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Arborescence packing and restricted b-matchings

Ph.D. thesis

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Notation

Undirected graphs

$G = (V, E)$	An undirected graph G on node set V with edge set E.
$H = (V(H), E(H))$	A subgraph H of G with node set $V(H)$ and edge set $E(H)$.
$G = (S, T; E)$	A bipartite graph with colour classes S and T and edge set E.
$G[X]$	The subgraph of G induced by $X \subseteq V$.
$G - X$	$G[V \setminus X]$ for $X \subseteq V$ and $G' = (V, E \setminus X)$ for $X \subseteq E$.
$E[X]$	The set of edges induced by $X \subseteq V$.
$E[X, Y]$	The set of edges between $X - Y$ and $Y - X$.
$\delta_G(X)$	The set of edges having exactly one end in $X \subseteq V$.
$\dot{\delta}_G(v)$	Family of edges incident to $v \in V$ in which loops are included twice.
$\ell(v)$	The set of loops at $v \in V$.
$\ell(X)$	The set of loops induced by $X \subseteq V$.
$d_G(v)$	$= \lvert\dot{\delta}_G(v)\rvert = \lvert\delta_G(v)\rvert + 2\lvert\ell(v)\rvert$ for $v \in V$.
$d_G(X)$	$= \lvert\delta_G(X)\rvert$ for $X \subseteq V, \lvert X\rvert \geq 2$.
$d_G(X, Y)$	$= \lvert E[X, Y]\rvert$.
$\overline{d_G}(X, Y)$	The number of edges between $X \cap Y$ and $V - (X \cup Y)$.
$i_G(X)$	The number of edges with both endnodes in X.
$I_G(X)$	The set of edges with both endnodes in X.
$e_G(X)$	The number of edges with at least one endnode in X.
$\bar{G}$	The complement of G.
K_n	Complete graph on n nodes.
$K_{s,t}$	Complete graph with colour classes having sizes s and t, respectively.
$h_F(X)$	$= \sum_{v \in X} d_F(v)$.
$\Gamma_G(X)$	The set of nodes in $V - X$ adjacent to X.
(G, w)	A graph G with weight function $w : E \to \mathbb{R}$.

Directed graphs

$D = (V, A)$	A directed graph (shortly, digraph) on node set V with edge set A.
$t(a), h(a)$	The tail and head of arc a, respectively.
$\varrho_D(X)$	The number of edges entering $X \subseteq V$.
$\Delta_D^{in}(X)$	The set of edges entering $X \subseteq C$.
$\delta_D(X)$	The number of edges leaving $X \subseteq V$.
$\Delta_D^{out}(X)$	The set of edges leaving $X \subseteq V$.
$\delta_D(X, Y)$	The number of directed edges from $X - Y$ to $Y - X$.
$d_D(X, Y)$	$= \delta_D(X, Y) + \delta_D(Y, X)$.
$\lambda_D(u, v)$	The maximum number of edge-disjoint directed paths from u to v.
$\kappa_D(r, v)$	The maximum number of internally node-disjoint directed paths from u to v.

$\Gamma^-(X)$	The entrance of X, that is, $\{v \in X : \exists uv \in A,\ u \in V - X\}$.

Matroids

$\mathcal{M} = (S, r_{\mathcal{M}})$	A matroid on ground set S with rank function $r_{\mathcal{M}}$.
$\texttt{cl}(Z)$	The closure of $Z \subseteq S$.

Bi-sets

$X = (X_O, X_I)$	A bi-set $X_I \subseteq X_O \subseteq V$ with outer member X_O and inner member X_I.
$\mathcal{P}_2(V) = \mathcal{P}_2$	The set of all bi-sets on ground-set V.
$X \cap Y$	$= (X_O \cap Y_O, X_I \cap Y_I)$ for $X, Y \in \mathcal{P}_2$.
$X \cup Y$	$= (X_O \cup Y_O, X_I \cup Y_I)$ for $X, Y \in \mathcal{P}_2$.
$X \subseteq Y$	This means $X_O \subseteq Y_O, X_I \subseteq Y_I$.
$\varrho_D(X)$	The number of edges entering bi-set X.
$\Delta_D^{in}(X)$	The set of edges entering bi-set X.
$\delta_D(X)$	The number of edges leaving bi-set X.
$\Delta_D^{out}(X)$	The set of edges leaving bi-set X.

Restricted b-matchings

V_K	The node set of subgraph K.
E_K	The edge set of subgraph K.
$V_{\mathfrak{K}}$	The set of nodes contained by subgraphs in $\mathfrak{K}$.
$E_{\mathfrak{K}}$	The set of edges contained by subgraphs in $\mathfrak{K}$.
e^u, e^v	End nodes of edge $e \in E$.
e_{ij}^T	Edge of triangle T between i and j (resp. t_i and t_j) if $V_T = \{u, v, w\}$ (resp. $V_T = \{t_1, t_2, t_3\}$).
$\mathcal{T}_K^1$	The set of triangles in $\mathcal{T}$ 1-fitting K.
$\mathcal{T}_K^2$	The set of triangles in $\mathcal{T}$ 2-fitting K.
$\mathcal{T}_K$	$= \mathcal{T}_K^1 \cup \mathcal{T}_K^2$.
$\texttt{def}(K, F, \mathfrak{T})$	$= \lfloor \frac{1}{2}(b(K) + \lvert F\rvert + 3\lvert\mathfrak{T}\rvert) \rfloor - \big(x(E[K]) + x(F) + \sum_{T \in \mathfrak{T}} x(E_T)\big)$.
F_u	Set of non self-loop edges in F incident to u.

Miscellaneous

$\mathbb{Z}_+, \mathbb{R}_+$	The sets of non-negative integers and reals.
$X - v$	$= X \setminus \{v\}$ for a set X and single element v.
$X + v$	$= X \cup \{v\}$ for a set X and single element v.
$b(U)$	$= \sum_{v \in U} b(v)$ for a function $b : V \to \mathbb{R}$ and $U \subseteq V$.
$x \prec y$	$x \preceq y$ and $x \neq y$ for a partial order $\preceq$.

Instead of 'G' and 'D' we sometimes use the above notations with subscripts denoting a subset of edges. In such a case the quantity in question has to be computed by considering only the subset showed by the subscript.

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Chapter 1

Introduction

Two families of problems are considered in the thesis the first of which is arborescence packing. An arborescence is a directed tree with a root in which the edges are directed ‘away’ from the root node (sometimes this is called an out-arborescence in the literature; in an in-arborescence the edges are directed ‘toward’ the root node). The packing problem consists of finding disjoint copies of arborescences satisfying certain conditions. The motivation of these problems comes from real-life applications such as survivable network or evacuation plan design. A cornerstone in graph theory is Edmonds’ theorem characterizing the existence of k edge-disjoint spanning arborescences rooted at the same root node in a directed graph [34]. In fact, Edmonds proved a stronger version of his result in which branchings are considered instead of arborescences. This result implied great many extensions, but the condition requiring the branchings to be spanning was not weakened for almost three decades. The reason for that is that even a slight modification of the spanning constraint may result in difficult problems, as was shown in [10].

In 2008, Kamiyama, Katoh and Takizawa gave a surprising extension of Edmonds’ theorem in which arborescences spanning only nodes that are reachable from the given root nodes are considered [82]. In [6], we showed that the abstract theorem of Szegő on covering intersecting families can be extended to bi-set systems and proved that the theorem of Kamiyama et al. is a special case of our result.

Another approach to extend Edmonds result is due to Colussi, Conforti and Zambelli who introduced the notion of strongly edge-disjoint arborescences [18]. They conjectured the existence of k spanning arborescences under more strict restrictions than that of Edmonds’ theorem. For the very special case when two arborescences are needed the conjecture has been verified. We extended the notion of strongly edge-disjointness in [13] and showed that the conjecture is also true for two dicycle-disjoint arborescences, while gave a disproof of the conjecture in general.

In some applications not only out-arborescences but also in-arborescences are needed. Unfortunately, even the problem of finding an in- and an out-arborescence with the same root node that are disjoint is NP-complete. However, for acyclic digraphs the problem becomes tractable as in this special case both the set of in- and out-arborescences form a matroid on the edges. In [11], we gave a linear time algorithm for finding a pair of disjoint in- and out-arborescences in an acyclic digraph. Chapter 2 gives an overview of the above mentioned results.

Chapter 3 reveals the connection between the problem of packing arborescences and covering intersecting bi-set families. The introduction of bi-sets made it possible to give a simpler proof for the theorem of Kamiyama et al. and the very special bi-set families appearing in the proof turned out to be really useful. We extended Schrijver’s strongly polynomial time algorithm [114] for packing branchings under capacity restrictions [10]. The usage of bi-sets here is essential; the running time could not be bounded without the deep understanding of the structure of bi-set families in question. We also gave a polyhedral description of arborescence-packable digraphs based on bi-sets.

The second part of the thesis deals with algorithmic and polyhedral aspects of restricted b-matchings. The motivation of the problem comes from node-connectivity augmentation. It is an easy observation, that the problem of increasing the node-connectivity of an undirected graph on n nodes from $n-4$ to $n-3$ is equivalent to finding a maximum 2-matching in the complement of the graph not containing a cycle of length 4. This latter problem is called the square-free 2-matching problem, and was the starting point of our investigations as discussed in Chapter 4.

Much is known about square-free 2-matchings, although the mentioned problem in general is still unsolved. For a list $\mathcal{K}$ of forbidden subgraphs, a $\mathcal{K}$-free b-matching is a b-matching containing no member if $\mathcal{K}$. Here $\mathcal{K}$ may contain concrete subgraphs of a digraph D by defining their node and edge sets, or may be given by describing a class of graphs in general. As the most important special cases, the C_k-free or $C_{\leq k}$-free 2-matching problems ask for a 2-matching with maximum size not containing cycles of length k or at most k, respectively. Clearly, these problems can be considered as relaxations of the Hamiltonian cycle problem and so are well investigated. Unfortunately, we can not go to far with the values of k: the problems are NP-hard when $k \geq 5$ as was shown by Papadimitriou (see eg. [22]). From the positive side of results, Hartvigsen [59] gave an augmenting path algorithm for the case $k = 3$. Hence only the C_4-free and $C_{\leq 4}$-free 2-matching problems are left open.

The weighted versions of these problems can be defined in a straightforward manner. However, there is a firm difference in complexity between the unweighted and the weighted versions: the weighted square-free 2-matching problem is NP-hard even in bipartite graphs and $0-1$ weights [87]. This difference will be important when we would like to give a polyhedral description of the corresponding polytopes.

The problems becomes significantly easier if the graph is subcubic, that is, each node has degree at most three. Note that this is the case in the node-connectivity augmentation problem if an $(n-4)$-connected graph is given and one would like to increase its node-connectivity to $n-3$. In [12], we gave a polynomial time algorithm for the square-free 2-matching problem in subcubic graphs and for the case of node-induced weight functions as well. It is worth mentioning that the problem of increasing the node-connectivity of a graph by one was solved in general by Végh [129]. Algorithms for the weighted C_3-free 2-matching (also called triangle-free 2-matching) problem in subcubic graphs were given by Hartvigsen and Li [62], and Kobayashi [88]. However, the problem for $k = 3$ in general graphs with arbitrary weights is still open.

As a triangle and a square can be considered as a K_3 and a $K_{2,2}$, respectively, the $C_{\leq 4}$-free 2-matching problem admits a natural generalization. The $K_{t,t}$- and K_{t+1}-free t-matching problem asks for a subgraph with maximum size not containing a $K_{t,t}$ or a K_{t+1} as a subgraph. The problem was first considered in bipartite graphs [41,103]. In [14], we extended the algorithm of [12] to $K_{t,t}$- and K_{t+1}-free t-matchings in degree bounded graphs. The degree bound is essential here, the problem is still open for general graphs.

The polyhedral descriptions of the corresponding polytopes are also of interest, forming the topic of Chapter 6. By the NP-hardness result of Király [86], we may not expect a 'nice' description for the $C_{\leq k}$-free or C_k-free 2-matching polytopes for $k \geq 4$, where 'nice' means that we can separate the inequalities appearing in the description. Hartvigsen and Li gave a polyhedral description of the triangle-free 2-factor polytope for subcubic simple graphs in [62]. They also showed that, somewhat surprisingly, triangle-free 2-matchings in subcubic graphs admit a more complicated description. This is a strange phenomenon as results on b-matchings and b-factors are typically can be derived from each other. They also proposed a description of the triangle-free 2-matching polytope and gave a sketch of the proof, which was finally published in [63]. The proof is quite difficult and complicated, but provides an algorithm for finding a maximum triangle-free 2-matching in a subcubic graph. In [7], based on the description proposed in [62], we gave another proof of this result. Our motivation was to find a simpler, clearer proof, but to be honest it finally grew into something rather complicated.

Considering the above, a natural question arises: what can we say about the maximum size or polyhedral description of a triangle-free subgraph, that is, if the upper bound b on the nodes is left out. Yannakakis showed [136] that the problem in general is NP-complete, hence we may not expect a nice polyhedral description again. Conforti et al. proved that the problem remains NP-complete even in chordal graphs, but given a fixed upper bound on the maximum size of a clique in the graph the problem becomes polynomially solvable [19, 20].

Determining the maximum size of a triangle-free subgraph is equivalent to determine the minimum size of an edge-set covering each triangle at least once. In 1981, Tuza proposed the following conjecture [127]: Given a simple undirected graph G, let $\nu(G)$ denote the maximum number of pairwise edge-disjoint triangles, while $\tau(G)$ denote the minimum number of edges covering each triangles in G. Then $\tau(G) \leq 2\nu(G)$. It is easy to see that the inequality holds with 3 instead of 2. The conjecture has been verified for various classes of graphs, but is still unsolved in general. The first non-trivial bound was given by Haxell [64], who proved that the inequality is true with factor $(3 - \frac{3}{23})$.

The problem can be generalized in two sense: weights on the edges might be given, and -looking at a triangle as a clique again- a clique version of the conjecture can be formalized. In [8], we proposed an extension of Tuza's conjecture combining these ideas, and proved a fractional weakening of the conjecture which can be considered as a generalization of Krivelevich's result. Our approach uses the notion of Turán numbers, and basically builds on the so-called splitting property of maximal antichains.

The rest of the thesis is organized as follows. In the remaining part of this chapter, in Sections 1.1-1.5, we give a short overview of the definitions and results that form the background of our work. Chapters 2 and 3 can be considered as a continuation of the work started in [6]; we present here the results of [10, 11, 13] on packing arborescences, and show its connection to covering intersecting bi-set families. Chapter 4 introduces the second main topic of the thesis and presents the algorithm and the min-max result of [12] for the square-free 2-matching problem in subcubic graphs. This result is then further generalized to $K_{t,t}$- and K_{t+1}-free t-matchings in degree bounded graphs in Chapter 5, which contains the results of [14]. Chapter 6 presents the most technical part of the thesis based on [7]. Through the example of b-factors we introduce a new shrinking operation which is then extended to give a complete description of the triangle-free 2-matching polytope of subcubic graphs. This part of the thesis contains many technical computations; the most of them is left to the end of the chapter. Finally, Chapter 7 contains the result of [8]. It introduces the notion of shadow systems and verifies that a special class of maximal antichains has the splitting property. This result is then used to give an upper bound on a weighted version of the Turán number and to prove a fractional weakening of a weighted extension of Tuza's conjecture to clique packing.

1.1 Packing arborescences

Let $D = (V, A)$ be a directed graph with designated root-node r. An **arborescence** is a directed tree in which every node is reachable from a given root node. We sometimes identify an arborescence (U, F) with its edge-set F and will say that the arborescence F spans U. An arborescence F with root node r is called an **r-arborescence**. We call D **rooted k-edge-connected** if for each $v \in V$, there exist k edge-disjoint directed paths from r to v. By Menger's theorem, this is equivalent to $\varrho(X) \geq k$ whenever $\emptyset \subset X \subseteq V - r$. A fundamental theorem on packing arborescences is due to Edmonds who gave a characterization of the existence of k edge-disjoint spanning arborescences rooted at the same node [34].

Theorem 1.1.1 (Edmonds' theorem, weak form)**.** *Let $D = (V, A)$ be a digraph with root r. D has k edge-disjoint spanning r-arborescences if and only if D is rooted k-edge-connected.*

This result inspired great many extensions in the last three decades. Edmonds actually proved his theorem in a stronger form where the goal was packing k edge-disjoint branchings of given root-sets. A **branching** is a directed forest in which the in-degree of each node is at most one. The set of nodes of in-degree 0 is called the **root-set** of the branching. Note that a branching with root-set R is the union of $|R|$ node-disjoint arborescences (where an arborescence may consist of a single node and no edge but we always assume that an arborescence has at least one node). For a digraph $D = (V, A)$ and root-set $\emptyset \subset R \subseteq V$ a branching (V, B) is called a **spanning R-branching** of D if its root-set is R. In particular, if R is a singleton consisting of an element r, then a spanning branching is a spanning r-arborescence.

Theorem 1.1.2 (Edmonds' theorem, strong form I.)**.** *In a digraph $D = (V, A)$, let $\mathcal{R} = \{R_1, \dots, R_k\}$ be a family of k non-empty (not necessarily disjoint or distinct) subsets of V. There are k edge-disjoint spanning branchings of D with root-sets $R_1, \dots, R_k$, respectively, if and only if*

$$\varrho_D(X) \geq p(X) \text{ for all } \emptyset \subset X \subseteq V \tag{1.1}$$

where $p(X)$ denotes the number of root-sets R_i disjoint from X.

Observe that in the special case of Theorem 1.1.2 when each root-set R_i is a singleton consisting of the same node r, we are back at Theorem 1.1.1. Conversely, when the R_i's are singletons (which may or may not be distinct), then Theorem 1.1.2 easily follows from Theorem 1.1.1. However, for general R_i's no reduction is known.

Theorem 1.1.2 can be reformulated as follows.

Theorem 1.1.3 (Edmonds' theorem, strong form II.)**.** *Let $D = (V, A)$ be a digraph whose node set is partitioned into a root-set $R = \{r_1, \dots, r_k\}$ (of distinct roots) and a terminal set T. Suppose that no edge of D enters any node of R. There are k disjoint arborescences $F_1, \dots, F_k$ in D so that F_i is rooted at r_i and spans $T + r_i$ for each $i = 1, \dots, k$ if and only if $\varrho_D(X) \geq |R - X|$ for every subset $X \subseteq V$ for which $X \cap T \neq \emptyset$.*

Indeed, this follows easily by applying Theorem 1.1.2 to the subgraph D' of D induced by T with choice $R_i = \{v : \text{ there is an edge } r_i v \in A\}$ $(i = 1, \dots, k)$. The same construction shows the reverse implication, too.

The following proper extension of Theorem 1.1.3 was derived in [9] with the help of a theorem of Frank and Tardos [46] on covering supermodular functions by digraphs.

Theorem 1.1.4 (Frank and Tardos). *Let $D = (V, A)$ be a digraph whose node set is partitioned into a root-set $R = \{r_1, \dots, r_q\}$ and a terminal set T. Suppose that no edge of D enters any node of R. Let $m : R \to \mathbb{Z}_+$ be a function and let $k = m(R)$. There are k disjoint arborescences in D so that $m(r)$ of them are rooted at r and spanning $T + r$ for each $r \in R$ if and only if*

$$\varrho_D(X) \geq m(R - X) \text{ for every subset } X \subseteq V \text{ for which } X \cap T \neq \emptyset. \tag{1.2}$$

One way to extend Edmonds' theorems is to decrease the size of the node sets spanned by the arborescences in question. However, it is not easy to find such a generalization as one can easily run into difficult questions. In Section 2.1, we show that a variant of Theorem 1.1.4 and even an apparently slight weakening of the reachability conditions result in NP-complete problems (Theorems 2.1.6 and 2.1.7).

In 2009, Kamiyama, Katoh and Takizawa [82] were able to find a surprising new proper extension of Edmonds' strong theorem which implies Theorem 1.1.4 as well.

Theorem 1.1.5 (Kamiyama, Katoh and Takizawa). *Let $D = (V, A)$ be a digraph and $R = \{r_1, \dots, r_k\} \subseteq V$ a list of k (possibly not distinct) root-nodes. Let S_i denote the set of nodes reachable from r_i. There are edge-disjoint r_i-arborescences F_i spanning S_i for $i = 1, \dots, k$ if and only if*

$$\varrho_D(Z) \geq p_1(Z) \text{ for every subset } Z \subseteq V \tag{1.3}$$

where $p_1(Z)$ denotes the number of sets S_i for which $S_i \cap Z \neq \emptyset$ and $r_i \notin Z$.

The original proof of Theorem 1.1.5 is more complicated than that of Theorem 1.1.2 due to the fact that the corresponding set function p_1 in the theorem is no more supermodular. Based on Theorem 1.1.5, Fujishige [48] found a further extension. For two disjoint subsets X and Y of V of a digraph $D = (V, A)$, we say that Y is **reachable** from X if there is a directed path in D whose first node is in X and last node is in Y. We call a subset U of nodes **convex** if there is no node v in $V \setminus U$ so that U is reachable from v and v is reachable from U.

Theorem 1.1.6 (Fujishige). *Let $D = (V, A)$ be a directed graph and let $R = \{r_1, \dots, r_k\} \subseteq V$ be a list of k (possibly not distinct) root-nodes. Let $U_i \subseteq V$ be convex sets with $r_i \in U_i$. There are edge-disjoint r_i-arborescences F_i spanning U_i for $i = 1, \dots, k$ if and only if*

$$\varrho_D(Z) \geq p_1(Z) \text{ for every subset } Z \subseteq V \tag{1.4}$$

where $p_1(Z)$ denotes the number of sets U_i's for which $U_i \cap Z \neq \emptyset$ and $r_i \notin Z$.

Note that the set of nodes reachable from an r_i form a convex set, hence Theorem 1.1.5 immediately follows from Theorem 1.1.6. It has been showed recently in [84] that these results are in fact equivalent.

In [32], Edmonds' theorems was extended in another direction. Let $D = (V, A)$ be a digraph, $\mathcal{M} = (S, r_{\mathcal{M}})$ a matroid on ground set S with rank function $r_{\mathcal{M}}$ and $\pi : S \to V$ a (not necessarily injective) map. For $Z \subseteq S$ the **closure** of Z is denoted by $\mathtt{cl}(Z)$, that is, $\mathtt{cl}(Z) = \{s \in S : r_{\mathcal{M}}(Z + s) = r_{\mathcal{M}}(Z)\}$. A triple (D, S, π) is called a **digraph with roots**. The map π is called **$\mathcal{M}$-independent** if $\pi^{-1}(v)$ is independent in $\mathcal{M}$ for each $v \in V$. For $X \subseteq V$, S_X denotes $\pi^{-1}(X)$.

A digraph with roots (D, S, π) is called $\mathcal{M}$**-connected**, if

$$\varrho(X) \geq r_{\mathcal{M}}(S) - r_{\mathcal{M}}(S_X) \tag{1.5}$$

holds for each $\emptyset \neq X \subseteq V$.

An $\mathcal{M}$**-basic packing of arborescences** in (D, S, π) is a set $\{F_1, \ldots, F_{|S|}\}$ of pairwise edge-disjoint (not necessarily spanning) arborescences in D such that F_i has root at $\pi(s_i)$ for $i = 1, \ldots, |S|$ and the set $\{s_j \in S : v \in V(F_j)\}$ forms a base of $\mathcal{M}$ for each $v \in V$. The result of [32] is the following.

Theorem 1.1.7 (Gevigney, Nguyen and Szigeti)**.** *Let (D, S, π) be a digraph with roots and $\mathcal{M}$ be a matroid on S. There exists an $\mathcal{M}$-basic packing of arborescences in (D, S, π) if and only if π is $\mathcal{M}$-independent and (D, S, π) is $\mathcal{M}$-connected.*

Theorem 1.1.2 can be easily derived from Theorem 1.1.7. Indeed, let $\mathcal{R} = \{R_1, \ldots, R_k\}$ be a family of k non-empty (not necessarily disjoint or distinct) subsets of V. Define $S = \bigcup_{R \in \mathcal{R}} R$ to be a multiset in which each $v \in V$ is included as many times as the number of R_i's containing v, and let $\pi(v) = v$. If we take the partition matroid $\mathcal{M}$ on S in which a set $Z \subseteq S$ is independent if and only if $|Z \cap R_i| \leq 1$ for $1 \leq i \leq k$, then an $\mathcal{M}$-basic packing of arborescences corresponds to a collection of edge-disjoint spanning R_i-arborescences and vice versa. Note that π is clearly $\mathcal{M}$-independent and (1.1) is equivalent to (1.5), hence Edmonds' result follows from that of Szigeti et al.

It is a natural question that whether there is a common generalization of Theorems 1.1.5 and 1.1.7. In [84], Cs. Király gave a common extension of these theorems. Using the notation of [84], we call an R-branching **maximal** if it spans all the nodes that are reachable from R in D. For non-empty sets $X, Y \subseteq V$, let $Z \mapsto X$ denote that X and Z are disjoint and X is reachable from Z. Let $P(X) = X \cup \{v \in V \setminus X : v \mapsto X\}$. A set $\{F_1, \ldots, F_{|S|}\}$ of pairwise edge-disjoint arborescences is called a **maximal** $\mathcal{M}$**-independent packing of arborescences** if F_i has root $\pi(s_i)$ for $i = 1, \ldots, |S|$, the set $\{s_j \in S : v \in V(F_j)\}$ is independent in $\mathcal{M}$ and $|\{s_j \in S : v \in V(F_j)\}| = r_{\mathcal{M}}(S_{P(v)})$.

Theorem 1.1.8 (Cs. Király)**.** *Let (D, S, π) be a digraph with roots and $\mathcal{M}$ be a matroid on S with rank function $r_{\mathcal{M}}$. There exists a maximal $\mathcal{M}$-independent packing of arborescences in (D, S, π) if and only if π is $\mathcal{M}$-independent and*

$$\varrho(X) \geq r_{\mathcal{M}}(S_{P(X)}) - r_{\mathcal{M}}(S_X) \tag{1.6}$$

holds for each $X \subseteq V$.

A natural idea is to reformulate Edmonds' theorem to the node-connected case. Let D and r denote a digraph and a root-node as previously, then D is called **rooted** k**-node-connected** (or rooted k-connected, for short) if there exist k internally node-disjoint directed paths from r to v for each $v \in V$, that is, any two of the paths have only r and v in common. The maximum number of node-disjoint $r - v$ paths is denoted by $\kappa(r, v)$. For an r-arborescence F, a node u is an F**-ancestor** of another node v if there is a directed path from u to v in F. We denote this unique path by $F(u, v)$. For example, the root is the F-ancestor of all other nodes. The maximum number of edge-disjoint $r - v$ paths is denoted by $\lambda(r, v)$. We say that a node w **dominates** a node v if every path from r to v includes w. We denote the set of nodes dominating v by $\mathtt{dom}(v)$. Clearly, r and v are in $\mathtt{dom}(v)$.

Note that two r-arborescences F_1 and F_2 are edge-disjoint if and only if for each $v \in V$ the two paths $F_1(r, v)$ and $F_2(r, v)$ are edge-disjoint. That gives the idea of the following definition: we call two spanning r-arborescences F_1 and F_2 **independent** if $F_1(r,v)$ and $F_2(r,v)$ are internally node-disjoint for each $v \in V$.

As a node-disjoint counterpart of Edmonds' theorem, Frank conjectured that in a rooted k-connected graph there exist k independent arborescences (see eg. [112]). The case $k = 2$ was verified by Whitty [135], but for $k \geq 3$ the statement does not hold as was shown by Huck [73]. However, Huck also proved that the conjecture is true for simple acyclic graphs [74] and verified it for planar multigraphs except for a few values of k [75].

Theorem 1.1.9.

(i) (Whitty) Let $D = (V, A)$ be a digraph with root r. D has two independent spanning r-arborescences if and only if D is rooted 2-connected.

(ii) (Huck) Let $D = (V, A)$ be an acyclic digraph with root r such that $D - r$ is simple. D has k independent spanning r-arborescences if and only if D is rooted k-connected.

(iii) (Huck) Let $D = (V, A)$ be a directed multigraph with root r and $k \in \{1, 2\} \cup \{6, 7, 8, \ldots\}$ such that D is planar if $k \geq 6$. D has k independent spanning r-arborescences if and only if D is rooted k-connected.

In [18], Colussi, Conforti and Zambelli introduced another type of disjointness concerning arborescences, which put slightly stronger restrictions on the paths than edge-disjointness. In a digraph we call two arcs **symmetric** if they share the same endnodes but have opposite orientations. Two edge-disjoint arborescences F_1, F_2 rooted at r are called **strongly edge-disjoint** if the paths $F_1(r,v), F_2(r,v)$ do not contain a pair of symmetric arcs. In [18], the following strengthening of Edmonds' theorem was proposed.

Conjecture 1.1.10 (Colussi, Conforti, Zambelli)**.** *Let $D = (V, A)$ be a digraph with root r. D has k strongly edge-disjoint spanning r-arborescences if and only if D is rooted k-edge-connected.*

For $k = 2$, the conjecture was verified in [18]. As Colussi et al. note, the motivation of the problem is the following. It is easy to see that a similar statement holds for strongly edge-disjoint directed $s - t$ paths. Hence the conjecture, if it were true, could be considered as a common generalization of Edmonds' disjoint arborescences theorem and Menger's theorem. Note that the arborescences in the conjecture are allowed to contain pairs of symmetric arcs, only the paths in question are required not to do so. In Section 2.2 we give a generalization of the case $k = 2$ (Theorem 2.2.8) and show that the conjecture does not hold for $k \geq 3$ (Section 2.2.3). As a side result, we get a new proof of a theorem of Georgiadis and Tarjan [55].

Let now $D = (V, A)$ be a digraph without loops, but D may have parallel arcs. We assume that D is weakly connected, i.e., $|V| - 1 \leq |A|$ holds. For each $a \in A$, we denote by $t(a)$ and $h(a)$ the tail and the head of a, respectively. From now on we distinguish two types of arborescences: **in-** and **out-arborescences**. An r-out-arborescence is just the same as an r-arborescence defined earlier, that is, it is a directed tree in which the edges are directed away from the root node r. An r-in-arborescence is a directed tree in which the edges are directed toward the root node r, so the reversal of its edges results in an out-arborescence.

The problem of finding k arc-disjoint spanning r-out-arborescences for a given root $r \in V$ is very important not only from the theoretical viewpoint but also from practical viewpoints, and it has been extensively studied. It is known [15, 52, 101, 122, 124] that this problem can be solved in polynomial time, and several extensions have been considered in [9, 48, 82]. However, in many situations, we have to simultaneously consider not only an in-arborescence but also an out-arborescence. For example, in evacuation situations, an in-arborescence represents roads which refugees use. On the other hand, an out-arborescence represents roads used by emergency vehicles. Unfortunately, it is known [5] that the problem of finding a pair of arc-disjoint spanning r_1-in-arborescence and r_2-out-arborescence for given roots $r_1, r_2 \in V$ is NP-complete even if $r_1 = r_2$. As a special case, it is only known [5] that this problem in a tournament can be solved in polynomial time. In Section 2.3, we consider this problem in a directed acyclic graph and we give a linear time algorithm for solving it (Theorem 2.3.1).

1.2 Covering intersecting bi-set systems

Sub- and supermodular set functions are known to be useful tools in graph optimization but in the last fifteen years it turned out that several results can be extended to functions defined on pairs of sets or on bi-sets. Given a ground-set V, we call a pair $X = (X_O, X_I)$ of subsets a **bi-set** if $X_I \subseteq X_O \subseteq V$ where X_O is the **outer member** and X_I is the **inner member** of X. By a **bi-set function** we mean a function defined on the set of bi-sets of V. We will tacitly identify a bi-set $X = (X_O, X_I)$ for which $X_O = X_I$ with the set X_I and hence bi-set functions may be considered as straight generalizations of set functions. The set of all bi-sets on ground-set V is denoted by $\mathcal{P}_2(V) = \mathcal{P}_2$. The **intersection** $\cap$ and the **union** $\cup$ of bi-sets is defined in a straightforward manner: for $X, Y \in \mathcal{P}_2$ let $X \cap Y := (X_O \cap Y_O, X_I \cap Y_I)$, $X \cup Y := (X_O \cup Y_O, X_I \cup Y_I)$. We write $X \subseteq Y$ if $X_O \subseteq Y_O, X_I \subseteq Y_I$ and this relation is a partial order on $\mathcal{P}_2$. Accordingly, when $X \subseteq Y$ or $Y \subseteq X$, we call X and Y **comparable**. A family of pairwise comparable bi-sets is called a **chain**. Two bi-sets X and Y are **independent** if $X_I \cap Y_I = \emptyset$ or $V = X_O \cup Y_O$. A set of bi-sets is independent if its members are pairwise independent. We call a set of bi-sets a **ring-family** if it is closed under taking union and intersection. Two bi-sets are **intersecting** if $X_I \cap Y_I \neq \emptyset$ and **properly intersecting** if, in addition, they are not comparable. Note that $X_O \cup Y_O = V$ is allowed for two intersecting bi-sets. In particular, two sets X and Y are properly intersecting if none of $X \cap Y, X - Y, Y - X$ is empty. A family of bi-sets is called **laminar** if it has no two properly intersecting members. A family $\mathcal{F}$ of bi-sets is **intersecting** if both the union and the intersection of any two intersecting members of $\mathcal{F}$ belong to $\mathcal{F}$. In particular, a family $\mathcal{L}$ of subsets is intersecting if $X \cap Y, X \cup Y \in \mathcal{L}$ whenever $X, Y \in \mathcal{L}$ and $X \cap Y \neq \emptyset$. A laminar family of bi-sets is obviously intersecting. Two bi-sets are **crossing** if $X_I \cap Y_I \neq \emptyset$ and $X_O \cup Y_O \neq V$ and **properly crossing** if they are not comparable. A bi-set (X_O, X_I) is **trivial** if $X_I = \emptyset$ or $X_O = V$. We will assume throughout Chapter 3 that the bi-set functions in question are integer-valued and that their value on trivial bi-sets is always zero. In particular, set functions are also integer-valued and zero on the empty set and on the ground-set.

A directed edge **enters** or **covers** X if its head is in X_I and its tail is outside X_O. The set of edges entering a bi-set X is denoted by $\Delta^{in}_D(X) = \Delta^{in}(X)$. An edge set **covers** a family of bi-sets if it covers each member of the family. For a bi-set function p, a digraph $D = (V, A)$ is said to **cover** p

if $\varrho_D(X) \geq p(X)$ for every $X \in \mathcal{P}_2(V)$ where $\varrho_D(X)$ denotes the number of edges of D covering X. For a vector $z : A \to \mathbb{R}$, let $\varrho_z(X) := \sum[z(a) : \ a \in A, a \text{ covers } X]$. A vector $z : A \to \mathbb{R}$ **covers** p if $\varrho_z(X) \geq p(X)$ for every $X \in \mathcal{P}_2(V)$.

A bi-set function p is said to satisfy the **supermodular inequality** on $X, Y \in \mathcal{P}_2$ if

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y). \tag{1.7}$$

If the reverse inequality holds, we speak of the **submodular** inequality. p is said to be **fully supermodular** or supermodular if it satisfies the supermodular inequality for every pair of bi-sets X, Y. If (1.7) holds for intersecting (resp. crossing) pairs, we speak of **intersecting** (resp. **crossing**) **supermodular** functions. Analogous notions can be introduced for submodular functions. Sometimes (1.7) is required only for pairs with $p(X) > 0$ and $p(Y) > 0$ in which case we speak of **positively supermodular** functions. Positively intersecting or crossing supermodular functions are defined analogously. A typical way to construct a positively supermodular function is replacing each negative value of a fully supermodular function by zero. An easy example for a submodular bi-set function is the in-degree function.

Proposition 1.2.1. *The in-degree function ϱ_D on $\mathcal{P}_2$ is submodular.*

There is another line of extending Theorem 1.1.1 in which, rather than working directly with arborescences, one considers disjoint edge-coverings of certain families of sets or bi-sets. In [40], Frank proved the following.

Theorem 1.2.2 (Frank)**.** *Let $D = (V, A)$ be a digraph and $\mathcal{F}$ an intersecting family of subsets of V. It is possible to partition A into k coverings of $\mathcal{F}$ if and only if the in-degree of every member of $\mathcal{F}$ is at least k.*

Obviously, when $\mathcal{F}$ consists of every non-empty subset of $V - r$, we obtain the weak form of Edmonds' theorem. A disadvantage of Theorem 1.2.2 is that it does not imply the strong version of Edmonds' theorem. The following result of Szegő [120], however, overcame this difficulty.

Theorem 1.2.3 (Szegő)**.** *Let $\mathcal{F}_1, \dots, \mathcal{F}_k$ be intersecting families of subsets of nodes of a digraph $D = (V, A)$ with the following mixed intersection property:*

$$X \in \mathcal{F}_i,\ Y \in \mathcal{F}_j,\ X \cap Y \neq \emptyset \ \Rightarrow \ X \cap Y \in \mathcal{F}_i \cap \mathcal{F}_j.$$

Then A can be partitioned into k subsets $A_1, \dots, A_k$ such that A_i covers $\mathcal{F}_i$ for each $i = 1, \dots, k$ if and only if $\varrho_D(X) \geq p_1(X)$ for all non-empty $X \subseteq V$ where $p_1(X)$ denotes the number of $\mathcal{F}_i$'s containing X.

However, Theorem 1.2.3 does not imply Theorem 1.1.5. In [9], we derived an extension of Szegő's theorem to bi-set families.

The bi-set families $\mathcal{F}_1, \dots, \mathcal{F}_k$ said to satisfy the **mixed intersection** property if

$$X \in \mathcal{F}_i,\ Y \in \mathcal{F}_j,\ X_I \cap Y_I \neq \emptyset \ \Rightarrow \ X \cap Y \in \mathcal{F}_i \cap \mathcal{F}_j.$$

For a bi-set X, let $p_2(X)$ denote the number of indices i for which $\mathcal{F}_i$ contains X. For $X \in \mathcal{F}_i$, $Y \in \mathcal{F}_j$, the inclusion $X \subseteq Y$ implies $X = X \cap Y \in \mathcal{F}_j$ and hence p_2 is monotone non-increasing in the sense that $X \subseteq Y$, $p_2(X) > 0$ and $p_2(Y) > 0$ imply $p_2(X) \geq p_2(Y)$.

Theorem 1.2.4. *Bérczi and Frank Let $D = (V, A)$ be a digraph and $\mathcal{F}_1, \ldots, \mathcal{F}_k$ be intersecting families of bi-sets on ground set V satisfying the mixed intersection property. The edges of D can be partitioned into k subsets $A_1, \ldots, A_k$ such that A_i covers $\mathcal{F}_i$ for each $i = 1, \ldots, k$ if and only if*

$$\varrho_D(X) \geq p_2(X) \text{ for every bi-set } X.$$

The proof of Theorem 1.2.4 went along the same line as Lovász' original proof for Edmonds' theorem and was based on the following property.

Lemma 1.2.5. *If $p_2(X) > 0$, $p_2(Y) > 0$ and $X_I \cap Y_I \neq \emptyset$, then $p_2(X)+p_2(Y) \leq p_2(X\cap Y)+p_2(X\cup Y)$. Moreover, if there is an $\mathcal{F}_i$ for which $X \cap Y \in \mathcal{F}_i$ and $X, Y \notin \mathcal{F}_i$, then strict inequality holds.*

Using Theorem 1.2.4, we give a new proof of Theorem 1.1.6 in Section 3.1. The application of bi-sets gives a new insight into the structure of convex sets. By using the special bi-set families appearing in the proof, we are able to give a strongly polynomial time algorithm for finding rooted branchings spanning given convex sets under edge capacity constraints (Theorem 3.2.2). We also give a polyhedral description of arborescence packable subgraphs based on a connection with bi-set families (Lemma 3.3.5), and prove that the corresponding system of inequalities is TDI (Theorem 3.3.7).

1.3 Restricted b-matchings

Let $G = (V, E)$ be an undirected graph and let $b : V \to \mathbb{Z}_+$ be an upper bound on the nodes. An edge set $F \subseteq E$ is called a ***b*-matching** if $d_F(v)$, the number of edges in F incident to v, is at most $b(v)$ for each node v. This is often called **simple** b-matching in the literature, since multiple copies of the same edge are not allowed. If not stated otherwise, all b-matchings considered will be simple throughout Sections 1.3-1.4 and Chapters 4-6. For some integer $t \geq 2$, by a ***t*-matching** we mean a b-matching with $b(v) = t$ for every $v \in V$. A closely related concept is ***b*-factor**, where instead of $d_F(v) \leq b(v)$ strictly $d_F(v) = b(v)$ is required.

Let $\mathcal{K}$ be a list of forbidden subgraphs. The node-set and the edge-set of a subgraph $K \in \mathcal{K}$ are denoted by V_K and E_K, respectively. By a **$\mathcal{K}$-free *b*-matching** we mean a b-matching not containing any member of $\mathcal{K}$. The maximum $\mathcal{K}$-free b-matching problem asks for a $\mathcal{K}$-free b-matching in G with maximum size (that is, a $\mathcal{K}$-free b-matching $F \subseteq E$ with maximum cardinality).

The most important special cases of $\mathcal{K}$-free b-matchings are the so-called $C_{\leq k}$-free and C_k-free 2-matching problems. A 2-matching M is **C_k-free** if it contains no cycle of length k, and it is **$C_{\leq k}$-free**-free if it contains no cycle of length k or less. The motivation of these problems is twofold. On the one hand, they have been studied as relaxations of the Hamiltonian cycle problem. The case $k \leq 2$ is exactly the classical simple 2-matching problem, which can be solved efficiently. Papadimitriou showed that the problems are NP-hard when $k \geq 5$ [22], and Hartvigsen [59] gave an augmenting path algorithm for the case $k = 3$. The C_4-free and $C_{\leq 4}$-free 2-matching problems are left open.

The other motivation comes from undirected node-connectivity augmentation. For an integer k, a graph (resp. digraph) is ***k*-connected** if it contains more than k nodes and it remains connected (resp. strongly connected) when we delete at most $k - 1$ nodes from the graph (resp. digraph). The k-connectivity augmentation problem is the following: make a given graph or digraph k-connected by adding a minimum number of new edges. Concerning the directed case, Frank and Jordán gave a min-max formula and also an algorithm relying on the ellipsoid method for finding the minimum [43]. In [44], they also provided a combinatorial algorithm to make a $(k-1)$-connected digraph k-connected. However, their algorithm is polynomial only for fixed k's, that is, the running time is polynomial in the size of the digraph but exponential in k. Végh and Benczúr gave a combinatorial algorithm for the general case whose running time is polynomial also in k [130].

There are only partial results for the undirected case. The solution is trivial when $k = 1$. Eswaran and Tarjan solved the problem for $k = 2$ in [38], while Watanabe and Nakamura found a characterization for the case of $k = 3$ [132]. Later, Hsu and Ramachandran [71,72] gave linear time algorithms for both of these problems. For $k = 4$, a polynomial algorithm was developed by Hsu [70]. It is also known that near-optimal solutions can be found in polynomial time for every k, see [76,77]. In [78], Jackson and Jordán gave an algorithm which provides an optimal solution in polynomial time for every fixed k. If the size of an optimal solution is large compared to k, their algorithm is polynomial for all k. They also obtained a min-max formula for this special case, and completely solved the problem for a new family of graphs called **k-independence free graphs**. However, the complexity of the node-connectivity augmentation problem is still open, and it is certainly one of the most interesting unsolved questions in this area.

An interesting special case consists of increasing the connectivity by one, that is, when the starting graph is already $(k-1)$-connected. We call this problem the **k-connectivity augmentation by one problem**. Hsu gave an almost linear time algorithm to increase the connectivity from three to four in [115]. Hence a linear time algorithm for $k = 1, 2, 3$, an almost linear time algorithm for $k = 4$ and a polynomial time algorithm provided by [78] for fixed k are at hand. A polynomial time algorithm was given when the graph has a certain condition [100], and approximation algorithms are proposed in [80,81]. The general case was solved by Végh [129], see later.

On the other hand, values of k close to n are also of interest. If $k = n - 1$, then the graph should be simply extended to a complete graph and the answer is trivial since every augmenting set consists of the edges of $\bar{G}$ where $\bar{G}$ denotes the complement of G. An easy argument shows that a graph G is $(n-2)$-connected if and only if each node has degree at most one in $\bar{G}$. This implies that for $k = n-2$ the k-connectivity augmentation problem is equivalent to finding a maximum matching in the complement of the graph. It can be verified that a graph G is $(n-3)$-connected if and only if the edge set of $\bar{G}$ is a C_4-free 2-matching, also called a **square-free 2-matching**. Moreover, an obvious but important observation is that if G is $(n-4)$-connected then its complement $\bar{G}$ is a subcubic graph (i.e. each node has degree at most three). Therefore, the $(n-3)$-connectivity augmentation by one problem can be reduced to the problem of finding a square-free 2-matching of maximum size in a subcubic graph.

The main result of Chapter 4 is a polynomial time algorithm for the square-free 2-matching problem in simple subcubic graphs (Theorem 4.3.1), which leads to a polynomial time algorithm for the $(n-3)$-connectivity augmentation problem (Theorem 4.3.2). Our algorithm is based on the theorem that square-free 2-matchings in a simple subcubic graph have a matroid-like structure called a jump system (Theorem 4.3.3). With the aid of known results on jump systems, we show that some optimization problems are also solvable in polynomial time. We also give a faster algorithm for the square-free 2-matching problem in simple subcubic graphs, which runs in $O(n^{\frac{3}{2}})$ time (Theorem 4.3.9).

We also discuss the weighted versions of the problems. Given a $(k-1)$-connected graph $G = (V, E)$

and a weight function $w : \bar{E} \to \mathbb{R}_+$, where $\bar{E}$ is the complement of E, the **weighted k-connectivity augmentation by one problem** is the problem of finding a set of edges of minimum total weight that should be added to the original graph to obtain a simple k-connected graph. This problem is known to be NP-hard for fixed $k \geq 2$ [38]. A 2-approximation algorithm is given for $k = 3$ [4], and also a 3-approximation algorithm exists for $k = 4, 5$ [27]. For an arbitrary k, an algorithm with the approximation ratio $2(1 + \frac{1}{2} + \cdots + \frac{1}{k})$ is given in [111], and further improvement is given in [109]. See [97] for an overview of the known results.

Of course the weighted $(n-3)$-connectivity augmentation by one problem can be reduced to the problem of finding a square-free 2-matching maximizing the total weight of its edges, which we call the **weighted square-free 2-matching problem**. Z. Király proved that the weighted square-free 2-matching problem in bipartite graphs is NP-hard even for $0-1$ weights [87]. This problem is, however, polynomially solvable in bipartite graphs if the weight function is node-induced on every square [103,121]. For a subgraph $H = (V(H), E(H))$ of G, we say that w is **node-induced on** H if there exists a function $\pi_H : V(H) \to \mathbb{R}$ such that $w(e) = \pi_H(u) + \pi_H(v)$ for every edge $e = uv \in E(H)$. We show that the weighted square-free 2-matching problem in simple subcubic graphs can be solved in polynomial time if the weight function is node-induced on every square (Theorem 4.6.1), whereas the problem is NP-hard for general weights (Theorem 4.5.1). In our algorithm for the weighted problem, we use the theory of M-concave (M-convex) functions on constant-parity jump systems introduced by Murota [107]. Hartvigsen and Li [62], and Kobayashi [88] gave polynomial time algorithms for the weighted C_3-free 2-matching problem in subcubic graphs with an arbitrary weight function. However, the problem for $k = 3$ in general graphs with arbitrary weights is still open.

