Graph Structures from Combinatorial Optimization and Rigidity Theory

PhD Thesis

Author: Csaba Király

Eötvös Loránd University
Institute of Mathematics
Department of Operations Research

Doctoral School: Mathematics
Director: Miklós Laczkovich, member of the Hungarian Academy of Sciences

Doctoral Program: Applied Mathematics
Director: György Michaletzky, Doctor of Sciences

Supervisor: András Frank, Doctor of Sciences

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Notations

We list some notations used throughout this thesis. We note that some sub- or superscripts in the notations will be omitted later if it is clear from the context.

Graphs and digraphs

All the graphs considered here are loopless but may contain parallel edges if not stated otherwise.

$G = (V, E)$	A graph with node set V and edge set E.
$G = (S, T; E)$	A bipartite graph with color classes S and T and edge set E.
$G = (V; E, E')$	A graph with node set V and edge set partitioned to E and E'.
$D = (V, A)$	A digraph with node set V and edge set (also called arc set) A.
$N_G(v)$	The set of the neighbors of a node v in G.
$d_G(v)$	The degree of a node $v \in V$.
$E_G(X, Y)$	The edges of G with one endpoint in $X - Y$ and another in $Y - X$ for $X, Y \subseteq V$.
$d_G(X, Y)$	$= \lvert E_G(X - Y, Y - X)\rvert$.
$d_G(X)$	$= d_G(X, V - X)$.
$\delta_D(v)$	The out-degree of a node $v \in V$.
$\varrho_D(v)$	The in-degree of a node $v \in V$.
$\delta_D(X)$	The out-degree of a set $X \subseteq V$.
$\varrho_D(X)$	The in-degree of a set $X \subseteq V$.
$\varrho^D_x(X)$	$= \sum\{x(e) : e \text{ enters } X\}$ for $X \subseteq V$ and $x : A \to \mathbb{R}$.
$\delta^D_x(X)$	$= \sum\{x(e) : e \text{ leaves } X\}$ for $X \subseteq V$ and $x : A \to \mathbb{R}$.
$G[X] = (X, E[X])$	The subgraph induced by a set $X \subseteq V$.
$D[X] = (X, A[X])$	The subdigraph induced by a set $X \subseteq V$.

G/X	The graph that arises from G by *contracting* $X \subseteq V$ into a single node.
D/X	The digraph that arises from D by *contracting* $X \subseteq V$ into a single node.
$G_1 \subseteq G_2$	The graph G_1 is a subgraph of G_2.
$i_G(X)$	$= \|E[X]\|$.
$e_G(\mathcal{P})$	The number of edges connecting distinct members of a partition $\mathcal{P}$ of V.
$G - X$	$= G[V - X]$ for $X \subseteq V$.
$G - F$	$= (V, E - F)$ for $F \subseteq E$.
$G + F$	$= (V, E \cup F)$ for another edge set F on the node set V.
$G + G'$	$= (V \cup V', E \cup E')$ for another graph $G' = (V', E')$.
$G * v$	$= G + G'$ with $G' = (V + v, \{uv : u \in V\})$ for $v \notin V$, the **cone graph** of G.
kG	The (multi)graph obtained from G by replacing each edge e of G by k parallel copies of e.
G^k	The graph obtained from G by connecting every pair of nodes whose distance is at most k in G.
K_n	A complete graph on n nodes.
$K(X)$	A complete graph on set X.
$K_{s,t}$	A complete bipartite graph with color classes of size s and t.
P_n	A path with node set $\{v_1, \dots, v_n\}$ and edge set $\{v_i v_{i+1} : 1 \leq i \leq n-1\}$.
C_n	$= P_n + v_1 v_n$, a cycle on n nodes.

Matroids

$\mathcal{M} = (S, \mathcal{F})$	A matroid on ground set S. The members of $\mathcal{F}$ are the independent sets of $\mathcal{M}$.
$r_{\mathcal{M}}$	The rank function of the matroid $\mathcal{M}$.
$r(\mathcal{M})$	$= r_{\mathcal{M}}(S)$.

Miscellaneous

$\mathbb{R}_+, \mathbb{Z}_+$	The sets of non-negative reals and integers.
$\widetilde{x}(X)$	$= \sum_{v \in X} x(v)$ for a function $x : V \to \mathbb{R}$ and a set $X \subseteq V$.
$\widetilde{h}(\mathcal{F})$	$= \sum_{F \in \mathcal{F}} h(F)$ for a set function $h : 2^V \to \mathbb{R}$ and a family $\mathcal{F} \subseteq 2^V$.
$X - v$	$= X - \{v\}$ for a set X and an element v.
$X + v$	$= X \cup \{v\}$ for a set X and an element v.
$\langle .,. \rangle$	The Euclidean scalar product in $\mathbb{R}^d$.
$\|\|.\|\|$	The Euclidean norm in $\mathbb{R}^d$.

Chapter 1

Introduction

Combinatorial optimization is a fast-growing area of discrete applied mathematics, dealing with algorithmic approaches to finding optimal objects in discrete structures. For the development of efficient algorithms, it requires the mathematical understanding of the underlying structures. One of the main structures of combinatorial optimization are graphs. Starting from the works of Edmonds in the 1960s, the main tools of combinatorial optimization were developed by examining the properties of special graph structures such as trees and arborescences.

Rigidity theory is an area with lots of industrial and bioinformatical applications in the border of combinatorial optimization and algebraic geometry. Its main problem is to determine whether a framework (in $\mathbb{R}^d$ for a given dimension d) consisting of joints connected by rigid bars is rigid or not. It is easy to see that, for $d = 1$, the connectivity of the underlying graph is a sufficient and necessary condition of the rigidity of the framework. However, this question turns to be NP-hard for $d \geq 2$. But, if we assume that the joints are in generic positions (see the definition later), the rigidity of such a framework only depends on the underlying graph structure where the joints are the nodes of the graph and the bars are the edges. In 1970, Laman [77] solved this special case for $d = 2$. One of the main open problem in rigidity theory is three dimensional analogue of this question.

With the same thread, lots of interesting graph structures come from rigidity theory. In this thesis, we focus on these structures together with some other related structures from combinatorial optimization.

Now, we give a a brief introduction to the topics investigated in this thesis. We note that we do not introduce matroids although we must use some basic properties of them at some points. However, all properties of matroids used here are part of any basic matroid theory course. We refer to [31, Chapter 5] for more details.

1.1 Connectivity of graphs and digraphs

One of the most studied property of graphs is the connectivity. In this section, we list some notions that measure the connectivity of graphs.

Let $G = (V, E)$ be a graph. For a subset $X \subset V$, $E(X, V - X)$ is the **edge-cut** corresponding to X. We also say that X corresponds to the edge-cut. If $2 \leq |X| \leq |V| - 2$ then the edge-cut is **nontrivial**.

A graph $G = (V, E)$ is k**-edge-connected** if $d_G(X) \geq k$ for every $\emptyset \neq X \subset V$. A digraph $D = (V, A)$ is k**-edge-connected** if $\varrho_D(X) \geq k$ for every $\emptyset \neq X \subset V$. By Menger's theorem, these are equivalent to the following: for any two nodes u and v there are k edge-disjoint paths from u to v where a path in a digraph means a one-way path. A 1-edge-connected digraph is called **strongly connected**. A graph G is **essentially** k**-edge-connected** if all nontrivial edge-cuts of G contain at least k edges. A graph or digraph on at least $k + 1$ nodes is k**-connected** if it remains connected or strongly connected, respectively, after deleting any set of at most $k - 1$ nodes. By Menger's theorem, these are equivalent to the following: for any two nodes u and v there exist k internally node-disjoint paths from u to v.

An undirected graph $G = (V, E)$ is called (k, ℓ)**-partition-connected** if

$$e_G(\mathcal{P}) \geq k(|\mathcal{P}| - 1) + \ell$$

for every partition $\mathcal{P}$ of V, where $e_G(\mathcal{P})$ denotes the number of edges that are not induced by any set of the partition. A $(k, 0)$-partition-connected graph is called k**-partition-connected**. Note that (k, ℓ)-partition-connectivity coincides with $k+\ell$-connectivity when $\ell \geq k$. $G = (V, E)$ is called (k, ℓ)-tree-connected if $G - F$ contains k edge-disjoint spanning trees for all $F \subseteq E$ with $|F| \leq \ell$. When $\ell = 0, 1$ or 2 a (k, ℓ)-tree-connected graph is also called k**-tree-connected, highly** k**-tree-connected** or **doubly-highly** k**-tree-connected**, respectively. The famous results of Tutte [105] and Nash-Williams [86] imply a connection between these notions, as follows.

Theorem 1.1.1. *A graph $G = (V, E)$ is (k, ℓ)-partition-connected for a positive integer k and a non-negative integer ℓ if and only if G is (k, ℓ)-tree-connected.* □

An arborescence is an oriented tree $T = (V, A)$ where every node $v \in V - r_0$ is reachable on a one-way path from its root $r_0 \in V$. One could ask for a characterization similar to Theorem 1.1.1 for the case where one wants to pack arborescences. This characterization has been found by Edmonds [21] and uses the following connectivity property. For positive integers k and ℓ, a digraph $D = (V, A)$ with a root node $r_0 \in V$ is called r_0**-rooted** (k, ℓ)**-edge-connected** if $\delta_D(X) \geq k$ and $\varrho_D(X) \geq \ell$ for every set $X \subset V$ with $r_0 \in X$. By Menger's theorem, this is equivalent to the following: there are k arc-disjoint paths from

r_0 to every other node and there are ℓ arc-disjoint paths from every node to r_0. A rooted $(k, 0)$-edge-connected digraph is called **rooted k-edge-connected**. Edmonds' result is the following.

Theorem 1.1.2 (Edmonds' weak arborescence theorem [21]). *In a digraph $D = (V, A)$, there are k edge-disjoint spanning arborescences with root r_0 if and only if D is r_0-rooted k-edge-connected.* □

We summarize the generalizations of Theorem 1.1.2 and give a common generalization of these results in Chapter 3.

One interesting problem of combinatorial optimization is to orient a given graph (or mixed graph which has both directed and non-directed edges) such that we have some connectivity prescriptions for the output. We present some results on this topic in Chapter 2. We note that such orientation results can be used to prove Theorem 1.1.1 (for $\ell = 0$) from Theorem 1.1.2.

1.2 Rigidity of graphs and frameworks

As the most of the graph structures of this thesis are somehow related to rigidity theory, we give a brief introduction to (combinatorial) rigidity theory. We refer to [48, 108] for more details.

A **d-dimensional framework** is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^d$. We will also refer to (G, p) as a **realization** of G. Two realizations (G, p) and (G, q) of G are **equivalent** if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $uv \in E$. Frameworks (G, p) and (G, q) are **congruent** if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $u, v \in V$.

We say that (G, p) is **globally rigid** in $\mathbb{R}^d$ if every d-dimensional framework which is equivalent to (G, p) is also congruent to (G, p). A framework (G, p) is **rigid** if there exists an $\varepsilon > 0$ such that, if (G, q) is equivalent to (G, p) and $||p(u) - q(u)|| < \varepsilon$ for all $v \in V$, then (G, q) is congruent to (G, p).

Intuitively, we think of a d-dimensional framework (G, p) as a collection of bars and joints where the image of the node set correspond to joints and each edge to a rigid bar (with fixed length) between its end-points. In this model, the framework is globally rigid if its bar lengths determine its realization up to congruence, and it is rigid if it has no non-trivial continuous deformations where the bar lengths are preserved.

We assign to (G, p) a matrix, called a **rigidity matrix** $R(G, p) \in \mathbb{R}^{|E| \times d|V|}$ that is defined as follows. We assign a row of $R(G, p)$ for each edge $uv \in E$ and d columns for each $v \in V$. Let the row of $R(G, p)$ assigned to $uv \in E$ consists of 0 elements except on the d columns assigned to u and on the d columns assigned to v where we write the

d coordinates of $p(u) - p(v)$ and $p(v) - p(u)$, respectively. The linear matroid $\mathcal{R}(G, p)$ on the rows of $R(G, p)$ is the **(d-dimensional) rigidity matroid of the framework** (G, p).

An **infinitesimal motion** of a bar-joint framework (G, p) is an assignment $m : V \to \mathbb{R}^d$ of infinitesimal velocities to the nodes, such that

$$\langle p(u) - p(v), m(u) - m(v)\rangle = 0 \text{ for all edges } uv \in E, \tag{1.1}$$

that is, $R(G, p)\, m = 0$. An infinitesimal motion is **trivial** if it can be obtained as the derivative of a rigid congruence of all of $\mathbb{R}^d$ restricted to the nodes of (G, p), that is, if it is a map for which $m(v) = S\, p(v) + t$ holds for all $v \in V$, for some $d \times d$ skew-symmetric matrix S and some $t \in \mathbb{R}^d$. It is easy to see that these are indeed infinitesimal motions. (G, p) is **infinitesimally rigid** in $\mathbb{R}^d$ if all of its infinitesimal motions are trivial. Infinitesimal rigidity of (G, p) follows its rigidity but the other direction is not true (see Figure 1.1 for an example). We also note that the dimension of the vector space of the trivial infinitesimal

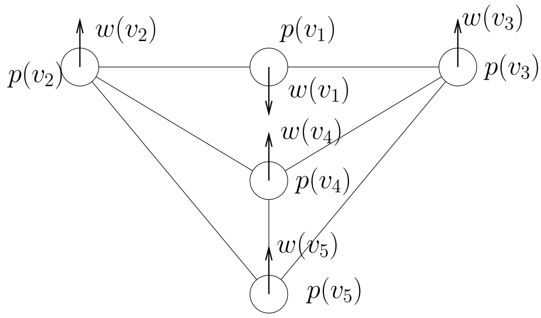


Figure 1.1: A rigid framework with a non-trivial infinitesimal motion w.

motions of a d-dimensional framework is $\binom{d+1}{2}$. Hence the dimension of the vector space of infinitesimal motions is at least $\binom{d+1}{2}$, that is, $\dim(\mathrm{Ker}(R(G, p))) \geq \binom{d+1}{2}$. Therefore, (G, p) is infinitesimally rigid if and only if $\mathrm{rank}(R(G, p)) = d|V| - \binom{d+1}{2}$.

It is easy to see that, for $d = 1$, the connectivity of G is a sufficient and necessary condition of the rigidity of any realization (G, p). However, it is usually a hard problem to decide if a given framework is rigid or globally rigid: Saxe [92] showed that it is NP-hard to decide whether even a 1-dimensional framework is globally rigid and Abbot [1] showed that the rigidity problem is NP-hard for 2-dimensional frameworks. These problems become more tractable, however, if we consider **generic** realizations in which the elements of the set $\{p(v_i)_j : i = 1, ..., |V|,\ j = 1, ..., d\}$ are algebraically independent over $\mathbb{Q}$.

If p_0 is generic, then it follows by the definition of generic realizations that

$$\mathrm{rank}(R(G', p_0)) = \max\{\mathrm{rank}(R(G', p)) : p : V \to \mathbb{R}^d\}$$

for any subgraph G' of G. It is known that a generic framework is rigid if and only if it is infinitesimally rigid. Thus the rigidity of frameworks in $\mathbb{R}^d$ is a generic property, that is, the rigidity of (G, p) depends only on the graph G and not the particular realization p, if (G, p) is generic (see [108]). We say that the graph G is **rigid** in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is rigid. $G = (V, E)$ is said to be **minimally rigid** if $G - e$ is not rigid for any $e \in E$. The d**-dimensional rigidity matroid** $\mathcal{R}_d(G)$ of a graph G is the rigidity matroid of any generic realization (G, p) of G. (We usually consider this matriod such that its ground set is the edge set of the graph by using the bijection between the rows of the rigidity matrix and the edges.) A circuit (that is, a minimal dependent set) of $\mathcal{R}_d(G)$ is called an **M-circuit** of G. We call an edge of G an **M-bridge** if the deletion of e reduces the rank of $\mathcal{R}_d(G)$, that is, $r(\mathcal{R}_d(G - e)) = r(\mathcal{R}_d(G)) - 1$.

The characterization of the rigidity of a graph is rigid in $\mathbb{R}^d$ has been solved for $d = 1, 2$, and is a major open problem for $d \geq 3$. As we have noticed before, a graph is rigid on the line if and only if it is connected. On the plane, the following characterization was given by Laman [77].

Theorem 1.2.1 ([77])**.** *A graph $G = (V, E)$ is minimally rigid in $\mathbb{R}^2$ if and only if*
(i) $|E| = 2|V| - 3$,
(ii) $i(X) \leq 2|X| - 3$ *for every* $X \subseteq V$ *with* $|X| \geq 2$. □

For higher dimensions the problem of characterizing rigidity is still open, and, for $d = 3$, it is one of the main open questions in rigidity theory. It is easy to deduce from our previous observations for the rigidity matrix that some Laman-type conditions are necessary for minimal rigidity however these conditions are not sufficient (see Figure 1.2 for a 3-dimensional counterexample).

Theorem 1.2.2 ([108])**.** *Let $G = (V, E)$ be minimally rigid in $\mathbb{R}^d$ with $|V| \geq d$. Then*
(i) $|E| = d|V| - \binom{d+1}{2}$,
(ii) $i(X) \leq d|X| - \binom{d+1}{2}$ *for every* $X \subseteq V$ *with* $|X| \geq d$. □

Before we turn to the investigation of global rigidity, we note that rigidity is not always a generic property. For example, the rigidity of *tensegrity frameworks* (see the definition in Chapter 8) is not a generic property in the sense that there are *tensegrity graphs* which have rigid generic realizations but not all of their generic realizations are rigid.

Gortler, Healy and Thurston [47] proved that the global rigidity of d-dimensional frameworks is a generic property for all $d \geq 1$. We say that a graph G is **globally rigid** in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is globally rigid. We note that, contrary to (infinitesimal) rigidity, it is not enough to have a globally rigid realization for the global rigidity of a graph (for example, C_4 is not globally rigid in $\mathbb{R}^2$,

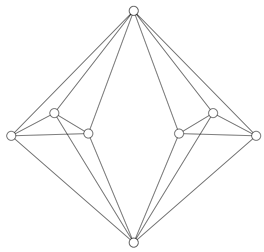


Figure 1.2: The so called **double banana** graph holds both conditions of Theorem 1.2.2 but is not rigid in $\mathbb{R}^3$.

but its realization with $p(v_i) = (0, i)$ $(i = 1, \dots, 4)$ is globally rigid). However, Connelly and Whiteley [17, Corollary 14] proved the following.

Theorem 1.2.3 ([17])**.** *If a graph G has an globally rigid realization (G, p) in $\mathbb{R}^d$ such that (G, p) is also infinitesimally rigid, then G is globally rigid in $\mathbb{R}^d$.* □

Hendrickson [52] proved two key necessary conditions for the global rigidity of a graph. We say that G is **redundantly rigid in** $\mathbb{R}^d$ if removing any edge of G results in a rigid graph.

Theorem 1.2.4 ([52])**.** *Let G be a globally rigid graph in $\mathbb{R}^d$. Then either G is a complete graph on at most $d+1$ nodes, or G is*
(i) $(d+1)$-connected, and
(ii) redundantly rigid in $\mathbb{R}^d$. □

Hendrickson conjectured that the necessary conditions of Theorem 1.2.4 are also sufficient to imply the global rigidity of the graph in $\mathbb{R}^d$. It is indeed so for $d = 1, 2$: It is not hard to verify that a 1-dimensional generic framework (G, p) is globally rigid if and only if either G is K_2 or G is 2-connected. For $d = 2$, the characterization was given by Jackson and Jordán [57], as follows.

Theorem 1.2.5 ([57])**.** *Let G be a graph. Then G is globally rigid in $\mathbb{R}^2$ if and only if either G is a complete graph on two or three nodes, or G is 3-connected and redundantly rigid in $\mathbb{R}^2$.* □

However, there exist counterexamples to Hendrickson's conjecture for $d \geq 3$ (see [11]). Connelly [14, 17] conjectured that only the complete bipartite graph $K_{5,5}$ is counterexample for the 3-dimensional version of this conjecture. In Chapter 7, we disprove this conjecture by showing infinitely many counterexamples for every dimension $d \geq 3$.

1.2.1 Special classes of graphs

Although the d-dimensional rigidity and global rigidity of graphs are not characterized for $d \geq 3$, the characterization has been found for several important graph classes.

Body-bar frameworks and graphs

Body-bar frameworks consist of full-dimensional rigid bodies connected by disjoint bars. The bodies are free to move continuously in $\mathbb{R}^d$ subject to the constraint that the distance between their points connected by bars must be constant. In the underlying (multi)graph of the framework, the nodes correspond to the bodies and the edges correspond to the bars. We can obtain an equivalent bar-joint framework by replacing each body by a bar-joint realization of a large enough complete graph such that the bars of the body-bar framework are represented by disjoint edges between these large complete graphs. The graph of such a bar-joint framework is a body-bar graph. More precisely, for a graph $H = (V, E_B)$, the (d-dimensional) **body-bar graph induced by** H, denoted by G_H^{BB}, is obtained from H by replacing each node $v \in V$ by a complete graph $B(v)$ (the body of v) on $d_H(v) + d + 1$ nodes, in which $d + 1$ nodes inducing a K_{d+1} form the **core** $C(v)$ of the body and the remaining nodes are the end of the bars connecting the bodies. For each edge $e = uv$ of H, we add an edge to G_H^{BB} between $B(u) - C(u)$ and $B(v) - C(v)$ such that these edges form a (perfect) matching on $\bigcup_{v \in V} B(v) - C(v)$. The bodies are pairwise disjoint. (See Figure 1.3.)

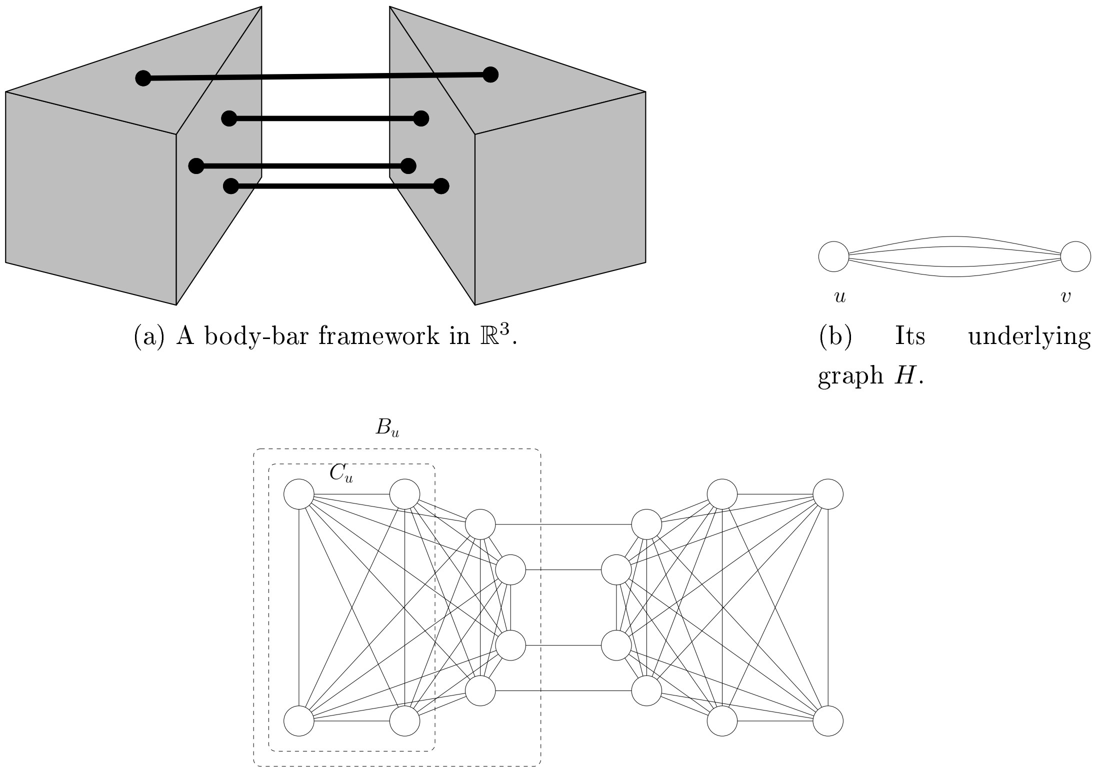
Tay [100, 102] provided a characterization of rigid d-dimensional body-bar graphs, as follows.

Theorem 1.2.6 ([100, 102]). *Let $H = (V, E)$ be a graph. Then the body-bar graph G_H^{BB} is rigid in $\mathbb{R}^d$ if and only if H is $\binom{d+1}{2}$-tree-connected.* □

It is not hard to get from Theorem 1.2.6 that G_H^{BB} is redundantly rigid if and only if H is highly $\binom{d+1}{2}$-tree-connected and in this case the $d+1$-connectivity of G_H^{BB} also follows. Therefore, the following result of Connelly, Jordán, and Whiteley [15] – that gives a combinatorial characterization of globally rigid body-bar graphs in $\mathbb{R}^d$ with high $\binom{d+1}{2}$-tree-connectivity of the underlying graph H – shows that Hendrickson's conjecture is true for body-bar graphs.

Theorem 1.2.7 ([15]). *Let $H = (V, E)$ be a graph. Then the body-bar graph G_H^{BB} is globally rigid in $\mathbb{R}^d$ if and only if H is highly $\binom{d+1}{2}$-tree-connected.* □

It is known that testing high k-tree-connectivity is polynomial (see Frank [28]). In Chapter 2, we also give a simple and fast algorithm for this problem.



(a) A body-bar framework in $\mathbb{R}^3$.

(b) Its underlying graph H.

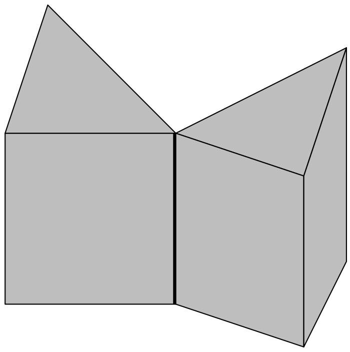
(c) The body-hinge graph G_H^{BB} of H.

Figure 1.3: Body-bar frameworks and graphs.

Body-hinge frameworks and graphs

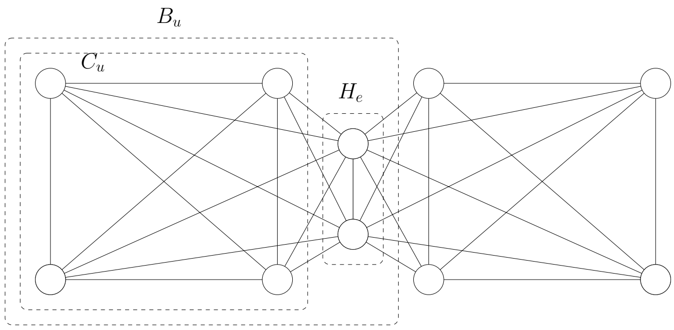
A d-dimensional **body-hinge framework** is a structural model consisting of rigid bodies and hinges. Each hinge is a $(d-2)$-dimensional affine subspace that joins some pair of bodies. The bodies are free to move continuously in $\mathbb{R}^d$ subject to the constraint that the relative motion of any two bodies joined by a hinge is a rotation about the hinge. The framework is rigid if every such motion preserves the distances between all pairs of points belonging to different rigid bodies, that is, the motion extends to an isometry of $\mathbb{R}^d$. In the underlying graph of the framework, the nodes correspond to the bodies and the edges correspond to the hinges. We can obtain an equivalent bar-joint framework by replacing each body by a bar-joint realization of a large enough complete graph in such a way that two bodies joined by a hinge share $d-1$ joints. The graph of such a bar-joint framework is a body-hinge graph. More precisely, for a graph $H = (V, E_H)$, the (d-dimensional) **body-hinge graph induced by** H, denoted by G_H^{BH}, is obtained from H by replacing each node $v \in V$ by a complete graph $B(v)$ (the body of v) on $(d-1)d_H(v) + d + 1$ nodes, in which $d+1$ nodes inducing a K_{d+1} form the **core** $C(v)$ of the body and the remaining nodes are partitioned into sets of $d-1$ nodes so that each set is assigned to one edge incident with v. For each edge $e = uv$ of H, the bodies $B(u)$

and $B(v)$ share the $d-1$ nodes assigned to e in these bodies. This set of $d-1$ nodes assigned to e, denoted by $H(e)$, is a **hinge** between the corresponding bodies. The cores of the bodies are pairwise disjoint. (See Figure 1.4.)



(a) A body-hinge framework in $\mathbb{R}^3$.

(b) Its underlying graph H.



(c) The body-hinge graph G_H^{BH} of H.

Figure 1.4: Body-hinge frameworks and graphs.

Body-hinge frameworks (and body-hinge graphs) are extensively studied objects in rigidity theory with various applications. Among others, they can be used to investigate the flexibility of molecules, due to the fact that molecular conformations can be modeled by body-hinge frameworks with certain additional geometric constraints, see [67, 108]. Tay [101, 103] and Whiteley [107] characterized rigid d-dimensional body-hinge graphs in terms of their underlying graphs.

Theorem 1.2.8 ([101, 103], [107])**.** *Let $H=(V,E)$ be a graph. Then the body-hinge graph G_H^{BH} is rigid in $\mathbb{R}^d$ if and only if $\left(\binom{d+1}{2}-1\right)H$ is $\binom{d+1}{2}$-tree-connected.* □

Connelly, Jordán and Whiteley [15] conjectured a sufficient condition for the global rigidity of body-hinge graphs. We give an affirmative answer to their conjecture in Chapter 7. Furthermore, we show that the conjectured sufficient condition is also necessary.

Square graphs

In a molecule, not only the distances between the atoms connected by chemical bounds are determined, but also the angle between the chemical bounds. We can add this condition to our graph by taking its square, that is, a molecule represented by a graph G (in which the nodes represent the atoms and the edges represent the chemical bounds) is (generically) rigid (or globally rigid, respectively) if and only if its square G^2 is rigid (or globally rigid, respectively). Square graphs are similar to 3-dimensional body-hinge graphs with underlying graph G but the hinges incident a body go through the same point. However, the rigidity of G^2 can be characterized similar as the rigidity of the body-hinge graph of G. This was conjectured by Whiteley [110] and was proven recently by Katoh and Tanigawa [67].

Theorem 1.2.9 ([67]). *Let G be a graph with minimum degree 2. Then G^2 is rigid in R^3 if and only if $5G$ is 6-tree-connected.* □

Connelly, Jordán and Whiteley [15] conjectured that G^2 is globally if $5G$ is highly 6-tree-connected and G has no cycles of length ≤ 5. However, we can disprove this conjecture easily by using Theorem 1.2.4. If $5G$ is highly 6-tree-connected and G has no cycles of length ≤ 5 and v is a node in G of degree 2, then the conditions of the conjecture remain valid for the graph G' that we get by gluing two copies of G at v but G'^2 is not 4-connected (see Figure 1.5).

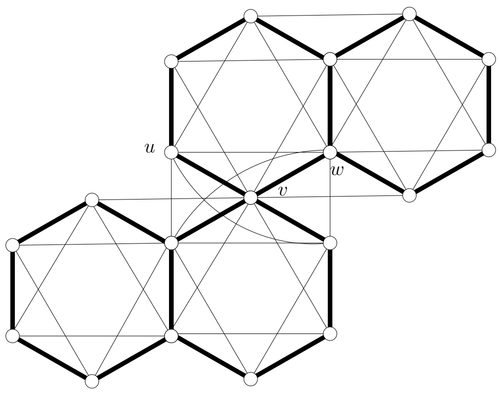


Figure 1.5: A counterexample for the conjecture of [15] on the global rigidity of square graphs. The square graph of the bold graph G is not connected after removing u, v and w, but $5G$ is highly 6-tree-connected.

Body-bar-and-hinge frameworks and graphs

A d-dimensional **body-bar-and-hinge framework** is consisting of rigid bodies connected by bars and hinges. The motions of such a framework can be defined analogously to body-bar and body-hinge frameworks. Now the edge set of the underlying graph $H = (V; E_B, E_H)$ is partitioned into two sets E_B and E_H so that the nodes correspond to the bodies and the edges correspond to the hinges. (In what follows, we shall call a partition of the edge set of a graph into k classes a **k-labeling** and say that the graph is **k-edge-labeled**. Thus H is 2-edge-labeled.) We can obtain an equivalent bar-joint framework by replacing each body by a bar-joint realization of a large enough complete graph, by combining the ideas of body-bar and body-hinge graphs. More precisely, for a 2-edge-labeled graph $H = (V; E_B, E_H)$, the (d-dimensional) **body-bar-and-hinge graph induced by** H, denoted by G_H^{BBH}, is obtained from H by replacing each node $v \in V$ by a complete graph $B(v)$ (the body of v) on $d_{E_B}(v)+(d-1)d_{E_H}(v)+d+1$ nodes, in which $d+1$ nodes inducing a K_{d+1} form the **core** $C(v)$ of the body, $d_{E_B}(v)$ nodes are the endpoints of the bars incident to that body and are denoted by $R(v)$, and the remaining nodes are partitioned into sets of $d-1$ nodes so that each set is assigned to one E_H-edge incident to v. For each edge $e = uv \in E_B$, we add an edge to G_H^{BBH} between $R(u)$ and $R(v)$ such that these edges form a (perfect) matching on $\bigcup_{v \in V} R(v)$. For each edge $e = uv \in E_H$, the bodies $B(u)$ and $B(v)$ share the $d-1$ nodes assigned to e in these bodies. This set of $d-1$ nodes assigned to e, denoted by $H(e)$, is a **hinge** between the corresponding bodies. The sets $C(v), R(v)$ $(v \in V)$ and $H(e)$ $(e \in E_H)$ are pairwise disjoint.

Jackson and Jordán [55] characterized the rigidity of body-bar-and-hinge frameworks, as follows.

Theorem 1.2.10 ([55]). *Let $H = (V; E_B, E_H)$ be a 2-edge-labeled graph. The body-bar-and-hinge graph G_H^{BBH} of H is rigid in $\mathbb{R}^d$ if and only if $H' = \left(V, E_B \cup \left(\binom{d+1}{2} - 1\right) E_H\right)$ is $\binom{d+1}{2}$-tree-connected.* □

We will characterize global rigidity of body-bar-and-hinge graphs in Chapter 7.

1.2.2 Node- and edge-redundantly rigid graphs

If one wants to consider not only rigid but somehow safe frameworks, it is obvious to define some notions similar to k-connectivity and k-edge-connectivity. A graph $G = (V, E)$ is called **k-rigid** in $\mathbb{R}^d$, or simply **$[k, d]$-rigid**, if $|V| \geq k+1$ and, for any $U \subseteq V$ with $|U| \leq k-1$, the graph $G - U$ is rigid in $\mathbb{R}^d$. In this context, we will call graphs that are rigid in $\mathbb{R}^d$ $[1, d]$-rigid. A graph $G = (V, E)$ is called **k-edge-rigid** in $\mathbb{R}^d$, or simply **$[k, d]$-edge-rigid**, if $|V| \geq k+1$ and, for any $F \subseteq E$ with $|F| \leq k-1$, the graph $G - F$ is rigid in $\mathbb{R}^d$. In this context, we will call graphs that are redundantly rigid in $\mathbb{R}^d$ $[2, d]$-edge-rigid. Similarly, one can define **globally** $[k, d]$**-rigid graphs** and **globally** $[k, d]$**-edge-rigid graphs** by substituting globally rigid instead of rigid in the definitions. Every $[k, d]$-rigid graph is $[\ell, d]$-rigid by definition for $1 \leq l \leq k$. We remark that another equivalent definition of $[k, d]$-rigidity is also used. By this equivalent definition a graph is $[k,d]$-rigid if $|V| \geq k+1$ and the deletion of any set of $k-1$ nodes results in a graph that is rigid in $\mathbb{R}^d$. The following observation [111] shows the equivalence of these two definitions.

Lemma 1.2.11 ([111])**.** *A graph $G = (V, E)$, with $|V| \geq k+1$, is $[k,d]$-rigid if and only if the deletion of any set of $k-1$ nodes results in a graph that is rigid in $\mathbb{R}^d$.* □

One can observe a similar statement for $[k,d]$-edge-rigidity, global $[k,d]$-rigidity and global $[k,d]$-edge-rigidity. In this thesis, both definitions will be used.

G is called **minimally** $[k,d]$**-rigid** if it is $[k,d]$-rigid but $G-e$ fails to be $[k,d]$-rigid for every $e \in E$. G is said to be **strongly minimally** $[k,d]$**-rigid** if it is minimally $[k,d]$-rigid and there is no (minimally) $[k,d]$-rigid graph on the same node set with less edges. If G is minimally $[k,d]$-rigid but not strongly minimally $[k,d]$-rigid, then it is called **weakly minimally** $[k,d]$**-rigid**. Similarly, one can define **weakly/strongly minimally globally** $[k,d]$**-rigid** graphs, and so on.

The investigation of $[k,d]$-rigid graphs was commenced in the plane by B. Servatius [94] and was continued recently in higher dimensions and also for global $[k,d]$-rigidity by Anderson, Montevallian, Summers and Yu [80, 81, 82, 96, 97] motivated by multi-agent formations and sensor networks. In Chapter 6, we continue this work by proving upper and lower bounds on the edge number of minimally (globally) $[k,d]$-rigid graphs.

A recent result of Tanigawa [99] shows an interesting property of $[2,d]$-rigid graphs and gives a useful tool to investigate global rigidity.

Theorem 1.2.12 ([99])**.** *Let $G = (V, E)$ be a graph and let $x \in V$. Suppose that $G-x$ is rigid in $\mathbb{R}^d$ and $G - x + K(N_G(x))$ is globally rigid in $\mathbb{R}^d$. Then G is globally rigid in $\mathbb{R}^d$.* □

Theorem 1.2.13 ([99])**.** *If G is 2-rigid in $\mathbb{R}^d$, then it is globally rigid in $\mathbb{R}^d$.* □

We note that the converse direction is not true, as the **wheel graph** $C_n * v$ (that is, the cone graph of a cycle) is globally rigid in $\mathbb{R}^2$ but C_n is not rigid for any $n \geq 4$ in R^2. The following observation follows inductively from Theorem 1.2.13.

Corollary 1.2.14. *If G is k-rigid in $\mathbb{R}^d$ then it is globally $(k-1)$-rigid in $\mathbb{R}^d$.* □

1.2.3 Inductive techniques on rigid and globally rigid graphs

Constructive characterizations are useful tools in combinatorial rigidity. Even though we do not have a constructive characterization theorem for all of the classes of rigid and globally rigid graphs it can be very useful to find operations that preserve rigidity. In this section we mention some of these operations.

The d-**dimensional Henneberg-0 extension**, or simply **0-extension**, on G adds a new node and connects it to d distinct nodes of G. The d-**dimensional Henneberg-1-extension**, or simply **1-extension**, deletes an edge $uw \in E$, adds a new node v and connects it to u, v and $d-1$ other nodes of G. The d-dimensional 0-extension is also called d-**valent node addition** and the d-dimensional 1-extension is also called $d+1$-**valent edge split**.

Theorem 1.2.15 ([104]). *If G is rigid in $\mathbb{R}^d$ and G' is obtained from G by a d-dimensional 0-extension or 1-extension, then G' is rigid in $\mathbb{R}^d$.* □

As a 0-extension adds a node of degree d to the graph, it does not preserve global rigidity. However, Connelly [13] observed that 1-extension preserves global rigidity.

Theorem 1.2.16 ([13]). *If $G = (V, E)$ is globally rigid in $\mathbb{R}^d$ with $|V| \geq d+2$ and G' is obtained from G by a d-dimensional 1-extension, then G' is rigid in $\mathbb{R}^d$.* □

It is well known that a graph is minimally rigid graph in $\mathbb{R}^2$ if and only if it can be built up from an edge by a sequence of 0- and 1-extensions. There is no similar constructive characterization result for minimally rigid graphs in higher dimensions. Still, there are some operations that are known to preserve rigidity in higher dimensions. An example for such an operation is the following that we call a **(**d**-dimensional) simplex-based X-replacement**. Let $d \geq 2$ and let $a, b, w_1, \ldots, w_{d-2}$ be a complete subgraph of G and $cd \in E$ an edge which is node-disjoint from the simplex. The d-dimensional simplex-based X-replacement deletes ab, cd, adds a new node v and connects it to $a, b, c, d, w_1, \ldots, w_{d-2}$. A 3-dimensional simplex-based X-replacement is also called a **triangle-based X-replacement**. The following lemma shows that this operation preserves rigidity (for a proof see [65]).

Lemma 1.2.17. *Let G be rigid in $\mathbb{R}^d$ and let G' be the result of a d-dimensional simplex-based X-replacement applied to G. Then G' is rigid in $\mathbb{R}^d$.* □

We shall also mention the well-known node splitting operation. Let G be a graph, let $v_1 \in V$, let $v_1v_2, ..., v_1v_d$ be $d-1$ designated edges incident with v_1, and let $v_1v_{d+1}, ..., v_1v_{d+k_1}$ and $v_1v_{d+k_1+1}, ..., v_1v_{d+k_1+k_2}$ be a bipartition of the remaining edges incident with v_1. The **(**d**-dimensional) node splitting** operation at v_1 removes the edges $v_1v_{d+1}, ..., v_1v_{d+k_1}$, adds a new node v_0, and adds the new edges $v_0v_1, v_0v_2, ..., v_0v_d, v_0v_{d+1}, ..., v_0v_{d+k_1}$. Whiteley [?] proved that this operation preserves the rigidity of graphs in $\mathbb{R}^d$.

Theorem 1.2.18 ([?]). *Let G be a rigid graph in $\mathbb{R}^d$ and let G' be the result of a d-dimensional node spitting applied to G. Then G' is rigid in $\mathbb{R}^d$.*

The node splitting operation is **non-trivial** if $k_1 \geq 1$ and $k_2 \geq 1$ hold. The new edge v_0v_1 is called the **bridging edge** in the resulting graph. The following result is due to Connelly [14, Section 11].

