05	0.75	3.710	43.335	59.693	53.682
0.05	1.00	3.458	43.955	66.987	61.234
0.50	1.00	7.501	41.990	64.005	57.892
0.50	1.50	6.300	43.864	74.149	67.773
0.50	2.00	5.767	42.334	77.901	68.834
1.00	1.50	5.747	41.592	70.074	63.541
1.00	2.50	4.153	41.538	75.789	62.364
1.00	3.00	3.745	35.402	68.278	48.191
2.00	2.50	2.962	41.656	75.953	67.562
2.00	2.75	2.705	42.967	76.661	66.328
2.00	3.00	2.453	43.743	75.830	62.404

Table 7.4.2: Percentage of relative risk improvements of Bayes estimators for sample sizes $(m, n) = (5, 4)$

μ_1/σ	μ_2/σ	$R(d_{UB})$	$R(d_B)$	$R(d_{CB1})$	$R(d_{CB2})$
0.00	0.25	1.329	40.530	41.521	33.016
0.00	0.50	1.745	42.833	51.134	43.913
0.00	0.75	0.950	44.786	59.493	53.905
0.00	1.00	1.100	45.010	66.968	62.020
0.05	0.25	3.756	40.188	42.341	34.601
0.05	0.50	4.316	42.390	51.888	44.675
0.05	0.75	3.238	44.558	60.107	54.518
0.05	1.00	3.265	45.163	67.443	62.311
0.50	1.00	7.136	42.583	64.302	58.221
0.50	1.50	6.782	44.351	74.805	68.278
0.50	2.00	5.745	42.382	78.057	68.121
1.00	1.50	5.105	42.322	70.269	63.623
1.00	2.50	3.631	41.885	75.732	61.797
1.00	3.00	3.350	33.696	66.901	45.766
2.00	2.50	2.525	42.839	76.259	67.406
2.00	2.75	2.908	43.763	76.811	66.310
2.00	3.00	2.590	44.692	76.137	62.273

Table 7.4.3: Percentage of relative risk improvements of Bayes estimators for sample sizes $(m, n) = (10, 5)$

μ_1/σ	μ_2/σ	$R(d_{UB})$	$R(d_B)$	$R(d_{CB1})$	$R(d_{CB2})$
0.00	0.25	0.772	26.839	27.358	20.485
0.00	0.50	1.246	28.743	37.856	32.160
0.00	0.75	1.651	29.058	47.356	40.899
0.00	1.00	0.603	29.925	53.643	47.598
0.05	0.25	2.737	26.519	28.727	21.499
0.05	0.50	3.240	28.493	38.730	32.833
0.05	0.75	3.885	29.212	48.476	42.194
0.05	1.00	2.933	29.685	54.246	48.105
0.50	1.00	3.385	28.569	50.252	43.550
0.50	1.50	2.502	29.523	59.512	50.836
0.50	2.00	1.924	26.731	60.954	48.637
1.00	1.50	2.114	28.873	55.258	47.837
1.00	2.50	0.680	26.569	58.888	43.916
1.00	3.00	0.379	22.295	51.127	28.965
2.00	2.50	2.098	28.439	61.090	51.318
2.00	2.75	1.631	29.592	61.811	49.908
2.00	3.00	1.123	29.754	60.417	46.385

Table 7.4.4: Percentage of relative risk improvements of Bayes estimators for sample sizes $(m, n) = (5, 10)$

μ_1/σ	μ_2/σ	$R(d_{UB})$	$R(d_B)$	$R(d_{CB1})$	$R(d_{CB2})$
0.00	0.25	2.971	25.860	27.878	20.479
0.00	0.50	1.581	27.028	36.825	30.367
0.00	0.75	0.903	27.926	44.733	38.267
0.00	1.00	-0.022	29.008	50.903	44.355
0.05	0.25	4.744	25.519	29.208	22.444
0.05	0.50	3.272	26.702	38.105	30.099
0.05	0.75	1.581	28.088	44.488	38.317
0.05	1.00	1.039	29.045	50.914	44.685
0.50	1.00	4.204	26.794	48.992	41.965
0.50	1.50	3.435	28.257	58.142	49.908
0.50	2.00	3.149	28.170	61.816	50.067
1.00	1.50	2.387	26.926	54.289	46.416
1.00	2.50	1.467	27.791	60.105	45.906
1.00	3.00	1.472	24.112	54.188	35.605
2.00	2.50	0.948	27.041	60.059	49.449
2.00	2.75	0.637	28.281	60.803	48.286
2.00	3.00	0.399	28.605	59.743	45.977