Let us now consider the special case of C_4-free 2-matchings in bipartite graphs. This problem was solved by Hartvigsen [60, 61] and Király [86]. A generalization of the problem to maximum $K_{t,t}$-free t-matchings in bipartite graphs was given by Frank [41] who observed that this is a special case of covering positively crossing supermodular functions on set pairs, solved by Frank and Jordán in [43]. Makai [103] generalized Frank's theorem for the case when a list $\mathcal{K}$ of forbidden $K_{t,t}$'s is given (that is, a t-matching may contain $K_{t,t}$'s not in $\mathcal{K}$.) He gave a min-max formula based on a polyhedral description for the minimum cost version for node-induced cost functions. Pap [110] gave a further generalization of the maximum cardinality version for excluded complete bipartite subgraphs and developed a simple, purely combinatorial algorithm. For node induced cost functions, such an algorithm was given by Takazawa [121] for $K_{t,t}$-free t-matching.

The C_4-free 2-matching problem admits two natural generalizations. The first one is $K_{t,t}$-free t-matchings considered in Chapter 5, while the second is t-matchings containing no complete bipartite graph $K_{a,b}$ with $a + b = t + 2$. This latter problem is equivalent to connectivity augmentation for $k = n - t - 1$. The complexity of connectivity augmentation for general k is yet open, while connectivity augmentation by one, that is, when the input graph is already $(k-1)$-connected was solved in [129] (this corresponds to the case when the graph contains no $K_{a,b}$ with $a + b = t + 3$, in particular, $d(v) \leq t + 1$).

Let $\mathcal{K}$ be a set consisting of $K_{t,t}$'s, complete bipartite subgraphs of G on two colour classes of size t, and K_{t+1}'s, complete subgraphs of G on $t + 1$ nodes. We give a min-max formula (Theorem 5.1.4) on the size of $\mathcal{K}$-free b-matchings and a polynomial time algorithm (Section 5.4) for finding one with maximum size under the assumptions that for any $K \in \mathcal{K}$ and any node v of K,

$$V_K \text{ spans no parallel edges} \tag{1.8}$$
$$b(v) = t \tag{1.9}$$
$$d_G(v) \leq t + 1. \tag{1.10}$$

Note that this is a generalization of the maximum C_3-free, C_4-free and $C_{\leq 4}$-free 2-matching problems in subcubic graphs. Among our assumptions, (1.8) and (1.9) may be considered as natural ones as they hold for the maximum $K_{t,t}$-free t-matching problem in a simple graph. We exclude parallel edges on the node sets of members of $\mathcal{K}$ in order to avoid having two different $K_{t,t}$'s on the same two colour classes or two K_{t+1}'s on the same ground set. However, the degree bound (1.10) is a restrictive assumption and dissipates essential difficulties. Our proof strongly relies on this and the theorem cannot be straightforwardly generalized as it can be shown by using the example in Chapter 6 of [129]. The proof and algorithm use the contraction technique of [87], [110] and [12]. The contribution of Chapter 5 on the one hand is the extension of this technique for $t \geq 2$ and forbidding K_{t+1}'s as well, while on the other hand the argument is significantly simpler than the argument in Chapter 4.

Kobayashi and Yin considered the problem of finding a maximum t-matching not containing H as a subgraph for a fixed graph H, called the H-free t-matching problem [95]. They generalized the results of [14] by solving the case when H is a t-regular complete partite graph. They also showed that the problem is NP-complete when H is a connected t-regular graph that is not complete partite.

It is worth mentioning that the polynomial solvability of the above problems seems to show a strong connection with jump systems. In [119], Szabó proved that for a list $\mathcal{K}$ of forbidden $K_{t,t}$ and K_{t+1} subgraphs the degree sequences of $\mathcal{K}$-free t-matchings form a jump system in any graph. Concerning bipartite graphs, Kobayashi and Takazawa showed [92] that the degree sequences of $C_{\leq k}$-free 2-matchings do not always form a jump system for $k \geq 6$. These results are consistent with the polynomial solvability of the $C_{\leq k}$-free 2-matching problem, even when restricting it to bipartite graphs. Similar results are known about even factors due to [91]. Although Szabó's result suggests that finding a maximum $\mathcal{K}$-free t-matching should be solvable in polynomial time for a list $\mathcal{K}$ of forbidden $K_{t,t}$ and K_{t+1} subgraphs, the problem is still open. Concluding the above, jump systems and M-concave (M-convex) functions are understood as a natural framework of efficiently solvable problems. Besides studies of these structures themselves [89, 102, 107, 116], their relation to efficiently solvable combinatorial optimization problems has been revealed (see [2, 29, 88, 90, 93, 94, 107, 119]). The results of Chapters 4 and 5 present such examples and enforces the importance of these structures.

1.4 Polyhedral descriptions

A cornerstone of matching theory is Edmonds' [33] description of the perfect matching polytope, the convex hull of incidence vectors of perfect matchings of a graph $G = (V, E)$.

Theorem 1.4.1 (Edmonds). *The perfect matching polytope is determined by*

$$\begin{aligned} &(i)\ x_e \geq 0 && (e \in E), \\ &(ii)\ x(\delta(v)) = 1 && (v \in V), \\ &(iii)\ x(\delta(K)) \geq 1 && (K \subseteq V, |K| \text{ odd}). \end{aligned} \tag{P_1}$$

Observe that the incidence vector of a perfect matching satisfies all these conditions. The theorem yields that the set of vertices of the above polytope is identical to the set of incidence vectors of perfect matchings.

A natural generalization of perfect matchings are b-factors, with 1-factors being perfect matchings. Recall that $b(K) = \sum_{v \in K} b(v)$, while $\dot{\delta}(v)$ denotes the family of edges incident to $v \in V$, that is, any loop at v occurs twice in $\dot{\delta}(v)$. The set of loops at $v \in V$ is denoted by $l(v)$. We call $K \subseteq V, F \subseteq \delta(K)$ a **pair** if F does not contain loops (by notation, this only means restriction in case of $|K| = 1$). The pair is **odd** if $b(K) + |F|$ is odd. The b**-factor polytope** is the convex hull of the incidence vectors of b-factors of G. In the same paper [33], Edmonds gave the following characterization of the b-factor polytope.

Theorem 1.4.2 (Edmonds). *The b-factor polytope is determined by*

$$\begin{aligned} &(i)\ 0 \leq x_e \leq 1 && (e \in E), \\ &(ii)\ x(\dot{\delta}(v)) = b(v) && (v \in V), \\ &(iii)\ x(\delta(K) \setminus F) - x(F) \geq 1 - |F| && ((K, F) \text{ odd}). \end{aligned} \tag{P_2}$$

A polyhedral description of b-matchings can easily be derived from Theorem 1.4.2.

Theorem 1.4.3. *The b-matching polytope is determined by*

$$\begin{aligned} &(i)\ 0 \leq x_e \leq 1 && (e \in E), \\ &(ii)\ x(\dot{\delta}(v)) \leq b(v) && (v \in V), \\ &(iii)\ x(E[K]) + x(F) \leq \lfloor \tfrac{b(K)+|F|}{2} \rfloor && ((K, F) \text{ odd}). \end{aligned} \tag{P_3}$$

We refer the reader to Part III, in particular, Chapters 30-33 of Schrijver [114] for a detailed discussion of b-matchings and b-factors.

Results on b-factors can be reduced to perfect matchings via a simple construction. Given a graph $G = (V, E)$, construct a new graph $G' = (V', E')$ as follows. Introduce $b(v)$ nodes for each node $v \in V$. For each edge $e = uv \in E$, introduce two nodes $p_{e,u}$ and $p_{e,v}$, an edge $p_{e,u}p_{e,v}$, and edges connecting $p_{e,u}$ to all $b(u)$ copies of u and connecting $p_{e,v}$ to all $b(v)$ copies of v. It is not difficult to see that G' contains a perfect matching if and only if G contains a b-factor. Using this correspondence, results on matchings can be extended to b-factors, including Theorem 1.4.2, which thus deduces from Theorem 1.4.1. To the extent of our knowledge, all previous proofs of Theorem 1.4.3 used this correspondence.

An important subclass of b-factors are 2-factors, decompositions of a graph to disjoint union of cycles. Hamiltonian cycles being 2-factors, it is a natural question looking at special 2-factors not containing short cycles which led to the notion of $C_{\leq k}$-free or C_k-free 2-matchings or factors. We have already mentioned that determining the maximum size of such a subgraph is NP-complete for $k \geq 5$.

Considering the maximum weight version of the C_k-free 2-factor problem, there is a firm difference between triangle- and square-free 2-factors. Z. Király showed [87] that finding a maximum weight square-free 2-factor is NP-hard even in bipartite graphs with $0-1$ weights. For subcubic graphs, polynomial time algorithms were given by Hartvigsen and Li [62], and by Kobayashi [88] for the weighted C_3-free 2-factor problem with an arbitrary weight function. The former result implies that we should not expect a nice polyhedral description of the square-free 2-factor polytope. However, solvability of the triangle-free case was a main motivation of our investigation.

Deciding the existence of a triangle-free 2-factor becomes significantly harder without assuming the graph is subcubic. Yet if instead of (simple) 2-factors, we look at the problem of **uncapacitated 2-factors**, when we are allowed to use two copies of the same edge, there exists a polyhedral description for arbitrary graphs, given by Cornuéjols and Pulleyblank [23]. Let $\mathcal{T}$ be a set consisting of triangles of G. The node-set and the edge-set of a triangle $T \in \mathcal{T}$ are denoted by V_T and E_T, respectively. An (uncapacitated) 2-factor is called **$\mathcal{T}$-free** if it contain at most two edges (counted by multiplicity) of any member of $\mathcal{T}$. Cornuéjols and Pulleyblank proved the following.

Theorem 1.4.4 (Cornuéjols and Pulleyblank)**.** *The convex hull of characteristic vectors of $\mathcal{T}$-free uncapacitated 2-factors is determined by*

$$\begin{aligned} &(i)\ 0 \le x_e && (e \in E), \\ &(ii)\ x(\dot{\delta}(v)) = 2 && (v \in V), \\ &(iii)\ x(E_T) \le 2 && (T \in \mathcal{T}). \end{aligned} \tag{P_4}$$

Moreover, description (P_4) is totally dual integral.

Returning to our subject, Hartvigsen and Li gave a polyhedral description of the triangle-free 2-factor polytope for subcubic simple graphs [62].

Theorem 1.4.5 (Hartvigsen and Li)**.** *The $\mathcal{T}$-free 2-factor polytope of a simple subcubic graph is determined by*

$$\begin{aligned} &(i)\ 0 \le x_e \le 1 && (e \in E), \\ &(ii)\ x(\delta(v)) = 2 && (v \in V), \\ &(iii)\ x(\delta(K) \setminus F) - x(F) \ge 1 - |F| && (K \subseteq V, F \subseteq \delta(K), |F| \text{ odd}), \\ &(iv)\ x(E_T) = 2 && (T \in \mathcal{T}). \end{aligned} \tag{P_5}$$

Their proof is based on shrinking triangles and on a variation of the Basic Polyhedral Theorem of [21]. In the same paper, they gave a description of the $\mathcal{T}$-free 2-matching polytope as well and gave a sketch of the proof, which was published in its full version in [63].

As we have seen, the b-matching and b-factor polytopes have a similar description. Unexpectedly, the same does not hold in the triangle-free case. We say that a triangle T **1-fits** (resp. **2-fits**) a set $K \subseteq V$ if $|V_T \cap K| = 1$ (resp. 2). The **special edge** of a triangle T 1-fitting (resp. 2-fitting) the set K is the edge $e \in E_T$ having exactly 0 (resp. 2) endnodes in K, and is denoted by e_T. Given a set $\mathcal{T}$ of forbidden triangles, the set of triangles 1-fitting (resp. 2-fitting) K is denoted by $\mathcal{T}_K^1$ (resp. $\mathcal{T}_K^2$) while $\mathcal{T}_K$ stands for $\mathcal{T}_K^1 \cup \mathcal{T}_K^2$.

Definition 1.4.6. $(K, F, \mathfrak{T})$ is called a **tri-comb of Type i** if

1. $K \subseteq V$, $F \subseteq \delta(K)$, $\mathfrak{T} \subseteq \mathcal{T}_K^i$.

2. $F \cap E_{\mathfrak{T}} = \emptyset$.

3. The triangles in $\mathfrak{T}$ are edge-disjoint.

A tri-comb is called **odd** if $b(K) + |F| + |\mathfrak{T}|$ is odd. The **deficiency** of a tri-comb is defined as $\texttt{def}(K, F, \mathfrak{T}) = x(E[K]) + x(F) + \sum_{T \in \mathfrak{T}} x(E_T) - \lfloor \frac{1}{2}(b(K) + |F| + 3|\mathfrak{T}|) \rfloor$.

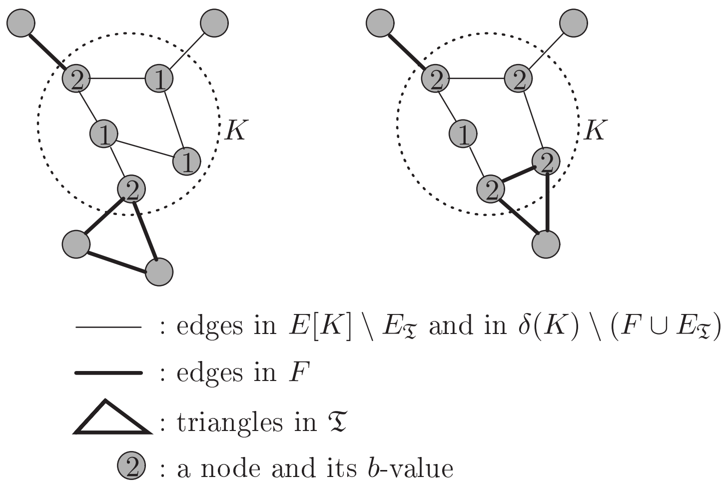

Figure 1.1: Odd tri-combs of Type 1 and 2

The fundamental result of Hartvigsen and Li is the following (see [62, 63]).

Theorem 1.4.7 (Hartvigsen and Li). *The $\mathcal{T}$-free 2-matching polytope of a simple subcubic graph is determined by*

$$
\begin{aligned}
&(i)\ 0 \le x_e \le 1 && (e \in E),\\
&(ii)\ x(\delta(v)) \le 2 && (v \in V),\\
&(iii)\ x(E[K]) + x(F) + \textstyle\sum_{T \in \mathfrak{T}} x(E_T) \le |K| + \lfloor \frac{|F| + 3|\mathfrak{T}|}{2} \rfloor && ((K, F, \mathfrak{T}) \textit{ odd tri-comb of Type 2}),\\
&(iv)\ x(E_T) \le 2 && (T \in \mathcal{T}).
\end{aligned}
\qquad (P_6)
$$

Their proof is algorithmic and uses, in some sense, an Edmonds-style matching algorithm consisting of clever triangle alteration and alternating forest growing. The algorithm alternates between a primal and a dual phase, and a quite complex dual change is performed whenever no improvement is found during the forest growing. The algorithm stops when the primal and dual solutions (that are feasible throughout) satisfy complementary slackness.

We give new proofs of Theorems 1.4.5 and 1.4.7 in a slightly more general form (Theorems 6.1.1 and 6.1.2). Our proof is a natural extension of the proof of Theorem 1.4.1 given by Aráoz, Cunningham, Edmonds, and Green-Krótki [3] and Schrijver [113]. It is based on a new shrinking operation that hopefully could be extended to the non-subcubic case as well which is the sole remaining open problem concerning triangle-free 2-matchings.

1.5 Splitting property

Let $\mathcal{P} = (P, \prec)$ be a finite partially ordered set. For a subset $H \subseteq P$, sets $\mathcal{U}(H) = \{x \in P : \exists h \in H : x \succeq h\}$ and $\mathcal{L}(H) = \{x \in P : \exists h \in H : x \preceq h\}$ are called the upper and lower shadows of H, respectively. An antichain $A \subseteq P$ is maximal if and only if $\mathcal{U}(A) \cup \mathcal{L}(A) = P$. We say that a maximal antichain A has the **splitting property** if it can be partitioned into two disjoint parts $A_1 \cup A_2 = A$ such that $\mathcal{U}(A_1) \cup \mathcal{L}(A_2) = P$. This property was introduced and first studied by Ahlswede et al. [1]. They gave the following sufficient condition for the splitting property. A maximal antichain $A \subseteq P$ is called **dense** if it satisfies the following: whenever $x \prec a \prec y$ for some $a \in A$ and $x, y \in P$, there exists an $a' \in A \setminus \{a\}$ also satisfying $x \prec a' \prec y$. They proved the following theorem.

Theorem 1.5.1 (Ahlswede, Erdős and Graham)**.** *Every dense maximal antichain in a finite poset satisfies the splitting property.*

The poset $\mathcal{P}$ itself has the **splitting property** if every maximal antichain in $\mathcal{P}$ satisfies the splitting property. The following negative result in [1] shows that this property is NP-hard to decide.

Theorem 1.5.2 (Ahlswede, Erdős and Graham)**.** *It is NP-hard to decide whether a given poset $\mathcal{P} = (P, \prec)$ has the splitting property.*

On the other hand, Duffus and Sands [31] gave a complete characterization of finite distributive lattices with the splitting property.

Theorem 1.5.3 (Duffus and Sands)**.** *If $\mathcal{P}$ is a finite distributive lattice with the splitting property, then it is either a Boolean lattice, or one of three other lattices.*

We consider the poset of multisets of k colours. Formally, let us use the elements of the group $\mathbb{Z}_k$ as colours, denoted by $\{1, \ldots, k\}$. We call the vectors $\mathbb{Z}_k \to \mathbb{Z}$ k**-colour vectors**, and denote their set by M_k. We can define a natural partial ordering on M_k: for $a, c \in M_k$, $a \prec c$ if $a_i \leq c_i$ for every $i \in \mathbb{Z}_k$ and $a \neq c$. If $a \prec c$, we also say that a is a **shadow** of c. $(M_k, \prec)$ is a distributive lattice, however, it is not finite and therefore Theorem 1.5.3 is not applicable. Let

$$M_k^r = \{x \in M_k : \sum_{i \in \mathbb{Z}_k} x_i = r\}$$

denote the set of k-colour vectors whose coordinates sum up to r. The main result of Chapter 7 shows the splitting property of this antichain for $r = k$ (Theorem 7.1.1). It is easy to verify that M_k^k is not dense and therefore Theorem 1.5.1 does not imply our result. Indeed, take an arbitrary $x \in M_k^{k-1}$ and let $y_1 = x_1 + 2$ and $y_i = x_i$ if $i \neq 1$. Then M_k^k contains exactly one element a with $x \prec a \prec y$.

For $r \leq t \leq n$, a **Turán** (n, t, r)**-system** is an r-uniform hypergraph on n nodes such that every t-element subset of the nodes spans at least one edge of the hypergraph. The Turán number $T(n, t, r)$ asks for the minimum size of such a family; determining the exact values is a problem posed by Pál Turán [125]. The simplest case $t = 3$, $r = 2$ asks for the minimum number of edges of a graph such that every subset of 3 nodes contains at least one edge. This is equivalent to determining the maximum number of edges in a triangle free graph on n nodes, a problem solved by Mantel in 1907. The optimal $(n, 3, 2)$-Turán system is the disjoint union of two cliques on node sets of size $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$.

The limit

$$t(t,r) = \lim_{n\to\infty} \frac{T(n,t,r)}{\binom{n}{r}}$$

expresses the fraction of all r-element subsets needed for a Turán (n,t,r)-system. No exact value is known for any $t > r > 2$ - in 1981, Pál Erdős offered a bounty of \$500 for even a single special case and \$1000 for resolving the general case [36]. For surveys on Turán numbers, see [49, 83, 118]. De Caen [26] gave the lower bound $t(t,r) \geq \frac{1}{\binom{t-1}{r-1}}$. The best currently known upper bound is due to Sidorenko [117].

Theorem 1.5.4 (Sidorenko)**.** *For any integers $t > r$,*

$$t(t,r) \leq \left(\tfrac{r-1}{t-1}\right)^{r-1}. \tag{1.11}$$

We give a new interpretation of Sidorenko's construction in terms of shadow systems, and reprove the theorem using a combinatorial colouring result (Theorem 7.1.2).

We also introduce the natural weighted extension of Turán numbers: we are given a nonnegative weight function w on the r-element subsets of V, and let w^* denote the total weight of all subsets. The **Turán weight** $T_w(n,t,r)$ is the minimum weight of a Turán (n,t,r)-system. Analogously to $t(t,r)$ we may define

$$tw(t,r) = \lim_{n\to\infty} \sup_{w} \frac{T_w(n,t,r)}{w^*}.$$

Somewhat surprisingly, we show that $tw(t,r) = t(t,r)$, that is, the bound is not affected by the weight, and the bound on $tw(t,r)$ can be derived from Theorem 7.1.2 the same way as the bound on $t(t,r)$ (Theorem 7.2.1).

The notion of weighted Turán numbers enables us to establish a connection between Turán systems and Tuza's [127] famous conjecture asserting that in every graph the minimum number of edges covering every triangle is at most twice the maximum number of pairwise edge-disjoint triangles. Finding a minimum number of edges in a graph $G = (V,E)$ covering every triangle is equivalent to computing the weighted Turán number $T_w(n,3,2)$ with $n = |V|$, and $w(e) = 1$ if $e \in E$ and $w(e) = 0$ otherwise. We propose a weighted hypergraphic version of Tuza's conjecture (Conjecture 7.3.2), and prove its fractional relaxation (Theorem 7.3.3). This extends the result of Krivelevich [99] on the fractional version of Tuza's original conjecture and also makes use of our construction on shadow systems.

Chapter 2

Packing arborescences

2.1 Extending Edmonds' theorem

Let $D = (V, A)$ be a digraph. We call a vector $z : V \to \{0, 1, \dots, k\}$ a **root-vector** if there are k edge-disjoint spanning arborescences in D so that each node v is the root of $z(v)$ arborescences. From Edmonds' theorem one easily gets the following characterization of root-vectors.

Theorem 2.1.1. *Given a digraph $D' = (V', A')$, a vector z is a root-vector if and only if $z(V') = k$ and $z(X) \geq k - \varrho_{D'}(X)$ for every non-empty subset $X \subseteq V'$.*

Proof. The necessity of both conditions is evident. For the sufficiency, extend D' with a node r and $z(v)$ parallel edges from r to v for each $v \in V$. In the resulting digraph D the out-degree of r is exactly k and $\varrho_D(X) = z(X) + \varrho_{D'}(X) \geq k$ holds for every non-empty $X \subseteq V'$. By Edmonds' theorem, D contains k edge-disjoint spanning arborescences of root r. Since $\delta_D(r) = k$, each of these arborescences must have exactly one edge leaving r and therefore their restrictions to A' form k arborescences of D' of root-vector z. □

For an intersecting supermodular function p with finite $p(S)$, let

$$B'(p) = \{x \in \mathbb{R}^S : \ x(S) = p(S), \ x(A) \geq p(A) \text{ for every } A \subseteq S\}.$$

This is called a base polyhedron. The following result appeared in an equivalent form in [45].

Theorem 2.1.2 (Frank and Tardos)**.** *Let p be an intersecting supermodular function for which $p(S)$ finite and let $f : S \to \mathbb{R} \cup \{-\infty\}$, $g : S \to \mathbb{R} \cup \{\infty\}$ be two functions for which $f \leq g$.*

(i) The polyhedron $\{x \in B'(p) : f \leq x\}$ is non-empty if and only if

$$f(S) \leq p(S) \tag{2.1}$$

and

$$f(X_0) + \sum_{i=1}^{t} p(X_i) \leq p(S) \tag{2.2}$$

for every partition $\{X_0, X_1, \dots, X_t\}$, $(t \geq 1)$ of S in which only X_0 may be empty.

(ii) The polyhedron $\{x \in B'(p) : x \leq g\}$ is non-empty if and only if

$$g(X) \geq p(X) \text{ for every } X \subseteq S. \tag{2.3}$$

(iii) The base-polyhedron $\{x \in B'(p) : f \leq x \leq g\}$ *is non-empty if and only if neither* $\{x \in B'(p) : f \leq x\}$ *nor* $\{x \in B'(p) : x \leq g\}$ *is empty.*

If, in addition, each of p*,* f *and* g *is integer-valued, then the corresponding polyhedra are integral.*

Let $D = (V, A)$ be a digraph. Define the set function p by $p(X) = k - \varrho_D(X)$ for non-empty subsets X. Then p is intersecting supermodular and Theorem 2.1.1 implies that the root vectors of D are exactly the integral elements of the bases polyhedron $B'(p)$. By combining this with Theorem 2.1.2, one arrives at the following result appeared in [39, 104].

Theorem 2.1.3 (Cai, Frank)**.** *In a digraph* $D = (V, A)$ *there exist* k *edge-disjoint spanning arborescences so that*

(i) each node v *is the root of at most* $g(v)$ *of them if and only if*

$$\sum_{i=1}^{t} \varrho_D(X_i) \geq k(t-1) \tag{2.4}$$

holds for every subpartition $\{X_1, \dots, X_t\}$ *of* V*, and*

$$g(X) \geq k - \varrho_D(X) \tag{2.5}$$

for every $\emptyset \subset X \subseteq V$*;*

(ii) each node v *is the root of at least* $f(v)$ *of them if and only if* $f(V) \leq k$ *and*

$$\sum_{i=1}^{t} \varrho_D(X_i) \geq k(t-1) + f(X_0) \tag{2.6}$$

holds for every partition $\{X_0, X_1, \dots, X_t\}$ *of* V *for which* $t \geq 1$ *and only* X_0 *may be empty;*

(iii) each node v *is the root of at least* $f(v)$ *and at most* $g(v)$ *of them if and only if the lower bound problem and the upper bound problem have separately solutions.*

Two interesting special cases are as follows.

Corollary 2.1.4. *A digraph* $D = (V, A)$ *includes* k *edge-disjoint spanning arborescences (with no restriction on their roots) if and only if* $\sum_{i=1}^{t} \varrho_D(X_i) \geq k(t-1)$ *for every subpartition* $\{X_1, \dots, X_t\}$ *of* V*.*

Corollary 2.1.5. *A digraph* $D = (V, A)$ *includes* k *edge-disjoint spanning arborescences whose roots are distinct if and only if* $|X| \geq k - \varrho_D(X)$ *holds for every non-empty subset* $X \subseteq V$ *set and* $\sum_{i=1}^{t} \varrho_D(X_i) \geq k(t-1)$ *for every subpartition* $\{X_1, \dots, X_t\}$ *of* V*.*

Theorem 2.1.3 characterized root-vectors satisfying upper and lower bounds. One may be interested in a possible generalization for the framework described in Theorem 1.1.4. We show that this problem is NP-complete. Indeed, let $D = (V, A)$ be a digraph whose node set is partitioned into a root-set $R = \{r_1, \dots, r_q\}$ and a terminal set T. Suppose that no edge of D enters any node of R.

Theorem 2.1.6. *The problem of deciding whether there are k disjoint arborescences so that they are rooted at distinct nodes in R and each of them spans T is NP-complete.*

Proof. Let T be a set with even cardinality and let $\mathcal{R} = \{R_1, \ldots, R_q\}$ be subsets of T so that $|R_i| \geq 2$ for $i = 1, \ldots, q$. It is well-known that the problem of deciding whether T can be covered with k members of $\mathcal{R}$ is NP-complete. Let D_T be a directed graph on T with $\varrho_{D_T}(Z) = k - 1$ for each $Z \subseteq T$, $|Z| = 1$ or $|Z| = |T| - 1$ and $\varrho_{D_T}(Z) \geq k$ otherwise. Such a graph can be constructed easily as follows. Take the same directed Hamilton cycle on the nodes $k-2$ times, then add the arcs $v_i v_{i+\frac{|T|}{2}}$ to the graph for each $i = 0, \ldots, |T| - 1$ where $v_0, \ldots, v_{|T|-1}$ denote the nodes according to their order around the cycle (the indices are meant modulo $|T|$). The arising digraph satisfies the in-degree conditions.

Extend the graph with $R = \{r_1, \ldots, r_q\}$ and with a new arc $r_i v$ for each $v \in R_i$. Let $r_{i_1}, \ldots, r_{i_k} \in R$ be a set of distinct root-nodes. Edmonds' disjoint branchings theorem implies that there are edge-disjoint r_i-arborescences F_i spanning $r_i + T$ for $i = i_1, \ldots, i_k$ if and only if $\varrho_{D_T}(Z) \geq p(Z)$ for each $\emptyset \subset Z \subseteq T$ where $p(Z)$ denotes the number of R_i's (with $i \in \{i_1, \ldots, i_k\}$) disjoint from Z. For a subset Z with $|Z| \geq 2$ the inequality holds automatically because of the structure of D_T and $|R_i| \geq 2$. Hence one only has to care about sets containing a single node and so the existence of the arborescences is equivalent to cover T with $R_{i_1}, \ldots, R_{i_k}$.

The observation above means that T can be covered with k members of $\mathcal{R}$ if and only if the digraph includes k arborescences rooted at different nodes in R. □

A natural idea to extend Edmonds' results would be to somehow decrease the set of nodes to be spanned by the arborescences. However, as the following theorem shows, one may easily face difficult questions if doing so.

Theorem 2.1.7. *Let $D = (V, A)$ be a digraph with $u_1, u_2, v_1, v_2 \in V$ and let $U_1 = V$, $U_2 = V - v_1$. The problem of finding two edge-disjoint arborescences rooted at u_1, u_2 and spanning U_1, U_2, respectively, is NP-complete.*

Proof. Let D' be a digraph with $u_1, u_2, v_1, v_2 \in V$. It is well-known that the problem of finding edge-disjoint $u_1 v_1$ and $u_2 v_2$ paths is NP-complete. We may suppose that the in-degree of v_1 and v_2 is one. Let D denote the graph arising from D' by adding arcs $v_1 v$ and $v_2 v$ to A for each $v \in V$ except for the arc $v_2 v_1$. Clearly, there are edge-disjoint directed $u_1 v_1$ and $u_2 v_2$ paths in D' if and only if there are two arborescences F_1, F_2 in D such that F_i is rooted at u_i and spans U_i. □

2.2 Dicycle-disjoint arborescences

2.2.1 Disjoint Steiner-arborescences

For a digraph $D = (V + r, A)$ with root r and terminal set $T \subseteq V$, an r-arborescence spanning T is called a **Steiner-arborescence**. Two Steiner-arborescences F_1 and F_2 are called **edge-independent** if the paths $F_1(r, t), F_2(r, t)$ are edge-disjoint for every terminal $t \in T$. **Independent** Steiner-arborescences can be defined in a straightforward manner. Note that paths corresponding to non-terminal nodes are allowed to violate the disjointness condition hence the arborescences are not necessarily edge-disjoint.

Z. Király asked [85] whether the existence of k edge-independent Steiner-arborescences is ensured by $\lambda(r, t) \geq k$ for each $t \in T$. As Frank's conjecture on independent arborescences would follow from such a result, Huck's counterexample shows that $k = 2$ is the only case when this statement may hold. The following example shows that even acyclicity is not satisfactory for the existence of edge-independent Steiner-arborescences [98].

Theorem 2.2.1 (Kovács)**.** *There is an acyclic graph for which there are three internally node-disjoint paths to all of the terminals but there are no three edge-independent Steiner-arborescences.*

Proof. The terminal set of the example consists of two nodes t_1, t_2 (see Figure 2.1). It can be easily checked that three edge-disjoint paths can be chosen only one way for both terminals but these cannot be partitioned into three arborescences. □

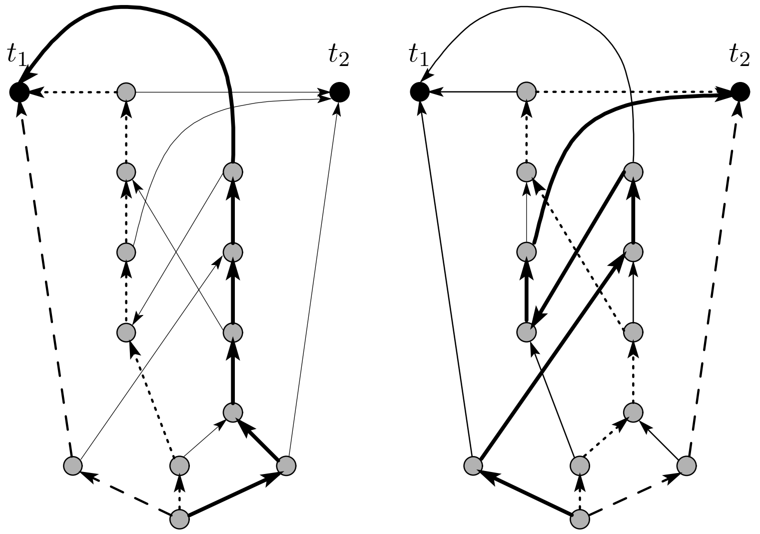

Figure 2.1: An example without three edge-independent Steiner-arborescences

Concerning the case when $k = 2$, the following theorem appeared in [98].

Theorem 2.2.2 (Kovács)**.** *Let $D = (V + r, A)$ be a digraph with root r, terminal set $T \subseteq V$ and $\lambda(r, t) \geq 2$ for each $t \in T$. Then there exist two edge-independent Steiner-arborescences.*

The node-independent version of the theorem is also of interest. However, the result of Georgiadis and Tarjan in [55] is a generalization of Theorem 1.1.9 (i).

Theorem 2.2.3 (Georgiadis and Tarjan)**.** *Let $D = (V + r, A)$ be a digraph with root r, terminal set $T \subseteq V$ and $\kappa(r, t) \geq 2$ for each $t \in T$. Then there exists two independent Steiner-arborescences.*

In fact, it can be showed that Theorems 2.2.2 and 2.2.3 are equivalent. The proof of Theorem 2.2.3 in [55] uses the properties of depth-first search (DFS) to find the two arborescences in question. Whitty's proof of Theorem 1.1.9 (i) is based on the following special ordering of the nodes.

Lemma 2.2.4. *Let $D = (V + r, A)$ be a digraph with root r and $\kappa(r, v) \geq 2$ for each $v \in V$. There is an ordering $r = v_0, v_1, \ldots, v_n, v_{n+1} = r$ of the nodes so that, for each $v_i \in V$, there is an edge $v_h v_i$ with $h < i$ and an edge $v_i v_j$ with $i < j$.*

This very special ordering proved to be useful. Huck's proof for Theorem 1.1.9 (ii) is based on the following lemma which is a variant of Lemma 2.2.4 for acyclic graphs.

Lemma 2.2.5. *Let $D = (V + r, A)$ be a simple acyclic graph with $\varrho(r) = 0$ and $\varrho(v) \geq 1$ for each $v \in V$. There is an ordering $o : V + r \to \mathbb{Z}$ of the nodes and an r-arborescence F such that for each $uv \in A$, we have $uv \in F$ if and only if $o(u) < o(v)$, that is, the set of edges going forward is exactly F.*

With the help of Lemma 2.2.4 and using the idea of the proof of Theorem 2.2.2, the following ordering of the nodes immediately shows the existence of proper Steiner-arborescences [98].

Theorem 2.2.6 (Kovács)**.** *Let $D = (V + r, A)$ be a digraph with root r, $\varrho(v) = \lambda(r, v) \leq 2$ for each $v \in V$ and assume that the set of nodes with in-degree 1 is stable. Then there exists an ordering $v_0, v_1, \dots, v_{n+1}$ of the nodes for which*

(i) $v_0 = v_{n+1} = r$

(ii) Cutting nodes appear twice, other nodes appear once.

(iii) Entering edges of nodes with in-degree 1 appear twice, other edges appear once.

(iv) For a cutting node p, if $v_i = v_j = p$ and $i < j$ then there is an edge entering v_i from the left and there is an edge entering v_j from the right, and all the copies of nodes cut by p from r lie between them.

(v) For every non-cutting node v, there is an edge entering v from the left and one from the right.

(vi) If F_1 and F_2 denote the sets of edges going forward and backward, respectively, then F_1 and F_2 are independent Steiner-arborescences with terminal set $T = \{v \in V : \lambda(r, v) = 2\}$.

The most important consequence of the existence of the above ordering is the following. Note, that each non-cutting node appears only once in the ordering. This observation immediately implies the following theorem, which was also proved in [55].

Theorem 2.2.7 (Georgiadis and Tarjan, Kovács)**.** *Let $D = (V, A)$ be a digraph with root r. There exist two arborescences F_1 and F_2 such that for each $v \in V - r$, the paths $F_1(r, v)$ and $F_2(r, v)$ intersect only at the nodes of $\mathtt{dom}(v)$.*

This theorem is the base of our proof for a slight generalization of Conjecture 1.1.10 when $k = 2$.

2.2.2 A generalization

Note that a pair of symmetric arcs can be considered as a directed cycle. This gives the idea of the following definition. Let $D = (V + r, A)$ be a digraph with root r and terminal set $T \subseteq V$. We call two edge-independent Steiner-arborescences F_1 and F_2 **dicycle-disjoint** if for each $t \in T$ the union $F_1(r, t) \cup F_2(r, t)$ does not contain a directed cycle. The motivation of this definition is the following: if $T = V$ and the arborescences are dicycle-disjoint then they are also strongly edge-disjoint.

The following theorem generalizes the theorem of Colussi, Conforti and Zambelli for $k = 2$.

Theorem 2.2.8. *Let $D = (V, A)$ be a directed graph with root r and terminal set T. There exist two dicycle-disjoint Steiner-arborescences if and only if $\lambda(r, t) \geq 2$ for each $t \in T$.*

Proof. The necessity is clear, we prove sufficiency. Consider the arborescences provided by Theorem 2.2.7. We claim that these arborescences are dicycle-disjoint.

Assume indirectly that there is a node $t \in T$ such that the union of the paths $F_1(r,t)$ and $F_2(r,t)$ contains a directed cycle. Let $r = x_1, x_2, \ldots, x_p = t$ and $r = y_1, y_2, \ldots, y_q = t$ denote the nodes along these paths. As the union of the paths contains a cycle, there are indices i_1, i_2, j_1, j_2 such that $x_{i_1} = y_{j_2}$, $x_{i_2} = y_{j_1}$ and $i_1 < i_2, j_1 < j_2$. Let $x_{i_1} = y_{j_2} = w$ and $x_{i_2} = y_{j_1} = z$. The choice of F_1 and F_2 implies $w, z \in \mathtt{dom}(t)$. Now consider the graph $G - z$. Then the union $F_1(r, w) \cup F_2(w, t)$ contains a path from r to t, which contradicts to $z \in \mathtt{dom}(t)$. □

2.2.3 Disproof of Conjecture 1.1.10 for $k \geq 3$

We give a counterexample for $k = 3$ based on a graph given by Huck [73], for other values a similar construction works. Let D be the graph of Figure 2.2. It is easy to check that D is rooted 3-edge-connected. The set of nodes in $V - r$ is partitioned into three blocks B_1, B_2 and B_3. There is one arc from r to B_i, and there are two arcs from B_i to B_{i+1} for each i (the indices are meant modulo 3 plus 1) such that together they form two directed cycles of length three. The edges of these triangles are denoted by e_{12}, e_{23}, e_{31} and f_{12}, f_{23}, f_{31}, respectively (see Figure 2.2).

Assume that there exist three strongly edge-disjoint arborescences F_1, F_2 and F_3. Clearly, each F_i contains an edge from r to one of the blocks, say F_i contains the one that goes to B_i, and it uses exactly one of e_{ii+1} and f_{ii+1} and the same holds for e_{i+1i+2} and f_{i+1i+2}. Also, at least one of the arborescences has to use the pair e_{ii+1}, f_{i+1i+2} or f_{ii+1}, e_{i+1i+2}. Assume that F_1 does so. But that implies that F_1 and F_2 can not be strongly edge-disjoint as they have to share a symmetric pair in B_2 that they use when going to B_3, so for any node $v \in B_3$ the paths $F_1(r, v)$ and $F_2(r, v)$ contain a pair of symmetric arcs.

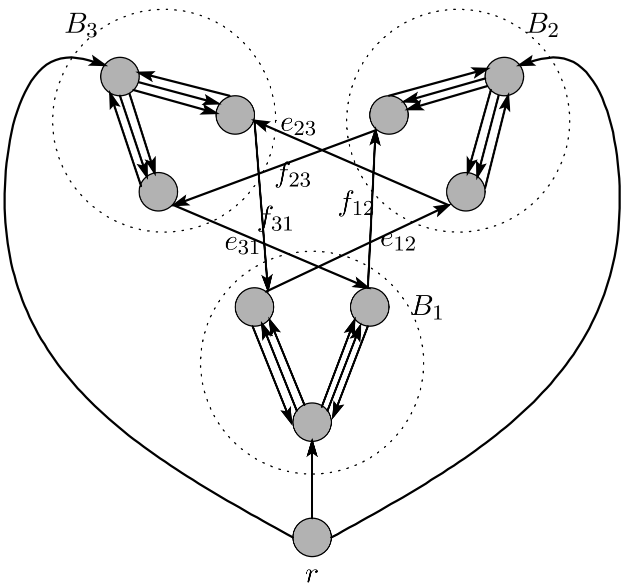

Figure 2.2: Counterexample for Conjecture 1.1.10

2.2.4 Further remarks

Edmonds' theorem gives a characterization of the existence of k edge-disjoint arborescences. On the other hand, we have seen that the analogue statement about independent arborescences does not hold. The notion of strongly edge-disjointness somehow lies between these two types of disjointness, but, as we showed, the conditions of Edmonds' theorem do not ensure the existence of such arborescences. So a natural idea is to turn to the other 'extremity' concerning the necessary conditions, and formulate the following conjecture.

Conjecture 2.2.9. *Let $D = (V + r, A)$ be a digraph with root r and assume that $\kappa(r, v) \geq k$ for each $v \in V$. Then there exist k dicycle-disjoint arborescences.*

2.3 In- and out arborescences

The aim of this section is to prove the following theorem.

Theorem 2.3.1. *Given a directed acyclic graph $D = (V, A)$ with roots $r_1, r_2 \in V$, we can discern the existence of a pair of arc-disjoint spanning r_1-in-arborescence and r_2-out-arborescence, and find such arborescences if they exist, in $O(|A|)$ time.*

2.3.1 An associated bipartite graph

We define a bipartite graph $G_D = (X, Y; E)$ associated with our problem for a directed acyclic graph $D = (V, A)$, and we show that our problem in D is equivalent to the problem of finding a matching that covers all nodes of Y in G_D. In the sequel, we assume without loss of generality that $\delta_D(r_1) = 0$ and $\varrho_D(r_2) = 0$ holds. Note that if $\delta_D(r_1) \neq 0$ or $\varrho_D(r_2) \neq 0$ holds, there exists no feasible solution since D is acyclic.

Define a bipartite graph $G_D = (X, Y; E)$ with two node sets X and Y and an edge set E between X and Y as follows.

(i) Node set X is given by $X = \{x(a) \mid a \in A\}$, where $|X| = |A|$.

(ii) Node set Y consists of two disjoint sets Y^+ and Y^- given by $Y^+ = \{y^+(v) \mid v \in V \setminus \{r_1\}\}$ and $Y^- = \{y^-(v) \mid v \in V \setminus \{r_2\}\}$.

(iii) For each $a \in A$, we have two edges in E: one connects $x(a)$ and $y^+(t(a))$ and the other connects $x(a)$ and $y^-(h(a))$. That is, $E = \{(x(a), y^+(t(a))) \mid a \in A\} \cup \{(x(a), y^-(h(a))) \mid a \in A\}$.

For example, for a directed graph D in Figure 2.3 (a) the bipartite graph G_D becomes the one as illustrated in Figure 2.3 (b).

Here we introduce notations to be used in the subsequent arguments (see Figure 2.4). For each $e \in E$, let $\partial_X(e)$ (resp. $\partial_Y(e)$) be the endpoint of e belonging to X (resp. Y). For each $e \in E$, we denote by $p(e)$ the edge $e' \in E$ with $e \neq e'$ and $\partial_X(e) = \partial_X(e')$. Notice that since $d_{G_D}(x) = 2$ holds for each $x \in X$ by the definition of G_D, $p(e)$ is uniquely determined for each $e \in E$.

Now we are ready to show the equivalence between our problem for D and the problem of finding a matching in G_D which covers all nodes of Y.

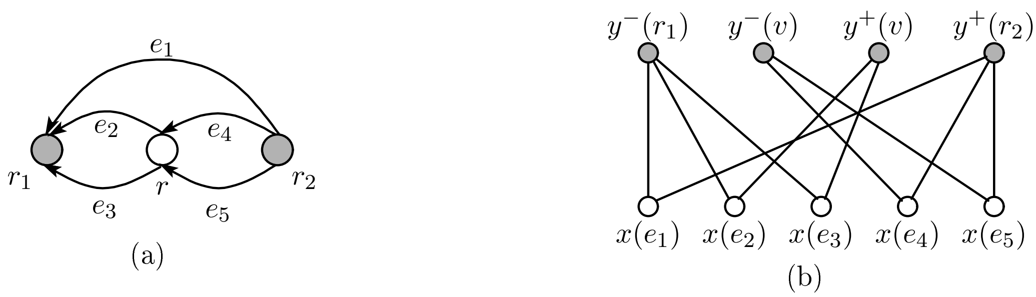

Figure 2.3: (a) An input directed graph D. (b) The bipartite graph G_D associated with D.

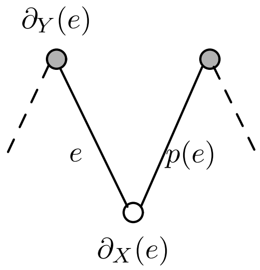

Figure 2.4: An illustration of notations.

Lemma 2.3.2. *Given a directed acyclic graph $D = (V, A)$ with roots $r_1, r_2 \in V$, there exists a pair of arc-disjoint spanning r_1-in-arborescence F_1 and r_2-out-arborescence F_2 if and only if there exists a matching M in $G_D = (X, Y; E)$ which covers all nodes of Y. Furthermore, we can construct a pair of such F_1 and F_2 from a matching M in $O(|A|)$ time.*

Proof. Since it is not difficult to see the 'only if' part of the lemma, we show the 'if' part. Let M be a matching in G_D which covers all nodes of Y. Let A^+ (resp. A^-) be the set of arcs $a \in A$ such that $x(a)$ is connected with some node of Y^+ (resp. Y^-) by an edge of M. Let T_1 (resp. T_2) be the subgraph (V, A^+) (resp. (V, A^-)) of D. Since M covers all nodes of Y, $|\delta_{T_1}(v)| = 1$ (resp. $|\varrho_{T_2}(v)| = 1$) holds for each $v \in V \setminus \{r_1\}$ (resp. $V \setminus \{r_2\}$). Thus, since D is acyclic, T_1 and T_2 are a spanning r_1-in-arborescence and a spanning r_2-out-arborescence, respectively. Furthermore, since M is a matching, A^+ and A^- are disjoint, which implies T_1 and T_2 are arc-disjoint. This completes the proof of the 'if' part.

The latter half of the lemma immediately follows from the proof of the 'if' part. □

By Lemma 2.3.2, we can discern the existence of a pair of arc-disjoint spanning r_1-in-arborescence and r_2-out-arborescence, and find such arborescences if they exist, by computing a maximum matching of G_D. Hence, we can solve our problem in polynomial time by using bipartite-matching algorithms such as in [69]. However, we show in the subsequent section that we can discern the existence of a matching of G_D which covers all nodes of Y and find such a matching if one exists, in $O(|A|)$ time.

2.3.2 A linear time algorithm

Our goal is to show the following theorem, which implies Theorem 2.3.1 by Lemma 2.3.2.

Theorem 2.3.3. *Given a directed acyclic graph $D = (V, A)$ with roots $r_1, r_2 \in V$, we can discern the existence of a matching in $G_D = (X, Y; E)$ which covers all nodes of Y and find such a matching if one exists, in $O(|A|)$ time.*

In the subsequent arguments, we assume without loss of generality that $d_{G_D}(y) \geq 1$ holds for every $y \in Y$ since if there exists a node $y \in Y$ with $d_{G_D}(y) = 0$, there exists no solution. We divide the proof into two parts corresponding to the following two cases.

Case 1: For every $y \in Y$, $d_{G_D}(y) \geq 2$ holds.
Case 2: There exists $y \in Y$ with $d_{G_D}(y) = 1$.