Theorem 1.2.19 ([14]). *Let G be globally rigid in $\mathbb{R}^d$ and let G' be obtained from G by a non-trivial node splitting operation. If the bridging edge e is redundant (i.e. $G' - e$ is rigid in $\mathbb{R}^d$) then G' is also globally rigid in $\mathbb{R}^d$.*

We shall also use another type of operation that not only preserves the rigidity or global rigidity of a graph but augments it for higher dimension. This operation is the **coning**. We recall that the cone graph $G * v$ of G arises from G by adding a new node v and edges vu for every $u \in V$. The following two results of Whiteley [106], and Connelly and Whiteley [17] shows how coning can be used in the investigation of rigid and globally rigid graphs.

Theorem 1.2.20 ([106]). *A graph G is rigid in $\mathbb{R}^d$ if and only if the cone graph $G * v$ is rigid in $\mathbb{R}^{d+1}$.* □

Theorem 1.2.21 ([17]). *A graph G is globally rigid in $\mathbb{R}^d$ if and only if the cone graph $G * v$ is globally rigid in $\mathbb{R}^{d+1}$.* □

1.3 Sparse graphs

A graph $G = (V, E)$ is called (k, ℓ)**-sparse** if $i_G(X) \leq k|X| - \ell$ for all $X \subseteq V$ with $|X| \geq 2$, where k and ℓ are integers with $k > 0$ and $\ell < 2k$. A (k, ℓ)-sparse graph is called (k, ℓ)**-tight** if $|E| = k|V| - \ell$. A well-known class of (k, ℓ)-sparse graphs are the (k, k)-sparse graphs that are exactly the graphs that can be partitioned into k forests by the famous result of Nash-Williams [85]. Thus (k, k)-tight graphs are the minimal k-tree-connected graphs. Sparsity properties are also important in rigidity theory as they can be used in the characterization of many rigidity classes. By Theorem 1.2.6 and by or previous observation, a graph is the underlying graph of a generically minimally rigid body-bar framework if and only if it is $\left(\binom{d+1}{2}, \binom{d+1}{2}\right)$-tight. By Laman's theorem (Theorem 1.2.1), a graph is $(2, 3)$-tight if and only if it is minimally rigid in $\mathbb{R}^2$. ($(2, 3)$-tight graphs are also called **Laman graphs**). Following the appellations of rigidity theory, we call G (k, ℓ)**-rigid** if G has a (k, ℓ)-tight spanning subgraph. We will call an edge e of G a (k, ℓ)**-redundant edge** if $G - e$ is (k, ℓ)-rigid. A graph G will be called a (k, ℓ)**-redundant graph** if each edge of G is (k, ℓ)-redundant.

Sparsity is also defined in a more general way that we will also need later. Let $G = (V, E)$ be a graph and $m : V \to \mathbb{Z}_+$ be a function with $\ell < m(u) + m(v)$ for every edge $uv \in E$. A graph $G = (V, E)$ is called $(\mathbf{m}, \ell)$**-sparse** if $i_G(X) \leq \widetilde{m}(X) - \ell$ for all $X \subseteq V$

with $|X| \geq 2$. An $(\mathbf{m}, \ell)$-sparse graph is called **$(\mathbf{m}, \ell)$-tight** if $|E| = \widetilde{m}(V) - \ell$. A graph G is called **$(\mathbf{m}, \ell)$-rigid** if G has an $(\mathbf{m}, \ell)$-tight spanning subgraph. We will call an edge e of G an **$(\mathbf{m}, \ell)$-redundant edge** if $G - e$ is $(\mathbf{m}, \ell)$-rigid. A graph G will be called an **$(\mathbf{m}, \ell)$-redundant graph** if each edge of G is $(\mathbf{m}, \ell)$-redundant.

It is known that the edge sets of the $(\mathbf{m}, \ell)$-sparse subgraphs of a given graph form a matroid, called the **$(\mathbf{m}, \ell)$-sparsity matroid** or **count matroid** (see [31, 108]). A circuit of this matroid is called an **$(\mathbf{m}, \ell)$-circuit**. Here, the notion of $(\mathbf{m}, \ell)$-circuit will be used for graphs and not only for edge sets. Following [5], we call a $(2, 3)$-circuit a **generic circuit**. (The term is based on the fact that the $(2, 3)$-sparsity matroid is isomorphic to the 2-dimensional (generic) rigidity matroid.)

It is well-known that $(\mathbf{m}, \ell)$-sparsity can be tested in polynomial time (see [31, Section 13.5.4] for a survey). (This algorithm is also called **pebble game** algorithm [78] when $\ell \geq 0$.) The algorithm can also determine the rank of any edge set in the $(\mathbf{m}, \ell)$-sparsity matroid and output a maximum independent set.

1.4 Some other notions from combinatorial optimization

In this section, we collect some other related definitions used in combinatorial optimization. (These notions will be used only in Chapters 2 and 3.) We also list some well-known basic theorems. For more details, we refer to [31].

Sets and set families

Two subsets A and $B \subseteq V$ are **crossing** if $A - B \neq \emptyset$, $B - A \neq \emptyset$, $A \cap B \neq \emptyset$, $V - (A \cup B) \neq \emptyset$. A and $B \subseteq V$ are **intersecting** if $A - B \neq \emptyset$, $B - A \neq \emptyset$ and $A \cap B \neq \emptyset$. A family $\mathcal{F}$ of subsets of V is an **intersecting** or **crossing family** respectively if $A \cap B \in \mathcal{F}$ and $A \cup B \in \mathcal{F}$ for any two intersecting or crossing members $A, B \in \mathcal{F}$, respectively. A family $\mathcal{F}$ is called **laminar** if there are no intersecting pairs in $\mathcal{F}$. A family $\mathcal{F}$ is called **cross-free** if there are no crossing pairs in $\mathcal{F}$.

Let $T = (U, A)$ be a directed tree. For an edge $e \in A$, let U_e denote the component of $T - e$ that contains the head of e. Edmonds and Giles [22] proved the following representation of cross-free families:

Lemma 1.4.1 ([22]). *For every cross-free family $\mathcal{F}$ on a ground set V, there exists a directed tree $T = (U, A)$ and a map $\varphi : V \to U$ so that the sets in $\mathcal{F}$ and the edges of T are in a one-to-one correspondence, as follows. For every edge $e \in A$, the corresponding set $\mathcal{F}$ is $\varphi^{-1}(U_e)$.* □

We denote the edge of the tree representing a set F by e_F.

Remark 1.4.2.
1. It is easy to see that for $F \in \mathcal{F}$ a representing tree of $\mathcal{F} - \{F\}$ is T/e_F with the map φ' that arises from φ by combining it with the natural identifying map $\pi : U \to U/e_F$.

2. It can be proved by induction that a directed tree $F = (U, A)$ admits a **level function** $\pi : U \to \mathbb{Z}_+$ so that $\pi(v) - \pi(u) = 1$ for every $uv \in A$. If (T, φ) is a tree-representation of a cross-free family $\mathcal{F}$ on the ground set V and π is a level function of T, then one can get $\pi(\varphi(v)) - \pi(\varphi(v')) = d_{\mathcal{F}}(v) - d_{\mathcal{F}}(v')$ for every $v, v' \in V$ after proving the statement when $\varphi(v)\varphi(v') \in A$.

A family $\mathcal{K}$ on the ground set V is a **composition of** $\emptyset \neq X \subseteq V$ if there is an integer $\Delta \in \mathbb{Z}_+$ for which every element of X is contained in exactly $\Delta + 1$ members of $\mathcal{K}$ and every element of $V - X$ is contained in exactly Δ members of $\mathcal{K}$. Note that a composition of V is a regular hypergraph while a composition of a proper subset $\emptyset \neq Z \subset V$ becomes a regular hypergraph by adding $V - Z$ to it. The integer Δ is called the **ground-degree of** $\mathcal{K}$ and is denoted by $\Delta(\mathcal{K})$. Thus the ground-degree of an r-regular hypergraph is $r - 1$. Observe that the difference between the ground-degree and the maximum degree of a composition is 1.

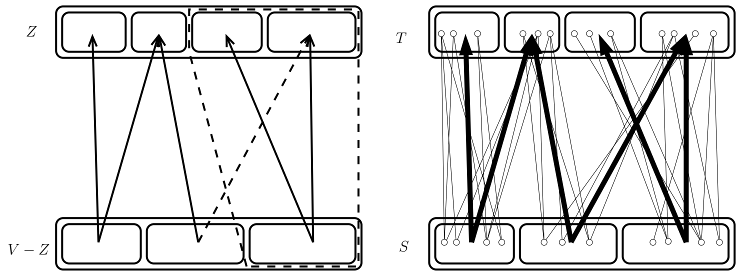
Special compositions are the following. A **partition** of $Z \subseteq V$ is a family formed by disjoint sets $Z_1, Z_2, ..., Z_t$ with $\bigcup_{i=1}^{t} Z_i = Z$. We call a partition of any subset of V a **subpartition** of V. If $\{V_1, V_2, ..., V_t\}$ is a partition of V, then $\{V - V_1, V - V_2, ..., V - V_t\}$ is called a **co-partition** of V. Lemma 1.4.1 and Remark 1.4.2 imply the following observation [35]:

Lemma 1.4.3. *Let $\mathcal{K}$ be a cross-free composition of its ground set V. Then $\mathcal{K}$ can be partitioned into partitions and co-partitions of V.* □

Let $\{Z_1, Z_2, ..., Z_t\}$ be a partition of $Z \subseteq V$, and let $\{Z_i^1, Z_i^2, ..., Z_i^{t_i}\}$ be a partition of $V - Z_i$ $(i = 1, ..., t)$. Then the set-system $\mathcal{D} := \{V - Z_i^j : 1 \leq i \leq t, 1 \leq j \leq t_i\}$ is called a **double-partition of** Z. This $\mathcal{D}$ is a composition of Z with ground-degree $\sum_{i=1}^{t}(t_i - 1)$. If each $t_i = 1$, then $\mathcal{D}$ is a partition of Z.

For $\emptyset \neq Z \subset V$, a cross-free double-partition $\mathcal{T}$ of Z on the ground set V is called a **tree-composition of** Z if $u \in \varphi(V - Z)$, $v \in \varphi(Z)$ for each edge uv of a representing directed tree of $\mathcal{T}$. By Remark 1.4.2.2 and the definition of double-partitions, $\pi(u) = \pi(u') = \pi(v) + 1$ holds for every $u, u' \in \varphi(Z)$ and $v \in \varphi(V - Z)$ where π denotes the level function of a representing tree (T, φ). Thus $\varphi(Z)$ and $\varphi(V - Z)$ are disjoint. Moreover, $\varphi^{-1}(u) \neq \emptyset$ for any node u of the tree since a tree is connected and each edge of the representing tree connects $\varphi(Z)$ and $\varphi(V - Z)$. Therefore, a tree-representation can be constructed by taking the members a partition of Z and a partition of $V - Z$ as its node set, mapping each node with φ to the set containing it and taking the edges with tails in $\varphi(V-Z)$ and heads in $\varphi(Z)$ (see Figure 1.6a). From now on we will use this tree-representation of a tree-composition. Note that a cross-free family $\mathcal{T}$ with such a tree-representation is always a tree-composition. To prove this, one needs to prove that $\mathcal{T}$ is a double-partition. This follows from the fact that the edges of the tree entering a node $u \in \varphi(z)$ represents the complement of a co-partition of $V - \varphi^{-1}(u)$. We will say that the partitions and co-partitions of the ground set V are the **tree-compositions of** V. While a double-partition may consist of $\Omega(|V|^2)$ elements, a tree-composition always has at most $|V|-1$ elements.

Assume that we are also given a bipartite graph $G=(S,T;E)$ and $V = S \cup T$. We say that a **tree-composition $\mathcal{T}$ of T complies with** G if $\varphi(s)\varphi(t) \in A$ for every edge $st \in E$ with $s \in S,\ t \in T$, where $F=(U,A)$ is a directed tree representing $\mathcal{T}$ with the surjective map $\varphi : (S \cup T) \to U$ (see Figure 1.6b).



(a) The tree-representation of a tree-composition $\mathcal{T}$ of Z where the node set of the tree is formed by a partition of Z and a partition of $V-Z$ and the dashed edge of the tree represents the dashed member of $\mathcal{T}$. (For simpleness the other members of $\mathcal{T}$ are not shown in this figure.)

(b) The thick directed edges give the tree-representation of a tree-composition of T that complies with the bipartite graph formed by the thin edges.

Figure 1.6: Tree-compositions.

Set-functions

Unless otherwise stated, we assume that a set-function is zero on the empty set. For a vector $x \in \mathbb{R}^V$ and $X \subseteq V$, let $\widetilde{x}(X) := \sum_{v\in X} x(v)$. We also use the notation $\widetilde{h}(\mathcal{F}) := \sum_{i=1}^{q} h(F_i)$ for a set-function $h : 2^V \to \mathbb{R}$ and for a family $\mathcal{F} := \{F_1, F_2, ..., F_q\}$ of subsets $F_i \subseteq V$. A set-function $b : 2^V \to \mathbb{R} \cup \{\infty\}$ is **(crossing) submodular** if

$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y)$ for every (crossing) $X, Y \subseteq V$. A set-function $p : 2^V \to \mathbb{R} \cup \{-\infty\}$ is **(crossing) supermodular** if $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$ for every (crossing) $X, Y \subseteq V$. In some cases when it is important to highlight that the function is submodular (respectively supermodular) on every pair of sets, we call it **fully submodular** (respectively **fully supermodular**).

Let b be a set-function on V for which $b(V) < \infty$. The **base-polyhedron** $B(b)$ defined by b is as follows.

$$B(b) := \{x \in \mathbb{R}^V : \ \widetilde{x}(X) \leq b(X) \ (X \subset V), \ \widetilde{x}(V) = b(V)\}.$$

This polyhedron is used mainly when b is a fully (intersecting, crossing) *sub*modular function. For a supermodular function p, a related polyhedron $B'(p)$ is considered:

$$B'(p) := \{x \in \mathbb{R}^V : \ \widetilde{x}(X) \geq p(X) \ (X \subset V), \ \widetilde{x}(V) = p(V)\}.$$

For an arbitrary set-function $h : 2^V \to \mathbb{R} \cup \{+\infty, -\infty\}$ for which $h(V)$ is finite, we define the **complement** $\overline{h}$ of h by the following formula.

$$\overline{h}(X) := h(V) - h(V - X).$$

Obviously, $\overline{h}(\emptyset) = 0$, $\overline{h}(V) = h(V)$ and $\overline{\overline{h}} = h$. The complement p of a (crossing) submodular function b is (crossing) supermodular and $B'(p) = B(b)$. Therefore, the results on (crossing) submodular functions can automatically be transformed into ones on (crossing) supermodular functions by using the complement.

It is known that $B(b)$ is a non-empty integer polyhedron whenever b is fully submodular and integer-valued. A basic theorem of Fujishige [40] characterizes non-emptiness of $B(b)$ for crossing submodular b.

Theorem 1.4.4 ([40])**.** *Let b be a crossing submodular function on the subsets of V for which $b(V)$ is finite. The polyhedron $B(b)$ is non-empty if and only if both*

$$\sum_i b(V_i) \geq b(V) \quad \text{and} \quad \sum_i \overline{b}(V_i) \leq \overline{b}(V) \tag{1.2}$$

hold for every partition $\mathcal{F} = \{V_1, \ldots, V_t\}$ of V. □

There is another important result on crossing submodular functions. If b is a crossing submodular function on the subsets of V and $B(b) \neq \emptyset$, then there exists a fully submodular function $b^{\downarrow}$ for which $B(b) = B(b^{\downarrow})$. This fact appeared implicitly in an algorithm for finding submodular flows confined by crossing submodular functions [29] and was formulated explicitly by Fujishige in [40]. The function $b^{\downarrow}$ is unique by the following theorem (see for example in [20]).

Theorem 1.4.5. *Let b^* be a submodular function on the subsets of V. Then $b^*(Z) = \max\{\widetilde{m}(Z) : m \in B(b^*)\}$ for every $Z \subseteq V$. If b^* is integer-valued, then the maximum is achieved by an integer vector. An analogous statement holds for supermodular functions.* □

This unique submodular function is the **full (lower) truncation of** b. It is also known [30, 38, 40] that if $b(V) = 0$ and $B(b) \neq \emptyset$, then $b^{\downarrow}$ can be expressed by the following formula for every $\emptyset \neq Z \subseteq V$:

$$b^{\downarrow}(Z) = \min\{\widetilde{b}(\mathcal{D}) : \mathcal{D} \text{ a double-partition of } Z\}. \tag{1.3}$$

Similarly, if p is a crossing supermodular function on the subsets of V and $B'(p) \neq \emptyset$, then there exists a unique fully supermodular $p^{\uparrow}$, called the **full (upper) truncation of** p, for which $B'(p) = B'(p^{\uparrow})$. If $B'(p) \neq \emptyset$, then $p^{\uparrow}$ can be expressed by the following formula of [40] for every $\emptyset \neq Z \subseteq V$:

$$p^{\uparrow}(Z) = \max\{\widetilde{p}(\mathcal{D}) - \Delta(\mathcal{D})p(V) : \mathcal{D} \text{ a double-partition of } Z\}.$$

Chapter 2

Tree-compositions and graphs with connected orientations

Several results and algorithms in submodular optimization are based on the following fact [29, 40]. For every crossing submodular function b for which the base polyhedron $B(b)$ is non-empty, there exists a unique fully submodular function $b^{\downarrow}$ for which $B(b^{\downarrow}) = B(b)$. The function $b^{\downarrow}$ is called the full (lower) truncation of b. The following result of Frank [30] provides a relatively simple formula for $b^{\downarrow}$. Here and in some other theorems of this chapter, we assume $b(V) = 0$. The general case can be derived from the case $b(V) = 0$, though the formulas get a bit more involved.

Theorem 2.0.1. *Let b be a crossing submodular set-function on the subsets of a ground set V for which $b(V) = 0$ and $B(b) \neq \emptyset$. Then, for $\emptyset \neq Z \subseteq V$,*

$$b^{\downarrow}(Z) = \min\Big\{ \sum_{X \in \mathcal{T}} b(X) : \mathcal{T} \text{ a tree-composition of } Z \Big\}. \tag{2.1}$$

This result was originally proved by using the uncrossing procedure. In Section 2.2, we exhibit a different approach in which the minimizing tree-composition is given directly without using uncrossing. This simplifies the way how a minimizing tree-composition can be found algorithmically (see Section 2.3). The proof is based on a structural result on tree-compositions that can be used to find a sub-tree-composition of a special family (see Section 2.1).

In Section 2.4, after recalling the original applications of Frank [30], we give a new proof of a graph orientation theorem of [28]. One of its main application is a connection between (k, ℓ)-partition-connected graphs and graphs that are rooted (k, ℓ)-edge-connected orientable.

By Theorem 1.2.7, the body-bar graph G_H^{BB} of a graph H is globally rigid in $\mathbb{R}^d$ if and only if H is highly $\binom{d+1}{2}$-tree-connected. This result motivates us to give a simple algorithm for testing high k-tree-connectivity, that is $(k, 1)$-partition-connectivity. Although, this is the special case of $\ell = 1$ of the upper result, first we give a new algorithm for the case of $\ell = 0$ in Section 2.4.4 as this algorithm will be used as a subroutine in the latter algorithms. Next, we give a new proof of Theorem 2.4.4 for the case where $\ell = 1$ in Section 2.4.2. This proof shows a simple algorithm for testing high k-tree-connectivity. By combining the ideas of these two algorithms, we obtain an efficient algorithm for finding a $(k, 1)$-edge-connected orientation in Section 2.4.3.

Finally in Section 2.5, a new application is described in which we show that the maximum size of a tree-composition of T complying with a 2-edge-connected bipartite graph $G = (S, T; E)$ is equal to the minimum number of edges entering T in a strongly connected orientation of G.

In this chapter, the number of nodes and edges in a graph will be denoted by n and m, respectively.

The results of this chapter appeared in a joint work with András Frank [32] (Sections 2.1–2.4 and 2.5) and in [73] (Sections 2.4.1–2.4.3).

2.1 Tree-compositions

By a $z\bar{s}$**-set** we mean a set containing z and not containing s. Let Z be a non-empty proper subset of a ground set V. A family $\mathcal{Z}$ of subsets of V is Z**-separating** if it contains a $z\bar{s}$-set for every pair $\{z, s\}$ of elements with $z \in Z$ and $s \in V - Z$. Here, $\mathcal{Z}$ is said to be **minimal** for this property if no proper subfamily of $\mathcal{Z}$ is Z-separating.

Our first goal is to show that a crossing Z-separating family $\mathcal{F}$ always includes a cross-free Z-separating subfamily $\mathcal{Z}$. We prove this by describing a direct construction of $\mathcal{Z}$ that does not rely on the uncrossing technique. Since $\mathcal{F}$ is crossing, the intersection $M_{z\bar{s}}$ of all $z\bar{s}$-sets of $\mathcal{F}$ belongs to $\mathcal{F}$ for every choice of $z \in Z$, $s \in V - Z$. Also, if some members of $\mathcal{F}$ form a connected hypergraph on a subset $U \subset V$, then U belongs to $\mathcal{F}$. It follows for every $s \in V - Z$ that the connected components of the hypergraph $H_s := (V, \{M_{z\bar{s}} : z \in Z\})$ intersecting Z form a subpartition $\mathcal{P}_s$ of $V - s$. By construction, $\mathcal{P}_s \subseteq \mathcal{F}$ and $\mathcal{P}_s$ covers Z.

Theorem 2.1.1. *Let Z be a non-empty proper subset of a ground set V and let $\mathcal{F}$ be a crossing Z-separating family of subsets of V. Then $\mathcal{F}$ includes a cross-free Z-separating subfamily. Namely, $\mathcal{Z} := \bigcup\{\mathcal{P}_s : s \in V - Z\}$ is such a subfamily.*

Proof. Since $\mathcal{P}_s$ is a subpartition of $V - s$ covering Z for each $s \in V - Z$, the family $\mathcal{Z}$ is Z-separating. We have to prove that $\mathcal{Z}$ is cross-free.

Claim 2.1.2. *Let $s_1, s_2 \in V - Z$ and $z \in Z$ be elements for which $M_{z\bar{s}_1} \neq M_{z\bar{s}_2}$. Then $s_1 \in M_{z\bar{s}_2}$ and $s_2 \in M_{z\bar{s}_1}$.*

Proof. Suppose by contradiction that, say, $s_1 \notin M_{z\bar{s}_2}$. Since $M_{z\bar{s}_1}$ is a minimal $z\bar{s}_1$-set in $\mathcal{F}$, $M_{z\bar{s}_1} \subseteq M_{z\bar{s}_2}$. But $M_{z\bar{s}_2}$ is also a minimal $z\bar{s}_2$-set in $\mathcal{F}$ from which $M_{z\bar{s}_1} = M_{z\bar{s}_2}$, contradicting the hypothesis of the claim. □

For a contradiction, suppose that $\mathcal{Z}$ contains two crossing sets F_1 and F_2. Since the members of $\mathcal{P}_s$ for a given $s \in V-Z$ are disjoint, there are distinct elements $s_1, s_2 \in V-Z$ so that $F_1 \in \mathcal{P}_{s_1}$ and $F_2 \in \mathcal{P}_{s_2}$.

Case 1 $s_2 \in F_1$ and $s_1 \in F_2$. Then there is an element $z \in Z \cap F_1$ for which $s_2 \in M_{z\bar{s}_1}$. Since $s_2 \notin M_{z\bar{s}_2}$, we have $M_{z\bar{s}_1} \neq M_{z\bar{s}_2}$ and hence Claim 2.1.2 implies that $s_1 \in M_{z\bar{s}_2}$. Since F_2 is a connected component of H_{s_2} containing s_1, we conclude that $M_{z\bar{s}_2} \subseteq F_2$, and in particular, $z \in F_2$. As F_1 and F_2 are crossing, $F_1 \cap F_2 \in \mathcal{F}$, contradicting the minimality of $M_{z\bar{s}_1}$.

Case 2 $s_2 \notin F_1$ or $s_1 \notin F_2$. By symmetry, we may assume that $s_2 \notin F_1$. By Claim 2.1.2, $M_{z\bar{s}_1} = M_{z\bar{s}_2}$ for every $z \in F_1 \cap Z$. Hence, as F_1 is a connected component of H_{s_1}, $F_1 = \cup(M_{z\bar{s}_1} : z \in F_1 \cap Z) = \cup(M_{z\bar{s}_2} : z \in F_1 \cap Z)$ and therefore there is a component of H_{s_2} including F_1, contradicting the assumption that the connected component F_2 of H_{s_2} crosses F_1. □ □

Remark In the construction of the subpartition $\mathcal{P}_s$ for a given $s \in V - Z$, we considered the connected components of the hypergraph H_s intersecting Z. One may feel that it would be more natural, and certainly simpler, to define a subpartition $\mathcal{P}'_s$ of $V - s$ by considering the maximal members of $\mathcal{F}$ not containing s and intersecting Z. Figure 2.1, however, shows that the family $\mathcal{Z}' := \bigcup\{\mathcal{P}'_s : s \in V - Z\}$ is not cross-free.

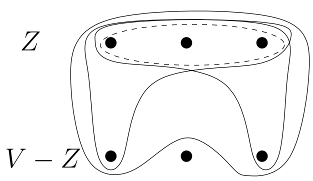


Figure 2.1: A family of 4 sets where the maximal $z\bar{s}$-sets do not form a cross-free family.

We claim that a tree-composition $\mathcal{T}$ of Z is a cross-free Z-separating family. Indeed for $s \in V - Z$ and $z \in Z$, there is an undirected path between $\varphi(s)$ and $\varphi(z)$ on the representing tree that contains at least one edge with the proper direction (namely the edge of the path exiting $\varphi(s)$) that represents a $z\bar{s}$-set. Moreover, $\mathcal{T}$ is minimal since an edge uv of the tree represents the only $z\bar{s}$-set for $z \in \varphi^{-1}(v)$ and $s \in \varphi^{-1}(u)$. The following result shows that the converse is also true.

Theorem 2.1.3. *For a non-empty proper subset Z of V, a minimal cross-free Z-separating family $\mathcal{Z}$ is a tree-composition of Z.*

Proof. By Lemma 1.4.1, $\mathcal{Z}$ has a tree-representation (T, φ) where $T = (U, A)$ is a directed tree. Since $\mathcal{Z}$ is Z-separating, we must have $\varphi(s) \neq \varphi(z)$ whenever $s \in V - Z$ and $z \in Z$.

Claim 2.1.4. *No edge of T enters $\varphi(s)$ for every $s \in V - Z$. No edge of T leaves $\varphi(z)$ for every $z \in Z$.*

Proof. We prove only the first half as it immediately implies the second one by reversing the orientation of T and complementing each member of $\mathcal{Z}$.

Assume for a contradiction that there is an edge $f = u\varphi(s) \in A$ of T entering $\varphi(s)$. Let Z_f be the member of $\mathcal{Z}$ which is represented by f. By Remark 2.2.1, $T' := T/f$ represents the family $\mathcal{Z}' := \mathcal{Z} - \{Z_f\}$. The minimality of $\mathcal{Z}$ shows that $\mathcal{Z}'$ is not Z-separating, that is, there are elements $z \in Z$ and $s' \in V - Z$ such that Z_f is the only member of $\mathcal{Z}$ for which $z \in Z_f$ and $s' \in V - Z_f$. It follows that, going along the unique (undirected) path P of T from $\varphi(s')$ to $\varphi(z)$, the only forward edge is f. Therefore, each edge of the subpath of P connecting $\varphi(s)$ and $\varphi(z)$ is oriented toward $\varphi(s)$, implying that $\mathcal{Z}$ does not contain a $z\bar{s}$-set, a contradiction. □

Let $A' \subseteq A$ denote the subset of edges of T leaving $\varphi(V - Z)$. Let $s \in V - Z$ and $z \in Z$ and consider the unique path P of T connecting $\varphi(s)$ and $\varphi(z)$. By Claim 2.1.4, its first edge f at $\varphi(s)$ leaves $\varphi(s)$ and hence Z_f is a $z\bar{s}$-set where Z_f denotes the member of $\mathcal{Z}$ represented by f. Therefore, the minimality of $\mathcal{Z}$ implies that $A' = A$, that is, every edge of T leaves $\varphi(V - Z)$. Analogously, every edge of T enters $\varphi(Z)$. Therefore, T is a tree such that each of its edges is of form $\varphi(s)\varphi(z)$ for some $s \in V - Z$ and $z \in Z$, that is, $\mathcal{Z}$ is a tree-composition of Z. □ □

By combining Theorems 2.1.1 and 2.1.3, we obtain the following corollary.

Theorem 2.1.5. *For a given non-empty proper subset Z of V, a crossing and Z-separating family $\mathcal{Z}$ of subsets of V includes a tree-composition of Z.* □

2.2 Computing the full truncation of b

As an application of Theorem 2.1.5, we provide a simple proof of Theorem 2.0.1.

Proof. Since a tree-composition is a special double-partition by definition, (1.3) implies that $b^{\downarrow}(Z) \leq \min\{\sum_{F \in \mathcal{T}} b(F) : \mathcal{T}$ a tree-composition of $Z\}$, therefore, we need to show a tree-composition for which equality holds.

Obviously, $b^{\downarrow}(V) = b(V)$, and $b(V) = \min\{\widetilde{b}(\mathcal{T}) : \mathcal{T}$ a tree-composition of $V\}$ by Theorem 1.4.4. Hence from now on we can assume that Z is a proper subset of V.

$B(b) = B(b^{\downarrow})$ by definition. Theorem 1.4.5 implies that there is an element m of $B(b)$ for which $\widetilde{m}(Z)(:= \sum_{z \in Z} m(z)) = b^{\downarrow}(Z)$. Call a subset $X \subset V$ m-**tight** if $\widetilde{m}(X) = b(X)$

and let $\mathcal{F}$ be the family of m-tight sets. Then $\mathcal{F}$ is a crossing set system by submodularity: $\widetilde{m}(X)+\widetilde{m}(Y) = b(X)+b(Y) \geq b(X\cap Y)+b(X\cup Y) \geq \widetilde{m}(X\cap Y)+\widetilde{m}(X\cup Y) = \widetilde{m}(X)+\widetilde{m}(Y)$ whenever $X, Y \subseteq V$ are crossing.

Claim 2.2.1. *There exists an m-tight $t\overline{s}$-set for every $s \in V - Z$, $t \in Z$, so $\mathcal{F}$ is Z-separating.*

Proof. If there is an $s \in V - Z$, $t \in Z$ for which no m-tight $t\overline{s}$-set exists, then for $\varepsilon := \min\{b(X) - \widetilde{m}(X) : X \text{ is a } t\overline{s}\text{-set}\}$ the vector m' for which $m'(s) := m(s) - \varepsilon$, $m'(t) := m(t) + \varepsilon$, $m'(v) := m(v)$ $(v \in V - \{s,t\})$ would belong to $B(b)$ but would not belong to $B(b^{\downarrow})$ that is a contradiction. □

By Theorem 2.1.5 there is a tree-composition $\mathcal{T}$ of Z consisting of m-tight sets. Thus for this tree-composition, $\sum_{X\in\mathcal{T}} b(X) = \sum_{X\in\mathcal{T}} \widetilde{m}(X) = \Delta(\mathcal{T})\widetilde{m}(V-Z) + (\Delta(\mathcal{T})+1)\widetilde{m}(Z) = \Delta(\mathcal{T})\widetilde{m}(V) + \widetilde{m}(Z) = \Delta(\mathcal{T})b(V) + \widetilde{m}(Z) = 0 + b^{\downarrow}(Z)$. □ □

With a similar proof one can get the following.

Theorem 2.2.2. *Let p be a crossing supermodular function for which $B'(p) \neq \emptyset$. Then*

$$p^{\uparrow}(Z) = \max\left\{\widetilde{p}(\mathcal{T}) - \Delta(\mathcal{T})p(V)\right\},$$

where the maximum is taken over all tree-compositions $\mathcal{T}$ of Z. □

2.3 Algorithmic aspect

With the bi-truncation algorithm of Frank and Tardos [38, 83] one can compute the value of the full truncation of a crossing sub- or supermodular function on a set $Z \subseteq V$ provided a subroutine is available that computes $\min\{b(Y) - \widetilde{a}(Y) : Y \subseteq X\}$ and the minimizing set for a vector $a \in \mathbb{R}^V$ and a subset $X \subseteq V$. In several applications this minimizing oracle can be obtained via a flow-algorithm hence its running time is usually $\vartheta = O(n^3)$. If $B(b) \neq \emptyset$, then the algorithm outputs a vector $m \in B(b)$ with maximum value of $\widetilde{m}(Z)$, hence $\widetilde{m}(Z) = b^{\downarrow}(Z)$ by Theorem 1.4.5. However, the algorithm does not compute the minimizing tree-composition.

Now we will give an algorithm for computing the minimizing tree-composition in formula (2.1) if we are given a vector $m \in B(b)$ with maximum value of $\widetilde{m}(Z)$. The proof of Theorem 2.0.1 implies that if $\widetilde{m}(Z)$ is maximum for $m \in B(b)$, then the m-tight sets form a crossing Z-separating family $\mathcal{F}$. By using the method described in the beginning of Section 2.1, one can get a cross-free Z-separating family $\mathcal{Z}$ if the minimal m-tight sets are computable. Therefore, assume that the minimizing oracle used in the bi-truncation algorithm can compute also the minimizing set of $\min\{b(Y) - \widetilde{a}(Y) : v \in Y \subseteq X\}$ for a vector $a \in \mathbb{R}^V$ and a subset $X \subseteq V$ in running time ϑ. Then the minimal $z\bar{s}$-set can be calculated using this oracle with $a := m$, $X := V - s$ and $v := z$. To get $\mathcal{Z}$ one needs to run the minimizing oracle $O(n^2)$ times and after that some search algorithm is needed that runs in $O(n^3)$ time. Thus we get $\mathcal{Z}$ in $O(n^2\vartheta + n^3)$ time. After that we need to omit some elements of $\mathcal{Z}$ to get a minimal Z-separating family that needs $O(n^4)$ running time. The family we get will be a tree-composition of Z by Theorem 2.1.3. Therefore, the total running time of calculating the minimizing tree-composition is $O(n^2\vartheta + n^4)$ that is $O(n^5)$ in practice if the minimizing oracle is given by a flow-algorithm.

Remark: With the bi-truncation algorithm one can get a double-partition that minimizes (1.3). One can also read out an algorithm from the proof of Theorem 2.0.1 given in [30] to obtain a minimizing tree-composition, but this algorithm needs to uncross the double-partition provided by the bi-truncation algorithm. The **uncrossing method** means that we keep replacing two crossing members of $\mathcal{F}$ with their union and intersection until the current family gets cross-free. An uncrossing step increases the value of $\sum_{F \in \mathcal{F}} |F|^2$, and hence the running time of the uncrossing method on a double-partition of size $\Omega(n^2)$ is more than the running time of the bi-truncation algorithm. More precisely, the bi-truncation algorithm (for finite-valued b) runs in time $O(n^2\vartheta + n^3)$ where ϑ is the running time of the minimizing oracle used in the algorithm, while the uncrossing procedure for such a family needs $O(n^7)$ running time. (If b can have infinite values, then the running time of the bi-truncation algorithm is $O(n^3\vartheta)$.) With an adaptation of Fleiner's method [23], this bound for the running time of the uncrossing procedure can be lowered to $O(n^5)$.

2.4 Orientations covering submodular functions

Robbins' theorem [90] states that a graph G has a strongly connected orientation if and only if G is 2-edge-connected. A natural extension to mixed graphs was given by Boesch and Tindell in [7].

A significantly deeper extension of Robbins' theorem is due to Nash-Williams [84] who proved that a $2k$-edge-connected graph has a k-edge-connected orientation. Perhaps surprisingly, the problem of finding a k-edge-connected orientation of a mixed graph is much more complex since in this case the necessary and sufficient condition relies on tree-compositions.

Suppose that $M = (V, E \cup F)$ is a mixed graph that consists of an undirected graph $G = (V, E)$ and a digraph $H = (V, F)$. We want to find an orientation $\overrightarrow{G} = (V, \overrightarrow{E})$ of G so that the digraph $(V, \overrightarrow{E} \cup F)$ is k-edge-connected. This is equivalent to requiring that the orientation of G covers h_k where

$$h_k(X) := \begin{cases} k - \varrho_H(X) & \text{if } \emptyset \neq X \subset V, \\ 0 & \text{if } X \in \{\emptyset, V\}, \end{cases}$$

and an orientation $\vec{G}$ is said to **cover** h if $\varrho_{\vec{G}}(X) \geq h(X)$ for all $X \subseteq V$.

For a graph $G = (V, E)$, a set-function $h : 2^V \to \mathbb{R}$ is **(crossing) G-supermodular** if $h(X) + h(Y) \leq h(X \cap Y) + h(X \cup Y) + d_G(X, Y)$ for every (crossing) $X, Y \subseteq V$. It is easy to see that h_k is crossing G-supermodular, hence the following theorem of [30] gives a necessary and sufficient condition to the k-edge-connected orientability problem for mixed graphs.

For an edge $e = uv$ and for a family $\mathcal{F}$ let $w_e(\mathcal{F})$ denote the maximum of the number of $u\overline{v}$-sets and the number of the $v\overline{u}$-sets.

Theorem 2.4.1 ([30]). *Let $G = (V, E)$ be a graph and let $h : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a crossing G-supermodular function for which $h(V) = 0$. There is an orientation of G covering h if and only if*

$$\sum_{e \in E} w_e(\mathcal{T}) \geq \widetilde{h}(\mathcal{T}), \tag{2.2}$$

holds for every tree-composition $\mathcal{T}$ of each subset of V. □

Theorem 2.4.1 was derived in [30] from the submodular flow feasibility theorem. Submodular flows were introduced and investigated by Edmonds and Giles in [22]. Let $D = (V, A)$ be a directed graph, $f : A \to \mathbb{Z} \cup \{-\infty\}$, $g : A \to \mathbb{Z} \cup \{\infty\}$ two integer-valued bounding functions for which $f \leq g$. Moreover, we are given a crossing submodular set-function $b : 2^V \to \mathbb{Z} \cup \{\infty\}$ for which $b(\emptyset) = 0$ and $b(V)$ is finite. A function (or vector) $x : A \to \mathbb{R}$ is called a **submodular flow** or **subflow** confined by b if $\Psi_x(Z) := \varrho_x(Z) - \delta_x(Z) \leq b(Z)$ for every $Z \subseteq V$, where $\varrho_x(Z) := \sum\{x(uv) : u \in V - Z, v \in Z, uv \in A\}, \delta_x(Z) := \sum\{x(uv) : u \in Z, v \in V - Z, uv \in A\}$. Since $\Psi_x(V) = 0$ we can assume that $b(V) = 0$. A subflow x is **feasible** if $f \leq x \leq g$.

The basic case of the submodular flow feasibility theorem is when b^* is fully submodular and integer-valued with $b^*(\emptyset) = b^*(V) = 0$ was proved in [29]: *given a digraph $D = (V, A)$ and bounding functions $f : A \to \mathbb{Z}$, $g : A \to \mathbb{Z}$ with $f \leq g$, there is an integer feasible submodular flow confined by b^* if and only if*

$$\varrho_f - \delta_g \leq b^*. \tag{2.3}$$

By combining this result with Theorem 2.0.1, one obtains the general characterization:

Theorem 2.4.2 ([30]). *Let b be a crossing submodular function for which $b(V) = 0$. There is an integer feasible submodular flow confined by b if and only if*

$$\varrho_f(Z) - \delta_g(Z) \leq \widetilde{b}(\mathcal{T}) \tag{2.4}$$

for every nonempty $Z \subseteq V$ and every tree-composition $\mathcal{T}$ of Z. □

Note that if b is fully supermodular, then there are several algorithms for finding a feasible (integer) submodular flow (for a survey see [42]). These algorithms can also be applied to compute a feasible subflow when b is crossing submodular. But, in the case when no feasible submodular flow exists, extra work is needed to compute a violating tree-composition (see [30, 36]). With the algorithm described in Section 2.3, it is simpler to find this and hence the present method simplifies the finding of an obstacle if a mixed graph has no k-edge-connected orientation.

In [28] it was proved that in the special case when $h \geq 0$, (2.2) is required only for tree-compositions of V, that is, for partitions and co-partitions of V. Here we show how Theorem 2.4.1 implies this special case.

Theorem 2.4.3. *Let $G = (V, E)$ be a graph and let $h : 2^V \to \mathbb{Z}_+$ be a crossing G-supermodular function for which $h(V) = 0$. There is an orientation of G covering h if and only if*

$$e(\mathcal{P}) \geq \sum_{i=1}^{q} h(V_i) \quad \text{and} \quad e(\mathcal{P}) \geq \sum_{i=1}^{q} h(V - V_i) \tag{2.5}$$

hold for every partition $\mathcal{P} = \{V_1, V_2, ..., V_q\}$ of V.

Proof. As the necessity of (2.5) is straightforward, we only prove its sufficiency. Assume that G has no orientation covering h. By Theorem 2.4.1 there is a tree-composition $\mathcal{T}$ of $X \subseteq V$ with $\sum_{e \in E} w_e(\mathcal{T}) < \widetilde{h}(\mathcal{T})$. Then $\mathcal{T}' := \mathcal{T} \cup \{V - X\}$ is a composition of V. Assume for a contradiction that $w_e(\mathcal{T}') > w_e(\mathcal{T})$ for an edge $e = uv$. Then e may have exactly one endpoint, say u, in $V - X$. However, in this case, $d_{\mathcal{T}}(u) = d_{\mathcal{T}}(v) - 1$, thus after subtracting the number of sets containing both u and v, we get that the number of $v\bar{u}$-sets is more than the number of $u\bar{v}$-sets in $\mathcal{T}$. Hence $w_e(\mathcal{T}') = w_e(\mathcal{T})$, a contradiction. Thus $\sum_{e \in E} w_e(\mathcal{T}') = \sum_{e \in E} w_e(\mathcal{T})$. Therefore, since h is non-negative, $\sum_{e \in E} w_e(\mathcal{T}') = \sum_{e \in E} w_e(\mathcal{T}) < \widetilde{h}(\mathcal{T}) \leq \widetilde{h}(\mathcal{T}')$.