Table 7.4.5: Percentage of relative risk improvements of Bayes estimators for sample sizes $(m, n) = (12, 16)$

μ_1/σ	μ_2/σ	$R(d_{UB})$	$R(d_B)$	$R(d_{CB1})$	$R(d_{CB2})$
0.00	0.25	0.840	14.761	15.524	11.867
0.00	0.50	-0.093	15.850	22.555	18.095
0.00	0.75	-0.111	16.486	29.146	23.915
0.05	0.25	2.475	14.476	17.770	13.727
0.05	0.50	1.677	15.465	24.935	19.582
0.05	0.75	1.546	16.052	30.578	25.169
0.05	1.00	1.443	16.403	35.396	29.181
0.50	1.00	0.612	15.622	32.491	25.990
0.50	1.50	0.463	16.246	39.030	31.006
0.50	2.00	0.447	15.908	41.243	30.739
1.00	1.50	0.183	15.407	36.369	29.121
1.00	2.50	0.023	16.086	40.544	28.257
1.00	3.00	0.024	13.396	35.539	21.546
2.00	2.50	0.164	15.527	40.522	31.643
2.00	2.75	0.033	16.175	40.924	31.026
2.00	3.00	0.004	16.297	40.091	28.067

Table 7.4.6: Percentage of relative risk improvements of Bayes estimators for sample sizes $(m, n) = (16, 12)$

μ_1/σ	μ_2/σ	$R(d_{UB})$	$R(d_B)$	$R(d_{CB1})$	$R(d_{CB2})$
0.00	0.25	0.971	15.000	16.059	10.898
0.00	0.50	0.545	15.780	23.946	18.716
0.00	0.75	0.030	16.318	29.873	24.698
0.00	1.00	-0.223	16.785	34.617	28.978
0.05	0.25	2.911	14.692	18.451	13.624
0.05	0.50	2.248	15.660	26.132	20.570
0.05	0.75	2.001	15.924	31.801	25.902
0.05	1.00	1.610	16.616	35.884	29.877
0.50	1.00	0.519	16.071	32.648	26.629
0.50	1.50	0.183	16.829	39.197	30.979
0.50	2.00	0.180	14.903	40.638	30.333
1.00	1.50	0.378	15.823	36.675	29.664
1.00	2.50	0.005	15.661	40.166	28.096
1.00	3.00	0.004	12.711	34.705	19.734
2.00	2.50	0.387	15.723	40.859	31.967
2.00	2.75	0.093	16.282	40.984	30.467
2.00	3.00	0.021	16.324	39.930	28.466

Table 7.4.7: Percentage of relative risk improvements of Bayes estimators for sample sizes $(m, n) = (15, 15)$

μ_1/σ	μ_2/σ	$R(d_{UB})$	$R(d_B)$	$R(d_{CB1})$	$R(d_{CB2})$
0.00	0.25	1.365	13.808	16.146	10.431
0.00	0.50	0.286	14.706	22.344	16.824
0.00	0.75	0.024	15.289	28.145	22.717
0.00	1.00	-0.175	15.793	32.647	27.206
0.05	0.25	2.657	13.634	17.409	12.663
0.05	0.50	1.517	14.794	23.212	18.651
0.05	0.75	1.552	15.223	29.366	23.854
0.05	1.00	1.464	15.563	33.951	27.979
0.50	1.00	0.383	14.580	31.157	24.818
0.50	1.50	0.197	15.562	37.170	29.535
0.50	2.00	0.194	14.766	39.107	29.112
1.00	1.50	0.174	14.727	34.600	27.730
1.00	2.50	0.005	14.638	38.256	26.639
1.00	3.00	0.005	12.816	34.050	19.770
2.00	2.50	0.177	14.994	38.969	30.218
2.00	2.75	0.030	15.214	38.942	28.912
2.00	3.00	0.004	15.683	38.551	27.111

Table 7.4.8: Percentage of relative risk improvements of Bayes estimators for sample sizes $(m, n) = (25, 25)$