We first show that in Case 1, there always exists a matching in G_D which covers all nodes of Y, and we can find such a matching in $O(|A|)$ time. Then, we show that in Case 2, we can discern the existence of a matching in G_D which covers all nodes of Y, and reduce the problem to Case 1 if any such matching exists, in $O(|A|)$ time.

Case 1

We prove the following lemma for Case 1.

Lemma 2.3.4. *Given a directed acyclic graph $D = (V, A)$ with roots $r_1, r_2 \in V$, if $d_{G_D}(y) \geq 2$ holds for every $y \in Y$, there always exists a matching in $G_D = (X, Y; E)$ which covers all nodes of Y, and we can find one such matching in $O(|A|)$ time.*

Proof. Let $\hat{G}_D = (X \cup \{s\}, Y; \hat{E})$ be the bipartite graph obtained from G_D by adding a new node s and connecting edges between s and each odd-degree node $y \in Y$ (see Figure 2.5 (a)). It is easy to see that $|\hat{E}| \leq |E| + |Y| = |E| + 2(|V| - 1)$. Furthermore, since $d_{G_D}(x) = 2$ holds for every $x \in X$, we have $|E| = 2|X| = 2|A|$. Hence, $|\hat{E}| = O(|A|)$ holds, and our goal is to find a desired matching in $O(|\hat{E}|)$ time.

Since the sum of the degrees of all nodes $x \in X$ is even, the degree of s in $\hat{G}_D$ is even. This implies that $\hat{G}_D$ is an Eulerian graph. Hence, $\hat{G}_D$ consists of several edge-disjoint cycles (see Figure 2.5 (b)), which can be computed in $O(|\hat{E}|)$ time by using an algorithm for finding Eulerian walk (for a standard algorithm, see [96]). Let $\hat{M}$ be the set of edges of $\hat{G}_D$ obtained from all the cycles by choosing every other edges along the cycles (see Figure 2.5 (b)). Then every node v of $\hat{G}_D$ has $\frac{1}{2}d_{\hat{G}_D}(v)$ edges in $\hat{M}$ that are incident to v. It should be noted that for each odd degree node v in G_D we have $d_{\hat{G}_D}(v) \geq 4$, so that such a node v is incident to at least two edges in $\hat{M}$. Hence, letting $M = \hat{M} \cap E$, M satisfies the following conditions. (Note that M is obtained by removing from $\hat{M}$ the edges incident to s in $\hat{G}_D$.)

A1. M covers all nodes of Y.

A2. Each $x \in X$ is covered by exactly one edge in M.

By Conditions A1. and A2., we can obtain a matching in G_D which covers all nodes of Y by appropriately removing edges from M. This completes the proof. □

Case 2

We show that in Case 2 we can discern the existence of a feasible solution of our problem and reduce the problem to Case 1 if one exists, in $O(|A|)$ time. This will complete the proof of Theorem 2.3.3.

The following lemma asserts that we can reduce Case 2 to Case 1 by greedily removing nodes with degree one.

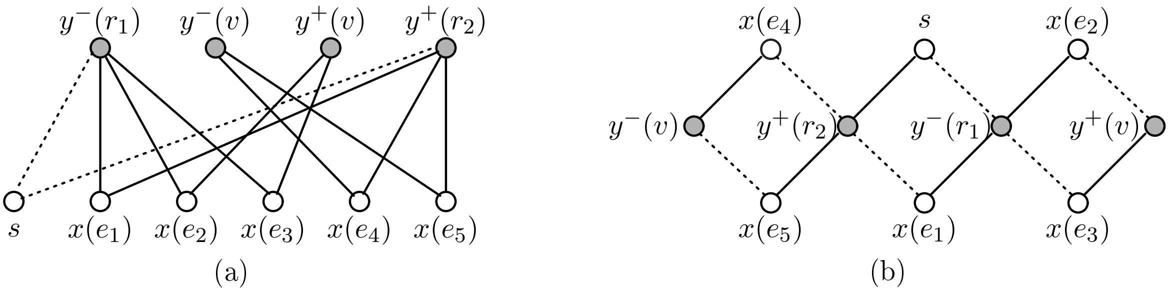

Figure 2.5: (a) A bipartite graph $\hat{G}_D$ obtained from G_D in Figure 2.3 (b). (b) Cycles C_1, C_2 and C_3 in $\hat{G}_D$. The set of dotted lines represents $\hat{M}$.

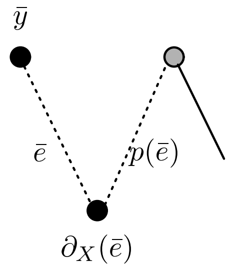

Figure 2.6: Black nodes and dotted edges are removed from G_D.

Lemma 2.3.5. *Suppose that we are given a directed acyclic graph $D = (V, A)$ with roots $r_1, r_2 \in V$, and a node $\bar{y} \in Y$ with $d_{G_D}(\bar{y}) = 1$, denoting by $\bar{e} \in E$ the single edge incident to $\bar{y}$. Let $\bar{G}_D = (\bar{X}, \bar{Y}; \bar{E})$ be the bipartite graph obtained from $G_D = (X, Y; E)$ by removing nodes $\bar{y}$ and $\partial_X(\bar{e})$ and edges $\bar{e}$ and $p(\bar{e})$ (see Figure 2.6). Then, there exists a matching M in G_D which covers all nodes of Y if and only if there exists a matching $\bar{M}$ in $\bar{G}_D$ which covers all nodes of $\bar{Y}$.*

Proof. We first prove the 'if' part. Assume that there exists a matching $\bar{M}$ in $\bar{G}_D$ which covers all nodes of $\bar{Y}$. Then, we can construct a matching M in G_D which covers all nodes of Y by adding $\bar{e}$ to $\bar{M}$.

Next we prove the 'only if' part. Assume that there exists a matching M in G_D which covers all nodes of Y. Since $d_{G_D}(\bar{y}) = 1$, $\bar{e}$ must be included in M, and $p(\bar{e})$ is not included in M. Hence, we can construct a matching $\bar{M}$ in $\bar{G}_D$ which covers all nodes of $\bar{Y}$ by removing $\bar{e}$ from M. □

By Lemma 2.3.5, we can describe the procedure in which we can discern the existence of a feasible solution of our problem, and reduce the problem to Case 1 if one exists, in $O(|A|)$ time as in Procedure 1.

Procedure 1 Processing degree one nodes

1: Compute $d_{G_D}(y)$ for all $y \in Y$, and set $Q = \{y \in Y \mid d_{G_D}(y) = 1\}$ and $M_0 = \emptyset$.
2: **while** $Q \neq \emptyset$ **do**
3: Choose $\bar{y} \in Q$. We denote by $\bar{e}$ the single edge incident to $\bar{y}$. Put $M_0 \leftarrow M_0 \cup \{\bar{e}\}$ and remove $\bar{y}$ from Q. Then, we remove nodes $\bar{y}$ and $\partial_X(\bar{e})$, and edges $\bar{e}$ and $p(\bar{e})$ from G_D. Furthermore, if the degree of $\partial_Y(p(\bar{e}))$ in the updated G_D is equal to one, we add $\partial_Y(p(\bar{e}))$ to Q; if it is equal to zero, we remove $\partial_Y(p(\bar{e}))$ from Q.
4: **end while**
5: **return** G_D and M_0.

It should be noted that since Q contains all nodes $y \in Y$ with $d_{G_D}(y) = 1$ in each iteration of Step 3, the procedure is correct. Furthermore, we can easily see the following lemma, due to Lemma 2.3.5.

Lemma 2.3.6. *Given a directed acyclic graph $D = (V, A)$ with roots $r_1, r_2 \in V$, Procedure 1 always terminates in $O(|A|)$ time. Suppose that Procedure 1 returns a bipartite graph $G'_D = (X', Y'; E')$ and a matching M_0. Then, we have $d_{G'_D}(x) = 2$ for every $x \in X'$ and $d_{G'_D}(y) \neq 1$ for every $y \in Y'$. If there exists a node y in G'_D such that $d_{G'_D}(y) = 0$, then there does not exist a pair of arc-disjoint spanning r_1-in-arborescence and r_2-out-arborescence. Otherwise we can construct a matching M in G_D which covers all nodes of Y, from a matching M' in G'_D which covers all nodes of Y', by putting $M \leftarrow M' \cup M_0$.*

A full description of our algorithm

We are now ready to describe a linear time algorithm for our problem.

1. If there exists $y \in Y$ with $d_{G_D}(y) = 1$, apply Procedure 1 and let G'_D and M_0 be the output of Procedure 1. If there exists a node whose degree is equal to zero in G'_D, return NULL (there exists no feasible solution). Otherwise, put $G_D \leftarrow G'_D$ and go to Step 2.
2. Find a matching M in G_D covering all nodes of Y as described in the proof of Lemma 2.3.4, and put $M \leftarrow M \cup M_0$.
3. Using the matching M in G_D, compute a pair of arc-disjoint spanning r_1-in-arborescence F_1 and r_2-out-arborescence F_2 and return F_1 and F_2.

It follows from Lemmas 2.3.4 and 2.3.6 that the above algorithm can find a matching in G_D which covers all nodes of Y if one exists in $O(|A|)$ time. This completes the proof of Theorem 2.3.3.

2.3.3 An extension to multiple roots

Now we consider the case where we have multiple roots for in-arborescences and out-arborescences, respectively. Suppose that we are given a directed acyclic graph $D = (V, A)$, two disjoint finite index sets I_1 and I_2, and a root $r_i \in V$ for each $i \in I_1 \cup I_2$, where we allow $r_i = r_j$ for distinct i, j. We assume without loss of generality that $\delta_D(r_i) = 0$ (resp. $\varrho_D(r_i) = 0$) holds for each $i \in I_1$ (resp. $i \in I_2$). Let R_1 (resp. R_2) be the set $\{r_i \mid i \in I_1\}$ (resp. $\{r_i \mid i \in I_2\}$). Then we consider the problem of discerning the existence of a set of arc-disjoint r_i-in-arborescences F_i ($i \in I_1$) and r_i-out-arborescences F_i ($i \in I_2$) such that for each $i \in I_1$ (resp. $i \in I_2$) the node set of F_i is $(V \setminus R_1) \cup \{r_i\}$ (resp. $(V \setminus R_2) \cup \{r_i\}$).

In the same manner as in Section 2.3.1, we can see that there exist desired arborescences if and only if there exists a matching which covers all nodes of Y in a bipartite graph $G_D = (X, Y; E)$ defined as follows.

(i′) Node set $|X|$ is given by $X = \{x(a) \mid a \in A\}$, where $|X| = |A|$.

(ii′) Node set Y consists of disjoint sets Y_i^+ ($i \in I_1$) and Y_i^- ($i \in I_2$). For each $i \in I_1$ (resp. $i \in I_2$) , Y_i^+ (resp. Y_i^-) is given by $\{y_i^+(v) \mid v \in V \setminus R_1\}$ (reps., $\{y_i^-(v) \mid v \in V \setminus R_2\}$).

(iii′) The edge set E consists of two sets E^+ and E^-. For each $a \in A$ with $h(a) \notin R_1$ (resp. $t(a) \notin R_2$) and $i \in I_1$ (resp. $i \in I_2$), we connect $x(a)$ and $y_i^+(t(a))$ (resp. $y_i^-(h(a))$) by an edge in E^+ (resp. E^-). For each $a \in A$ with $h(a) \in R_1$ (resp. $t(a) \in R_2$), we connect $x(a)$ and $y_i^+(t(a))$ (resp.

$y_i^-(h(a)))$ for $i \in I_1$ with $h(a) = r_i$ (resp. $i \in I_2$ with $t(a) = r_i$). The edge sets E^+ and E^- contain no other edge.

We can discern the existence of desired arborescences and find them if they exist, by computing a maximum matching in G_D. However, notice that $d_{G_D}(x) \geq 3$ may hold for each $x \in X$, which is different from the case of the problem of finding a pair of an in-arborescence and an out-arborescence. It is left open whether we can find desired arborescences more efficiently than by using existing bipartite matching algorithms.

2.3.4 Thomassen's conjecture

As we have already mentioned, the problem of finding disjoint in- and out-arborescences for a given root node is NP-complete. The following conjecture was proposed by Thomassen [123]. Recall that a digraph D is k-edge-connected if $\kappa(u,v) \geq k$ for each $u, v \in V$.

Conjecture 2.3.7 (Thomassen)**.** *There exists a value k so that in every k-edge-connected directed graph $D = (V, A)$ and for every node $v \in V$, there are disjoint spanning in- and out-arborescences rooted at v.*

It is known that Conjecture 2.3.7 is not true for $k = 2$, but it is still open for $k = 3$. Assume that $D = (V, A')$ is a directed graph and $r \in V$ is a designated root-node for which $D - r$ is acyclic. Then the existence of disjoint spanning in- and out-arborescences rooted at r can be decided easily with a slight modification of the bipartite graph defined in 2.3.1.

Define a bipartite graph $G = (V^+ \cup V^-, A; E)$ where V^+ and V^- are two copies of $V - r$, each node in A corresponds to an arc of D and E contains the edges av^+ and au^- for each $uv = a \in A'$ (if $u, v \neq r$, in other case one of the edges is missing from E). Since $D - r$ is acyclic, a matching covering $V^+ \cup V^-$ corresponds to a pair of disjoint spanning in- and out-arborescences, hence Hall's theorem gives a necessary and sufficient condition. However, as each node in A has degree at most 2, it is easy to see that -for example- $\varrho(v), \delta(v) \geq 2\ \forall v \in V - r$ ensures the existence of such arborescences in this very special case.

Hence a natural idea would be the following. Leave out edges from a highly-edge-connected directed graph in such a way that the resulting graph contains a node covering each directed cycle and every other node has in- and out-degree at least 2. Then the above would imply the existence of disjoint in- and out-arborescences rooted at r. Unfortunately this approach does not work in general. Take the same directed cycle $v_1, \dots, v_{2k}$ k times, do the same with another directed cycle $w_1, \dots, w_{2k}$ and finally add the edges $v_{2i-1}w_{2i-1}$, $w_{2i}v_{2i}$ for $i = 1, \dots, k$. The resulting digraph is clearly k-edge-connected. In order to make each directed cycle going through the same node we have to completely cut through at least one of the cycles by leaving out edges. Then in this cycle a node with in- or out-degree at most 1 certainly appears.

2.4 Covering by arborescences

When can a digraph $D = (V, A)$ be covered by k spanning arborescences of root r? For any subset X of nodes, let $\Gamma^-(X) = \{v \in X$: there is an edge $uv \in A$ for which $u \in V \setminus X\}$ and call this set the

entrance of X. That is, the entrance consists of the head nodes of edges entering X. The following result of [131] may be considered as a covering counterpart of Edmonds' disjoint arborescences theorem.

Theorem 2.4.1 (Vidyasankar). *Let r be a root node of a digraph $D = (V, A)$ so that no edge enters r. It is possible to cover the edge set of D by k r-arborescences if and only if*

$$\varrho(v) \leq k \text{ for every } v \in V - r \tag{2.7}$$

and

$$k - \varrho(X) \leq \sum[k - \varrho(v) : v \in \Gamma^-(X)] \tag{2.8}$$

for every $\emptyset \subset X \subseteq V - r$, where $\Gamma^-(X)$ is the entrance of X.

Theorem 2.4.1 can be proved by using Edmonds' weak theorem. One may be interested in a similar covering counterpart of Theorems 1.1.5 and 1.1.6 as well. The following theorem from [10] shows that such a generalization of Theorem 2.4.1 is indeed valid.

Theorem 2.4.2. *Let $D = (V, A)$ be a digraph and $\{r_1, \dots, r_k\} = R \subseteq V$ be a set of (not necessary distinct) root-nodes. Let $U_i \subseteq V$ be convex sets with $r_i \in U_i$. The edge set A can be covered by r_i-arborescences F_i not leaving U_i if and only if*

$$\varrho(v) \leq p_1(v) \text{ for each } v \in V \tag{2.9}$$

and

$$p_1(X) - \varrho(X) \leq \sum[p_1(v) - \varrho(v) : \ v \in \Gamma^-(X)] \tag{2.10}$$

for every $\emptyset \subset X \subseteq V$, where $\Gamma^-(X)$ denotes the entrance of X and $p_1(X)$ denotes the number of sets U_i's for which $U_i \cap X \neq \emptyset$ and $r_i \notin X$.

Proof. First we prove necessity. Suppose that there are k proper arborescences covering A. We may suppose that F_i spans U_i for each $i \in \{1, \dots, k\}$. Since an arborescence F_i contains no edge entering v if $v = r_i$ or $v \notin U_i$, and one edge entering v if $v \neq r_i$ and $v \in U_i$, the necessity of (2.9) follows immediately.

Necessity of (2.10) can be seen as follows. For each $e \in A$ let $z(e)$ denote the number of arborescences covering e minus 1. Then $z \geq 0$, moreover $\varrho_z(X) + \varrho(X) \geq p_1(X)$ for each $\emptyset \subset X \subseteq V$ and $\varrho_z(v) + \varrho(v) = p_1(v)$ for each $v \in V$. Since each edge entering X has its head in $\Gamma^-(X)$, we have $\varrho_z(X) \leq \sum[\varrho_z(v) : v \in \Gamma^-(X)]$ and these imply

$$p_1(X) - \varrho(X) \leq \varrho_z(X) \leq \sum[\varrho_z(v) : \ v \in \Gamma^-(X)] = \sum[p_1(v) - \varrho(v) : \ v \in \Gamma^-(X)].$$

Now we turn to sufficiency. For every node $v \in V$, give a copy of v to D denoted by v'. For a subset X of V let X' be the copy of X. Add $p_1(v)$ parallel edges from v to v', $p_1(v) - \varrho(v)$ parallel edges from v' to v, and finally $p_1(v)$ parallel edges from u to v' for every edge $uv \in A$. Let D' denote the directed graph thus arising.

If there exist $F'_1, \dots, F'_k$ disjoint arborescences in D' such that F'_i is rooted at r_i and F'_i is spanning $U_i \cup U'_i$ (where U'_i denotes the copy of U_i), then these determine k proper arborescences of D covering A. It is easy to see that for every convex set $X \subseteq V$ in D the union $X \cup X' \subseteq V \cup V'$ is also convex in D'.

In other case, by Fujishige's theorem, there is a subset W of $V \cup V'$ such that $p'(W) > \varrho'(W)$ where $p'(W) = |\{i \in \{1, \dots, k\} :\ (U_i \cup U_i') \cap W \neq \emptyset,\ r_i \notin W\}|$ and $\varrho' = \varrho_{D'}$. We define the following subsets of W: $X = \{v \in V :\ v \in W\}$, $Y = \{v \in V :\ v' \notin W\}$, and $Z = \{v' \in W :\ v \notin W\}$. We have

$$p'(W) \leq p_1(X) + \sum[p_1(v) :\ v' \in Z].$$

On the other hand

$$\varrho_{D'}(W) \geq \varrho(X) + \sum[p_1(v) - \varrho(v) :\ v \in Y] + \sum[p_1(v) :\ v \in \Gamma^-(X) - Y] + \sum[p_1(v) :\ v' \in Z].$$

The explanation of the second sum is that if $v \in \Gamma^-(X) - Y$, then $v' \in W$ also holds. Moreover, there exists, since v is in the entrance, $u \notin W$ such that $uv \in A$, hence there are $p_1(v)$ arcs from u to v'.

From these inequalities we get

$$\begin{aligned} p_1(X) &> \varrho(X) + \sum[p_1(v) - \varrho(v) :\ v \in Y] + \sum[p_1(v) :\ v \in \Gamma^-(X) - Y] \\ &\geq \varrho(X) + \sum[p_1(v) - \varrho(v) :\ v \in \Gamma^-(X)], \end{aligned}$$

contradicting condition (2.10). □

As we have seen, most of the theorems about packing arborescences admit a covering counterpart. It would be natural to find such an extension corresponding to Theorem 1.1.8. A set $\{F_1, \dots, F_{|S|}\}$ of -not necessarily edge-disjoint- arborescences is called a **capacitated maximal $\mathcal{M}$-independent packing of arborescences** if F_i has root $\pi(s_i)$ for $i = 1, \dots, |S|$, the set $\{s_j \in S :\ v \in V(F_j)\}$ is independent in $\mathcal{M}$ and $|\{s_j \in S :\ v \in V(F_j)\}| = r_{\mathcal{M}}(S_{P(v)})$. We propose the following conjecture.

Conjecture 2.4.3. *Let (D, S, π) be a digraph with roots and $\mathcal{M}$ be a matroid on S with rank function $r_{\mathcal{M}}$. It is possible to cover the edge set of D by a capacitated maximal $\mathcal{M}$-independent packing of arborescences if and only if*

$$\varrho(v) \leq r_{\mathcal{M}}(S_{P(v)}) - r_{\mathcal{M}}(S_v) \text{ for every } v \in V \tag{2.11}$$

and

$$\begin{gathered} r_{\mathcal{M}}(S_{P(X)}) - r_{\mathcal{M}}(S_X) - \varrho(X) \leq \\ \sum[r_{\mathcal{M}}(S_{P(v)}) - r_{\mathcal{M}}(S_v) - \varrho(v) : v \in \Gamma^-(X)] \end{gathered}$$

for every $\emptyset \subset X \subseteq V$, where $\Gamma^-(X)$ is the entrance of X.

We only prove necessity.

Proof of necessity. Suppose that there exists a proper covering. Clearly, at most $r_{\mathcal{M}}(S_{P(v)}) - r_{\mathcal{M}}(S_v)$ arborescences not rooted at v contain v, hence (2.11) follows.

Necessity of (2.12) can be seen as follows. For each $e \in A$ let $z(e)$ denote the number of arborescences covering e minus 1. Clearly, $z \geq 0$. As there exists a capacitated maximal $\mathcal{M}$-independent packing of arborescences, we have $\varrho_z(X) + \varrho(X) \geq r_{\mathcal{M}}(S_{P(X)}) - r_{\mathcal{M}}(S_X)$ for each $\emptyset \subset X \subseteq V$ by Theorem 1.1.8.

Moreover, $\varrho_z(v) + \varrho(v) = r_{\mathcal{M}}(S_{P(v)}) - r_{\mathcal{M}}(S_v)$ for each $v \in V$ by the maximality of the packing. Since each edge entering X has its head in $\Gamma^-(X)$, we have $\varrho_z(X) \leq \sum[\varrho_z(v) : \ v \in \Gamma^-(X)]$ and these imply

$$\begin{aligned} r_{\mathcal{M}}(S_{P(X)}) - r_{\mathcal{M}}(S_X) - \varrho(X) &\leq \varrho_z(X) \\ &\leq \sum[\varrho_z(v) : \ v \in \Gamma^-(X)] \\ &= \sum[r_{\mathcal{M}}(S_{P(v)}) - r_{\mathcal{M}}(S_v) - \varrho(v) : \ v \in \Gamma^-(X)]. \end{aligned}$$

□

Chapter 3

Covering intersecting bi-set families

3.1 Proof of Theorem 1.1.6

We start this section by proving Fujishige's theorem (Theorem 1.1.6) based on Theorem 1.2.4.

Proof of Theorem 1.1.6. If the node set of an arborescence F of root r_i intersects a subset $Z \subseteq V - r_i$, then F contains an element entering Z. Therefore if the k edge-disjoint arborescences exist, then Z admits as many entering edges as the number of sets U_i for which $Z \cap U_i \neq \emptyset$ and $r_i \notin Z$, that is, (1.4) is indeed necessary.

Now we prove sufficiency. For brevity, we call a strongly connected component of D an **atom**. It is known that the atoms form a partition of the node set of D and that there is a so-called topological ordering of the atoms so that there is no edge from a later atom to an earlier one. By a **subatom** we mean a subset of an atom. Clearly, a subset $X \subseteq V$ is a subatom if and only if any two elements of X are reachable in D from each other. The following observation is obvious from the definitions.

Proposition 3.1.1. *If a subatom X intersects a convex set U, then $X \subseteq U$.*

Define k bi-set families $\mathcal{F}_i$ for $i = 1, \dots, k$ as follows. Let

$$\mathcal{F}_i := \{(X_O, X_I) : \; X_O \subseteq V - r_i, \; X_I = X_O \cap U_i \neq \emptyset, \; X_I \text{ is a subatom}\}. \tag{3.1}$$

For each bi-set X, let $p_2(X)$ denote the number of $\mathcal{F}_i$'s containing X. It follows immediately that $\mathcal{F}_i$ is an intersecting bi-set family.

Proposition 3.1.2. *The bi-set families $\mathcal{F}_i$ satisfy the mixed intersecting property.*

Proof. Let $X = (X_O, X_I)$ and $Y = (Y_O, Y_I)$ be members of $\mathcal{F}_i$ and $\mathcal{F}_j$, respectively, and suppose that X and Y are intersecting, that is, $X_I \cap Y_I \neq \emptyset$. By Proposition 3.1.1, we have that $X_I = X_O \cap U_i \subseteq U_i \cap U_j$ and $Y_I = Y_O \cap U_j \subseteq U_i \cap U_j$. This implies for sets $Z_O := X_O \cap Y_O$ and $Z_I := X_I \cap Y_I$ that $Z_O \cap U_i = X_O \cap U_i \cap Y_O = X_O \cap U_i \cap Y_O \cap U_j = Z_I$ and also $Z_O \cap U_j = X_O \cap Y_O \cap U_j = X_O \cap U_i \cap Y_O \cap U_j = Z_I$ from which $Z_I \subseteq U_i \cap U_j$ and $(Z_O - Z_I) \cap (U_i \cup U_j) = \emptyset$. Hence $X \cap Y = (Z_O, Z_I) \in \mathcal{F}_i \cap \mathcal{F}_j$, as required. ☐

Proposition 3.1.3. $\varrho(X) \geq p_2(X)$ *for each bi-set X.*

Proof. Let $q := p_2(X)$ and suppose that X belongs to $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_q$. Let $V' := V - (U_1 \cup \ldots \cup U_q)$ and $Z := X_I \cup \{v \in V' : X_I \text{ is reachable from } v\}$.

Let $e = uv$ be an edge of D entering the set Z. Then u cannot be in $V' - Z$ for otherwise X_I would be reachable from u and then u should belong to Z. Therefore u is in $(U_1 \cup \ldots \cup U_q) - Z$. Let U_i be one of the sets $U_1, \dots, U_q$ containing u. We claim that the head v of e must be in X_I. For otherwise we are in a contradiction with the hypothesis that U_i is convex since v is reachable from U_i (along the edge uv) and U_i is also reachable from v since $X_I \subseteq U_i$ is reachable from v.

It follows that the edge e entering the set Z also enters the bi-set $X = (X_O, X_I)$. Therefore $\varrho(X) \geq \varrho(Z)$. By (1.4), we have $\varrho(Z) \geq p_1(Z)$. It follows from the definition of Z that $p_1(Z) \geq q = p_2(X)$, and hence $\varrho(X) \geq p_2(X)$ □

Therefore Theorem 1.2.4 applies and hence the edges of D can be partitioned into subsets $A_1, \dots, A_k$ so that A_i covers $\mathcal{F}_i$ for $i = 1, \dots, k$.

Proposition 3.1.4. *Each A_i includes an r_i-arborescence F_i which spans U_i.*

Proof. If the requested arborescence does not exist for some i, then there is a non-empty subset Z of $U_i - r_i$ so that A_i contains no edge from $U_i - Z$ to Z. Consider a topological ordering of the atoms and let Q be the earliest one intersecting Z. Since no edge leaving a later atom can enter Q, no edge with tail in Z enters Q.

Let $X_O := (V - U_i) \cup (Z \cap Q)$ and $X_I := X_O \cap U_i$. Then $X_I = Z \cap Q$ is a subatom and $X = (X_O, X_I)$ belongs to $\mathcal{F}_i$. Therefore there is an edge $e = uv$ in A_i which enters X. It follows that $v \in X_I \subseteq Z$ and that $u \in U_i - X_I$. Since u is not in Z and not in $V - U_i$, it must be in $U_i - Z$, that is, e is an edge from $U_i - Z$ to $X_I \subseteq Z$, contradicting the assumption that no such edge exists. □

□

It is worth mentioning that Theorem 1.2.4 has an equivalent form that uses T-intersecting families instead of bi-sets [9]. For a subset T of V, we call the set families $\mathcal{F}_1, \dots, \mathcal{F}_k$ **T-intersecting** if

$$X, Y \in \mathcal{F}_i,\ X \cap Y \cap T \neq \emptyset \ \Rightarrow\ X \cap Y, X \cup Y \in \mathcal{F}_i.$$

We say that $\mathcal{F}_1, \dots, \mathcal{F}_k$ satisfy the **mixed T-intersection property** if

$$X \in \mathcal{F}_i,\ Y \in \mathcal{F}_j,\ X \cap Y \cap T \neq \emptyset \ \Rightarrow\ X \cap Y \in \mathcal{F}_i \cap \mathcal{F}_j.$$

Then the equivalent form is as follows.

Theorem 3.1.5. *Let $D = (V, A)$ be a digraph and T a subset of V that contains the head of every edge of D. Let $\mathcal{F}_1, \dots, \mathcal{F}_k$ be T-intersecting families also satisfying the mixed T-intersection property. Then A can be partitioned into subsets $A_1, \dots, A_k$ so that A_i covers $\mathcal{F}_i$ if and only if $\varrho(X) \geq p(X)$ for each non-empty subset X of V where $p(X)$ denotes the number of $\mathcal{F}_i$'s containing X.*

3.2 The capacitated case

Fujishige's theorem can also be reformulated in terms of root-sets and branchings.

Theorem 3.2.1. *Let $D = (V, A)$ be a directed graph and let $\mathcal{R} = \{R_1, \dots, R_k\}$ be a list of k (possibly not distinct) root-sets. Let $U_i \subseteq V$ be convex sets with $R_i \subseteq U_i$. There are edge-disjoint R_i-branchings B_i spanning U_i for $i = 1, \dots, k$ if and only if*

$$\varrho_D(Z) \geq p_1(Z) \text{ for every subset } Z \subseteq V \tag{3.2}$$

where $p_1(Z)$ denotes the number of sets U_i's for which $U_i \cap Z \neq \emptyset$ and $R_i \cap Z = \emptyset$.

In [114] (pp. 920–921), Schrijver presented a strongly polynomial time algorithm to find maximum number of r-arborescences under capacity restrictions. By following his approach, one can find disjoint branchings satisfying the conditions of Theorem 3.2.1 in strongly polynomial time even in the more general case when a demand function is given on the set of root-sets. The approach of [114] does not work directly as it strongly relies on the supermodularity of the set function $p(Z) = \sum[m(R_i) : \ R_i \in \mathcal{R}, \ R_i \cap Z = \emptyset]$. It is easy to see that p_1 is no more supermodular (for that very reason the original proof of Theorem 3.2.1 was far more complicated than the one Lovász gave to Edmonds' theorem).

Theorem 3.2.2. *Let $D = (V, A)$ be a digraph, $g : A \to \mathbb{Z}_+$ a capacity function, $\mathcal{R} = \{R_1, \dots, R_k\}$ a list of root-sets, $\mathcal{U} = \{U_1, \dots, U_k\}$ a set of convex sets with $R_i \subseteq U_i$ and $m : \mathcal{R} \to \mathbb{Z}_+$ a demand function. There is a strongly polynomial time algorithm that finds (if there exist) $m(\mathcal{R})$ disjoint branchings so that $m(R_i)$ of them are spanning U_i with root-set R_i and each edge $e \in A$ is contained in at most $g(e)$ branchings.*

Proof. Define the bi-set function

$$p_2(X) = \begin{cases} \sum[m(R_i) : \ R_i \in \mathcal{R}, \ X_I \cap R_i = \emptyset, \ X_I = X_O \cap U_i] \text{ if } X_I \neq \emptyset \text{ is a subatom,} \\ 0 \text{ otherwise.} \end{cases}$$

By replacing every arc a by $g(a)$ parallel arcs, it follows from the proof of Theorem 1.1.6 using bi-sets that (3.2) is equivalent to requiring that

$$\varrho_g(X) \geq p_2(X) \text{ for every bi-set } X \in \mathcal{P}_2. \tag{3.3}$$

The algorithm gradually increases the set of triples $(R_i, U_i, m(R_i))$ during the algorithm, that is, new root sets may appear and we always assign one of the convex sets to a newly appearing root-set. We may assume that g and m are strictly positive everywhere and (3.3) is satisfied.

We are done if $R_i = U_i$ for each triple so assume that, say, $R_1 \subset U_1$. Let $e = uv$ be an arc with $u \in R_1, \ v \in U_1 \setminus R_1$ and define the following parameter.

$$\mu = \min\big\{g(e), \ m(R_1), \ \min\{\varrho_g(Z) - p_2(Z) : \ e \text{ enters } Z, \ R_1 \cap Z_I \neq \emptyset \text{ or } Z_O \cap U_1 \neq Z_I\}\big\}. \tag{3.4}$$

Proposition 3.2.3. *The value of μ can be determined in strongly polynomial time.*

Proof. Let S denote the atom containing v. Delete those arcs of D that enter a node not in S. Note that if e enters a bi-set Z with $p_2(Z) > 0$ then $\varrho_g(Z)$ does not change during this step. Extend the graph with a new node v_{R_i} for each root set $R_i \in \mathcal{R}$. Add the arcs $v_{R_i}w$ for each $R_i \in \mathcal{R}$ and $w \in U_i \setminus (S \setminus R_i)$ with capacity $m(R_i)$. Moreover, add a source node s with outgoing arcs sv_{R_i} with capacity $m(R_i)$ for $R_i \in \mathcal{R}$, and finally an arc su with infinite capacity. Let $D' = (V', A')$ and g' denote the graph and capacity function thus arising.

Compute a maximum flow in D' from s to v and let C denote a minimum cut containing v. The construction of D' implies that e enters C and if $C \cap R_i \neq \emptyset$ or $C \cap U_i \neq C \cap S$ then $v_{R_i} \in C$ may be assumed. Hence for the bi-set $Z = (Z_O, Z_I) = (C, C \cap S)$ we have

$$\varrho_{g'}(Z) = \varrho_g(Z) + \sum[m(R_i): \ R_i \in \mathcal{R}, \ Z_I \cap R_i \neq \emptyset \text{ or } Z_O \cap U_i \neq Z_I].$$

That is,

$$\begin{aligned}\varrho_{g'}(Z) &= \varrho_g(Z) - \sum[m(R_i): \ R_i \in \mathcal{R}, \ Z_I \cap R_i = \emptyset, \ Z_O \cap U_i = Z_I] + \sum[m(R_i): \ R_i \in \mathcal{R}] \\ &= \varrho_g(Z) - p_2(Z) + m(\mathcal{R}).\end{aligned}$$

Hence a minimum cut defines a bi-set Z such that e enters Z and $\varrho_g(Z) - p_2(Z)$ is minimal. To ensure $R_1 \cap Z_I \neq \emptyset$ or $Z_O \cap U_1 \neq Z_I$, we can run the maximum flow algorithm for each case when v is shrunk together with a node in $U_i \setminus (S \setminus R_i)$. The minimum of these values gives the minimum appearing in (3.4). □

By Theorem 3.2.1, there is an arc e for which μ is strictly positive. Delete $(R_1, U_1, m(R_1))$ from the set of triples, and add the triple $(R_1, U_1, m(R_1) - \mu)$ instead if $m(R_1) - \mu > 0$. Moreover, delete the triple $(R_1 + v, U_1, m(R_1 + v))$ if exists and add the triple $(R_1 + v, U_1, m(R_1 + v) + \mu)$ instead. Finally, revise $g(e)$ by $g(e) - \mu$. Due to the definition of μ, the revised problem also meets (3.3) and we can apply the basic step recursively.

Now we turn to the running time. First we consider phases when the minimum in (3.4) is taken on $g(e)$ or $m(R_1)$. If the minimum is taken on $g(e)$ for some arc e, then the number of arcs with positive capacity decreases which may happen at most $|A|$ times. Note that the set of $(R_i, U_i, m(R_i))$ triples may increase only in these phases. Otherwise, the minimum is taken on $m(R_1)$ meaning that $(R_1, U_1, m(R_1))$ gets out from the set of observed triples. The size of each root-set increases at most $|V|$ times and the set of triples may increase, according to the above, at most $|A|$ times, hence the total number of phases is bounded by $(k + |A|)|V|$.

It only remains to take into account those phases when the minimum is taken on $\min\{\varrho_g(W) - p_1(W): \ e \text{ enters } W, \ R_1 \cap W \neq \emptyset\}$. The advantage of using bi-sets is that p_2 is positively intersecting supermodular on $\mathcal{P}_2$ (this can be seen similarly to Lemma 1.2.5). The collection $\mathcal{C} = \{X \in \mathcal{P}_2 : \ \varrho_g(X) = p_2(X) > 0\}$ of tight bi-sets increases in the considered phases ($\varrho_g(X) > 0$ may be assumed, otherwise the minimum in (3.4) is also taken on $g(e)$ and such phases are already counted).

Let $\mathcal{C}_O(a) = \{X_O : \ X \in \mathcal{C}, \ a \text{ enters } X\}$ for an arbitrary $a \in A$. However, $|\mathcal{C}_O(a)| = |\{X \in \mathcal{C} : a \text{ enters } X\}|$ holds for each a. Indeed, for an arbitrary set Z_O containing v, there is at most one set Z_I such that $v \in Z_I$ and $p_2((Z_O, Z_I)) > 0$. Namely, Z_I must be a subatom and it must arise as the intersection of Z_O and the atom containing v. Hence for each $Z_O \in \mathcal{C}_O(a)$ the corresponding inner set Z_I is uniquely determined. This implies that if a bi-set X becomes tight during the revision step then $X_O \notin \mathcal{C}_O(a)$ before the revision step as otherwise $X \in \mathcal{C}$, a contradiction.

The above immediately implies that if $\mathcal{C}$ increases then also $\mathcal{C}_O(a)$ increases for some $a \in A$. If an edge a enters both $X, Y \in \mathcal{C}$, then $\varrho_g(X \cap Y) > 0$ and $\varrho_g(X \cup Y) > 0$. The submodularity of ϱ_g and positively intersecting supermodularity of p_2 implies that $\mathcal{C}_O(a)$ is a lattice family. As a lattice family $\mathcal{L}$ is uniquely determined by the preorder defined as

$$s \preceq t \ \Leftrightarrow \ \text{each set in } \mathcal{L} \text{ containing } t \text{ also contains } s,$$

if $\mathcal{L}$ increases then $\preceq$ decreases, which can happen at most $|V|^2$ times. Hence $\mathcal{C}_O(a)$ increases at most $|V|^2$ times for each $a \in A$, and the number of phases is $\mathrm{O}(|A||V|^2)$.

Concluding the above, the total number of phases is bounded by $\mathrm{O}((k+|A|)|V|+|A||V|^2)$, which is dominated by $\mathrm{O}(k|V|+|A||V|^2)$. $\square$

3.3 Polyhedral description

Let $D=(V,A)$ be a digraph, $R=\{r_1,\dots,r_k\}$ a set of root-nodes and $\mathcal{U}=\{U_1,\dots,U_k\}$ a set of convex sets with $r_i \in U_i$ for each i. We say that the digraph is **arborescence-packable** (with respect to $\mathcal{U}$) if there are k disjoint arborescences $F_1,\dots,F_k$ so that F_i is an r_i-arborescences spanning U_i. Our next goal is to describe the convex hull of the incidence vectors of arborescence-packable subgraphs of D.

We may suppose that the root nodes $r_1,\dots,r_k$ are distinct, each having exactly one leaving edge and no entering ones. Let $R=\{r_1,\dots,r_k\}$ and $T=V\setminus R$, so $U_i\cap R=\{r_i\}$ for each $r_i\in R$. For every non-empty subset Z of T, let $p_1(Z)$ denote the number of roots r_i for which $Z\cap U_i\neq\emptyset$. In particular, for every $v\in T$, $p_1(v)$ is the number of roots r_i for which $v\in U_i$.

Theorem 1.1.6 can be reformulated as follows.

Theorem 3.3.1. *Let $D=(V,A)$ be a digraph in which R is a set of k root-nodes so that the out-degree and the in-degree of each root-node is one and zero, respectively. Let $T=V\setminus R$ and for each root-node r_i let U_i be a convex set for which $U_i\cap R=\{r_i\}$. Then D is arborescence-packable if and only if $\varrho(Z)\geq p_1(Z)$ for every subset $Z\subseteq T$.*

Define k bi-set families $\mathcal{F}_i$ for $i=1,\dots,k$ as follows. Let

$$\mathcal{F}_i := \{(X_O,X_I):\ X_O\subseteq T,\ X_I=X_O\cap U_i\neq\emptyset,\ X_I \text{ is a subatom}\}.$$

For each bi-set X, let $p_2(X)$ denote the number of $\mathcal{F}_i$'s containing X. It follows immediately that $\mathcal{F}_i$ is an intersecting bi-set family.

Remark 3.3.2. Suppose that the out-degree of the root nodes in R may be larger than one. Let $\mathcal{U}=\{U_1,\dots,U_k\}$ be a set of convex sets so that $U_i\cap R=\{r_i\}$ for each $r_i\in R$. Furthermore, let $m:R\to\mathbb{Z}_+$ be a demand function on the root nodes so that $m(R)=t$. By Fujishige's theorem, there are t disjoint arborescences so that r_i is the root of m_i arborescences spanning U_i if and only if $\varrho(Z)\geq p_1(Z)$ for every subset $Z\subseteq V$ where

$$p_1(Z)=\sum\{m(r_i)|\ r_i\notin Z,\ Z\cap U_i\neq\emptyset\}.$$

In this case the bi-set families should be defined as follows. Let

$$\mathcal{F}_i^j := \{(X_O,X_I):\ X_O\cap T\neq\emptyset,\ X_I=X_O\cap U_i,\ \emptyset\neq X_I\subseteq T \textit{ is a subatom}\},$$

where $i=1,\dots,k$ and $j=1,\dots,m(r_i)$. It is easy to see that $\mathcal{F}_i^j$ is an intersecting bi-set family. However, this form follows from Theorem 3.3.1 by an easy construction. Since the statements are simpler when root nodes has out-degree one, we will use this special form when formulating our result.

Before formulating our result, we prove two useful lemmas exhibiting an interrelation between sets and bi-sets.

Lemma 3.3.3. *For every bi-set $X = (X_O, X_I)$ there is a subset $Z \subseteq T$ for which $p_1(Z) \geq p_2(X)$ and $\Delta^{in}(Z) \subseteq \Delta^{in}(X)$.*

Proof. Let $q := p_2(X)$. If $q = 0$, then $Z := \emptyset$ will do. Suppose that $q \geq 1$ and X belongs to $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_q$. Let $V' := V \setminus (U_1 \cup \ldots \cup U_q)$. We claim that the set $Z := X_I \cup \{v \in V' : X_I \text{ is reachable from } v\}$ satisfies the properties required by the lemma.

One obviously has $p_1(Z) \geq q = p_2(X)$ since Z intersects each of $U_1, \ldots, U_q$. Consider now an edge $e = uv$ of D entering Z. The tail u of e cannot be in $V' \setminus Z$ for otherwise X_I would be reachable from u and then u should belong to Z. Therefore u must be in $(U_1 \cup \ldots \cup U_q) \setminus Z$. Let U_i be one of the sets $U_1, \ldots, U_q$ containing u. Then the head v of e must be in X_I, for otherwise v is reachable from U_i (along the edge uv) and X_I is also reachable from v by the definition of Z but this contradicts the convexity of U_i since $X_I \subseteq U_i$. Hence the edge e entering the set Z also enters the bi-set $X = (X_O, X_I)$. □

Lemma 3.3.4. *For every subset $Z \subseteq T$,there are bi-sets $X_1, \ldots, X_t$ so that $\sum[p_2(X_j) : \ j = 1, \ldots, t] = p_1(Z)$ and $\{\Delta^{in}(X_j) : j = 1, \ldots, t\}$ is a partition of $\Delta^{in}(Z)$.*

Proof. Let $\mathcal{C}_Z := \{C_1, \ldots, C_t\}$ denote the set of atoms of D intersecting Z and assume that its members are arranged in a topological ordering, that is, no edge of D leaving a C_j enters a C_i for which $i < j$. For each $j = 1, \ldots, t$, let $X_j = (X_O^j, X_I^j)$ where $X_O^j := Z \cap (C_1 \cup \ldots \cup C_j)$ and $X_I^j := Z \cap C_j$. We claim that these bi-sets X_j satisfy the properties required by the lemma.

If an edge $e = uv$ enters a bi-set X_j, then its head v is in $Z \cap C_j$ while its tail u must be outside Z by the property of the topological ordering, that is, e enters Z, too. This and the obvious fact that $\{X_I^j : \ j = 1, \ldots, t\}$ forms a partition of Z imply $\{\Delta^{in}(X_j) : \ j = 1, \ldots, t\}$ forms a partition of $\Delta^{in}(Z)$.

Let $\mathcal{U}_Z := \{U \in \mathcal{U} : U \text{ intersects } Z\}$. Note that $|\mathcal{U}_Z|$ has been denoted by $p_1(Z)$ and recall that an atom is either disjoint from or included by a convex set. For $j = 1, \ldots, t$, let

$$\mathcal{U}_Z^j := \{U \in \mathcal{U}_Z : \ j \text{ is the smallest subscript for which } C_j \in \mathcal{C}_Z \text{ and } C_j \subseteq U\}.$$

Some of the $\mathcal{U}_Z^j$'s may be empty but the non-empty ones form a partition of $\mathcal{U}_Z$. For each $j = 1, \ldots, t$, one has $p_2(X_j) = |\mathcal{U}_Z^j|$ and hence

$$p_1(Z) = |\mathcal{U}_Z| = \sum_{j=1}^{t} |\mathcal{U}_Z^j| = \sum_{j=1}^{t} p_2(X_j),$$

as required. □

Consider the following two polyhedra.

$$R_1 := \{x \in \mathbb{R}^A : \ 0 \leq x, \ \varrho_x(Z) \geq p_1(Z) \text{ for every non-empty } Z \subseteq T\}, \tag{3.5}$$

$$R_2 := \{x \in \mathbb{R}^A : \ 0 \leq x, \ \varrho_x(X) \geq p_2(X) \text{ for every non-trivial bi-set } X = (X_O, X_I) \text{ with } X_O \subseteq T\}. \tag{3.6}$$

Lemma 3.3.5. $R_1 = R_2$.

Proof. Suppose first that $x \in R_1$. Let X be an arbitrary bi-set for which $p(X) > 0$. By Lemma 3.3.3 there is a subset $Z \subseteq T$ for which $p_1(Z) \geq p_2(X)$ and $\Delta^{in}(Z) \subseteq \Delta^{in}(X)$. This and the non-negativity of x imply that $\varrho_x(X) \geq \varrho_x(Z) \geq p_1(Z) \geq p_2(X)$ from which $x \in R_2$ follows.

Second, suppose that $x \in R_2$. Let Z be an arbitrary set for which $p_1(Z) > 0$. By Lemma 3.3.4 there are bi-sets $X_1, \dots, X_t$ so that $\sum[p_2(X_j) : j = 1, \dots, t] = p_1(Z)$ and $\{\Delta^{in}(X_j) : j = 1, \dots, t\}$ is a partition of $\Delta^{in}(Z)$. This and the non-negativity of x imply that $\varrho_x(Z) \geq \sum[\varrho_x(X_j) : j = 1, \dots, t] \geq [p_2(X_j) : j = 1, \dots, t] = p_1(Z)$ from which $x \in R_1$ follows. □

The following result was proved in [42].

Theorem 3.3.6 (Frank and Jordán). *Let $D = (V, A)$ be a digraph and p a positively intersecting supermodular bi-set function on V. Let $g : A \to \mathbb{Z}_+ \cup \{\infty\}$ be a capacity function on A so that $\varrho_g(X) \geq p(X)$ for every bi-set. The following linear system for $x \in \mathbb{R}_+$ is totally dual integral (TDI):*

$$\{0 \leq x \leq g, \varrho_x(X) \geq p(X) \text{ for every bi-set } X\}.$$

From this we derive the following.