By uncrossing $\mathcal{T}'$, we get a cross-free composition $\mathcal{K}$ of V for which $\sum_{e \in E} w_e(\mathcal{K}) < \widetilde{h}(\mathcal{K})$ holds since h is crossing-G-supermodular. By Lemma 1.4.3, $\mathcal{K}$ can be partitioned into partitions and co-partitions of V and hence at least one of them violates (2.5). □

Note that $\mathcal{T}'$ has only $O(n)$ members hence the running time of the uncrossing procedure is less than the running time of the bi-truncation algorithm. Hence with the present method one can find an orientation covering h or a partition violating (2.5) in the running time of the bi-truncation algorithm. This is the best known running time for this problem that can also be achieved by the algorithm that can be read out from the following proof of Theorem 2.4.3 given in [34]:

Proof. Let $p(X) = h(X)+i(X)$. It can be shown that p is crossing supermodular and that an integer vector in the base-polyhedron $B'(p)$ is the in-degree vector of an orientation covering h. A little calculation shows that (2.5) implies the conditions of Fujishige's theorem (see Theorem 1.4.4). Hence there exists an integer vector $z \in B'(p)$. By the Orientation lemma of Hakimi [51] there exists an orientation of G with in-degree vector z, completing the proof. □

From Theorem 2.4.3 it is easy to show that (k, ℓ)-partition-connectivity and (k, ℓ)-edge-connected orientability are equivalent for $k \geq \ell$. (To prove this, just use the theorem the following function: $h(X) = k$ if $r_0 \notin X \neq \emptyset$, $h(X) = \ell$ if $r_0 \in X \neq V$, $l(X) = 0$ if $X \in \{\emptyset, V\}$.)

Theorem 2.4.4. *For k, ℓ positive integers with $k \geq \ell$ a graph with a root node r_0 has an r_0-rooted (k, ℓ)-edge-connected orientation if and only if it is (k, ℓ)-partition-connected.* □

The proof of Theorem 2.4.3 in [28] gives rise to an algorithm for finding a rooted (k, ℓ)-edge-connected orientation (and by this for testing (k, ℓ)-partition-connectivity); the algorithm is rather involved.

For the case where $\ell = 0$, when we want to give a rooted k-edge-connected orientation of a graph, the proof presented in [26] and in [31, Section 9.1] gives rise to a simpler algorithm. Yet another algorithm was given by Gabow and Manu in [43]. These algorithms output either a rooted k-edge-connected orientation of the input graph $G = (V, E)$ or a partition $\mathcal{P}$ of V for which

$$e_G(\mathcal{P}) < k(|\mathcal{P}| - 1),$$

showing that G is not k-partition-connected.

A more efficient algorithm for finding a (k, ℓ)-edge-connected orientation can be read out from our second proof of Theorem 2.4.3 (that used the technique of [34]). The complexity of this algorithm is $O(n^5 + n^2m)$. The involved part of this algorithm is that it uses the bi-truncation algorithm of Frank and Tardos [38, 83].

Moreover, this idea of using the Orientation lemma leads to a more efficient algorithm for the case where $\ell = 0$ because then only the truncation algorithm of Frank and Tardos [38] is needed (see [71] and [31, Section 15.4.4]) thus we get a bound $O(n^4 + n^2m)$ for the running time. Since we use the truncation algorithm instead of the bi-truncation algorithm, this algorithm is less involved. The previously mentioned algorithm of Gabow and Manu [43] has a time bound of $O(\min\{n, \log(N)\}n^2m^* \log(n^2/m^*))$ where m^* is the edge number of the underlying simple graph and N is the maximum number of parallel copies of an edge.

As noted in the introduction of this chapter, our next aim is to give a simple and efficient algorithm for the case of Theorem 2.4.4 where $\ell = 1$, but we begin by giving a new algorithm for the case of $\ell = 0$ as this algorithm will be used as a subroutine in the latter algorithms. This algorithm, that arises from modifying Frank's earlier algorithm [26] for finding rooted k-edge-connected orientations – using a simple data structure – has a time bound of $O(n^4+n^3k)$. Next, we turn to giving a new proof of Theorem 2.4.4 for the case where $\ell = 1$. This proof shows a simple algorithm for testing high k-tree-connectivity. The algorithm uses the rooted k-edge-connected orientation algorithm as a subroutine. (This algorithm can be extended to hypergraphs.) Finally, by combining the ideas of these two algorithms, we obtain another algorithm for finding a $(k, 1)$-edge-connected orientation of graphs that runs in time $O(n^4 + n^3k)$.

We note that to have a (k, ℓ)-edge-connected orientation (for $k \geq \ell$), the graph needs a maximum of $2k$ parallel edges between two nodes, and thus $m = O(n^2k)$. However, at some points this can be reduced to the edge number of the underlying simple graph, that is, to maximum $O(n^2)$. To achieve this in the following algorithms, the graph (respectively, its orientation) will be stored via its *adjacency matrix* $((a_{i,j}))$ where $a_{i,j}$ is the number of ij-edges (respectively arcs).

2.4.1 A more efficient algorithm for rooted k-edge-connected orientability

As noted before there exists an algorithm for finding a rooted k-edge-connected orientation of a graph that has a running time of $O(n^4 + n^2m)$. However, this algorithm uses a polyhedral technique. Here we describe another algorithm based on Frank's algorithm [26] with a running time of $O(n^4 + n^3k)$ that uses only basic graph theoretical arguments.

First we sketch Frank's algorithm:

Algorithm 2.4.5 (Frank [26])**.** INPUT: A graph $G = (V, E)$ and a root $r_0 \in V$.
OUTPUT: $\overrightarrow{G} = (V, \overrightarrow{E})$ an r_0-rooted k-edge-connected orientation of G OR a partition $\mathcal{P}$ of V with $e_G(\mathcal{P}) < k(|\mathcal{P}| - 1)$.

Phase 1: We add new edges to the graph from the root r_0 to some nodes so as to obtain a rooted k-edge-connected orientation $D = (V, A)$ of the extended graph with $\varrho_D(r_0) = 0$. More precisely, we add k parallel r_0v-edges for every $v \in V - r_0$, we orient all r_0v-edges towards v for every $v \in V - r_0$ and we orient the remaining edges arbitrarily.

Phase 2: We try to omit one by one the newly added arcs from D in such a way that the rooted k-edge-connected orientation is preserved. If the omission of a new arc r_0t decreases the in-degree of a $t\bar{r_0}$**-set**, (that is, a set containing t but avoiding r_0) below k, then we reverse a path starting at t and ending at a node $v \in V$ such that the digraph becomes rooted k-edge-connected. (Frank showed in [26, 31] that one can find such a v if the graph is k-partition-connected.) We will call this step of the algorithm an **elimination step.** If there is no appropriate v for a new edge, the elimination step fails and the algorithm returns a partition violating $e_G(\mathcal{P}) \geq k(|\mathcal{P}| - 1)$ that can be calculated in $O(n^4)$ running time (see [26]). ♣

We need to resolve two issues to reduce the running time. First, an elimination step is relatively slow since one needs to run a flow algorithm $O(n)$ times to determine whether there is any reversible path. Second, there are $O(nk)$ new edges resulting in $O(nk)$ elimination steps that need to run a flow algorithm $O(n^2k)$ times. To reduce the running time of the elimination steps, we present here a new simple data structure where k arc-disjoint one-way paths will be maintained from r_0 to v for every $v \in V - r_0$. Using these one-way paths it will be easy to check the reversibility of a path in the second phase of Algorithm 2.4.5 and it will be readily usable in a latter algorithm. Usually, these paths can be built up by running $n - 1$ flow algorithms; however in our cases the task will be simpler. Moreover, these paths can be updated easily when an arc is omitted or a path is reversed (after an omission of an arc), as follows. When a new arc r_0t is omitted, we omit the r_0v-path containing this arc from these k arc-disjoint paths if any exists and find k paths – if possible – using one augmenting path-searching step of the Ford–Fulkerson algorithm [25]. Note that if we update the data structure with this method, then there may remain only $k - 1$ paths to some of the nodes and our algorithms will make a path reversal step to restore the rooted k-edge-connectivity. After a path P_0 is reversed, we first modify the P_0-arc-intersecting r_0v-paths, that is, the r_0v-paths intersecting P_0 in at least one arc. Where a path P enters P_0 at a point x we modify P such that from x we follow the reversed path $\overleftarrow{P_0}$ until another P_0-arc-intersecting r_0v-path P' leaves P_0 or we arrive at the start point of P_0. Thus we obtain at least $k - 2$ one-way paths from r_0 to v (along with some circuits and some other paths) – as there were at least $k - 1$ r_0v-paths before the reversal of P_0. With these $k - 1$ paths we need only run one augmenting path-searching step of the Ford–Fulkerson algorithm to find k r_0v-paths. We call both methods – that is, both the one we perform after the omission of an arc and the one we perform after a path reversal – a v**-path update** or a v**-path check**, when we just want to check whether after reversing a path there remain k arc-disjoint r_0v-paths. One can see that the running time of a v-path update or check, respectively, is $O(n^2)$. We call the method when we call a v-path update for all $v \in V - r_0$ an **all-path update**.

Using this data structure we modify Algorithm 2.4.5 as follows.

Algorithm 2.4.6. INPUT: A graph $G = (V, E)$ and a root $r_0 \in V$.
OUTPUT: $\overrightarrow{G} = (V, \overrightarrow{E})$ an r_0-rooted k-edge-connected orientation of G along with the data structure of the k arc-disjoint r_0v-paths for every $v \in V - r_0$ OR a partition $\mathcal{P}$ of V with $e_G(\mathcal{P}) < k(|\mathcal{P}| - 1)$.

Phase 1: The same as *Phase 1* of Algorithm 2.4.5.

Phase 2:

Step 1: Initialize the data structure of k one-way paths: take the k newly added arcs to every $v \in V - r_o$ as the k r_0v-paths. Label every node in $V - r_0$ with *non-inspected*.

Step 2: Let t be a non-inspected node.

Step 3: As long as there is a newly added r_0t-arc, omit it, and do an all-path update; and if there remain only $k-1$ arc-disjoint paths from r_0 to t, then for each $v \in V$ that is reachable from t in a path P_0, do a v-path check with the reversed path P_0 until we find a node to which there remain k arc-disjoint paths and we can finish with doing an all-path update for the current reversed path. If there is no appropriate v, then the elimination step fails and we can return a partition violating $e_G(\mathcal{P}) \geq k(|\mathcal{P}| - 1)$ as in Algorithm 2.4.5.

Step 4: Modify the label of t to *inspected*. If there is a non-inspected node go to *Step 2*; otherwise return the current orientation and the current state of the data structure. ♣

Algorithm 2.4.6 works since it is similar to Algorithm 2.4.5. It is easy to see that with the data structure the running time of an elimination step is reduced from $O(n^4)$ (that is, the running time of $O(n)$ flow algorithms) to $O(n(n+m)) \leq O(n^3)$ (that is, the running time of $O(n)$ augmenting path-searching steps of the Ford–Fulkerson algorithm). The main point in *Step 3* is that we omit the new arcs going to the same node sequentially; thus we can prove the following lemma.

Lemma 2.4.7. *If the omission of γ r_0t-edges for a node $t \in V - r_0$ is possible in Step 3 of Algorithm 2.4.6, then in these elimination steps we need to check the reversibility of $O(n+\gamma)$ paths starting at t. The reversibility of a tv-path can be checked by a v-path check thus the omission can be done with $O(n+\gamma)$ path checks and $O(\gamma)$ all-path updates.*

Proof. Let T denote the set of nodes (currently) reachable from t. It is easy to see that after reversing a tv-path no nodes in $V - T$ become reachable from t. However, T could become a smaller set.

After omitting one of these new edges, a path reversal is needed if there arises a set not containing r_0 with in-degree $k-1$. This set must contain t since the single omitted edge is r_0t and before its omission the digraph was rooted k-edge-connected. Hence by Menger's theorem, if there remain k arc-disjoint r_0t-paths, then the digraph remains rooted k-edge-connected. Thus after the omission we must do an all-path update and check whether there are still k arc-disjoint r_0t-paths. If there are not, then a path reversal is needed.

The reversal of a tv-path is sufficient to restore the rooted k-edge-connectivity if after its reversal the in-degrees of the $t\bar{r_0}$-sets become at least k and the in-degrees of the $v\bar{r_0}$-sets are not reduced under k since the in-degrees of the other subsets of $V - r_0$ do not change. Thus for a node v, there is a reversible path if $v \in T$, the in-degree of any $v\bar{r_0}$-set is at least k and the in-degree of any set containing v and not containing t and r_0 is at least $k+1$. Since the reversal of a path can increase the in-degree of any set by at most 1, if there was a $t\bar{r_0}$-set X with in-degree $k-1$ after removing some r_0t-edges and we restore its in-degree with a path reversal, then the omission of the next r_0t-edge reduces $\varrho(X)$ to $k-1$. The in-degree of a set not containing t and r_0 cannot increase and, as noted before, T becomes smaller and smaller. Therefore, if after the omission of some r_0t-edges there is no reversible tv-path, then it is not necessary to check v again. Thus every node v is checked at most once plus as many times as there have been tv-path reversals. Hence we need to check $O(n+\gamma)$ times. As we omitted γ edges and reversed at most γ paths the number of all-path updates is clearly $O(\gamma)$. □

Therefore, we have the following corollary as we need to omit k r_0t-edges for each node $t \in V - r_0$.

Corollary 2.4.8. *The running time of Algorithm 2.4.6 is* $O(n(n+k)n^2 + n^4) = O(n^4 + n^3k)$. □

2.4.2 A simple algorithm for rooted $(k,1)$-edge-connected orientability

In this section we give a new algorithmic proof for Theorem 2.4.4 when $\ell = 1$. The proof will be based on the following simple lemmas. We will denote by r_R the new (contracted) node in G/R or D/R for $R \subseteq V$.

Lemma 2.4.9. *Let $D = (V, A)$ be a digraph with a root node $r_0 \in V$. Let $R \subset V$ be a set of nodes containing r_0 for which $D[R]$ is r_0-rooted k-edge-connected and D/R is r_R-rooted k-edge-connected. Then D is r_0-rooted k-edge-connected.*

Proof. For a set $r_0 \in X \subset V$, if $R \not\subseteq X$, then $\delta_D(X) \geq \delta_D(X, R-X) \geq \delta_D(X \cap R, R-X) = \delta_{D[R]}(X \cap R) \geq k$ where the last inequality holds because of the r_0-rooted k-edge-connectivity of $D[R]$.

For a set $r_0 \in X \subset V$, if $R \subseteq X$, we get $\delta_D(X) = \delta_{D/R}(X - R + r_R) \geq k$ by the r_0-rooted k-edge-connectivity of D/R. □

The next lemma holds for general ℓ, although we will need it only for $\ell = 1$.

Lemma 2.4.10. *Let $D = (V, A)$ be a digraph with a root $r_0 \in V$. Let $R \subset V$ be a set of nodes that contains r_0 and for which $D[R]$ is r_0-rooted (k, ℓ)-edge-connected and D/R is r_R-rooted (k, ℓ)-edge-connected. Then D is r_0-rooted (k, ℓ)-edge-connected.*

Proof. By using Lemma 2.4.9 both for D and for the reverse digraph $\overleftarrow{D}$ (for ℓ in place of k), we get that D is r_0-rooted k-edge-connected and $\overleftarrow{D}$ is r_0-rooted ℓ-edge-connected. Hence D is r_0-rooted (k, ℓ)-edge-connected. □

In the special case $\ell = 1$, Theorem 2.4.4 is as follows.

Theorem 2.4.11. *A graph $G = (V, E)$ with a root node r_0 has an r_0-rooted $(k, 1)$-edge-connected orientation if and only if it is $(k, 1)$-partition-connected.*

Proof. As the necessity of $(k, 1)$-partition-connectivity is straightforward (after observing that in an r_0-rooted $(k, 1)$-edge-connected orientation of G, the in-degree of a member of a partition of V is at least 1 if it contains r_0, and at least k otherwise), we only prove sufficiency. We will use induction on $|V|$. Let $e_0 = r_0u \in E$ be an arbitrary edge. By the $(k, 1)$-partition-connectivity of G, $G - e_0$ is k-partition-connected. Hence $G - e_0$ has an r_0-rooted k-edge-connected orientation. This orientation gives us an orientation D of G if we orient e_0 towards r_0. Let R be the set of nodes in D from which there is a path to r_0. Thus $\varrho(R) = 0$ and $\varrho_{D[R]}(X) \geq 1$ whenever $r_0 \in X \subset R$. By the rooted k-edge-connectivity, there are k arc-disjoint paths from r_0 to v for every $v \in R - r_0$. Since $\varrho_D(R) = 0$, these paths cannot leave R. Therefore, by Menger's theorem, $\delta_{D[R]}(X) \geq k$ whenever $r_0 \in X \subset R$. Hence $D[R]$ is r_0-rooted $(k, 1)$-edge-connected. We also see that $|R| \geq 2$ because $r_0, u \in R$. If $R = V$, then we are done.

If $R \neq V$, we do the following. If $\mathcal{P} = \{X_1, X_2, ...X_t\}$ is a partition of $V - R + r_R$, where $r_R \in X_1$, then $e_{\mathcal{P}}(G/R) = e_{\mathcal{P}'}(G) \geq k(|\mathcal{P}| - 1) + 1$ for the partition $\mathcal{P}' = \{X_1 - r_R \cup R, X_2, X_3, ...X_t\}$ of V. Hence G/R is $(k, 1)$-partition-connected. By induction there is an r_R-rooted $(k, 1)$-edge-connected orientation D' of G/R. Let $\overrightarrow{G}$ be the new orientation of G obtained by keeping the orientation of D on R and by orienting the other edges like in D'. Using Lemma 2.4.10 for $\overrightarrow{G}$ we get that $\overrightarrow{G}$ is r_0-rooted $(k, 1)$-edge-connected. □

We note that this approach does not seem to work in the case where $\ell > 1$ since $D[R]$ need not be (k, ℓ)-edge-connected in this case. If with another definition we define $D[R]$ to be the maximal r_0-rooted (k, ℓ)-edge-connected subgraph of D, then the approach will also fail for $\ell > 1$ as $D[R]$ will be able to consist of the single node r_0. From the proof presented above, one can obtain the following algorithm.

Algorithm 2.4.12. Input: A graph $G = (V, E)$ and a root $r_0 \in V$.

Output: $\vec{G} = (V, \vec{E})$ an r_0-rooted $(k, 1)$-edge-connected orientation of G OR a partition $\mathcal{P}$ of V with $e_G(\mathcal{P}) < k(|\mathcal{P}| - 1) + 1$.

Let $e_0 \in E$ be an arbitrary edge adjacent to r_0. Run Algorithm 2.4.5 (or 2.4.6) on $G - e_0$ to decide whether there is an r_0-rooted k-edge-connected orientation of $G - e_0$. If no such orientation exists, the subroutine outputs a partition $\mathcal{P}$ with $e_G(\mathcal{P}) - 1 \leq e_{G-e_0}(\mathcal{P}) < k(|\mathcal{P}| - 1)$ and we return this partition.

Suppose now that this subroutine has found an orientation and let $D = (V, \vec{E}')$ be the digraph that we get by taking this orientation on $G - e_0$ and orienting e_0 towards r_0. We will denote the oriented mate of $e \in E$ in D with $\vec{e}'$. Let R be the set of nodes from which r_0 is reachable in D that we could get by running any search algorithm. If $R = V$, then D is $(k, 1)$-edge-connected so we can return $\vec{G} := D$. Otherwise, for $e \in E[G]$, let $\vec{e} \in \vec{E}$ be $\vec{e}' \in \vec{E}'$ and run the algorithm recursively on G/R with root r_R (see Remark 2.4.13) and orient the edges not in $G[R]$ as this algorithm does. ♣

Remark 2.4.13. In Algorithm 2.4.12, $|R| > 1$ because r and the other endpoint of e_0 are in R. Hence we can run the algorithm on G/R recursively.

It is easy to see that if the subroutine outputs a partition $\mathcal{P}'$ of the node set of the possibly contracted graph G' with $e_{G'-e_0}(\mathcal{P}') < k(|\mathcal{P}'| - 1)$, then we can modify it easily to get a partition $\mathcal{P}$ of V with $e_G(\mathcal{P}) < k(|\mathcal{P}| - 1) + 1$. Namely, let $\mathcal{P}$ be the partition that we get by changing the new node r^* that represents the contracted set to the set that it represents in the member of $\mathcal{P}'$ containing r^*.

One can see that the algorithm runs the subroutine and the search algorithm $O(n)$ times; hence the running time of the algorithm is $O(n(\vartheta + n + m))$ where ϑ is the running time of the subroutine.

Extension to hypergraphs

To extend the algorithm to hypergraphs we need to consider directed hypergraphs. A directed hypergraph or **dypergraph** $\mathcal{D} = (V, \mathcal{A})$ consists of the node set V and the set $\mathcal{A} \subseteq 2^V$ of directed hyperedges. Here, as in [34], a directed hyperedge, called a **dyperedge**, has one head node while all of its other nodes are the tails. (We assume that a dyperedge of a dypergraph and a hyperedge of a hypergraph consist of at least two nodes.)

One can define (k, ℓ)-partition-connectivity of hypergraphs and rooted (k, ℓ)-edge-connectivity of dypergraphs like for graphs (see [34]). We note that Menger's theorem can be extended to dypergraphs by using Menger's theorem for the digraph that we get by changing every dyperedge to a new node and arcs from every tail of the dyperedge to this node and one arc from the new node to the head of the dyperedge. Frank, T. Király and

Z. Király [34] extended Theorem 2.4.4 to hypergraphs with an algorithmic proof. The case where $\ell = 0$ can be solved by using Edmonds' matroid partition algorithm [19]. To extend Algorithm 2.4.12 one needs this algorithm as a subroutine. Observe that the proof of the lemmas and Theorem 2.4.11 is nearly the same as for graphs. The single issue is the following. In the proof of the extension of Lemma 2.4.10 we cannot use the extended Lemma 2.4.9 for the reverse dypergraph as it cannot be well defined. Hence we need to prove a hypergraphic counterpart of Lemma 2.4.9 where rooted $(0, \ell)$-edge-connectivity is considered. Fortunately, the same proof works; one only needs to substitute δ with ϱ in the proof. Therefore, the extension of Algorithm 2.4.12 works well for hypergraphs.

2.4.3 A quicker algorithm for rooted $(k, 1)$-edge-connected orientability

In this section we modify Algorithm 2.4.12 to achieve a running time of $O(n^4 + n^3k)$ for graphs. The main idea of this modification is the following. Instead of reorienting every edge of G/R in each step, we keep the orientation given by D/R and augment it using the idea of *Step 3* of Algorithm 2.4.6. As in Section 2.4.1 for every $v \in V - r_0$, k arc-disjoint one-way paths will be stored from r_0 to v. It is easy to see that these paths give the same structure on D/R for a set R with $r_0 \in R \subseteq V$ if we cut down the first part of them from r_0 to their last node in R. Now we are ready to describe the algorithm.

Algorithm 2.4.14. Input: A graph $G = (V, E)$; and a root $r_0 \in V$.
Output: $\overrightarrow{G} = (V, \overrightarrow{E})$ an r_0-rooted $(k, 1)$-edge-connected orientation of G OR a partition $\mathcal{P}$ of V with $e_G(\mathcal{P}) < k(|\mathcal{P}| - 1) + 1$.

Step 1: Let $e_0 = r_0u \in E$ be an arbitrary edge. Decide whether there is an r_0-rooted k-edge-connected orientation of $G - e_0$ with Algorithm 2.4.6. If no such orientation exists, the subroutine outputs a partition $\mathcal{P}$ with $e_G(\mathcal{P}) - 1 \leq e_{G-e_0}(\mathcal{P}) < k(|\mathcal{P}| - 1)$ and we return this partition.

Step 2: Suppose now that Algorithm 2.4.6 has found an orientation and let $D = (V, \overrightarrow{E}')$ be the digraph that we get by taking this orientation on $G - e_0$ and orienting e_0 towards r_0. We will denote the directed pair of $e \in E$ in D with $\overrightarrow{e}$. Note that the k arc-disjoint r_0v-paths for every $v \in V - r_0$ given by Algorithm 2.4.6 for the orientation of $G - e_0$ are still present in D.

Step 3: Run any search algorithm to find the set of nodes R from which r_0 is reachable in D. If $R = V$, then D is $(k, 1)$-edge-connected so we can return $\overrightarrow{G} := D$. Otherwise, go to *Step 4*.

Step 4: Let $\overrightarrow{e}$ be an arc of D/R leaving r_R. Run an elimination step for $\overrightarrow{e}$ as in *Step 3* of Algorithm 2.4.6. If the elimination step fails output the partition that is given by *Step*

3 of Algorithm 2.4.6 (after substituting r_R with all the nodes in R in the member of the partition containing r_R). Otherwise, update D to this new graph along with adding the reversed pair $\overleftarrow{e}$ of $\overrightarrow{e}$ to it and go to *Step 3*. ♣

It is easy to see that Algorithm 2.4.14 works as it is just a modification of Algorithm 2.4.12. Hence we only prove that its running time is $O(n^4 + n^3k)$.

Theorem 2.4.15. *Algorithm 2.4.14 runs in time $O(n^4 + n^3k)$.*

Proof. As we have seen in Corollary 2.4.8, *Step 1-2* runs in time $O(n^4 + n^3k)$. *Step 3-4* runs $O(n)$ times since $|R|$ increases in each run of *Step 3*. In *Step 3* we run a search algorithm hence the running time of this step is $O(n^2)$. By Lemma 2.4.7 the running time of *Step 4* is $O(n^3)$ if we find an augmenting path and $O(n^4)$ otherwise, but this case could only happen once, when the algorithm terminates by outputting a partition. Thus the total running time of *Step 3-4* in the whole algorithm is $O(n^4)$. Therefore, the algorithm runs in time $O(n^4 + n^3k)$. □

2.5 Tree-compositions of bipartite graphs

A theorem of [37] gives a min-max formula for the minimum in-degree of T in a strongly connected orientation of a bipartite graph $G = (S, T; E)$. We sharpen this theorem by using the notion of tree-compositions, as follows.

Theorem 2.5.1. *Let $G = (S, T; E)$ be a 2-edge connected bipartite graph. Then*

$$\min\left\{\varrho_{\overrightarrow{G}}(T) : \overrightarrow{G} \text{ strongly connected}\right\} =$$
$$\max\left\{|\mathcal{T}| : \mathcal{T} \text{ a tree-composition of } T \text{ complying with } G\right\}.$$

We are going to derive this theorem from the following more general result.

Theorem 2.5.2. *Let $G = (V, E)$ be a graph and let $h : 2^V \to \mathbb{Z}_+$ be a crossing G-supermodular function with $h(V) = 0$ for which (2.5) holds. For a given subset $\emptyset \neq T \subset V$,*

$$\min\left\{\varrho_{\overrightarrow{G}}(T) : \overrightarrow{G} \text{ covers } h\right\} =$$
$$\max\left\{\widetilde{h}(\mathcal{T}) + \widetilde{i}(\mathcal{T}) - \Delta(\mathcal{T})|E| - i(T) : \mathcal{T} \text{ a tree-composition of } T\right\}.$$

In Section 2.4 the problem of finding an orientation covering a G-supermodular function was formulated as a submodular flow problem. Therefore, the fundamental result of Edmonds and Giles on total dual integrality of the linear system describing a submodular flow polyhedron implies a min-max result for the minimum of $\varrho_{\overrightarrow{G}}(T)$. The point in Theorem 2.5.2 is that a min-max formula could be given in a relatively simple and compact form.

Proof. In the second proof of Theorem 2.4.3, and let $p := i + h$ it was shown that $B'(p) \neq \emptyset$ follows by (2.5) for $p = i + h$. Observe that an integer vector $x \in B'(p)$ for which $\widetilde{x}(T) = \min\{\widetilde{m}(T) : m \in B'(p)\}$, forms the in-degree vector of an orientation of G covering h for which $\varrho(T)$ is minimal. Thus the theorem follows basically from Theorem 1.4.5 and Theorem 2.2.2. □

Theorem 2.5.2 can be used to prove the most important corollaries of Theorem 2.4.3. For example (with $h(X) := k$ for $\emptyset \neq X \subset V$) we can find a k-edge-connected orientation of a $2k$-edge-connected graph in which the in-degree of a given subset of nodes $T \subseteq V$ is minimal. We get the following theorem:

Theorem 2.5.3. *Let $G = (V, E)$ be a $2k$-edge-connected graph, and let $\emptyset \neq T \subset V$. Then*

$$\begin{aligned} &\min\left\{\varrho_{\vec{G}}(T) : \vec{G} \text{ } k\text{-edge-connected}\right\} = \\ &\max\left\{\widetilde{i}(\mathcal{T}) + k|\mathcal{T}| - \Delta(\mathcal{T})|E| - i(T) : \mathcal{T} \text{ a tree-composition of } T\right\}. \end{aligned} \tag{2.6}$$

Proof of Theorem 2.5.1. Consider the case of Theorem 2.5.3 when $G = (S, T; E)$ is a bipartite graph and $k = 1$. In this case we show that there is a tree-composition of T complying with G maximizing (2.6). Note that if one edge e of G is induced by $u_{\mathcal{T}}(e)$ members of a tree-composition $\mathcal{T}$ of T, then $u_{\mathcal{T}}(e) \leq \Delta(\mathcal{T})$ since its endpoint in S is covered by $\Delta(\mathcal{T})$ members of $\mathcal{T}$. If $\mathcal{T}$ is a tree-composition that complies with the graph, then $u_{\mathcal{T}}(e) = \Delta(\mathcal{T})$ for every $e \in E$. Let the deficit of an edge be $\Delta(\mathcal{T}) - u_{\mathcal{T}}(e)$ and $\gamma(\mathcal{T}) := \sum_{F \in \mathcal{T}} \Delta(\mathcal{T})|E| - i(F)$. Therefore, $\gamma(\mathcal{T})$ is the sum of the deficits. Note also that $i(T) = 0$.

Let $\mathcal{T}$ be a tree-composition of T maximizing (2.6) for which $\gamma(\mathcal{T})$ is minimum. Let $F = (U_S \cup U_T, A)$ be the directed tree representing $\mathcal{T}$ along with the map $\varphi : V \to U_S \cup U_T$.

Claim 2.5.4. *$\mathcal{T}$ is complying with G.*

Proof. For a contradiction, assume that $\varphi(s^*)\varphi(t^*) \notin A$ for $s^* \in S$, $t^* \in T$, $s^*t^* \in E$. Let P be the undirected path of length $2r + 1$ between $\varphi(s^*)$ and $\varphi(t^*)$. Shrink the set $V(P) \cap U_T$ in F and let F' be the resulting tree and $\mathcal{T}'$ the tree-composition of T that is represented by F'. It is easy to see that $|\mathcal{T}'| = |\mathcal{T}| - r$ and $\gamma(\mathcal{T}') \leq \gamma(\mathcal{T}) - r$, since the deficit of the edge s^*t^* becomes 0 from r and the deficit of the other edges does not increase. Therefore, $\widetilde{i}(\mathcal{T}) + |\mathcal{T}| - \Delta(\mathcal{T})|E| = |\mathcal{T}| - \gamma(\mathcal{T}) \leq |\mathcal{T}'| + r - (\gamma(\mathcal{T}') + r) = \widetilde{i}(\mathcal{T}') + |\mathcal{T}'| - \Delta(\mathcal{T}')|E|$ thus $\mathcal{T}'$ is also a maximizing tree-composition of T with smaller deficit, a contradiction. □

This completes the proof of Theorem 2.5.1. □ □

Finally, we note that using Theorem 2.5.1 one can simplify formula (2.6) for $k = 1$ with the following idea. Let $G = (V, E)$ be a 2-edge-connected graph, $\emptyset \neq T \subset V$ and $\mathcal{C}$ be the set system formed by the components of $G[T]$ and $G[V - T]$. Now the graph $G/\mathcal{C}$ (that is the graph that arises from G by contracting each member of $\mathcal{C}$) is bipartite and a strongly connected orientation of G determines a strongly connected orientation of $G/\mathcal{C}$. Conversely, any strongly connected orientation of $G/\mathcal{C}$ can be extended to a strongly connected orientation of G. This follows immediately from a theorem of Boesch and Tindell [7] stating that a mixed graph has a strongly connected orientation if and only if there is no cut-edge and there is no one-way cut. Therefore, it suffices to find a strongly connected orientation of the bipartite graph $G/\mathcal{C}$ where the in-degree of $T/\mathcal{C}$ is minimum.

Chapter 3

Arborescence-packings

Recent research in rigidity theory showed that some extensions of the well known results of Tutte [105] and Nash-Williams [86, 85] on packing trees and covering with trees can be applied to some rigidity classes. A recent result is from Katoh and Tanigawa [66] who proved that minimal rigidity of 'bar-slider frameworks' is equivalent to some colored rooted-forest packing properties. This result inspired an extensive research on the possible extensions of Tutte's and Nash-Williams' results. Katoh and Tanigawa [68] proved a theorem on the existence of colored rooted-forest packings. In [69], they generalized this result with some constrains arising from matroids along with showing a wide overview of possible applications in rigidity theory.

Frank [26] showed how to derive Nash-Williams' and Tutte's result [105, 86] (the $\ell = 0$ case of Theorem 1.1.1) from Edmonds' theorem [21] (Theorem 1.1.2) using the special case of Theorem 2.4.4 where $\ell = 0$. Following this idea Durand de Gevigney, Nguyen and Szigeti [18] generalized Edmonds' weak theorem to give an alternative proof of the packing part of [69]. One can find that [18] also generalizes the strong form of the result of Edmonds. This raises the question whether the earlier extensions of [21] such as the one of Kamiyama, Katoh and Takizawa [64] and of Fujishige [41] can be generalized to such a form. We answer the question positively by extending [64] and showing that [41] is an easy consequence of [64]. (For a survey on tree and arborescence packing, see [3] and [31, Chapter 10].)

We conclude the introduction by introducing some definitions used throughout this chapter. For a non-empty set $R \subseteq V$, $B = (V, A')$ is said to be an R-**branching** if it consists of $|R|$ node-disjoint arborescences whose roots are in R. Let $D = (V, A)$ be a digraph. Then an R-branching is said to be **spanning** if it spans the node set V and it is said to be **maximal** if it spans all the nodes that are reachable from R in D. For non-empty sets $X, Z \subseteq V$, let $Z \mapsto X$ denote that X and Z are *disjoint* and X is reachable from Z, that is, there is a directed path from Z to X.

Throughout this chapter, $D = (V, A)$ is a digraph, $\mathcal{M}$ is a matroid on S with rank function $r_{\mathcal{M}}$ and $\pi : \mathsf{S} \to V$ is a (not necessarily injective) map. The independent sets of $\mathcal{M}$ are the sets $\mathsf{P} \subseteq \mathsf{S}$ with $|\mathsf{P}| = r_{\mathcal{M}}(\mathsf{P})$. For $\mathsf{P} \subseteq \mathsf{S}$, we will denote by $\mathrm{Span}_{\mathcal{M}}(\mathsf{P})$ the subset of S spanned by P, that is, the maximal set $\mathsf{X} \subseteq \mathsf{S}$ for which $\mathsf{P} \subseteq \mathsf{X}$ and $r_{\mathcal{M}}(\mathsf{X}) = \mathsf{r}_{\mathcal{M}}(\mathsf{P})$. For related definitions and properties of matroids we refer to [31]. We say that the quadruple $(D, \mathcal{M}, \mathsf{S}, \pi)$ is a **matroid-based rooted-digraph**. As in [18], π is called **$\mathcal{M}$-independent** if $\pi^{-1}(v)$ is independent in $\mathcal{M}$ for each $v \in V$. For $X \subseteq V$, we will denote by S_X the set $\pi^{-1}(X)$.

In the next section, we summarize previous results on arborescence packings that we generalize in Section 3.2. The results of this chapter are based on [72].

3.1 Preliminaries

The weak form of the result of Edmonds [21] (Theorem 1.1.2) asserts that a digraph has k edge-disjoint spanning arborescences with root r_0 if and only if

$$\varrho(X) \geq k \tag{3.1}$$

holds for every $\emptyset \neq X \subset V - r_0$.

The strong form of Edmonds' theorem considers the case when we are given k edge-disjoint subarborescences of D with root r_0 and we want to extend these arborescences to edge-disjoint spanning arborescences. We formulate this theorem in another but equivalent form using the notion of branchings. For a family $\mathcal{R} := \{R_1, ..., R_k\}$ of non-empty subsets of V and $X \subseteq V$, let us denote by $p_{\mathcal{R}}(X)$ the number of the members of $\mathcal{R}$ disjoint from X and let us denote by $p'_{\mathcal{R}}(X)$ the number of R_i's for which $R_i \mapsto X$.

Theorem 3.1.1 ([21])**.** *In a digraph $D = (V, A)$, let $\mathcal{R} := \{R_1, ..., R_k\}$ be a family of non-empty subsets of V. Then there are edge-disjoint spanning R_i-branchings for $i = 1, ...k$ if and only if*

$$\varrho(X) \geq p_{\mathcal{R}}(X) \tag{3.2}$$

holds for every $X \subseteq V$. □

Kamiyama, Katoh and Takizawa [64] extended the result of Edmonds. We formulate their theorem in the following form (as [31]), that seems to be a bit stronger at first sight. However, it is easy to see that it is equivalent to the original form where each R_i consists of a single node.

Theorem 3.1.2 ([64])**.** *In a digraph $D = (V, A)$, let $\mathcal{R} := \{R_1, ..., R_k\}$ be a family of non-empty subsets of V. Then there are k edge-disjoint maximal R_i-branchings in D if and only if*

$$\varrho(X) \geq p'_{\mathcal{R}}(X) \tag{3.3}$$

holds for every $X \subseteq V$. □

Fujishige [41] extended this theorem. We present here a proof for Fujishige's theorem which shows that in fact it follows easily from Theorem 3.1.2. A set of nodes U is called convex if there is no node $v \in V - U$ for which $v \mapsto U$ and $U \mapsto v$.

Theorem 3.1.3 ([41]). *In a digraph $D = (V, A)$, let $R := \{r_1, ..., r_k\} \subseteq V$ be a list of (possibly not distinct) nodes and let $U_i \subseteq V$ be convex sets with $r_i \in U_i$. Then there are edge-disjoint arborescences with root r_i spanning U_i in D for $i = 1, ..., k$ if and only if*

$$\varrho(X) \geq p_R^{\{U_1,...,U_k\}}(X) \tag{3.4}$$

holds for every $X \subseteq V$, where $p_R^{\{U_1,...,U_k\}}(X)$ denotes the number of U_i's for which $U_i \cap X \neq \emptyset$ and $r_i \notin X$.

Proof. As the proof of the necessity of (3.4) is straightforward we only prove its sufficiency.

Let Z_i be the set of nodes reachable from r_i and let $R_i := Z_i - (U_i - r_i)$ for $i = 1, ..., k$. Then we claim that a maximal R_i-branching consists of the single nodes as roots in $R_i - \{r_i\}$ and an arborescence with root r_i spanning U_i for $i = 1, ..., k$. This statement follows from the following two observations. First, for $i = 1, ..., k$, $\delta_D(Z_i) = 0$ by definition thus $\delta_D(R_i - r_i) = \delta_D(Z_i - U_i) = 0$ by convexity of U_i as $U_i \mapsto v$ for $v \in R_i - r_i$. Second, for $i = 1, ..., k$, it is easy to see that there is an arborescence with root r_i spanning U_i by (3.4). Thus the existence of edge-disjoint R_i-branchings is equivalent to the existence of edge-disjoint arborescences with root r_i spanning U_i.

It is easy to see that $p_R^{\{U_1,...,U_k\}}(X) \geq p'_{\{R_1,...,R_k\}}(X)$ for $X \subseteq V$. Thus for $X \subseteq V$, if (3.4) holds, then (3.3) also holds. Therefore, the theorem follows by Theorem 3.1.2. □

Next we present the recent result of Durand de Gevigney, Nguyen and Szigeti [18] that generalizes Edmonds' results [21] in another direction. Following [18] we use the following definitions. The matroid-based rooted-digraph $(D, \mathcal{M}, \mathsf{S}, \pi)$ is called **rooted-$\mathcal{M}$-connected**, if

$$\varrho(X) \geq r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X) \tag{3.5}$$

holds for each $X \subseteq V$. An **$\mathcal{M}$-based packing of arborescences** in $(D, \mathcal{M}, \mathsf{S}, \pi)$ is a set $\{T_1, ..., T_{|\mathsf{S}|}\}$ of pairwise edge-disjoint arborescences in D such that T_i has root $\pi(\mathsf{s}_i)$ for $i = 1, ..., |\mathsf{S}|$ and the set $\{\mathsf{s}_j \in \mathsf{S} : v \in V(T_j)\}$ forms a base of $\mathcal{M}$ for each $v \in V$. The result of [18] is the following:

Theorem 3.1.4 ([18]). *Let $(D, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-based rooted-digraph. There exists an $\mathcal{M}$-based packing of arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$ if and only if π is $\mathcal{M}$-independent and $(D, \mathcal{M}, \mathsf{S}, \pi)$ is rooted-$\mathcal{M}$-connected.* □

Let $\mathcal{R} = \{R_1, ..., R_k\}$ be a multiset such as in Theorem 3.1.1. If $\mathsf{S} := \bigcup \mathcal{R}$ (as a multiset), π maps each occurrence of r in S to the node $r \in V$, and $\mathcal{M}$ is the partition matroid on S given by $\mathcal{R}$ where a set $\mathsf{P} \subseteq \mathsf{S}$ is independent if and only if $|\mathsf{P} \cap R_i| \leq 1$ for $i = 1, ..., k$, then the problem of $\mathcal{M}$-based packing of arborescences and that of edge-disjoint spanning R_i-branchings for $i = 1, ...k$ coincide. Moreover, in this case π is always $\mathcal{M}$-independent and (3.5) is equivalent to (3.2). Therefore, Theorem 3.1.1 follows from Theorem 3.1.4. However, Theorem 3.1.2 cannot be deduced from this theorem. In the next section we will extend Theorem 3.1.4 to a theorem from which Theorem 3.1.2 follows.