μ_1/σ	μ_2/σ	$R(d_{UB})$	$R(d_B)$	$R(d_{CB1})$	$R(d_{CB2})$
0.00	0.25	0.038	8.568	8.495	6.360
0.00	0.50	0.095	9.004	14.468	10.786
0.00	0.75	-0.047	9.538	18.311	14.475
0.00	1.00	-0.157	9.831	21.723	17.574
0.05	0.25	1.601	8.357	11.957	8.268
0.05	0.50	1.276	8.891	16.414	12.447
0.05	0.75	1.247	9.294	20.127	15.947
0.05	1.00	1.152	9.680	22.958	18.297
0.50	1.00	0.011	9.137	20.340	15.650
0.50	1.50	0.005	9.427	24.886	18.925
0.50	2.00	0.005	9.028	26.255	19.078
1.00	1.50	0.005	9.109	23.023	17.782
1.00	2.50	0.000	9.244	25.991	17.409
1.00	3.00	0.000	7.461	22.449	13.257
2.00	2.50	0.005	9.226	26.105	19.430
2.00	2.75	0.000	9.645	26.535	18.466
2.00	3.00	0.000	9.576	25.858	17.811

7.5 Conclusions

We have considered the estimation of common scale parameter σ of two exponential populations under the assumption that the location parameters follow the ordering $0 < \mu_1 \leq \mu_2$. Previously, the problem of estimation of common scale parameter σ has been considered by Madi and Leonard (1996) and Madi and Tsui (1990), but without any restriction on the location parameters. However, from a practical point of view it is very much reasonable to have an order relation on the location parameters, as it indicates the minimum guarantee period or time for a specific product in practice. Taking the advantages of order restrictions on the location parameters μ_1 and μ_2, we could able to derive the Bayes estimators namely d_{UB} and d_{CB}, which improve upon some of the existing estimators (without order restriction on the location parameters) obtained by Madi and Leonard (1996). We have shown using a numerical study that, the proposed Bayes estimators perform quite satisfactorily in comparison to the estimator proposed by Madi and Leonard (1996), under the order restriction on the location parameters. The amount of risk reduction as well as the percentage of relative risk improvements have been shown in Tables 7.4.1-7.4.8. The current research work definitely will shed some light on finding better estimators for the common scale parameter σ, when the location parameters known to follow some ordering.

Chapter 8

Conclusion and Future Work

This chapter presents the conclusion of our contributed works given in different chapters of the thesis and a discussion on future research plan, which have culminated from our present study.

In this thesis, we have studied the problems of estimating parameters and quantiles of two or more normal and exponential populations under equality and inequality restrictions. In a nutshell, we have studied the problems from a decision theoretic view point using either full or censored samples. In certain cases we could able to obtain the Bayes estimators, which has importance when the prior information regarding the parameters is known in advance. For the sake of completeness, below we give the conclusions and suggestions for the future work chapter-wise.

- In Chapter 3, we have revisited the problem of estimating common mean of two normal populations when it is known apriori that the variances are ordered. Taking the advantage of order restriction on the variances, we could able to propose certain alternative estimators (including the restricted MLE which has been obtained numerically) for the common mean. It is worth mentioning that under the same set up Elfessi and Pal (1992) proposed a new estimator that dominates stochastically and hence universally the Graybill-Deal estimator (Graybill and Deal (1959)) for both equal and unequal sample sizes. Further, their results have been generalized to $k(\geq 2)$ normal populations by Misra and van der Meulen (1997). We have proposed improved estimators that dominate some other well known estimators for the common mean such as the estimators proposed by Khatri and Shah (1974), Moore and Krishnamoorthy (1997), Tripathy and Kumar (2010) and Brown and Cohen (1974) stochastically as well as in terms of Pitman measure of closeness criterion for both equal and unequal sample sizes. It has been seen from our simulation study that none of the improved estimators beats other in the whole parameter space like the case when the variances do not follow the ordering. Finally, we have given our comments regarding the use of the estimators which is important from an application point of view. **In future our target is to extend these results to a general $k(\geq 2)$ normal populations and obtain some decision theoretic results. Further, a Bayesian estimation of the common mean under the same set up can be considered**.

- In Section 4.2, we have considered the problem of estimating quantiles for $k(\geq 2)$ normal populations with a common mean. First, we estimate the quantile of the first population, when the other $k-1$ populations are available, with respect to the quadratic loss function. We have proved a general result which helps in obtaining better estimators for the quantiles. As a consequence, some improved estimators have been constructed. Then we introduce the concept of invariance and derive sufficient conditions for improving estimators in these classes. A detailed simulation study has been carried out in order to numerically compare the performances of all the proposed estimators for the case $k = 3$

and 4. The percentage of relative risk improvements for all the proposed estimators have been tabulated and recommendations have been made which is important for application point of view. In Section 4.3, we consider the same model and estimate the quantile vector with respect to sum of the quadratic losses. A similar type of results have been derived that to Section 4.2. The inadmissibility results have been obtained only for the case $k = 2$ normal populations. **In future our target is to prove the inadmissibility result for a general $k(\geq)$ populations. Further an application of IERD (Integral Expression for Risk Difference) approach of Kubokawa (1994) can be done to obtain new improved classes of estimators**.