Theorem 3.3.7. *The linear system written for $x \in \mathbb{R}^A$*

$$\{0 \leq x \leq g,\ \varrho_x(Z) \geq p_1(Z) \text{ for every non-empty } Z \subseteq T\} \tag{3.7}$$

is totally dual integral (TDI). In particular, the convex hull of arborescence-packable subgraphs of D is equal to the following polyhedron:

$$\{x \in \mathbb{R}^A :\ 0 \leq x \leq 1,\ \varrho_x(Z) \geq p_1(Z) \text{ for every non-empty } Z \subseteq T\}. \tag{3.8}$$

Proof. By theorem 3.3.6, the system

$$\{0 \leq x \leq g,\ \varrho_x(X) \geq p_2(X) \text{ for every bi-set } X\} \tag{3.9}$$

is TDI. By Lemma 3.3.5, this and (3.7) define the same polyhedron.

We say that an inequality $qx \geq \beta$ is an integer consequence of a inequality system $Qx \geq p$ if there is an integer vector y so that $yQ = q$ and $yp = \beta$. By elementary properties of TDI systems, it suffices to show that each inequality from (3.9) is an integer combination of inequalities of (3.7). By Lemma 3.3.3, for a bi-set $X = (X_O, X_I)$, there is a subset $Z \subseteq T$ for which $p_1(Z) \geq p_2(X)$ and $\Delta^{in}(Z) \subseteq \Delta^{in}(X)$. Therefore the inequality $\varrho_x(X) \geq p_2(X)$ is indeed a integer consequence of (3.7).

A general result of Edmonds and Giles [35] implies that the polyhedron defined by (3.8) is integral and hence its vertices are $0-1$ vectors. By Theorem 3.3.1, these vertices correspond to the arborescence-packable subgraphs of D. □

3.4 Further remarks

Theorem 1.2.4 gives a common generalization of Szegő's theorem on covering intersecting set families (Theorem 1.2.3) and the theorem of Fujishige on packing disjoint arborescences spanning convex sets (Theorem 1.1.6). Unfortunately, it does not imply the result of Cs. Király (Theorem 1.1.8), hence it would be interesting to formulate a generalization of covering bi-set families using matroids.

We conjecture that some -maybe rather modified- variant of the following conjecture holds.

Conjecture 3.4.1. *Let $D = (V, A)$ be a digraph, $\mathcal{F}_1, \dots, \mathcal{F}_k$ be intersecting families of bi-sets on ground set V satisfying the mixed intersection property, and $\mathcal{M} = (\{1, \dots, k\}, r_{\mathcal{M}})$ be a matroid on ground set $\{1, \dots, k\}$ with rank function $r_{\mathcal{M}}$. For a bi-set X, let $I_X = \{i : X \in \mathcal{F}_i\}$ and assume that $\varrho(X) \geq r_{\mathcal{M}}(I_X)$ for each bi-set X with $X_I = X_O$. Then there are sets $I'_X \subseteq I_X$ for each bi-set X satisfying the following conditions:*

(i) the families $\mathcal{F}'_i = \{X \in \mathcal{F}_i : i \in I'_X\}$ are intersecting and satisfy the mixed intersection property;

(ii) if $I_X \subseteq I_Y$ then $I'_Y \cap I_X \subseteq I'_X$;

(iii) $\varrho(X) \geq |I'_X|$ for each bi-set X;

(iv) $|I'_X| = r_{\mathcal{M}}(I_X)$ for each bi-set X with $X_I = X_O$.

The above conjecture, if it is true, would imply Theorem 1.1.8. Indeed, let (D, S, π) be a digraph with roots and $\mathcal{M}$ be a matroid on $S = \{s_1, \dots, s_k\}$ with rank function $r_{\mathcal{M}}$. Let U_i be the set of nodes reachable from $\pi(s_i)$ in D. Define $\mathcal{F}_i$ as in (3.1). It is easy to see that (1.6) implies $\varrho(X) \geq r_{\mathcal{M}}(I_X)$ for each bi-set X with $X_I = X_O$. By (i), (iii) and Theorem 1.2.4, the edge set can be partitioned in k parts $A_1, \dots, \mathcal{A}_k$ such that A_i covers $\mathcal{F}'_i$. Let $U'_i = \bigcup\{X_I : i \in I'_X\}$. The choice of the $\mathcal{F}_i$'s and (ii) imply that U'_i is convex for each i. However, without (iv) the choice $I'_X = \emptyset$ would satisfy the conditions. If we apply (iv) to non-trivial bi-sets consisting of a single node we get that each node v is contained in $r_{\mathcal{M}}(\{i : v \in U_i\})$ members of the new convex sets. These together imply that A_i contains an arborescences spanning U'_i for each i, and by (iv) these gives a maximal $\mathcal{M}$-independent packing of arborescences.

Chapter 4

Square-free 2-matchings

4.1 Connectivity and square-free 2-matchings

Let $G = (V, E)$ be an undirected graph with node set V and edge set E, and n and m denote the number of nodes and the number of edges, respectively. A cycle C, which is denoted by $C = (v_1, v_2, \ldots, v_l)$, is a subgraph consisting of distinct nodes $v_1, \ldots, v_l$ and edges $v_1v_2, \ldots, (v_{l-1}v_l, v_lv_1$. For a subgraph H of G, the node set and the edge set of H are denoted by V_H and E_H, respectively. Recall that for an integer k, we say that a graph $G = (V, E)$ is k-**connected** if $|V| \geq k + 1$ and $G - X$ is connected for every $X \subseteq V$ with $|X| \leq k - 1$. The **complement graph** of $G = (V, E)$ is the simple graph $\bar{G} = (V, \bar{E})$ such that $uv \in \bar{E}$ if and only if $uv \notin E$ for distinct $u, v \in V$.

The **degree** of a node $v \in V$ in G is the number of edges incident with v. The **degree sequence** of an edge set $F \subseteq E$ is the vector $d_F \in \mathbb{Z}^V$ such that $d_F(v)$ is the number of edges in F incident with v. Note that if a self-loop e is incident with v, e is counted twice. We say that a graph $G = (V, E)$ is **subcubic** (resp. **cubic**) if $d_E(v) \leq 3$ (resp. $d_E(v) = 3$) for every $v \in V$. An edge set $M \subseteq E$ is said to be a **2-matching** (resp. **2-factor**) if $d_M(v) \leq 2$ (resp. $d_M(v) = 2$) for every $v \in V$. In other words, a 2-matching is a node-disjoint collection of paths and cycles. For a simple undirected graph $G = (V, E)$, an edge set $M \subseteq E$ is a **square-free 2-matching** if M is a 2-matching that contains no cycle of length four as a subgraph.

We now look at the properties of the complement graphs of $(n - t)$-connected graphs.

Claim 4.1.1.

1. *G is $(n-2)$-connected if and only if $\bar{G}$ contains no $K_{1,2}$, that is, $\bar{E}$ is a matching.*

2. *G is $(n-3)$-connected if and only if $\bar{G}$ contains neither $K_{1,3}$ nor $K_{2,2}$, that is, $\bar{E}$ is a square-free 2-matching.*

3. *G is $(n-4)$-connected if and only if $\bar{G}$ contains neither $K_{1,4}$ nor $K_{2,3}$, in particular $\bar{G}$ is subcubic.*

Proof. By the definition of k-connectivity, for an integer t, a simple graph $G = (V, E)$ is $(n-t)$-connected if and only if $\bar{G}$ contains no complete bipartite graph with $t+1$ nodes. Since a graph has no $K_{1,d}$ if and only if its maximum degree is at most $d - 1$, we obtain the results. □

In what follows, we deal with simple graphs when we consider the $(n-3)$-connectivity augmentation problem and the square-free 2-matching problem, and so we often omit to declare that the graph is simple. Non-simple graphs appear only when we **shrink** graphs.

Definition 4.1.2 (Shrinking a square)**.** Let $C = (v_1, v_2, v_3, v_4)$ be a cycle of length four in $G = (V, E)$. **Shrinking** of C in G consists of the following operations:

- identify v_1 with v_3, and denote the corresponding node by u_1,
- identify v_2 with v_4, and denote the corresponding node by u_2, and
- identify all edges between u_1 and u_2.

In the obtained graph, the edge between u_1 and u_2 corresponding to E_C is called a **square-edge**.

Let $C_1, C_2, \ldots, C_q$ be node-disjoint cycles of length four, and let $G^\circ = (V^\circ, E^\circ)$ be the graph obtained from $G = (V, E)$ by shrinking $C_1, C_2, \ldots, C_q$. Note that G° might have self-loops and parallel edges, whereas G does not. We also note that if G is subcubic, G° is also subcubic. In a shrunk graph G°, a **square** is a cycle of length four whose nodes are not incident to a square-edge. In other words, a cycle in G° is a square if its corresponding edges in G form a cycle of length four. We say that an edge set in a shrunk graph G° is **square-free** if it contains no square.

4.2 Jump systems

Let V be a finite set. For $u \in V$, we denote by χ_u the **characteristic vector** of u, with $\chi_u(u) = 1$ and $\chi_u(v) = 0$ for $v \in V \setminus \{u\}$. For $x, y \in \mathbb{Z}^V$, a vector $s \in \mathbb{Z}^V$ is called an (x, y)**-increment** if $x(u) < y(u)$ and $s = \chi_u$ for some $u \in V$, or $x(u) > y(u)$ and $s = -\chi_u$ for some $u \in V$.

A **jump system**, introduced by Bouchet and Cunningham [16], is defined as follows.

Definition 4.2.1 (Jump system)**.** A nonempty set $J \subseteq \mathbb{Z}^V$ is said to be a **jump system** if it satisfies an exchange axiom, called the **2-step axiom**:

> For any $x, y \in J$ and for any (x, y)-increment s with $x + s \notin J$, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$.

A set $J \subseteq \mathbb{Z}^V$ is a **constant-parity system** if $x(V) - y(V)$ is even for any $x, y \in J$. Here $x(S) = \sum_{v \in S} x(v)$ for $x \in \mathbb{Z}^V$ and $S \subseteq V$. For constant-parity jump systems, Geelen observed a stronger exchange property:

> **(EXC)** For any $x, y \in J$ and for any (x, y)-increment s, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$ and $y - s - t \in J$.

This property characterizes a constant-parity jump system (see [107] for details).

Theorem 4.2.2 (Geelen)**.** *A nonempty set J is a constant-parity jump system if and only if it satisfies (EXC).*

A constant-parity jump system is a generalization of the base family of a matroid, an even delta-matroid [133, 134], and a base polyhedron of an integral polymatroid (or a submodular system) [47].

The degree sequences of all subgraphs in an undirected graph form a typical example of a constant-parity jump system [16, 102]. Cunningham [25] showed that the set of degree sequences of all C_k-free

2-matchings is a jump system for $k \leq 3$, but not a jump system for $k \geq 5$. Kobayashi, Szabó, and Takazawa [90, 119] showed that it is also a jump system when $k = 4$.

Efficient algorithms for optimization problems on jump systems are studied in [108, 116]. For a set $S \subseteq \mathbb{Z}^V$, we define $\Phi(S) = \max_{v \in V}\{\max_{y \in S} y(v) - \min_{y \in S} y(v)\}$.

Theorem 4.2.3 (Shioura and Tanaka)**.** *Let $J \subseteq \mathbb{Z}^V$ be a finite jump system, and $c \in \mathbb{R}^V$ be a vector. Suppose that a vector $x_0 \in J$ is given, and we can check whether $x \in J$ or not in γ time. Then, we can find a vector $x \in J$ maximizing cx in $O(n^3 \log \Phi(J)\gamma)$ time.*

We can also find a vector maximizing the sum of univariate concave functions efficiently. A univariate function $\phi : \mathbb{Z} \to \mathbb{R}$ is **concave** if it satisfies

$$2\phi(x) \geq \phi(x-1) + \phi(x+1)$$

for any $x \in \mathbb{Z}$. A univariate function ϕ is **convex** if $-\phi$ is **concave**. The following result appeared in [108].

Theorem 4.2.4 (Murota and Tanaka)**.** *Let $J \subseteq \mathbb{Z}^V$ be a finite jump system, and $\phi_v : \mathbb{Z} \to \mathbb{R}$ be a univariate concave function for each $v \in V$. Suppose that a vector $x_0 \in J$ is given, and we can check whether $x \in J$ or not in γ time. Then, we can find a vector $x \in J$ maximizing $\sum_{v \in V} \phi_v(x)$ in $O(n^3\Phi(J)\gamma)$ time.*

Note that Shioura and Tanaka [116] gave an algorithm for the problem that runs in $O(n^4(\log \Phi(J))^2\gamma)$ time. However, if $\Phi(J)$ is fixed, it is slower than the algorithm in Theorem 4.2.4.

4.3 Polynomial time algorithms for the problems

Let γ_1 denote the time to solve the b-factor problem when $b(v) \leq 2$. That is, for a not necessarily simple graph $G = (V, E)$ with $|V| = n$ and a vector $b \in \{0, 1, 2\}^V$, we can determine whether there exists an edge set $F \subseteq E$ such that $d_F = b$ in γ_1 time. It is of the same order as the running time of finding a maximum cardinality matching, and γ_1 is bounded by $O(\sqrt{n}m \log_n \frac{n^2}{m})$ [57]. In subcubic graphs, since $m = O(n)$, we have $\gamma_1 = O(n^{\frac{3}{2}})$.

Our first results are the following.

Theorem 4.3.1. *In subcubic graphs, the square-free 2-matching problem can be solved in $O(n^3\gamma_1)$ time.*

Theorem 4.3.2. *The $(n-3)$-connectivity augmentation problem is solvable in $O(n^3\gamma_1)$ time.*

Theorem 4.3.2 obviously follows from Theorem 4.3.1. Note that we can construct the complement graph in $O(n^2)$ time, which is shorter than $O(n^3\gamma_1)$ time. Our proof for Theorem 4.3.1 is based on the fact that the degree sequences of all square-free 2-matchings in a subcubic graph form a jump system. Let $J_{\rm sq}(G) \subseteq \mathbb{Z}^V$ denote the set of all degree sequences of square-free 2-matchings in G, that is,

$$J_{\rm sq}(G) = \{d_M \mid M \text{ is a simple square-free 2-matching in } G\}.$$

Then the following theorem holds [90, 119].

Theorem 4.3.3 (Kobayashi, Szabó, and Takazawa)**.** *For any subcubic graph G, $J_{\mathrm{sq}}(G)$ is a constant-parity jump system.*

Although a stronger result is given in [90, 119], we give a new proof for this theorem in Section 4.4 which can be extended to the weighted version.

On the other hand, the membership problem of $J_{\mathrm{sq}}(G)$ can be solved in polynomial time, whose proof is given in Section 4.3.1.

Lemma 4.3.4. *Given a subcubic graph $G = (V, E)$ and a vector $x \in \mathbb{Z}^V$, we can determine whether $x \in J_{\mathrm{sq}}(G)$ or not in $O(\gamma_1)$ time.*

By combining Theorems 4.2.3 and 4.3.3 and Lemma 4.3.4, we obtain Theorem 4.3.1. Note that $(0, 0, \dots, 0) \in \mathbb{Z}^V$ is a vector contained in $J_{\mathrm{sq}}(G)$.

We give a faster algorithm for the square-free 2-matching problem in Section 4.3.2, which does not use jump systems. However, the advantage of using a jump system is that we can immediately extend the result to optimization problems with the aid of some results on jump systems.

When the weight function is node-induced on V, the weighted square-free 2-matching problem is the problem of finding a square-free 2-matching M maximizing a linear function of d_M. Therefore, by Theorems 4.2.3 and 4.3.3 and Lemma 4.3.4, we obtain the following corollaries.

Corollary 4.3.5. *The weighted square-free 2-matching problem in subcubic graphs is solvable in $O(n^3\gamma_1)$ time if the weight function is node-induced on V.*

Corollary 4.3.6. *The weighted $(n-3)$-connectivity augmentation problem is solvable in $O(n^3\gamma_1)$ time if the weight function is node-induced on V.*

In the same way as these corollaries, we obtain the following by Theorem 4.2.4.

Corollary 4.3.7. *Let $\phi_v : \mathbb{Z} \to \mathbb{R}$ be a univariate concave function for each $v \in V$. For a subcubic graph $G = (V, E)$, we can find a square-free 2-matching M maximizing*

$$\sum_{v \in V} \phi_v(d_M(v))$$

in $O(n^3\gamma_1)$ time.

Corollary 4.3.8. *Let $\phi_v : \mathbb{Z} \to \mathbb{R}$ be a univariate convex function for each $v \in V$. For an $(n-4)$-connected graph $G = (V, E)$, we can find in $O(n^3\gamma_1)$ time an edge set $E' \subseteq \bar{E}$ minimizing*

$$\sum_{v \in V} \phi_v(d_{E \cup E'}(v))$$

such that $G' = (V, E \cup E')$ is a simple $(n-3)$-connected graph.

4.3.1 Proof of Lemma 4.3.4

In what follows we give a proof for Lemma 4.3.4.

Take a maximal family of node-disjoint cycles $C_1, C_2, \ldots, C_q$ of length four such that $x(v) = 2$ for each $v \in \bigcup V(C_i)$. Obviously, if there is a cycle C_i such that $V(C_i)$ spans a K_4 then $x \notin J_{\mathrm{sq}}(G)$. Thus, we may assume that $V(C_i)$ does not span a K_4.

Let $G^\circ = (V^\circ, E^\circ)$ denote the graph obtained from $G = (V, E)$ by shrinking $C_1, C_2, ..., C_q$ as in Definition 4.1.2. Define $E_1 \subseteq E$ as the set of all shrunk edges, that is, $E_1 = E(C_1) \cup \cdots \cup E(C_q)$, and let $E_0 = E \setminus E_1$. Similarly, define $V_1 \subseteq V$ as the set of all shrunk nodes, that is, $V_1 = V(C_1) \cup \cdots \cup V(C_q)$, and let $V_0 = V \setminus V_1$. Therefore E_0 and V_0 are also subsets of E° and V°, respectively. Note that E° may contain self-loops and also parallel edges.

Let $x^\circ \in \mathbb{Z}^{V^\circ}$ be the vector obtained from x by setting

$$x^\circ(v) = \begin{cases} x(v) & \text{if } v \in V_0, \\ 2 & \text{if } v \in V^\circ \setminus V_0. \end{cases}$$

We will show that $x \in J_{\mathrm{sq}}(G)$ if and only if x° is the degree sequence of some 2-matching in G°.

Let $x \in J_{\mathrm{sq}}(G)$ and let M be a square-free 2-matching in $G = (V, E)$ with $d_M = x$. Note that $|E(C_i) \cap M| = 2$ or $|E(C_i) \cap M| = 3$ for $i = 1, 2, \ldots, p$, because G is subcubic. Let u_1^i and u_2^i denote the nodes arising when shrinking $C_i = (v_1^i, v_2^i, v_3^i, v_4^i)$. Let I denote the set of indices for which $|E(C_i) \cap M| = 3$. Then define M° as

$$M^\circ = (M \cap E_0) \cup \left(\bigcup_{i \in I} \{u_1^i u_2^i\} \right).$$

One can see easily that M° is a 2-matching in G° with $d_{M^\circ} = x^\circ$.

Conversely, let M° be a 2-matching in $G^\circ = (V^\circ, E^\circ)$ with $d_{M^\circ} = x^\circ$. Let $C = (v_1, v_2, v_3, v_4)$ be one of the shrunk cycles and let u_1, u_2 be the corresponding nodes in G°. If $u_1 u_2 \notin M^\circ$ then either $\{v_1 v_2, v_3 v_4\}$ or $\{v_1 v_4, v_2 v_3\}$ can be added to $M^\circ \cap E_0$ without forming a square since G is subcubic (we use here the assumption that $V(C_i)$ does not span a K_4). One can also see that if $u_1 u_2 \in M^\circ$ then three properly chosen edges of C can be added to $M^\circ \cap E_0$ without forming a square (see Figure 4.1). What we do exactly is that we blow up the cycles one by one. In each step we extend the actual 2-matching to a new one in the extended graph using one of the two extensions described above in such a way that the arising 2-matching has no square. Recall that a square is defined as a cycle of length four whose all four nodes are contained in V_0. In this way $M^\circ \cap E_0$ can be extended to a square-free 2-matching M of $G = (V, E)$ with $d_M = x$.

The above reduction can be done in linear time and we can determine whether x° is a degree sequence of a 2-matching or not in $O(\gamma_1)$ time which proves the lemma.

4.3.2 Faster algorithm

In this section we give another algorithm for the square-free 2-matching problem that runs in $O(\gamma_1)$ time. A faster algorithm for the $(n-3)$-connectivity augmentation problem follows from the algorithm. However, in this case, we have to consider the time to construct the complement graph, which is denoted by γ_0. Obviously, γ_0 is bounded by $O(n^2)$, but it depends on how the input graph is represented.

Theorem 4.3.9. *The square-free 2-matching problem in subcubic graphs can be solved in $O(\gamma_1)$ time. The $(n-3)$-connectivity augmentation problem is solvable in $O(\gamma_0 + \gamma_1)$ time, where γ_0 is the time to construct the complement graph.*

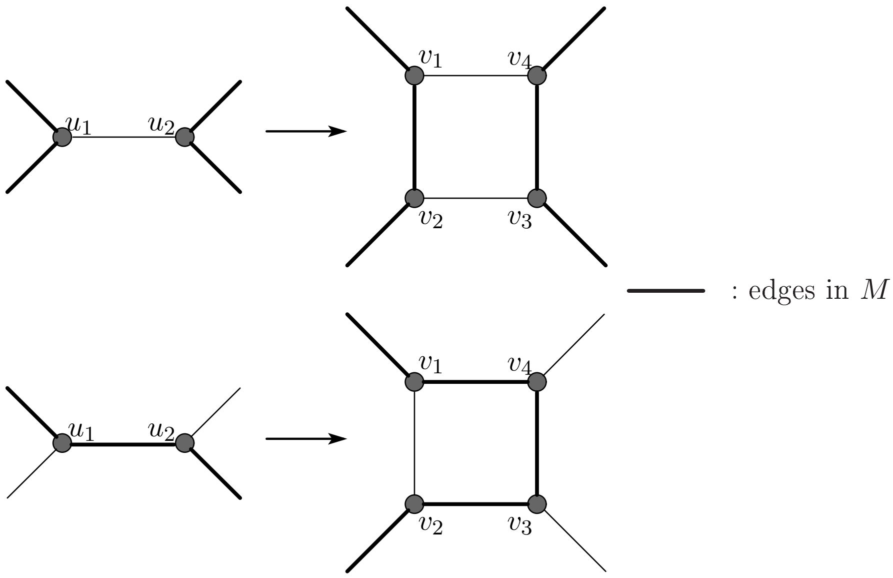

Figure 4.1: Constructing M from M°

Proof. Let $G = (V, E)$ be a subcubic graph. If G contains a complete graph on four nodes then this K_4 forms a component of G since the graph is subcubic. Clearly, a maximum square-free 2-matching contains exactly three edges of such a component. By handling these components separately, we may assume that G contains no K_4.

Take a maximal family of node-disjoint cycles $C_1, C_2, \dots, C_q$ of length four. Our first observation is that for any maximum square-free 2-matching M in G either $|M \cap C_i| = 2$ or $|M \cap C_i| = 3$ for every $C_i = (v_1^i, v_2^i, v_3^i, v_4^i)$. Moreover, we may assume the following:

(A) If $|M \cap C_i| = 2$ then $M \cap C_i = \{v_1^i v_2^i, v_3^i v_4^i\}$ or $\{v_1^i v_4^i, v_2^i v_3^i\}$.

Let $G^\circ = (V^\circ, E^\circ)$ denote the graph obtained from $G = (V, E)$ by shrinking $C_1, C_2, \dots, C_q$. Define E_0, E_1 and V_0, V_1 on the same lines with Lemma 4.3.4.

We will show that for any maximum square-free 2-matching M in G satisfying condition (A) we can find a 2-matching M° in G° with $|M^\circ| = |M| - 2q$. Conversely, for any maximum 2-matching M° in G° we can define a square-free 2-matching M in G so that $|M| = |M^\circ| + 2q$. Since a 2-matching in G° with maximum cardinality can be found in $O(\gamma_1)$ time that would prove the theorem.

The correspondence described in Lemma 4.3.4 works again. Namely, let M be a maximum square-free 2-matching in G satisfying condition (A) and let I denote the set of indices for which $|E(C_i) \cap M| = 3$. Then define M° as

$$M^\circ = (M \cap E_0) \cup \left(\bigcup_{i \in I} \{u_1^i u_2^i\} \right).$$

One can see easily that M° is a 2-matching in G° and the observation above implies $|M^\circ| = |M| - 2q$.

Conversely, let M° be a maximum 2-matching in G°. Let $C = (v_1, v_2, v_3, v_4)$ be one of the shrunk cycles and let u_1, u_2 be the corresponding nodes in G°. If $u_1 u_2 \notin M^\circ$ then either $\{v_1 v_2, v_3 v_4\}$ or $\{v_1 v_4, v_2 v_3\}$ can be added to $M^\circ \cap E_0$ without forming a square since G is subcubic (again, we use here the assumption that G contains no K_4). One can also see that if $u_1 u_2 \in M^\circ$ then three properly chosen edges of C can be added to $M^\circ \cap E_0$ without forming a square. In both cases, the size of the 2-matching increases by two. Hence $M^\circ \cap E_0$ can be extended to a square-free 2-matching M of $G = (V, E)$ with $|M| = |M^\circ| + 2q$.

Now it is understandable why K_4's are handled differently. If we let G contain a K_4 then after shrinking the cycles the K_4 corresponds to an edge with two self-loops at the end-nodes in G°. However, a maximum 2-matching in G° contains the two self-loops and a maximum square-free 2-matching in G contains three edges from the K_4 so in this case the size of the 2-matching increases only by one when blowing back the corresponding cycle.

As above, the square-free 2-matching problem can be reduced to the ordinary maximum 2-matching problem, which can be solved in $O(\gamma_1)$ time.

The latter half of the theorem is immediately derived from the first half. □

4.4 Proof of Theorem 4.3.3

This section is devoted to the proof of Theorem 4.3.3, that is, we show that $J_{\mathrm{sq}}(G)$ is a constant-parity jump system for any subcubic graph G. Recall that G is simple. In this section, we give an algorithm for finding an $(x + s, y)$-increment t such that $x + s + t \in J_{\mathrm{sq}}(G)$ and $y - s - t \in J_{\mathrm{sq}}(G)$. Without loss of generality, we assume that $s = -\chi_u$ for some $u \in V$.

Let M and N be edge sets in an undirected (not necessarily simple) graph. We say that a path $P = (v_0, v_1, v_2, \ldots, v_l)$ is an (M, N)-**alternating path** if

- $v_i v_{i+1} \in M \setminus N$ if i is even,
- $v_i v_{i+1} \in N \setminus M$ if i is odd, and
- $v_i v_{i+1} \neq v_j v_{j+1}$ for $i \neq j$.

Obviously, $d_{M \Delta E(P)} = d_M - \chi_{v_0} + (-1)^l \chi_{v_l}$ and $d_{N \Delta E(P)} = d_N + \chi_{v_0} - (-1)^l \chi_{v_l}$. By taking the longest (M, N)-alternating path, we can see the following.

Lemma 4.4.1. *For 2-matchings M, N in an undirected graph and for a (d_M, d_N)-increment $s = -\chi_u$, there exists an (M, N)-alternating path P beginning with $v_0 = u$ such that both $M \Delta E(P)$ and $N \Delta E(P)$ are 2-matchings (not necessarily square-free), $d_{M \Delta E(P)} = d_M + s + t$, and $d_{N \Delta E(P)} = d_N - s - t$ for some $(d_M + s, d_N)$-increment t.*

Let L be a subset of edges and let $C_1, C_2, \ldots, C_q$ be node-disjoint cycles of length four such that $|E(C_i) \cap L| = 3$ for $i = 1, 2, \ldots, p$. If an edge set $L^\circ \subseteq E^\circ$ is obtained from $L \subseteq E$ by shrinking $C_1, C_2, \ldots, C_q$, we say that L° is the **shrunk edge set** of L, and L is an **expanded edge set** of L°. Note that the shrunk edge set L° contains all square-edges in G°.

We now define a map $\phi : \mathbb{Z}^V \to \mathbb{Z}^{V^\circ}$ by

$$(\phi(x))(u) = \sum \{x(v) \mid v \in V,\ v \text{ corresponds to } u\} - 2|\{\text{square-edges incident to } u\}| \tag{4.1}$$

for $x \in \mathbb{Z}^V$ and $u \in V^\circ$. One can see that for an edge set $L \subseteq E$ satisfying that $|E(C_i) \cap L| = 3$ for $i = 1, 2, \ldots, p$, $\phi(d_L)$ is the degree sequence of the shrunk edge set of L. Conversely, the following lemma holds [93].

Lemma 4.4.2 (Kobayashi and Takazawa). *Let $L^\circ \subseteq E^\circ$ be a 2-matching in G° that contains all square-edges and x be a vector in $\{0,1,2\}^V$. If $\phi(x)$ is the degree sequence of L°, there exists an expanded edge set L of L° in G such that $d_L = x$. Furthermore, such L is unique.*

4.4.1 Finding an $(x+s, y-s)$-increment

Although we need an $(x+s, y)$-increment t to prove Theorem 4.3.3, in this subsection, we give a procedure to find an $(x+s, y-s)$-increment t such that $x+s+t \in J_{\mathrm{sq}}(G)$ and $y-s-t \in J_{\mathrm{sq}}(G)$. After that, we modify the procedure to obtain an $(x+s, y)$-increment t in Section 4.4.2.

For given degree sequences $x, y \in J_{\mathrm{sq}}(G)$, take edge sets $M, N \subseteq E$ such that $d_M = x$ and $d_N = y$. Let $s = -\chi_u$ be an (x, y)-increment for some $u \in V$. Let $C_1, C_2, \ldots, C_q$ be node-disjoint cycles of length four in G such that $E(C_i) \subseteq M \cup N$ and $|E(C_i) \cap M| = |E(C_i) \cap N| = 3$ for $i = 1, 2, \ldots, p$. We take such $C_1, C_2, \ldots, C_q$ maximally, and shrink them. Let $G^\circ = (V^\circ, E^\circ)$ be the obtained graph, and let $M^\circ, N^\circ, x^\circ, y^\circ, u^\circ$ and s° be counterparts in G° to M, N, x, y, u and s, respectively.

If $s^\circ = -\chi_{u^\circ}$ is not an (x°, y°)-increment, then G has a square $C = (u, v_1, v_2, v_3)$ such that $d_M(u) = 2$, $d_N(u) = 1$, $d_M(v_2) = 1$, $d_N(v_2) = 2$, and C is shrunk in G°. In this case, $t = \chi_{v_2}$ is an $(x+s, y)$-increment such that $x+s+t \in J_{\mathrm{sq}}(G)$ and $y-s-t \in J_{\mathrm{sq}}(G)$ by Lemma 4.4.2.

Thus, in what follows in this subsection, we only consider the case when $s^\circ = -\chi_{u^\circ}$ is an (x°, y°)-increment. Recall that a square is a cycle of length four whose nodes are not incident to a square-edge. Then, G° satisfy the following condition.

(B) Both edge sets M° and N° contain all square-edges in G°, and G° has no square C such that $E(C) \subseteq M^\circ \cup N^\circ$ and $|E(C) \cap M^\circ| = |E(C) \cap N^\circ| = 3$.

In order to obtain an $(x+s, y-s)$-increment t, it suffices to find an $(x^\circ + s^\circ, y^\circ - s^\circ)$-increment t° and edge sets M^*, N^* in the shrunk graph G° such that M^* and N^* are square-free 2-matchings in G°, $d_{M^*} = x^\circ + s^\circ + t^\circ$, and $d_{N^*} = y^\circ - s^\circ - t^\circ$. This is because a unit vector t corresponding to t° is a desired $(x+s, y-s)$-increment by Lemma 4.4.2. Thus, in what follows, we describe a procedure that finds an $(x^\circ + s^\circ, y^\circ - s^\circ)$-increment t° and edge sets M^*, N^* in G°.

Let $P = (v_0, v_1, v_2, \ldots, v_l)$ be an (M°, N°)-alternating path beginning with $v_0 = u^\circ$ such that both $M^\circ \Delta E(P)$ and $N^\circ \Delta E(P)$ are 2-matchings, $d_{M^\circ \Delta E(P)} = d_{M^\circ} + s^\circ + t^\circ$, and $d_{N^\circ \Delta E(P)} = d_{N^\circ} - s^\circ - t^\circ$ for some $(x^\circ + s^\circ, y^\circ)$-increment t°. The existence of such a path is guaranteed by Lemma 4.4.1. We choose v_1 such that $N + v_0 v_1$ is square-free if possible. Furthermore, we assume the minimality of P, that is, any subpath $(v_0, v_1, v_2, \ldots, v_p)$ does not satisfy the above conditions for $1 \le p \le l-1$. Let $P^{(p)}$ be the subpath $(v_0, v_1, v_2, \ldots, v_p)$ of P, and define $M^{(p)} = M^\circ \Delta E(P^{(p)})$ and $N^{(p)} = N^\circ \Delta E(P^{(p)})$.

If $M^{(l)}$ and $N^{(l)}$ are square-free, then $t^\circ := d_{M^{(l)}} - d_{M^\circ} - s^\circ$ is an $(x^\circ + s^\circ, y^\circ)$-increment by the definition of P, and $M^{(l)}, N^{(l)}$, and t° are the desired outputs. Otherwise, let p be the integer such that $M^{(0)}, M^{(1)}, \ldots, M^{(p)}$ and $N^{(0)}, N^{(1)}, \ldots, N^{(p)}$ are square-free, and $M^{(p+1)}$ or $N^{(p+1)}$ contains a square.

We consider the case when p is even, that is, $M^{(p+1)}$ is square-free and $N^{(p+1)}$ has a square containing $v_p v_{p+1}$. The case when p is odd can be dealt with in the same way. Let $C_1 = (v_{p+1}, v_p, u_1, u_2)$ be the square in $N^{(p+1)}$. When $p \ge 1$, by the minimality of l, $M^{(p)}$ is not a 2-matching, that is, $d_{M^{(p)}}(v_p) = 3$. Therefore $\{v_p v_{p+1}, v_p u_1\} \subseteq M^{(p)}$, because G° is subcubic. Furthermore, $\{v_p v_{p+1}, v_p u_1\} \subseteq M^{(p)}$ is also true when $p = 0$ by the following claim and the definition of P.

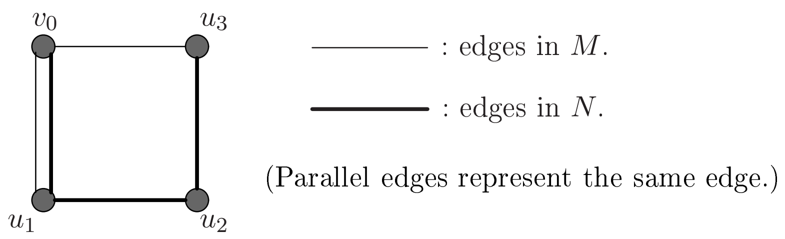

Figure 4.2: An illustration of Claim 4.4.3.

Claim 4.4.3. *One of the followings holds:*

- *there exists an edge $e \in \delta(v_0) \cap (M^\circ \setminus N^\circ)$ such that $N^\circ \cup \{e\}$ is square-free, or*
- *G° has a square $C = (v_0, u_1, u_2, u_3)$ such that $\{v_0u_1, v_0u_3\} \subseteq M^\circ$ and $\{v_0u_1, u_1u_2, u_2u_3\} \subseteq N^\circ$ (see Figure 4.2).*

Proof. It is obvious because $d_{M^\circ}(v_0) > d_{N^\circ}(v_0)$. □

Then, by the condition (B), $v_{p+1}u_2, u_1u_2 \notin M^{(p)}$. Since the graph is subcubic and $v_{p+1}u_2, u_1u_2 \notin M^{(p)}$, we have $d_{M^{(p)}}(u_2) \leq 1$.

Now we define

$$M' = M^{(p)} - v_pv_{p+1} + v_{p+1}u_2,$$
$$N' = N^{(p)} + v_pv_{p+1} - v_{p+1}u_2$$

(see Figure 4.3). Obviously, N' is square-free. Since $d_{M^{(p)}}(u_2) \leq 1$ and $d_{N^{(p)}}(u_2) = 2$, M' and N' are 2-matchings and $d_{M'} - d_{M^\circ} - s^\circ = \chi_{u_2}$ is a $(d_{M^\circ} + s^\circ, d_{N^\circ} - s^\circ)$-increment. Therefore, if M' is square-free, then M' and N' are the desired 2-matchings and $t^\circ = \chi_{u_2}$ is the desired unit vector.

Otherwise, M' has a square $C_2 = (v_{p+1}, u_2, u_3, u_4)$ containing $v_{p+1}u_2$. Then, the following claim holds.

Claim 4.4.4. *$u_3 \neq v_p$.*

Proof. Assume that $u_3 = v_p$. Since $v_pu_1 \in M'$, we have $u_1 = u_4$ and $u_1v_{p+1} \in M'$. Then, $|M^\circ \cap E[C_2]| + |N^\circ \cap E[C_2]| = |M' \cap E[C_2]| + |N' \cap E[C_2]| = 7$, where $E[C_2]$ is the set of edges whose end-nodes are both in $V(C_2)$. This contradicts that M° and N° are square-free 2-matchings. □

By this claim, $\{u_3, u_4\} \cap \{v_p, v_{p+1}\} = \emptyset$. Now we define

$$M'' = M' - u_2u_3, \qquad N'' = N' + u_2u_3$$

(see Figure 4.4). Obviously, M'' is a square-free 2-matching. Furthermore, N'' is square-free, because N'' contains u_3u_2, u_2u_1, u_1v_p, v_pv_{p+1}, which means that it has no square containing u_2u_3. If $d_{N'}(u_3) \leq 1$, then M'' and N'' are the desired 2-matchings and $t^\circ = -\chi_{u_3}$ is the desired unit vector, because $d_{M'}(u_3) = 2$.

Otherwise, $d_{N'}(u_3) = 2$ and $d_{N''}(u_3) = 3$. Since G° is subcubic, $u_3u_4 \in N'$.

Claim 4.4.5. *$u_4v_{p+1} \notin N'$.*

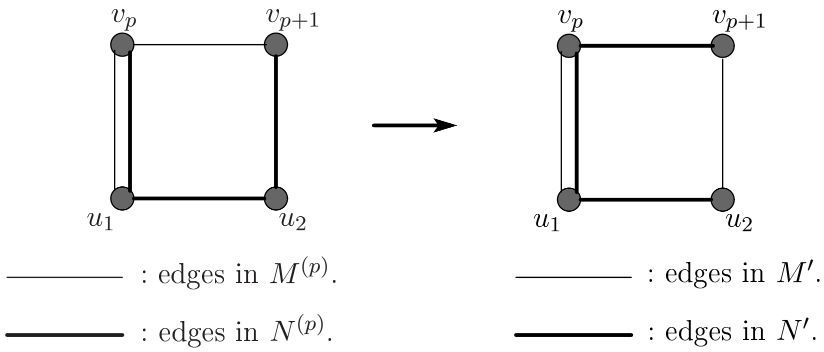

Figure 4.3: Definitions of M' and N'.

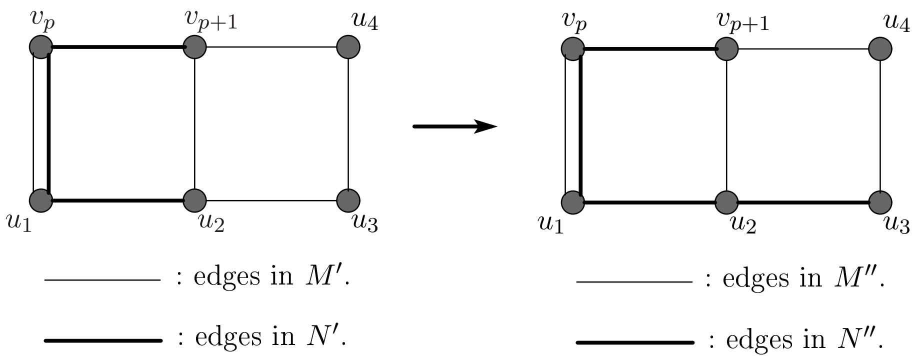

Figure 4.4: Definitions of M'' and N''.

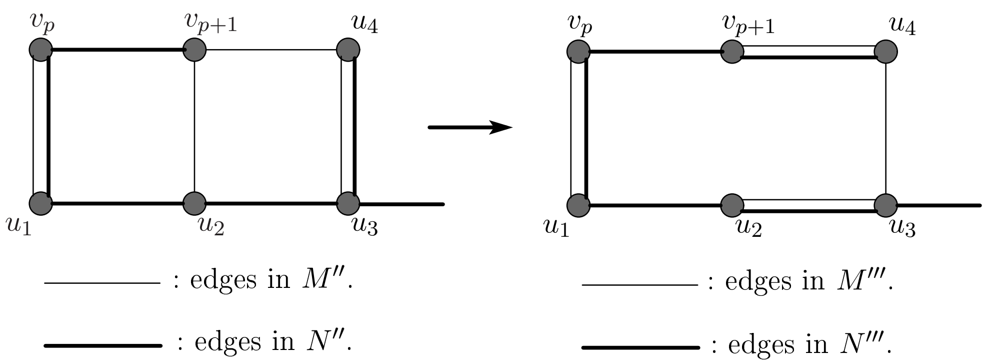

Figure 4.5: Definitions of M''' and N'''.

Proof. If $u_4v_{p+1} \in N'$, then $|M^\circ \cap E(C_2)| + |N^\circ \cap E(C_2)| = |M' \cap E(C_2)| + |N' \cap E(C_2)| = 6$, which contradicts the condition (B). □

We define

$$M''' = M'' - u_2v_{p+1} + u_2u_3,$$
$$N''' = N'' - u_3u_4 + u_4v_{p+1}$$

(see Figure 4.5). Then, $\delta(v_{p+1}) \cap M''' = \{v_{p+1}u_4\}$ and $\delta(v_{p+1}) \cap N''' = \{v_pv_{p+1}, v_{p+1}u_4\}$. Hence M''' and N''' are square-free 2-matchings and $t^\circ = d_{M'''} - d_{M^\circ} - s^\circ = -\chi_{v_{p+1}}$ is a $(d_{M^\circ} + s^\circ, d_{N^\circ} - s^\circ)$-increment.

4.4.2 Finding an $(x+s, y)$-increment

We have already presented a procedure to find an $(x+s, y-s)$-increment. To obtain an $(x+s, y)$-increment t, we choose M and N satisfying the following assumption.

Assumption 4.4.6. For $x, y \in J_{\rm sq}(G)$, let M and N be square-free 2-matchings with $d_M = x$ and $d_N = y$ maximizing $|M \cap N|$.

We show that under Assumption 4.4.6 we can find an $(x+s, y)$-increment by the procedure in the previous subsection. It suffices to show that we can find an $(x^\circ + s^\circ, y^\circ)$-increment t° in the shrunk graph G°. Note that an $(x^\circ + s^\circ, y^\circ - s^\circ)$-increment t° is not an $(x^\circ + s^\circ, y^\circ)$-increment if and only if $t^\circ = -s^\circ$. We also note that, by Assumption 4.4.6, M° and N° maximize $|M^\circ \cap N^\circ|$ among all square-free 2-matchings in G° such that both of them contain all square-edges and their degree sequences are x° and y°, respectively. Clearly, the modified 2-matchings in our proof contain all square-edges in each step, since the path is alternating and we modify in squares, where a square is a cycle of length four whose nodes are not incident to a square-edge.

Suppose that the output (M^*, N^*, t°) in the previous subsection satisfies that $t^\circ = -s^\circ$, that is, $d_{M^*} = d_{M^\circ}$ and $d_{N^*} = d_{N^\circ}$. Then, either $|M^* \cap N^*| > |M^\circ \cap N^\circ|$ holds or a pair of square-free 2-matchings (M^*, N°) satisfies that $d_{M^*} = x^\circ$, $d_{N^\circ} = y^\circ$, and $|M^* \cap N^\circ| > |M^\circ \cap N^\circ|$. More precisely, the following claims hold.

- If p is even and $(M^*, N^*) = (M', N')$, then $|M^* \cap N^\circ| - |M^\circ \cap N^\circ| \geq |E(P^{(p)}) \cap N^\circ| = \frac{p}{2}$.
- If p is odd (in this case, we alternate M and N in the procedure in the last subsection) and $(M^*, N^*) = (M'', N'')$, then $|M^* \cap N^\circ| - |M^\circ \cap N^\circ| \geq |E(P^{(p+1)}) \cap N^\circ| = \frac{p+1}{2}$.
- If p is odd (in this case, we alternate M and N in the procedure in the last subsection) and $(M^*, N^*) = (M''', N''')$, then $|M^* \cap N^*| - |M^\circ \cap N^\circ| = 1$, because $M^* \cap N^* = ((M^\circ \cap N^\circ) \cup \{(u_2, u_3), (v_{p+1}, u_4)\}) \setminus \{(u_3, u_4)\}$.

This contradicts Assumption 4.4.6.

Thus the output t° is an $(x^\circ + s^\circ, y^\circ)$-increment and its corresponding unit vector $t \in \mathbb{Z}^V$ is an $(x+s, y)$-increment, which completes the proof of Theorem 4.3.3.

4.5 NP-hardness of the weighted problem

The objective of this section is to show the NP-hardness of the weighted square-free 2-matching problem in subcubic graphs. Actually, we show the following stronger result, which extends Z. Király's result for bipartite graphs.

Theorem 4.5.1. *The weighted square-free* 2*-matching problem is NP-hard even if the given graph is cubic, bipartite, and planar.*

First, we show the NP-hardness of the problem of finding a square-free 2-factor of maximum total weight, called the **weighted square-free 2-factor problem**. After that we derive Theorem 4.5.1 from this result.

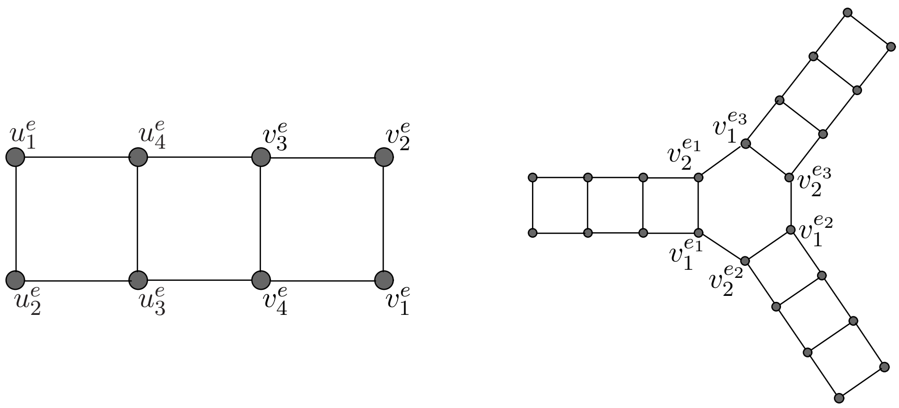

Figure 4.6: Definitions of V^e, E^e, and E^v.

Theorem 4.5.2. *The weighted square-free* 2*-factor problem is NP-hard even if the given graph is cubic, bipartite, and planar.*

Proof. We give a polynomial reduction from the independent set problem in planar cubic graphs to the weighted square-free 2-factor problem. For a graph $G = (V, E)$, a node set $I \subseteq V$ is **independent** if there exists no edge in E connecting two nodes in I. The independent set problem is to find an independent set I of maximum size, and this problem is NP-hard even if the input graph is cubic and planar [54].