In our proof, we will use the following technical claim (implicitly) proved in [18]:

Claim 3.1.5 ([18]). *Let $\mathcal{M}$ be a matroid on S with rank function $r_{\mathcal{M}}$ and $\mathsf{P}, \mathsf{Q} \subseteq \mathsf{S}$ such that $r_{\mathcal{M}}(\mathsf{P}) + r_{\mathcal{M}}(\mathsf{Q}) = r_{\mathcal{M}}(\mathsf{P} \cap \mathsf{Q}) + r_{\mathcal{M}}(\mathsf{P} \cup \mathsf{Q})$. Then $Span_{\mathcal{M}}(\mathsf{P}) \cap Span_{\mathcal{M}}(\mathsf{Q}) \subseteq Span_{\mathcal{M}}(\mathsf{P} \cap \mathsf{Q})$.* □

3.2 Main result

In this section we prove our main result. Let $P(X) := X \cup \{v \in V - X : v \mapsto X\}$. We call a **maximal $\mathcal{M}$-independent packing of arborescences** a set $\{T_1, ..., T_{|\mathsf{S}|}\}$ of pairwise edge-disjoint arborescences for which T_i has root $\pi(\mathsf{s}_i)$ for $i = 1, ..., |\mathsf{S}|$, the set $\{\mathsf{s}_j \in \mathsf{S} : v \in V(T_j)\}$ is independent in $\mathcal{M}$ and $|\{\mathsf{s}_j \in \mathsf{S} : v \in V(T_j)\}| = r_{\mathcal{M}}(\mathsf{S}_{P(v)})$. (We will also say that s_i is the root of T_i.)

Theorem 3.2.1. *Let $(D, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-based rooted digraph. There exists a maximal $\mathcal{M}$-independent packing of arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$ if and only if π is $\mathcal{M}$-independent and*

$$\varrho(X) \geq r_{\mathcal{M}}(\mathsf{S}_{P(X)}) - r_{\mathcal{M}}(\mathsf{S}_X) \tag{3.6}$$

holds for each $X \subseteq V$.

One can see that Theorem 3.1.2 follows from this theorem in the same way as Theorem 3.1.1 did from Theorem 3.1.4.

Proof. As the necessity of (3.6) and $\mathcal{M}$-independency is straightforward we only prove the sufficiency.

For $X \subseteq V$, let $p(X) := r_{\mathcal{M}}(\mathsf{S}_{P(X)}) - r_{\mathcal{M}}(\mathsf{S}_X)$. X is called **tight** if $p(X) = \varrho(X)$. We call two sets X and Y **intersecting** if $X - Y, Y - X$ and $X \cap Y$ are non-empty sets. Although the supermodularity of p cannot be proven, in the proof of the next lemma we will prove it in a special case and use it to prove that in this special case the intersection of two tight sets is tight.

Lemma 3.2.2. *Let X and Y be two intersecting tight subsets of V. If $v \mapsto X \cap Y$ for every $v \in Y - X$, then $X \cap Y$ is tight and $Span_{\mathcal{M}}(\mathsf{S}_X) \cap Span_{\mathcal{M}}(\mathsf{S}_Y) \subseteq Span_{\mathcal{M}}(\mathsf{S}_{X\cap Y})$.*

Proof. Let X and Y be two intersecting tight subsets of V for which $Y - X \mapsto X \cap Y$. First we prove that

$$p(X) + p(Y) \leq p(X \cup Y) + p(X \cap Y). \tag{3.7}$$

As $v \mapsto X \cap Y$ for every $v \in Y - X$ and the reachability is transitive, we get $P(Y) \subseteq P(X \cap Y)$. Furthermore, it is easy to see that $P(X) \subseteq P(X \cup Y)$. Thus by the monotonicity of the rank function,

$$r_{\mathcal{M}}(\mathsf{S}_{P(X)}) + r_{\mathcal{M}}(\mathsf{S}_{P(Y)}) \leq r_{\mathcal{M}}(\mathsf{S}_{P(X\cup Y)}) + r_{\mathcal{M}}(\mathsf{S}_{P(X\cap Y)}). \tag{3.8}$$

It is easy to see that $\mathsf{S}_X \cap \mathsf{S}_Y = \mathsf{S}_{X\cap Y}$ and $\mathsf{S}_X \cup \mathsf{S}_Y = \mathsf{S}_{X\cup Y}$. Thus by the submodularity of the rank function,

$$r_{\mathcal{M}}(\mathsf{S}_X) + r_{\mathcal{M}}(\mathsf{S}_Y) \geq r_{\mathcal{M}}(\mathsf{S}_{X\cup Y}) + r_{\mathcal{M}}(\mathsf{S}_{X\cap Y}). \tag{3.9}$$

Subtracting (3.9) from (3.8) we get (3.7).

From the tightness of X and Y, (3.7), (3.6) and submodularity of ϱ, we get

$$\varrho(X) + \varrho(Y) = p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y) \leq$$

$$\leq \varrho(X \cap Y) + \varrho(X \cup Y) \leq \varrho(X) + \varrho(Y). \tag{3.10}$$

Hence $p(X \cap Y) + p(X \cup Y) = \varrho(X \cap Y) + \varrho(X \cup Y)$. Thus $X \cap Y$ and $X \cup Y$ are tight. Moreover, $p(X)+p(Y) = p(X\cap Y)+p(X\cup Y)$. Hence in (3.9) equality must hold. Thus by Claim 3.1.5 and $\mathsf{S}_X \cap \mathsf{S}_Y = \mathsf{S}_{X\cap Y}$, we get $\mathrm{Span}_{\mathcal{M}}(\mathsf{S}_X) \cap \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_Y) \subseteq \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_{X\cap Y})$. □

The following claim follows easily for Lemma 3.2.2

Claim 3.2.3. *Let X and Y be two intersecting subsets of V such that X is tight and $\varrho(Y) = 0$. If $v \mapsto X \cap Y$ for every $v \in Y - X$, then $X \cap Y$ is tight and $Span_{\mathcal{M}}(\mathsf{S}_X) \cap Span_{\mathcal{M}}(\mathsf{S}_Y) \subseteq Span_{\mathcal{M}}(\mathsf{S}_{X\cap Y})$.*

Proof. By (3.6) and the monotonicity of the rank function, we get

$$0 = \varrho(Y) \geq r_{\mathcal{M}}(\mathsf{S}_{P(Y)}) - r_{\mathcal{M}}(\mathsf{S}_Y) \geq 0.$$

Thus Y is tight and the claim follows by Lemma 3.2.2. □

Following [18], we introduce some definitions. For $X, Y \subseteq V$, Y **dominates** X if $\mathsf{S}_X \subseteq \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_Y)$. It is easy to see that domination is a transitive relation. An edge $uv \in A$ is said to be **good** if v dominates u, otherwise it is **bad**. We call a bad edge **s-bad**, if $\pi(\mathsf{s}) = u$ and $\mathsf{s} \notin \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_v)$. Note that a bad edge is s-bad for at least one $\mathsf{s} \in \mathsf{S}$. For $\mathsf{s} \in \mathsf{S}$, we call a *tight* set $X \subseteq V$ **s-critical** if $\mathsf{s} \in \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_X)$ and there exists a node v for which $\pi(\mathsf{s})v$ is an s-bad edge of D that enters X.

By transitivity of the domination, if there is no bad edge, then $\mathsf{S}_{P(v)} \subseteq \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_v)$. Thus the arborescences consisting of the roots, (that is, each node v is an arborescence consisting of this single node $|\mathsf{S}_v|$ times), form a maximal $\mathcal{M}$-independent packing of arborescences. (Note that the $\mathcal{M}$-independency of π ensures the independency of the roots.) From now on, we will prove by induction on $|A|$.

Let $\pi(\mathsf{s})v$ be an s-bad edge such that it enters no s-critical set where $\mathsf{s} \in \mathsf{S}$. Let $D' := D - \pi(\mathsf{s})v$, $\mathsf{S}' := \mathsf{S} \cup \{\mathsf{s}'\}$ where $\mathsf{s}' \notin \mathsf{S}$, $\pi' : \mathsf{S}' \to V$ such that $\pi'|_{\mathsf{S}} \equiv \pi$ and $\pi'(\mathsf{s}') = v$ and let $\mathcal{M}'$ be the matroid on S' that is obtained from $\mathcal{M}$ by considering s' as an element parallel to s. For $X \subseteq V$, let $P'(X) := X \cup \{v \in V : v \mapsto_{D'} X\}$ and $\mathsf{S}'_X := \pi^{-1}(X)$.

Now π' is $\mathcal{M}'$-independent since $\pi(\mathsf{s})v$ was an s-bad edge. We claim that

$$\varrho_{D'}(X) \geq r_{\mathcal{M}'}(\mathsf{S}'_{P'(X)}) - r_{\mathcal{M}'}(\mathsf{S}'_X) \tag{3.11}$$

holds for every $X \subseteq V$, hence there exist a maximal $\mathcal{M}'$-independent packing of arborescences $\mathcal{P}'$ in $(D', \mathcal{M}', \mathsf{S}', \pi')$ by induction. Take an arbitrary set $X \subseteq V$. To prove (3.11), first observe that $P'(X) \subseteq P(X)$ and they are not equal if and only if $v \in P'(X)$ and $\pi(\mathsf{s}) \notin P'(X)$ both hold. Thus $r_{\mathcal{M}'}(\mathsf{S}'_{P'(X)}) \leq r_{\mathcal{M}}(\mathsf{S}_{P(X)})$ by the definition of $\mathcal{M}'$. Also by definition, $r_{\mathcal{M}}(\mathsf{S}'_X) \geq r_{\mathcal{M}}(\mathsf{S}_X)$. Thus the right side of (3.6) does not increase in (3.11). Therefore, if $X \subseteq V$ is not tight, then (3.11) holds trivially as $\varrho_{D'}(X)+1 \geq \varrho(X)$ and (3.6) holds with '>'; if $X \subseteq V$ is tight but $\pi(\mathsf{s})v$ does not enter X, then (3.11) holds trivially as $\varrho_{D'}(X) = \varrho(X)$. If X is tight and $\pi(\mathsf{s})v$ enters X, then $r_{\mathcal{M}}(\mathsf{S}'_X) > r_{\mathcal{M}}(\mathsf{S}_X)$ because $\mathsf{s} \in \mathrm{Span}_{\mathcal{M}'}(\mathsf{S}'_X)$ as $s' \in \mathsf{S}'_X$ but $\mathsf{s} \notin \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_X)$ since $\pi(\mathsf{s})v$ enters no s-critical set. Thus in this case, $\varrho(X) = \varrho_{D'}(X) + 1$ and $r_{\mathcal{M}}(\mathsf{S}_{P(X)}) - r_{\mathcal{M}}(\mathsf{S}_X) \geq r_{\mathcal{M}'}(\mathsf{S}'_{P'(X)}) - r_{\mathcal{M}'}(\mathsf{S}'_X) + 1$ hence (3.11) is again a consequence of (3.6).

Since s and s' are parallel in $\mathcal{M}'$, the arborescences $T, T' \in \mathcal{P}'$ rooted at s and s' are node-disjoint. Therefore, $T \cup T' \cup \pi(\mathsf{s})v$ is an arborescence rooted at $\pi(\mathsf{s})$ and $\mathcal{P} = \mathcal{P} - \{T, T'\} \cup \{T \cup T' \cup \pi(\mathsf{s})v\}$ is a packing of arborescences rooted at S in D. To see that $\mathcal{P}$ is a maximal $\mathcal{M}$-independent packing of arborescences, first observe that the $\mathcal{M}$-rank of the root set of the arborescences in $\mathcal{P}$ covering an arbitrary node u is the same as the $\mathcal{M}'$-rank of the root set of the arborescences in $\mathcal{P}'$ covering u by the definitions of $\mathcal{M}'$ and $\mathcal{P}'$. As $\mathcal{P}'$ is a maximal $\mathcal{M}'$-independent packing of arborescences this latter value is equal to $r_{\mathcal{M}'}(\mathsf{S}'_{P'(u)})$.

To prove that $\mathcal{P}$ is a maximal $\mathcal{M}$-independent packing of arborescences, we show that

$r_{\mathcal{M}'}(\mathsf{S}'_{P'(u)}) = r_{\mathcal{M}}(\mathsf{S}_{P(u)})$ for all $u \in V$. Observe that $r_{\mathcal{M}'}(\mathsf{S}'_{P'(u)}) \leq r_{\mathcal{M}}(\mathsf{S}_{P(u)})$ is obvious hence we only prove '$\geq$'. Suppose to the contrary, that $r_{\mathcal{M}'}(\mathsf{S}'_{P'(u)}) < r_{\mathcal{M}}(\mathsf{S}_{P(u)})$ for a given $u \in V$. Since $r_{\mathcal{M}'}(\mathsf{S}'_Q) \geq r_{\mathcal{M}}(\mathsf{S}_Q)$ holds for any $Q \subseteq V$, $P'(u) \neq P(u)$ in this case. Hence $v \in P'(u)$ but $\pi(\mathsf{s}) \notin P'(u)$ because D and D' differ only on the edge $\pi(\mathsf{s})v$. Therefore, $\pi(\mathsf{s})v$ is the single edge of D that enters $P'(u)$. Thus inequality (3.6) for $X = P'(u)$ is

$$1 = \varrho(P'(u)) \geq r_{\mathcal{M}}(\mathsf{S}_{P(P'(u))}) - r_{\mathcal{M}}(\mathsf{S}_{P'(u)}) = r_{\mathcal{M}}(\mathsf{S}_{P(u)}) - r_{\mathcal{M}}(\mathsf{S}_{P'(u)})$$

and hence, by our assumption,

$$r_{\mathcal{M}}(\mathsf{S}_{P'(u)}) + 1 \geq r_{\mathcal{M}}(\mathsf{S}_{P(u)}) \geq r_{\mathcal{M}'}(\mathsf{S}'_{P'(u)}) + 1 \geq r_{\mathcal{M}}(\mathsf{S}_{P'(u)}) + 1.$$

Therefore, equality holds. From $r_{\mathcal{M}}(\mathsf{S}_{P'(u)}) + 1 = r_{\mathcal{M}}(\mathsf{S}_{P(u)})$, we get that $P'(u)$ is tight, and by $r_{\mathcal{M}'}(\mathsf{S}'_{P'(u)}) = r_{\mathcal{M}}(\mathsf{S}_{P'(u)})$, we get that $\mathsf{s} \in \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_{P'(u)})$. Thus $P'(u)$ is s-critical, a contradiction.

To finish our proof, we must show that there exists an $\mathsf{s} \in \mathsf{S}$ and an s-bad edge of the form $\pi(\mathsf{s})v$ such that it enters no s-critical set. First we show the following lemma where $D[X]$ denotes the subgraph of D spanned by X:

Lemma 3.2.4. *For $\mathsf{s} \in \mathsf{S}$, let $\pi(\mathsf{s})v$ be an s-bad edge and X be a minimal s-critical set with $v \in X$. Then $X \subseteq P(v)$. Moreover, v is reachable from all points of X in $D[X]$.*

Proof. Assume for a contradiction that X is not a subset of $Y := P(v)$. Then X and Y are intersecting sets, $\varrho(Y) = 0$ and $v \in X \cap Y$ is reachable from all points of $Y - X$. As X is tight, $X \cap Y$ is also tight by Claim 3.2.3. Moreover, X is s-critical hence $\mathsf{s} \in \mathrm{Span}_{\mathcal{M}}\mathsf{S}_X$; furthermore, $\mathsf{s} \in \mathrm{Span}_{\mathcal{M}}\mathsf{S}_Y$ as $\pi(\mathsf{s}) \in Y$. Thus $\mathsf{s} \in \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_{X\cap Y})$ also holds by Claim 3.2.3. Therefore, $X \cap Y \subset X$ is an s-critical set such that $\pi(\mathsf{s})v$ enters it, contradicting the minimality of X.

To prove the second part, assume for a contradiction that v is not reachable from all nodes of X in $D[X]$. Let Y' denote the subset of X from which v is reachable in $D[X]$. Then $\varrho(Y') \leq \varrho(X)$. Furthermore, $P(Y') = P(X) = Y$ by the first part. As Y' is not s-critical by the minimality of X, $\mathsf{s} \notin \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_{Y'})$ and thus $r_{\mathcal{M}}(\mathsf{S}_{Y'}) < r_{\mathcal{M}}(\mathsf{S}_X)$. Therefore,

$$\varrho(Y') \leq \varrho(X) = r_{\mathcal{M}}(\mathsf{S}_{P(X)}) - r_{\mathcal{M}}(\mathsf{S}_X) < r_{\mathcal{M}}(\mathsf{S}_{P(Y')}) - r_{\mathcal{M}}(\mathsf{S}_{Y'}),$$

contradicting (3.6). □

Suppose that for all $\mathsf{s} \in \mathsf{S}$ and for each s-bad edge, there exists an s-critical set that is entered by $\pi(\mathsf{s})v$. We call a set **critical** if there exists an $\mathsf{s} \in \mathsf{S}$ for which it is s-critical. Choose a minimal critical set X. We can assume that, say, X is s-critical for $\mathsf{s} \in \mathsf{S}$ and $\pi(\mathsf{s})v$ is an s-bad edge entering X.

If there is no bad edge spanned by X, then $\mathrm{Span}_{\mathcal{M}}(\mathsf{S}_X) \subseteq \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_v)$ by the minimality of X and Lemma 3.2.4. Moreover, as X is s-critical $\mathsf{s} \in \mathrm{Span}_{\mathcal{M}}\mathsf{S}_X$. Thus $\mathsf{s} \in \mathrm{Span}_{\mathcal{M}}\mathsf{S}_v$, that is, v dominates $\pi(\mathsf{s})$ contradicting that $\pi(\mathsf{s})v$ is an s-bad edge.

Thus there is a bad edge $u'v'$ spanned by X. By our assumption, there is an s'-critical set X' for any $\mathsf{s}' \in \mathsf{S}_{u'} - \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_{v'})$ such that $u'v'$ enters X'. Let us use the notation $Z := P(v')$. Then using Claim 3.2.3, one can prove that $Z \cap X$ is tight as $\varrho(Z) = 0$ and $v' \in Z \cap X$ is reachable from all points of $Z - X$. Moreover, since $(Z \cap X) - X' \subseteq Z - v' = P(v') - v' \mapsto \{v'\} \subseteq Z \cap X \cap X'$, we get that $Z \cap X \cap X'$ is tight by Lemma 3.2.2. Thus $Z \cap X \cap X'$ is s'-critical and $Z \cap X \cap X' \subset X$ because $u' \in X - X'$ and $v' \in Z \cap X \cap X'$. Therefore, $Z \cap X \cap X'$ is a proper critical subset of X, a contradiction.

This finishes the proof of Theorem 3.2.1. □ □

One can see that the proof of Theorem 3.2.1 gives rise to an algorithm if the matroid is given by an oracle for the rank function.

3.3 Concluding remarks

Durand de Gevigney, Nguyen and Szigeti [18] also gave an algorithm for the weighted case of their problem using polyhedral techniques. This, along with their proof for the undirected case, uses the fact that the right-hand side function of (3.5) is supermodular. As noted at the beginning of the proof of Theorem 3.2.1, the supermodularity right-hand side of (3.6) cannot be proved but in a very special case. Moreover, we do not see any meta-property of this function. However, developing such a property could help to give an algorithm for the weighted case. Bérczi, T. Király and Kobayashi [4] proved recently Theorem 3.2.1 by extending the method of Frank [27], Szegő [98] and Frank and Bérczi [2]. Their proof is more complicated than the proof given in this chapter however it also gives rise to an algorithm for the weighted case as in their extension the bounding function is a positively intersecting supermodular function on set pairs.

Chapter 4

Balanced generic circuits without long paths

All graphs considered in this chapter are simple (that is, may not contain loops or multiple edges). We recall the result of Nash-Williams [85] that a graph G is decomposable into k spanning forests if and only if $i(X) \leq k(|X|-1)$ for all non-empty subsets $X \subseteq V$. From this theorem it follows that every generic circuit is decomposable into two spanning trees, moreover we obtain the following proposition.

Proposition 4.0.1. *A graph G is a generic circuit if and only if it can be decomposed into two spanning trees such that no pair of proper subtrees, except single nodes, spans the same node set.* □

Having this result one may ask whether a tree decomposition with some special properties exists. Graver, B. Servatius and H. Servatius posed the following problem.

Open question 4.0.2 ([48], Exercise 4.69)**.** *Does every generic circuit with nodes of degrees 3 and 4 only, have a two tree decomposition into two Hamiltonian paths?*

One can extend this question for $(k, k+1)$-circuits that has nodes with degrees $2k-1$ and $2k$ only. Such a graph in which the difference between the maximum and minimum degree is at most one is called **balanced**. Note that the smallest (simple) $(k, k+1)$-circuit is K_{2k} since a graph with $k|V|-k$ edges cannot be simple if it has less than $2k$ nodes. A balanced $(k, k+1)$-circuit has nodes with degrees $2k-1$ and $2k$ only and the number of nodes with degree $2k-1$ is exactly $2k$. Hence one may ask if it is decomposable into k Hamiltonian paths.

In this chapter, we show balanced $(k, k+1)$-circuits for all $k \geq 2$ which do not contain a Hamiltonian path. Thus a balanced generic circuit does not always have a decomposition demanded in Open question 4.0.2. Moreover, we have a stronger result on the length of the longest paths in balanced $(k, k+1)$-circuits. For a graph G, let $h(G)$ and $h^*(G)$

denote the length of a maximum cycle and the number of nodes in a maximum path of G, respectively. Following [50] we define the **shortness exponent** $\sigma(\mathcal{G})$ for a family $\mathcal{G}$ of graphs, as follows:

$$\sigma(\mathcal{G}) = \liminf_{G \in \mathcal{G}} \frac{\log h(G)}{\log |V(G)|}.$$

Concerning paths instead of cycles we also define the parameter $\sigma^*(\mathcal{G})$ in a similar way but with $h^*(G)$ in place of $h(G)$. Let $\mathcal{C}^{bal.}_{k,k+1}$ denote the family of balanced $(k, k+1)$-circuits. What we prove is as follows:

Theorem 4.0.3. *$\sigma^*(\mathcal{C}^{bal.}_{k,k+1}) \leq \frac{\log 8}{\log 9}$ for all $k \geq 2$.*

In the next section, we reformulate the question to a similar question on graphs with connectivity prescriptions. This along with a result of Grünbaum and Malkevitch [49] immediately gives a negative answer to Open question 4.0.2. In Section 4.2, we summarize some properties of 3-regular 3-connected graphs. Bondy and Simonovits [10] showed that the shortness exponent of this graph class is at most $\frac{\log 8}{\log 9}$. We use this result for the proof of Theorem 4.0.3 in Section 4.3. Finally in Section 4.4, we show a small generic circuit without a Hamiltonian path and we give a constructive characterization theorem for 4-regular essentially 6-edge-connected graphs.

The results of this chapter are joint with Ferenc Péterfalvi [75].

4.1 Preliminaries

We recall the definition of essential k-edge-connectivity. A graph G is essentially k-edge-connected if all nontrivial edge-cuts of G contain at least k edges. Our construction will be based on the observation that balanced $(k, k+1)$-circuits are just the $2k$-regular essentially $(2k+2)$-edge-connected graphs minus a node in the following sense:

Lemma 4.1.1.

(i) Let $G = (V, E)$ be a balanced $(k, k+1)$-circuit. If we add a new node s to G and connect it to the nodes of G with degree $2k-1$ then the obtained graph $G' = (V', E')$ is $2k$-regular and essentially $(2k+2)$-edge-connected.

(ii) Let $G' = (V', E')$ be a $2k$-regular essentially $(2k+2)$-edge-connected graph. If we omit an arbitrary node s of G' then the obtained graph $G = (V, E)$ is a balanced $(k, k+1)$-circuit.

Proof. (i) It is clear that G' is $2k$-regular. Suppose that it is not essentially $(2k+2)$-edge-connected. Then there is a subset $X \subseteq V'$, $2 \leq |X| \leq |V'| - 2$ with $d_{G'}(X) \leq 2k$. (As G' is an Eulerian graph $d_{G'}(X) = 2k + 1$ cannot hold.) We may assume that $s \notin X$. Then $2i_G(X) = 2i_{G'}(X) = (\sum_{v \in X} d_{G'}(v)) - d_{G'}(X) \geq 2k|X| - 2k$, a contradiction.

(ii) Let $X \subseteq V$ be a subset with $2 \leq |X| \leq |V| - 1$. Then the same calculation as above admits that $i_G(X) = i_{G'}(X) \leq k|X| - k - 1$. □

We call the graph G' obtained from a balanced $(k, k+1)$-circuit G as described in Lemma 4.1.1 (i) the **underlying regular graph** of G.

In the case of balanced generic circuits we obtain 4-regular essentially 6-edge-connected graphs. Now we can easily prove that not all balanced generic circuits can be decomposed into two Hamiltonian paths. It is clear that if a balanced generic circuit G admits such a decomposition, then the four end-nodes of the two Hamiltonian paths must be disjoint and can only be the four nodes with degree 3. Thus in the underlying 4-regular graph G' they extend to a decomposition into two Hamiltonian cycles. Therefore, it is sufficient to show a 4-regular essentially 6-edge-connected graph which does not have a decomposition into two Hamiltonian cycles.

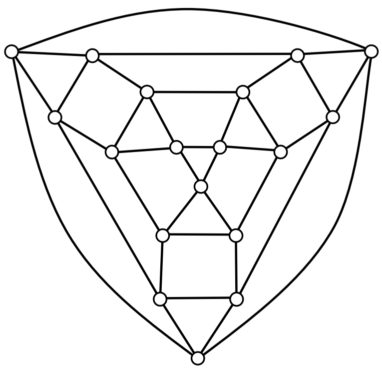


Figure 4.1: The graph given by Grünbaum and Malkevitch.

Grünbaum and Malkevitch [49] gave an example for a 4-regular 4-connected planar graph without a Hamiltonian decomposition (see Figure 4.1). One can easily check that it is essentially 6-edge-connected therefore this is a good example for our problem as well.[1] (For an elegant proof that this graph is not decomposable, using the fact that it is the medial graph of the Herschel graph, see [8].)

[1] This result was observed independently in [70].

4.2 3-regular 3-connected graphs without long paths

In our further constructions, we will use 3-regular 3-connected graphs as a starting point. In this section, we review some of their properties. Let us denote the set of these graphs with $\mathcal{B}$.

Examining their shortness parameters Bondy and Simonovits [10] constructed graphs certifying that $\sigma(\mathcal{B}) \leq \frac{\log 8}{\log 9} \simeq 0.9464$ and Jackson [53] gave the lower bound $\sigma(\mathcal{B}) \geq \log_2(1+\sqrt{5}) - 1 \simeq 0.6942$. The lower bound was recently improved in [6] to $\lambda \simeq 0.753$, where λ is the real root of $4^{1/x} - 3^{1/x} = 2$. Now considering the case of longest paths we have the same lower bound for $\sigma^*(\mathcal{B})$ as $h^*(G) \geq h(G)$ for any graph G. On the other hand Bondy and Locke [9] showed that $h^*(G) \leq \frac{3}{2}h(G) - 2$ for all 3-regular 3-connected graphs G, which implies $\sigma^*(\mathcal{B}) = \sigma(\mathcal{B})$. We can also observe that the construction of Bondy and Simonovits actually works for paths as well. The fact that a graph constructed in such a way does not contain a Hamiltonian path either was also used by Singleton [95]. In what follows we will briefly sketch the construction.

First we need to define the inverse operation of contraction. For this, we recall the definition of contraction. For a graph $G = (V \cup X, E)$ with $V \cap X = \emptyset$, let s be a new node. We say that the (multi)graph $G' = (V + s, E')$ is obtained from G by **contracting** X into s if E' is obtained from E by deleting the edges $E[X]$ and replacing all edges in $\{uv : u \in X,\ v \in V\}$ with an edge sv. Note that this definition admits G' to have multiple edges but in our case the obtained graphs will be simple. We will also denote with G/X the graph obtained from G by contracting X. To well define the inverse operation we need to specify from which edges of G are obtained the edges of G'. Let F be a graph on the node set $X + t$ with $t \notin V \cup X + s$. We say that G is obtained from G' by **inflating** (F, X) **into** s if $G/X = G'$ and $G/V = F$.

Considering 3-regular 3-connected graphs we can establish some special properties of these operations. First observe that if G is 3-regular then it is 3-connected if and only if it is 3-edge-connected.

Proposition 4.2.1 ([53]). *Let $G = (V, E)$ be a 3-regular 3-connected graph and let $C = E(X, V - X)$ be a nontrivial edge-cut of size 3. Then G/X and $G/(V - X)$ are also 3-regular 3-connected (simple) graphs.*

Proof. They are simple because C consists of 3 independent edges by the 3-(node)-connectivity of G. The 3-connectivity follows from the fact that any edge-cut in G/X or $G/(V - X)$ is also an edge-cut in G. □

Proposition 4.2.2. *Let G' and F be 3-regular 3-connected graphs on node sets $V + s$ and $X + t$ respectively. Let G be a graph on $V \cup X$ obtained from G' by inflating (F, X) into s. Then G is also 3-regular and 3-connected.*

Proof. First observe that $G[X] = F[X]$ and $G[V] = G'[V]$ are 2-connected because they are both obtained from a 3-connected graph by a deletion of a node. Now let $u, v \in G$ be different nodes. We need to show that $G - \{u, v\}$ is connected. If $\{u, v\} \subseteq V$ then $(G - \{u, v\})/X$ is connected by the 3-connectivity of G'. Since $F - t$ is connected, inflating (F, X) into s preserves connectivity. The case $\{u, v\} \subseteq X$ is similar. If $|\{u, v\} \cap V| = 1$, say $u \in V$ and $v \in X$ then $G - \{u, v\}$ consists of the two connected subgraphs $G[V] - u$ and $G[X] - v$ and at least one edge connecting them. □

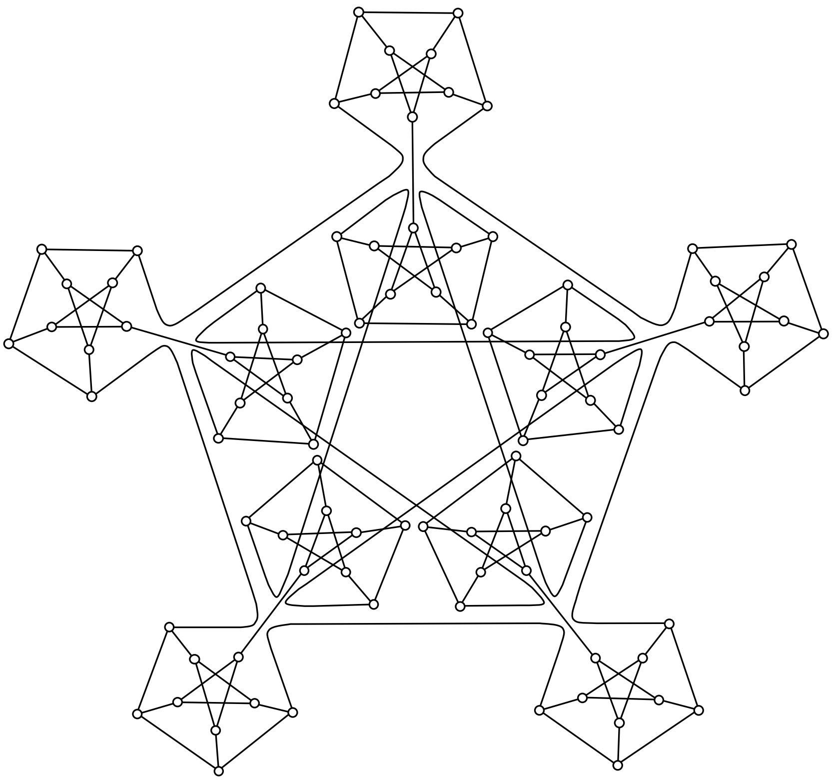


Figure 4.2: The graph S_1 constructed by Bondy and Simonovits.

Let $G_P = (V_P, E_P)$ be the Petersen graph and let t be an arbitrary node of G_P. Now we can describe the sequence of graphs $S_0, S_1, \ldots, S_i, \ldots$ constructed by Bondy and Simonovits. Let $S_0 = G_P$ and if S_i is already constructed let S_{i+1} be obtained from S_i by inflating $(G_P, V_P - t)$ into each node (see Figure 4.2). By Proposition 4.2.2 all S_i are 3-connected. Observe that if for a graph $G = (V \cup X, E)$, $G/X \cong G_P$ and P is a path in G with both of its ends in X, then P avoids at least one node of V. Otherwise, P/X is a Hamiltonian cycle in G/X which is isomorphic to the Petersen graph. Hence $h^*(S_{i+1}) \leq 8h^*(S_i) + 2$. Since $|V(S_i)| = 10 \cdot 9^i$ finally we get $\lim_{i\to\infty} \frac{\log h^*(S_i)}{\log |V(S_i)|} = \frac{\log 8}{\log 9}$.

Now we consider other properties of 3-regular 3-connected graphs. According to Petersen's theorem [88] all these graphs have a perfect matching. Moreover, by Schönberger's result [93] all edges of the graph are included in some perfect matching. We call a perfect matching M **admissible** if $|M \cap C| \leq 1$ for all edge-cuts C of size 3. The following lemma is a consequence of a result of Kaiser and Škrekovski [63] where the statement's complement is proved. For completeness we give here a direct proof.

Lemma 4.2.3. *Let $G = (V, E)$ be a 3-regular 3-connected graph and let $e \in E$ be an arbitrary edge of G. Then G has an admissible perfect matching which includes e.*

Proof. If the size of every nontrivial edge-cut of G is greater than 3 then every perfect matching is admissible. Now suppose that there is a nontrivial edge-cut $C = E(X, V - X)$ of size 3. By Proposition 4.2.1 $G_1 = G/X$ and $G_2 = G/(V - X)$ are 3-regular and 3-connected too so by induction they both have an admissible perfect matching including an arbitrary designated edge.

Case 1: $e \in C$. Consider an admissible perfect matching M_i in G_i including e for $i = 1, 2$. We claim that the union M of M_1 and M_2 (with one copy of e) is an admissible perfect matching in G. To prove this let D be an edge-cut with the corresponding node set Z. If Z or $V - Z$ is a subset of X or $V - X$ then D is also an edge-cut in G_1 or G_2 so it does not violate admissibility. Now suppose that D cuts both X and $V - X$. As we already observed in the proof of Proposition 4.2.2, $G[X]$ and $G[V - X]$ are 2-connected subgraphs. So $|D| \geq |E(X \cap Z, X - Z)| + |E((V - X) \cap Z, (V - X) - Z)| \geq 2 + 2 = 4$.

Case 2: $e \notin C$. We may assume by symmetry that $e \in I(V - X)$. Now consider an admissible perfect matching M_1 in G_1 which includes e. Let f be the edge in M_1 incident with the node in G_1 corresponding to X. Now let M_2 be an admissible perfect matching in G_2 which includes f. Just as in *Case 1* the perfect matching combining M_1 and M_2 is admissible in G. □

4.3 Our construction

Now we present our construction of balanced $(k, k+1)$-circuits which do not contain long paths. It is clear that $h^*(G) \leq h^*(G')$ for a $(k, k+1)$-circuit G and its underlying $2k$-regular graph G'. Therefore, to prove Theorem 4.0.3 it suffices to show the same proposition for $2k$-regular essentially $(2k+2)$-edge-connected graphs by Lemma 4.1.1.

For the construction we use a special operation similar to inflating that we call **blackberrizing**. This operation 'substitutes' all nodes of a t-regular graph with a special auxiliary graph that we call a **berry**. A berry is a simple graph that consists of two types of nodes that we call **inner nodes** and **outer nodes**, respectively (see Figure 4.3). We will use a berry with t outer nodes for blackberrizing a t-regular graph $G = (V, E)$. We take $|V|$ disjoint copies of the berry and correspond them to the nodes of G. If two nodes are adjacent in G we choose one outer node of both corresponding berries and identify

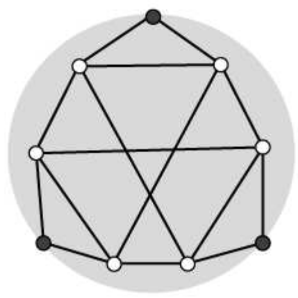


Figure 4.3: A copy of a berry which can be used in the case $k = 2$. The black nodes are the outer nodes, the white ones are the inner nodes.

them. For the t neighbors of a node we use the t different outer nodes (in any order). Finally, the inner nodes of a berry correspond to a certain node of the basic graph and an outer node correspond to an edge of the basic graph (see Figure 4.4).

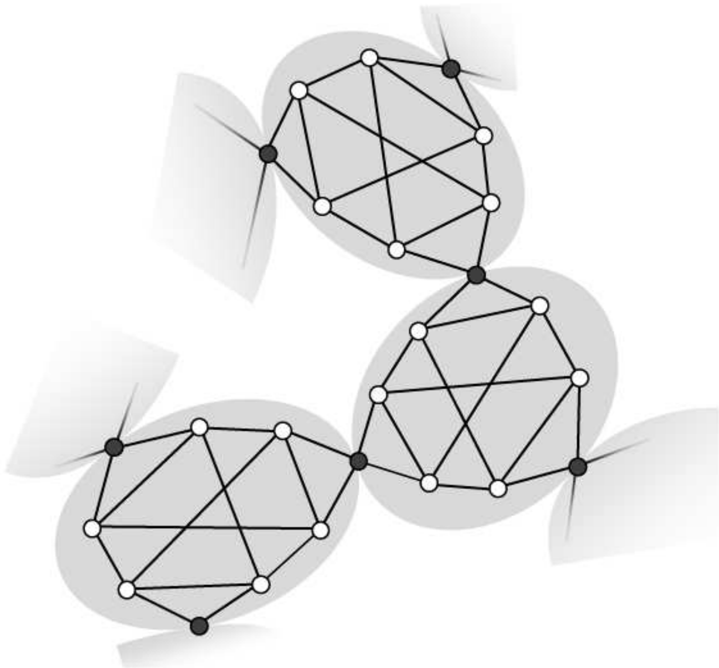


Figure 4.4: Adjacencies of berries in the blackberry graph.

To get $2k$-regular graphs by blackberrizing we need that the inner nodes of the berry are of degree $2k$ and the outer nodes are of degree k. In our construction we will use the graphs S_i described in Section 4.2 as basic graphs.

Since for odd k there are no graphs with three nodes of degree k while the other nodes are of degree $2k$, we need another operation that we call **parallel blackberrizing**. This operation substitutes the nodes of a t-regular graph that has a perfect matching. The substitution will be practically as before, but we use the copies of two types of berries.

The designated outer node of a berry that is substituted to a node v will be identified with the designated outer node of the berry that is substituted to the other endpoint of the matching edge incident to v. Therefore, when k is odd we will take a basic graph S_i (described in Section 4.2) that has an admissible perfect matching M by Lemma 4.2.3; and we will *parallel blackberrize* S_i *along* M. We will use two berries which have inner nodes of degree $2k$, two non-designated outer nodes of degree k and a designated outer node of degrees $k-1$ and $k+1$, respectively. Though it is not a requirement of this operation, from now on we will assume that the two berries used in parallel blackberrizing have the same number of nodes.

A graph that arises by (parallel) blackberrizing a regular graph is called a **blackberry graph**. Therefore, in both cases we get a $2k$-regular graph H as blackberry graph.

Examining the properties of this operation we first show that if the basic graph does not contain a long path, then neither does its blackberry graph. The proof is similar to the one in [87].

Proposition 4.3.1. *Let S be a 3-regular graph and let H be its blackberry graph. Let b be the number of nodes in the berries used in the (parallel) blackberrization. Then*

(i) $h^*(H) \leq (b-1) \cdot h^*(S) + 1$,

(ii) $h^*(H) \leq |V(H)| - (|V(S)| - h^*(S))(b-3)$.

Proof. Take an arbitrary path P in H and let us consider the sequence of its edges $e_1, e_2, e_3, \ldots, e_m$. To this sequence we associate a sequence of nodes of S in the following way: we replace every edge by the node of S which the berry including the edge corresponds to. If the resulting sequence contains identical neighboring elements then we delete some until only one remains consecutively. Let $v_1, v_2, \ldots, v_r$ denote the remaining sequence.

It is easy to check that the consecutive members of this sequence are neighbors in S and thus it describes a walk in S. We claim that if $\{i, j\} \cap \{1, r\} = \emptyset$ with $i < j$ then $v_i \neq v_j$. Suppose for contradiction that $v_i = v_j$, let B be the berry corresponding to this node of S and let e_p and e_q be the edges of H corresponding to v_{i-1} and v_{j+1}, respectively. Then in the original path in H, between e_p and e_q we enter and leave B two times, which contradicts that B has only 3 outer nodes.

Therefore, if a node occurs more than once in the sequence $v_1, \ldots, v_r$ then it is v_1 or v_r. In this case delete v_1 or v_r or both until every node occurs exactly once. Now we obtained a path P' in S. For any edge e in H the node of S corresponding to the berry including e is present on P'. We conclude that P contains edges from at most $h^*(S)$ berries. Since a path contains at most $b-1$ edges from a berry we have (i). On the other hand, if a path does not use any edge from a berry then it avoids all its inner nodes, which implies (ii). ∎

Now we show that if we choose appropriate berries the blackberry graph is essentially $(2k+2)$-edge-connected. To obtain this, we prove more general results for (parallel) blackberrizing.