- In Section 5.2, we have considered the estimation of quantiles from two exponential populations under equality assumption on the location parameters using type-II censored samples. The findings of this section generalizes results obtained by Sharma and Kumar (1994) where they have studied the problem for full sample case. In addition to this a detailed numerical comparison of all the proposed estimators have been done which is handy for practitioner. **In future, we want to study the problem using progressive type-II censoring scheme which is a generalization of type-II censoring sampling scheme. Further the results obtained in this section can be considered for a general $k(\geq 2)$ exponential populations with a common location parameter**. **In a similar way, the results obtained in Section 5.3 can be extended to the case of progressively censored samples. Further generalization of these results for $k(\geq 2)$ populations can be done in future**.

- In Section 6.2, we have considered the problem of estimating ordered quantiles of two exponential populations under equality assumption on the location parameters. The loss function is taken as quadratic loss. It is worth mentioning that, so far in the literature, we have not come across the problem of estimating function of ordered parameters. First we have obtained certain baseline estimators without assuming ordering of quantiles. Under order restriction, we propose a new estimator which is the isotonic version of the MLE, call it, restricted MLE. A sufficient condition for improving equivariant estimators are derived under order restriction on quantiles. Consequently, estimators improving upon the baseline estimators such as the MLE, a modification to the MLE, the UMVUE and the restricted MLE have been improved. The percentage of risk improvements have been calculated and presented in the form of tables. **In our future study, we intend to obtain some Bayes estimators and will aim to derive minimax estimators. Further generalization to $k(\geq)$ populations using progressive censored samples may be done**. In Section 6.3, we have considered the estimation of ordered quantiles under equality assumption on the scale parameters, using type-II censored samples. First we have derived some basic estimators such as the MLE, a modification to the MLE, the UMVUE, and the best affine equivariant estimator without considering ordering of the quantiles. Under order restriction on the quantiles, isotonized version of all these estimators have been proposed. Then Bayes estimators have been derived for the quantiles assuming order restriction on the quantiles. For this purpose we have considered two types of priors namely the non-informative prior and the conditional prior. **In our future study we intend to consider classes of mixed estimators and prove some inadmissibility results using Brewster and Zidek (1974) technique**.

- In Chapter 7, we have considered the estimation of the common scale parameter of two exponential populations when the location parameters satisfy the simple ordering.

First, we discuss some basic results for the common scale parameter without assuming ordering of location parameters. Under order restriction on the location parameters, we propose the restricted MLE. Further we have obtained Bayes estimators of the common scale using uniform prior and a conditional inverse gamma prior, taking into account the order restriction on the location parameters. We have numerically compared the risk of estimators with that of Madi and Leonardo (1996). **In future our target will be to obtain an improved class of estimators that may dominate the usual estimators by an application of either Brewster and Zidek (1974) technique or IERD approach of Kubokawa (1994)**.

- In addition to the above research problems, we also intend to study the following problems in future. Suppose $X \sim \frac{1}{\sigma_1} f(\frac{x-\mu}{\sigma_1})$ and $Y \sim \frac{1}{\sigma_2} f(\frac{y-\mu}{\sigma_2})$. The aim will be to estimate the ordered quantiles $\theta_1 \leq \theta_2$; where $\theta_i = \mu + \eta\sigma_i$ using a decision theoretic approach. The problem can be studied when the scale parameters are same and location follow certain ordering. These models can be extended to the case of $k(\geq 2)$ populations. Further let $X \sim Ex(\mu_1, \sigma_1)$ and $Y \sim Ex(\mu_2, \sigma_2)$, where $Ex(\mu_i, \sigma_i)$ denotes the exponential population with location parameter μ_i and scale parameter σ_i. In this model we intend to study the estimation of ordered quantiles using decision theoretic approach.

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- Adarsha Kumar Jena and Manas Ranjan Tripathy (2018). Estimating Ordered Quantiles of Two Exponential Populations with a Common Scale Under Type-II Censoring. (Under Review)

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Other Articles

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