Let $G = (V, E)$ be a cubic planar graph which is an instance of the independent set problem. We construct a new graph $G' = (V', E')$ as follows. As shown in Figure 4.6, define a node set V^e and an edge set E^e corresponding to $e = uv \in E$ by

$$\begin{aligned} V^e &= \{u_1^e, u_2^e, u_3^e, u_4^e, v_1^e, v_2^e, v_3^e, v_4^e\}, \\ E^e &= \{u_1^e u_2^e, u_2^e u_3^e, u_3^e u_4^e, u_4^e u_1^e, \\ &\qquad v_1^e v_2^e, v_2^e v_3^e, v_3^e v_4^e, v_4^e v_1^e, u_3^e v_4^e, v_3^e u_4^e\}. \end{aligned}$$

For any node $v \in V$ with $\delta(v) = \{e_1, e_2, e_3\}$, define an edge set E^v by

$$E^v = \{v_1^{e_1} v_2^{e_2}, v_1^{e_2} v_2^{e_3}, v_1^{e_3} v_2^{e_1}\},$$

and define

$$V' = \bigcup_{e \in E} V^e, \qquad E' = \left(\bigcup_{e \in E} E^e\right) \cup \left(\bigcup_{v \in V} E^v\right).$$

Note that E^v is depending on the ordering of e_1, e_2, and e_3, and if three edges in $\delta(v)$ are arranged in an appropriate order for each $v \in V$, then G' is planar. It is obvious that G' is cubic and bipartite.

Set $L = 3|V| + 1$, and define the weight $w : E' \to \mathbb{R}_+$ by

$$w(e') = \begin{cases} L & \text{if } e' = u_1^e u_2^e, v_1^e v_2^e, u_3^e v_4^e, v_3^e u_4^e \text{ for some } e = uv \in E, \\ 1 & \text{if } e' \in E^v \text{ for some } v \in V, \\ 0 & \text{otherwise.} \end{cases}$$

Then the following claim holds.

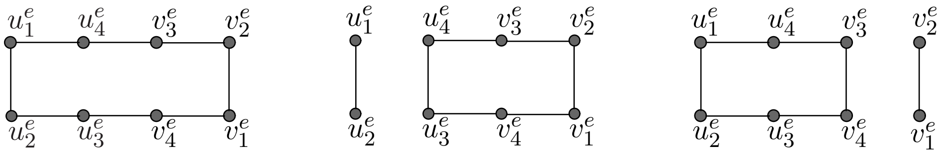

Figure 4.7: Three patterns of $M \cap E^e$.

Claim 4.5.3. *The original graph $G = (V, E)$ has an independent set of size k if and only if $G' = (V', E')$ contains a square-free 2-factor whose total weight is $4|E|L + 3k$.*

Proof of Claim 4.5.3. Let $M \subseteq E'$ be a square-free 2-factor in G' whose total weight is at least $4|E|L$. We show that such a square-free 2-factor in G' and an independent set of G correspond to each other. First, by the definition of L, one can see that M contains all edges of weight L. Then, since M is a square-free 2-factor, we have the following three possibilities for each $e = uv \in E$ (see Figure 4.7):

$$M \cap E^e = \begin{cases} E^e \setminus \{u_3^e u_4^e, v_3^e v_4^e\}, \\ E^e \setminus \{u_1^e u_4^e, u_2^e u_3^e, v_3^e v_4^e\}, \\ E^e \setminus \{v_1^e v_4^e, v_2^e v_3^e, u_3^e u_4^e\}. \end{cases} \tag{4.2}$$

Note that a 2-factor is a collection of cycles covering all nodes.

For a node $v \in V$ with $\delta(v) = \{e_1, e_2, e_3\}$, let C^v be a cycle of length six in G' through $v_1^{e_1}, v_2^{e_1}, v_1^{e_2}, v_2^{e_2}, v_1^{e_3}$, and $v_2^{e_3}$. Then, each cycle in M is contained in E^e for some $e \in E$ or coincides with C^v for some $v \in V$.

Let $V_M \subseteq V$ be a node set defined by $V_M = \{v \mid v \in V,\ E(C^v) \subseteq M\}$. By (4.2), V_M is an independent set of G. On the other hand, when we are given an independent set I of G, we can construct a square-free 2-factor M in G' such that M contains C^v for $v \in I$ and $w(M) \geq 4|E|L$ by (4.2). As above, an independent set I of G and a square-free 2-factor M in G' with $w(M) \geq 4|E|L$ correspond to each other.

Since M contains $3|V_M|$ edges of weight 1, $w(M) = 4|E|L + 3|V_M|$, which shows the claim. □

By this claim, the independent set problem in G is equivalent to the weighted square-free 2-factor problem in (G', w). □

Now we can easily give a proof of Theorem 4.5.1.

Proof of Theorem 4.5.1. Let $G = (V, E)$ and w be an instance of the weighted square-free 2-factor problem. Define a new weight function $w' : E \to \mathbb{R}_+$ by $w'(e) = L + w(e)$, where $L = n(\max_{e \in E} w(e)) + 1$. We consider an instance (G, w') of the weighted square-free 2-matching problem. Then, by the definition of w', the optimal solution M of the weighted square free 2-matching problem must be a 2-factor if $w'(M) \geq nL$, and in this case M is also an optimal solution of the original problem. If $w'(M) < nL$, we can conclude that G has no 2-factors.

Therefore, we can reduce the weighted square-free 2-factor problem to the weighted square-free 2-matching problem, which means that Theorem 4.5.1 can be derived from Theorem 4.5.2. □

Since the graph G' in the proof of Theorem 4.5.2 contains no complete bipartite graph with five nodes (i.e. $K_{1,4}$ and $K_{2,3}$) as a subgraph, its complement graph is $(|V'|-4)$-connected. Hence, we also obtain the following theorem.

Theorem 4.5.4. *The weighted $(n-3)$-connectivity augmentation problem is NP-hard.*

4.6 Weighted square-free 2-matchings

We have already seen in Section 4.5 that the weighted square-free 2-matching problem in subcubic graphs is NP-hard for general weight functions. In this section, we show that the weighted square-free 2-matching problem is polynomially solvable if the weight function is node-induced on every square.

Suppose that for a weighted (not necessarily simple) graph (G, w) and for a vector $x \in \{0,1,2\}^V$, we can find in γ_2 time an edge set $F \subseteq E$ maximizing $w(F)$ such that $d_F = x$. Note that γ_2 is bounded by $O(n(m + n\log n))$ [51] and $O(m\log(nw(E))\sqrt{n\alpha(m,n)\log n})$ [53], where α is the inverse of the Ackermann function.

Theorem 4.6.1. *In a weighted subcubic graph (G, w), if w is node-induced on every square in G, then the weighted square-free 2-matching problem is solvable in $O(n^3\gamma_2)$ time.*

In what follows, we give a proof of Theorem 4.6.1. In our proof, we show the relation between the weighted square-free 2-matching problem and M-concave functions, which are a quantitative extension of jump systems.

4.6.1 M-concave functions

An **M-concave** (**M-convex**) function on a constant-parity jump system is a quantitative extension of a jump system, which is a generalization of valuated matroids [28, 30], valuated delta-matroids [29], and M-concave (M-convex) functions on base polyhedra [105, 106].

Definition 4.6.2 (M-concave function on a constant-parity jump system [107])**.** For $J \subseteq \mathbb{Z}^V$, we call $f : J \to \mathbb{R}$ an **M-concave function on a constant-parity jump system** if it satisfies the following exchange axiom:

(M-EXC) For any $x, y \in J$ and for any (x, y)-increment s, there exists an $(x+s, y)$-increment t such that $x+s+t \in J$, $y-s-t \in J$, and $f(x)+f(y) \leq f(x+s+t)+f(y-s-t)$.

It directly follows from (M-EXC) that J satisfies (EXC), and hence J is a constant-parity jump system. We call a function $f : J \to \mathbb{R}$ an **M-convex function** if $-f$ is an M-concave function on a constant-parity jump system. M-concave functions on constant-parity jump systems appear in many combinatorial optimization problems such as the weighted matching problem, the minsquare factor problem [2], and the weighted even factor problem in odd-cycle-symmetric digraphs [94]. Some properties of M-concave functions are investigated in [89], and efficient algorithms for maximizing an M-concave function on a constant-parity jump system are given in [108, 116].

Theorem 4.6.3 (Murota and Tanaka). *Let $J \subseteq \mathbb{Z}^V$ be a finite constant-parity jump system, and $f : J \to \mathbb{Z}$ be an M-concave function on J. Suppose that a vector $x_0 \in J$ is given, and we can check whether $x \in J$ or not and evaluate $f(x)$ in γ time. Then we can find a vector $x \in J$ maximizing $f(x)$ in $O(n^3\Phi(J))\gamma)$ time.*

Note that $O(n^4(\log \Phi(J))^2\gamma)$ time algorithm is proposed in [116] also for this problem.

4.6.2 Relation with M-concave functions

We consider a generalization of Theorem 4.3.3. For a weighted subcubic graph (G, w), define a function $f_{\rm sq}$ on $J_{\rm sq}(G)$ by

$$f_{\rm sq}(x) = \max\left\{\sum_{e\in M} w(e) \;\middle|\; M \text{ is a square-free 2-matching, } d_M = x\right\}.$$

Then, the following theorem holds.

Theorem 4.6.4. *For a weighted subcubic graph (G, w), if w is node-induced on every square in G, $f_{\rm sq}$ is an M-concave function on the constant-parity jump system $J_{\rm sq}(G)$.*

In what follows, we give a proof of this theorem. In a similar way as Theorem 4.3.3, we use the procedure in Section 4.4.1 to find an $(x + s, y)$-increment t satisfying (M-EXC) for given x, y, and s. We now consider the weight of the output. Define $E_1 \subseteq E$ as the set of all shrunk edges, that is, $E_1 = E(C_1) \cup \cdots \cup E(C_q)$, and let $E_0 = E \setminus E_1$. Define $w(F) = \sum_{e\in F} w(e)$ for $F \subseteq E$. Then the following lemma holds.

Lemma 4.6.5. *Let M and N be square-free 2-matchings in G, whose shrunk edge sets in G° are M° and N°, respectively. Let M^*, N^* be square-free 2-matchings in G° obtained from M and N by the procedure in Section 4.4.1. Then, $w(M^* \cap E_0) + w(N^* \cap E_0) = w(M^\circ \cap E_0) + w(N^\circ \cap E_0)$.*

Proof. If $(M^*, N^*) = (M^{(l)}, N^{(l)}), (M', N'), (M'', N'')$, then $M^* + N^* = M^\circ + N^\circ$, where '+' means the union when we consider the multiplicity of the edges. Hence, $w(M^*\cap E_0)+w(N^*\cap E_0) = w(M^\circ\cap E_0)+ w(N^\circ\cap E_0)$. If $(M^*, N^*) = (M''', N''')$ then $M^*+N^* = M^\circ+N^\circ-\{u_2v_{p+1}, u_3u_4\}+\{u_2u_3, v_{p+1}u_4\}$, where '$-$' means the difference of sets when we consider the multiplicity of the edges. Since w is node-induced on $v_{p+1}u_2, u_3u_4$, we have $w(M^* \cap E_0) + w(N^* \cap E_0) = w(M^\circ \cap E_0) + w(N^\circ \cap E_0)$. □

Lemma 4.6.6. *Let M^*, N^* and t° be the outputs of the procedure in Section 4.4.1. Suppose that M^{**} and N^{**} are square-free 2-matchings which are expanded edge sets of M^* and N^*, respectively, and t is a $(d_M + s, d_N - s)$-increment corresponding to t° such that $d_{M^{**}} = d_M + s + t$ and $d_{N^{**}} = d_N - s - t$. Then, $w(M^{**}) + w(N^{**}) = w(M) + w(N)$.*

Proof. By Lemma 4.6.5, it suffices to show that

$$w(M^{**} \cap E(C_i)) + w(N^{**} \cap E(C_i)) = w(M \cap E(C_i)) + w(N \cap E(C_i)) \tag{4.3}$$

for any shrunk cycle C_i. Since $d_{M^{**}\cap E_0} + d_{N^{**}\cap E_0} = d_{M\cap E_0} + d_{N\cap E_0}$ and $d_{M^{**}} + d_{N^{**}} = d_M + d_N$, it holds that $d_{M^{**}\cap E(C_i)} + d_{N^{**}\cap E(C_i)} = d_{M\cap E(C_i)} + d_{N\cap E(C_i)}$. Then the equation (4.3) holds because w is node-induced on C_i. □

We are now ready to show Theorem 4.6.4.

Proof of Theorem 4.6.4. For $x, y \in J_{\rm sq}(G)$ and an (x,y)-increment s, let M and N be square-free 2-matchings such that $d_M = x$, $d_N = y$, $w(M) = f_{\rm sq}(x)$, and $w(N) = f_{\rm sq}(y)$. As with Assumption 4.4.6, we assume that M and N maximize $|M \cap N|$ among such 2-matchings.

Let M^{**}, N^{**}, and t be as in Lemma 4.6.6. If t is not an $(x+s, y)$-increment, then $d_{M^{**}} = d_M$ and $d_{N^{**}} = d_N$. Since $w(M^{**}) + w(N^{**}) = w(M) + w(N)$ by Lemma 4.6.6, $w(M^{**}) = w(M)$ and $w(N^{**}) = w(N)$. However, either $|M^{**} \cap N^{**}| > |M \cap N|$ or $|M^{**} \cap N| > |M \cap N|$ holds in the same way as Section 4.4, which contradicts the maximality of $|M \cap N|$. Thus, t is an $(x+s, y)$-increment.

On the other hand, by Lemma 4.6.6, we have

$$\begin{aligned} f_{\rm sq}(x) + f_{\rm sq}(y) &= w(M) + w(N) \\ &= w(M^{**}) + w(N^{**}) \\ &\leq f_{\rm sq}(x+s+t) + f_{\rm sq}(y-s-t). \end{aligned}$$

Hence $f_{\rm sq}$ is an M-concave function on $J_{\rm sq}$. ◻

4.6.3 Polynomial time algorithm

Now we are ready to give a proof of Theorem 4.6.1 with the aid of previous works on M-concave functions. As a generalization of Lemma 4.3.4, we show the following lemma.

Lemma 4.6.7. *Given a weighted subcubic graph (G, w) and a vector $x \in J_{\rm sq}(G)$, we can calculate $f_{\rm sq}(x)$ in $O(\gamma_2)$ time if w is node-induced on every square.*

Proof. Take a maximal family of node-disjoint cycles $C_1, C_2, \ldots, C_q$ of length four such that $x(v) = 2$ for each $v \in \bigcup V(C_i)$. Let $G^\circ = (V^\circ, E^\circ)$ denote the graph obtained from $G = (V, E)$ by shrinking $C_1, C_2, \ldots, C_q$. Let u_1^i and u_2^i denote the nodes arising when shrinking $C_i = (v_1^i, v_2^i, v_3^i, v_4^i)$. Let π_i be a function on $V(C_i)$ such that $w(e) = \pi_i(u) + \pi_i(v)$ for every edge $e = (u, v) \in E(C_i)$, and let π be the function on $\bigcup V(C_i)$ defined by $\pi(v) = \pi_i(v)$ for $v \in V(C_i)$. Since the cycles $C_1, \ldots, C_q$ are disjoint we can define such π. Let E_0, E_1, V_0, V_1 and x° be the same as in the proof of Lemma 4.3.4. We define $w^\circ : E^\circ \to \mathbb{R}$ as follows (see Figure 4.8):

$$w^\circ(e) = \begin{cases} w(e) & \text{when } u, v \in V_0, \\ w(e) - \pi(v) & \text{when } u \in V_0 \text{ and } v \in V^\circ \setminus V_0, \\ w(e) - \pi(u) - \pi(v) & \text{when } u, v \in V^\circ \setminus V_0, \end{cases}$$

for each $e = uv \in E_0$, and

$$w^\circ(e) = \pi(v_1^i) + \pi(v_2^i) + \pi(v_3^i) + \pi(v_4^i)$$

for each $e = u_1^i u_2^i \in E^\circ \setminus E_0$.

We will show that $f_{\rm sq}(x) = f(x^\circ) + \pi(V_1)$ where

$$f(x^\circ) = \max\left\{ \sum_{e \in M^\circ} w^\circ(e) \;\middle|\; M^\circ \text{ is a 2-matching in } G^\circ,\ d_{M^\circ} = x^\circ \right\}.$$

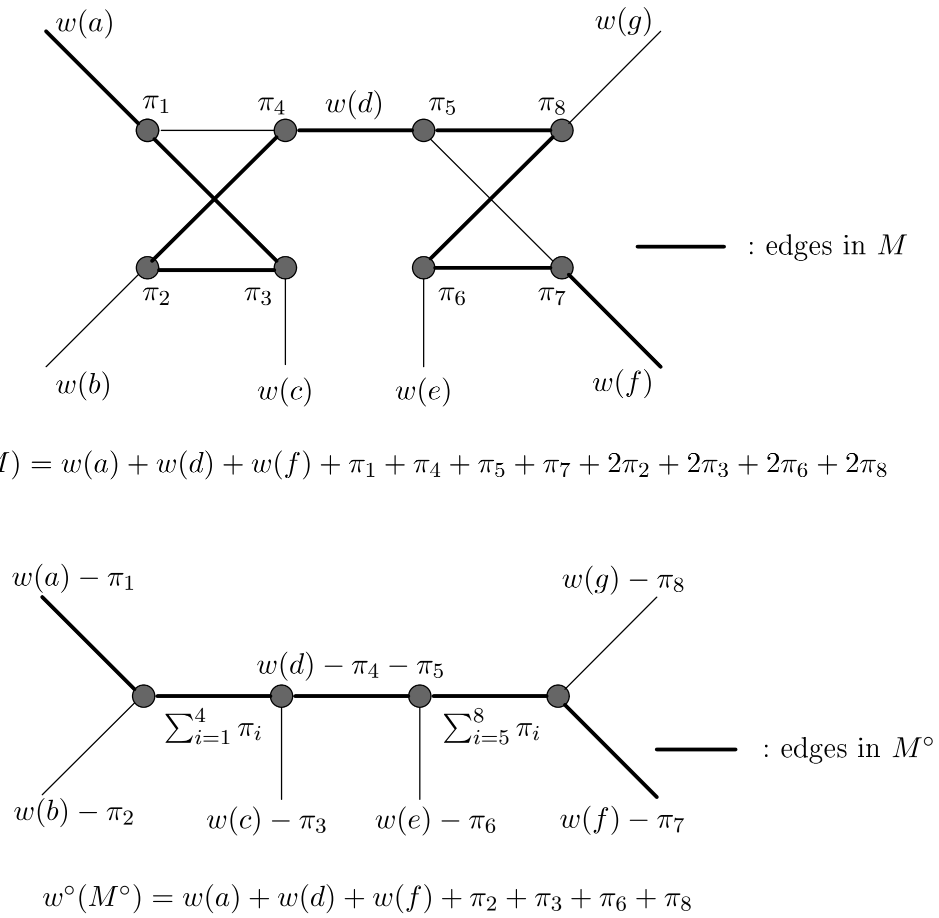

Figure 4.8: Example of $w^\circ(M^\circ)$

Clearly, that would prove the lemma since $f(x^\circ)$ can be calculated in $O(\gamma_2)$ time.

For a square-free 2-matching M with $d_M = x$ we can get a 2-matching M° in G° with $d_{M^\circ} = x^\circ$, and conversely, for any 2-matching M° of G° with $d_{M^\circ} = x^\circ$ we can define a square-free 2-matching M of G with $d_M = x$ as described in Lemma 4.3.4. One only has to observe that for a corresponding pair M, M°, we have $w(M) = w^\circ(M^\circ) + \pi(V_1)$. This means that for any M with $d_M = x$ and $w(M) = f_{\text{sq}}(x)$ we can find an M° with $w^\circ(M^\circ) = f_{\text{sq}}(x) - \pi(V_1)$, and conversely, for any M° with $d_{M^\circ} = x^\circ$ and $w^\circ(M^\circ) = f(x^\circ)$ we can find an M with $w(M) = f(x^\circ) + \pi(V_1)$, hence we are done. □

Theorem 4.6.1 follows from Lemma 4.6.7 and Theorems 4.6.3 and 4.6.4.

4.7 A min-max formula

In this section we give a min-max formula that characterizes the maximum size of a square-free 2-matching in a subcubic graph. The proof is based on the connection between square-free 2-matchings in G and 2-matchings in G° that was described in Section 4.3.

The following characterization of the maximum size of a 2-matching (not necessarily square-free) can be derived from a construction of Tutte [126].

Theorem 4.7.1. *Let $G = (V, E)$ be a graph. The maximum size of a 2-matching in G is equal to the* *minimum value of*

$$\tau_G(U,S) = |V| + |U| - |S| + \sum_T \lfloor \tfrac{1}{2}|E(T,S)|\rfloor, \tag{4.4}$$

where U and S are disjoint subsets of V, S is independent, and T ranges over the components of $G-U-S$.

We drop the subscript G if it is clear from the context. Our first observation is that U can be eliminated from the formula in the subcubic case.

Theorem 4.7.2. *Let $G=(V,E)$ be a subcubic graph. The maximum size of a 2-matching in G is equal to the minimum value of*

$$\tau'_G(S) = |V| - |S| + \sum_T \lfloor \tfrac{1}{2}|E(T,S)|\rfloor, \tag{4.5}$$

where S is an independent subset of V, and T ranges over the components of $G-S$.

Proof. Let U and S be disjoint subsets of V that minimize (4.4). If $U=\emptyset$, then we are done, otherwise take a node $u \in U$. As G is subcubic, $d(u) \leq 3$ and so we have the following cases.

- If u has all of its neighbors in $U \cup S$, then u is a component of $G-(U-u)-S$ and $\lfloor \frac{1}{2}|E(u,S)|\rfloor \leq 1$. Hence $\tau(U-u,S) \leq \tau(U,S)$.
- If u has exactly one neighbor in $V \setminus (U \cup S)$, then let T be the component of $G-U-S$ containing the neighbor of u. Then $\lfloor \frac{1}{2}|E(T+u,S)|\rfloor \leq \lfloor \frac{1}{2}|E(T,S)|\rfloor + 1$, hence $\tau(U-u,S) \leq \tau(U,S)$.
- If u has exactly two neighbors in $V \setminus (U \cup S)$, then we have two subcases. If these neighbors are contained in the same component T of $G-U-S$ then $\lfloor \frac{1}{2}|E(T+u,S)|\rfloor \leq \lfloor \frac{1}{2}|E(T,S)|\rfloor + 1$ so $\tau(U-u,S) \leq \tau(U,S)$. If the two neighbors are contained in T_1 and T_2, then T_1+T_2+u will form one component of $G-(U-u)-S$. It is easy to see that $\lfloor \frac{1}{2}|E(T_1+T_2+u,S)|\rfloor \leq \lfloor \frac{1}{2}|E(T_1,S)|\rfloor + \lfloor \frac{1}{2}|E(T_2,S)|\rfloor + 1$ which implies $\tau(U-u,S) \leq \tau(U,S)$ again.
- If u has three neighbors in $V \setminus (U \cup S)$, then, depending on the position of these neighbors in the components of $G-U-S$, we may get one from two or three components when leaving u out from U. One can easily check that the sum in (4.4) belonging to the components of $G-U-S$ may increase only by one in each case while the size of U always decreases by one. That means that $\tau(U-u,S) \leq \tau(U,S)$.

The observations above imply that if U and S attain the minimum in (4.4) and the graph is subcubic, then we can make U empty by trimming its nodes one by one so that the value $\tau(U,S)$ does not increase. At the end, we get an independent set S for which $\tau'(S) = \tau(U,S)$, and we are done. □

Now we turn to the min-max formula characterizing the maximum size of a square-free 2-matching. Let G be a subcubic graph, let S be an independent subset of V, and take a set $\mathcal{C}$ of node-disjoint cycles $C_1,\ldots,C_q$ of length four. We define the **$\mathcal{C}$-components** of $G-S$ as follows.

Definition 4.7.3 ($\mathcal{C}$-component)**.** We say that $u,v \in V \setminus S$ are in the same $\mathcal{C}$-component of $G-S$ if and only if one of the followings hold:

- u and v are in the same component of $G-S$, or

- $u \in V(T_1)$, $v \in V(T_2)$ (where T_1 and T_2 are components of $G - S$), and there is a cycle $C = (v_1, v_2, v_3, v_4) \in \mathcal{C}$ such that $v_1 \in V(T_1)$, $v_3 \in V(T_2)$, $v_2, v_4 \in S$.

We say that $C = (v_1, v_2, v_3, v_4) \in \mathcal{C}$ **fits** a $\mathcal{C}$-component T if $v_1, v_3 \in V(T)$ and $v_2, v_4 \in S$.

In other words, a $\mathcal{C}$-component is the union of some components of $G - S$ that are connected with cycles from $\mathcal{C}$ in a special configuration. Using this definition, we can formalize our result.

Theorem 4.7.4. *Let $G = (V, E)$ be a subcubic graph and let $\mathcal{C}$ be a maximal set of node-disjoint cycles of length four. The maximum size of a square-free 2-matching in G is equal to the minimum value of*

$$\tau_G(S) = |V| - |S| + \sum_T \lfloor \tfrac{1}{2}(|E(T, S)| - |\mathcal{C}_T|) \rfloor - |\mathcal{K}|, \tag{4.6}$$

where S is an independent subset of V, T ranges over the $\mathcal{C}$-components of $G - S$, $\mathcal{C}_T \subseteq \mathcal{C}$ denotes the set of cycles fitting T, and $\mathcal{K}$ is the set of K_4's in G.

Seemingly, the minimum value of (4.6) also depends on the choice of $\mathcal{C}$. The theorem implies that we can anyhow take node-disjoint cycles maximally, the minimum value of $\tau_G(S)$ will always be the same, namely, the maximum size of a square-free 2-matching.

Proof. As a K_4 forms a component of G, first we handle such a component separately. Let $K \in \mathcal{K}$ be a K_4-subgraph of G. For an independent set $S \subseteq V$, $|S \cap K| = 0$ or 1 by the definition of independence, and in both cases, $|S \cap K| = \lfloor \frac{1}{2}(|E(K - S, S)| - |\mathcal{C}_{K-S}|) \rfloor$. Thus, a square-free 2-matching M of maximum size satisfies that

$$|M \cap E(K)| = 3 = |K| - |S \cap K| + \lfloor \tfrac{1}{2}(|E(K - S, S)| - |\mathcal{C}_{K-S}|) \rfloor - 1,$$

and hence it suffices to consider the case when G has no K_4 as a subgraph.

First we show that the maximum is not more than the minimum. Let M be a square-free 2-matching and take an independent subset S of V. We claim that for each $\mathcal{C}$-component T of $G - S$, the number of edges in M spanned by $V(T) \cup S$ is at most $|V(T)| + \lfloor \frac{1}{2}(|E(T, S)| - |\mathcal{C}_T|) \rfloor$. Indeed,

$$\begin{aligned} 2|M \cap E(T + S)| &= 2|M \cap E(T)| + 2|M \cap E(T, S)| \\ &\leq 2|M \cap E(T)| + |M \cap E(T, S)| + |E(T, S)| - |\mathcal{C}_T| \\ &\leq 2|V(T)| + |E(T, S)| - |\mathcal{C}_T|. \end{aligned}$$

Here, $T + S$ denotes the graph induced by $V(T) \cup S$. Hence we have

$$\begin{aligned} |M| &\leq \sum_T (|V(T)| + \lfloor \tfrac{1}{2}(|E(T, S)| - |\mathcal{C}_T|) \rfloor) \\ &= |V| - |S| + \sum_T \lfloor \tfrac{1}{2}(|E(T, S)| - |\mathcal{C}_T|) \rfloor. \end{aligned}$$

Now we turn to the reverse inequality. According to the above mentioned, we may assume that G does not contain a K_4. Let $\mathcal{C} = \{C_1, \ldots, C_q\}$ and let $G^\circ = (V^\circ, E^\circ)$ denote the graph obtained from $G = (V, E)$ by shrinking $C_1, C_2, \ldots, C_q$. By Theorem 4.7.2, the maximum size of a 2-matching in G° is equal to the minimum value of

$$\tau'_{G^\circ}(S^\circ) = |V^\circ| - |S^\circ| + \sum_{T^\circ} \lfloor \tfrac{1}{2} |E^\circ(T^\circ, S^\circ)| \rfloor. \tag{4.7}$$

From now let $S^\circ \subseteq V^\circ$ be an independent set attaining the minimum in (4.7). In Section 4.3, we have already shown that the maximum size of a square-free 2-matching in G is equal to $\tau'_{G^\circ}(S^\circ) + 2q$. So we only have to find an independent subset S of V such that $\tau_G(S) = \tau'_{G^\circ}(S^\circ) + 2q$.

Let S denote the set of nodes in V that corresponds to S°. Since no self-loops are incident to nodes in S° by the definition of an independent set, S is obviously independent. We claim that $\tau_G(S) = \tau'_{G^\circ}(S^\circ) + 2q$. To see this, we will blow back the cycles one by one and show that (4.7) increases by two at each step. Assume that some of the cycles are already blown back, and G' and S' are the actual graph and an independent set, while G'' and S'' are those arising after blowing back the next square-edge. We also use the notation $\mathcal{C}'$ and $\mathcal{C}''$ for the set of cycles already blown back.

If the edge has both of its end-nodes in $V' \setminus S'$ then $|V''| = |V'| + 2$, $|S''| = |S'|$ and the set of edges going between S' and $V' \setminus S'$ does not change. Hence $\tau_{G''}(S'') = \tau_{G'}(S') + 2$. Now assume that the square-edge has one of its end-nodes in S' and the other in T' where T' is a $\mathcal{C}'$-component of $G' - S'$. Then we have $|V''| = |V'| + 2$, $|S''| = |S'| + 1$, and $|E(T'', S'')| - |\mathcal{C}''_{T''}| = |E(T', S')| - |\mathcal{C}'_{T'}| + 2$. Hence $\tau_{G''}(S'') = \tau_{G'}(S') + 2$ again, and we are done. □

Remark 4.7.5. It is easy to see that both an algorithm and a min-max theorem can be presented in the slightly more general case when a list of forbidden squares is given in the graph. That is, if we denote by $\mathcal{L}$ the list, we are looking for a maximum $\mathcal{L}$-free 2-matching M where $\mathcal{L}$-free means that M contains at most three edges from each square in $\mathcal{L}$. The only difference is that we have to take node-disjoint cycles of length four maximally from $\mathcal{L}$ and only shrink these cycles.

By using the min-max result, we can prove a special case of a conjecture of Jordán appeared in [79]. To describe the conjecture, first we give some definitions.

We call an ordered pair $L = (Z, \mathcal{P})$ a **clump** of G if Z is a cut of size $k-1$ and $\mathcal{P}$ is a partition of $V \setminus Z$ such that no edge of G joins two distinct member of $\mathcal{P}$. A clump L **covers** a pair of nodes u, v if u and v belong to distinct members of $\mathcal{P}$. A **bush** B is a set of clumps such that each pair of nodes is covered by at most two of them. A bush B covers a pair of nodes if it contains a clump covering them. Two bushes B_1 and B_2 are **disjoint** if no pair of nodes is covered by both of them. Let

$$\sigma(B) = \lceil \tfrac{1}{2} \sum_{(Z,\mathcal{P}) \in B} (|\mathcal{P}| - 1) \rceil.$$

It is easy to see that in order to make G k-connected, one must add a set of at least $\sum_{B \in \mathcal{D}} \sigma(B)$ edges to G for any collection $\mathcal{D}$ of disjoint bushes.

Conjecture 4.7.6 (Jordán)**.** *Let G be a $(k-1)$-connected graph. Then the minimum number of edges that must be added to G to make it k-connected is equal to the maximum value of $\sum_{B \in \mathcal{D}} \sigma(B)$, where the maximum is taken over all sets of pairwise disjoint bushes $\mathcal{D}$ of G.*

The conjecture can be easily verified for $k = n-1$ and $n-2$. Now we show how it follows from our min-max result when $k = n-3$.

Theorem 4.7.7. *Let G be an $(n-4)$-connected graph. Then the minimum number of edges that must be added to G to make it $(n-3)$-connected is equal to the maximum value of $\sum_{B \in \mathcal{D}} \sigma(B)$, where the maximum is taken over all sets of pairwise disjoint bushes $\mathcal{D}$ of G.*

Proof. Obviously, the maximum is at most the minimum. We prove the reverse inequality. Let $\bar{G} = (V, \bar{E})$ be the complement of the graph, which is a subcubic graph. We have already seen that a graph is $(n-3)$-connected if and only if its complement is a square-free 2-matching. Take a maximal family of node-disjoint cycles $C_1, \ldots, C_q$ of length four in $\bar{G}$. However, we know, by the min-max result, that the minimum number of edges that must be added to G to make it $(n-3)$-connected is equal to the maximum value of

$$|\bar{E}| - (|V| - |S| + \sum_T \lfloor \tfrac{1}{2}(|\bar{E}(T,S)| - |\mathcal{C}_T|) \rfloor - |\mathcal{K}|), \tag{4.8}$$

where S is an independent subset of V in $\bar{G}$, T ranges over the $\mathcal{C}$-components of $\bar{G} - S$, and $\mathcal{K}$ is the set of K_4's of $\bar{G}$. Assume that S attains the minimum in (4.8). Let $T_1, \ldots, T_t$ be the $\mathcal{C}$-components of $\bar{G} - S$ intersecting no K_4. We will define a set of disjoint bushes $\mathcal{D}$ of G such that

$$\sum_{B \in \mathcal{D}} \sigma(B) \geq |\bar{E}| - (|V| - |S| + \sum_T \lfloor \tfrac{1}{2}(|\bar{E}(T,S)| - |\mathcal{C}_T|) \rfloor - |\mathcal{K}|), \tag{4.9}$$

which would clearly prove the theorem.

For $i = 1, \ldots, t$, let B_i be the set of the following clumps:

- for $v \in T_i$ with $d_{\bar{G}}(v) = 3$, let L be the star of v, namely $L = (Z, \mathcal{P})$ where $Z = V \setminus (N_{\bar{G}}(v) \cup \{v\})$ and $\mathcal{P} = \{\{v\}, N_{\bar{G}}(v)\}$;
- for a cycle $C = (v_1, v_2, v_3, v_4) \in \mathcal{C}$ fitting T_i, let $L = (Z, \mathcal{P})$ be a clump such that $Z = V \setminus V(C)$ and $\mathcal{P} = \{\{v_1, v_3\}, \{v_2, v_4\}\}$.

Here $N_G(v)$ is the set of nodes adjacent to v in G.

Clearly, these pairs are clumps in G. Moreover, each pair of nodes is covered by at most two of them. Hence the B_i's form a set $\mathcal{D}$ of pairwise disjoint bushes of G. We have

$$\begin{aligned}
\sigma(B_i) &= \lceil \tfrac{1}{2} \sum_{(Z,\mathcal{P}) \in B_i} (|\mathcal{P}| - 1) \rceil \\
&= \lceil \tfrac{1}{2}(|\{v \in V(T_i) : \ d_{\bar{G}}(v) = 3\}| + |\mathcal{C}_{T_i}|) \rceil \\
&\geq \lceil \tfrac{1}{2}(\sum_{v \in T_i} (d_{\bar{G}}(v) - 2) + |\mathcal{C}_{T_i}|) \rceil \\
&= \lceil \tfrac{1}{2}(2|\bar{E}(T_i)| + |\bar{E}(T_i, S)| - 2|V(T_i)| + |\mathcal{C}_{T_i}|) \rceil \\
&= |\bar{E}(T_i)| - |V(T_i)| + \lceil \tfrac{1}{2}(|\bar{E}(T_i, S)| + |\mathcal{C}_{T_i}|) \rceil \\
&= |\bar{E}(T_i + S)| - |V(T_i)| - \lfloor \tfrac{1}{2}(|\bar{E}(T_i, S)| - |\mathcal{C}_{T_i}|) \rfloor
\end{aligned}$$

Note that for a subgraph T of $\bar{G} = (V, \bar{E})$, $\bar{E}(T)$ is the set of edges of T.

For $T \in \mathcal{K}$, the bush B_T will contain a single clump twice. Namely, if $V(T) = \{v_1, v_2, v_3, v_4\}$, then $L = (Z, \mathcal{P})$ is defined by $Z = V \setminus V(T)$ and $\mathcal{P} = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}\}$. Clearly, $\sigma(B_T) = 3$. By summing these values over the bushes defined above we get

$$\begin{aligned}
\sum_{B \in \mathcal{D}} \sigma(B) &\geq \sum_{i=1}^{t} (|\bar{E}(T_i + S)| - |V(T_i)| - \lfloor \tfrac{1}{2}(|\bar{E}(T_i, S)| - |\mathcal{C}_{T_i}|) \rfloor) + 3|\mathcal{K}| \\
&= \sum_T (|\bar{E}(T + S)| - |V(T)| - \lfloor \tfrac{1}{2}(|\bar{E}(T, S)| - |\mathcal{C}_T|) \rfloor) + |\mathcal{K}| \\
&= |\bar{E}| - (|V| - |S| + \sum_T \lfloor \tfrac{1}{2}(|\bar{E}(T,S)| - |\mathcal{C}_T|) \rfloor - |\mathcal{K}|),
\end{aligned}$$

where T ranges over the $\mathcal{C}$-components of $G - S$ and the second equality follows from $|\bar{E}(T + S)| = 6$, $|V(T)| = 4$ if $T \in \mathcal{K}$ and $|\bar{E}(T+S)| = 6$, $|V(T)| = 3$, $|\bar{E}(T, S)| = 3$ if $T + v \in \mathcal{K}$ for some $v \in S$. $\square$

Chapter 5

$K_{t,t}$- and K_{t+1}-free t-matchings

Let $\mathcal{K}$ be a list of forbidden $K_{t,t}$ and K_{t+1} subgraphs where $t \geq 2$ is assumed throughout the chapter. For disjoint subsets X, Y of V we denote by $\mathcal{K}[X]$ and $\mathcal{K}[X, Y]$ the members of $\mathcal{K}$ contained in X and having edges only between X and Y, respectively. That is, $\mathcal{K}[X, Y]$ stands for forbidden $K_{t,t}$'s whose colour classes are subsets of X and Y. Recall that V_K and E_K denote the node-set and edge-set of the forbidden graph $K \in \mathcal{K}$, respectively.

5.1 Main theorem

Before stating our theorem, let us recall the well-known min-max formula on the maximum size of a b-matching (see e.g. [114, Vol A, p. 562.]).

Theorem 5.1.1 (Maximum size of a b-matching)**.** *Let $G = (V, E)$ be a graph with an upper bound $b : V \to \mathbb{Z}_+$. The maximum size of a b-matching is equal to the minimum value of*

$$b(U) + |E[W]| + \sum_T \left\lfloor \tfrac{1}{2}(b(T) + |E[T, W]|) \right\rfloor \tag{5.1}$$

where U and W are disjoint subsets of V, and T ranges over the connected components of $G - U - W$.

Let us now formulate our theorem. There are minor technical difficulties when $t = 2$ that do not occur for larger t. In order to make both the formulation and the proof simpler it is worth introducing the following definitions. We refer to forbidden $K_{2,2}$ and K_3 subgraphs as squares and triangles, respectively.

Definition 5.1.2. For $t = 2$, we call a complete subgraph on four nodes **square-full** if it contains three forbidden squares.

Note that, by assumption (1.10), every square-full subgraph is a connected component of G. We denote the number of square-full components of G by $S(G)$ for $t = 2$, and define $S(G) = 0$ for $t > 2$. It is easy to see that a $\mathcal{K}$-free b-matching contains at most three edges from each square-full component of G. The following definition will be used in the proof of the theorem.

Definition 5.1.3. For $t = 2$, a forbidden triangle is called **square-covered** if its node set is contained in the node set of a forbidden square, otherwise **uncovered**.

The theorem is as follows.

Theorem 5.1.4. *Let $G = (V, E)$ be a graph with an upper bound $b : V \to \mathbb{Z}_+$ and $\mathcal{K}$ be a list of forbidden $K_{t,t}$ and K_{t+1} subgraphs of G so that (1.8), (1.9) and (1.10) hold. Then the maximum size of a $\mathcal{K}$-free b-matching is equal to the minimum value of*

$$b(U) + |E[W]| - |\dot{\mathcal{K}}[W]| + \sum_{T\in\mathcal{P}} \left\lfloor \tfrac{1}{2}(b(T) + |E[T, W]| - |\dot{\mathcal{K}}[T, W]|) \right\rfloor - S(G) \tag{5.2}$$

where U and W are disjoint subsets of V, $\mathcal{P}$ is a partition of the connected components of $G - U - W$ and $\dot{\mathcal{K}} \subseteq \mathcal{K}$ is a collection of node-disjoint forbidden subgraphs.

For fixed $U, W, \mathcal{P}$ and $\dot{\mathcal{K}}$ the value of (5.2) is denoted by $\tau(U, W, \mathcal{P}, \dot{\mathcal{K}})$. It is easy to see that the contribution of a square-full component to (5.2) is always 3 and a maximum $\mathcal{K}$-free b-matching contains exactly 3 of its edges. Hence we may count these components of G separately, so the following theorem immediately implies the general one.

Theorem 5.1.5. *Let $G = (V, E)$ be a graph with an upper bound $b : V \to \mathbb{Z}_+$ and $\mathcal{K}$ be a list of forbidden $K_{t,t}$ and K_{t+1} subgraphs of G so that (1.8), (1.9) and (1.10) hold. Furthermore, if $t = 2$, assume that G has no square-full component. Then the maximum size of a $\mathcal{K}$-free b-matching is equal to the minimum value of*

$$b(U) + |E[W]| - |\dot{\mathcal{K}}[W]| + \sum_{T\in\mathcal{P}} \left\lfloor \tfrac{1}{2}(b(T) + |E[T, W]| - |\dot{\mathcal{K}}[T, W]|) \right\rfloor \tag{5.3}$$

where U and W are disjoint subsets of V, $\mathcal{P}$ is a partition of the connected components of $G - U - W$ and $\dot{\mathcal{K}} \subseteq \mathcal{K}$ is a collection of node-disjoint forbidden subgraphs.

Proof of max $\leq$ min in Theorem 5.1.5. Let M be a $\mathcal{K}$-free b-matching. Then clearly $|M \cap (E[U] \cup E[U, V \setminus U])| \leq b(U)$ and $|M \cap E[W]| \leq |E[W]| - |\dot{\mathcal{K}}[W]|$. Moreover, for each $T \in \mathcal{P}$ we have

$$\begin{aligned} 2 \cdot |M \cap (E[T] \cup E[T, W])| &= 2 \cdot |M \cap E[T]| + 2 \cdot |M \cap E[T, W]| \\ &\leq 2 \cdot |M \cap E[T]| + |M \cap E[T, W]| \\ &+ |E[T, W]| - |\dot{\mathcal{K}}[T, W]| \\ &\leq b(T) + |E[T, W]| - |\dot{\mathcal{K}}[T, W]|. \end{aligned}$$

These together prove the inequality. □

5.2 Shrinking

In the proof of max $\geq$ min we use two shrinking operations to get rid of the $K_{t,t}$ and K_{t+1} subgraphs in $\mathcal{K}$.

Definition 5.2.1 (Shrinking a $K_{t,t}$ subgraph)**.** Let K be a $K_{t,t}$ subgraph of $G = (V, E)$ with colour classes K_A and K_B. **Shrinking** K in G consists of the following operations (see Figure 5.1:

- identify the nodes in K_A, and denote the corresponding node by k_a,
- identify the nodes in K_B, and denote the corresponding node by k_b, and

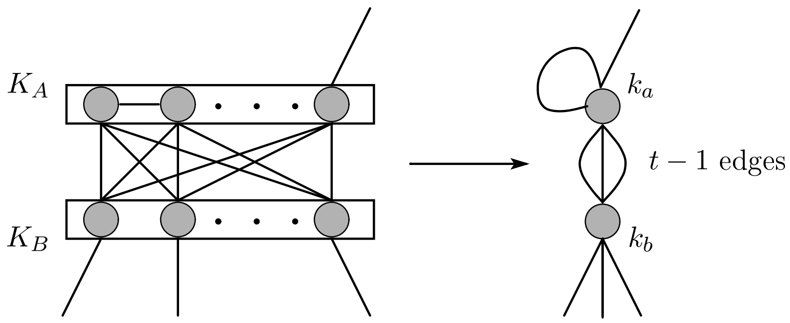

Figure 5.1: Shrinking a $K_{t,t}$ subgraph

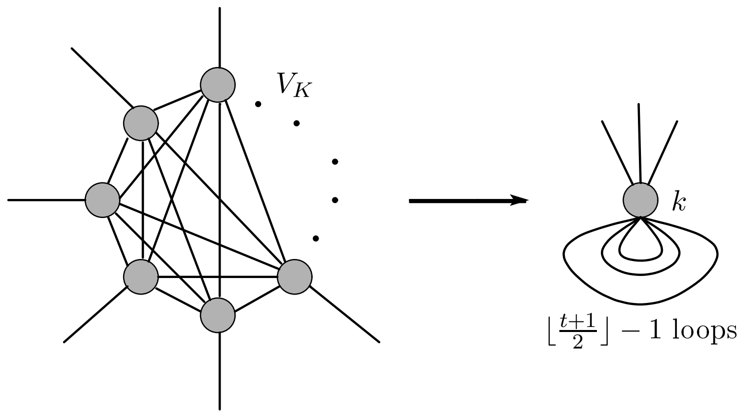

Figure 5.2: Shrinking a K_{t+1} subgraph

- replace the edges between K_A and K_B with $t-1$ parallel edges between k_a and k_b (we call the set of these edges a **shrunk bundle between** k_a **and** k_b).

When identifying the nodes in K_A and K_B, the edges (and also loops) spanned by K_A and K_B are replaced by loops on k_a and k_b, respectively. Each edge $e \in E \setminus E_K$ is denoted by e again after shrinking a $K_{t,t}$ subgraph and is called the **image** of the original edge. By abuse of notation, for an edge set $F \subseteq E \setminus E_K$, the corresponding subset of edges in the contracted graph is also denoted by F. Hence for an edge set $F \subseteq E \setminus E_K$ we have $h_F(K_A) = d_F(k_a)$, $h_F(K_B) = d_F(k_b)$.

Definition 5.2.2 (Shrinking a K_{t+1} subgraph)**.** Let K be a K_{t+1} subgraph of $G = (V, E)$. **Shrinking** K in G consists of the following operations (see Figure 5.2:

- identify the nodes in V_K, and denote the corresponding node by k,

- replace the edges in E_K by $\lfloor \frac{t+1}{2} \rfloor - 1$ loops on the new node k.

Again, for an edge set $F \subseteq E \setminus E_K$, the corresponding subset of edges in the contracted graph is also denoted by F.

We usually denote the graph obtained by applying one of the shrinking operations by $G^\circ = (V^\circ, E^\circ)$. Throughout the section, the graph G, the function b and the list $\mathcal{K}$ of forbidden subgraphs are supposed to satisfy the conditions of Theorem 5.1.5. It is easy to see, by using (1.10), that two members of $\mathcal{K}$ are edge-disjoint if and only if they are also node-disjoint, hence we simply call such pairs **disjoint**.

The following two lemmas give the connection between the maximum size of a $\mathcal{K}$-free b-matching in G and a $\mathcal{K}^\circ$-free b°-matching in G° where b° is a properly defined upper bound on V° and $\mathcal{K}^\circ$ is a list of forbidden subgraphs in the contracted graph.