Lemma 4.3.2. *Let $t \geq 3$ and $\alpha \geq 3$ be two integers and let G be a 3-connected t-regular simple graph and let β be a positive integer with $\beta \leq \frac{\alpha}{3}$. Let B be a berry with t outer nodes. Let $\mathcal{O}$ and $\mathcal{I}$ denote the set of the outer and the inner nodes of B respectively. Assume that B satisfies the following properties:*

(i) $d_B(v) = \alpha - \beta,\ \forall v \in \mathcal{O}$,

(ii) $d_B(v) = 2\alpha - 2\beta,\ \forall v \in \mathcal{I}$,

(iii) *if* $X \subseteq \mathcal{I}, |X| \geq 2$ *then* $d_B(X) \geq 2\alpha$.

(iv) *if* $X \subseteq \mathcal{I} \cup \mathcal{O}, 2 \leq |X| \leq |\mathcal{I} \cup \mathcal{O}| - 2$ *then* $d_B(X) \geq \alpha + \beta$.

Then the graph H obtained from G by blackberrizing with B is $(2\alpha - 2\beta)$-regular and essentially 2α-edge-connected.

Lemma 4.3.3. *Let $t \geq 3$ and $\alpha \geq 4$ be two integers. Let G be a 3-connected t-regular simple graph with an admissible perfect matching M and let β be a positive integer with $\beta \leq \frac{\alpha-1}{3}$. Let B_1 and B_2 be two berries with t outer nodes. Let $\mathcal{O}_i$ and $\mathcal{I}_i$ denote the set of outer and inner nodes of B_i respectively for $i = 1, 2$. Assume that B_i satisfies the following properties for $i = 1, 2$:*

(i)' *for the designated outer node* $u_i \in \mathcal{O}_i$: $d_{B_i}(u_i) = \alpha - \beta + (-1)^i$ *and* $d_{B_i}(v) = \alpha - \beta,\ \forall v \in \mathcal{O}_i - \{u_i\}$,

(ii)' $d_{B_i}(v) = 2\alpha - 2\beta,\ \forall v \in \mathcal{I}_i$,

(iii)' *if* $X \subseteq \mathcal{I}_i, |X| \geq 2$ *then* $d_{B_i}(X) \geq 2\alpha$.

(iv)' *if* $X \subseteq \mathcal{I}_i \cup \mathcal{O}_i, 2 \leq |X| \leq |\mathcal{I}_i \cup \mathcal{O}_i| - 2$ *then* $d_{B_i}(X) \geq \alpha + \beta + 1$.

Then the graph H obtained from G by parallel blackberrizing along M with B_1 and B_2 is $(2\alpha - 2\beta)$-regular and essentially 2α-edge-connected.

To use the above lemmas the following propositions will be useful.

Proposition 4.3.4. *Let B be a berry with **3** outer nodes. Let $\mathcal{O}$ and $\mathcal{I}$ denote the set of the outer and the inner nodes of B respectively. If B satisfies properties (i)-(iii) from Lemma 4.3.2, then it also satisfies property (iv).*

Proposition 4.3.5. *Let B_1 and B_2 be two berries with* **3** *outer nodes. Let $\mathcal{O}_i$ and $\mathcal{I}_i$ denote the set of outer and inner nodes of B_i respectively for $i = 1, 2$. Assume that B_i satisfies the properties (i)'-(ii)' from Lemma 4.3.3 and it satisfies the following property for $i = 1, 2$:*

(iii) if $X \subseteq \mathcal{I}_i, |X| \geq 2$ then $d_{B_i}(X) \geq 2\alpha + 2$.*

Then it also satisfies property (iv)' for $i = 1, 2$.

We prove Proposition 4.3.4 and Proposition 4.3.5 parallel. If we need some special argument for the proof of Proposition 4.3.5 we will put it between brackets.

Let $B := B_i$, $\mathcal{O} := \mathcal{O}_i$, $\mathcal{I} := \mathcal{I}_i$. By taking a complement we can assume that $|X \cap \mathcal{O}| \leq 1$. If $|X \cap \mathcal{O}| = 0$, then (iv) [(iv)'] follows by (iii) [(iii)*]. Otherwise there are two cases. If $|X \cap \mathcal{I}| \geq 2$, then $d_B(X \cap \mathcal{I}) \geq 2\alpha[+2]$ by (iii) [(iii)*]. Since $d_B(X) \geq d_B(X \cap \mathcal{I}) - d_B(X \cap \mathcal{O})$, (iv) [(iv)'] follows by (i) [(i)']. If $|X \cap \mathcal{I}| = 1$, then $|X| = 2$, $d_B(X \cap \mathcal{I}) = 2\alpha - 2\beta$ by (ii) [(ii)'] and $d_B(X \cap \mathcal{O}) \geq \alpha - \beta[-1]$ by (i) [(i)']. Since the berries are simple graphs $d_B(X) \geq d_B(X \cap \mathcal{I}) + d_B(X \cap \mathcal{O}) - 2 \geq d_B(X \cap \mathcal{I}) \geq 2\alpha - 2\beta \geq \alpha + \beta[+1]$ by (i)–(ii) [(i)'–(ii)'], $\beta \leq \frac{\alpha[-1]}{3}$ and $\alpha \geq 3$. □

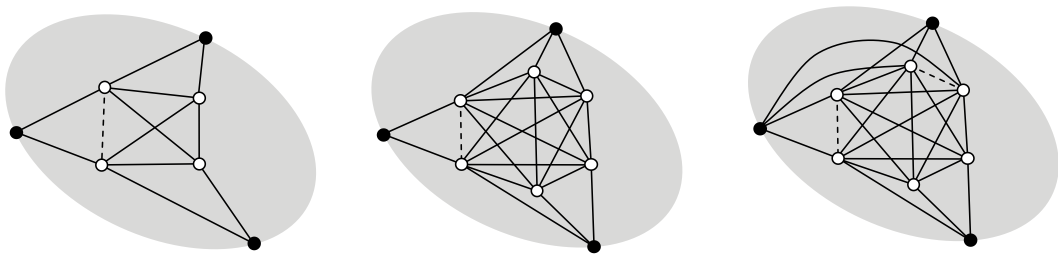
Now we will prove Lemma 4.3.2 and Lemma 4.3.3. If we need some special argument for the proof of Lemma 4.3.3 we will put it between brackets again. In the proof we will also use properties (iv) and (iv)'.

Proof. For $X \subset V(H)$ we call the corresponding edge-cut **proper** if both of $G[X]$ and $G[V - X]$ contain a berry. By 3-connectivity of G, in a proper edge-cut at least 3 berries must be cut. An edge-cut that cuts at least 3 berries includes at least $3(\alpha - \beta) \geq 2\alpha$ edges by (i)-(iv) and $\beta \leq \frac{\alpha}{3}$. [In (i)' it may happen that a cut intersects only $\alpha - \beta - 1$ edges for a berry B, namely the edges exiting a designated outer node of B. But then the edge of G associated with this outer node is included in M. Since M is admissible there could be just one such a berry-cut if we cut only 3 berries. Hence if we cut 3 berries, then the edge cut includes at least $3(\alpha - \beta) - 1 \geq 2\alpha$ edges by (i)'-(iv)' and $\beta \leq \frac{\alpha - 1}{3}$. If we cut at least 4 berries, then the edge-cut includes at least $4(\alpha - \beta - 1) \geq 2\alpha$ edges by (i)'-(iv)', $\beta \leq \frac{\alpha - 1}{3}$ and $\alpha \geq 4$.] Therefore we only need to consider those cases when the edge-cut is non-proper and it cuts at most two berries.

If a nontrivial edge-cut intersects only one berry B', then for one of its corresponding node sets, say for X, $X \subseteq \mathcal{I}'$ where $\mathcal{I}'$ denotes the set of inner nodes of B'. Therefore since $|X| \geq 2$, $d(X) \geq 2\alpha$ by (iii) [(iii)'].

Now consider the case when a nontrivial edge-cut cuts two berries B' and B''. Let $\mathcal{I}'$, $\mathcal{I}''$, $\mathcal{O}'$ and $\mathcal{O}''$ be the set of the inner and the outer nodes of B' and B'' respectively and let X be the set corresponding to the cut that is included by $V(B') \cup V(B'')$. Note that X contains an outer node if and only if it is an outer node of both B' and B'', because otherwise the edge-cut would cut another berry. Hence if $|X \cap V(B')| = |X \cap V(B'')| = 1$, then $|X \cap \mathcal{I}'| = |X \cap \mathcal{I}''| = 1$ because otherwise $|X| = 1$, a contradiction. Therefore in this subcase $d(X) = 2(2\alpha - 2\beta) \geq 2\alpha$ by (ii) [(ii)'] and $\beta \leq \frac{\alpha[-1]}{3}$. Now by changing the indices we can assume that $|X \cap V(B')| \geq 2$. Then by (iv) [(iv)'] $d_{B'}(X \cap V(B')) \geq \alpha + \beta[+1]$, and by (i)–(iv) [(i)'–(iv)'], $d_{B''}(X \cap V(B'')) \geq \alpha - \beta[-1]$. Hence $d(X) \geq 2\alpha$. □

By **pinching** together k edges of a graph we mean the following operation: take k independent edges $u_1v_1, u_2v_2, \ldots, u_kv_k$ (that is they have no end-nodes in common), delete them, and add a new node s and the $2k$ edges $su_1, sv_1, su_2, sv_2, \ldots, su_k, sv_k$ to the graph.



(a) Berry B^2. (b) Berry B^3_1. (c) Berry B^3_2.

Figure 4.5: Some berries used in the proof of Lemma 4.3.2 and Lemma 4.3.3. Outer nodes in black, inner nodes in white. The dashed edges are the pinched edges that are not included in the berry.

Now we define the berries that we use to blackberrize the graphs described in Section 4.2. For any $k \geq 2$ we take the complete graph K_{2k} and we take a partition $\{V_2, V_3\}$ of its nodes with $|V_2| = |V_3| = k$. We add two new nodes v_2 and v_3 and we connect v_i to all nodes of V_i for $i = 1, 2$. For even k we pinch together $\frac{k}{2}$ independent edges of K_{2k} with a new node v_1. We define the resulting graph as the berry B^k with the outer nodes v_1, v_2, v_3. (See Figure 4.5a.) For odd k we pinch together $\frac{k+(-1)^i}{2}$ independent edges of K_{2k} with a new node v_1. We define the resulting graph as the berry B^k_i with the designated outer node v_1 and non-designated outer nodes v_2, v_3 for $i = 1, 2$, respectively. (See Figure 4.5b and 4.5c.) (We note that in the case $k = 2$ it is easy to find smaller berries, e.g. a triangle which is a berry without any inner nodes works and using this berry the blackberry graph is in fact the line graph of the original one. One can see that if $k > 2$ the berry holding properties (i) [(i)'] and (ii) [(ii)'] from Lemma 4.3.2 [Lemma 4.3.3] must have at least $2k$ inner nodes hence there are no simpler 'good' berries than the ones described here.)

We show that these berries hold the conditions of Lemma 4.3.2 [Lemma 4.3.3].

Proposition 4.3.6. *Let $k \geq 2$ be an even integer. Then the berry $B := B^k$ holds the conditions of Lemma 4.3.2 with $t = 3$, $\alpha = k + 1$ and $\beta = 1$.*

Proposition 4.3.7. *Let $k \geq 3$ be an odd integer. Then the berries $B_1 := B_1^k$ and $B_2 := B_2^k$ hold the conditions of Lemma 4.3.3 with $t = 3,\ \alpha = k+1$ and $\beta = 1$.*

We give again a parallel proof for the two propositions.

Proof. By Proposition 4.3.4 [Proposition 4.3.5] it is enough to show that properties (i) [(i)'], (ii) [(ii)'] and (iii) [(iii)*] hold. (i) [(i)'] and (ii) [(ii)'] follows by definition hence we will show only (iii) [(iii)*]. Let $X \subseteq \mathcal{I}_{[i]}$ with $|X| \geq 2$. We need to show that $d_{B_{[i]}^k}(X) \geq 2\alpha[+2]$. It is easy to show that K_{2k} is essentially $(4k-4)$-edge-connected hence $d_{B_{[i]}^k}(X) \geq d_{K_{2k}}(X) + |X| \geq 4k - 4 + 2 = 4\alpha - 6$. For $\alpha = 3$, $4\alpha - 6 = 2\alpha$ and for $\alpha \geq 4$, $4\alpha - 6 \geq 2\alpha + 2 \geq 2\alpha[+2]$. □

Now we are ready to prove Theorem 4.0.3. Let us denote by G_i^k the graph obtained from the basic graph S_i by [parallel] blackberrizing it with the berry [berries] B^k [B_1^k and B_2^k]. [For even k, $|V(B_1^k)| = |V(B_2^k)|$ and $|\mathcal{I}(B_1^k)| = |\mathcal{I}(B_2^k)|$ so we may simply denote these values with $|V(B^k)|$ and $|\mathcal{I}(B^k)|$]. By Lemma 4.3.2 [Lemma 4.3.3] G_i^k is $2k$-regular and essentially $(2k+2)$-edge-connected for all i. By Proposition 4.3.1 (i), $h^*(G_i^k) \leq |V(B^k)| \cdot h^*(S_i)$. Hence

$$\liminf_{i\to\infty} \frac{\log h^*(G_i^k)}{\log |V(G_i^k)|} \leq \lim_{i\to\infty} \frac{\log(|V(B^k)| \cdot h^*(S_i))}{\log(|\mathcal{I}(B^k)| \cdot |V(S_i)|)} = \lim_{i\to\infty} \frac{\log |V(B^k)| + \log h^*(S_i)}{\log |\mathcal{I}(B^k)| + \log |V(S_i)|} =$$

$$= \lim_{i\to\infty} \frac{\log h^*(S_i)}{\log |V(S_i)|} = \frac{\log 8}{\log 9}.$$

We note that according to [53] this upper bound cannot be essentially improved by our method that is it remains $|V|^\lambda$ for a constant $\lambda > \log_2(1+\sqrt{5}) - 1$ for any 3-connected 3-regular basic graph and for any berry [berries] for which the conditions of Lemma 4.3.2 [4.3.3] hold.

4.4 Concluding remarks

4.4.1 Small $(k, k+1)$-circuits without long paths

We call the **deficit** of a graph $G = (V, E)$ the value $\mathrm{df}(G) = |V(G)| - h^*(G)$. Our graphs G_i^k are rather big graphs: even S_1 has 90 nodes before the blackberrization. They proved to be useful for showing Theorem 4.0.3 but do not provide small examples for $(k, k+1)$-circuits without Hamiltonian paths or with a fixed small deficit. Here we give a method to obtain some small graphs with this property.

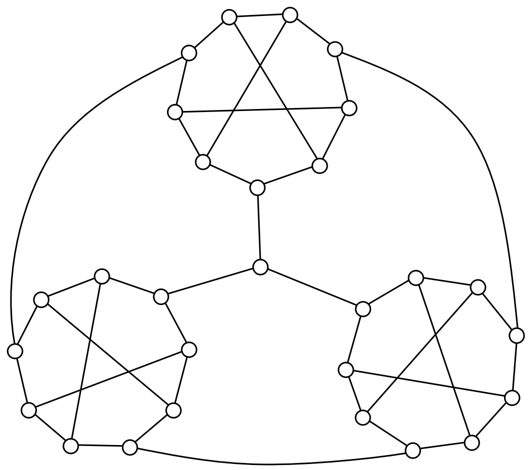


Figure 4.6: The Singleton-graph.

First we note that similar to the proof used for the Bondy-Simonovits-graphs in Section 4.2 one can prove that given an arbitrary 3-regular 3-connected graph G and a positive integer $t \leq |V(G)|$, inflating $(G_P, V_P - s)$ into each of t nodes of G, $\mathrm{df}(G_P^t) \geq \mathrm{df}(G) + t - 2 + 8 \cdot \max\{t - h^*(G), 0\}$ for the obtained graph G_P^t. If we choose K_4 as the initial graph and $t = 3$ we get a 3-regular 3-connected graph G with only 28 nodes which does not include a Hamiltonian path (see Figure 4.6 or [95, Figure 5.4]). We call this graph the Singleton-graph. Using this as the basic graph in the blackberrization we obtain relatively small $(k, k+1)$-circuits without a Hamiltonian path. Counting precisely their deficit is $|\mathcal{I}| - 1$ by Proposition 4.3.1 (ii).

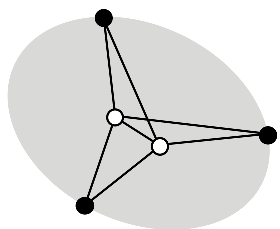


Figure 4.7: A copy of a berry with two adjacent inner nodes which are both adjacent to all 3 outer nodes. This berry can be used in case $k = 2$.

In the case $k = 2$ the smallest possible berry including inner nodes is the one with two adjacent inner nodes which are both adjacent to all 3 outer nodes (see Figure 4.7). With this berry we obtain a blackberry graph with 98 nodes and deficit 2 from the Singleton-graph (see Figure 4.8). The resulting generic circuit with 97 nodes is the smallest generic circuit without Hamiltonian paths that the authors know.

Let c be a positive integer. We give a method for deriving a $(k, k+1)$-circuit with deficit

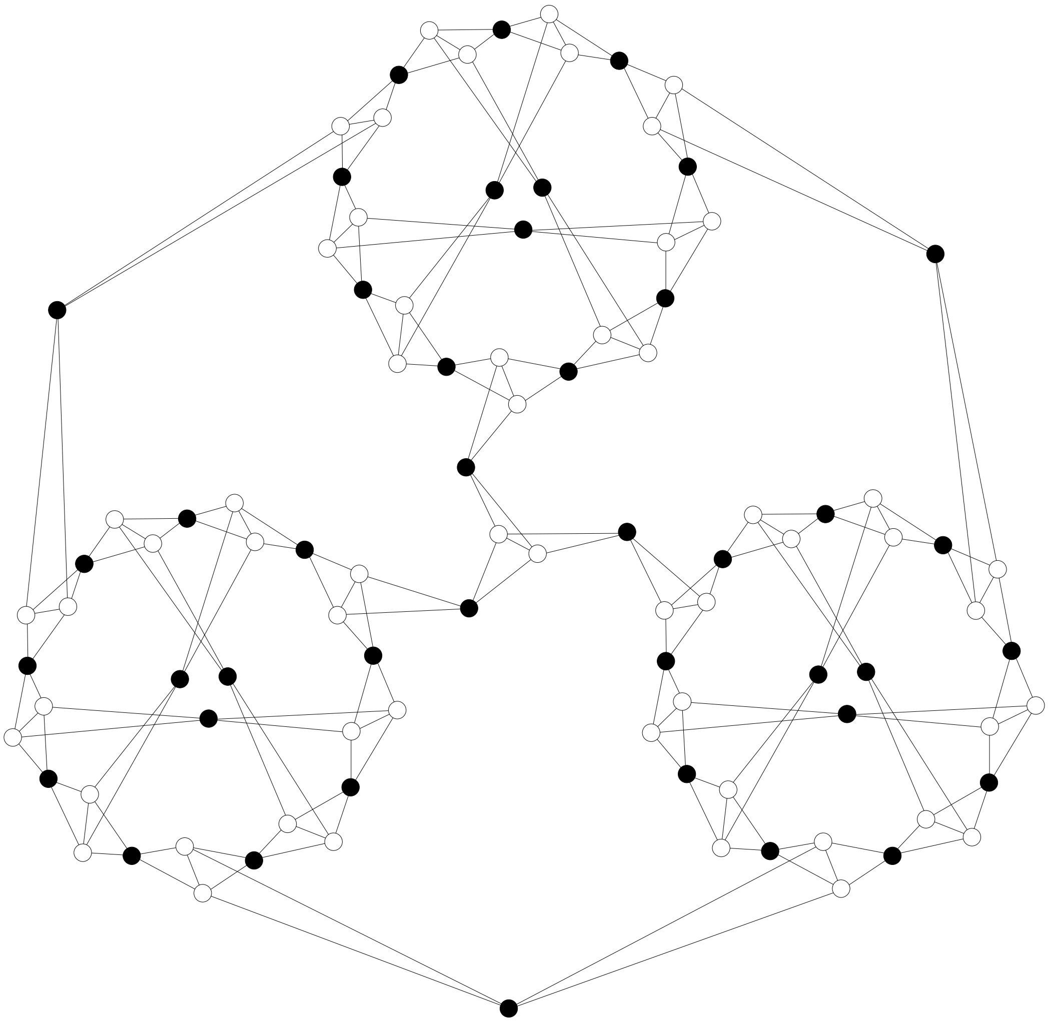


Figure 4.8: The blackberry graph of the Singleton-graph.

$\mathrm{df}(G)+c$ from a $(k,k+1)$-circuit G constructed with berries including inner nodes. First observe that the pinching operation preserves $2k$-regularity and also essential $(2k+2)$-edge-connectivity.

Lemma 4.4.1. *Let $k \geq 2$. If we pinch together k edges $u_1v_1, u_2v_2, \ldots, u_kv_k$ of a $2k$-regular, essentially $(2k+2)$-edge-connected graph $G=(V,E)$, then the obtained graph $G'=(V',E')$ is also $2k$-regular and essentially $(2k+2)$-edge-connected.*

Proof. Let $E'(X, V'-X)$ be a nontrivial edge-cut in G'. We may suppose that the new node s is included in $V'-X$. If $|V'-X| \geq 3$ then $E(X, V-X)$ is a nontrivial edge-cut in G, and thus contains at least $2k+2$ edges. Suppose that the cut contains t edges $u_{i_1}v_{i_1}, \ldots, u_{i_t}v_{i_t}$ out of the pinched ones. We may assume that $u_{i_1}, \ldots, u_{i_t} \in X$. Then $su_{i_1}, \ldots, su_{i_t}$ and the $2k+2-t$ remaining original edges are $2k+2$ edges of the cut in G'. To complete the proof observe that in a $2k$-regular graph for a subset of nodes X with

$|X| = 2$ the inequality $d(X) \geq 4k - 2 \geq 2k + 2$ always holds ($k \geq 2$). □

Now let G be a blackberry graph. If we successively perform c arbitrary pinching operations in every berry, the proof of Proposition 4.3.1 remains valid and we obtain that the lower bound of the deficit increased by c. The fact that the obtained graph is $2k$-regular and essentially $(2k+2)$-edge-connected follows directly from Lemma 4.4.1 without using Lemmas 4.3.2 or 4.3.3.

4.4.2 A constructive characterization

We only showed in Lemma 4.4.1 that pinching preserves essential $(2k + 2)$-edge-connectivity. In fact a much stronger theorem is true in the case $k = 2$. It is easy to check that 'replacing' a triangle with the berry in Figure 4.7 also preserves 4-regularity and essential 6-edge-connectivity. This, together with [24, Lemma 3.2] provides the following constructive characterization:

Theorem 4.4.2. *A graph $G = (V, E)$ is 4-regular and essentially 6-edge-connected if and only if it can be obtained from K_5 by the following operations (see also Figure 4.9):*

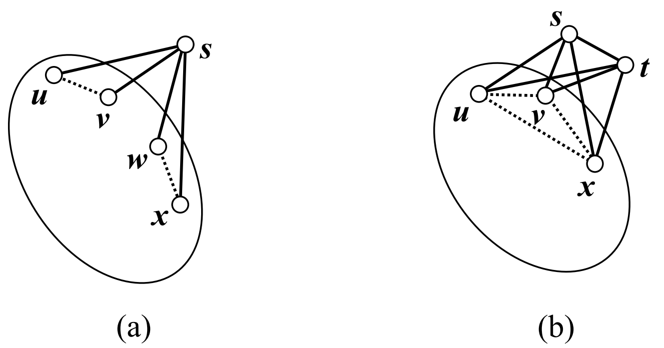


Figure 4.9: The two operations used in Theorem 4.4.2. The dashed edges are deleted from, the solid ones are added to the graph.

(a) pinch together two independent edges,

(b) take a triangle uvw (which means that uv, uw, vw are all edges), delete its edges and add two new nodes s, t and edges $st, su, sv, sw, tu, tv, tw$ to the graph. □

This theorem is also an immediate consequence of the constructive characterization of generic circuits given in [5] and of Lemma 4.1.1.

Chapter 5

How to augment rigidity

We recall that a graph $G = (V, E)$ is called (k, ℓ)-rigid if it has a spanning (k, ℓ)-tight subgraph and we call G (k, ℓ)-redundant if $G - e$ is still (k, ℓ)-rigid for any edge $e \in E$. We consider the following augmentation problem that we call here the **general (augmentation) problem**.

Problem. *Let k and ℓ be integers with $k \geq 0$ and let $G = (V, E)$ be a loopless (k, ℓ)-rigid graph. Find a graph $H = (V, F)$ on the same node set with minimum number of edges, such that $G + H = (V, E \cup F)$ is (k, ℓ)-redundant.*

We call the special case of this problem, where the input graph G is (k, ℓ)-tight, the **reduced (augmentation) problem**.

We recall from Section 1.3 that sparsity properties are important in rigidity theory. It is natural to ask how many new edges are needed to make a rigid graph redundantly rigid, that is, the augmented graph remains rigid if we omit an arbitrary edge of it. García and Tejel [44] showed that this is NP-hard for $(2, 3)$-rigid graphs but can be solved polynomially for minimally rigid graphs, that is, when G is $(2, 3)$-tight. (The idea of this algorithm was also used later by Kohta et al. [76] to give a 2-approximate solution for the redundantly rigid 'Truss Topology Design' problem.) Frank and T. Király [33] gave a polynomial algorithm to augment a graph to a (k, h)-tree-connected graph using polyhedral techniques. It follows by a result of Nash-Williams [85] that the graphs that the k-tree-connected graphs are the (k, k)-rigid graphs. Therefore, the algorithm of Frank and T. Király, with parameters $k \in \mathbb{Z}_+$ and $h = 1$, can be used to give a polynomial algorithm for the general problem when $\ell = k$. This algorithm with input parameter $k = \binom{d+1}{2}$, together with Theorem 1.2.6 shows that there is a polynomial algorithm that finds an edge set F of minimum cardinality for a graph H (with rigid d-dimensional body-bar graph G_H^{BB}) such that the body-bar graph induced by $H' = (V, E \cup F)$ is redundantly rigid. (Moreover, $G_{H'}^{\mathrm{BB}}$ is globally rigid by Theorem 1.2.7.) The algorithm, that will be presented here, will be a rather simple solution for these problems however it does not deal with the case of $h \geq 2$.

We will use the idea of García and Tejel [44] to give a polynomial algorithm that solves the reduced problem for $\ell \leq \frac{3}{2}k$ in Section 5.2. We will use this algorithm to give a polynomial algorithm that solves the general problem for $\ell \leq k$ in Section 5.3. We give some further extensions in Section 5.4.1 On the other hand, [44] showed that the general problem is NP-hard for $k = 2$ and $\ell = 3$. This result will be extended for the case where k is even and $\ell = \frac{3}{2}k$ in Section 5.4.2.

To obtain the solution for the general problem, we need the more general concepts of /tight/rigid/redundant graphs. Throughout this chapter, we will assume that

$$m(v) \geq \ell \text{ for all } v \in V \text{ or } m \equiv k \text{ for a positive integer } k \text{ for which } k < \ell \leq \frac{3}{2}k. \quad (5.1)$$

Note that, when $m \equiv k$, an $(\mathbf{m}, \ell)$-sparse/tight/rigid/redundant graph is (k, ℓ)-sparse/tight/rigid/redundant, respectively.

The results of this chapter are based on [74].

5.1 Preliminaries

In this section, we list some well-known properties of $(\mathbf{m}, \ell)$-sparse graphs. We sketch their proofs for completeness. See [31, 108] for more details. It follows from the definition that an $(\mathbf{m}, \ell)$-tight subgraph of a $(\mathbf{m}, \ell)$-sparse graph is always an induced subgraph. Therefore, if $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ both are $(\mathbf{m}, \ell)$-tight subgraphs of an $(\mathbf{m}, \ell)$-sparse graph G, then $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$ is also a induced subgraph of G.

Lemma 5.1.1. *Let $G = (V, E)$ be an $(\mathbf{m}, \ell)$-sparse graph, where (5.1) holds for $m : V \to \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$, and let $G_i = (V_i, E_i)$ be $(\mathbf{m}, \ell)$-tight subgraphs of G for $i = 1, 2, 3$.*

(a) If $\widetilde{m}(V_1 \cap V_2) \geq \ell$, then $G_1 \cup G_2$ and $G_1 \cap G_2$ are $(\mathbf{m}, \ell)$-tight graphs and there are no edges between $V_1 - V_2$ and $V_2 - V_1$. Otherwise, $G_1 \cup G_2$ is not $(\mathbf{m}, \ell)$-tight.

(b) If $V_1 \cap V_2 = \{i_1\}$, $V_2 \cap V_3 = \{i_2\}$ and $V_3 \cap V_1 = \{i_3\}$ where i_1, i_2, i_3 are three different nodes of V, then $m \equiv k = \frac{2}{3}\ell$ and $G_1 \cup G_2 \cup G_3$ is (k, ℓ)-tight.

We note that the assumption of (a) holds always when $E_i \cap E_j \neq \emptyset$.

Proof. (a) The statements follow easily by the following. $|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2| = \widetilde{m}(V_1) - \ell + \widetilde{m}(V_2) - \ell = \widetilde{m}(V_1 \cup V_2) - \ell + \widetilde{m}(V_1 \cap V_2) - \ell$.
(b) The statements follow easily by the following. $|E_1 \cup E_2 \cup E_3| = |E_1| + |E_2| + |E_3| = \widetilde{m}(V_1) - \ell + \widetilde{m}(V_2) - \ell + \widetilde{m}(V_3) - \ell = \widetilde{m}(V_1 \cup V_2 \cup V_3) + \widetilde{m}(\{i_1, i_2, i_3\}) - 3\ell$. Thus $3 \min_{v \in V} m(v) \leq \widetilde{m}(\{i_1, i_2, i_3\}) \leq 2\ell$. □

With the same idea as in Lemma 5.1.1(a), we get the following.

Lemma 5.1.2. *Let $\ell > k > 0$, let $G = (V, E)$ be a (k, ℓ)-sparse graph and let $G_i = (V_i, E_i)$ be (k, ℓ)-sparse subgraphs of G for $i = 1, 2$. If $|V_1 \cap V_2| \leq 1$, then $G_1 \cup G_2$ cannot be a (k, ℓ)-tight graph. Therefore, a (k, ℓ)-tight graph is 2-connected for $\ell > k$.* □

Lemma 5.1.3. *If $\ell \geq 0$, then the minimum degree of a (k, ℓ)-tight graph $G = (V, E)$ with $n \geq 3$ nodes is between k and $2k - 1$.*

Proof. The upper bound follows from the fact that the average degree in G is less than $2k$. To prove the lower bound, let v be a node of minimum degree. Then $G - v$ has at most $k(n - 1) - \ell$ edges thus the degree of v is at least k as $|E| = kn - \ell$. □

We recall that the edge sets of the $(\mathbf{m}, \ell)$-sparse subgraphs of a given graph form a matroid when $m(u) + m(v) \geq \ell$ for each edge uv and a circuit of this matroid is called an $(\mathbf{m}, \ell)$-circuit.. (However, this assumption on m and ℓ follows by (5.1).) We summarize some properties of $(\mathbf{m}, \ell)$-circuits in the following lemma. The proof will be skipped as it directly follows by some matroid properties.

Lemma 5.1.4. *Let $G = (V, E)$ be an $(\mathbf{m}, \ell)$-tight graph and $i, j \in V$.*

(a) The graph $G + ij$ contains an unique $(\mathbf{m}, \ell)$-circuit, that will be denoted by $C^G_{(\mathbf{m},\ell)}(ij)$. Hence the subgraph of G $T^G_{(\mathbf{m},\ell)}(ij) := C^G_{(\mathbf{m},\ell)}(ij) - ij$ is $(\mathbf{m}, \ell)$-tight.

(b) For every edge e' of $C^G_{(\mathbf{m},\ell)}(ij)$, $G' = G + ij - e'$ is also $(\mathbf{m}, \ell)$-tight and the unique $(\mathbf{m}, \ell)$-circuit of $G' + e'$ is again $C^G_{(\mathbf{m},\ell)}(ij)$. Moreover, if $e'' \notin E(C^G_{(\mathbf{m},\ell)}(ij))$, then, $G' + ij - e''$ is not $(\mathbf{m}, \ell)$-tight.

(c) If $G' = (V', E')$ is an $(\mathbf{m}, \ell)$-tight subgraph of G with $i, j \in V'$, then $T^G_{(\mathbf{m},\ell)}(ij) \subseteq G'$. Hence $T^G_{(\mathbf{m},\ell)}(ij) = \bigcap\{T_h : T_h$ an $(\mathbf{m}, \ell)$-tight subgraph of G spanning both i and $j\}$. □

As it will be usually clear from the context, we will omit the subscript $(\mathbf{m}, \ell)$ and the superscript G when we speak about $C^G_{(\mathbf{m},\ell)}(ij)$ or $T^G_{(\mathbf{m},\ell)}(ij)$. From now on, let $E(ij) := E(T(ij))$ and $V(ij) := V(T(ij))$. We will call an $(\mathbf{m}, \ell)$-tight subgraph G' of G **generated** if there are nodes i, j such that $T(ij) = G'$; in this case, i and j will be called the **generators** of G'. We will use the appellation of **($(\mathbf{m}, \ell)$-)MGT subgraph** for the maximal generated $(\mathbf{m}, \ell)$-tight subgraphs of $G = (V, E)$ that are the generated subgraphs $T(ij)$ such that there is no $i', j' \in V$ with $T(ij) \subset T(i'j')$.

Let $R^G_{(\mathbf{m},\ell)}(i_1j_1, \ldots, i_rj_r) = (V(i_1j_1, \ldots, i_rj_r), E(i_1j_1, \ldots, i_rj_r))$ denote the subgraph induced by the $(\mathbf{m}, \ell)$-redundant edges of $G = (V, E)$ in the graph $G + \{i_1j_1, \ldots, i_rj_r\}$ for $i_1, \ldots, i_r, j_1, \ldots, j_r \in V$. (Again we usually omit the subscript $(\mathbf{m}, \ell)$ and the superscript G as it will be clear from the context.) Note that $R(ij) = T(ij)$ for any $i, j \in V$. The following lemma shows that if G is $(\mathbf{m}, \ell)$-tight, then $R(i_1j_1, \ldots, i_rj_r) = T(i_1j_1) \cup$

$\cdots \cup T(i_rj_r)$. Therefore, to make G $(\mathbf{m},\ell)$-redundant, we need the minimum number of generated $(\mathbf{m},\ell)$-tight subgraphs of G such that their edges cover all edges of G. We can assume that these generated $(\mathbf{m},\ell)$-tight subgraphs are MGT subgraphs of G.

Lemma 5.1.5. *If G is $(\mathbf{m},\ell)$-tight, then $R(i_1j_1,\dots,i_rj_r) = T(i_1j_1)\cup\cdots\cup T(i_rj_r)$.*

Proof. $R(i_1j_1) = T(i_1j_1)$ by definition hence $T(i_1j_1)\cup\cdots\cup T(i_rj_r) \subseteq R(i_1j_1,\dots,i_rj_r)$. For the other direction, let $e \in E(i_1j_1,\dots,i_rj_r)$ be an arbitrary edge. Now, $G-e$ is $(\mathbf{m},\ell)$-sparse and $|E-e| = \widetilde{m}(V)-\ell-1$. $G+\{i_1j_1,\dots,i_rj_r\}-e$ is $(\mathbf{m},\ell)$-rigid hence $E\cup\{i_1j_1,\dots,i_rj_r\}-e$ has a rank of $\widetilde{m}(V)-\ell$ in the $(\mathbf{m},\ell)$-sparsity matroid. Thus there is an edge f in $\{i_1j_1,\dots,i_rj_r\}$ for which $E-e+f$ is a basis of the $(\mathbf{m},\ell)$-sparsity matroid. Since $E-e+f$ is independent in the $(\mathbf{m},\ell)$-sparsity matroid, we must have $e\in T(f)$. □

The following lemma shows that every (k,ℓ)-MGT subgraph induces at least 3 nodes. (A similar statement holds for the general $(\mathbf{m},\ell)$ case, but we will not use it.)

Lemma 5.1.6. *Assume that $G=(V,E)$ is a (k,ℓ)-tight graph with $k<\ell\le\frac{3}{2}k$. If $T(ij)$ is (k,ℓ)-MGT and $n\ge 4$, then $|V(ij)|\ge 3$.*

Proof. $(2k-\ell)K_n$ is always (k,ℓ)-redundant for $n \ge \frac{k+1}{2k-\ell}+1$ (that holds for $n\ge 4$ by $\ell\le\frac{3}{2}k$). Hence $R(E((2k-\ell)K_n)-E) = G$. For any $ij\in E((2k-\ell)K_n)-E$, $|V(ij)|\ge 3$ since $|V(ij)|=2$ if and only if there are $2k-\ell$ parallel edges between i and j in G. By Lemma 5.1.5, $G = R(E((2k-\ell)K_n)-E) = \bigcup_{ij\in E((2k-\ell)K_n)-E} T(ij)$. Hence the edges of G can be covered by generated $(\mathbf{m},\ell)$-tight subgraphs with at least 3 nodes. Therefore, each MGT subgraph of G must induce at least 3 nodes. □

5.2 Augmenting an $(\mathbf{m},\ell)$-tight graph to an $(\mathbf{m},\ell)$-redundant graph

Before we present the algorithm, we prove the following lemmata.

Lemma 5.2.1. *Let $G=(V,E)$ be an $(\mathbf{m},\ell)$-tight graph (where (5.1) holds for $m: V\to\mathbb{Z}_+$ and $\ell\in\mathbb{Z}$) with at least 4 nodes and let $T(ij)$ be an MGT subgraph of G. Then $\widetilde{m}(V(ij)\cap V(ij'))\ge\ell$ and $T(ij)\cup T(ij')$ is $(\mathbf{m},\ell)$-tight for any $j'\in V-i$.*

Proof. The lemma follows by Lemma 5.1.1(a) for the case where $m\ge\ell$. Assume now that $m\equiv k<\ell\le\frac{3}{2}k$. Then the statement holds for $j'\in V(ij)$ because then $T(ij')\subseteq T(ij)$ by Lemma 5.1.4(c).

Suppose that $j'\notin V(ij)$ and suppose for a contradiction, that $|V(ij)\cap V(ij')|<2$.

Case 1: If $|V(ij')\cap V(jj')|\ge 2$, then $T(ij')\cup T(jj')$ is a (k,ℓ)-tight graph by Lemma 5.1.1(a), and as it contains i and j, we have $T(ij)\subseteq T(ij')\cup T(jj')$ by Lemma 5.1.4(c). But as $E(ij)\cap E(ij')=\emptyset$, we have $T(ij)\subset T(jj')$ contradicting the maximality of $T(ij)$.

Case 2: If $|V(ij) \cap V(jj')| \geq 2$, then $T(ij) \cup T(jj')$ is a (k, ℓ)-tight graph by Lemma 5.1.1(a), and as it contains i and j', we have $T(ij') \subseteq T(ij) \cup T(jj')$ by Lemma 5.1.4(c). But as $E(ij) \cap E(ij') = \emptyset$, we have $T(ij') \subset T(jj')$ and we are again at *Case 1*.

Case 3: Finally if we are neither in *Case 1* nor in *Case 2*, it is because $G_1 = T(ij)$, $G_2 = T(ij')$ and $G_3 = T(jj')$ satisfy the conditions of Lemma 5.1.1(b), hence $\ell = \frac{3}{2}k$ and $G_1 \cup G_2 \cup G_3$ is (k, ℓ)-tight. Moreover, by the maximality of G_1 and $|V| \geq 4$, there is at least one additional node $i' \notin \{i, j\}$ in G_1 by Lemma 5.1.6. Now, $T(i'j') \subseteq G_1 \cup G_2 \cup G_3$ by Lemma 5.1.4(c). Since any path between i' and j' in $G_1 \cup G_2 \cup G_3$ must contain i or j; and since a (k, ℓ)-tight graph is 2-connected for $\ell > k$ by Lemma 5.1.2, i and j must be in $V(i'j')$. Hence $T(ij) \subset T(i'j')$ by Lemma 5.1.4(c), contradicting again the maximality of $T(ij)$.

As $E(ij) \cap E(ij') \neq \emptyset$, $k|V(ij) \cap V(ij')| \geq \ell$ and hence $T(ij) \cup T(ij')$ is (k, ℓ)-tight by Lemma 5.1.1(a). □

Lemma 5.2.2. *Let $k < \ell \leq \frac{3}{2}k$ and let G be a (k, ℓ)-tight graph with at least 4 nodes and let i be a node of degree between k and $2k-1$. Then $E(ij) \cap E(ij') \neq \emptyset$ and $T(ij) \cup T(ij')$ is (k, ℓ)-tight if $j, j' \in V - i$, $j \neq j'$ and $\min(|V(ij)|, |V(ij')|) \geq 3$.*

Proof. As the minimum degree in a (k, ℓ)-tight graph with more then 3 nodes is at least k by Lemma 5.1.3, $d_{T(ij)}(i) \geq k$ and $d_{T(ij')}(i) \geq k$. Thus there is at least one edge incident to i that is in both $T(ij)$ and $T(ij')$ since $d_G(i) \leq 2k - 1$.

As $E(ij) \cap E(ij') \neq \emptyset$, $T(ij) \cup T(ij')$ is (k, ℓ)-tight by Lemma 5.1.1(a). □

5.2.1 Algorithm to find a small covering rooted at a single node

Now we show that the subroutine of the algorithm of García and Tejel works similarly for $(\mathbf{m}, \ell)$-tight graphs as it worked for Laman graphs. This subroutine outputs a covering of the node set of G with minimum number of sets of the form $V(i_1 j)$ for a given node i_1 that we call the root of the covering.

Algorithm 5.2.3. INPUT: $m : V \to \mathbb{Z}_+, \ell \in \mathbb{Z}$ holding (5.1) and an $(\mathbf{m}, \ell)$-tight graph $G = (V, E)$ with a given node $i_1 \in V$ and a set $L \subseteq V$.
OUTPUT: A list $V'(i_1, L) = \{j_2, \dots, j_r\}$ of nodes such that $V = \bigcup_{s=2}^{r} V(i_1 j_s)$. [If L is $\emptyset$ then we will refer to this set as $V'(i_1)$.]