Lemma 5.2.3. *Let $G^\circ = (V^\circ, E^\circ)$ be the graph obtained by shrinking a $K_{t,t}$ subgraph K. Let $\mathcal{K}^\circ$ be the set of forbidden subgraphs disjoint from K and define b° as $b^\circ(v) = b(v)$ for $v \in V \setminus V_K$ and $b^\circ(k_a) = b^\circ(k_b) = t$. Then the difference between the maximum size of a $\mathcal{K}$-free b-matching in G and the maximum size of a $\mathcal{K}^\circ$-free b°-matching in G° is exactly $t^2 - t$.*

Lemma 5.2.4. *Let $G^\circ = (V^\circ, E^\circ)$ be the graph obtained by shrinking a K_{t+1} subgraph $K \in \mathcal{K}$ where K is uncovered if $t = 2$. Let $\mathcal{K}^\circ$ be the set of forbidden subgraphs disjoint from K and define b° as $b^\circ(v) = b(v)$ for $v \in V \setminus V_K$, $b^\circ(k) = t$ if t is even and $b^\circ(k) = t+1$ if t is odd. Then the difference between the maximum size of a $\mathcal{K}$-free b-matching in G and the maximum size of a $\mathcal{K}^\circ$-free b°-matching in G° is exactly $\left\lfloor \frac{t^2}{2} \right\rfloor$.*

The proof of Lemma 5.2.3 is based on the following claim.

Claim 5.2.5. *Assume that $K \in \mathcal{K}$ is a $K_{t,t}$ subgraph with colour classes K_A and K_B and M' is a $\mathcal{K}$-free b-matching of $G - E_K$. Then M' can be extended to a $\mathcal{K}$-free b-matching M of G with $|M| = |M'| + t^2 - \max\{1, h_{M'}(K_A), h_{M'}(K_B)\}$.*

Proof. First we consider the case $t \geq 3$. Let P be a minimum size matching of K covering each node $v \in V_K$ with $d_{M'}(v) = 1$ (note that $d_{M'}(v) \leq 1$ for $v \in V_K$ as $d(v) \leq t+1$). If there is no such node, then let P consist of an arbitrary edge in E_K. We claim that $M = M' \cup (E_K \setminus P)$ satisfies the above conditions. Indeed, M is a b-matching and $|M \cap E_K| = t^2 - \max\{1, h_{M'}(K_A), h_{M'}(K_B)\}$ clearly holds, so we only have to verify that it is also $\mathcal{K}$-free.

Assume that there is a forbidden $K_{t,t}$ subgraph K' in M with colour classes K'_A, K'_B. $E_{K'}$ must contain an edge $uv \in E_K \cap M$ with $u \in K'_A$ and $v \in K'_B$. By symmetry, we may assume that $u \in K_A$. As $b(u) = t$, $\Gamma_M(u) = K'_B$ and also $|\Gamma_M(u) \cap K_B| \geq t - 1$. Hence $|K'_B \cap K_B| \geq t-1$. Consider a node $z \in K_A$. Since $d_M(z, K_B) \geq t-1$ and $t \geq 3$, we get $d_M(z, K'_B) > 0$, thus $K_A \subseteq \Gamma_M(K'_B)$. Because of $\Gamma_M(K'_B) = K'_A$, this gives $K_A = K'_A$. $K_B = K'_B$ follows similarly, giving a contradiction.

If there is a forbidden K_{t+1} subgraph K' in M, then $E_{K'}$ must contain an edge $uv \in E_K \cap M$, $u \in K_A$. As above, $|V_{K'} \cap K_B| \geq t-1$. Using $t \geq 3$ again, $K_A \subseteq \Gamma_M(V_{K'} \cap K_B) \subseteq V_{K'}$. But $K_A \subseteq V_{K'}$ is a contradiction since $t + 1 = |V_{K'}| \geq |V_{K'} \cap K_A| + |V_{K'} \cap K_B| \geq t + t - 1 = 2t - 1 > t + 1$.

Now let $t = 2$ and $K_A = \{v_1, v_3\}$, $K_B = \{v_2, v_4\}$. If $\max\{h_{M'}(K_A), h_{M'}(K_B)\} \leq 1$, then we may assume by symmetry that $d_{M'}(v_1) = d_{M'}(v_2) = 0$. Clearly, $M = M' \cup \{v_1v_2, v_1v_4, v_2v_3\}$ is a $\mathcal{K}$-free 2-matching. If $\max\{h_{M'}(K_A), h_{M'}(K_B)\} = 2$, we claim that at least one of $M_1 = M' \cup \{v_1v_2, v_3v_4\}$ and $M_2 = M' \cup \{v_1v_4, v_2v_3\}$ is $\mathcal{K}$-free. Assume M_1 contains a forbidden square or triangle K'; by symmetry assume it contains the edge v_1v_2. If K' contains v_3v_4 as well, then K' is the square $v_1v_3v_4v_2$. Otherwise, it consists of v_1v_2 and a path L of length 2 or 3 between v_1 and v_2, not containing v_3 and v_4. In the first case, the only forbidden subgraph possibly contained in M_2 is the square $v_1v_3v_2v_4$, implying that $\{v_1, v_2, v_3, v_4\}$ is a square-full component, a contradiction. In the latter case, it is easy to see that M_2 cannot contain a forbidden subgraph. □

Proof of Lemma 5.2.3. First we show that if M is a $\mathcal{K}$-free b-matching in G then there is a $\mathcal{K}^\circ$-free b°-matching M° in G° with $|M^\circ| \geq |M| - (t^2 - t)$. Let $M' = M \setminus E_K$. Clearly, $|M \cap E_K| \leq t^2 - \max\{1, h_{M'}(K_A), h_{M'}(K_B)\}$. In G°, let M° be the union of M' and $t - \max\{1, d_{M'}(k_a), d_{M'}(k_b)\}$ parallel edges from the shrunk bundle between k_a and k_b. Is is easy to see that M° is a $\mathcal{K}^\circ$-free b°-matching in G° with $|M^\circ| \geq |M| - (t^2 - t)$.

The proof is completed by showing that for an arbitrary $\mathcal{K}^\circ$-free b°-matching M° in G° there exists a $\mathcal{K}$-free b-matching M in G with $|M| \geq |M^\circ| + (t^2 - t)$. Let H denote the set of parallel edges in the shrunk bundle between k_a and k_b, and let $M' = M^\circ \setminus H$. Now $|M^\circ \cap H| \leq t - \max\{1, d_{M'}(k_a), d_{M'}(k_b)\}$ and, by Claim 5.2.5, M' may be extended to a $\mathcal{K}$-free b-matching in G with $|M \cap E_K| = t^2 - \max\{1, h_{M'}(K_A), h_{M'}(K_B)\}$, that is

$$\begin{aligned}|M| = |M^\circ| - |M^\circ \cap H| + |M \cap E_K| \geq |M^\circ| - (t - \max\{1, d_{M'}(k_a), d_{M'}(k_b)\})\\ + (t^2 - \max\{1, h_{M'}(K_A), h_{M'}(K_B)\}) \geq |M^\circ| + (t^2 - t).\end{aligned}$$

□

Lemma 5.2.4 can be proved in a similar way by using the following claim.

Claim 5.2.6. *Assume that $K \in \mathcal{K}$ is a K_{t+1} subgraph and M' is a $\mathcal{K}$-free b-matching of $G - E_K$. If $t = 2$, then assume that K is uncovered. Then M' can be extended to obtain a $\mathcal{K}$-free b-matching M of G with $|M| = |M'| + \binom{t+1}{2} - \left\lceil \frac{\max\{1, h_{M'}(V_K)\}}{2} \right\rceil$.*

Proof. Let P be a minimum size subgraph of K covering each node $v \in V_K$ with $d_{M'}(v) = 1$ (so P is a matching or a matching and one more edge in E_K). If there is no such node, then let P consist of an arbitrary edge in E_K. For $t = 2$ and 3, we will choose P in a specific way, as given later in the proof. We show that $M = M' \cup (E_K \setminus P)$ satisfies the above conditions. Indeed, M is a b-matching and $|M \cap E_K| = \binom{t+1}{2} - \left\lceil \frac{\max\{1, h_{M'}(K)\}}{2} \right\rceil$ clearly holds, so we only have to show that it is also $\mathcal{K}$-free.

Assume that there is a forbidden K_{t+1} subgraph K' in M. $E_{K'}$ must contain an edge $uv \in E_K \cap M$. By the minimal choice of P at least one of $|\Gamma_M(u) \cap V_K| \geq t - 1$ and $|\Gamma_M(v) \cap V_K| \geq t - 1$ is satisfied which implies $|V_{K'} \cap V_K| \geq t - 1$. For $t \geq 3$ this immediately implies $V_K \subseteq \Gamma_M(V_{K'} \cap V_K) \subseteq V_{K'}$, a contradiction.

If $t = 2$, then $|V_{K'} \cap V_K| \geq 1$ does not imply $V_K \subseteq V_{K'}$ and an improper choice of P may enable M to contain a forbidden K_3. The only such case is when $h_{M'}(V_K) = 3$, $V_K = \{v_1, v_2, v_3\}$, $V_{K'} = \{v_2, v_3, v_4\}$, $v_2v_4, v_3v_4 \in M'$ and $P = \{v_1v_2, v_1v_3\}$ (Figure 5.3). In this case, we may leave the edge incident to v_1 from M' and then $P = \{v_2v_3\}$ is a good choice. Indeed, the only problem could be that $v_1v_2v_3v_4$ is a forbidden square, contradicting K being uncovered.

Otherwise $h_{M'}(V_K) \leq 2$ implies $|P| \leq 1$. Hence at least one of $|\Gamma_M(u) \cap V_K| = 2$ and $|\Gamma_M(v) \cap V_K| = 2$ is satisfied meaning $K' = K$, a contradiction again.

Now assume that there is a forbidden $K_{t,t}$ subgraph K' in M with colour classes K'_A, K'_B. The same argument gives a contradiction for $t \geq 4$. However, in case of $t = 3$, choosing P arbitrarily may enable M to contain a forbidden $K_{3,3}$ in the following single configuration: $V_K = \{v_1, v_2, v_3, v_4\}$, $K'_A = \{v_1, v_2, x\}$, $K'_B = \{v_3, v_4, y\}$, $xv_3, xv_4, yv_1, yv_2, xy \in M'$ and $P = \{v_1v_2, v_3v_4\}$ (Figure 5.4). In this case, $P = \{v_1v_4, v_2v_3\}$ is a good choice.

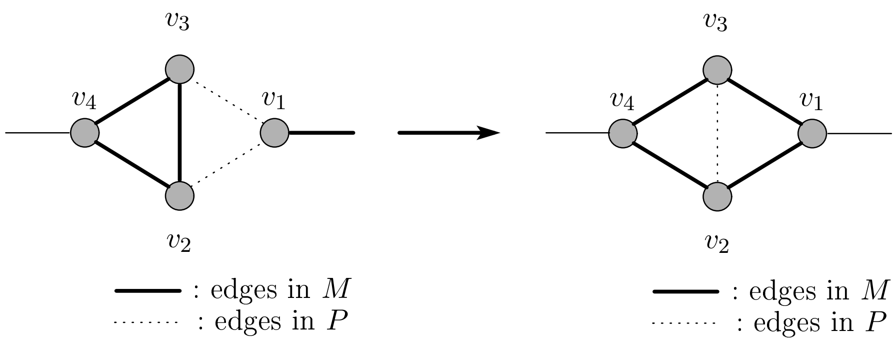

Figure 5.3: Choice of P for $t = 2$ in the proof of Claim 5.2.6

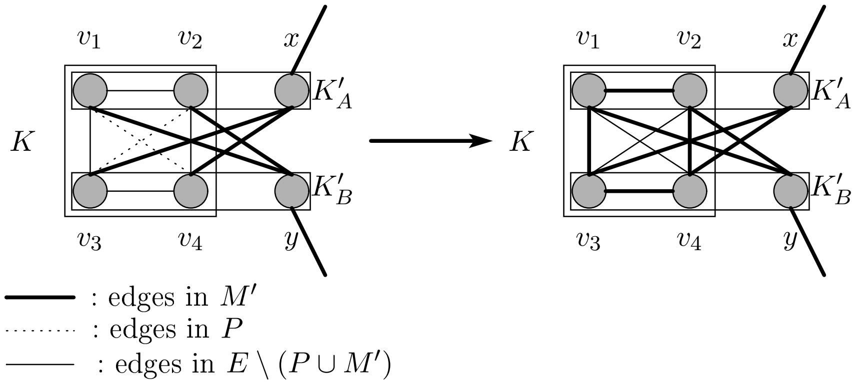

Figure 5.4: Choice of P for $t = 3$ in the proof of Claim 5.2.6

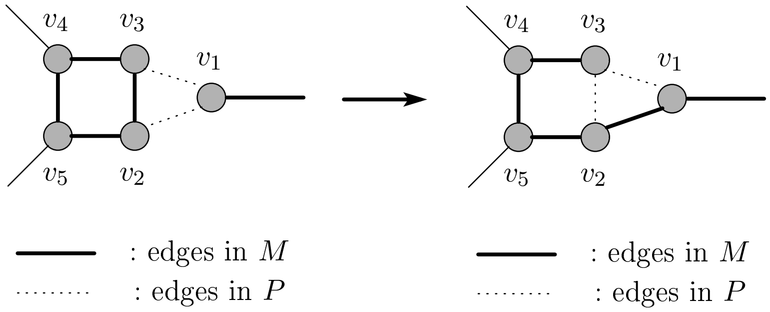

Figure 5.5: Choice of P for $t = 2$ in the proof of Claim 5.2.6

Finally, for $t = 2$ no forbidden square appears if $h_{M'}(K) \leq 2$ as otherwise K would be a square-covered triangle. If $h_{M'}(K) = 3$, then such a square K' may appear only if $V_K = \{v_1, v_2, v_3\}$, $V_{K'} = \{v_2, v_3, v_4, v_5\}$, $v_3v_4, v_4v_5, v_5v_2 \in M'$, $P = \{v_1v_2, v_1v_3\}$ ($v_1 \neq v_4, v_5$ as K is uncovered). In this case both $P = \{v_1v_2, v_2v_3\}$ and $P = \{v_1v_3, v_2v_3\}$ give a proper M (Figure 5.5). □

Proof of Lemma 5.2.4. First we show that if M is a $\mathcal{K}$-free b-matching in G then there is a $\mathcal{K}^\circ$-free b°-matching M° in G° with $|M^\circ| \geq |M| - \left\lfloor \frac{t^2}{2} \right\rfloor$. Let $M' = M \setminus E_K$. Clearly, $|M \cap E_K| \leq \binom{t+1}{2} - \left\lceil \frac{\max\{1, h_{M'}(V_K)\}}{2} \right\rceil$. In G°, let M° be the union of M' and $\left\lfloor \frac{t - \max\{1, d_{M'}(k)\}}{2} \right\rfloor$ or $\left\lfloor \frac{t+1-\max\{1, d_{M'}(k)\}}{2} \right\rfloor$ loops on k depending on whether t is even or not, respectively. Is is easy to see that M° is a $\mathcal{K}^\circ$-free b°-matching in G° with $|M^\circ| \geq |M| - \left\lfloor \frac{t^2}{2} \right\rfloor$.

The proof is completed by showing that for an arbitrary $\mathcal{K}^\circ$-free b°-matching M° in G° there exists a $\mathcal{K}$-free b-matching M in G with $|M| \geq |M^\circ| + \left\lfloor \frac{t^2}{2} \right\rfloor$. Let H denote the set of loops on k obtained when shrinking K, and let $M' = M^\circ \setminus H$. Now $|M^\circ \cap H| \leq \left\lfloor \frac{t-\max\{1,d_{M'}(k)\}}{2} \right\rfloor$ if t is even and $|M^\circ \cap H| \leq \left\lfloor \frac{t+1-\max\{1,d_{M'}(k)\}}{2} \right\rfloor$ if t is odd. By Claim 5.2.5, M' can be extended modified as to get a $\mathcal{K}$-free b-matching in G with $|M \cap E_K| = \binom{t+1}{2} - \left\lceil \frac{\max\{1,h_{M'}(V_K)\}}{2} \right\rceil$, that is

$$|M| = |M^\circ| - |M^\circ \cap H| + |M \cap E_K| \geq |M^\circ| - \left\lfloor \frac{t-\max\{1,d_{M'}(k)\}}{2} \right\rfloor + \binom{t+1}{2} - \left\lceil \frac{\max\{1,h_{M'}(V_K)\}}{2} \right\rceil \geq |M^\circ| + \left\lfloor \frac{t^2}{2} \right\rfloor$$

if t is even and

$$|M| = |M^\circ| - |M^\circ \cap H| + |M \cap E_K| \geq |M^\circ| - \left\lfloor \frac{t+1-\max\{1,d_{M'}(k)\}}{2} \right\rfloor + \binom{t+1}{2} - \left\lceil \frac{\max\{1,h_{M'}(V_K)\}}{2} \right\rceil \geq |M^\circ| + \left\lfloor \frac{t^2}{2} \right\rfloor$$

if t is odd. □

5.3 Proof of Theorem 5.1.5

We prove max $\geq$ min by induction on $|\mathcal{K}|$. For $\mathcal{K} = \emptyset$, this is simply a consequence of Theorem 5.1.1.

Assume now that $\mathcal{K} \neq \emptyset$ and let K be a forbidden subgraph such that K is uncovered if $t = 2$. Let $G^\circ = (V^\circ, E^\circ)$ denote the graph obtained by shrinking K, let b° be defined as in Lemma 5.2.3 or 5.2.4 depending on whether K is a $K_{t,t}$ or a K_{t+1}. We denote by $\mathcal{K}^\circ$ the list of forbidden subgraphs disjoint from K.

By induction, the maximum size of a $\mathcal{K}^\circ$-free b°-matching in G° is equal to the minimum value of $\tau(U^\circ, W^\circ, \mathcal{P}^\circ, \dot{\mathcal{K}}^\circ)$. Let us choose an optimal $U^\circ, W^\circ, \mathcal{P}^\circ, \dot{\mathcal{K}}^\circ$ so that $|U^\circ|$ is minimal. The following claim gives a useful property of U°.

Claim 5.3.1. *Assume that $v \in U$ is such that $d(v, W) + |\Gamma(v) \cap (V \setminus W)| \leq b(v) + 1$. Then $\tau(U - v, W, \mathcal{P}', \dot{\mathcal{K}}) \leq \tau(U, W, \mathcal{P}, \dot{\mathcal{K}})$ where $\mathcal{P}'$ is obtained from $\mathcal{P}$ by replacing its members incident to v by their union plus v.*

Proof. By removing v from U, $b(U)$ decreases by $b(v)$. $|E[W]| - |\dot{\mathcal{K}}[W]|$ remains unchanged, while the bound on $d(v, W) + |\Gamma(v) \cap (V \setminus W)|$ implies that the increment in the sum over the components of $G - U - W$ is at most $b(v)$. □

Case 1: K is a $K_{t,t}$ with colour classes K_A and K_B.

By Lemma 5.2.3, the difference between the maximum size of a $\mathcal{K}$-free b-matching in G and the maximum size of a $\mathcal{K}^\circ$-free b°-matching in G° is exactly $t^2 - t$. We will define $U, W, \mathcal{P}$ and $\dot{\mathcal{K}}$ such that

$$\tau(U, W, \mathcal{P}, \dot{\mathcal{K}}) = \tau(U^\circ, W^\circ, \mathcal{P}^\circ, \dot{\mathcal{K}}^\circ) + t^2 - t. \tag{5.4}$$

The shrinking replaces K_A and K_B by two nodes k_a and k_b with $t-1$ parallel edges between them. Let U, W and $\mathcal{P}$ denote the pre-images of $U^\circ, W^\circ, \mathcal{P}^\circ$ in G, respectively and let $\dot{\mathcal{K}} = \dot{\mathcal{K}}^\circ \cup \{K\}$. By (1.10), $d_{G^\circ - k_b}(k_a), d_{G^\circ - k_a}(k_b) \leq t$. Since $b^\circ(k_a) = b^\circ(k_b) = t$, Claim 5.3.1 and the minimal choice of $|U^\circ|$ implies that if $k_a \in U^\circ$, then $k_b \in W^\circ$.

Hence we have the following cases (T° denotes a member of $\mathcal{P}^\circ$). In each case we are only considering those terms in $\tau(U^\circ, W^\circ, \mathcal{P}^\circ, \dot{\mathcal{K}}^\circ)$ that change when taking $\tau(U, W, \mathcal{P}, \dot{\mathcal{K}})$ instead.

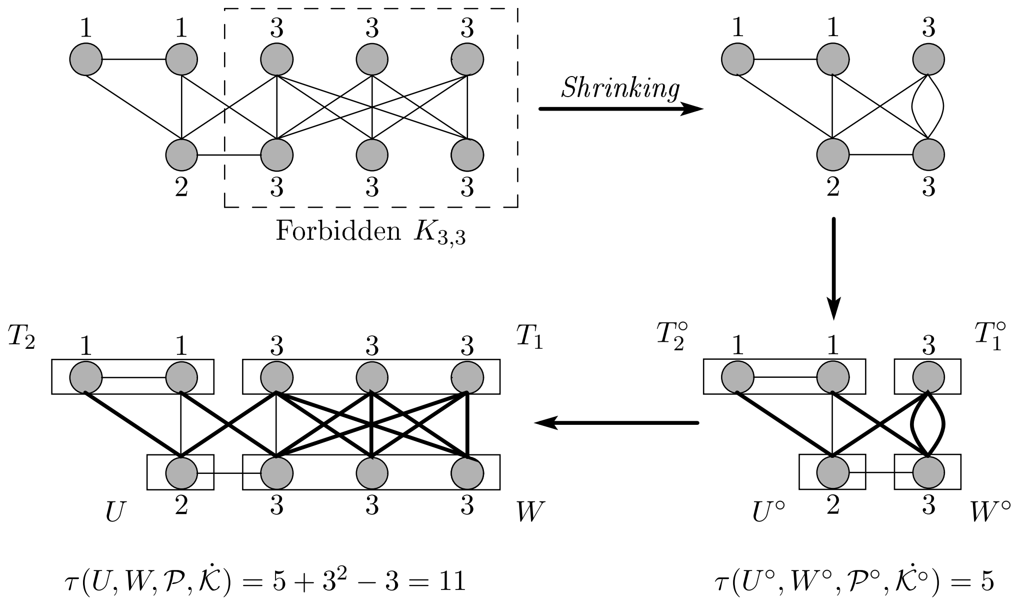

Figure 5.6: Extending M°

- $k_a \in U^\circ,\ k_b \in W^\circ$: $b(U) = b^\circ(U^\circ) + t^2 - t$.
- $k_a, k_b \in W^\circ$: $|E[W]| = |E^\circ[W^\circ]| + t^2 - t + 1$ and $|\dot{\mathcal{K}}[W]| = |\dot{\mathcal{K}}^\circ[W^\circ]| + 1$.
- $k_a \in W^\circ,\ k_b \in T^\circ$: $|E[T, W]| = |E^\circ[T^\circ, W^\circ]| + t^2 - t + 1$, $b(T) = b^\circ(T^\circ) + t^2 - t$ and $|\dot{\mathcal{K}}[T, W]| = |\dot{\mathcal{K}}^\circ[T^\circ, W^\circ]| + 1$ (see Figure 6.9 for an example).
- $k_a \in T^\circ,\ k_b \in W^\circ$: similar to the previous case.
- $k_a, k_b \in T^\circ$: $b(T) = b^\circ(T^\circ) + 2t^2 - 2t$.

(5.4) is satisfied in each of the above cases, hence we are done. Note that in the first and the last case we may leave out K from $\dot{\mathcal{K}}$ as it is not counted in any term.

Case 2: K is a K_{t+1}.

By Lemma 5.2.4, the difference between the maximum size of a $\mathcal{K}$-free b-matching in G and the maximum size of a $\mathcal{K}^\circ$-free b°-matching in G° is $\left\lfloor \frac{t^2}{2} \right\rfloor$. We show that for the pre-images U, W and $\mathcal{P}$ of U°, W° and $\mathcal{P}^\circ$ with $\dot{\mathcal{K}} = \dot{\mathcal{K}}^\circ \cup \{K\}$ satisfy

$$\tau(U, W, \mathcal{P}, \dot{\mathcal{K}}) = \tau(U^\circ, W^\circ, \mathcal{P}^\circ, \dot{\mathcal{K}}^\circ) + \left\lfloor \frac{t^2}{2} \right\rfloor. \tag{5.5}$$

After shrinking $K = (V_K, E_K)$ we get a new node k with $\left\lfloor \frac{t+1}{2} \right\rfloor - 1$ loops on it. (1.10) implies that there are at most $t+1$ non-loop edges incident to k. Since $b^\circ(k) \geq t$, Claim 5.3.1 implies $k \notin U$. Hence we have the following two cases (T° denotes a member of $\mathcal{P}^\circ$).

- $k \in W^\circ$: $|E[W]| = |E^\circ[W^\circ]| + \binom{t+1}{2} - \left\lfloor \frac{t+1}{2} \right\rfloor + 1$ and $|\dot{\mathcal{K}}[W]| = |\dot{\mathcal{K}}^\circ[W^\circ]| + 1$.
- $k \in T^\circ$: $b(T) = b^\circ(T^\circ) + t^2$ if t is even and $b(T) = b^\circ(T^\circ) + t^2 - 1$ for an odd t.

(5.5) is satisfied in both cases, hence we are done. We may also leave out K from $\dot{\mathcal{K}}$ in the second case as it is not counted in any term.

5.4 Algorithm

In this section we show how the proof of Theorem 5.1.5 immediately yields an algorithm for finding a maximum $\mathcal{K}$-free b-matching in strongly polynomial time. In such problems, an important question from an algorithmic point of view is how $\mathcal{K}$ is represented. For example, in the $\mathcal{K}$-free b-matching problem for bipartite graphs solved by Pap in [110], the set of excluded subgraphs may be exponentially large. Therefore Pap assumes that $\mathcal{K}$ is given by a membership oracle, that is, a subroutine is given for determining whether a given subgraph is a member of $\mathcal{K}$. However, with such an oracle there is no general method for determining whether $\mathcal{K} = \emptyset$. Fortunately, we do not have to tackle such problems: by the next claim, we may assume that $\mathcal{K}$ is given explicitly, as its size is linear in n. We use $n = |V|$, $m = |E|$ for the number of nodes and edges of the graph, respectively.

Claim 5.4.1. *If the graph $G = (V, E)$ satisfies (1.8) and (1.10), then the total number of $K_{t,t}$ and K_{t+1} subgraphs is bounded by $\frac{(t+3)n}{2}$.*

Proof. Assume that $v \in V$ is contained in a forbidden subgraph and so $d_G(v) = t + 1$. If we select an edge incident to v, the remaining t edges may be contained in at most one K_{t+1} subgraph hence the number of K_{t+1}'s containing v is at most $t + 1$. However, these t edges also determine one of the colour classes of those $K_{t,t}$'s containing them. If we pick a node v' from this colour class (implying $d_G(v') = t + 1$), pick an edge incident to v' (but not to v), then the remaining t edges, if they do so, exactly determine the other colour class of a $K_{t,t}$ subgraph. Therefore the number of $K_{t,t}$ subgraphs containing v is bounded by $(t+1)t = t^2+t$. Hence the total number of forbidden $K_{t,t}$ and K_{t+1} subgraphs is at most $\frac{(t^2+t)n}{2t} + \frac{(t+1)n}{t+1} = \frac{(t+3)n}{2}$. □

Now we turn to the algorithm. First we choose an inclusionwise maximal subset $\mathcal{H} = \{H_1, \dots, H_k\}$ of disjoint forbidden subgraphs greedily. For $t = 2$, let us always choose squares as long as possible and then go on with triangles. This can be done in $O(t^3 n)$ time as follows. Maintain an array of size m that encodes for each edge whether it is used in one of the selected forbidden subgraphs or not. When increasing $\mathcal{H}$, one only has to check whether any of the edges of the examined forbidden subgraph is already used, which takes $O(t^2)$ time. This and Claim 5.4.1 together give an $O(t^3 n)$ bound.

Let us shrink the members of $\mathcal{H}$ simultaneously (this can be easily done since they are disjoint), resulting in a graph $G' = (V', E')$ with a bound $b' : V' \to \mathbb{Z}_+$ and no forbidden subgraphs since $\mathcal{H}$ was maximal. One can find a maximal b'-matching M' in G' in $O(|V'||E'| \log |V'|) = O(nm \log m)$ time as in [50]. Using the constructions described in Lemmas 5.2.3 and 5.2.4 for $H_k, ..., H_1$, this can be modified into a maximal $\mathcal{K}$-free b-matching M. Note that, for $t = 2$, H_i is always uncovered in the actual graph by the selection rule. A dual optimal solution $U, W, \mathcal{P}, \dot{\mathcal{K}}$ can be constructed simultaneously as in the proof of Theorem 5.1.5. The best time bound of the shrinking and extension steps may depend on the data structure used and the representation of the graph. In any case, one such step may be performed in $O(m)$ time and $|\mathcal{H}| = O(n)$, hence the total running time is $O(t^3 n + nm \log m)$.

Chapter 6

Polyhedral descriptions

6.1 Main results

Let $G = (V, E)$ be a graph and $b : V \to \mathbb{Z}_+$ an upper bound on the node set such that for any $T \in \mathcal{T}$ and any node v of T,

$$d_G(v) \leq 3, \tag{6.1}$$

$$b(v) = 2. \tag{6.2}$$

These settings clearly includes and generalizes the triangle-free 2-factor and 2-matching problems in subcubic graphs.

In this chapter we give new proofs of Theorems 1.4.5 and 1.4.7 in a slightly more general form, based on a newly introduced contraction operation. The proof easily extends to the polyhedral description of $\mathcal{T}$-free b-factors under assumptions (6.1) and (6.2). Hartvigsen and Li showed that the polyhedral description of $\mathcal{T}$-free 2-matchings is far more complicated, and proved their fundamental characterization in [63]. We give a slight generalization of their nice result by extending our contraction techniques.

Yet giving a polyhedral description of triangle-free (or, more generally, $\mathcal{T}$-free) 2-factors and 2-matchings of arbitrary graphs is still open. One might wonder whether the description for subcubic graphs could be a valid description for the general case. Unfortunately, the answer is negative as shown by the counterexample of Figure 6.9.

As the considered graphs may contain parallel edges and self-loops, it may happen that two non-identical triangles share the same node-set, that is, T_1 and T_2 are triangles with $V_{T_1} = V_{T_2}$ but $E_{T_1} \neq E_{T_2}$. We call these triangles **node-identical**. If there exists a pair of node-identical triangles in G then, by (6.1) and (6.2), no b-factor exists.

Theorem 6.1.1. *Let $G = (V, E)$, $b : V \to \mathbb{Z}_+$ and $\mathcal{T}$ a collection of triangles satisfying (6.1) and (6.2). Assume that there are no node-identical triangles in G. The $\mathcal{T}$-free b-factor polytope is determined by*

$$\begin{aligned}
&(i)\ 0 \leq x(e) \leq 1 && (e \in E),\\
&(ii)\ x(\dot{\delta}(v)) = b(v) && (v \in V),\\
&(iii)\ x(\delta(K) \setminus F) - x(F) \geq 1 - |F| && ((K, F)\ \textit{odd}),\\
&(iv)\ x(E_T) = 2 && (T \in \mathcal{T}).
\end{aligned} \tag{P_7}$$

Our main result is the proof of the following theorem which gives a slight generalization of Theorem 1.4.7. The method we use is also inspired by Edmonds' matching algorithm, but different from that of [63] and is based on a new shrinking approach.

Theorem 6.1.2. *Let $G = (V, E)$, $b : V \to \mathbb{Z}_+$ and $\mathcal{T}$ a collection of triangles satisfying (6.1) and (6.2). The $\mathcal{T}$-free b-matching polytope is determined by*

$$
\begin{aligned}
&(i)\ 0 \le x(e) \le 1 && (e \in E),\\
&(ii)\ x(\dot{\delta}(v)) \le b(v) && (v \in V),\\
&(iii)\ x(E[K]) + x(F) + \textstyle\sum_{T\in\mathfrak{T}} x(E_T) \le && ((K, F, \mathfrak{T})\ \text{odd}\\
&\qquad \left\lfloor \tfrac{b(K)+|F|+3|\mathfrak{T}|}{2} \right\rfloor && \text{tri-comb of Type 2}),\\
&(iv)\ x(E_T) \le 2 && (T \in \mathcal{T}),\\
&(v)\ x(E_{T_1} \cup E_{T_2}) \le 2 && (T_1, T_2 \in \mathcal{T}, V_{T_1} = V_{T_2}).
\end{aligned}
\qquad (P_8)
$$

Assumption (6.1) here is essential: the theorem is false if we remove the degree bound $d_G(v) \le 3$ on nodes of forbidden triangles. An example is shown in Section 6.9.

6.2 Shrinking odd pairs

We prove Theorem 1.4.2 by induction on $b(V)$, $|V|$ and $|E|$. In the proof we use a shrinking operation to get a smaller graph on which the induction step can be applied. Note that condition *(iii)* in Theorems 1.4.2 and 6.1.1 is required for odd pairs. If $b(V)$ is odd then $(V, \emptyset)$ is an odd pair and thus (P_2) and (P_7) are infeasible. In the sequel we assume that $b(V)$ is even.

Definition 6.2.1 (Shrinking an odd pair)**. Shrinking** an odd pair (K, F) consists of the following operations (see Figure 6.1):

- replace K by an edge pq with $b^\circ(p) = |F|$ and $b^\circ(q) = 1$,
- define $b^\circ(v) = b(v)$ for each $v \in V \setminus K$,
- replace each edge e with $e^u \in K, e^v \in V \setminus K$ by an edge pe^v if $e \in F$, otherwise by qe^v.

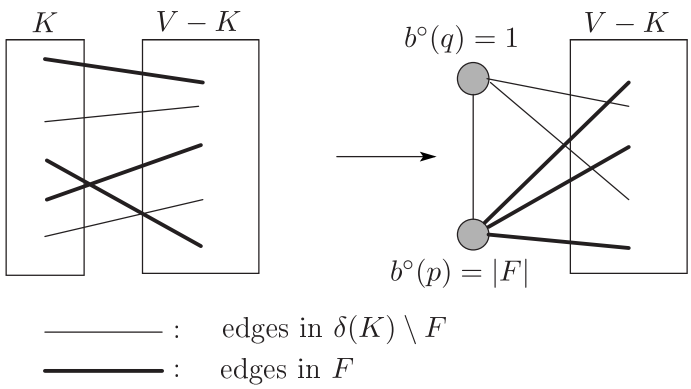

Figure 6.1: Shrinking an odd pair (K, F)

We usually denote the graph obtained by shrinking an odd pair by $G^\circ = (V^\circ, E^\circ)$. By abuse of notation, each edge $e \in \delta(K)$ is denoted by e again after shrinking the pair and is called the **image** of the original edge. Hence the intersection $E \cap E^\circ$ stands for the set of all edges not induced by K, in other words, $E^\circ - pq \subseteq E$. Similarly, $V^\circ \setminus \{p, q\} \subseteq V$.

Assume that $x \in \mathbb{R}^E$ satisfies (P_2). An odd pair (K, F) is called **x-tight** if it satisfies *(iii)* with equality. When shrinking an x-tight pair, we use the notation x° for the image of x, namely

$$x^\circ(e) = \begin{cases} x(e) & \text{if } e \in E^\circ - pq, \\ |F| - x(F) & \text{if } e = pq. \end{cases}$$

The main advantage of the shrinking operation is the following.

Lemma 6.2.2. *Let $G = (V, E)$ be a graph with $b : V \to \mathbb{Z}_+$. Assume that $x \in \mathbb{R}^E$ satisfies (P_2) and (K, F) is an x-tight pair. Then x° satisfies (P_2) in $G^\circ = (V^\circ, E^\circ)$ with b°.*

Proof. *(i)* clearly holds for edges different from pq. Concerning pq, $x^\circ(pq) = |F| - x(F) \geq 0$. The tightness of (K, F) implies $x^\circ(pq) = |F| - x(F) = 1 - x(\delta(K) \setminus F) \leq 1$.

For a node v in $V^\circ \setminus \{p, q\}$, by the definition of shrinking, $x^\circ(\dot{\delta}(v)) = x(\dot{\delta}(v)) = b(v) = b^\circ(v)$. Also, $x^\circ(\dot{\delta}(p)) = x(F) + x^\circ(pq) = |F| = b^\circ(p)$. By the tightness of (K, F), $x^\circ(\dot{\delta}(q)) = x(\delta(K) \setminus F) + x^\circ(pq) = 1 = b^\circ(q)$.

It only remains to show that x° satisfies *(iii)* in G°. First, observe that -assuming $b(V)$ is even- (Z, H) is an odd pair if and only if $(\bar{Z}, H)$ is also an odd pair. For these two pairs, condition *(iii)* is identical.

(iii) immediately follows for odd pairs (Z, H) with $Z \subseteq V^\circ \setminus \{p, q\}$ as x satisfied *(iii)* in the original problem. By taking $(\bar{Z}, H)$ instead, it also holds if $p, q \in Z$. Again by possibly changing Z to $\bar{Z}$, it remains to show that *(iii)* is satisfied if $p \in Z, q \notin Z$.

If $pq \in H$, then add q to Z and delete pq from H. We have previously seen that the odd pair $(Z', H') = (Z + q, H - pq)$ satisfies *(iii)*, thus

$$\begin{aligned} x(\delta(Z) \setminus H) - x(H) &\geq x(\delta(Z') \setminus H') - x(H') - x(\delta(q)) \\ &\geq (1 - |H'|) - 1 \\ &= 1 - |H|. \end{aligned}$$

If $pq \notin H$, then first consider the case when $F \cap (\delta(Z) \setminus H) \neq \emptyset$. Let f be an edge in this set. Define $(Z', H') = (Z + q, H + f)$, which is again an odd pair satisfying *(iii)*. Then

$$\begin{aligned} x(\delta(Z) \setminus H) - x(H) &\geq x(\delta(Z') \setminus H') - x(H') + 2x(pq) - x(\delta(q)) + 2x(f) \\ &\geq (1 - |H'|) + 2(x(pq) + x(f)) - 1 \\ &= 1 - |H| + 2(x(pq) + x(f) - 1) \\ &\geq 1 - |H|. \end{aligned}$$

For the last inequality, we use that $x(\delta(p)) = |F|$, and the degree of p is $|F| + 1$. Hence pq and f, two edges incident to p must have x value together at least 1.

If $F \cap (\delta(Z) \setminus H) = \emptyset$, then let $F_1 = F \cap H$, $F_2 = F \setminus H$. Define $Z' = Z - p$, $H' = (H \setminus F_1) \cup F_2$. (Z', H') is odd since $b(Z') + |H'| = b(Z) + |H| - |F| - |F_1| + |F_2| = b(Z) + |H| - 2|F_1|$. As we have seen, the pair (Z', H') satisfies (iii), so

$$\begin{aligned} x(\delta(Z) \setminus H) - x(H) &\geq x(\delta(Z') \setminus H') - x(H') + x(F_2) + x(pq) - x(F_1) \\ &\geq (1 - |H'|) + x(\dot{\delta}(p)) - 2x(F_1) \\ &\geq (1 - |H'|) + |F| - 2|F_1| \\ &= 1 - |H|. \end{aligned}$$

This completes the proof. □

6.3 Proof of Theorem 1.4.2

It is easy to see that each b-factor satisfies (i) and (ii). To show that (iii) indeed holds for a b-factor $M \subseteq E$, add all equalities $d_M(v) = b(v)$ for $v \in K$. This gives

$$2|M \cap E[K]| + |M \cap \delta(K)| = b(K). \tag{6.3}$$

Adding the inequalities $|M \cap F| \leq |F|$ and $-|M \cap (\delta(K) \setminus F)| \leq 0$, we get $2|M \cap E[K]| + 2|M \cap F| \leq b(K) + |F|$. This yields $|M \cap E[K]| + |M \cap F| \leq \lfloor \frac{1}{2}(b(K) + |F|) \rfloor = \frac{1}{2}(b(K) + |F| - 1)$ since (K, F) is odd. Subtracting the double of this from (6.3), we get $|M \cap (\delta(K) \setminus F)| - |M \cap F| \geq 1 - |F|$, as required.

Recall that we may assume that $b(V)$ is even since otherwise there exists no b-factor and the polytope (P_2) is empty.

It remains to show that $(i), (ii)$ and (iii) completely determine the b-factor polytope, that is, any $x \in \mathbb{R}^E$ satisfying (P_2) is a convex combination of incidence vectors of b-factors. Assume that this does not hold. Let us choose x to be a vertex of the polytope described by (P_2) not contained in the b-factor polytope.

We choose this counterexample in such a way that $(|\ell(V)|, b(V), |V|, |E|)$ is lexicographically minimal. This implies that $0 < x < 1$ as edges with $x(e) = 0$ could be deleted, while if $x(e) = 1$ we can delete e and decrease the b values on its ends by one (if e is a loop on v then decrease $b(v)$ by 2). It is easy to see that the x' and b' thus obtained would satisfy $(i) - (iii)$ thus giving a smaller counterexample, a contradiction. Also, it can be shown that, in presence of parallel edges, the total x value of parallel edges between two nodes should be strictly smaller than one.

As $b(v) \geq 1$ for each $v \in V$, each node has degree at least 2 in G, so $|E| \geq |V|$. G is connected, otherwise one of its components would be a smaller counterexample. If $|E| = |V|$, then G is an even cycle as it implies that $b \equiv 1$ and $b(V)$ is even. By (ii), x is alternately μ and $1 - \mu$ for some value $0 < \mu < 1$ on the edges of this cycle, hence it is the convex combination of the two perfect matchings of the graph, a contradiction.

So $|E| > |V|$. As x is a vertex, it satisfies $|E|$ linearly independent constraints among (P_2) with equality. From $|E| > |V|$, there is a tight odd pair (K, F) linearly independent from the equalities of form (ii).

Proposition 6.3.1. *For any tight odd pair (K, F) independent from equalities of form (ii), the shrinking of (K, F) results in a lexicographically smaller problem, and the same holds for $(\bar{K}, F)$.*

Proof. The second part follows by complementing K and by the observation that (K,F) is independent from equalities of form (ii) if and only if $(\bar{K},F)$ does so.

What we have to prove is that either **(A)** $\ell(K)\neq\emptyset$, or **(B)** $\ell(K)=\emptyset$ and $b(K)>|F|+1$, or **(C)** $\ell(K)=\emptyset, b(K)=|F|+1$ and $|K|>2$, or **(D)** $\ell(K)=\emptyset, b(K)=|F|+1, |K|=2$ and $E[K]>1$ as $(|\ell(V)|, b(V), |V|, |E|)$ decreases only in these cases. However, we will show that either **(A)**, **(B)** or **(C)** is satisfied.

We claim that $G[K]$ is connected. Indeed, assume indirectly that $K=K_1\cup K_2$ where $K_1\cap K_2=\emptyset$ and there is no edge between K_1 and K_2. Define $F_i=F\cap\delta(K_i)$ for $i=1,2$. Then one of the pairs $(K_1,F_1),(K_2,F_2)$ is odd while the other is not, say (K_1,F_1) is odd. We have

$$\begin{aligned}1-|F| &= x(\delta(K)\setminus F)-x(F)\\ &= x(\delta(K_1)\setminus F_1)-x(F_1)+x(\delta(K_2)\setminus F_2)-x(F_2)\\ &\geq 1-|F_1|-|F_2|\\ &= 1-|F|,\end{aligned}$$

thus we have equality everywhere. That means that $x(\delta(K_2)\setminus F_2)-x(F_2)=-|F_2|$, which is only possible (by $0<x<1$) if $\delta(K_2)=\emptyset$, contradicting the connectivity of G. Hence $G[K]$ must be connected.

Assume that **(A)** does not hold, so $\ell(K)=\emptyset$ and (**B**) does not hold either, so $b(K)\leq|F|+1$. We show that $b(K)=|F|+1$ in this case. Otherwise $b(K)\leq|F|-1$ as (K,F) is an odd pair. As $x(F)\geq|F|-1$, only $b(K)=|F|-1$ is possible. By $0<x<1$, $E[K]=\emptyset$ and so $|K|=1$ by the previous observation. If $F=\delta(v)$, the tightness of (K,F) is identical to $x(\dot{\delta}(v))=b(v)$, contradicting linear independence. Hence $\delta(v)\setminus F\neq\emptyset$ and thus $x(\delta(v)\setminus F)>0$. Also, $x(F)\leq b(v)\leq|F|-1$. Consequently, $x(\delta(v)\setminus F)-x(F)>1-|F|$, a contradiction.

Now we show that $|K|\geq 2$. If $K=\{v\}$ then $x(\delta(v)\setminus F)\geq 1$ as $x(\dot{\delta}(v))=|F|+1$ and $\ell(v)=\emptyset$. If $F\neq\emptyset$ then $x(F)<|F|$ as $x<1$, so (iii) cannot hold with equality. Hence $F=\emptyset$ and $x(\delta(v))=1=b(v)$, so the tightness of (K,F) is identical to $x(\dot{\delta}(v))=b(v)$, contradicting independence.

Assume that **(C)** does not hold either, so $\ell(K)=\emptyset, b(K)=|F|+1$ and $|K|=2$. We show that this leads to contradiction. Let $K=\{u,v\}$, and let C be the set of parallel edges between u and v. Then we have

$$x(\delta(K)\setminus F)-x(F)=b(u)+b(v)-2x(C)-2x(F_u)-2x(F_v).$$

As $b(u)+b(v)=|F|+1$, either $b(u)\leq|F_u|$ or $b(v)\leq|F_v|$, say the first holds. In this case $x(C)+x(F_u)\leq b(u)\leq|F_u|$, so $x(C)+x(F_u)+x(F_v)\leq|F_u|+|F_v|$. Here $F_v=\emptyset$, otherwise strict inequality holds by $x<1$, contradicting the tightness of (K,F), and also $b(u)=|F_u|$ follows. Then the tightness of the pair can be reformulated as $x(\delta(u)\setminus C)-2x(F_u)=1-|F_u|$. By subtracting this from equality $2x(C)+x(\delta(K))=|F|+1$, we get $2x(C)+x(\delta(K)\setminus\delta(u))+2x(F_u)=2|F_u|=2b(u)$. But $x(C)+x(F_u)\leq b(u)$, hence $\delta(K)\setminus\delta(u)=\emptyset$ and $x(C)+x(F_u)=x(C)+x(\delta(u))=b(u)=|F_u|$, $b(v)=1$. That means that the tightness of (K,F) is identical to $x(\delta(u))=b(u)$, contradicting linear independence. □

Note that $(\bar{K},F)$ is also x-tight. Let $G_1^\circ=(V_1^\circ,E_1^\circ),b_1^\circ,x_1^\circ$ and $G_2^\circ=(V_2^\circ,E_2^\circ),b_2^\circ,x_2^\circ$ denote the problems we get after shrinking (K,F) and $(\bar{K},F)$, respectively. By Proposition 6.3.1, the induction step can be applied, and -by the minimality of G- x_i° is the convex combination of incidence vectors of b_i°-factors of G_i°. Note, that a b_i°-factor contains either each edge of F and exactly one edge from

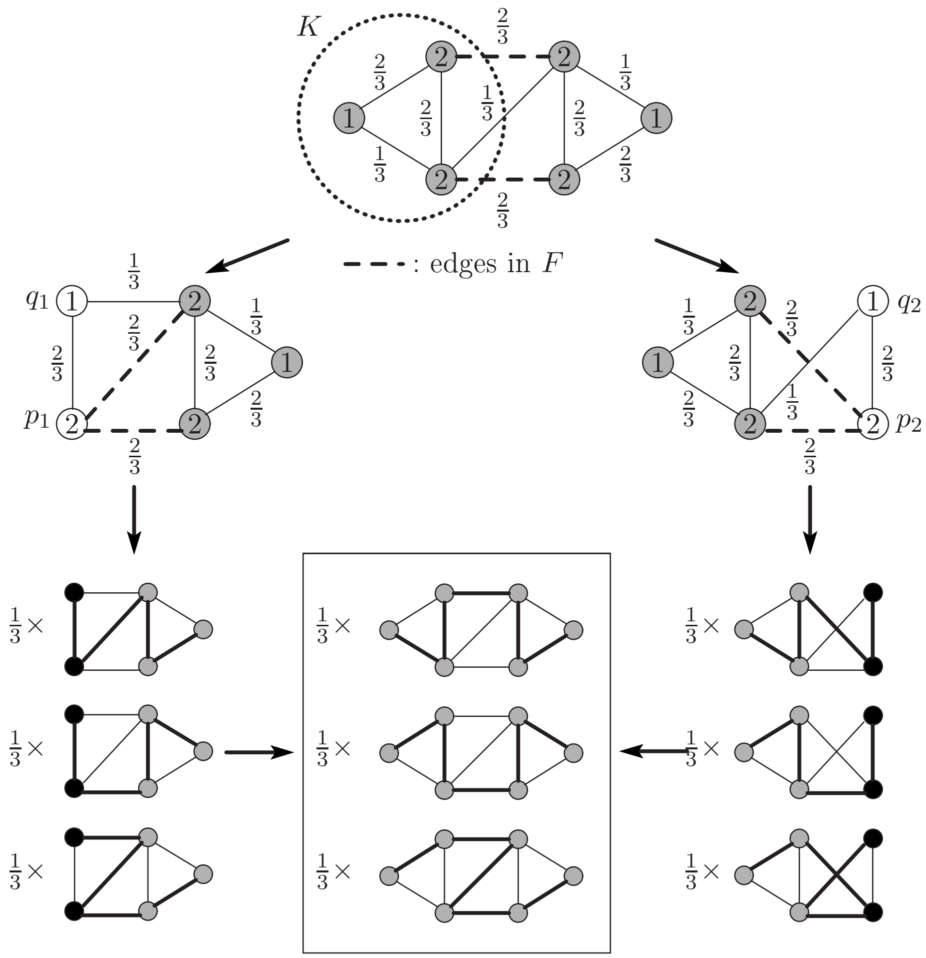

Figure 6.2: Illustration of the shrinking method

$\delta(K) \setminus F$, or all but one edges of F, the edge p_iq_i and none of the edges of $\delta(K) \setminus F$. We can write these combinations in the form $x_1^\circ = \frac{1}{k} \sum \chi_{M_i}$ and $x_2^\circ = \frac{1}{k} \sum \chi_{N_j}$ for some $k \in \mathbb{Z}_+$, where the M_i's and N_j's are (not necessarily distinct) b_1°- and b_2°-factors, respectively (note that x° is rational, being a vertex of a rational polytope).

Then each edge $e \in \delta(K) \setminus F$ is contained in exactly $kx(e)$ number of M_i's and N_j's. Each of them contains the entire F. We can pair these b-factors and 'glue' them together to get $kx(e)$ b-factors of G containing the edge e. This can be done for each edge $e \in \delta(K) \setminus F$. Similarly, for each edge $e \in F$ there are exactly $k(1 - x(e))$ M_i's and N_j's that does not contain e. Notice that these contain all edges in $F - e$ and none in $\delta(K) - F$. Again, pair and glue these together to get b-factors of G not containing e. For an illustration of this step, see Figure 6.2.

These b-factors altogether yield x as a convex combination of b-factors of G, a contradiction.

Remark 6.3.2. Note that the above proof also gives a new proof of Theorem 1.4.3 by using the well-known construction given below.

Take a copy of G denoted by G' and for each $v \in V$ add $b(v)$ new edges between v and v'. Let G^* be the graph thus arising and define $b^*(v) = b^*(v') = b(v)$. Theorem 1.4.3 follows as the restriction of a b^*-factor of G^* to G gives a b-matching in G, and the restriction of the b^*-factor polytope of G^* to G gives exactly the polytope described by P_3.

6.4 Triangle-free b-factors

In this section, we extend the proof of Theorem 1.4.2 to Theorem 6.1.1. Besides shrinking odd pairs, we also need to shrink triangles. The following shrinking operation appeared in [12].

Definition 6.4.1 (Shrinking a triangle)**.** Assume G, b and $\mathcal{T}$ satisfy (6.1) and (6.2). **Shrinking** a triangle $T \in \mathcal{T}$ consists of the following operations (see Figure 6.3):

- replace T by a node t,
- replace each edge $e \in E \setminus E_T$ with $e^u \in V_T, e^v \in V \setminus V_T$ by an edge te^v, and each edge $e \in E \setminus E_T$ with $e^u, e^v \in V_T$ by a loop e on t,
- let $b^\circ(t) = 2$ and define $b^\circ(v) = b(v)$ if $v \neq t$,
- let $\mathcal{T}^\circ$ denote the set of triangles in $\mathcal{T}$ node-disjoint from T.

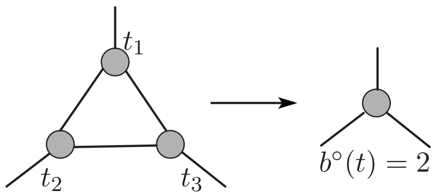

Figure 6.3: Shrinking a triangle

Similarly to Definition 6.2.1, we use the notation $G^\circ = (V^\circ, E^\circ)$ for the shrunk graph with $E^\circ \subseteq E$ and $V^\circ - t \subseteq V$. It is easy to see that G°, b° and $\mathcal{T}^\circ$ also satisfy (6.1) and (6.2).

Assume that $x \in \mathbb{R}^E$ satisfies (P_7). When shrinking a triangle, we use the notation x° for the image of x, that is, $x^\circ(e) = x(e)$ for each $e \in E^\circ$.

Lemma 6.4.2. *Let $G = (V, E)$, $b : V \to \mathbb{Z}_+$ and $\mathcal{T}$ a collection of triangles satisfying (6.1) and (6.2). Assume that there are no node-identical forbidden triangles in $\mathcal{T}$. If $x \in \mathbb{R}^E$ satisfies (P_7) and $T \in \mathcal{T}$ is a forbidden triangle, then x° satisfies (P_7) in $G^\circ = (V^\circ, E^\circ)$ with b° and $\mathcal{T}^\circ$.*

Proof. $(i), (iii)$ and (iv) easily follow from the same inequalities in the original graph. Also, (ii) holds for nodes different from t. As T is x-tight, $x^\circ(\dot{\delta}(t)) = x(\delta(V_T)) = \sum x(\dot{\delta}(t_i)) - 2x(E_T) = 2 = b^\circ(t)$. $\square$

Now we turn to the proof of Theorem 6.1.1. It is clear that a $\mathcal{T}$-free b-factor satisfies $(i) - (iv)$ ((iii) can be verified as in the proof of Theorem 1.4.2).

It remains to show that $(i) - (iv)$ completely determine the polytope in question, that is, any $x \in \mathbb{R}^E$ satisfying (P_7) is a convex combination of incidence vectors of $\mathcal{T}$-free b-factors. Assume that this does not hold. Let us choose x to be a vertex of the polytope described by (P_7) not contained in the $\mathcal{T}$-free b-factor polytope.

We choose this counterexample in such a way that $(|V|, |E|)$ is lexicographically minimal. This immediately implies that $\mathcal{T} = \emptyset$. Indeed, if there is a triangle $T \in \mathcal{T}$ then it is automatically tight, that is, $x(E_T) = 2$. Shrink T to a single node t as in Definition 6.4.1, obtaining G°, b°, $\mathcal{T}^\circ$, x°. By Lemma 6.4.2, these satisfy (P_7). As $|V^\circ| < |V|$, x° is a convex combination of $\mathcal{T}^\circ$-free b°-factors M_i of

G°. Note that $b^\circ(t) = 2$ and $d_{G^\circ}(t) \leq 3$ follows by (6.1). Let $x^\circ = \frac{1}{k}\sum \lambda_i \chi_{M_i^\circ}$. For each i, $|M_i^\circ \cap \delta(t)| = 2$. Moreover, $|M_i^\circ \cap \delta(t_j)| \leq 1$ for $j = 1, 2, 3$. We extend M_i° to a $\mathcal{T}$-free b-matching of G as follows: if $|M_i^\circ \cap \delta(t_j)| = |M_i^\circ \cap \delta(t_{j+1})| = 1$ (indices are meant modulo 3) then $M_i = M_i^\circ \cup \{e^T_{j,j+2}, e^T_{j+1,j+2}\}$.

Proposition 6.4.3. *M_i is a $\mathcal{T}$-free b-factor of G.*

Proof. Assume that $|M_i^\circ \cap \delta(t_1)| = |M_i^\circ \cap \delta(t_2)| = 1$. M_i cannot contain a triangle in $\mathcal{T}^\circ$, and neither contains T due to the construction. It suffices to check that it does not contain a triangle $T' \in \mathcal{T}$ which shares a node with T. By (6.1), T and T' must have an edge in common. If the common edge is e^T_{12}, then M_i does not contain T' since $e^T_{12} \notin M_i$. If the common edge is e^T_{13} then $e^T_{13}, e^T_{23} \in M_i$ and (6.2) implies that the edge of T' not incident to t_1 is not in M_i. The same argument works if the common edge of T and T' is e^T_{23}. $\square$

As $b(t_j) = 2$ for $j = 1, 2, 3$ and $x(E_T) = 2$, an easy computation shows that $x(e^T_{j,j+1}) = x(\dot{\delta}(t_{j+2}) \setminus E_T)$. This implies that $x = \frac{1}{k}\sum \chi_{M_i}$, a contradiction. So $\mathcal{T} = \emptyset$ indeed holds and the theorem follows from Theorem 1.4.2.

6.5 Extending the shrinking operations

Theorem 6.1.1 turned out to easily follow from Theorem 1.4.2 due to the fact that a forbidden triangle is always tight if (6.1) and (6.2) hold. Not surprisingly, this does not hold for b-matchings. In this section, we extend the notion of shrinking to tri-combs. To prove Theorem 6.1.2, we also need to slightly modify the notion of shrinking a triangle. We start with the latter one.

Definition 6.5.1 (Shrinking a triangle - extended)**.** Assume G, b and $\mathcal{T}$ satisfy (6.1) and (6.2). **Shrinking** a triangle $T \in \mathcal{T}$ consists of the following operations (see Figure 6.4):

- replace T by two nodes t, t',
- replace each edge $e \in E \setminus E_T$ with $e^u \in V_T, e^v \in V \setminus V_T$ by an edge te^v, and each edge $e \in E \setminus E_T$ with $e^u, e^v \in V_T$ by a loop e on t,
- add three edges between t and t' denoted by g_1, g_2 and g_3,
- let $b^\circ(t) = 2$, $b^\circ(t') = 2$ and define $b^\circ(v) = b(v)$ if $v \neq t, t'$,
- let $\mathcal{T}^\circ$ denote the set of triangles in $\mathcal{T}$ node-disjoint from T.

We use the notation $G^\circ = (V^\circ, E^\circ)$ for the shrunk graph with $E^\circ \setminus \{g_1, g_2, g_3\} \subseteq E$ and $V^\circ \setminus \{t, t'\} \subseteq V$. It is easy to see that G°, b° and $\mathcal{T}^\circ$ also satisfy (6.1) and (6.2).

Assume that $x \in \mathbb{R}^E$ satisfies (P_8). A triangle $T \in \mathcal{T}$ is called **x-tight** if it satisfies (iv) with equality. Let $T \in \mathcal{T}$ be a tight triangle with $V_T = \{t_1, t_2, t_3\}$ and $\delta(t_1) \setminus E_T = f_1$, $\delta(t_2) \setminus E_T = f_2$ and $\delta(t_3) \setminus E_T = f_3$ (two of these edges may coincide). When shrinking T, we use the notation x° for the image of x, namely

$$x^\circ(e) = \begin{cases} x(e) & \text{if } e \in E^\circ \setminus E^\circ[t, t'], \\ x(e^T_{i+1,i+2}) - x(f_i) & \text{if } e = g_i \text{ for } i = 1, 2, 3. \end{cases}$$

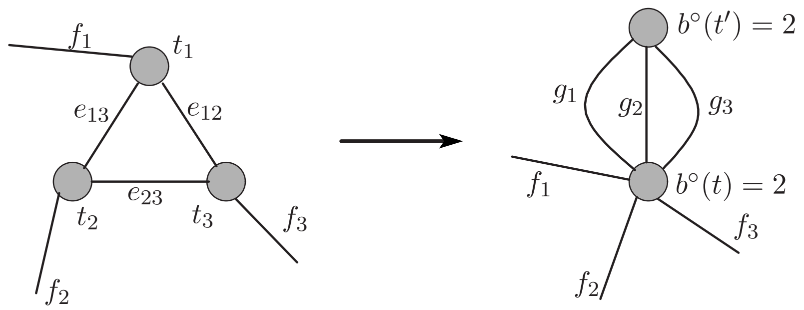

Figure 6.4: Shrinking a triangle - extended

Remark 6.5.2. In case of x being a b-factor, $x(g_i) = 0$ for each i, making the presence of edges g_1, g_2, g_3 unnecessary. That is the reason for the simpler definition of shrinking a triangle when proving Theorem 6.1.1.

Lemma 6.5.3. *Let $G = (V, E)$, $b : V \to \mathbb{Z}_+$ and $\mathcal{T}$ a collection of triangles satisfying (6.1) and (6.2). Assume that $x \in \mathbb{R}^E$ satisfies (P_8) and T is an x-tight triangle. Then x° satisfies (P_8) in $G^\circ = (V^\circ, E^\circ)$ with b° and $\mathcal{T}^\circ$.*

Proof. Let $V_T = \{t_1, t_2, t_3\}$ and $\delta(t_1) \setminus E_T = f_1$, $\delta(t_2) \setminus E_T = f_2$ and $\delta(t_3) \setminus E_T = f_3$ again. Then $(i), (iv)$ and (v) easily follow from the same inequalities in the original graph and from $x(g_i) = x(e^T_{i+1,i+2}) - x(f_i) \geq 0$. Also, (ii) holds for nodes different from t and t'. Clearly, $x^\circ(\dot{\delta}(t)) = x(E_T) = 2 = b^\circ(t)$. As for t', $x^\circ(\dot{\delta}(t')) = x(E_T) - \sum_i x(\delta(t_i) \setminus E_T) \leq 2 = b^\circ(t')$.

Concerning (iii), for a tri-comb $(Z, H, \mathfrak{R})$ with $Z \subseteq V^\circ, H \subseteq \delta(Z), \mathfrak{R} \subseteq \mathcal{T}^\circ$ the required inequality follows from the same inequality for $(Z \setminus \{t, t'\},\ H \setminus (\delta(t) \cup \delta(t')),\ \mathfrak{R})$ in the original graph. □

As mentioned earlier, forbidden triangles are not automatically tight in case of b-matchings. This phenomenon lead us to extend the notion of shrinking to more complex structures than odd pairs, namely to tri-combs, already introduced in Section 1.4.

Definition 6.5.4 (Shrinking a tri-comb of Type 1)**.** **Shrinking** a tri-comb $(K, F, \mathfrak{T})$ of Type 1 consists of the following operations (see Figure 6.5):

- replace K by an edge pq with $b^\circ(p) = |F| + |\mathfrak{T}|$ and $b^\circ(q) = 1$,
- replace each triangle $T \in \mathfrak{T}$ with $V_T = \{u, v, w\}$ and $V_T \cap K = \{u\}$ by edges $pr_T, r_Ts_T, r_Tt_T, s_Tv, t_Tw$ where r_T, s_T and t_T are new nodes with $b^\circ(r_T) = 2, b^\circ(s_T) = b^\circ(t_T) = 1$, and we also set $b^\circ(v) = b^\circ(w) = 1$,
- define $b^\circ(v) = b(v)$ for each $v \in V \setminus (K \cup V_{\mathfrak{T}})$,
- replace each edge $e \in E$ with $e^u \in K, e^v \in V \setminus K$ by an edge pe^v if $e \in F$, and by qe^v if $e \in \delta(K) \setminus (F \cup E_{\mathfrak{T}})$,
- let $\mathcal{T}^\circ$ denote the set of triangles in $\mathcal{T}$ node-disjoint from $K \cup V_{\mathfrak{T}}$.

We usually denote the graph obtained by shrinking a tri-comb of Type 1 by $G^\circ = (V^\circ, E^\circ)$. By abuse of notation, each edge $e \in \delta(K) \setminus E_{\mathfrak{T}}$ is denoted by e again after shrinking the tri-comb and is

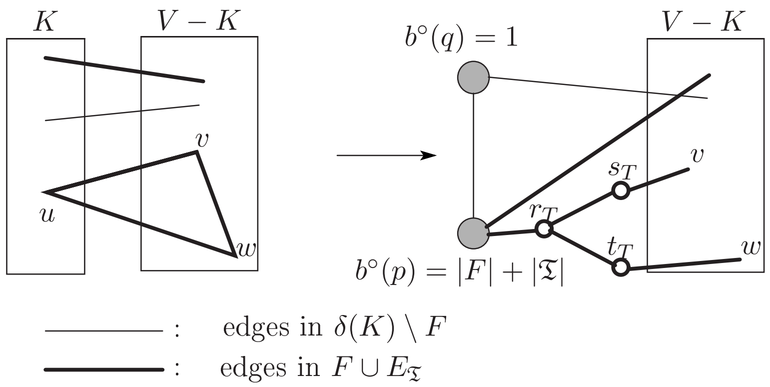

Figure 6.5: Shrinking a tri-comb of Type 1

called the **image** of the original edge. Hence the intersection $E \cap E^\circ$ stands for the set of all edges not induced by K nor by a triangle in $\mathfrak{T}$.

Assume that $x \in \mathbb{R}^E$ satisfies (P_8). When shrinking a tri-comb of Type 1, we use the notation x° for the image of x, namely

- for an edge $e \in E \cap E^\circ$ let $x^\circ(e) = x(e)$,
- for a triangle $T \in \mathfrak{T}$ with $V_T = \{u, v, w\}$ and $V_T \cap K = \{u\}$ consider the new edges mentioned in Definition 6.5.4, and define

$$\begin{aligned} x^\circ(pr_T) &= 2x(e^T_{vw}) + x(e^T_{uv}) + x(e^T_{uw}) - 2, \\ x^\circ(r_T s_T) &= 2 - x(e^T_{vw}) - x(e^T_{uv}), \\ x^\circ(r_T t_T) &= 2 - x(e^T_{vw}) - x(e^T_{uw}), \\ x^\circ(s_T v) &= x(e^T_{vw}) + x(e^T_{uv}) - 1, \\ x^\circ(t_T w) &= x(e^T_{vw}) + x(e^T_{uw}) - 1, \end{aligned}$$

- define $x^\circ(pq) = |F| + 3|\mathfrak{T}| - x(F) - \sum_{T\in\mathfrak{T}} x(E_T) - \sum_{T\in\mathfrak{T}} x(e_T)$.

Recall that e_T denotes the special edge of triangle T, that is, the edge in E_T having no end in K.

Definition 6.5.5 (Shrinking an odd tri-comb of Type 2)**.** **Shrinking** a tri-comb $(K, F, \mathfrak{T})$ of Type 2 consists of the following operations (see Figure 6.6):

- replace K by an edge pq with $b^\circ(p) = |F| + |\mathfrak{T}|$ and $b^\circ(q) = 1$,
- replace each triangle $T \in \mathfrak{T}$ with $V_T = \{u, v, w\}$ and $V_T \cap K = \{u, v\}$ by an edge pr_T, a loop l_T on r_T, and two parallel edges between r_T and w_T (denoted by $r_T w^1$ and $r_T w^2$) where r_T is a new node with $b^\circ(r) = 2$,
- define $b^\circ(v) = b(v)$ for each $v \in V \setminus K$,
- replace each edge $e \in E$ with $e^u \in K, e^v \in V \setminus K$ by an edge pe^v if $e \in F$, and by qe^v if $e \in \delta(K) \setminus (F \cup E_{\mathfrak{T}})$,
- let $\mathcal{T}^\circ$ denote the set of triangles in $\mathcal{T}$ node-disjoint from K.

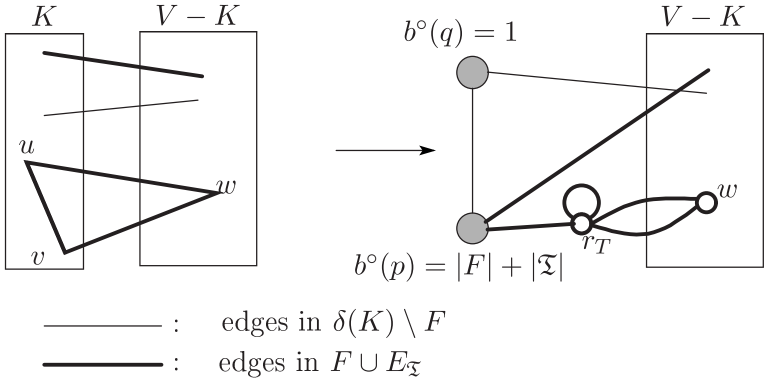

Figure 6.6: Shrinking a tri-comb of Type 2

We usually denote the graph obtained by shrinking a tri-comb of Type 2 by $G^\circ = (V^\circ, E^\circ)$. Again, each edge $e \in \delta(K) \setminus E_{\mathfrak{T}}$ is denoted by e again after shrinking the tri-comb.

Assume that $x \in \mathbb{R}^E$ satisfies (P_8). When shrinking a tri-comb of Type 2, we use the notation x° for the image of x, namely

- for an edge $e \in E \cap E^\circ$ let $x^\circ(e) = x(e)$,
- for a triangle $T \in \mathfrak{T}$ with $V_T = \{u, v, w\}$ and $V_T \cap K = \{u, v\}$ consider the new edges mentioned in Definition 6.5.5, and define

$$\begin{aligned} x^\circ(pr_T) &= 2x(e^T_{uv}) + x(e^T_{vw}) + x(e^T_{uw}) - 2, \\ x^\circ(l_T) &= 2 - x(e^T_{uv}) - x(e^T_{vw}) - x(e^T_{uw}), \\ x^\circ(r_T w^1) &= x(e^T_{uw}), \\ x^\circ(r_T w^2) &= x(e^T_{vw}), \end{aligned}$$

- define $x^\circ(pq) = |F| + 3|\mathfrak{T}| - x(F) - \sum_{T\in\mathfrak{T}} x(E_T) - \sum_{T\in\mathfrak{T}} x(e_T)$.

Recall that e_T denotes the special edge of triangle T, that is, the edge in E_T having both ends in K.

An odd tri-comb $(K, F, \mathfrak{T})$ of Type 2 is called **x-tight** (or **tight**, for short) if it satisfies (*iii*) with equality. A tri-comb $(K, F, \mathfrak{T})$ of Type 1 is called **tight** if $(\bar{K}, F, \mathfrak{T})$ is a tight tri-comb of Type 2. If $\mathfrak{T} = \emptyset$ then (K, F) is called a **tight pair** instead.

The following simple observation will be useful later.

Proposition 6.5.6. *Let $(K, F, \mathfrak{T})$ be an x-tight tri-comb of any type for some $0 < x < 1$ satisfying (P_8). For any $F' \subseteq F, \mathfrak{T}' \subseteq \mathfrak{T}, \mathfrak{T}'' \subseteq \mathfrak{T}$ and $H \subseteq \delta(K) \setminus (F \cup E_{\mathfrak{T}})$ we have*

$$x(H) \leq 1$$

and

$$|F'| + 2|\mathfrak{T}'| + |\mathfrak{T}''| - 1 \leq x(F') + \sum_{T\in\mathfrak{T}'} x(E_T) + \sum_{T\in\mathfrak{T}''} x(e_T) \leq |F'| + 2|\mathfrak{T}'| + |\mathfrak{T}''|.$$

Moreover, if at least one of F' and $\mathfrak{T}''$ is nonempty then the upper bound hold with strict inequality.

Proof. We may assume that the tri-comb is of Type 2. Summing up inequalities $x(\dot{\delta}(v)) \leq b(v)$ for $v \in K$, $x(e) \leq 1$ for $e \in F$, $x(E_T) \leq 2$ and $x(e_T) \leq 1$ for $T \in \mathfrak{T}$ gives

$$2x(E[K]) + x(\delta(K)) + x(F) + \sum_{T\in\mathfrak{T}} x(E_T) + \sum_{T\in\mathfrak{T}} x(e_T) \leq b(K) + |F| + 3|\mathfrak{T}|.$$

As $(K, F, \mathfrak{T})$ is x-tight, we have

$$x(E[K]) + x(F) + \sum_{T\in\mathfrak{T}} x(E_T) = \tfrac{b(K)+|F|+3|\mathfrak{T}|-1}{2}.$$

These together imply $x(\delta(K) \setminus (F \cup E_{\mathfrak{T}})) \leq 1$, hence proving the first part. The upper bound in the second part follows from $x < 1$ (from what strict inequality immediately follows if F' or $\mathfrak{T}''$ is not empty). On the other hand, the tightness of the tri-comb means that we may loose at most 1 when summing up the inequalities as described above, hence

$$x(F) + \sum_{T\in\mathfrak{T}} x(E_T) + \sum_{T\in\mathfrak{T}} x(e_T) \geq |F| + 3|\mathfrak{T}| - 1,$$

from what the lower bound follows by $x < 1$. □

In the sequel, we will refer to the following special case of Proposition 6.5.6 several times.

Corollary 6.5.7. *If v is a node without loops and $x(\delta(v)) = b(v) = d(v) - 1$ then $x(F) \geq |F| - 1$ for any $F \subseteq \delta(v)$.*

Proof. The tri-comb $(v, \delta(v), \emptyset)$ is odd as $b(v) + |\delta(v)| = b(v) + d(v) = 2d(v) - 1$ and is also tight as $x(\delta(v)) = d(v) - 1 = \frac{b(v)+|\delta(v)|-1}{2}$. The statement follows from Proposition 6.5.6. □

The main advantage of shrinking odd pairs was that the arising graph G° and vector x° still satisfied (P_2). The above definitions also have this useful property, as shown in the following lemma. The proof is rather technical and needs a lot of computation, hence is left to the end of this chapter. The reader may skip it in order to follow the main idea of the proof of Theorem 6.1.2.

Lemma 6.5.8. *Let $G = (V, E)$, $b : V \to \mathbb{Z}_+$ and $\mathcal{T}$ a collection of triangles satisfying (6.1) and (6.2). Assume that $x \in \mathbb{R}^E$, $0 < x < 1$ satisfies (P_8) and $(K, F, \mathfrak{T})$ is an x-tight tri-comb of Type 2. Then either shrinking $(K, F, \mathfrak{T})$ or $(\bar{K}, F, \mathfrak{T})$, (6.1) and (6.2) hold for $G^\circ = (V^\circ, E^\circ)$. Moreover, $b^\circ, \mathcal{T}^\circ$ and x° satisfies (P_8).*

Remark 6.5.9. In the above, we only defined shrinking for tri-combs either of Type 1 or 2. The definition could be easily generalized to shrink gadgets having both triangles 1-fitting and 2-fitting them. The reason for not introducing shrinking in that way was the form of description (P_8).

6.6 Proof of Theorem 6.1.2

It is easy to see that each $\mathcal{T}$-free b-matching satisfies (*i*), (*ii*), (*iv*) and (*v*). To show that (*iii*) indeed holds for a $\mathcal{T}$-free b-matching $M \subseteq E$, take an odd tri-comb $(K, F, \mathfrak{T})$ and add up inequalities $d_M(v) \leq b(v)$ for $v \in K$, $|M \cap F| \leq |F|$, $|M \cap E_T| \leq 2$ and $|M \cap e_T| \leq 1$ for $T \in \mathfrak{T}$. This gives

$$2|M \cap E[K]| + |M \cap \delta(K)| + |M \cap F| + \sum_{T \in \mathfrak{T}} (|M \cap E_T| + |M \cap e_T|) \leq b(K) + |F| + 3|\mathfrak{T}|.$$

Clearly, $|M \cap F| + |M \cap E_{\mathfrak{T}}| \leq |M \cap \delta(K)| + \sum_{T \in \mathfrak{T}} |M \cap e_T|$, so $|M \cap E[K]| + |M \cap F| + \sum_{T \in \mathfrak{T}} |M \cap E_T| \leq \lfloor \frac{1}{2}(b(K) + |F| + 3|\mathfrak{T}|) \rfloor$, as required. The above proof easily implies that (iii) is also valid for even tri-combs, where a tri-comb $(K, F, \mathfrak{T})$ is called **even** if $b(K) + |F| + |\mathfrak{T}|$ is even.

It remains to show that $(i) - (v)$ completely determine the $\mathcal{T}$-free b-matching polytope, that is, any $x \in \mathbb{R}^E$ satisfying (P_8) is a convex combination of incidence vectors of $\mathcal{T}$-free b-matchings. Assume that this does not hold. Let us choose x to be a vertex of the polytope described by (P_8) not contained in the $\mathcal{T}$-free b-matching polytope.

We choose this counterexample in such a way that $(|\mathcal{T}|, |\ell(V)|, b(V), |V|, |E|)$ is lexicographically minimal. G is connected, otherwise one of its components would be a smaller counterexample. As x is a vertex, it satisfies $|E|$ linearly independent constraints among (P_8) with equality. We call a node, a tri-comb or a triangle x**-tight** (or simply **tight** for short) if the corresponding inequality, which is of type (ii), (iii) or (iv), respectively, is satisfied with equality. Also, the corresponding inequality is called x-tight. We also use this notation for even tri-combs satisfying (iii) with equality.

From now on, our aim is to show that there is a tight tri-comb or triangle whose shrinking results in a lexicographically smaller problem. Then we show that a proper convex combination for the smaller problem can be transformed into a convex combination for the original problem giving x, thus leading to contradiction. However, this latter step requires much more work than it did in case of b-factors.

We start with some technical observations.

Proposition 6.6.1. *For each $T \in \mathcal{T}$, V_T does not span parallel edges.*

Proof. Assume to the contrary that $V_T = \{u, v, w\}$ spans parallel edges, say between v and w as on Figure 6.7. By (6.1), $d(u), d(v), d(w) \leq 3$. We claim that G is in fact consists of these three nodes, or these three nodes plus an edge incident to u. Indeed, $d(u) \leq 3$ implies that if $|V| \geq 4$ then u has a third neighbour different from v and w, say z, and uz is a cutting edge in G. Let G_1 and G_2 denote the graphs consisting of a component of $G - uz$ plus uz. We denote by $x_1, b_1, \mathcal{T}_1$ and $x_2, b_2, \mathcal{T}_2$ the natural restriction of x, b and $\mathcal{T}$ to G_1 and G_2, respectively. If both of these graphs have at least two nodes then we get two lexicographically smaller instances, hence x_i is a convex combination of $\mathcal{T}_i$-free b_i-matchings of G_i. These could be glued together as to get a convex combination of $\mathcal{T}$-free b-matchings of G giving x, a contradiction.

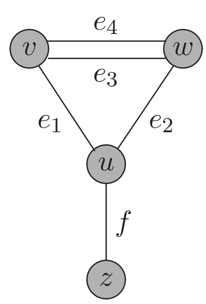

Figure 6.7: V_T spanning parallel edges

So G is in fact consists of four or three nodes. Let us consider the first case, the second can be handled similarly (by using (v) of (P_8)). We use the notation of Figure 6.7. First assume that both triangles are forbidden. Delete z from G. The graph thus arising is not a counterexample, hence the restriction of x to $G-z$ is a convex combination of $\mathcal{T}$-free b-matchings of $G-z$. Let $\frac{1}{k}\sum \chi_{M_i}$ denote this combination and let $\lambda_I = \frac{1}{k}|\{i: \ M_i = \{e_j: \ j \in I\}\}|$ for $I \subseteq \{1,2,3,4\}$. Moreover, take a convex combination with λ_{12} as small as possible. That means that $\lambda_{12} = 0$ or $\lambda_3 = \lambda_4 = \lambda_{34} = 0$. Indeed, assume to the contrary that both $\lambda_{12} > 0$ and $\lambda_{34} > 0$ hold. Take an M_i with $e_1, e_2 \in M_i$ and an M_j with $e_3, e_4 \in M_j$ and exchange the edges e_1 and e_3 between them. Then we get $\mathcal{T}$-free b-matchings still giving the restriction of x to $G-z$ but the value of λ_{12} decreased, a contradiction. The other cases can be proved similarly.

If $\lambda_{12} = 0$ then f can be added to any of these b-matchings, a contradiction. So $\lambda_3 = \lambda_4 = \lambda_{34} = 0$ and $\lambda_{12}+\lambda_{13}+\lambda_{14}+\lambda_{23}+\lambda_{24}+\lambda_1+\lambda_2 = 1$. If $\lambda_{12} \leq 1-x(f)$ then we can add the edge f to some of these b-matchings with coefficients in total equals $x(f)$ and so get a proper convex combination in the original graph, a contradiction. Hence $x(\dot{\delta}(u)) = x(f)+2\lambda_{12}+\lambda_{13}+\lambda_{14}+\lambda_{23}+\lambda_{24}+\lambda_1+\lambda_2 > 2$, a contradiction.

Now assume that only one of the triangles, say $\{e_1, e_2, e_3\}$, is forbidden. Delete z from G. The graph thus arising is not a counterexample, hence the restriction of x to $G-z$ is a convex combination of $\mathcal{T}$-free b-matchings of $G-z$. Let $\frac{1}{k}\sum \chi_{M_i}$ denote this combination and let $\lambda_I = \frac{1}{k}|\{i: \ M_i = \{e_j: \ j \in I\}\}|$ for $I \subseteq \{1,2,3,4\}$. Moreover, take a convex combination with λ_{12} as small as possible, and beside this, λ_{124} as small as possible. That means that $\lambda_{12} = 0$ or $\lambda_3 = \lambda_4 = \lambda_{34} = 0$, and also $\lambda_{124} = 0$ or $\lambda_3 = \lambda_4 = 0$. If both $\lambda_{12} = \lambda_{124} = 0$ then f can be added to any of these b-matchings, a contradiction. Otherwise if $\lambda_{12}+\lambda_{124} \leq 1-x(f)$ then we can add the edge f to some of these b-matchings with total coefficients $x(f)$ and so get a proper convex combination in the original graph, a contradiction again. Hence $\lambda_{12}+\lambda_{124} > 1-x(f)$ and $\lambda_{12}+\lambda_{13}+\lambda_{14}+\lambda_{23}+\lambda_{24}+\lambda_{34}+\lambda_1+\lambda_2 = 1$. We have

$$\begin{aligned} x(E[V_T]) + x(f) &= 3\lambda_{124} + 2\lambda_{12} + 2\lambda_{13} + 2\lambda_{14} + 2\lambda_{23} + 2\lambda_{24} + 2\lambda_{34} + \lambda_1 + \lambda_2 + x(f) \\ &= \lambda_{124} + 2 + x(f) \\ &> 3 - \lambda_{12}. \end{aligned}$$

As x satisfies (iii) of (P_8) for the odd pair (V_T, f), $\lambda_{12} > 0$ must hold. But then $\lambda_{34} = 0$ and so $x(\dot{\delta}(u)) = x(f)+2\lambda_{124}+2\lambda_{12}+\lambda_{13}+\lambda_{14}+\lambda_{23}+\lambda_{24}+\lambda_1+\lambda_2 > 2$, a contradiction. □

Proposition 6.6.2. $0 < x(e) < 1$ *for each* $e \in E$.

Proof. Clearly, edges with $x(e) = 0$ could be deleted, contradicting minimality.

If $x(e) = 1$ and $\mathcal{T} = \emptyset$, delete e and decrease b on its endnodes by 1 (if e is a loop on v then decrease $b(v)$ by 2). However, the situation is more complicated if $\mathcal{T} \neq \emptyset$. If $e \in E_T$ for some $T \in \mathcal{T}$, it may happen that there is a proper convex combination in the smaller graph, but it can not be extended to the original problem because a triangle may arise. Hence we use a simple trick here to show $x(e) < 1$.

Assume that $x(uv) = 1$ and let $\mathcal{T}_{uv} \subseteq \mathcal{T}$ denote the set of triangles containing uv (there are at most two such triangles as (6.1) holds). Note that the edge uv is well-defined as there exist no parallel edges between u and v by Proposition 6.6.1. For a triangle $T \in \mathcal{T}_{uv}$, let t_T denote its third node.

By (6.1), t_T has at most one neighbour different from u and v, denoted by z_T (if exists). Delete $e = uv$ from G, decrease $b(u)$ and $b(v)$ by one, for each $T \in \mathcal{T}_{uv}$ decrease $b(t_T)$ by one, delete -if exists-

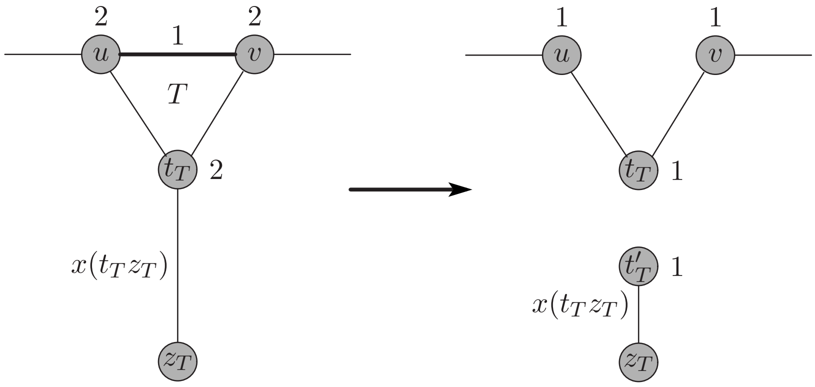

Figure 6.8: Excluding saturated edges

$t_T z_T$ and add a new edge $t'_T z_T$ where t'_T is a new node. The graph thus arising will be denoted by $G' = (V', E')$. The modified degree prescription is denoted by b' (with $b'(t'_T) = 1$ for a new node) and the natural image of x on E' is denoted by x' (that is, $x'(t'_T z_T) = x(t_T z_T)$). Let $\mathcal{T}' \subseteq \mathcal{T}$ denote the set of triangles disjoint from the triangles in $\mathcal{T}_{uv}$. The degree condition implies that two triangles are node-disjoint if and only if they are edge-disjoint. It is easy to check that x' satisfies (P_8) in G' with b' and $\mathcal{T}'$.

As $|\mathcal{T}'| < |\mathcal{T}|$, x' is a convex combination of incidence vectors of $\mathcal{T}'$-free b'-matchings of G', say $x' = \frac{1}{k}\sum \chi_{M'_i}$. These b'-matchings use at most one of $e^T_{ut_T}, e^T_{vt_T}$ for each $T \in \mathcal{T}_{uv}$. If we extend M'_i by uv and edges $\{t_T z_T : \; T \in \mathcal{T}_{uv},\; t'_T z_T \in M'_i\}$, we get a $\mathcal{T}$-free b-matching M_i of G by (6.2) and Proposition 6.6.1.

An easy computation shows that $x = \frac{1}{k}\sum \chi_{M_i}$, hence x is a convex combination of $\mathcal{T}$-free b-matchings of G, a contradiction. □

So we may assume that $0 < x(e) < 1$ for each edge $e \in E$.

Proposition 6.6.3. *For each $u, v \in V$, $x(E[u,v]) < 1$.*

Proof. If $|E[u,v]| = 1$ then the proposition follows from Proposition 6.6.2. Otherwise no edge in $E[u,v]$ is contained in a forbidden triangle by Proposition 6.6.1 and we can decrease the x-values on them by one in total and also decrease $b(u), b(v)$ by one, thus obtaining a smaller counterexample, a contradiction. □

Claim 6.6.4. *There is no x-tight triangle $T \in \mathcal{T}$.*

Proof. Assume that there exists a tight triangle T and let $V_T = \{t_1, t_2, t_3\}$. Shrink T to a single node t as in Definition 6.5.1, obtaining G°, b°, $\mathcal{T}^\circ$, x°. By Lemma 6.5.3, these satisfy (P_8).

As $|\mathcal{T}^\circ| < |\mathcal{T}|$, x° is a convex combination of $\mathcal{T}^\circ$-free b°-matchings M_i° of G°. Let $x^\circ = \frac{1}{k}\sum \chi_{M_i^\circ}$ and let $\alpha_{jl} = \frac{1}{k}|\{i : \; f_j, f_l \in M_i\}|$, $\beta_{jl} = \frac{1}{k}|\{i : \; f_j, g_l \in M_i\}|$ and finally $\gamma_{jl} = \frac{1}{k}|\{i : \; g_j, g_l \in M_i\}|$ where $f_1, f_2, f_3, g_1, g_2, g_3$ are as in Definition 6.5.1. As $x^\circ(\dot{\delta}(t)) = 2$, we have $\sum \alpha_{jl} + \sum \beta_{jl} + \sum \gamma_{jl} = 1$.

Proposition 6.6.5. *There exist a proper convex combination for what $\sum \beta_{jj} = 0$.*

Proof. Take a combination in which $\sum \beta_{jj}$ is minimal and assume that $\beta_{11} > 0$. This immediately implies that $\beta_{22}, \beta_{23}, \beta_{32}, \beta_{33}, \gamma_{23} = 0$ as otherwise we could easily modify the b°-matchings and decrease $\sum \beta_{jj}$.

We have the following equalities.

$$\begin{aligned}
\alpha_{12}+\alpha_{13}+\beta_{11}+\beta_{12}+\beta_{13} &= x(f_1),\\
\alpha_{12}+\alpha_{23}+\beta_{21} &= x(f_2),\\
\alpha_{13}+\alpha_{23}+\beta_{31} &= x(f_3),\\
\beta_{11}+\beta_{21}+\beta_{31}+\gamma_{12}+\gamma_{13} &= x(t_2t_3)-x(f_1),\\
\beta_{12}+\gamma_{12} &= x(t_1t_3)-x(f_2),\\
\beta_{13}+\gamma_{13} &= x(t_1t_2)-x(f_3).
\end{aligned}$$

From these and from $x(E_T)=2$ we get $\alpha_{23}-\beta_{11}=1-x(t_2t_3)>0$. Hence there is an M_i, say M_1, with $f_1,g_1\in M_1$ and another one, say M_2, with $f_2,f_3\in M_2$. The proof of Theorem 4.1 of [88] implies that we can take an alternating path P in $M_1\triangle M_2$ starting at t' such that $M_1\triangle P$ and $M_2\triangle P$ are also $\mathcal{T}^\circ$-free b°-matchings of G°. Hence β_{11} can be decreased while β_{22} and β_{33} do not change, so in total $\sum\beta_{ii}$ can be decreased, and the proposition follows. □

Take a convex combination $\frac{1}{k}\sum\chi_{M_i}$ as in Proposition 6.6.5. We extend the M_i°'s to $\mathcal{T}$-free b-matchings of G as follows: if $M_i^\circ\cap\delta(t)=\{f_j,f_l\}$ or $\{f_j,g_l\}$ or $\{g_j,g_l\}$ where $j\neq l$ then define $M_i=M_i^\circ\cup(E_T-e^T_{j,l})$.

It suffices to verify that the b-matchings thus arising are $\mathcal{T}$-free b-matchings of G. Indeed, they cannot contain any triangle in $\mathcal{T}^\circ$, and neither contain T due to the construction. For a triangle $T'\in\mathcal{T}$ which shares a node with T, by (6.1), T and T' must have an edge in common. By Proposition 6.6.1, they do not have the same node-set but then (6.2) implies that at least one of the edges of T' is not in M_i.

The convex combination of the M_i's gives x. To see this, it suffices to check that the combination gives $x(e^T_{j,j+1})$ in total for each $j=1,2,3$. This is assured by the choice of the coefficients as T is tight. □

If x is a b-factor, that is, $x(\dot\delta(v))=b(v)$ for each $v\in V$ then each $T\in\mathcal{T}$ is tight. By Theorem 1.4.2 and Claim 6.6.4, x is not a b-factor. So our aim is now to show that there is an x-tight odd tri-comb $(K,F,\mathfrak{T})$ of Type 2 whose shrinking lexicographically decreases $(|\mathcal{T}|,b(V),\ell(V),|V|,|E|)$, and the same holds for $(\bar K,F,\mathfrak{T})$.

The next proposition states that, as one would expect, $b\le d$ can be assumed.

Proposition 6.6.6. *$b(v)\le\min\{d(v),\lceil x(\dot\delta(v))\rceil+1\}$ for each $v\in V$.*

Proof. Assume that $b(v)>d(v)$ for some $v\in V$. By (6.1) and (6.2), v is not a node of a triangle. Set $b(v):=d(v)$. We claim that the inequalities of (P_8) remain valid, contradicting the minimal choice of the counterexample. Assume indirectly that there is a tri-comb $(K,F,\mathfrak{T})$ with $v\in K$ violating (iii) after the modification. However, for the tri-comb $(K-v,F\setminus F_v\cup E[v,K-v],\mathfrak{T})$ the left hand side of (iii) decreases by $x(\ell(v))+x(F_v)$ while the right decreases by exactly $\frac{1}{2}(d(v)+|F_v|-|E[v,K-v]|)=|\ell(v)|+|F_v|$ (compared to $(K,F,\mathfrak{T})$ after the modification) implying that $(K-v,F\setminus F_v\cup E[v,K-v],\mathfrak{T})$ is a violating odd tri-comb in the original problem, a contradiction.

If we set $b'(v) := \lceil x(\dot\delta(v)) \rceil$ for each $v \in V$ then $(i), (ii), (iv)$ and (v) clearly remains valid in (P_8). Assume that there is an odd tri-comb $(K, F, \mathfrak{T})$ violating (iii) after the modification. Inequalities of form (iii) are obtained by summing up inequalities of from (i) and (ii), then dividing by two and taking the floor of the right hand side. But until the very last step the inequality remains valid, so the violation, that is, the deficiency of the tri-comb can be at most $\frac{1}{2}$. Hence setting $b'(v) := \min\{b(v), \lceil x(\dot\delta(v)) \rceil + 1\}$ assures that no violating tri-comb arises.

The proposition follows by the choice of the counterexample. □

Since G is connected, $|E| \geq |V| - 1$. If $|E| = |V| - 1$ or $|E| = |V|$ and G does not contain triangles then x is a convex combination of b-matchings by Theorem 1.4.3, a contradiction. Assume that $|E| = |V|$ and $\mathcal{T} \neq \emptyset$. This is only possible if G is obtained from a tree by replacing a node with a triangle (where the degree of a node of the triangle should not exceed 3). If after deleting the edges of the triangle at least one of the connected components has size larger than 2 then the G can be divided into two smaller graphs as in the proof of Proposition 6.6.1, giving a contradiction. So G is in fact a triangle with at most one extra edge at each of its nodes. These cases can be easily seen not to give a counterexample (similarly to the proof of Proposition 6.6.1), hence we may assume that $|E| > |V|$.

We call an even tri-comb $(K, F, \mathfrak{T})$ **tight** if $x(E[K]) + x(F) + \sum_{T \in \mathfrak{T}} x(E_T) = \frac{b(K)+|F|+3|\mathfrak{T}|}{2}$.

Proposition 6.6.7. *Let (K, F) be a tight pair (odd or even), $v \in \bar{K}$. If $b(v) \leq |F_v|$ then $(K + v, F \setminus F_v)$ is also tight. Moreover, $\ell(v) = \emptyset$ and $E[v, K] \setminus F = \emptyset$.*

Proof. By adding v to K, the left hand side of (iii) of (P_8) may only increase while the right hand side may only decrease. The second part follows by Proposition 6.6.2. □

If there is an x-tight odd tri-comb $(K, F, \mathfrak{T})$ such that $\mathfrak{T} \neq \emptyset$, then $|\mathcal{T}|$ decreases when shrinking either $(K, F, \mathfrak{T})$ or $(\bar{K}, F, \mathfrak{T})$, and we are done. So assume that this is not the case. Recall that a tight tri-comb $(K, F, \mathfrak{T})$ with $\mathfrak{T} = \emptyset$ was called a tight pair.

We have already seen that there is no tight constraint of form $(i), (iv)$ or (v), and now we assumed that neither of form (iii) with $\mathfrak{T} \neq \emptyset$. Let us call an x-tight constraint **bad** if it is of form (ii) for some $v \in V$, or it is of form (iii) for some odd pair (K, F) and at least one of the followings holds.