1. Initialize $V'(i_1, L) = \emptyset$. All nodes are unmarked. Mark i_1.

2. Explore all nodes $j \in L$ and after this all other nodes $j \in V - L$:

   If j is unmarked do

     Calculate $T(i_1 j)$;

Mark all unmarked nodes in $T(i_1 j)$;

$V'(i_1, L) := [V'(i_1, L) - V(i_1 j)] + j$.

With the algorithm of [31, Section 13.5.4] $T(i_1 j)$ can be calculated in polynomial time thus the following claim holds.

Claim 5.2.4. *The running time of Algorithm 5.2.3 is polynomial.* □

Next, we show how to use Algorithm 5.2.3 to cover its edge set with MGT subgraphs. The proofs will slightly differ for the case of $m \equiv k < \ell \leq \frac{3}{2}k$ and for the case of $m \geq \ell$.

First we show that the subgraphs $T(i_1 j_s)$ $(s = 2, \ldots, r)$ cover the edges of G if i_1 is a node of minimum degree.

Lemma 5.2.5. *Let $G = (V, E)$ be an $(\mathbf{m}, \ell)$-tight graph with at least 4 nodes (where (5.1) holds for $m : V \to \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$). Suppose that i_1 has minimum degree in G. Then $E = \bigcup_{j_s \in V'(i_1)} E(i_1 j_s)$ and, if $m \equiv k < \ell$, $|V(i_1 j_s)| \geq 3$ for every $j_s \in V'(i_1)$.*

Proof. If $r = |V(i_1)| + 1 = 2$, then $V(i_1 j_2) = V$ thus $T(i_1 j_2) = G$ that proves the lemma. From now on assume that $r \geq 3$.

Assume first that $m \geq \ell$. Then $\bigcup_{j_s \in V'(i_1)} T(i_1 j_s)$ is $(\mathbf{m}, \ell)$-tight by Lemma 5.1.1(a) as $i_1 \in V(i_1 j_s) \cap V(i_1 j_{s'})$ for any $j_s, j_{s'} \in V'(i_1)$. As G is $(\mathbf{m}, \ell)$-tight and $\bigcup_{j_s \in V'(i_1)} T(i_1 j_s)$ is an induced $(\mathbf{m}, \ell)$-tight subgraph of G, these two graphs must coincide.

Next assume that $m \equiv k < \ell$. As the minimum degree of G is between k and $2k - 1$ by Lemma 5.1.3, $E(i_1 j_s) \cap E(i_1 j_{s'}) \neq \emptyset$ and $T(i_1 j_s) \cup T(i_1 j_{s'})$ is (k, ℓ)-tight if $|V(i_1 j_s)| \geq 3$ and $|V(i_1 j_{s'})| \geq 3$ for $j_s, j_{s'} \in V'(i_1)$ by Lemma 5.2.2. As $d(i_1) \leq 2k - 1$ and $\ell \leq \frac{3}{2}k$, i_1 has at most 3 neighbors v' with $d(i_1, v') = 2k - \ell$. Thus there is at least one node $j \in V - i_1$ such that $d_G(i_1, j) < 2k - \ell$. (Note that in the case of $n = 4$, if $d_G(i_1, v') = 2k - \ell$ for every $v' \in V - i_1$, then by the minimality of the degree of i_1 and $\ell \leq \frac{3}{2}k$, $|E| \geq \frac{3(2k-\ell)4}{2} \geq \frac{9}{2}k - \ell > 4k - \ell$ thus G is not (k, ℓ)-sparse.) Therefore, there is at least one $j_s \in V'(i_1)$ such that $T(i_1 j_s)$ has at least 3 nodes. Let $T := \bigcup \{T(i_1 j_s) : j_s \in V'(i_1), |V(i_1 j_s)| \geq 3\}$. This is a (k, ℓ)-tight graph as we have seen before.

Since T is (k, ℓ)-tight and $|V(T)| \geq 3$, $d_T(v) \geq k$ by Lemma 5.1.3. As $d_G(i_1, v') = 2k - \ell$ for $v' \in V - V(T)$, $d_G(i_1) \geq k + |V - V(T)|(2k - \ell) \geq k + |V - V(T)|\frac{k}{2}$ by $\ell \leq \frac{3}{2}k$. Therefore, $|V - V(T)| \leq 1$ by Lemma 5.1.3 as the degree of i_1 is minimum. Moreover, if $V - V(T) = \{v'\}$ then $d_G(v') \geq d_G(i_1) \geq 3k - \ell$. Thus if $|V - V(T)| = 1$ then $kn - \ell = |E| \geq (k + 2k - \ell) + |E(T)| = (2k - \ell) + k|V(T)| - \ell = kn - \ell + (k - \ell) > kn - \ell + 1$, a contradiction. Hence $|V - V(T)| = 0$ and T is a (k, ℓ)-tight subgraph of G covering V. Therefore, $G = T = \bigcup_{j_s \in V'(i_1)} T(i_1 j_s)$, as we claimed. □

From the proof of the previous lemma we also get the following.

Corollary 5.2.6. *Let $G = (V, E)$ be an $(\mathbf{m}, \ell)$-tight graph with at least 4 nodes (where (5.1) holds for $m : V \to \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$). Suppose that i_1 has minimum degree in G. Then $T(i_1j) \cup T(i_1j')$ is $(\mathbf{m}, \ell)$-tight for every $j, j' \in V(i_1)$.* □

We say that a set $J \subseteq V$ is an **MGT-generator of** G if $T(jj')$ is MGT for every two distinct elements $j, j' \in J$; $j'' \notin T(jj')$ for every three distinct elements $j, j', j'' \in J$; and for every MGT subgraph T of G there is a pair $j, j' \in J$ for which $T = T(jj')$. The following lemma gives a constructive proof for the existence of MGT-generators.

Lemma 5.2.7. *Let $G = (V, E)$ be an $(\mathbf{m}, \ell)$-tight graph with at least 4 nodes (where (5.1) holds for $m : V \to \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$). Suppose that i_1 has minimum degree in G. Then $V'(i_1)$ or $V'(i_1) \cup \{i_1\}$ is an MGT-generator of G.*

Proof. If $r = 2$, then $G = T(i_1j_2)$ is MGT hence $\{i_1, j_2\}$ is an MGT-generator of G. Assume that $r \geq 3$.

By the definition of $V'(i_1)$, $j_s \notin V(i_1j_{s'})$ for $s, s' \in \{2, \dots, r\}$, $s \neq s'$. As by Lemma 5.2.5 $E = \bigcup_{s=2}^{r} E(i_1j_s)$, all the edges incident to j_s are in $T(i_1j_s)$.

Let $T(vv')$ be an MGT subgraph of G. Let $j \in V'(i_1)$ such that $v \in V(i_1j)$. If $v' \in V(i_1j)$, then $T(vv') \subseteq T(i_1j)$ hence $T(vv') = T(i_1j)$ by maximality. Thus we can assume that $v' \in V(i_1j')$ for $j' \in V'(i_1)$, $j' \neq j$. $E(i_1j) \cap E(jj') \neq \emptyset$ since $T(i_1j)$ spans all edges incident to j. Thus $T(i_1j) \cup T(jj')$ is $(\mathbf{m}, \ell)$-tight by Lemma 5.1.1(a). As $i_1, j' \in V(i_1j) \cup V(jj')$, $T(i_1j') \subset T(i_1j) \cup T(jj')$. Since $v' \in V(i_1j')$ but $v' \notin V(i_1j)$, $v' \in V(jj')$. Similarly, $v \in V(jj')$ and hence $T(vv') \subseteq T(jj')$. Thus $T(vv') = T(jj')$ by the maximality.

$T(i_1j) \cup T(i_1j')$ is $(\mathbf{m}, \ell)$-tight by Corollary 5.2.6. Hence $T(jj') \subseteq T(i_1j) \cup T(i_1j')$. Thus $j'' \notin V(vv')$ for $j'' \in V'(i_1)$, $j'' \neq j, j'$.

Therefore, each MGT subgraph of G, that covers at least 2 elements of $V'(i_1)$, covers exactly 2 of them and these 2 elements are its generators. As any two node of G is covered by an MGT subgraph of G, we get that $T(jj')$ is MGT for every $j, j' \in V'(i_1)$. Moreover, we have seen that every MGT subgraph of G has the form of $T(jj')$ for some $j, j' \in V'(i_1) + i_1$.

Now, we turn to show that if $T(i_1j)$ is MGT for a $j \in V'(i_1)$, then $T(i_1j')$ is MGT for every $j' \in V'(i_1)$. Assume for a contradiction that $T(i_1j')$ is not MGT for a $j' \in V'(i_1)$. Then $T(i_1j') \subseteq T(j'j'')$ for some $j'' \in V'(i_1)$ since all (possible) MGT subgraph of G covering j' has the form of $T(j'j'')$ with $j'' \in V'(i_1) + i_1$. By Lemma 5.2.1, we have $T(j'j'') \subseteq T(jj') \cup T(jj'')$ since these latter two graphs are MGT or $j = j''$.

Now $i_1 \in V(j'j'') \subseteq V(jj') \cup V(jj'')$ hence $i_1 \in V(jj^*)$ where $j^* = j'$ or j'' (but $j^* \neq j$). Therefore, $T(i_1j) \subseteq T(jj^*)$. But by the construction of $V'(i_1)$, $j^* \notin V(i_1j)$ that contradicts the maximality of $T(i_1j)$. Moreover, this argument show that if $i_1 \in V(jj^*)$ for $j, j^* \in V'(i_1)$, then $T(i_1j)$ is not an MGT subgraph of G. □

Now we have the following two corollaries.

Corollary 5.2.8. *Let $G = (V, E)$ be an $(\mathbf{m}, \ell)$-tight graph with at least 4 nodes (where (5.1) holds for $m : V \to \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$). Suppose that i_1 has minimum degree in G and $d_G(i_1) = m(i_1)$. Then $V'(i_1) + i_1$ is an MGT-generator of G.*

Proof. As $d_G(i_1) = m(i_1)$, if T is an $(\mathbf{m}, \ell)$-tight subgraph of G with $i_1 \in V(T)$ and $|V(T)| \geq 3$, $T - i_1$ is also $(\mathbf{m}, \ell)$-tight. Thus i_1 is the generator of all MGT subgraphs containing i_1 and we are done by Lemma 5.2.7. □

If Corollary 5.2.8 cannot be applied, then we will determine whether i_1 is needed to MGT subgraphs with another run of Algorithm 5.2.3.

Corollary 5.2.9. *Let $G = (V, E)$ be an $(\mathbf{m}, \ell)$-tight graph with at least 4 nodes (where (5.1) holds for $m : V \to \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$). Suppose that i_1 has minimum degree in G. Let $j_2 \in V'(i_1)$ and let $V'' = V'(j_2, V'(i_1) - j_2 + i_1)$. Then $V'' + j_2$ is an MGT-generator of G. Moreover, $G = \bigcup_{j \in V''} T(j_2 j)$.*

Proof. By Lemma 5.2.7 and by the construction of V'', $V'' = V'(i_1) - j_2$ or $V'' = (V'(i_1) - j_2) + i_1$ and we have the first part of the corollary. But then we have $G = \bigcup_{j \in V''} T(j_2 j)$ by Lemma 5.2.1 and by the construction of V''. □

5.2.2 Algorithm to find the minimum covering of an $(\mathbf{m}, \ell)$-tight graph

The following two properties of MGT subgraphs will be useful to reduce the number of MGT subgraphs covering G.

Lemma 5.2.10. *Let $G_1 := T(i_1 j_1) = (V_1, E_1)$ and $G_2 := T(i_2 j_2) = (V_2, E_2)$ be two MGT subgraphs of an $(\mathbf{m}, \ell)$-tight graph G.*

(a) Suppose that $m \equiv k < \ell$ and $E_1 \cap E_2 = \emptyset$. Then $E(i_1 i_2) \cap E(j_1 j_2) \neq \emptyset$ and similarly $E(i_1 j_2) \cap E(j_1 i_2) \neq \emptyset$.

(b) Suppose that $m \geq \ell > 0$ and $V_1 \cap V_2 = \emptyset$. Then $V(i_1 i_2) \cap V(j_1 j_2) \neq \emptyset$ and similarly $V(i_1 j_2) \cap V(j_1 i_2) \neq \emptyset$.

Moreover in both cases, $G_1 \cup G_2 \subset T(i_1 i_2) \cup T(j_1 j_2)$ and $T(i_1 i_2) \cup T(j_1 j_2) = T(i_1 j_2) \cup T(j_1 i_2)$.

Proof. Case (a): As $E_1 \cap E_2 = \emptyset$ and G_1, G_2 are both MGT, i_1, i_2, j_1, j_2 are 4 different nodes by Lemma 5.2.1.

Let us choose one of the (k, ℓ)-tight graphs $T(i_1i_2), T(j_1j_2), T(i_1j_2), T(j_1i_2)$, say $T(i_1i_2)$. By Lemma 5.2.1, $E' := E(i_1i_2) \cap E_1 \neq \emptyset$ and $E'' := E(i_1i_2) \cap E_2 \neq \emptyset$. Since $E_1 \cap E_2 = \emptyset$, $E' \cap E'' = \emptyset$. Thus $E' \cup E''$ cannot be the edge set of a (k, ℓ)-tight graph by Lemma 5.1.1(a) and hence $F(i_1i_2) := E(i_1i_2) - (E' \cup E'') \neq \emptyset$. Similarly, we can define the non-empty edge sets $F(j_1j_2), F(i_1j_2), F(j_1i_2)$.

By Lemma 5.2.1, $T := G_1 \cup T(i_1i_2) \cup G_2$ is a (k, ℓ)-tight graph with the edge set $E_1 \cup E_2 \cup F(i_1i_2)$. As $j_1, j_2 \in V(T)$, $T(j_1j_2) \subseteq T$ and hence $F(j_1j_2) \subseteq F(i_1i_2)$. With telling the same reasoning for the other pairs, we get $F(i_1i_2) = F(j_1j_2) = F(i_1j_2) = F(j_1i_2) =: F$.

Therefore, $E(i_1i_2) \cap E(j_1j_2) \neq \emptyset$ and hence $T(i_1i_2) \cup T(j_1j_2)$ is (k, ℓ)-tight containing all the 4 nodes i_1, i_2, j_1, j_2. Thus $G_1, G_2 \subseteq T(i_1i_2) \cup T(j_1j_2)$. Moreover, $(E(i_1i_2) \cup E(j_1j_2)) - (E_1 \cup E_2) = F \neq \emptyset$. The same properties can be proved for $T(i_1j_2) \cup T(j_1i_2)$.

Case (b): As $V_1 \cap V_2 = \emptyset$ and G_1 and G_2 are MGT graphs, i_1, i_2, j_1, j_2 are 4 different nodes.

Let us choose one of the $(\mathbf{m}, \ell)$-tight graphs $T(i_1i_2), T(j_1j_2), T(i_1j_2), T(j_1i_2)$, say $T(i_1i_2)$. Now, $i_1 \in V' := V(i_1i_2) \cap V_1 \neq \emptyset$ and $i_2 \in V'' := V(i_1i_2) \cap V_2 \neq \emptyset$. Moreover, both V' and V'' span an $(\mathbf{m}, \ell)$-tight subgraph of G. Since $V_1 \cap V_2 = \emptyset$, $V' \cap V'' = \emptyset$. Thus $V' \cup V''$ cannot span an $(\mathbf{m}, \ell)$-tight graph by Lemma 5.1.1(a) (as $\ell > 0$) and hence $W(i_1i_2) := V(i_1i_2) - (V' \cup V'') \neq \emptyset$. Similarly, we can define the non-empty node sets $W(j_1j_2), W(i_1j_2), W(j_1i_2)$.

By Lemma 5.1.1(a), $T := G_1 \cup T(i_1i_2) \cup G_2$ is an $(\mathbf{m}, \ell)$-tight graph with the node set $V_1 \cup V_2 \cup W(i_1i_2)$. As $j_1, j_2 \in V(T)$, $T(j_1j_2) \subseteq T$ and hence $W(j_1j_2) \subseteq W(i_1i_2)$. With telling the same reasoning for the other pairs, we get $W(i_1i_2) = W(j_1j_2) = W(i_1j_2) = W(j_1i_2) =: W$.

Therefore, $V(i_1i_2) \cap V(j_1j_2) \neq \emptyset$ and hence $T(i_1i_2) \cup T(j_1j_2)$ is $(\mathbf{m}, \ell)$-tight containing all the 4 nodes i_1, i_2, j_1, j_2. Thus $G_1, G_2 \subseteq T(i_1i_2) \cup T(j_1j_2)$. Moreover, $(V(i_1i_2) \cup V(j_1j_2)) - (V_1 \cup V_2) = W \neq \emptyset$. The same properties can be proved for $T(i_1j_2) \cup T(j_1i_2)$. □

This general lemma will be applied in the following way.

Corollary 5.2.11. *Let $T(ij_1)$, $T(ij_2)$ and $T(ij_3)$ be three different MGT subgraphs of the $(\mathbf{m}, \ell)$-tight graph G (where (5.1) holds for $m : V \to \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$) and let T be the $(\mathbf{m}, \ell)$-tight graph $T(ij_1) \cup T(ij_2) \cup T(ij_3)$. Then at least two of the three graphs $T(ij_1) \cup T(j_2j_3)$, $T(ij_2) \cup T(j_1j_3)$ and $T(ij_3) \cup T(j_1j_2)$ coincide with T.*

Note that T is tight by Lemma 5.2.1 in the case of $m \equiv k < \ell \leq \frac{3}{2}k$ and by Lemma 5.1.1(a) in the case of $m \geq \ell$.

Proof. First note that if $\ell \leq 0$, then all the three graphs coincide with T by Lemmata 5.1.1(a) and 5.1.4(c). Let us assume now $\ell > 0$.

Note that if $E(ij_{s_1}) \cap E(j_{s_2}j_{s_3}) \neq \emptyset$ and $m \equiv k < \ell$ or if $V(ij_{s_1}) \cap V(j_{s_2}j_{s_3}) \neq \emptyset$ and $m \geq \ell > 0$, then $T(ij_{s_1}) \cup T(j_{s_2}j_{s_3}) \subseteq T$ is $(\mathbf{m},\ell)$-tight for every s_1, s_2, s_3 with $\{s_1, s_2, s_3\} = \{1,2,3\}$ by Lemma 5.1.1(a). Hence $T = T(ij_1) \cup T(ij_2) \cup T(ij_3) \subseteq T(ij_{s_1}) \cup T(j_{s_2}j_{s_3})$ by Lemma 5.1.4(c) as $i, j_1, j_2, j_3 \in V(ij_{s_1}) \cup V(j_{s_2}j_{s_3})$. Moreover, if for a triple $\{s_1, s_2, s_3\} = \{1,2,3\}$, $E(ij_{s_1}) \cap E(j_{s_2}j_{s_3}) = \emptyset$ and $m \equiv k < \ell$ or $V(ij_{s_1}) \cap V(j_{s_2}j_{s_3}) = \emptyset$ and $m \geq \ell$, then we can apply Lemma 5.2.10 and we get that the other two intersections are non-empty. □

Now we are ready to prove that the following algorithm (that extends the algorithm of García and Tejel [44] for $(\mathbf{m},\ell)$-tight graphs) gives the minimum covering of an $(\mathbf{m},\ell)$-tight graph with its MGT subgraphs.

Algorithm 5.2.12. INPUT: $m : V \to \mathbb{Z}_+, \ell \in \mathbb{Z}$ with (5.1) and an $(\mathbf{m},\ell)$-tight graph $G = (V,E)$ with at least 4 nodes. OUTPUT: A list of edges F for which $G + F$ is $(\mathbf{m},\ell)$-redundant.

1. Search for a node with minimum degree in G. Let v be a node of minimum degree.

If $d_G(v) = m(v)$ then let $i_1 := v$, $V' := \emptyset$ and go to step 3.

Otherwise, let $i_1 := v$.

2. Run Algorithm 5.2.3 with inputs m, ℓ, G, i_1 and $L := \emptyset$. Let the output $V'(i_1) = \{j_2, \ldots, j_r\}$.

If $r = 2$, then RETURN $F := \{i_1 j_2\}$.

Else let $L := V'(i_1) - j_2 + i_1$ and let $i_1 := j_2$.

3. Run Algorithm 5.2.3 with inputs m, ℓ, G, i_1 and L. Let the output be $V'(i_1, L) = \{i_2, \ldots, i_h\}$.

If $h = 2$ then RETURN $F := \{i_1 i_2\}$.

Else, let $V' := V'(i_1, L) + i_1 = \{i_1, \ldots, i_h\}$ and let $F = \emptyset$.

4. While $h \geq 4$ do:

Calculate $T(i_1 i_{h-2})$ and $T(i_{h-1} i_h)$.

If $\widetilde{m}(V(i_1 i_{h-2}) \cap V(i_{h-1} i_h)) \geq \ell$, then

$F := F + i_{h-1} i_h$.

Else

$F := F + i_{h-2} i_h$,

$i_{h-2} := i_{h-1}$.

$h := h - 2$.

5. Final step.

If $h = 2$, then $F := F + i_1 i_2$.

If $h = 3$, then $F := F \cup \{i_1 i_2, i_1 i_3\}$.

RETURN F.

Theorem 5.2.13. *There is a polynomial time algorithm to obtain a set of edges* $F = \{e_1, \dots e_t\}$ *of minimum cardinality for any input of* $m : V \to \mathbb{Z}_+, \ell \in \mathbb{Z}$ *with* (5.1) *and of an* $(\mathbf{m}, \ell)$*-tight graph* G*, such that* $G = T(e_1) \cup \dots \cup T(e_t)$.

Proof. We will prove that Algorithm 5.2.12 calculates an optimal covering of G and it runs in polynomial time. If the algorithm finishes in step 2 or 3, then it is because $G = T(e_1)$ thus we have found the minimum covering.

Otherwise, Corollaries 5.2.8 and 5.2.9 show that any MGT subgraph of G covering two different $i, i' \in V'' = \{i_1, \dots, i_h\}$ coincides with $T(ii')$ and does not contain any other $i'' \in V''$. Therefore, at least $\lceil \frac{h}{2} \rceil$ MGT subgraphs are needed to cover the node set V''.

By Lemma 5.2.5 and Corollary 5.2.9 $G = \bigcup_{s=2}^{h} T(i_1 i_s)$. The construction of E', Lemma 5.1.1(a) and Corollary 5.2.11 provides that after each iteration in step 4 the set of edges covered by $\bigcup_{s=2}^{h} T(i_1 i_s) \cup \bigcup_{e \in F} T(e)$ does not reduce. Therefore, $G = \bigcup_{e \in F} T(e)$ at the end of Algorithm 5.2.12. (Note that for $\ell \le 0$ the algorithm can be simplified in this point by taking an arbitrary perfect matching on L (or a matching that covers all but one node v and an additional edge covering v) that follows from the proof of Corollary 5.2.11.)

We have seen that Algorithm 5.2.3 is polynomial. Thus the running time of steps 1-3 is polynomial. The running time of step 4 is also polynomial since it has $O(n)$ iterations and in each iteration we need to calculate 2 generated $(\mathbf{m}, \ell)$-tight graphs. Finally, the running time of step 5 is $O(1)$. □

5.3 Augmenting an $(\mathbf{m}, \ell)$-rigid graph to $(\mathbf{m}, \ell)$-redundant for $m \ge \ell$

Throughout this section, $R = (V, \bar{E})$ will denote an $(\mathbf{m}, \ell)$-rigidgraph and $G = (V, E)$ will denote an $(\mathbf{m}, \ell)$-tight spanning subgraph of R. Obviously, every edge in $\bar{E} - E$ is $(\mathbf{m}, \ell)$-redundant in R. By Lemma 5.1.5, the $(\mathbf{m}, \ell)$-redundant edges of G in R are the edges of $R^G(\bar{E} - E) = \bigcup_{uv \in \bar{E} - E} T^G(uv)$. As we have solved the augmentation problem for $(\mathbf{m}, \ell)$-tight graphs, we assume that $\bar{E} - E \neq \emptyset$.

The idea of our proof comes from Jackson and Jordán [54] where the authors proved that the (k, k)-redundant edges of a (k, k)-rigid (that is, a k-tree-connected) graph R form induced subgraphs of R with disjoint node sets.

We divide the problem into two cases depending on whether $\ell < 0$ or $\ell \ge 0$.

The case of $\ell \geq 0$

First, let us consider the case where $m \geq \ell \geq 0$. In this case, the $(\mathbf{m},\ell)$-redundant edges of G form some node disjoint $(\mathbf{m},\ell)$-tight induced subgraphs of G by Lemma 5.1.1(a). By shrinking each of these subgraphs T_i's to a single node and by defining m' to be ℓ on the shrunken nodes and to be $m(v)$ on each non-shrunken node v, we get the shrunken graph $G' = (V', E')$. We claim the following.

Proposition 5.3.1. *G' is $(\mathbf{m}',\ell)$-tight. Moreover, the pre-image of any $(\mathbf{m}',\ell)$-tight subgraph of G' is $(\mathbf{m},\ell)$-tight and the shrunken image of an $(\mathbf{m},\ell)$-tight subgraph of G is $(\mathbf{m}',\ell)$-tight.*

Proof. G' is $(\mathbf{m}',\ell)$-tight as the number of G'-edges in $X' \subseteq V'$ equals the number of G-edges in its pre-image X (that is, at most $\widetilde{m}(X) - \ell$) minus the number of edges in the shrunken components T_i's of X (that is, $\sum(\widetilde{m}(V(T_i)) - \ell)$).

Now let T' be an $(\mathbf{m}',\ell)$-tight subgraph of G', that is, $widetildem'(V(T')) = \widetilde{m}(V(T') - S) + \ell|V(T') \cap S| - \ell = |E(T')|$ where S denotes the set of shrunken nodes in V'. Let T be the pre-image of T'. Then $|E(T)| = |E(T')| + \sum^* |E(T_i)|$ where the sum is on the shrunken components with image in $V(T')$. Therefore, $|E(T)| = \widetilde{m}(V(T') - S) + \ell|V(T') \cap S| - \ell + \sum^*(\widetilde{m}(V(T_i)) - \ell) = \widetilde{m}(V(T)) - \ell$, that is, T is $(\mathbf{m},\ell)$-tightas the $(\mathbf{m},\ell)$-sparsity follows by the $(\mathbf{m},\ell)$-sparsity of G.

For the last statement let T be an $(\mathbf{m},\ell)$-tight subgraph of G. As if we take the union of T with the shrunken T_i components whose node set is intersected by $V(T)$, we get another $(\mathbf{m},\ell)$-tight subgraph of G by Lemma 5.1.1(a) and hence its shrunken image, that coincides with the shrunken image of T, is $(\mathbf{m}',\ell)$-tightsimilarly as G'. □

Therefore, a covering of G with $(\mathbf{m},\ell)$-tight subgraphs gives a covering of G' with $(\mathbf{m}',\ell)$-tight subgraphs. Hence the minimum number of edges that we need to make G $(\mathbf{m},\ell)$-redundant is at least the minimum number of edges that we need to make G' $(\mathbf{m}',\ell)$-redundant. The following statement shows that these two values are equal.

Proposition 5.3.2. *Let F' denote an edge set of minimum cardinality on V' for which $G' \cup F'$ is $(\mathbf{m}',\ell)$-redundant. Let F be an arbitrary pre-image of F' with the same cardinality, that is, we get F' form F by shrinking the $(\mathbf{m},\ell)$-tight subgraphs of redundant edges of G. Then $R \cup F$ is $(\mathbf{m},\ell)$-redundant.*

Proof. The shrunken image of $T^G_{(\mathbf{m},\ell)}(uv)$ is an $(\mathbf{m}',\ell)$-tight subgraph of G' that spans the image u' and v' of u and v both by Proposition 5.3.1. Thus it is a subgraph of $T^{G'}_{(\mathbf{m}',\ell)}(u'v')$ by Lemma 5.1.4(c). Since the image of each non-$(\mathbf{m},\ell)$-redundant edge of R is in G' and the subgraphs $\{T^{G'}_{(\mathbf{m}',\ell)}(u'v') : u'v' \in F'\}$ cover the edge set of G', the subgraphs $\{T^G_{(\mathbf{m},\ell)}(uv) : uv \in F\}$ cover every non-$(\mathbf{m},\ell)$-redundant edge of R. Hence $R \cup F$ is $(\mathbf{m},\ell)$-redundant. □

We have reduced the problem of augmenting an $(\mathbf{m}, \ell)$-rigid graph to an $(\mathbf{m}, \ell)$-redundant graph to the problem of augmenting an $(\mathbf{m}', \ell)$-tight graph to an $(\mathbf{m}', \ell)$-redundant graph that we can solve with our previous algorithm as $m' \geq \ell$ holds obviously. We have got the following theorem.

Theorem 5.3.3. *There is a polynomial time algorithm to obtain a set of edges F of minimum cardinality for any input of $m : V \to \mathbb{Z}_+$, $\ell \in \mathbb{Z}_+$ with $m \geq \ell$ and of an $(\mathbf{m}, \ell)$-rigid graph R, such that $R \cup F$ is $(\mathbf{m}, \ell)$-redundant.* □

The case of $\ell < 0$

Now, we consider the case where $\ell < 0$. In this case, the $(\mathbf{m}, \ell)$-redundant edges of G form one $(\mathbf{m}, \ell)$-tight induced subgraph T of G by Lemma 5.1.1(a). By shrinking T to a single node and by defining m' to be 0 on the shrunken node and to be $m(v)$ on each non-shrunken node v, we get the shrunken graph $G' = (V', E')$. If $X \subseteq V$ is disjoint from $V(T)$, then $i_G(X) \leq \widetilde{m}(X)$ since $i_G(X) + \widetilde{m}(V(T)) - \ell = i_G(X) + i_G(V(T)) \leq i_G(X \cup V(T)) \leq \widetilde{m}(X) + \widetilde{m}(V(T)) - \ell$. Hence if $X \subseteq V'$ does not contain the shrunken node, then $i_{G'}(X) \leq \widetilde{m'}(X)$. Moreover, the number of G'-edges in $X' \subseteq V'$ containing the shrunken node equals to the number of G-edges in its pre-image X (that is at most $\widetilde{m}(X) - \ell$) minus the number of edges in T (that is $\widetilde{m}(V(T)) - \ell$). Therefore, G' is $(\mathbf{m}', 0)$-tight. Like in Proposition 5.3.1, the image of an $(\mathbf{m}, \ell)$-tight subgraph of G is $(\mathbf{m}', 0)$-tight(as if we take its union with the T, we get another $(\mathbf{m}, \ell)$-tight subgraph of G by Lemma 5.1.1(a)). Moreover, by a similar argument, the union of the pre-image of any $(\mathbf{m}', 0)$-tight subgraph of G' and T is $(\mathbf{m}, \ell)$-tight. (Note that an $(\mathbf{m}, \ell)$-tight subgraph of G must intersect T in this case.) Therefore, a covering of G with $(\mathbf{m}, \ell)$-tight subgraphs gives a covering of G' with $(\mathbf{m}', 0)$-tight subgraphs. Hence the minimum number of edges that we need to make G $(\mathbf{m}, \ell)$-redundant is at least the minimum number of edges that we need to make G' $(\mathbf{m}', 0)$-redundant. The following statement shows that these two numbers are equal.

Proposition 5.3.4. *Let F' denote an edge set of minimum cardinality on V' for which $G' \cup F'$ is $(\mathbf{m}', 0)$-redundant. Let F be an arbitrary pre-image of F' with the same cardinality, that is, we get F' form F by shrinking T. Then $R \cup F$ is $(\mathbf{m}, \ell)$-redundant.*

Proof. The shrunken image of $T^G_{(\mathbf{m},\ell)}(uv)$ is an $(\mathbf{m}', 0)$-tight subgraph of G' that spans the image u' and v' of u and v both. Thus it is a subgraph of $T^{G'}_{(\mathbf{m}',0)}(u'v')$ by Lemma 5.1.4(c). Since the image of each non-$(\mathbf{m}, \ell)$-redundant edge of R is in G' and the subgraphs $\{T^{G'}_{(\mathbf{m}',0)}(u'v') : u'v' \in F'\}$ cover the edge set of G', the subgraphs $\{T^G_{(\mathbf{m},\ell)}(uv) : uv \in F\}$ cover every non-$(\mathbf{m}, \ell)$-redundant edge of R. Hence $R \cup F$ is $(\mathbf{m}, \ell)$-redundant. □

We have reduced the problem of augmenting an $(\mathbf{m},\ell)$-rigid graph to an $(\mathbf{m},\ell)$-redundant graph to the problem of augmenting an $(\mathbf{m}',0)$-tight graph to an $(\mathbf{m}',0)$-redundant graph that we can solve with our previous algorithm as $m' \geq 0$ holds obviously. We have got the following theorem.

Theorem 5.3.5. *There is a polynomial time algorithm to obtain a set of edges F of minimum cardinality for any input of $m : V \to \mathbb{Z}_+$, $\ell \in \mathbb{Z}_-$ and of an $(\mathbf{m},\ell)$-rigid graph R, such that $R \cup F$ is $(\mathbf{m},\ell)$-redundant.* □

5.4 Concluding remarks

5.4.1 Further extensions

Let G be a graph such that cG is $(\mathbf{m},\ell)$-tight (for some m and ℓ with (5.1)). By Lemma 5.1.5, $R(i_1j_1,\dots,i_cj_c) = T(i_1j_1) \cup \dots \cup T(i_cj_c)$ hence $R(ij,\dots,ij) = T(ij) \cup \dots \cup T(ij) = T(ij)$. Thus if we get G' by adding some edges to G we get the same $(\mathbf{m},\ell)$-redundant edges in cG' as if we add just one (and not c) copy of these edges to cG. Hence our previous algorithms can be used to prove the following corollaries.

Corollary 5.4.1. *There is a polynomial time algorithm to obtain a set of edges F of minimum cardinality for any input of $m : V \to \mathbb{Z}_+, \ell \in \mathbb{Z}, c \in \mathbb{Z}_+$ with (5.1) and of a graph G for which cG is $(\mathbf{m},\ell)$-tight, such that $c(G+F)$ is $(\mathbf{m},\ell)$-redundant. Moreover, $cG+F$ is also $(\mathbf{m},\ell)$-redundant.* □

Corollary 5.4.2. *There is a polynomial time algorithm to obtain a set of edges F of minimum cardinality for any input of $m : V \to \mathbb{Z}_+, \ell \in \mathbb{Z}, c \in \mathbb{Z}_+$ with $m \geq \ell$ and of a graph G for which cG is $(\mathbf{m},\ell)$-rigid, $c(G+F)$ is $(\mathbf{m},\ell)$-redundant. Moreover, $cG+F$ is also $(\mathbf{m},\ell)$-redundant.* □

An application of these algorithms will be shown in Chapter 7.

5.4.2 An NP-hardness result

Finally, we show that the general problem is NP-hard in the case where k is even and $\ell = \frac{3}{2}k$.

Theorem 5.4.3. *Let k be a fixed positive even number. Then the general problem for $(k,\frac{3}{2}k)$-rigid graphs cannot be solved with a polynomial algorithm unless P=NP.*

Proof. García and Tejel [44] proved the statement of this theorem for $k = 2$. Now let $k \geq 4$. For a contradiction, let us assume that we have a polynomial algorithm $\mathcal{A}$ for the general problem for $(k,\frac{3}{2}k)$-rigid graphs. We will show that $\mathcal{A}$ can be used to solve the general problem for $(2,3)$-rigid graphs, contradicting [44]. To this end, let $G=(V,E)$ be a $(2,3)$-rigid graph. It is easy to check that $\frac{k}{2}G$ is $(k,\frac{3}{2}k)$-rigid. Let F be a set of new edges that $\mathcal{A}$ returns on the input graph $\frac{k}{2}G$. By using Lemma 5.1.5 for an arbitrary $(k,\frac{3}{2}k)$-tight subgraph of $\frac{k}{2}G$, we get that F does not include parallel edges. We will show that $G^*=G+F$ is $(2,3)$-redundant and there is no smaller set of new edges with this property.

Let $G'=(V,E')$ be a $(2,3)$-tight spanning subgraph of G. Then it is easy to see that $\frac{k}{2}G'$ is $(k,\frac{3}{2}k)$-tight. Let $e\in E'$ be an arbitrary edge. As $\frac{k}{2}G+F$ is $(k,\frac{3}{2}k)$-redundant, there is an edge $f\in\frac{k}{2}(E-E')\cup F$ such that the graph $\frac{k}{2}G'-e+f$ is $(k,\frac{3}{2}k)$-tight. We show that $\frac{k}{2}(G'-e+f)$ is also $(k,\frac{3}{2}k)$-tight and hence $G'-e+f$ is $(2,3)$-tight. To prove the $(k,\frac{3}{2}k)$-sparsity, it is enough to prove it on every $X\subseteq V$ for which f is induced by X as $i_{\frac{k}{2}(G'-e+f)}(X)=i_{\frac{k}{2}(G'-e)}(X)\le k|X|-\frac{3}{2}k$ whenever X does not induce f and $|X|\ge 2$. Since the graph $\frac{k}{2}G'-e+f$ is $(k,\frac{3}{2}k)$-sparse, $0<k|X|-\frac{3}{2}k-i_{\frac{k}{2}G'-e}(X)\le k|X|-\frac{3}{2}k-i_{\frac{k}{2}(G'-e)}(X)$. By the definition of $\frac{k}{2}(G'-e)$, $\frac{k}{2}\Big|i_{\frac{k}{2}(G'-e)}(X)$ and thus $\frac{k}{2}\Big|k|X|-\frac{3}{2}k-i_{\frac{k}{2}(G'-e)}(X)$ for every $X\subseteq V$. As $\frac{k}{2}\Big|k|X|-\frac{3}{2}k-i_{\frac{k}{2}(G'-e)}(X)>0$, $k|X|-\frac{3}{2}k-i_{\frac{k}{2}(G'-e)}(X)\ge\frac{k}{2}$ and hence $i_{\frac{k}{2}(G'-e)}(X)+\frac{k}{2}\le k|X|-\frac{3}{2}k$ for every $X\subseteq V$ for which f is induced in X. Therefore, $\frac{k}{2}(G'-e+f)$ is also $(k,\frac{3}{2}k)$-sparse. Thus, by counting the edges, we get that $\frac{k}{2}(G'-e+f)$ is $(k,\frac{3}{2}k)$-tight and hence $G'-e+f$ is $(2,3)$-tight. Therefore, G^* is $(2,3)$-redundant as e was arbitrarily chosen.

Finally, assume that $G^{**}=G+F'$ is $(2,3)$-redundant for a set of new edges F'. Then it follows easily that $\frac{k}{2}G^{**}$ is $(k,\frac{3}{2}k)$-redundant. By Lemma 5.1.5, we can conclude that $\frac{k}{2}G+F'$ is also $(k,\frac{3}{2}k)$-redundant, and we get $|F|\le|F'|$ by the minimality of $|F|$. $\square$

Chapter 6

Node-redundantly rigid and globally rigid graphs

Theorem 1.2.2 gives a formula for the edge number of minimally rigid graphs. As the edge set of a minimally $[1, d]$-rigid graph corresponds to the base of the d-dimensional rigidity matroid of the graph, it is not surprising that the edge sets of minimally $[1, d]$-rigid graphs on the same node set have the same cardinality. However, as we will see later, this is not true for $[k, d]$-rigid graphs when $k \geq 2$, there are minimally $[k, d]$-rigid graphs for all $k \geq 2$ and $d \geq 1$ with different edge numbers, that is, the set of weakly minimally $[k, d]$-rigid graphs is not empty for any d if $k \geq 2$.

To see a simple example, consider the case where $d = 1$. It is well known that G is rigid in $\mathbb{R}^1$ if and only if G is connected. Hence G is minimally $[k, 1]$-rigid if and only if it is minimally k-connected. It is easy to construct minimally k-connected graphs with different edge-numbers, for example, the complete bipartite graph $K_{n-k,k}$ is minimally k-connected with $k(n-k)$ edges and there exist (almost) k-regular minimally k-connected graphs with $\lceil kn/2 \rceil$ edges.

It was shown by B. Servatius [94] that the smallest possible number of edges in a $[2, 2]$-rigid graph is $2|V| - 1$ and this bound is sharp. Later, lower and upper bounds were provided for the edge number of minimally $[k, d]$-rigid graphs for $d = 2$ and 3 by Anderson et al. in [80, 82, 96, 97]). In [81], Montevallian, Yu and Anderson gave lower bounds for the edge number of minimally globally $[k, 2]$-rigid graphs. We summarize these results in the following theorem.

Theorem 6.0.1. *Let $G = (V, E)$ be a minimally $[k, d]$-rigid graph. Then there are some constants $N_{k,d}$ and $c_{k,d}$ for $k, d \in \mathbb{Z}_+$ such that*

(i) $|E| \geq 2|V| - 1$ *if* $k = d = 2$ *and* $|V| \geq N_{2,2}$. *This bound is sharp.*

(ii) $|E| \geq 2|V| + 2$ *if* $k = 3$, $d = 2$ $|V| \geq N_{3,2}$. *This bound is sharp.*

(iii) $|E| \geq \lceil \frac{k+1}{2}|V| \rceil$ *if* k *is arbitrary,* $d = 2$ *and* $|V| \geq N_{k,2}$.

(iv) $|E| \leq c_{k,d}\binom{k+d}{2}n$ *if* k *is arbitrary and* $d = 2$ *or 3.*

If a graph $G' = (V', E')$ *is minimally globally* $[k,2]$*-rigid, then there is a constant* N'_k *such that*

(v) $|E'| \geq \lceil \frac{k+2}{2}|V| \rceil$ *if* k *is arbitrary and* $|V| \geq N'_k$*. This bound is sharp when* $k = 2$ *or* 3. □

In this chapter, we present lower and upper bounds for the number of edges of minimally $[k,d]$-rigid graphs. Our lower bound is sharp for $k = 2$ for all d and for $k = 3,\ d \leq 3$ and our upper bound is sharp for every pair $[k,d]$. We also show that weakly minimally $[k,d]$-rigid graphs exist for every pair $[k,d]$. These results are joint with Viktória E. Kaszanitzky [65].

In [65], we noted that our proofs can be extended to obtain similar results for globally rigid graphs. It is easy to deduce from Theorems 1.2.2 and 1.2.4 that a globally rigid graph in $\mathbb{R}^d$ has at least $d|V| - \binom{d+1}{2} + 1$ edges if $|V| \geq d + 2$. Moreover, C_n is a sharp example for this result on the line and hence $C_n * v_1 * \cdots * v_{d-1}$ is a sharp example for $\mathbb{R}^d$ by Theorem 1.2.21. However, no upper bound is know not even for the edge number of minimally globally $[1,2]$-rigid graphs. (Again, the case of $\mathbb{R}^1$ is simple as globally k-rigid graphs on the line are the $k+1$-connected graphs.) As an extension of these results, we also give lower bounds for the number of edges of globally $[k,d]$-rigid graphs.

In the next section, we show how coning can be used in the investigation of $[k,d]$-rigid and globally $[k,d]$-rigid graphs. This is our main new proof technique that will be used in Section 6.3, to give the upper bound for the edge number of minimally $[k,d]$-rigid graphs, and in Section 6.5 to construct examples for (weakly) minimally $[k,d]$-rigid and globally $[k,d]$-rigid graphs. In Section 6.2, we give our lower bounds for the edge number of minimally $[k,d]$-rigid and globally $[k,d]$-rigid graphs. We show the sharpness of these bounds in some cases in Section 6.4.

6.1 The effect of coning on (globally) $[k,d]$-rigid graphs

First, we prove some important consequences of Theorems 1.2.20 and 1.2.21 that will be useful throughout this chapter.

Lemma 6.1.1. *Let* $e \in E$ *be an M-bridge in* $\mathcal{R}_d(G)$*. Then* e *is a M-bridge in* $\mathcal{R}_{d+1}(G * v)$*.*

Proof. We can assume that G is rigid in $\mathbb{R}^d$. (If it is not rigid then we add a minimum set of edges that makes it rigid and so e is still a bridge.) Then by Theorem 1.2.20 $G * v$ is rigid is $\mathbb{R}^{d+1}$ but $(G - e) * v = (G * v) - e$ is not. Hence e is a bridge in $\mathcal{R}_{d+1}(G * v)$ as we claimed. □

We remark that Theorem 1.2.20 (Theorem 1.2.21, respectively) cannot be generalized to (globally) k-rigid graphs. That is, if G is (globally) $[k,d]$-rigid for some $k \geq 2$, then $G*v$ is not necessarily (globally) $[k,d+1]$-rigid. For example, C_n is $[2,1]$-rigid, but C_n*v (which is the wheel graph with $n+1$ nodes) is not $[2,2]$-rigid. However, the following results show that coning can be used to construct (globally) $[k,d]$-rigid graphs.

Lemma 6.1.2. *Let G be a $[k,d+1]$-rigid graph. Then G is $[k+1,d]$-rigid.* □

Lemma 6.1.3. *Let $k \geq 2$ and $d \geq 1$ be integers and let $G=(V,E)$ be a $[k-1,d]$-rigid graph. Then $G*v$ is $[k,d]$-rigid.* □

We skip the proofs of Lemmata 6.1.2 and 6.1.3 (they can be found in [65]) as we give similar proofs for the following results on globally $[k,d]$-rigid graphs.

Lemma 6.1.4. *Let G be a globally $[k,d+1]$-rigid graph. Then G is globally $[k+1,d]$-rigid.*

Proof. Let G' be a globally $[1,d+1]$-rigid graph that we obtain from G by deleting $k-1$ arbitrary nodes. Suppose, for a contradiction, that there is a node $u \in V(G')$ such that $G'-u$ is not globally $[1,d]$-rigid. Then $(G'-u)*u$ is not globally $[1,d+1]$-rigid by Theorem 1.2.21 which contradicts the global $[1,d+1]$-rigidity of $G' \subseteq (G'-u)*u$. □

Lemma 6.1.5. *Let $k \geq 2$ and $d \geq 1$ be integers and let $G=(V,E)$ be a globally $[k-1,d]$-rigid graph. Then $G*v$ is globally $[k,d]$-rigid.* □

Proof. We need to show that, after deleting $k-1$ nodes, $G*v$ remains globally $[1,d]$-rigid. If v is omitted, then we are done by the global $[k-1,d]$-rigidity of G. Otherwise, let $u_1,...,u_{k-1}$ be the omitted nodes. $G-\{u_1,...,u_{k-2}\}$ is globally $[1,d]$-rigid and v is connected to every neighbor of v_{k-1}. Hence $(G*v)-\{u_1,...,u_{k-1}\}$ has a subgraph isomorphic to the globally $[1,d]$-rigid graph $G-\{u_1,...,u_{k-2}\}$ showing that it is globally $[1,d]$-rigid. □

6.2 Lower bounds for the number of edges in (globally) $[k,d]$-rigid graphs

In this section, we present several lower bounds for the number of edges in $[k,d]$-rigid and globally $[k,d]$-rigid graphs for arbitrary positive integers k and d. Theorem 6.0.1 (i)-(iii) and (v) summarizes the lower bounds that were known earlier. Again, we only prove the global rigidity versions of these results. (For the omitted proofs we refer to [65].) First we extend (i) and (ii) to every dimension d.

Theorem 6.2.1. *If a graph $G=(V,E)$ is $[k,d]$-rigid with $|V|\geq d^2+d+k$ then*

$$|E| \geq d|V| - \binom{d+1}{2} + (k-1)d + \max\left\{0, \left\lceil k-1-\frac{d+1}{2}\right\rceil\right\}. \tag{6.1}$$

□

Note that the bound given in (6.1) coincides with the bounds given in Theorem 6.0.1 (i)-(ii) for $[k,d]=[2,2],\ [3,2]$, and hence it is sharp for these values of k and d. In Section 6.4, we show that this lower bound is also sharp for $[k,d]=[2,d]$ where d is arbitrary, and for $[k,d]=[3,3]$.

Theorem 6.2.2. *If a graph $G=(V,E)$ is globally $[k,d]$-rigid with $|V|\geq d^2+d+k+1$ then*

$$|E| \geq d|V| - \binom{d+1}{2} + (k-1)d + 1 + \max\left\{0, \left\lceil k-1-\frac{d+1}{2}+\frac{1}{d}\right\rceil\right\}. \tag{6.2}$$

Proof. We prove this theorem by induction on k. For $k=1$, we have seen in the introduction of this chapter that the theorem follows from Theorems 1.2.2 and 1.2.4.

Now, let $G=(V,E)$ be a globally $[k,d]$-rigid graph for $k\geq 2$ with $|V|\geq d^2+d+k+1$ and assume that the theorem is true for $k-1$. Let $v\in V$ be a node of maximum degree in G. As $G-v$ is globally $[k-1,d]$-rigid with at least $d^2+d+k-1$ nodes,

$$|E(G-v)| \geq d(|V|-1) - \binom{d+1}{2} + (k-2)d + 1 + \max\left\{0, \left\lceil k-2-\frac{d+1}{2}+\frac{1}{d}\right\rceil\right\}$$

by induction. Using this inequality, we have

$$\begin{aligned} |E| &\geq d(|V|-1) - \binom{d+1}{2} + (k-2)d + 1 + \max\left\{0, \left\lceil k-2-\frac{d+1}{2}+\frac{1}{d}\right\rceil\right\} + \Delta(G) \\ &= d|V| - \binom{d+1}{2} + (k-1)d + 1 + \max\left\{0, \left\lceil k-2-\frac{d+1}{2}+\frac{1}{d}\right\rceil\right\} + (\Delta(G)-2d) \end{aligned}$$

where $\Delta(G)$ denotes the maximum degree in G. Here, $\max\left\{0, \left\lceil k-2-\frac{d+1}{2}+\frac{1}{d}\right\rceil\right\}=0=\max\left\{0, \left\lceil k-1-\frac{d+1}{2}+\frac{1}{d}\right\rceil\right\}$ if $k-1+\frac{1}{d}\leq\frac{d+1}{2}$; and $\max\left\{0, \left\lceil k-2-\frac{d+1}{2}+\frac{1}{d}\right\rceil\right\}+1=\left\lceil k-2-\frac{d+1}{2}+\frac{1}{d}\right\rceil+1=\left\lceil k-1-\frac{d+1}{2}+\frac{1}{d}\right\rceil=\max\left\{0, \left\lceil k-1-\frac{d+1}{2}+\frac{1}{d}\right\rceil\right\}$ if $k-1+\frac{1}{d}>\frac{d+1}{2}$. Therefore, we need to prove that $\Delta(G)\geq 2d$ for all k and $\Delta(G)\geq 2d+1$ also holds if $k-1+\frac{1}{d}>\frac{d+1}{2}$.

To prove that $\Delta(G)\geq 2d$ for all k, let us observe that if a graph $H=(V',E')$ is $[1,d]$-rigid with $|V'|\geq d^2+d+2$ then $\Delta(H)\geq 2d$. (To see this suppose that $\Delta(H)\leq 2d-1$. Then $|E'|\leq |V'|d-\frac{|V'|}{2}<|V'|d-\binom{d+1}{2}$ which contradicts Theorem 1.2.2.) Since a globally $[k,d]$-rigid graph is also $[1,d]$-rigid and we have $|V|\geq d^2+d+k+1$, we get that $\Delta(G)\geq 2d$. But then

$$|E| \geq d|V| - \binom{d+1}{2} + (k-1)d + \max\left\{0, \left\lceil k-2-\frac{d+1}{2}+\frac{1}{d}\right\rceil\right\}$$

and hence $|E| > d|V|$ if $k-1+\frac{1}{d} > \frac{d+1}{2}$. Therefore, we get $\Delta(G) \geq 2d+1$ if $k-1+\frac{1}{d} > \frac{d+1}{2}$ as we wanted. □

The following theorem gives a better lower bound if k is large compared to d. This result extends Theorem 6.0.1 (iii) for higher dimensions.

Theorem 6.2.3. *Let $k \geq d+2$ and let $G = (V, E)$ be a $[k, d]$-rigid graph with $|V| \geq d+k$. Then $|E| \geq \lceil \frac{d+k-1}{2}|V| \rceil$.* □

Again we only prove an analogue of this result for globally $[k, d]$-rigid graphs that extends Theorem 6.0.1 (v). (The proof of Theorem 6.2.3 is similar in [65].)

Theorem 6.2.4. *Let $k \geq d+1$ or let $k = d \in \{1, 2\}$ and let $G = (V, E)$ be a $[k, d]$-rigid graph with $|V| \geq d+k+1$. Then $|E| \geq \lceil \frac{d+k}{2}|V| \rceil$.*

Proof. If we delete $k-1$ neighbors of a node v we get a globally $[1, d]$-rigid graph with at least $d+2$ nodes. Since the minimum degree of such a graph is at least $d+1$ by Theorem 1.2.4, we get $d_G(v) \geq k+d$. Thus the minimum degree in G is at least $k+d$ hence $|E| \geq \lceil \frac{d+k}{2}|V| \rceil$. □

Note that the bound given in Theorem 6.2.4 coincides with the bound of Theorem 6.2.2 when $k = d = 2$. We also remark that the bound given in Theorem 6.2.4 coincides with the bound of Theorem 6.0.1 (v) when $d = 2$. Hence it is sharp for $d = 2$ and $k \in \{2, 3\}$. Geleji [46] proved the sharpness also for the case where $k+d$ is even and $k \geq 3d+6$. We also note that we cannot give a similar bound for $k = d = 3$ as it is not hard to see that $C_n^3 - v_1v_2$ is globally $[2, 3]$-rigid when n is sufficiently large. (One can use 1-extension, that preserves global rigidity by Theorem 1.2.16, to obtain a subgraph of $C_n^3 - v_1v_2 - v_i$ from the complete graph K_5; we skip the details as this not gives a sharp example for Theorem 6.2.2.)

6.3 Upper bound for the number of edges in minimally $[k, d]$-rigid graphs

In this section, we give an upper bound for the number of edges of minimally $[k, d]$-rigid graphs. First we prove the following lemma.

Lemma 6.3.1. *Suppose that G is a minimally $[k, d]$-rigid graph. Then G is independent in $\mathcal{R}_{d+k-1}(G)$.*

Proof. By the minimality of G, for each e, there is a set $U_e \subseteq V$ such that $|U_e| = k-1$, e is not incident to U_e and $G - U_e - e$ is not rigid. ($G - U_e$ is rigid by the $[k, d]$-rigidity of G.) Let $U_e = \{v_1, ..., v_{k-1}\}$. Then e is an M-bridge in $\mathcal{R}_d(G - U_e)$. By Lemma 6.1.1 e is an M-bridge in $\mathcal{R}_{d+k-1}((\dots((G - U_e) * v_1) * \dots) * v_{k-1})$ and so it is an M-bridge in $\mathcal{R}_{d+k-1}(G)$. Therefore, every edge of G is an M-bridge in $\mathcal{R}_{d+k-1}(G)$, that is, G is independent in $\mathcal{R}_{d+k-1}(G)$. □

By combining Lemma 6.3.1 and Theorem 1.2.2, we immediately get the following upper bound.

Theorem 6.3.2. *Let $G = (V, E)$ be a minimally $[k, d]$-rigid graph. Then*

$$|E| \leq (d + k - 1)|V| - \binom{d+k}{2}.$$

□

The sharpness of this bound for $d \geq 2$ will be proved later in Lemma 6.5.4. As a graph is $[k, 1]$-rigid if and only if it is k-connected Mader's sharp upper bound for the edge number of minimally k-connected graphs can be applied for the edge number of minimally $[k, 1]$-rigid graphs, see [79]. This gives us the following.

Theorem 6.3.3. *Let $G = (V, E)$ be a minimally $[k, 1]$-rigid graph with $|V| \geq 3k - 1$. Then*

$$|E| \leq k|V| - k^2$$

and this bound is sharp. □

If we try to prove a similar bound for global $[k, d]$-rigidity, then we will need an upper bound for the edge number of minimally globally $[1, d]$-rigid graphs. Jordán [58] asked if our upper bound for the edge number for minimally $[2, d]$-rigid graphs is valid for minimally globally rigid graphs in $\mathbb{R}^d$ as by Theorem 1.2.13 $[2, d]$-rigid graphs are globally rigid. We conjecture that the answer is yes, moreover, we conjecture the following that will answer Jordán's question positively with the same proof as we used for Theorem 6.3.2 from Lemma 6.1.1; moreover, it will induce an upper bound of $(d + k)|V| - \binom{d+k+1}{2}$ for the edge number of minimally $[k, d]$-rigid graphs.

Conjecture 6.3.4. *Let G be a globally rigid graph in $\mathbb{R}^d$ and let e be an edge of G such that $G - e$ is not globally rigid. Then e is an M-bridge in $\mathcal{R}_{d+1}(G)$.*

The idea behind Conjecture 6.3.4 is the following. If e is an edge of a rigid M-circuit of $\mathcal{R}_{d+1}(G)$, then its end nodes are in the node set of a $[1, d + 1]$-rigid subgraph of $G - e$ and hence this subgraph is globally $[1, d]$-rigid by Lemma 6.1.2 and Theorem 1.2.13. Therefore, $G - e$ is also globally $[1, d]$-rigid. However, an M-circuit is not necessarily rigid if $d \geq 2$ (for example the double banana graph (Figure 1.2) is a non-rigid M-circuit in the 3-dimensional space).

Note that for $d = 1$ we again get a better (sharp) upper bound by Mader's theorem [79] as a graph is globally $[k, 1]$-rigid if and only if it is $k + 1$-connected.

Theorem 6.3.5. *Let $G = (V, E)$ be a minimally globally $[k, 1]$-rigid graph with $|V| \geq 3k + 2$. Then*

$$|E| \leq (k + 1)|V| - (k + 1)^2$$

and this bound is sharp. □

6.4 Strongly minimally $[k, d]$-rigid graphs

In this section, we sketch why the the lower bound given in Theorem 6.2.1 is sharp for $k = 2$ in any dimension and for $k = d = 3$. (We refer to [65] for the full proofs.)

Consider graph C_n^d and its subgraph L_d induced by nodes $v_{n-d+1}, \ldots, v_n$. (Note that L_d is isomorphic to K_d.) Let $H_{n,2}^d = C_n^d - E(L_d)$. First we show that $H_{n,2}^d$ is $[2, d]$-rigid.

Lemma 6.4.1. *$H_{n,2}^d$ is $[2, d]$-rigid if $n \geq 3d$.*

Proof. Let $v_i \in V(H_{n,2}^d)$ be arbitrary. We can prove that $H_{n,2}^d - v_i$ is $[1, d]$-rigid by constructing it from a subgraph isomorphic to K_d using (d-dimensional) 0- and 1-extensions. We skip the details, see Figure 6.1 for a sketch. □

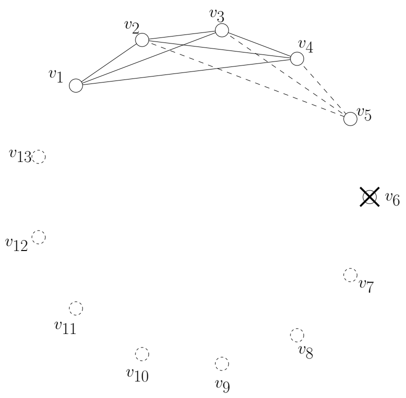
If $G = (V, E)$ is $[2, d]$-rigid then $|E| \geq d|V| - \binom{d+1}{2} + d = d|V| - \binom{d}{2}$ if $|V| \geq d^2 + d + 2$ by Theorem 6.2.1. $|E(H_{n,2}^d)| = dn - \binom{d}{2}$ since C_n^d has dn edges if $n \geq 2d + 1$ and the deleted edges form a complete subgraph with d nodes. Hence by Lemma 6.4.1 we get the following.

Theorem 6.4.2. *If $G = (V, E)$ is a strongly minimally $[2, d]$-rigid graph with $|V| \geq d^2 + d + 2$ then $|E| = d|V| - \binom{d}{2}$.* □

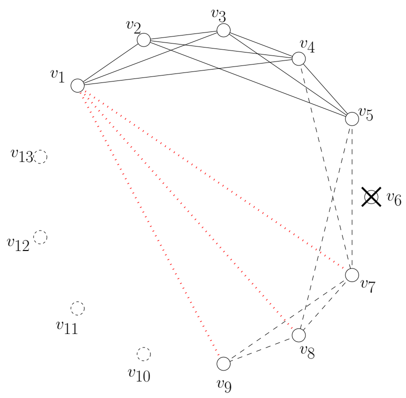
Next we show that the lower bound given in Theorem 6.2.1 is also sharp when $k = d = 3$.

Lemma 6.4.3. *C_n^3 is $[3, 3]$-rigid if $n \geq 9$.*

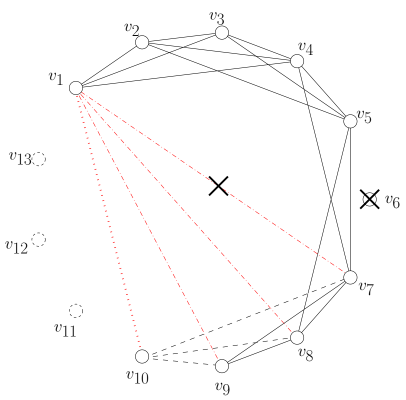
Proof. Let $v_i, v_j \in V(C_n^3)$ be arbitrary. We can assume that $j = n$ and $i \geq \lceil \frac{n}{2} \rceil$. Let $\sigma : \mathbb{Z} \to \{1, 2\}$ be a function with $\sigma(t) := 2$ if $t \equiv \ell - 2 \pmod 3$ and $\sigma(t) := 1$ otherwise. We will prove that $C_n^3 - \{v_i, v_j\}$ is $[1, 3]$-rigid by constructing it from a subgraph isomorphic to K_4 using 3-dimensional 0- and 1-extensions and triangle-based X-replacements. We skip the details, see Figure 6.2 for a sketch. □



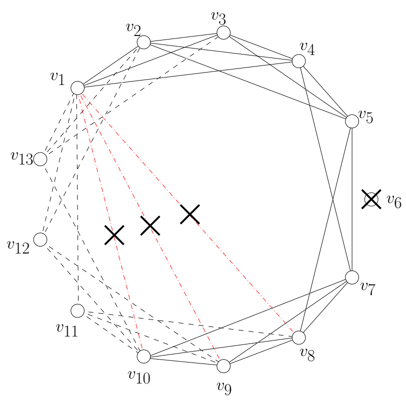
(a) Add v_5 with a 0-extension.



(b) Add v_7, v_8, v_9 with 0-extensions by adding the extra edges $v_j v_1$ for $7 \leq j \leq 9$.



(c) Add v_{10} with 1-extension on the extra edge $v_7 v_1$ by adding the extra edge $v_{10} v_1$



(d) Add v_{11}, v_{12}, v_{13} with 1-extensions on the edges $v_8 v_1, v_9 v_1, v_{10} v_1$, respectively.

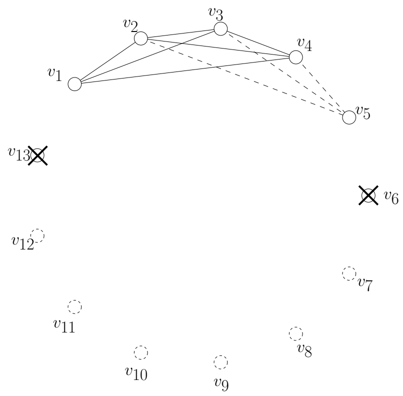
Figure 6.1: Building up $C_{13}^3 - E(L_3) - v_6$ using Henneberg operations.

We have proved that C_n^3 is $[3,3]$-rigid. It is easy to see that C_n^3 has $3n$ edges if $n \geq 7$. These together with Theorem 6.2.1 gives the following:

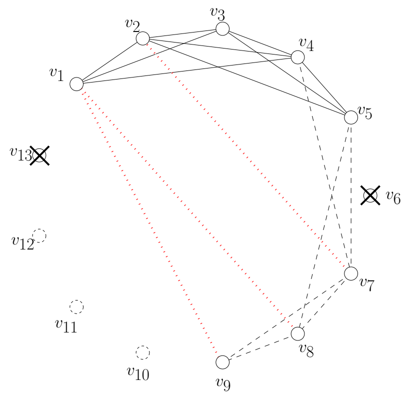
Theorem 6.4.4. *If $G = (V, E)$ is a strongly minimally $[3,3]$-rigid graph with $|V| \geq 15$, then $|E| = 3|V|$.* □

It remains open whether the lower bounds given in Theorems 6.2.1, 6.2.2, 6.2.3 and 6.2.4 are tight for some other pairs $[k, d]$ different from $[2, d]$, $[3, 2]$ and $[3, 3]$ in the rigidity case and from $[2, 2]$, $[3, 2]$ and $[k, d]$ with $k + d$ even and $k \geq 3d + 6$ for the global rigidity case. This question seems to be more complicated for larger values of k and d as there are just a few operations known that preserve rigidity in higher dimensions.

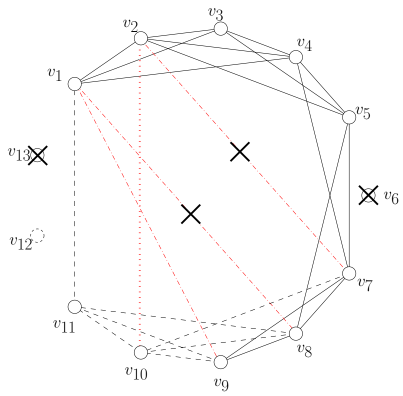
We note that a proof similar to that of Lemma 6.4.3 works if one wants to prove that C_n^d is $[3, d]$-rigid for any other $d \geq 4$. As the edge number of these graphs does not coincide



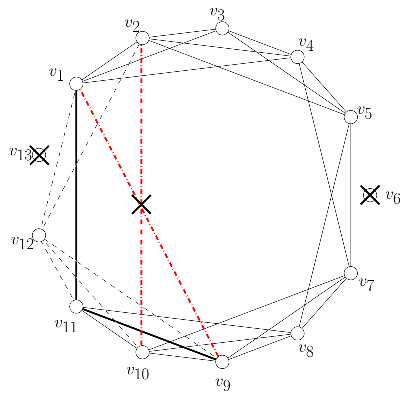
(a) Add u_5 with a 0-extension.



(b) Add v_7, v_8, v_9 with 0-extensions by adding extra edges $v_{i+m}v_{\sigma(i)}$.



(c) Add v_{10} and v_{11} with 1-extensions on edges v_7v_2, v_8v_1, respectively. By performing the first of these extensions we create the extra edge $v_{10}v_2$.



(d) Add v_{12} with triangle-based X-replacement on edges v_2v_{10}, v_1v_9. (Note that $v_1v_9v_{11}$ is a triangle.)

Figure 6.2: Building up $C_{12}^3 - \{u, v\}$.

with the bound given in Theorem 6.2.1, we skip the details. However, we conjecture that the lower bound given in Theorem 6.2.1 is sharp for $k = 3$ for all $d \geq 3$. To formulate the conjecture more precisely, recall that L_d denotes the complete subgraph of C_n^d spanned by nodes $v_{n-d+1}, \ldots, v_n$. Let L'_d denote the graph that we get from L_d by deleting the cycle that consists of edges v_iv_{i+1} for $n-d+1 \leq i \leq n-1$ and $v_{n-d+1}v_n$. Note that L'_3 is the empty graph on three nodes. Lemma 6.4.3 states that $C_n^d - L'_d$ is strongly minimally [3, 3]-rigid. $|E(C_n^d - L'_d)| = dn - \binom{d}{2} + d = dn - \binom{d+1}{2} + 2d$ which motivates the following conjecture.

Conjecture 6.4.5. *$C_n^d - L'_d$ is a strongly minimally $[3, d]$-rigid graph if n is sufficiently large. Thus the lower bound given in Theorem 6.2.1 is sharp for $k = 3$ and $d \geq 3$.*

Now we turn to the cases where $k \geq d+2$. Our conjecture is that the bound given in Theorem 6.2.3 is tight in these cases, that is, there are $k+d-1$-regular $[k,d]$-rigid graphs for every $k \geq d+2$. This would also imply the sharpness of the bound given in Theorem 6.2.4 for the case where $k \geq d+1$.

6.5 Examples for minimally (globally) $[k,d]$-rigid graphs

The question whether weakly minimally $[k,d]$-rigid graphs exist for every pair $[k,d]$ with $k \geq 2$ can still be solved without knowing the edge count of strongly minimally $[k,d]$-rigid graphs for $d \geq 2$. There are examples for weakly minimally $[2,2]$-rigid graphs in [94, 96, 97] but the existence of weakly minimally $[k,d]$-rigid graphs for other values of k and $d \geq 2$ was open so far. In this section, we will give examples for minimally $[k,d]$-rigid graphs with the same number of nodes but with different number of edges. Such a pair of graphs shows that the graph with the larger number of edges has to be weakly minimally $[k,d]$-rigid. We also give examples for minimally globally $[k,d]$-rigid graphs along with proving the existence of weakly minimally globally $[k,d]$-rigid graphs.

Let $H^d_{n,i}$ denote the cone graph of $H^d_{n,(i-1)}$ for $i \geq 3$. (For the definition of $H^d_{n,2}$ see Section 6.4.) By Lemma 6.1.3 and Lemma 6.4.1, we can get get a minimally $[k,d]$-rigid graph by deleting some edges of $H^d_{t,k}$ (to obtain minimality).

Corollary 6.5.1. *Let t, d and k be three positive integers such that $t \geq 3d$ and $k \geq 2$. Then there exists a minimally $[k,d]$-rigid graph $H^d_{t,k,\text{reduced}}$ with $n = t+k-2$ nodes and at most $(d+k-2)n - \binom{d}{2} + \binom{k-2}{2} - (d+k-2)(k-2)$ edges.* □

Let $C_n^{*^{d-1}} = C_n * w_1 * \cdots * w_{d-1}$.

Lemma 6.5.2. *Let n, d and k be three positive integers such that $d \geq 2$ and $n \geq d+k+1$. Then $C_{n-d-k+2}^{*^{d+k-2}}$ is a minimally globally $[k,d]$-rigid graph with n nodes and $(d+k-1)n - \binom{d+k}{2} + 1$ edges.* □

Proof. As $C_{n-d-k+2}$ is globally $[1,1]$-rigid, $C_{n-d-k+2}^{*^{d+k-2}}$ is globally $[1,d+k-1]$-rigid by Theorem 1.2.21. Hence $C_{n-d-k+2}^{*^{d+k-2}}$ is globally $[k,d]$-rigid by Lemma 6.1.4.

It follows by the definition that $C_{n-d-k+2}^{*^{d+k-2}}$ has n nodes. To get the edge-number, observe that $C_{n-d-k+2}$ has $n-d-k+2$, there are $(n-d-k+2)(d+k-2)$ edges between $C_{n-d-k+2}$ and $\{w_1,\dots,w_{d+k-2}\}$, and $\binom{d+k-2}{2}$ edges are spanned in $\{w_1,\dots,w_{d+k-2}\}$. Hence $C_{n-d-k+2}^{*^{d+k-2}}$ has $(n-d-k+2)(d+k-1)+\binom{d+k-2}{2} = (d+k-1)n-(d+k-1)(d+k-2)+\frac{(d+k-2)(d+k-3)}{2} = (d+k-1)n - \frac{(d+k-2)(d+k+1)}{2} = (d+k-1)n - \frac{(d+k-2)(d+k+1)}{2} = (d+k-1)n - \binom{d+k}{2} + 1$ edges. Therefore, we get the following by Theorem 6.2.2.

Claim 6.5.3. $C_{n-d-k+2}^{*^{d+k-2}}$ *is minimally globally $[1,d+k-1]$-rigid.* □

Finally, we prove that $C^{*d+k-2}_{n-d-k+2}$ is *minimally* globally $[k,d]$-rigid if $d \geq 2$. Suppose for contradiction that $C^{*d+k-2}_{n-d-k+2} - e$ is also globally $[k,d]$-rigid. By symmetry, we can assume $e = v_1w_1$ or $e = w_1w_2$. But then $C^{*d+k-2}_{n-d-k+2} - e - \{w_3, \dots, w_{k+1}\}$ is isomorphic to $C^{*d-1}_{n-d-k+2} - e$. But $C^{*d-1}_{n-d-k+2} - e$ is not globally rigid in $\mathbb{R}^d$ as $C^{*d-1}_{n-d-k+2}$ is minimally $[1,d]$-globally rigid by Claim 6.5.3, a contradiction. □ □

Define graph Y^c_t as follows for any integers c and t. Take the disjoint union of an independent set I_t of t nodes (on the node set $\{v_1, \dots, v_t\}$) and a complete graph K_c (on the node set $\{w_1, \dots, w_c\}$) and add edges v_iw_j for every pair $1 \leq i \leq t, 1 \leq j \leq c$ (see Figure 6.3).

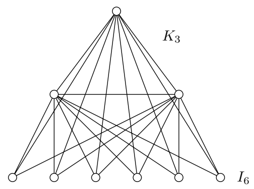


Figure 6.3: Y^3_6.

It follows easily from Theorem 1.2.20 and Lemma 6.1.2 that Y^{d+k-1}_t is $[k,d]$-rigid. Moreover, it is not hard to see that it is minimally $[k,d]$-rigid (for more details on this proof see [65]). By counting the edges of Y^{d+k-1}_t, we get that the upper bound given in Theorem 6.3.2 is sharp for $d \geq 2$.

Lemma 6.5.4. *Let $t \geq 1$, $k \geq 1$ and $d \geq 2$ be three integers. Then Y^{d+k-1}_t is a minimally $[k,d]$-rigid graph with $n = t+d+k-1$ nodes and $(d+k-1)n - \binom{d+k}{2}$ edges.* □

Corollary 6.5.5. *The upper bound given in Theorem 6.3.2 is sharp for all pair $[k,d]$ with $k, d \geq 2$.* □

As Y^{d+k}_t is $[k+1,d]$-rigid for $d \geq 2$, it is globally $[k,d]$-rigid by Corollary 1.2.14. However, it is not minimally globally $[k,d]$-rigid since the following lemma states that its subgraph, that arises by deleting all the edges of the form w_iw_j $(1 \leq i < j \leq d+k)$, is a minimally globally $[k,d]$-rigid when t is sufficiently large.

Lemma 6.5.6. *Let $k \geq 1$, $d \geq 1$ and $t \geq \binom{d+k}{2} + 1$ be three integers. Then the complete bipartite graph $K_{d+k,t}$ is a minimally globally $[k,d]$-rigid graph with $n = d+k+t$ nodes and $(d+k)n - (d+k)^2$ edges.*

Proof. First we prove that $K_{d+k,t} = (\{w_1, \dots, w_{d+k}\}, \{v_1, \dots, v_t\}; \{w_iv_j : i = 1, \dots, d+k;\ j = 1, \dots, t\})$ is globally $[k,d]$-rigid. $Y^{d+k}_{t-\binom{d+k}{2}}$ is $[2, d+k-1]$-rigid by Lemma 6.5.4 and thus it is globally $[1, d+k-1]$-rigid by Theorem 1.2.13. Next we can add $v_{t-\binom{d+k}{2}+1}, \dots v_t$ in this order with ($(d+k-1)$-dimensional) 1-extension on the edges w_iw_j $(1 \leq i < j \leq d+k)$

by connecting them to $w_1, \dots, w_{d+k}$. Thus we get that $K_{d+k,t}$ is globally $[1, d+k-1]$-rigid by Theorem 1.2.16. Therefore, $K_{d+k,t}$ is globally $[k,d]$-rigid by Lemma 6.1.4.

To prove the minimality, observe that $d_{K_{d+k,t}}(v_j) = k + d$ $(j = 1, \dots, t)$. As the minimum degree in a globally $[k,d]$-rigid is $k+d$, no edge can be omitted from $K_{d+k,t}$ such that global $[k,d]$-rigidity is preserve.

Finally, it is easy to see that $K_{d+k,t}$ has $n = d+k+t$ nodes and $(d+k)t = (d+k)(n-(d+k)) = (d+k)n - (d+k)^2$ edges. □

We can obtain the following result from Corollary 6.5.1 and Lemma 6.5.4 (see [65]).

Theorem 6.5.7. *Let d and k be positive integers with $k \geq 2$. Then there are weakly minimally $[k,d]$-rigid graphs, that is, there are minimally $[k,d]$-rigid graphs that are not strongly minimally $[k,d]$-rigid.* □

Again we only prove the following global $[k,d]$-rigidity analogue of Theorem 6.5.7 from Lemmata 6.5.2 and 6.5.6.

Theorem 6.5.8. *Let d and k be positive integers. Then there are weakly minimally globally $[k,d]$-rigid graphs, that is, there are minimally globally $[k,d]$-rigid graphs that are not strongly minimally globally $[k,d]$-rigid.*

Proof. We only prove the theorem for $d \geq 2$ as we have seen in the introduction of this chapter that there are minimally $(k+1)$-connected graphs on the same node set with different edge numbers for every positive k. By Lemma 6.5.2, there exists a minimally globally $[k,d]$-rigid graph on n nodes with $(d+k-1)n - \binom{d+k}{2} + 1$ edges if $n \geq d+k+1$. By Lemma 6.5.6, $K_{d+k,n-d-k}$ is a minimally globally $[k,d]$-rigid graph on n nodes with $(d+k)n-(d+k)^2$ edges if $n \geq \binom{d+k}{2}+d+k+1 = \binom{d+k+1}{2}+1$. Since $(d+k-1)n-\binom{d+k}{2}+1 < (d+k)n-(d+k)^2$ if n is sufficiently large, $K_{d+k,n-d-k}$ is a weakly minimally $[k,d]$-rigid graph for all pair $[k,d]$ with $d \geq 2$ if n is sufficiently large. □

6.6 Concluding remarks

The results presented in this chapter are about the edge numbers of minimally $[k,d]$-rigid and globally $[k,d]$-rigid graphs. Other version of the problem is $[k,d]$-edge-rigidity (and global $[k,d]$-edge-rigidity). Proving similar results on these variants of the problem is a possible direction of future research. Some of our methods (for example our lower bound for large k in Theorem 6.2.3) can be used easily for these graph classes. We note that a sharp upper bound for the edge number of minimally [2,2]-edge-rigid graphs was recently given by Jordán [60], as follows.

Theorem 6.6.1 ([60]). *Let $G = (V, E)$ be a minimally $[2, 2]$-edge-rigid simple graph with $|V| \geq 7$. Then*

$$|E| \leq 3|V| - 9.$$

The complete bipartite graph graph $K_{3,n-3}$ shows that this bound is sharp. □

A different direction is to characterize inductively the class of graphs mentioned above for some values of $[k, d]$ which seems to be an interesting and difficult open question.

Chapter 7

Globally rigid body-hinge and body-bar-and-hinge graphs

One of the important steps towards a possible characterization of global rigidity in higher dimensions is to identify new necessary or sufficient conditions for global rigidity and to characterize global rigidity of special graph classes. In particular, finding more counterexamples to Hendrickson's conjecture is a challenging problem. We say that a graph G is an **H-graph** in $\mathbb{R}^d$ if it satisfies Hendrickson's necessary conditions in $\mathbb{R}^d$ (see Theorem 1.2.4) but it is not globally rigid in $\mathbb{R}^d$. For $d = 3$, Connelly [11] showed that the complete bipartite graph $K_{5,5}$ is an H-graph. He presented H-graphs for all $d \geq 3$ as well. These H-graphs are all complete bipartite graphs on $\binom{d+2}{2}$ nodes. Frank and Jiang [39] found two more (bipartite) H-graphs in $\mathbb{R}^4$ and infinite families of H-graphs in $\mathbb{R}^d$ for $d \geq 5$. Some of their H-graphs in $\mathbb{R}^d$, $d \geq 5$, contain the complete graph K_{d+1} as a subgraph. We remark that a d-dimensional H-graph G can be turned into a $d+1$-dimensional H-graph by applying the coning operation, see [17, 39].

Connelly conjectured that $K_{5,5}$ is the only H-graph in $\mathbb{R}^3$ [14, 17]. Connelly and Whiteley [17] conjectured that there exist no H-graphs in $\mathbb{R}^d$ containing K_{d+1} as a subgraph. They also conjectured that the number of H-graphs is finite in $\mathbb{R}^d$, for all $d \geq 3$. Although the above mentioned examples [39] disproved the latter conjectures for $d \geq 5$, they remained open in the three- and four-dimensional cases. Tanigawa [99] noted that every body-hinge graph which is rigid in $\mathbb{R}^d$ satisfies Hendrickson's necessary conditions and contains K_{d+1} as a subgraph. This motivated him to conjecture that a body-hinge graph is globally rigid in $\mathbb{R}^d$ if and only if it is rigid in $\mathbb{R}^d$, that is, if for its underlying graph H, $\left(\binom{d+1}{2} - 1\right) H$ is $\binom{d+1}{2}$-tree-connected.

Connelly, Jordán, and Whiteley [15] also conjectured a stronger sufficient condition for the global rigidity of body-hinge graphs. We give an affirmative answer to their conjecture. Furthermore, we show that the conjectured sufficient condition is also necessary (that immediately disproves the conjecture of Tanigawa). Our main result of this chapter is as follows.

Theorem 7.0.1. *Let $H=(V,E)$ be a graph and $d \geq 3$. Then the body-hinge graph $G_H^{\rm BH}$ is globally rigid in $\mathbb{R}^d$ if and only if $(\binom{d+1}{2}-1)H$ is highly $\binom{d+1}{2}$-tree-connected.*

We prove Theorem 7.0.1 in Section 7.3 after summarizing some observations on the tree-connectivity and edge-connectivity of graphs in Section 7.1 and showing that the deletion of the cores of the bodies in body-hinge graphs does not really effect global rigidity. We also show a similar characterization for globally rigid body-bar-and-hinge graphs in Section 7.4.

As a by-product of Theorem 7.0.1 we shall disprove (the remaining cases of) each of the previously mentioned conjectures by constructing various infinite families of H-graphs for all $d \geq 3$. Some of these families are in fact body-hinge graphs.

We close the introduction of this chapter by analyzing one of our 3-dimensional H-graphs which will also illustrate some of our arguments. The graph G of Figure 7.1a is 4-connected and minimally rigid in $\mathbb{R}^3$. Minimal rigidity can be verified by using some of the well-known inductive constructions or by using Theorem 1.2.8. Note that it can be obtained from the body-hinge graph induced by a six-cycle by deleting the cores of the bodies.

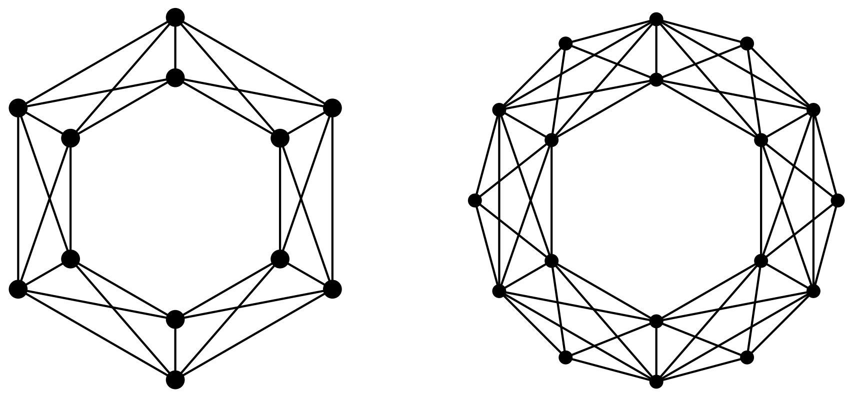
Theorem 1.2.4 implies that G is not globally rigid in $\mathbb{R}^3$. The graph $\hat{G}$ of Figure 7.1b is obtained from G by attaching a node of degree four to each of its six K_4 subgraphs. Thus $\hat{G}$ is 4-connected and redundantly rigid in $\mathbb{R}^3$. Since the new nodes are attached to complete subgraphs, the fact that G is not globally rigid implies that $\hat{G}$ is not globally rigid either in $\mathbb{R}^3$. We obtain that $\hat{G}$ is an H-graph in $\mathbb{R}^3$.

By using the same argument we can deduce that the body-hinge graph induced by the six-cycle is a 3-dimensional H-graph, too. Furthermore, as we shall see, we can construct an infinite family of H-graphs by replacing each node of the six-cycle by some graph M for which $5M$ is 6-tree-connected (and taking the induced body-hinge graph), see Section 7.3.1.

The results of this chapter are joint with Tibor Jordán and Shin-ichi Tanigawa [61].

7.1 Preliminaries

We have noted in the Introduction that (k,ℓ)-tree-connectivity coincides with (k,ℓ)-partition-connectivity by the famous results of Tutte [105] and Nash-Williams [86]. We will use here the appellation of tree-connectivity but sometimes we refer directly to the definition of partition-connectivity. To simplify the notation, D will denote the number $\binom{d+1}{2}$, where d (the dimension, in most cases) is clear from the context. G_H will denote



(a) A 4-connected minimally rigid graph G in $\mathbb{R}^3$.

(b) A 4-connected, redundantly rigid graph $\hat{G}$ in $\mathbb{R}^3$, which is not globally rigid in $\mathbb{R}^3$.

Figure 7.1: The construction of an H-graph in $\mathbb{R}^3$.

the body-hinge graph G_H^{BH} of H. For a graph H containing at least k copies of some edge e, we use ke to refer to k copies of e in H. The next lemma shows how a certain reduction step at some node with two neighbors preserves the tree-connectivity properties of the graph.

Lemma 7.1.1. *Let H be a graph and let v be a node of degree two in H with $N_H(v) = \{u, w\}$. Let $H_v = H - v + uw$ be obtained from H by removing v and adding a new edge uw. Suppose that $(D-1)H$ is highly D-tree-connected for some $d \geq 2$. Then $(D-1)H_v$ is highly D-tree-connected and $(D-1)H_v - 2(uw)$ is D-tree-connected.*

Proof. To prove the first statement, put $H' = (D-1)H_v$ and suppose, for a contradiction, that H' is not highly D-tree-connected. Then there is partition $\mathcal{P}' = \{X_1, X_2, ..., X_t\}$ of $V(H')$ with $t \geq 2$ for which $e_{H'}(\mathcal{P}') \leq D(t-1)$. If u and w belong to the same member, say P_1, then by adding v to P_1 we obtain a partition $\mathcal{P}$ of V with $e_{(D-1)H}(\mathcal{P}) = e_{H'}(\mathcal{P}') \leq D(t-1)$, contradicting the assumption that $(D-1)H$ is highly D-tree-connected. If u and v are in different members then by adding a new member $\{v\}$ to $\mathcal{P}'$ we obtain a partition $\mathcal{P}$ of V with $t+1$ members satisfying $e_{(D-1)H}(\mathcal{P}) = e_{H'}(\mathcal{P}') - (D-1) + 2(D-1) \leq D(t-1) + (D-1) = Dt - 1$, a contradiction.

The proof of the second statement is similar. □

The connectivity, edge-connectivity, and tree-connectivity parameters of H, $(D-1)H$, and G_H are related as follows. The proof of the next simple lemma is omitted.

Lemma 7.1.2. *(i) Suppose that $(D-1)H$ is D-tree-connected. Then H is 2-edge-connected.*

(ii) Suppose that H is k-edge-connected. Then G_H is $(d-1)k$-connected.

(iii) Suppose that $(D-1)H$ is D-tree-connected for some $d \geq 3$. Then G_H is $(d+1)$-connected. □

7.2 Truncated body-hinge graphs and skeletons

We shall consider graphs obtained from a body-hinge graph G_H, induced by some graph H, by deleting one node of the hinge set $H(e)$, for some $e \in E(H)$. In this graph that we denote by $G_{(H,e)}$, the two bodies associated with the endnodes of e share only $d-2$ nodes.

The following lemma is implicit in [99, Theorem 5.2].

Lemma 7.2.1. *Let $H = (V, E)$ be a graph and $e \in E(H)$. Suppose that $(D-1)H - 2e$ is D-tree-connected. Then $G_{(H,e)}$ is rigid in $\mathbb{R}^d$.* □

Let $H = (V, E)$ be a graph. The *skeleton* S_H of the induced body-hinge graph G_H is obtained by deleting the cores $C(v)$ for all $v \in V$. The rigidity, global rigidity, and connectivity properties of G_H and S_H are the same in the following sense.

Lemma 7.2.2. *Let H be a graph with minimum degree at least two and let $d \geq 3$ be an integer. Then G_H is globally rigid in $\mathbb{R}^d$ (rigid in $\mathbb{R}^d$, $(d+1)$-connected, resp.) if and only if S_H is globally rigid in $\mathbb{R}^d$ (rigid in $\mathbb{R}^d$, $(d+1)$-connected, resp.).*

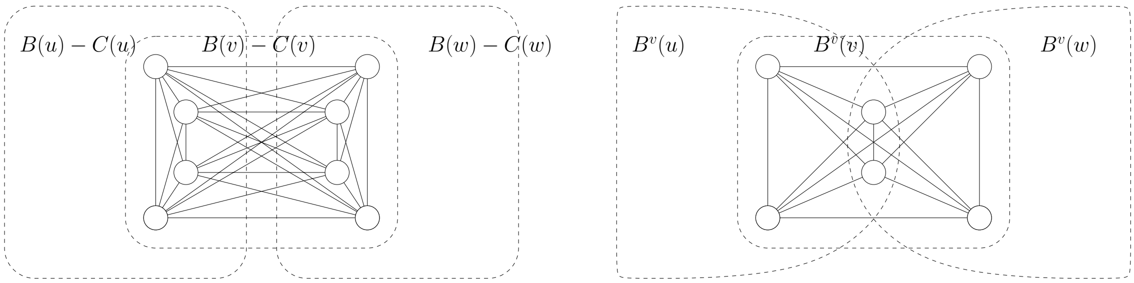
Proof. The lemma follows by observing that G_H is obtained from S_H by iteratively attaching complete subgraphs to complete subgraphs of size at least $2(d-1) \geq d+1$. □

The *skeleton* of $G_{(H,e)}$ is defined in the same manner as the graph obtained from $G_{(H,e)}$ by deleting the cores $C(v)$ for all $v \in V$.

7.3 Globally rigid body-hinge graphs

Before the proof of our main result, we prove one more lemma that we need only in the higher dimensional cases, when $d \geq 4$. To verify the three-dimensional case the next lemma and the preceding discussion can be skipped.

Let $H = (V, E)$ be a graph and suppose that H contains a node v of degree two with $N_H(v) = \{u, w\}$. Let $H(uv) = \{x_1, \ldots, x_{d-1}\}$ and $H(vw) = \{y_1, \ldots, y_{d-1}\}$ denote the hinge sets associated with uv and vw in the corresponding d-dimensional skeleton S_H. If $d = 3$ then we simply put $S_H^v = S_H$. Otherwise, when $d \geq 4$, we denote by S_H^v the graph obtained from S_H by contracting the edges $x_i y_i$ for all $3 \leq i \leq d-1$. See Figure 7.2. This operation changes the bodies of u, v, w and results in deformed hinge sets associated with edges uv and vw. We shall use $B^v(a)$ and $H^v(e)$ to denote the bodies and hinges in S_H^v associated with the nodes and edges of H, respectively. Thus $|B^v(u) \cap B^v(w)| = d-3$ and $B^v(v)$ induces a complete graph K_{d+1}.



(a) S_H (b) S_H^v

Figure 7.2: The construction of S_H^v from S_H in $\mathbb{R}^5$. The figure shows the bodies corresponding to node v with $N_H(v) = \{u, w\}$ of the underlying graph H before and after the edge contractions.

Lemma 7.3.1. *Let $H = (V, E)$ be a graph and v be a node of degree two in H. Suppose that $d \geq 3$. Then S_H is globally rigid in $\mathbb{R}^d$ if and only if S_H^v is globally rigid in $\mathbb{R}^d$.*

Proof. We shall prove the "only if" direction. The other direction can be proved in a similar fashion. As above, let $N_H(v) = \{u, w\}$, $H(uv) = \{x_1, \dots, x_{d-1}\}$, and $H(vw) = \{y_1, \dots, y_{d-1}\}$. Denote the node obtained by the contraction of edge $x_i y_i$ by z_i, for $3 \leq i \leq d-1$.

Consider a generic realization (S_H, p) of S_H in $\mathbb{R}^d$. Then the intersection L of the two affine subspaces spanned by the hinge sets $H(uv)$ and $H(vw)$, respectively, is a $(d-4)$-dimensional affine subspace of $\mathbb{R}^d$. Take $d-3$ points $a_3, \dots, a_{d-1}$ in such a way that their affine span is equal to L and define $p' : V(S_H^v) \to \mathbb{R}^d$ so that $p'(j) = p(j)$ for all $j \in V(S_H^v)$ with $j \neq z_i$ and by putting $p'(z_i) = a_i$ for $3 \leq i \leq d-1$. It follows that the affine subspaces spanned by $H(e)$ and $H^v(e)$ are the same for all $e \in E(H)$. Thus, informally speaking, the frameworks (S_H, p) and (S_H^v, p') give rise to the same body-hinge structure, since their underlying graphs as well as the corresponding hinge subspaces are all the same.

This equivalence is not entirely obvious in our bar-and-joint setting. The fact that these structures are indeed equivalent as far as global and infinitesimal rigidity are concerned can be made precise as follows.

Let (S_H^v, q') be a realization of S_H^v which is equivalent to (S_H^v, p'). Then for each $i \in V(H)$ there is an isometry f_i such that $q'(x) = f_i(p'(x))$ for each $x \in B^v(i)$. Note that for any edge $e = ij$ we have $f_i(s) = f_j(s)$ for all points s that belong to the affine span of

$p'(H^v(e))$. Hence, if we define a realization (S_H, q) such that $q(x) = f_i(p(x))$ for $x \in B(i)$ for each $i \in V(H)$, then q is well-defined. Note that (S_H^v, p') is congruent to (S_H^v, q') if and only if the isometries f_i are the same for all $i \in V(H)$, which is equivalent to saying that (S_H, p) is congruent to (S_H, q). Thus (S_H^v, p') is globally rigid if (and only if) (S_H, p) is globally rigid.

Similarly, one can check that (S_H^v, p') is infinitesimally rigid if (S_H, p) is infinitesimally rigid. Let $m' : V(S_H^v) \to \mathbb{R}^d$ be an infinitesimal motion of (S_H^v, p'). Since each rigid body affinely spans $\mathbb{R}^d$, for each $i \in V(H)$, there is a skew-symmetric matrix S_i and a vector $t_i \in \mathbb{R}^d$ such that $m'(x) = S_i p'(x) + t_i$ for $x \in B(i)$. Note that, for edge $e = ij$, we have $S_i s + t_i = S_j s + t_j$ for all points s that belong to the affine span of $p'(H^v(e))$. Hence, if we define an infinitesimal motion m of (S_H, p) such that $m(x) = S_i p(x) + t_i$ for $x \in B(i)$, for each $i \in V(H)$, then m is well-defined. Note that m is trivial if and only if the pairs (S_i, t_i) are the same for all $i \in V(H)$, which is equivalent to saying that m' is trivial.

The realization (S_H, p) is globally rigid and generic. Hence it is infinitesimally rigid. Thus the above arguments imply that (S_H^v, p'), which may not be generic, is also globally rigid and infinitesimally rigid. By Theorem 1.2.3, this implies that S_H^v is globally rigid, as required. □

We note that the "if" direction in Lemma 7.3.1 also follows from Theorem 1.2.19 by observing that S_H can be obtained from S_H^v by a sequence of node splitting operations and adding edges in such a way that in each iteration the bridging edge is part of complete subgraph K_{d+2} (and hence it is redundant) in the resulting graph.

We are now ready to prove our main result (Theorem 7.0.1) that we restate here for convenience.

Theorem 7.3.2. *Let $H = (V, E)$ be a graph and $d \geq 3$. Then the body-hinge graph G_H is globally rigid in $\mathbb{R}^d$ if and only if $(D-1)H$ is highly D-tree-connected.*

Proof. To prove necessity, suppose, for a contradiction, that G_H is globally rigid in $\mathbb{R}^d$ but $(D-1)H$ is not highly D-tree-connected. This implies that there is a partition $\mathcal{P}$ of V with $t \geq 2$ members satisfying $e_{(D-1)H}(\mathcal{P}) \leq D(t-1)$. By adding sufficiently many new edges inside the non-singleton partition classes to make their induced body-hinge graphs globally rigid and then contracting each of them into one node we may suppose that each partition member is a single node. This in turn implies that

$$(D-1)|E(H)| = D|V(H)| - D. \tag{7.1}$$

Hence there is a node v of degree two in H.

By Lemmas 7.2.2 and 7.3.1, S_H^v is globally rigid. Call an edge a hinge-edge if it is induced by some hinge set. Remove non-hinge edges from each body of S_H^v so that the sparsified skeleton, denoted by T_H, spans a minimally rigid graph on the node set of each body of S_H^v. This can be done, since each hinge set induces a small complete graph K_{d-1} and the hinge sets are pairwise disjoint (except in the body of v, which induces K_{d+1} and hence is already minimally rigid) and hence they can be extended to a spanning minimally rigid graph within each body. It is clear that T_H is rigid. We claim that T_H is minimally rigid. This follows by counting the edges and using (7.1):

$$\begin{aligned}
|E(T_H)| &= \sum_{u \in V(H)\setminus\{v\}} \left(d[(d-1)d_H(u)] - \binom{d+1}{2} \right) + \binom{d+1}{2} - \binom{d-1}{2}|E(H)| \\
&= \sum_{u \in V(H)} \left(d(d-1)d_H(u) - \binom{d+1}{2} \right) - d(d-3) - \binom{d-1}{2}|E(H)| \\
&= (d-1)\left(\frac{3}{2}d+1\right)|E(H)| - \binom{d+1}{2}|V(H)| - d(d-3) \\
&= d((d-1)|E(H)| - (d-3)) - \binom{d+1}{2} \\
&= d|V(S_H^v)| - D \\
&= d|V(T_H)| - D.
\end{aligned}$$

It follows that the edges of K_{d+1} induced by $B(v)$ are M-bridges (i.e. edges that belong to all rigid spanning subgraphs) in S_H^v. So S_H^v is not redundantly rigid. We can conclude, by using Theorem 1.2.4 and Lemma 7.3.1, that S_H is not globally rigid. By Lemma 7.2.2 this implies that G_H is not globally rigid either, a contradiction. This proves necessity.

To prove sufficiency, suppose that $(D-1)H$ is highly D-tree-connected. The proof is by induction on $|V|$. The statement is trivial for $|V| = 1$. If $(D-1)H$ is doubly highly D-tree-connected then we can use Lemma 7.2.1 to deduce that G_H is node-redundantly rigid in $\mathbb{R}^d$ and then it follows from Theorem 1.2.13 that G_H is globally rigid in $\mathbb{R}^d$, as required. Thus we may suppose that there is a partition $\mathcal{P} = \{P_1, ..., P_t\}$ of V with $t \geq 2$ and $(D-1)e(\mathcal{P}) = D(|\mathcal{P}|-1)+1$. We can also assume that $H[P_i]$ is highly D-tree-connected for all i (for otherwise we could refine $\mathcal{P}$ by an appropriate partition of P_i). By induction, $G_{H[P_i]}$ is globally rigid for all i. If one partition class, say P_1, is not a singleton then consider H' obtained from H by contracting P_1 to a node v_1. Then $(D-1)H'$ is highly D-tree-connected and hence, by induction, its body-hinge graph is globally rigid. So we may suppose that each partition class is a singleton and hence $(D-1)|E| = D|V| - D + 1$. Thus there is node in H with $d_H(v) = 2$.

By Lemmas 7.2.2 and 7.3.1, it suffices to show that S_H^v is globally rigid. Let $N_H(v) = \{u, w\}$. Let $H(uv) \setminus B^v(v) = \{x_1, x_2\}$ and $H(vw) \setminus B^v(v) = \{y_1, y_2\}$ be the two pairs of hinge-nodes not involved in the contractions when constructing S_H^v from S_H. Notice that x_1y_1 is incident to $d-1$ triangles in S_H^v, and hence the contraction of x_1y_1 corresponds to the inverse operation of node splitting. Let S' be the graph obtained from S_H^v by contracting x_1y_1. Then observe that $S' - x_2y_2$ is the skeleton of $G_{(H_v,uw)}$, where $H_v = H - v + uw$. Since H is highly D-tree-connected, Lemma 7.1.1 implies that $(D-1)H_v - 2(uw)$ is D-tree-connected. This in turn implies that $S' - x_2y_2$ is rigid by Lemma 7.2.1. It follows from Whiteley's node-splitting theorem that $S_H^v - x_2y_2$ is rigid, and x_iy_j is redundant in S_H^v for any $i = 1, 2$ and $j = 1, 2$ by symmetry.

Since $S' - x_2y_2$ is rigid, $S' - x_2$ is clearly rigid (since x_2 does not belong to any of the "hinge sets" in $S' - x_2y_2$). Observe that $S' - x_2 + K(N_{S'}(x_2))$ is the skeleton of G_{H_v}. Since $(D-1)H_v$ is highly D-tree-connected with $|V(H_v)| < |V(H)|$, $S' - x_2 + K(N_{S'}(x_2))$ is globally rigid by induction. Therefore by Theorem 1.2.12 S' is globally rigid.

Since S_H is constructed from S' by a node splitting operation and x_1y_1 is redundant in S_H^v, we can apply Theorem 1.2.19 to conclude that S_H is globally rigid. □

7.3.1 Infinite families of H-graphs

By using the necessary condition of Theorem 7.0.1 we can easily construct infinite families of d-dimensional H-graphs for all $d \geq 3$. Let H be a graph for which $(D-1)H$ is D-tree-connected but not highly D-tree-connected, or equivalently, for which $(D-1)H$ contains D edge-disjoint spanning trees and at the same time has a partition $\mathcal{P} = \{X_1, X_2, ..., X_t\}$ of V with $t \geq 2$ satisfying

$$e_H(\mathcal{P}) = \frac{D(t-1)}{D-1}.$$

For example we may obtain such graphs H from a cycle of length D by replacing each node by any subgraph H' for which $(D-1)H'$ contains D edge-disjoint spanning trees.

Then G_H is redundantly rigid (it follows by Theorem 1.2.8 and the fact that each edge belongs to a large enough complete subgraph), and $(d+1)$-connected (by Lemma 7.1.2(iii)), but not globally rigid in $\mathbb{R}^d$ (by Theorem 7.0.1). Thus it is a d-dimensional body-hinge graph which is also an H-graph.

It is possible to construct several other examples, including families which are not body-hinge graphs. For example, as noted earlier, the cone of an H-graph is also an H-graph. Another way to create examples is to take a body-hinge H-graph and then replace one (or more) of its bodies by a globally rigid graph, keeping the same nodes of attachment.

7.3.2 Globally rigid body-hinge graphs in two dimensions

Theorem 1.2.5 gives a complete description of globally rigid graphs in $\mathbb{R}^2$. We can use this result to characterize those graphs H that induce globally rigid body-hinge graphs in two dimensions. It turns out that the necessary and sufficient condition is different from that of the higher dimensional version in Theorem 7.0.1. The following result can be deduced from Theorems 1.2.4, 1.2.13 and Lemma 7.2.1.

Theorem 7.3.3. *Let H be a graph and let G_H be the two-dimensional body-hinge graph induced by H. Then G_H is globally rigid in $\mathbb{R}^2$ if and only if H is 3-edge-connected.* □

If H is 3-regular then S_H is a so-called *combinatorial zeolite* in the plane. A different proof for this special case was given in [59].

7.4 Globally rigid body-bar-and-hinge graphs

In this section, we prove an analogue of Theorem 7.0.1 for body-bar-and-hinge graphs. Our result is as follows.

Theorem 7.4.1. *Let $H = (V; E_B, E_H)$ be a 2-edge-labeled graph and $d \geq 3$. Then the body-bar-and-hinge graph G_H^{BBH} of H is globally rigid in $\mathbb{R}^d$ if and only if the graph $H' = \left(V; E_B \cup \left(\binom{d+1}{2} - 1\right) E_H\right)$ is highly $\binom{d+1}{2}$-tree-connected.*

Proof. Let us assume first that G_H^{BBH} is globally rigid in $\mathbb{R}^d$. Suppose, for a contradiction, that H' is not highly-D-tree-connected, that is, there is a partition $\mathcal{P}$ of V with $t \geq 2$ satisfying $e_{H'}(\mathcal{P}) \leq (t-1)|\mathcal{P}|$. If there is an E_B-edge e that connects two members of the partition then G_{H-e}^{BBH} is not rigid by Theorem 1.2.10 hence G_{H-e}^{BBH} is not redundantly rigid and hence not globally rigid by Theorem 1.2.4, a contradiction. Otherwise, we can use the same reasoning as in the proof of Theorem 7.0.1.

Next we prove the sufficiency by induction on $|E_B|$. Observe that the statement follows by Theorem 7.0.1 if $E_B = \emptyset$. Now, let $H = (V; E_B, E_H)$ be a 2-edge-labeled graph for which $E_B \neq \emptyset$ and $H' = (V, E_B \cup (D-1)E_H)$ is highly D-tree-connected. Let $e = uv \in E_B$ be an arbitrary E_B-edge. It is easy to see, by Theorem 1.2.10, that $G_H^{\mathrm{BBH}} - u_e$ is rigid. Hence, to fulfill the conditions of Theorem 1.2.12, it suffices to prove that $G' = G_H^{\mathrm{BBH}} - u_e + K(N_{G_H^{\mathrm{BBH}}}(u_e))$ is globally rigid. Again by Theorem 1.2.10 and Theorem 1.2.12, it suffices to prove that $G'' = G' - v_e + K(N_{G'}(v_e))$ is globally rigid. But G'', as a body-bar-and-hinge framework is the same as $G_{H/e}^{\mathrm{BBH}}$. More precisely the non-connected nodes of G'' in $B(u) \cup B(v) - u_e - v_e$ are globally linked in G'', hence we do not modify the global rigidity of the graph by connecting them; and after that we can delete the $d+1$ nodes of $C(v)$, resulting at $G_{H/e}^{\mathrm{BBH}}$, again without modifying the global rigidity of the graph. It is easy to see that H/e fulfills the conditions of this theorem and has less bar-edges. Thus $G_{H/e}^{\mathrm{BBH}}$ is globally rigid by induction. Therefore, as we have just seen, G_H^{BBH} is also globally rigid, finishing our proof. □

7.5 Concluding remarks

Theorem 7.0.1 gives rise to a polynomial time algorithm to determine whether a body-hinge graph is globally rigid in $\mathbb{R}^d$. This follows from that, a graph H is highly m-tree-connected if and only if $H - e$ contains m edge-disjoint spanning trees for all $e \in E(H)$. Thus efficient tree-packing algorithms can be used to test whether a given graph is highly m-tree-connected. An efficient algorithm for this problem was also shown in Section 2.4.3.

Another algorithmic observation is that one can easily test whether a given graph G is a body-hinge graph: the nodes of G with a non-complete neighbor set are the candidates for being the hinge nodes. If G is a body-hinge graph then these nodes are partitioned into classes in such a way that two nodes u, v are in the same class if and only if $uv \in E$ and $N(u) - v = N(v) - u$. This partition, if it exists, can be used to check whether G is indeed a body-hinge graph and if yes, to determine the underlying graph H.

One can ask how to augment a rigid body-hinge framework to a globally rigid one with minimum number of hinges. This question can be asked polynomially by Corollary 5.4.2 and Theorems 1.2.8 and 7.0.1. Moreover, it follows that it is enough to add bars instead of hinges by the last statement of Corollary 5.4.2 and Theorem 7.4.1. We note that the problem of augmenting a rigid body-bar-and-hinge framework to a globally rigid one with minimum numbers of bars/hinges can be solved similarly.

We did not directly refer to Hendrickson's $(d + 1)$-connectivity condition of Theorem 1.2.4 in our characterization for $d \geq 3$ in Theorem 7.0.1. This is because $(d + 1)$-node-connectivity follows for 'free' for body-hinge graphs G_H when $(D - 1)H$ is highly D-tree-connected, provided $d \geq 3$. Another related observation, which follows from Theorems 7.0.1 and 7.3.3, is that if G_H is $2d$-connected then it is globally rigid in $\mathbb{R}^d$. Thus the general conjecture, saying that sufficiently high connectivity implies global rigidity in $\mathbb{R}^d$ for all $d \geq 1$ [15], holds for body-hinge graphs.

Chapter 8

Strongly rigid tensegrity graphs on the line

A **tensegrity graph** $T = (V; C, S)$ is a 2-edge-labeled graph on node set $V = \{v_1, v_2 \ldots, v_n\}$ whose edge set is partitioned into two sets C and S, called **cables**, and **struts**, respectively. The elements of $E = C \cup S$ are the **members** of T. The **underlying graph** of T is the (unlabeled) graph $\overline{T} = (V; E)$. A d-dimensional **tensegrity framework** is a pair (T, p), where T is a tensegrity graph and p is a map from V to $\mathbb{R}^d$. We will also refer to (T, p) as a **realization** of T. If all pairs of adjacent nodes of T are joined by both a cable and a strut then we call T a **bar graph** and a realization (T, p) a **bar framework**.

An **infinitesimal motion** of a tensegrity framework (T, p) is an assignment $q : V \to \mathbb{R}^d$ of infinitesimal velocities to the nodes, such that

$$\begin{aligned} (p_i - p_j)(q_i - q_j) &\le 0 \quad \text{for all } v_iv_j \in C, \\ (p_i - p_j)(q_i - q_j) &\ge 0 \quad \text{for all } v_iv_j \in S, \end{aligned}$$

where $p(v_i) = p_i$ and $q(v_i) = q_i$ for all $1 \le i \le n$. An infinitesimal motion is **trivial** if it can be obtained as the derivative of a rigid congruence of all of $\mathbb{R}^d$ restricted to the nodes of (T, p). The tensegrity framework (T, p) is **infinitesimally rigid** in $\mathbb{R}^d$ if all of its infinitesimal motions are trivial. It is well-known that the infinitesimal rigidity of tensegrity frameworks is not a generic property: the same tensegrity graph may possess infinitesimally rigid as well as infinitesimally non-rigid generic realizations in any fixed dimension $d \ge 1$. Thus we may define two (different) families of tensegrity graphs in dimension d: we say that a tensegrity graph T is **rigid** in $\mathbb{R}^d$ if it has an infinitesimally rigid generic realization (T, p) in $\mathbb{R}^d$ and **strongly rigid** if all generic realizations (T, p) in $\mathbb{R}^d$ are infinitesimally rigid. We refer the reader to [12, 16, 91, 109] for more details on the rigidity of tensegrity frameworks.

It is not known whether the rigidity of tensegrity graphs can be tested in polynomial time for $d \geq 2$. (The solution for $d = 1$ can be found in [89].)

In this chapter, we show that it is co-NP-complete to test whether a tensegrity graph is strongly rigid in $\mathbb{R}^1$. These results are joint with Bill Jackson and Tibor Jordán [56].

8.1 Tensegrities on the line and alternating cycles

Let $G = (V; C, S)$ be a 2-edge-labeled graph which may contain parallel edges with different labels. A cycle in G is **alternating** if no two incident edges along the cycle have the same label. Note that a pair of parallel edges with different labels form an alternating cycle. We say that G has the **alternating cycle property** if for all proper bipartitions $(U, V-U)$ of V there is an alternating cycle in the bipartite subgraph $H = (V, E(U, V-U)$ of G induced by the bipartition. For such a bipartition of G let D be the directed graph obtained from H by orienting the edges in C from U to $V-U$ and the edges in S from $V-U$ to U. It is easy to verify that a cycle of H is alternating if and only if the corresponding oriented cycle in D is a directed cycle.

Theorem 8.1.1. *Let $G = (V; C, S)$ be a tensegrity graph. Then G is strongly rigid in $\mathbb{R}^1$ if and only if G, with the cable-strut labeling, has the alternating cycle property.*

Proof. Suppose first that G has a bipartition $(U, V-U)$ such that the graph $H = (V, E(U, V-U))$ contains no alternating cycle. Then the digraph obtained from H by orienting the edges in C from U to $V-U$ and the edges in S from $V-U$ to U, is acyclic and hence it has a topological ordering of the nodes, that is, an ordering $v_1, v_2, ..., v_n$ for which $i < j$ whenever v_iv_j is an arc. Thus, for each edge $e = v_iv_j$ of H, we have: $v_i \in U$ and $v_j \in V-U$ if $e \in C$; $v_i \in V-U$ and $v_j \in U$ if $e \in S$. Let (G,p) be a realization of G on the line that keeps this order, that is, $p(v_i) < p(v_j)$ if $i < j$, for all $1 \leq i < j \leq n$. Now, let $q : V \to \mathbb{R}$ be an infinitesimal motion with $q(v) = 1$ if $v \in U$ and $q(v) = 0$ otherwise. Then q is a non-trivial infinitesimal motion of (G,p). Hence (G,p) is not infinitesimally rigid. Therefore, G is not strongly rigid as p may be chosen to be generic.

Next suppose that G has the alternating cycle property. Let (G,p) be a 1-dimensional (generic) realization of G as a tensegrity framework and suppose that it has a non-trivial (that is, non-constant) infinitesimal motion q. Let $U = \{v \in V : q(v) > 0\}$. We may assume that $\emptyset \neq U \neq V$ as $q + a$ is also an infinitesimal motion of (G,p) for any real a. Let $X = u_1v_1u_2v_2...u_tv_tu_1$ be an alternating cycle in G with $u_i \in U$ and $v_i \in V-U$ for $1 \leq i \leq t$, and $u_1v_1 \in C$. The facts that X is alternating (with $u_iv_i \in C$ and $v_iu_{i+1} \in S$), $q(u_i) > 0$ and $q(v_i) \leq 0$ for all $1 \leq i \leq t$ imply that $p(u_i) < p(v_i) < p(u_{i+1})$ for all $1 \leq i \leq t$ where $u_{t+1} := u_1$. Thus $p(u_1) < p(v_1) < p(u_2) < ... < p(v_t) < p(u_1)$ is a strictly increasing sequence, a contradiction. ∎

8.2 The hardness proof

In this section we prove the hardness result. First we show that the following problem is co-NP-complete.

ACP. Given a 2-edge-labeled graph $G = (V; C, S)$, decide whether G has the alternating cycle property.

We shall reduce the following satisfiability problem to ACP. It is well-known that 3-SAT is NP-complete, see for example [45].

3-SAT. Given a Boolean formula in conjunctive normal form, in which each clause contains exactly three literals, decide whether there is a truth assignment for the variables which makes the formula true.

Theorem 8.2.1. *ACP is co-NP-complete.*

Proof. It is easy to see that ACP is in co-NP: given a bipartition $(U, V - U)$ of V, one can verify that $H = (V, E(U, V - U))$ contains no alternating cycles by showing that the directed graph D (defined in the proof of Theorem 8.1.1) is acyclic. It will be convenient to call a bipartition $(U, V - U)$ with no alternating cycles **pure**.

Consider a formula φ, an instance of 3-SAT. Suppose that φ contains c clauses and n variables and let c_i and $\bar{c}_i$ denote the number of occurrences of variable x_i and $\bar{x}_i$ in φ, respectively, for $1 \leq i \leq n$. Construct a 2-edge-labeled graph $G = (V; C, S)$ as follows. Take $2n + 2$ node-disjoint paths $T, F, P_1, ..., P_n, \bar{P}_1, ..., \bar{P}_n$ such that the number of nodes on T and F are $2n + 1 + 3c$ and $2n + 1$, respectively, while P_i ($\bar{P}_i$) has $3 + c_i$ (resp. $3 + \bar{c}_i$) nodes for $1 \leq i \leq n$. The paths $P_i, \bar{P}_i$ correspond to the variables and the long paths T, F correspond to the true and false assignments, as we shall see later. All nodes of G lie on these paths and we put an edge in C and an edge in S for each edge of the paths. This ensures that none of these paths can cross a pure bipartition of G.

Next we describe the additional edges of G that connect these paths. We connect the first two nodes of each path $P_i, \bar{P}_i$, $1 \leq i \leq n$, to the first node of path T and F, respectively, and label them so that edges $\{v_{i,1}t_1, \bar{v}_{i,1}t_1, v_{i,2}f_1, \bar{v}_{i,2}f_1\} \subseteq C$ and edges $\{v_{i,2}t_1, \bar{v}_{i,2}t_1, v_{i,1}f_1, \bar{v}_{i,1}f_1\} \subseteq S$. These edges ensure that T and F belong to different sides of a pure bipartition of G. We then connect the third node of each path $P_i, \bar{P}_i$, $1 \leq i \leq n$, to the nodes t_{2i}, t_{2i+1} and f_{2i}, f_{2i+1} of the path T and F, respectively, and label them so that $\{v_{i,3}t_{2i}, v_{i,3}f_{2i}, \bar{v}_{i,3}t_{2i+1}, \bar{v}_{i,3}f_{2i+1}\} \subset C$ and $\{v_{i,3}t_{2i+1}, v_{i,3}f_{2i+1}, \bar{v}_{i,3}t_{2i}, \bar{v}_{i,3}f_{2i}\} \subset S$ hold. These edges ensure that in a pure bipartition of G exactly one of the paths P_i and $\bar{P}_i$ will belong to the same side as path T, for all $1 \leq i \leq n$.

Finally, we add an alternating cycle of length six corresponding to each clause of φ. For example, for the clause $(x_2 \vee x_5 \vee \bar{x}_8)$ we add a cycle as follows: we take the next three unused nodes on T, say a, b, c, and the next unused node on each of the paths $P_2, P_5, \bar{P}_8$, call them x, y, z, respectively. We add the the edges ax, by, cz, label them C, and add az, bx, cy, and label them S. This ensures that in a pure bipartition at least one of $P_2, P_5, \bar{P}_8$ must belong to the same side as path T.

The construction of G and the arguments above imply that there is a truth assignment for the variables which makes formula φ true if and only if G has a pure bipartition, that is, G does not have the alternating cycle property. Therefore ACP is co-NP-hard. □

It follows from Theorems 8.1.1 and 8.2.1 that it is also hard to test whether a tensegrity graph is strongly rigid on the line. The decision problem is as follows.

STRONGLY RIGID TENSEGRITY. Given a tensegrity graph $G = (V; C, S)$, decide whether G is strongly rigid in $\mathbb{R}^1$.

Theorem 8.2.2. *STRONGLY RIGID TENSEGRITY is co-NP-complete.* □

8.3 Concluding remarks

The proof of Theorem 8.1.1 shows that if we replace 'generic' by 'general position' in the definition of strong rigidity in $\mathbb{R}^1$, then the family of strongly regular graphs will remain the same. This is not true when $d \geq 2$, even for bar graphs.

Let $G = (V; C, S)$ be a 2-edge-labeled graph and let B be a set of edges (bars) corresponding to the alternating 2-cycles of G. Theorem 8.1.1 implies that if G is strongly rigid in $\mathbb{R}^1$ then each node of G is incident with an edge of B. Let c be the number of components of the graph (V, B). Theorem 8.1.1 gives a polynomial algorithm for checking if G is strongly rigid when c is fixed since we need only check 2^c bipartite graphs for alternating cycles (which can be done by checking if the associated digraph is acyclic).

We can use Theorem 8.1.1 to construct graphs which are not strongly rigid in $\mathbb{R}^1$ and for which the connectivity of both (V, C) and (V, S) is arbitrarily large. For example let (V, B) consist of two large complete graphs and join them by $2k$ independent edges, half in C and half in S.

A natural open question is whether our hardness result concerning strong rigidity extends to higher dimensions $d \geq 2$.

Another question is whether the corresponding labeling problem is hard: given a graph G, decide whether the edges of G can be 2-edge-labeled so that the resulting tensegrity graph is strongly rigid in $\mathbb{R}^1$ (and find a good labeling, if it exists). If we replace strongly rigid by rigid, we obtain a labeling problem which is polynomially solvable in $\mathbb{R}^d$ for $d = 1, 2$, see [62].

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This thesis investigates several topics from combinatorial optimization and combinatorial rigidity theory. These topics are the following:

- A tree-composition is a tree-like family that serves to describe the obstacles to k-edge-connected orientability of mixed graphs. *We derive a structural result on tree-compositions* that gives rise to a simple algorithm for computing an obstacle when the orientation does not exist. *We also investigate the application of this result to the orientability of graphs* with several other connectivity prescriptions. With a different method *we give a simpler algorithm to find a rooted $(k,1)$-edge-connected orientation* of a graph.

- *We give a generalization of a result of Kamiyama, Katoh and Takizawa [64] on arborescence packings* where a matroid is given on the roots of the arborescences as in the paper of Durand de Gevigney, Nguyen and Szigeti [18].

- *We prove that there exist circuits of the 2-dimensional rigidity matroid with maximum degree 4 which do not contain any Hamiltonian path nor a path longer than $|V|^\lambda$ for $\lambda > \frac{\log 8}{\log 9} \simeq 0,9464$.* Moreover, we prove a similar statement for every $(k,k+1)$-sparsity matroid.

- *We extend the rigidity augmentation algorithm of García and Tejel [44] for every (k,ℓ)-sparsity matroid where $\ell \leq \frac{3}{2}k$.* That is, we give a polynomial algorithm to the following augmentation problem for $\ell \leq \frac{3}{2}k$. Given a (k,ℓ)-tight graph $G=(V,E)$, find a graph $H=(V,F)$ with minimum number of edges, such that $G+H-e$ has (k,ℓ)-tightspanning subgraph for every $e \in E$. We also give a polynomial algorithm for the case where $G=(V,E)$ is not (k,ℓ)-tight but has a spanning (k,ℓ)-tight subgraph when $\ell \leq k$ and show that this problem is NP-hard when $\ell = \frac{3}{2}k$.

- *We investigate the edge number of minimal highly node-redundantly rigid and globally rigid graphs.* We give several lower and upper bounds along with some tight examples.

- *We characterize the global rigidity of generic body-hinge and body-bar-and-hinge frameworks.* A surprising point in this result is that it disproves several well-known conjectures on global rigidity in $d \geq 3$ by giving infinitely many counterexamples for Hendrickson's conjecture in every dimension $d \geq 3$.

- Tensegrity frameworks are defined on a set of points in $\mathbb{R}^d$ and consist of bars, cables, and struts, which provide upper and/or lower bounds for the distance between their endpoints. *We prove that it is NP-hard to determine whether every generic realization of the (edge-labeled) graph of a tensegrity framework is rigid on the line.*

The results are based on [32, 56, 61, 65, 72, 73, 74, 75].

A dolgozat több, a kombinatorikus optimalizálás és a kombinatorikus merevségelmélet területéről érkező témával foglalkozik. Ezek az alábbiak:

- Egy fa-kompozíció egy fa-szerű halmazcsalád, mely segítségével leírható például a vegyes gráfok k-élösszefüggővé irányíhatóságára vonatkozó bizonyíték. *Fa-kompozíciók struktúrájáról bizonyítunk egy tétel,* mely segítségével egyszerű algoritmus adható annak a bizonyítékának megtalálására, hogy nem létezik ilyen irányítás. Az eredmény egyéb különböző összefüggőségi feltételeket teljesítő *gráf irányítási problémákra történő alkalmazhatóságát is vizsgáljuk.* Egy másik módszerrel egy *egyszerűbb algoritmust adunk gráfok gyökeres* $(k, 1)$*-élösszefüggő irányításának megtalálására.*

- *Általánosítjuk Kamiyama, Katoh és Takizawa [64] fenyőpakolási tételét* arra az esetre, amikor egy matroid adott a fenyők gyökerein, mint Durand de Gevigney, Nguyen és Szigeti [18] cikkében.

- *Belátjuk, hogy a kétdimenziós merevségi matroidban vannak olyan körök, melyekben a maximális fokszám 4, de nincs benne Hamilton-út, sőt* $|V|^\lambda$*-nál hosszabb út sem* $\lambda > \frac{\log 8}{\log 9} \simeq 0,9464$*-re.* Hasonló állítást bizonyítunk minden $(k, k+1)$-ritkasági matroidra.

- *Kiterjesztjük García és Tejel [44] merevség növelő algoritmusát minden olyan* (k, ℓ)*-ritkasági matroidra, ahol* $\ell \leq \frac{3}{2}k$*.* Azaz polinomiális algoritmust adunk a következő növelési problémára $\ell \leq k$ esetén: Adott (k, ℓ)-kritikus $G = (V, E)$ gráfhoz keressünk minimális élszámú $H = (V, F)$ gráfot, hogy tetszőleges $e \in E$-re $G + H - e$ tartalmazzon (k, ℓ)-kritikus feszítő részgráfot. Arra a bonyolultabb problémára is polinomiális algoritmust adunk $\ell \leq k$-ra, amikor G nem (k, ℓ)-kritikus, de tartalmaz ilyen feszítő részgráfot; valamint megmutatjuk, hogy ez a probléma $\ell = \frac{3}{2}k$ esetén már NP-nehéz.

- *Minimális többszörösen csúcs-redundánsan merev és globálisan merev gráfok élszámát vizsgáljuk.* Több alsó és felső korlátot adunk, néhány éles példával.

- *Karakterizáljuk a generikus test-zsanér és test-rúd-és-zsanér szerkezetek globális merevségét.* Az eredmény meglepő mellékterméke, hogy több korábbi jól-ismert globális merevségre vonatkozó sejtést is megcáfol azzal, hogy végtelen sok ellenpéldát szolgáltat a Hendrickson-sejtésre minden $d \geq 3$ dimenzióban.

- A rúd-csukló szerkezeteknél általánosabb tensegrity szerkezetekben a csuklók közt kötelek és dúcok is lehetnek, melyek csak felső illetve alsó korlátot adnak végpontjaik távolságára. *Belátjuk, hogy NP-nehéz eldönteni, hogy egy tensegrity szerkezethez tartozó (él-címkézett) gráf minden generikus realizációja merev-e az egyenesen.*

A dolgozat a [32, 56, 61, 65, 72, 73, 74, 75] cikkeken alapul.