(I) $\ell(K) = \emptyset, b(K) \leq |F|$

(II) $\ell(K) = \emptyset, b(K) = |F| + 1, |K| = 1$

(III) $\ell(K) = \emptyset, b(K) = |F| + 1, |K| = 2, |E[K]| \leq 1$

(IV) $\ell(\bar{K}) = \emptyset, b(\bar{K}) \leq |F|$

(V) $\ell(\bar{K}) = \emptyset, b(\bar{K}) = |F| + 1, |\bar{K}| = 1$

(VI) $\ell(\bar{K}) = \emptyset, b(\bar{K}) = |F| + 1, |\bar{K}| = 2, |E[\bar{K}]| \leq 1$

If the shrinking of (K, F) or the shrinking of $(\bar{K}, F)$ does not result in a lexicographically smaller problem then (K, F) must be bad (however, it may happen that we get a smaller problem even in case of a bad pair as $\mathcal{T}_K \neq \emptyset$ would also assure that).

As we may assume that $|E| > |V|$, the existence of a tight odd pair (K, F) whose shrinking results in a lexicographically smaller problem and the same holds for $(\bar{K}, F)$ is assured by the following fundamental lemma. The proof of the lemma is quite technical and is detailed in the end of the chapter.

Lemma 6.6.8. *Under the assumption that there is no tight constraint of form (iii) with $\mathfrak{T} \neq \emptyset$, the maximum number of linearly independent bad constraints is at most $|V|$.*

As $|E| > |V|$, Lemma 6.6.8 implies that there exists a tight odd tri-comb $(K, F, \mathfrak{T})$ whose shrinking lexicographically decreases the problem, and the same holds for $(\bar{K}, F, \mathfrak{T})$. More precisely, there is a tight tri-comb $(K, F, \mathfrak{T})$ with either $\mathfrak{T} \neq \emptyset$ or being independent from $\mathcal{L}$ defined earlier. Take such a tri-comb with $|K|$ being minimal and let $G_1^\circ = (V_1^\circ, E_1^\circ), b_1^\circ, x_1^\circ, \mathcal{T}_1^\circ$ and $G_2^\circ = (V_2^\circ, E_2^\circ), b_2^\circ, x_2^\circ, \mathcal{T}_2^\circ$ denote the problems arising through shrinking $(K, F, \mathfrak{T})$ and $(\bar{K}, F, \mathfrak{T})$, respectively. We refer to the new nodes p, q in these graphs by p_1, q_1 and p_2, q_2, respectively. By the minimality of the counterexample, x_i° is a convex combination of $\mathcal{T}^\circ{}_i$-free b_i°-matchings of G_i°, say, $x_1^\circ = \frac{1}{k}\sum \chi_{M_i}$ and $x_2^\circ = \frac{1}{2}\sum \chi_{N_j}$ for some $k \in \mathbb{Z}_+$ (note that x_i° is rational, being a vertex of a rational polytope). The following proposition is an easy observation.

Proposition 6.6.9. *The tightness of $(K, F, \mathfrak{T})$ implies that exactly one of the followings holds for each M_i:*

- $(\delta(p_1) - p_1q_1) \subseteq M_i,\ |(\delta(q_1) - p_1q_1) \cap M_i| \leq 1,$ *or*
- $|(\delta(p_1) - p_1q_1) \setminus M_i| = 1,\ p_1q_1 \in M_i,\ (\delta(q_1) - p_1q_1) \cap M_i = \emptyset.$

Similarly, for N_j's:

- $(\delta(p_2) - p_2q_2) \subseteq N_j,\ |(\delta(q_2) - p_2q_2) \cap N_j| \leq 1,$ *or*
- $|(\delta(p_2) - p_2q_2) \setminus N_j| = 1,\ p_2q_2 \in N_j,\ (\delta(q_2) - p_2q_2) \cap N_j = \emptyset.$

Each edge $e \in \delta(K) \setminus (F \cup E_{\mathfrak{T}})$ is contained in exactly $kx(e)$ number of M_i's and N_j's. By the above observation, each of these M_i's contains the entire F and edges pr_T, r_Tw^1 or pr_T, r_Tw^2 for each $T \in \mathfrak{T}$, while each of the N_j's contains the entire F and edges pr_T, r_Ts_T, t_Tw or pr_T, r_Tt_T, s_Tv. However, it is easy to see that, as they are parallel, the role of edges r_Tw^1 and r_Tw^2 can be 'exchanged' in such a way that the total number of M_i's with $pr_T, r_Tw^1 \in M_i$ is equal to the number of N_j's with $pr_T, r_Tt_T, s_Tv \in N_j$. This makes possible to pair these b_i°-matchings and 'glue' them together to get $kx(e)$ b-matchings of G containing the edge e. A b-matching obtained by gluing an M_i with $pr_T, r_Tw^1 \in M_i$ and an N_j with $pr_T, r_Tt_T, s_Tv \in N_j$ contains e_{vw}^T and e_{uw}^T from E_T; a b-matching obtained by gluing an M_i with $pr_T, r_Tw^2 \in M_i$ and an N_j with $pr_T, r_Ts_T, t_Tw \in N_j$ contains e_{vw}^T and e_{uv}^T from E_T. This can be done for each edge $e \in \delta(K) \setminus (F \cup E_{\mathfrak{T}})$.

Similarly, for each edge $e \in F$ there are exactly $k(1 - x(e))$ M_i's and N_j's that does not contain e. Notice that these contain all edges in $\delta(p_i) - e$ and none in $\delta(K) - (F \cup E_{\mathfrak{T}})$. Again, pair and glue these together to get b-matchings of G not containing e.

The number of M_i's with $l_T \in M_i$ or $r_Tw^1, r_Tw^2 \in M_i$ for some $T \in \mathfrak{T}$ is equal to the number of N_j's with $r_Ts_T, r_Tt_T \in N_j$. The idea is that a b-matching obtained by gluing an M_i with $l_T \in M_i$ and an N_j with $r_Ts_T, r_Tt_T \in N_j$ contains e_{vw}^T from E_T; a b-matching obtained by gluing an M_i with

$r_T w^1, r_T w^2 \in M_i$ and an N_j with $r_T s_T, r_T t_T \in N_j$ contains e^T_{uv} and e^T_{uw} from E_T. However, we have to pair these matchings together carefully. Note, that $\mathcal{T}_2^\circ$ consists of triangles disjoint from K. It may happen that there is a forbidden triangle $T' \in \mathcal{T}$ such that $V_{T'} \subseteq K$ for what a triangle $T \in \mathfrak{T}$ has $|V_T \cap V_{T'}| = 2$. In this case, we are not allowed to pair an M_i and an N_j together if $l_T \in M_i$ and the two remaining edges of T' not contained by T are in N_j. We can avoid this unless the sum of the coefficients of these N_j's is more than $1 - x_1^\circ(l_T) = x(E_T) - 1$. Consider a convex combination in which the sum of the coefficients of b_2°-matchings containing the edges of T' different from e_T is minimal. If this value is positive then there is no N_j containing none of these two edges. But this implies that $x(E_{T'}) > 2(x(E_T) - 1) + (1 - (x(E_T) - 1)) + x(e_T) = x(E_T) + x(e_T) \geq 2$, a contradiction. The last inequality follows from Proposition 6.5.6.

So the pairing can be done. However, it is left to prove that the b-matchings thus arising are also $\mathcal{T}$-free.

Lemma 6.6.10. *The b-matchings thus obtained are $\mathcal{T}$-free.*

Proof. The only triangles possibly contained in one of the b-matchings could be those in $\mathcal{T} - (\mathcal{T}_1^\circ \cup \mathcal{T}_2^\circ)$. Moreover, by the above, a bad triangle should have nodes both in K and $\bar{K}$.

Due to the construction, a triangle $T \in \mathfrak{T}$ is not contained in the b-matchings thus obtained. Also, a T with $E_T \cap E_{\mathfrak{T}} \neq \emptyset$ is not contained by (6.1), (6.2) and Proposition 6.6.9. Assume that T shares no edge with triangles in $\mathfrak{T}$.

If $|E_T \cap F| = 0$ then each M_i contains at most one of T's edges going between K and $\bar{K}$ as $|M_i \cap (\delta(K) \setminus (F \cup E_{\mathfrak{T}}))| \leq 1$, hence T is not contained by the b-matchings.

Let $V_T = \{r, s, t\}$. Recall that $(K, F, \mathfrak{T})$ is such that either $\mathfrak{T} \neq \emptyset$ or it is independent from $\mathcal{L}$. The following proposition will be useful.

Proposition 6.6.11. *There is no tight even tri-comb $(Z, H, \mathfrak{R})$ in G with $Z \neq \emptyset$.*

Proof. Assume to the contrary that $(Z, H, \mathfrak{R})$ is a tight even pair, that is, $x(E[Z]) + x(H) + \sum_{T \in \mathfrak{R}} x(E_T) = \frac{b(Z)+|H|+3|\mathfrak{R}|}{2}$. By $0 < x < 1$, this immediately implies $H = \delta(Z) = \emptyset$, which is only possible if $Z = V$ as G is connected. But $x(E) = \frac{b(V)}{2}$ means that x is a b-factor, a contradiction. □

We distinguish the following cases.

Case 1: $|E_T \cap F| = 1, |V_T \cap K| = 1$

Assume that $V_T \cap K = r$ and $rt \in F$. Let u be the third neighbour of r, if exists. If $u \in K$ then $x(E[K-r]) + x(F - rt + ru) + \sum_{T \in \mathfrak{T}} x(E_T) > x(E[K]) + x(F) + \sum_{T \in \mathfrak{T}} x(E_T) - 1$ while $b(K-r) + |F - rt + ru| + 3|\mathfrak{T}| = b(K) + |F| + 3|\mathfrak{T}| - 2$. Hence $(K - r, F - rt + ru, \mathfrak{T})$ would violate (iii), a contradiction.

If $u \in \bar{K}$ and $ru \in F$ then $x(E[K-r]) + x(F - rt - ru) + \sum_{T \in \mathfrak{T}} x(E_T) > x(E[K]) + x(F) + \sum_{T \in \mathfrak{T}} x(E_T) - 2$ while $b(K-r) + |F - rt - ru| + 3|\mathfrak{T}| = b(K) + |F| + 3|\mathfrak{T}| - 4$. Hence $(K - r, F \setminus \delta(r), \mathfrak{T})$ would violate (iii), a contradiction.

If $u \in \bar{K}$ and $ru \notin F$ or r has no third neighbour then $x(E[K-r]) + x(F - rt) + \sum_{T \in \mathfrak{T}} x(E_T) > x(E[K]) + x(F) + \sum_{T \in \mathfrak{T}} x(E_T) - 1$ while $b(K-r) + |F - rt| + 3|\mathfrak{T}| = b(K) + |F| + 3|\mathfrak{T}| - 3$, a contradiction as $(K-r, F-rt, \mathfrak{T})$ is an even tri-comb that would violate (iii) which is not possible.

Case 2: $|E_T \cap F| = 1, |V_T \cap K| = 2$

Assume that $K \cap V_T = \{r, s\}$ and $rt \in F$. Let u be the third neighbour of s, if exists. If $u \in K$ then $x(E[K-s]) + x(F+su+rs) + \sum_{T\in\mathfrak{T}} x(E_T) = x(E[K]) + x(F) + \sum_{T\in\mathfrak{T}} x(E_T)$ while $b(K-s) + |F+su+rs| + 3|\mathfrak{T}| = b(K) + |F| + 3|\mathfrak{T}|$. Hence $(K-s, F+su+rs, \mathfrak{T})$ is also tight and its tightness is identical to that of the original tri-comb. However, $|K|$ decreased, contradicting the minimality of K.

If $u \in \bar{K}$ and $su \in F$ then $x(E[K-s]) + x(F-su+rs) + \sum_{T\in\mathfrak{T}} x(E_T) > x(E[K]) + x(F) + \sum_{T\in\mathfrak{T}} x(E_T) - 1$ while $b(K-s) + |F-su+rs| + 3|\mathfrak{T}| = b(K) + |F| + 3|\mathfrak{T}| - 2$. Hence $(K-s, F-su+rs, \mathfrak{T})$ would violate (iii), a contradiction.

If $u \in \bar{K}$ and $su \notin F$ or s has no third neighbour then $x(E[K-s]) + x(F) + \sum_{T\in\mathfrak{T}} x(E_T) > x(E[K]) + x(F) + \sum_{T\in\mathfrak{T}} x(E_T) - 1$ while $b(K-s) + |F| + 3|\mathfrak{T}| = b(K) + |F| + 3|\mathfrak{T}| - 2$. Hence $(K-s, F, \mathfrak{T})$ would violate (iii), a contradiction.

Case 3: $|E_T \cap F| = 2, |V_T \cap K| = 1$

Assume that $V_T \cap K = r$ and $rs, rt \in F$. Let u be the third neighbour of r, if exists. If $u \in K$ then $x(E[K-r]) + x(F-rs-rt) + \sum_{T\in\mathfrak{T}} x(E_T) \geq x(E[K]) + x(F) + \sum_{T\in\mathfrak{T}} x(E_T) - 2$ while $b(K-r) + |F-rs-rt| + 3|\mathfrak{T}| \leq b(K) + |F| + 3|\mathfrak{T}| - 4$. Hence we must have equality everywhere, so $x(\delta(r)) = 2$ and $(K-r, F-rs-rt, \mathfrak{T})$ is tight. The tightness of $(K-r, F-rs-rt, \mathfrak{T})$ is identical to that of the original tri-comb. However, $|K|$ decreased, contradicting the minimality of K.

If $u \in \bar{K}$ and $ru \in F$ then $x(E[K-r]) + x(F-rs-rt-ru) + \sum_{T\in\mathfrak{T}} x(E_T) \geq x(E[K]) + x(F) + \sum_{T\in\mathfrak{T}} x(E_T) - 2$ while $b(K-r) + |F-rs-rt-ru| + 3|\mathfrak{T}| = b(K) + |F| + 3|\mathfrak{T}| - 5$. We must have equality everywhere as otherwise $(K-s, F-rs-rt-ru, \mathfrak{T})$ would be an even tri-comb violating (iii). That is, $x(\delta(r)) = 2$ and $(K-s, F-rs-rt-ru, \mathfrak{T})$ is tight. Note that $|K| \neq 1$ as otherwise $\mathfrak{T} \neq \emptyset$ or the tri-comb is not independent from $\mathcal{L}$. Hence $(K-s, F-rs-rt-ru, \mathfrak{T})$ is a tight even tri-comb with $K-s \neq \emptyset$, contradicting Proposition 6.6.11.

If $u \in \bar{K}$ and $ru \notin F$ or r has no third neighbour then $x(E[K-r]) + x(F-rs-rt) + \sum_{T\in\mathfrak{T}} x(E_T) > x(E[K]) + x(F) + \sum_{T\in\mathfrak{T}} x(E_T) - 2$ while $b(K-r) + |F-rs-rt| + 3|\mathfrak{T}| = b(K) + |F| + 3|\mathfrak{T}| - 4$. Hence $(K-r, F-rs-rt, \mathfrak{T})$ would violate (iii), a contradiction.

Case 4: $|E_T \cap F| = 2, |V_T \cap K| = 2$

Assume that $K \cap V_T = \{r, s\}$ and $rt, st \in F$. Let u be the third neighbour of r, if exists. If $u \in \bar{K}$ and $ru \in F$ then $x(E[K-r]) + x(F-ru-rt) + \sum_{T\in\mathfrak{T}} x(E_T) \geq x(E[K]) + x(F) + \sum_{T\in\mathfrak{T}} x(E_T) - 2$ while $b(K-r) + |F-ru-rt| + 3|\mathfrak{T}| = b(K) + |F| + 3|\mathfrak{T}| - 4$. Hence $x(\delta(r)) = 2$, $(K-r, F-ru-rt, \mathfrak{T})$ is also tight and is independent from $\mathcal{L}$ if the original tri-comb was so (note that $K-r \neq \emptyset$). However, $|K|$ decreased, contradicting the minimality of K.

If $u \in \bar{K}$ and $ru \notin F$ or r has no third neighbour then $x(E[K-r]) + x(F-rt+rs) + \sum_{T\in\mathfrak{T}} x(E_T) > x(E[K]) + x(F) + \sum_{T\in\mathfrak{T}} x(E_T) - 1$ while $b(K-r) + |F-rt+rs| + 3|\mathfrak{T}| = b(K) + |F| + 3|\mathfrak{T}| - 2$. Hence

$(K - r, F - rt + rs, \mathfrak{T})$ would violate (iii), a contradiction.

The same can be told about the third neighbour of s denoted by v, if exists. So the only remaining case is when both $u, v \in K$. Then $x(E[K-r-s]) + x(F - rs - rt + ru + sv) + \sum_{T\in\mathfrak{T}} x(E_T) > x(E[K]) + x(F) + \sum_{T\in\mathfrak{T}} x(E_T) - 2$ while $b(K-r-s) + |F - rs - rt + ru + sv| + 3|\mathfrak{T}| = b(K) + |F| + 3|\mathfrak{T}| - 4$. Hence $(K - r - s, F - rs - rt + ru + sv, \mathfrak{T})$ would violate (iii), a contradiction. □

By Lemma 6.6.10, the b-matchings constructed above altogether yield x as a convex combination of $\mathcal{T}$-free b-matchings of G, a contradiction. Hence x is indeed contained in the convex combination of the incidence vectors of $\mathcal{T}$-free b-matchings, finishing the proof.

6.7 Proof of Lemma 6.5.8

The validity of (6.1) and (6.2) can be checked easily in both cases. We discuss the second part separately for K and $\bar{K}$.

(I) Shrinking $(\bar{K}, F, \mathfrak{T})$, which is of **Type 1**:

We use the notation of Definition 6.5.5. (i) clearly holds for edges different from pq and not contained in $\delta(K) \cap E_{\mathfrak{T}}$. For the rest of the edges the required inequalities follow from Proposition 6.5.6. As an example, we show this for pq. We have

$$x(F) + \sum_{T\in\mathfrak{T}} x(E_T) + \sum_{T\in\mathfrak{T}} x(e_T) \leq |F| + 2|\mathfrak{T}| + |\mathfrak{T}| = |F| + 3|\mathfrak{T}|,$$

that is, $x^\circ(pq) \geq 0$. On the other hand,

$$x(F) + \sum_{T\in\mathfrak{T}} x(E_T) + \sum_{T\in\mathfrak{T}} x(e_T) \geq |F| + 2|\mathfrak{T}| + |\mathfrak{T}| - 1 = |F| + 3|\mathfrak{T}| - 1$$

by Proposition 6.5.6, so $x^\circ(pq) \leq 1$.

The validity of (ii) is straightforward for nodes different from q. However, the tightness of the tri-comb implies

$$\begin{aligned} x^\circ(\dot{\delta}(q)) &= x^\circ(pq) + x(\delta(K) \setminus (F \cup E_{\mathfrak{T}})) \\ &= |F| + 3|\mathfrak{T}| - x(F) - \sum_{T\in\mathfrak{T}} x(E_T) - \sum_{T\in\mathfrak{T}} x(e_T) + x(\delta(K) \setminus (F \cup E_{\mathfrak{T}})) \\ &= 2x(E[K]) + x(F) + \sum_{T\in\mathfrak{T}} x(E_T) + 1 - b(K) \\ &\quad - \sum_{T\in\mathfrak{T}} x(e_T) + x(\delta(K) \setminus (F \cup E_{\mathfrak{T}})) \\ &= 2x(E[K]) + x(\delta(K)) + 1 - b(K) \\ &\leq 1 \\ &= b^\circ(q). \end{aligned}$$

(iv) and (v) remain valid for triangles in $\mathcal{T}^\circ$ as the same inequalities were true in the original graph. So it remains to show that (iii) is indeed satisfied in G°. Choose an odd tri-comb $(Z, H, \mathfrak{R})$ of G° with

$(\mathtt{def}(Z,H,\mathfrak{R}), |\bar{Z} \cup \{p,q\}|, |H|)$ lexicographically maximal. Our aim is to show that $\mathtt{def}(Z,H,\mathfrak{R}) \le 0$, which would prove *(iii)* for all odd tri-combs.

Clearly, an even tri-comb has deficiency at most 0 in G°. Hence if we find an even tri-comb $(Z',H',\mathfrak{R}')$ with $\mathtt{def}(Z,H,\mathfrak{R}) \le \mathtt{def}(Z',H',\mathfrak{R}')$ then we are done. So assume that there is no such even tri-comb.

Proposition 6.7.1. *Let $v \in Z$ be a node with $\ell(v) = \emptyset$, $b^\circ(v) = d^\circ(v) - 1$ and $v \notin V^\circ_{\mathfrak{R}}$.*

(a) If $x^\circ(\dot\delta(v)) = b^\circ(v)$ and $v \ne p,q$, then $\delta(Z)_v \subseteq H$ and $|E^\circ[v, Z-v]| \ge 2$.

(b) If $v = p$ and $\delta(Z)_p \setminus H \ne \emptyset$ then $H_p = \emptyset$.

(c) If $v \ne p,q$ and $b^\circ(v) = d^\circ(v) - 1 = 1$ then $\delta(Z)_v = \emptyset$.

Proof. (a) The conditions on v imply that for any two edges $e, f \in \delta(v)$ we have $x^\circ(e) + x^\circ(f) \ge 1$. If $|\delta(Z)_v \setminus H| \ge 2$ then the addition of two of these edges to H would result in a lexicographically larger odd tri-comb, a contradiction.

Assume that $|\delta(Z)_v \setminus H| = 1$. Define $Z' = Z - v$, $H' = (H \setminus H_v) \cup E^\circ[v, Z-v]$. The tri-comb $(Z',H',\mathfrak{R})$ thus arising is odd and with deficiency

$$\begin{aligned}\mathtt{def}(Z',H',\mathfrak{R}) &= \mathtt{def}(Z,H,\mathfrak{R}) - x^\circ(H_v) + \tfrac{b^\circ(v)+|H_v|-|E^\circ[v,Z-v]|}{2}\\ &= \mathtt{def}(Z,H,\mathfrak{R}) - x^\circ(H_v) + \tfrac{b^\circ(v)+|H_v|-d^\circ(v)+|H_v|+1}{2}\\ &= \mathtt{def}(Z,H,\mathfrak{R}) - x^\circ(H_v) + |H_v|.\end{aligned}$$

That is, the deficiency is not decreased and $|Z \setminus \{p,q\}|$ decreased by 1, a contradiction.

So $|\delta(Z)_v \setminus H| = 0$. Assume that $|E[v, Z-v]| = 1$. Then $(Z-v, H \setminus H_v, \mathfrak{R})$ is an odd tri-comb with the same deficiency as $(Z,H,\mathfrak{R})$ but has larger $|Z \setminus \{p,q\}|$ value, a contradiction.

(b) The computation of part (a) shows that in case of $H_p \ne \emptyset$ the deficiency of the tri-comb would strictly decrease for the tri-comb $(Z-p, (H \setminus H_p) \cup E^\circ[p, Z-p], \mathfrak{R})$ as $x > 0$.

(c) The deletion of v from Z decreases $x^\circ(E^\circ[Z]) + x^\circ(H) + \sum_{T\in\mathfrak{R}} x^\circ(E^\circ_T)$ by at most 1 while $\lfloor \frac{1}{2}(b^\circ(Z) + |H| + 3|\mathfrak{R}|) \rfloor$ always decreases by 1 unless $|H_v| = 0$. If $|\delta(Z)_v| = 2$ then the deletion of v from Z gives an even tri-comb with deficiency not smaller than that of the original tri-comb; if $|\delta(Z)_v| = 1$ then the deletion of v from Z and the addition of the other edge incident to v to H would result in a lexicographically larger tri-comb, a contradiction. □

Proposition 6.7.1 indicate the following simple but useful observation.

Corollary 6.7.2. *Let $T \in \mathfrak{T}$ be a triangle with $V_T = \{u,v,w\}, V_T \cap K = \{u,v\}$. Then exactly one of the followings hold.*

1. *$p, r_T, s_T, t_T, u, v \notin Z$;*

2. *$p \notin Z$, $r_T, s_T, t_T, u, v \in Z$, $pr_T \in H$ and the third neighbours of u and v -if exist- are in Z;*

3. *$p \in Z$, $r_T, s_T, t_T, u, v \notin Z$;*

4. *$p, r_T, s_T, t_T, u, v \in Z$ and the third neighbours of u and v -if exist- are in Z;*

5. *$p, r_T, s_T, u \in Z$, $t_T, v \notin Z$, $r_Tt_T \in H$ and the third neighbour of u -if exist- is in Z;*

6. $p, r_T, t_T, v \in Z$, $s_T, u \notin Z$, $r_T s_T \in H$ and the third neighbour of v -if exist- is in Z.

Proof. Assume first that $p \notin Z$. If $r_T \in Z$ then (a) implies that both $s_T, t_T \in Z$ and $pr_T \in H$. However, (c) further implies $u, v \in Z$, and so their third neighbours are in Z.

If $r_T \notin Z$ then neither s_T, t_T and so nor u, v are by (c).

The proof of the cases when $p \in Z$ goes in a similar way. □

Corollary 6.7.2 reduces the number of cases to be checked. Let $\mathfrak{T}_i = \{T \in \mathfrak{T} :\ T$ satisfies $i.$ of Corollary 6.7.2$\}$. From now on, let $K' = V^\circ \setminus \{p, q\}$.

Case 1: $p, q \notin Z$

By Corollary 6.7.2, each $T \in \mathfrak{T}$ is of Type 1 or 2. Let $Z' = Z, H' = H \setminus \{pr_T :\ T \in \mathfrak{T}_2\}, \mathfrak{R}' = \mathfrak{R} \cup \mathfrak{T}_2$. It is easy to check that the tri-comb $(Z', H', \mathfrak{R}')$ is odd, hence satisfy (iii) of (P_8) in the original graph. However, both sides of (iii) remains unchanged when considering $(Z, H, \mathfrak{R})$ instead in G°, hence the validity of (iii) follows from the same inequality for $(Z', H', \mathfrak{R}')$ in the original graph.

Case 2: $p, q \in Z$

We prove Case 2 with the help of Case 1. First of all note that $|H_p| \geq |\delta(Z)_p| - 1$. To prove this, assume that $|\delta(Z)_p \setminus H| \geq 2$. We have $x^\circ(\dot{\delta}(p)) = |F| + |\mathfrak{T}|$, and the degree of p is $|F| + |\mathfrak{T}| + 1$. Hence any two edges incident to p must have x° value together at least 1. The addition of two of these edges to H would result in a lexicographically larger tri-comb, a contradiction.

We distinguish two subcases.

Subcase 2.1: $\delta(Z)_p = H_p$

If $|H_q| \geq 1$ then let $F_1 = H_p$, $F_2 = \delta(p) \setminus (F_1 + pq)$. Take $Z' = Z \cap K'$, $H' = (H \setminus (F_1 \cup H_q)) \cup F_2$. Then

$$
\begin{aligned}
&x^\circ(E^\circ[Z]) + x^\circ(H) + \sum_{T \in \mathfrak{R}} x^\circ(E^\circ_T) \\
&\quad = x^\circ(E^\circ[Z']) + x^\circ(H') + \sum_{T \in \mathfrak{R}} x^\circ(E^\circ_T) + x^\circ(pq) \\
&\qquad + x^\circ(E^\circ[q, Z']) + x^\circ(H_q) + x^\circ(F_1) \\
&\quad \leq \lfloor \tfrac{b^\circ(Z') + |H'| + 3|\mathfrak{R}|}{2} \rfloor + x^\circ(pq) + x^\circ(E^\circ[q, Z']) + x^\circ(H_q) + x^\circ(F_1) \\
&\quad \leq \lfloor \tfrac{b^\circ(Z) - 1 - |F| - |\mathfrak{T}| + |H| - |F_1| + |F_2| - 1 + 3|\mathfrak{R}|}{2} \rfloor \\
&\qquad + x^\circ(pq) + x^\circ(E^\circ[q, Z]) + x^\circ(H_q) + x^\circ(F_1) \\
&\quad = \lfloor \tfrac{b^\circ(Z) + |H| + 3|\mathfrak{R}|}{2} \rfloor - |F_1| - 1 + x^\circ(pq) + x^\circ(E^\circ[q, Z]) + x^\circ(H_q) + x^\circ(F_1) \\
&\quad \leq \lfloor \tfrac{b^\circ(Z) + |H| + 3|\mathfrak{R}|}{2} \rfloor,
\end{aligned}
$$

as $x^\circ(pq) + x^\circ(E^\circ[q, Z]) + x^\circ(H_q) \leq x^\circ(\delta(q)) \leq 1$. This implies $\texttt{def}(Z, H, \mathfrak{R}) \leq 0$.

Now assume that $|H_q| = 0$. If $Z = \{p, q\}$ then $\mathfrak{R} = \emptyset$ and $H = \delta(p) - pq$. Hence $x^\circ(E^\circ[Z]) + x^\circ(H) = x^\circ(\delta(p)) = |F| + |\mathfrak{T}| \leq \lfloor \frac{|F| + |\mathfrak{T}| + 1 + |F| + |\mathfrak{T}|}{2} \rfloor = \lfloor \frac{b^\circ(p) + b^\circ(q) + |H|}{2} \rfloor$, so (iii) holds.

So assume that $Z \neq \{p, q\}$ and let $Z' = Z \cap K'$. Define $F' = \delta(p) - pq$. It is easy to see that the tightness of $(K, F, \mathfrak{T})$ implies the tightness of (K', F'). Using this and that (iii) holds if $Z = \{p, q\}$, we have the following

$$\begin{aligned}
&x^\circ(E^\circ[K']) + x^\circ(F') + x^\circ(E^\circ[Z]) + x^\circ(H) + \sum_{T\in\mathfrak{R}} x^\circ(E_T^\circ) \\
&\quad = x^\circ(E^\circ[K'\setminus Z]) + x^\circ(E^\circ[Z\setminus K']) + x^\circ(H) + \sum_{T\in\mathfrak{R}} x^\circ(E_T^\circ) + x^\circ(F') \\
&\qquad + 2x^\circ(E^\circ[Z']) + x^\circ(E^\circ[K'\setminus Z', Z']) + x^\circ(E^\circ[\{p,q\}, Z']) \\
&\quad \le \lfloor \tfrac{b^\circ(K'\setminus Z)+|H|+3|\mathfrak{R}|}{2} \rfloor + \lfloor \tfrac{b^\circ(Z\setminus K')+|F'|}{2} \rfloor + 2x^\circ(E^\circ[Z']) + x^\circ(\delta(Z')) \\
&\quad = \tfrac{b^\circ(K')+|F'|-1}{2} + \tfrac{b^\circ(Z)+|H|+3|\mathfrak{R}|-1}{2} - b^\circ(Z') + 2x^\circ(E^\circ[Z']) + x^\circ(\delta(Z')) \\
&\quad \le \tfrac{b^\circ(K')+|F'|-1}{2} + \tfrac{b^\circ(Z)+|H|+3|\mathfrak{R}|-1}{2}.
\end{aligned}$$

The tightness of (K', F') implies $\texttt{def}(Z, H, \mathfrak{R}) \le 0$. In the proof we used that $(K' \setminus Z, H, \mathfrak{R})$ and $(Z\setminus K', F')$ are also odd. This can be seen by $b^\circ(K'\setminus Z)+|H|+3|\mathfrak{R}| = b^\circ(K')-b^\circ(Z)+1+|F'|+|H|+|\mathfrak{R}|$ which is odd as (K', F') and $(Z, H, \mathfrak{R})$ are odd, and $b^\circ(Z \setminus K') + |F'| = 1 + 2|F'|$.

Subcase 2.2: $|\delta(Z)_p| = |H_p| + 1$

By Proposition 6.7.1, $H_p = \emptyset$. Let $\delta(Z)_p = f$ and $F_2 = \delta(p) - f$. Take $Z' = Z \cap K'$, $H' = (H \setminus \delta(q)) \cup F_2$. Then

$$\begin{aligned}
&x^\circ(E^\circ[Z]) + x^\circ(H) + \sum_{T\in\mathfrak{R}} x^\circ(E_T^\circ) \\
&\quad = x^\circ(E^\circ[Z']) + x^\circ(H') + \sum_{T\in\mathfrak{R}} x^\circ(E_T^\circ) + x^\circ(pq) + x^\circ(E^\circ[q, Z']) + x^\circ(H_q) \\
&\quad \le \lfloor \tfrac{b^\circ(Z')+|H'|+3|\mathfrak{R}|}{2} \rfloor + x^\circ(pq) + x^\circ(E^\circ[q, Z']) + x^\circ(H_q) \\
&\quad = \lfloor \tfrac{b^\circ(Z)-1-|F|-|\mathfrak{T}|+|H|+|F_2|+3|\mathfrak{R}|}{2} \rfloor + x^\circ(pq) + x^\circ(E^\circ[q, Z']) + x^\circ(H_q) \\
&\quad = \lfloor \tfrac{b^\circ(Z)+|H|+3|\mathfrak{R}|}{2} \rfloor - 1 + x^\circ(pq) + x^\circ(E^\circ[q, Z'])) + x^\circ(H_q) \\
&\quad \le \lfloor \tfrac{b^\circ(Z)+|H|+3|\mathfrak{R}|}{2} \rfloor,
\end{aligned}$$

as $x^\circ(pq) + x^\circ(E^\circ[q, Z]) + x^\circ(H_q) \le x^\circ(\dot{\delta}(q)) \le 1$. This implies $\texttt{def}(Z, H, \mathfrak{R}) \le 0$.

Case 3: $p \in Z, q \notin Z$

If $pq \in H$, then add q to Z and delete H_q - including pq - from H. We have previously seen that the tri-comb $(Z', H', \mathfrak{R})$ thus obtained satisfies (iii), so

$$\begin{aligned}
&x^\circ(E^\circ[Z]) + x^\circ(H) + \sum_{T\in\mathfrak{R}} x^\circ(E_T^\circ) \\
&\quad = x^\circ(E^\circ[Z']) + x^\circ(H') + \sum_{T\in\mathfrak{R}} x^\circ(E_T^\circ) - x^\circ(E^\circ[q, Z]) + x^\circ(H_q) \\
&\quad \le \lfloor \tfrac{b^\circ(Z')+|H'|+3|\mathfrak{R}|}{2} \rfloor \\
&\quad \le \lfloor \tfrac{b^\circ(Z)+1+|H|-1+3|\mathfrak{R}|}{2} \rfloor \\
&\quad = \lfloor \tfrac{b^\circ(Z)+|H|+3|\mathfrak{R}|}{2} \rfloor.
\end{aligned}$$

This implies $\texttt{def}(Z, H, \mathfrak{R}) \le 0$.

If $pq \notin H$, then first consider the case when $\delta(Z)_p \setminus (H_p + pq) \neq \emptyset$. Let f be an edge in this set.

Define again $Z' = Z + q$, delete H_q from H and add f to it. For the new tri-comb $(Z', H', \mathfrak{R})$, we have

$$\begin{aligned} & x^\circ(E^\circ[Z]) + x^\circ(H) + \sum_{T \in \mathfrak{R}} x^\circ(E_T^\circ) \\ & = x^\circ(E^\circ[Z']) + x^\circ(H') + \sum_{T \in \mathfrak{R}} x^\circ(E_T^\circ) + x^\circ(H_q) - x^\circ(E^\circ[q, Z]) - x^\circ(f) \\ & \leq \lfloor \tfrac{b^\circ(Z') + |H'| + 3|\mathfrak{R}|}{2} \rfloor - x^\circ(pq) - x^\circ(f) \\ & \leq \lfloor \tfrac{b^\circ(Z) + 1 + |H| + 3|\mathfrak{R}|}{2} \rfloor - 1 \\ & \leq \lfloor \tfrac{b^\circ(Z) + |H| + 3|\mathfrak{R}|}{2} \rfloor. \end{aligned}$$

For the second last inequality, we used Corollary 6.5.7 ($x^\circ(\dot{\delta}(p)) = |F| + |\mathfrak{T}|$, and the degree of p is $|F| + |\mathfrak{T}| + 1$, hence pq and f, two edges incident to p must have x° value together at least 1). This implies $\texttt{def}(Z, H, \mathfrak{R}) \leq 0$.

If $\delta(Z)_p \setminus (H_p + pq) = \emptyset$, then let $F_1 = H_p - pq$, $F_2 = \delta(p) \setminus (H + pq)$. Define $Z' = Z - p$, $H' = (H \setminus F_1) \cup F_2$. Note that $(Z', H', \mathfrak{R})$ is odd since $b^\circ(Z') + |H'| + |\mathfrak{R}| = b^\circ(Z) + |H| - |F| - |\mathfrak{T}| - |F_1| + |F_2| + |\mathfrak{R}| = b^\circ(Z) + |H| + |\mathfrak{R}| - 2|F_1|$. Hence

$$\begin{aligned} & x^\circ(E^\circ[Z]) + x^\circ(H) + \sum_{T \in \mathfrak{R}} x^\circ(E_T^\circ) \\ & = x^\circ(E^\circ[Z']) + x^\circ(H') + \sum_{T \in \mathfrak{R}} x^\circ(E_T^\circ) + x^\circ(F_1) \\ & \leq \lfloor \tfrac{b^\circ(Z') + |H'| + 3|\mathfrak{R}|}{2} \rfloor + x^\circ(F_1) \\ & \leq \lfloor \tfrac{b^\circ(Z) - |F| - |\mathfrak{T}| + |H| - |F_1| + |F_2| + 3|\mathfrak{R}|}{2} \rfloor + x^\circ(F_1) \\ & \leq \lfloor \tfrac{b^\circ(Z) + |H| - 2|F_1| + 3|\mathfrak{R}|}{2} \rfloor + x^\circ(F_1) \\ & \leq \lfloor \tfrac{b^\circ(Z) + |H| + 3|\mathfrak{R}|}{2} \rfloor. \end{aligned}$$

This implies $\texttt{def}(Z, H, \mathfrak{R}) \leq 0$.

Case 4: $p \notin Z, q \in Z$

If $H_q \neq \emptyset$, then delete q from Z and H_q from H. Then

$$\begin{aligned} & x^\circ(E^\circ[Z]) + x^\circ(H) + \sum_{T \in \mathfrak{R}} x^\circ(E_T^\circ) \\ & = x^\circ(E^\circ[Z']) + x^\circ(E^\circ[q, Z - q]) + x^\circ(H') + x^\circ(H_q) + \sum_{T \in \mathfrak{R}} x^\circ(E_T^\circ) \\ & \leq \lfloor \tfrac{b^\circ(Z') + |H'| + 3|\mathfrak{R}|}{2} \rfloor + x^\circ(\delta(q)) \\ & \leq \lfloor \tfrac{b^\circ(Z) - 1 + |H| - 1 + 3|\mathfrak{R}|}{2} \rfloor + 1 \\ & = \lfloor \tfrac{b^\circ(Z) + |H| + 3|\mathfrak{R}|}{2} \rfloor. \end{aligned}$$

This implies $\texttt{def}(Z, H, \mathfrak{R}) \leq 0$.

If $H_q = \emptyset$, then first consider the case when $E^\circ[p, Z - q] \setminus H \neq \emptyset$. Let f be an edge in this set. Delete

q from Z and take $H' = H + f$. Then

$$\begin{aligned}
&x^\circ(E^\circ[Z]) + x^\circ(H) + \sum_{T\in\mathfrak{R}} x^\circ(E^\circ_T)\\
&\quad= x^\circ(E^\circ[Z']) + x^\circ(H') + \sum_{T\in\mathfrak{R}} x^\circ(E^\circ_T) + x^\circ(E^\circ[q, Z-q]) - x^\circ(f)\\
&\quad\le \lfloor \tfrac{b^\circ(Z')+|H'|+3|\mathfrak{R}|}{2} \rfloor + x^\circ(E^\circ[q, Z-q]) - x^\circ(f)\\
&\quad\le \lfloor \tfrac{b^\circ(Z)-1+|H|+1+3|\mathfrak{R}|}{2} \rfloor + x^\circ(\dot\delta(q)) - x^\circ(pq) - x^\circ(f)\\
&\quad\le \lfloor \tfrac{b^\circ(Z)+|H|+3|\mathfrak{R}|}{2} \rfloor
\end{aligned}$$

by Corollary 6.5.7. This implies $\mathtt{def}(Z, H, \mathfrak{R}) \le 0$.

If $E^\circ[p, Z-q] \setminus H = \emptyset$ then let $F_1 = H_p - pq$ and $F_2 = \delta(p) \setminus (H + pq)$. Define $Z' = Z + p$ and $H' = (H \setminus F_1) \cup F_2$. For the tri-comb $(Z', H', \mathfrak{R})$

$$\begin{aligned}
&x^\circ(E^\circ[Z]) + x^\circ(H) + \sum_{T\in\mathfrak{R}} x^\circ(E^\circ_T)\\
&\quad= x^\circ(E^\circ_1[Z']) + x^\circ(H') + \sum_{T\in\mathfrak{R}} x^\circ(E^\circ_T) - x^\circ(pq) - x^\circ(F_2)\\
&\quad\le \lfloor \tfrac{b^\circ(Z')+|H'|+3|\mathfrak{R}|}{2} \rfloor - x^\circ(pq) - x^\circ(F_2)\\
&\quad= \lfloor \tfrac{b^\circ(Z)+|F|+|H|-|F_1|+|F_2|+3|\mathfrak{R}|}{2} \rfloor - x^\circ(pq) - x^\circ(F_2)\\
&\quad\le \lfloor \tfrac{b^\circ(Z)+|H|+3|\mathfrak{R}|}{2} \rfloor + |F_2| - x^\circ(pq) - x^\circ(F_2)\\
&\quad\le \lfloor \tfrac{b^\circ(Z)+|H|+3|\mathfrak{R}|}{2} \rfloor
\end{aligned}$$

by Proposition 6.5.7. This implies $\mathtt{def}(Z, H, \mathfrak{R}) \le 0$.

(II) Shrinking $(K, F, \mathfrak{T})$, which is of **Type 2**:

The verification of $(i), (ii), (iv)$ and (v) goes in the same way as in the previous case. Choose an odd tri-comb $(Z, H, \mathfrak{R})$ of G° with $(\mathtt{def}(Z, H, \mathfrak{R}), |\bar{Z} \cup \{p, q\}|, |H|)$ lexicographically maximal. We start again with some technical propositions. These are only easy observations but they greatly help us to reduce the number of cases to be checked.

Again, an even tri-comb has deficiency at most 0 in G°. Hence if we find an even tri-comb $(Z', H', \mathfrak{R}')$ with $\mathtt{def}(Z, H, \mathfrak{R}) \le \mathtt{def}(Z', H', \mathfrak{R}')$ then we are done. So assume that there is no such even tri-comb.

Proposition 6.7.3. *Let $T \in \mathfrak{T}$ with $V_T = \{u, v, w\}$, $V_T \cap K = \{u, v\}$. Then $x(e^T_{uv}) + x(e^T_{uw}) \ge 1$ and $x(e^T_{uv}) + x(e^T_{vw}) \ge 1$.*

Proof. Assume that one of the mentioned sums, say $x(e^T_{uv}) + x(e^T_{uw})$, is strictly less than 1. Then $(K, F + e^T_{vw}, \mathfrak{T} - T)$ violates (iii), a contradiction. □

Proposition 6.7.4. *Let $T \in \mathfrak{T}$ with $V_T = \{u, v, w\}$, $V_T \cap K = \{u, v\}$. If both $p, w \notin Z$ then $r_T \notin Z$.*

Proof. If $|H_{r_T}| \ge 2$ then for the tri-comb $(Z - r_T, H \setminus H_{r_T}, \mathfrak{R})$ the left side of (iii) (P_8) decreases by at most 2 while the right decreases by 2, which means that the new tri-comb has no smaller deficiency and is either lexicographically larger or it is even, both leading to a contradiction.

If $|H_{r_T}| = 0$ then the left side of (iii) decreases by $x^\circ(l_T) < 1$ while the right decreases by 1, a contradiction.

If $H_{r_T} = r_T w^1$ then the left side of (iii) decreases by $x^\circ(l_T) + x^\circ(r_T w^1) = 2 - x(e^T_{uv}) - x(e^T_{uw}) - x(e^T_{vw}) + x(e^T_{uw}) = 2 - x(e^T_{uv}) - x(e^T_{vw}) \leq 1$ by Proposition6.7.3 while the right side decreases by 1, so $(Z - r_T, H \setminus H_{r_T}, \mathfrak{R})$ is an even tri-comb with deficiency no smaller than that of $(Z, H, \mathfrak{R})$, a contradiction. The other case when $H_{r_T} = r_T w^2$ leads to a contradiction similarly.

If $H_{r_T} = pr_T$ then the left side of (iii) decreases by $x^\circ(l_T) + x^\circ(pr_T) = 2 - x(e^T_{uv}) - x(e^T_{uw}) - x(e^T_{vw}) + 2x(e^T_{uv}) + x(e^T_{uw}) + x(e^T_{vw}) - 2 = x(e^T_{uv}) \leq 1$, hence $(Z - r_T, H \setminus H_{r_T}, \mathfrak{R})$ is an even tri-comb with deficiency no smaller than that of $(Z, H, \mathfrak{R})$, a contradiction. □

Proposition 6.7.5. *Let $T \in \mathfrak{T}$ with $V_T = \{u, v, w\}$, $V_T \cap K = \{u, v\}$. If $p, w \in Z$ then $r_T \in Z$.*

Proof. If $|H_{r_T}| \geq 2$ then for the tri-comb $(Z + r_T, H \setminus H_{r_T}, \mathfrak{R})$ the left side of (iii) strictly increases by $x > 0$ while the right does not change, which means that the new tri-comb has larger deficiency. So it is either a lexicographically larger odd tri-comb or it is even, both leading to a contradiction.

If $|H_{r_T}| = 0$ then the left side of (iii) increases by $x^\circ(\delta(r_T)) = x(E_T) \geq 1$ while the right increase by 1, a contradiction again.

If $H_{r_T} = r_T w^1$ then the left side of (iii) increases by $x^\circ(l_T) + x^\circ(r_T w^2) + x^\circ(pr_T) = x(e^T_{uv}) + x(e^T_{vw}) \geq 1$ by Proposition6.7.3 while the right side increases by 1, so $(Z + r_T, H \setminus H_{r_T}, \mathfrak{R})$ is an even tri-comb with deficiency no smaller than that of $(Z, H, \mathfrak{R})$, a contradiction. The other case when $H_{r_T} = r_T w^2$ leads to a contradiction similarly.

If $H_{r_T} = pr_T$ then the left side of (iii) increases by $x^\circ(l_T) + x^\circ(r_T w^1) + x^\circ(r_T w^2) = 2 - x(e^T_{uv}) > 1$ as $x < 1$, hence $(Z + r_T, H \setminus H_{r_T}, \mathfrak{R})$ is an even tri-comb with deficiency no smaller than that of $(Z, H, \mathfrak{R})$, a contradiction. □

Proposition 6.7.6. *Let $T \in \mathfrak{T}$ with $V_T = \{u, v, w\}$, $V_T \cap K = \{u, v\}$. If $p \notin Z$ but $w, r_T \in Z$ then $pr_T \notin H$.*

Proof. Let $wz = \delta(w) \setminus E_T$, if exists. If $pr_T \in H$ and $z \in Z$ then $(Z - r_T - w, H - pr_T + wz, \mathfrak{R})$, while if $pr_T \in H$ and $z \notin Z$ then $(Z - r_T - w, H \setminus \{pr_T, wz\}, \mathfrak{R})$ has deficiency at most $\texttt{def}(Z, H, \mathfrak{R})$ and smaller $|Z|$, a contradiction. □

Propositions 6.7.4, 6.7.5 and 6.7.6 imply the following.

Corollary 6.7.7. *Let $T \in \mathfrak{T}$ be a triangle with $V_T = \{u, v, w\}, V_T \cap K = \{w\}$. Then exactly one of the followings hold.*

1. $p, r_T, w \notin Z$;
2. $p, r_T \notin Z$, $w \in Z$;
3. $p \notin Z$, $r_T, w \in Z$ *and* $pr_T \in H$;
4. $p, r_T, w_T \in Z$;
5. $p, r_T \in Z$, $w \notin Z$;
6. $p \in Z$, $r_T, w \notin Z$ *and* $pr_T \in H$;
7. $p \in Z$, $r_T, w \notin Z$ *and* $pr_T \notin H$.

Let $\mathfrak{T}_i = \{T \in \mathfrak{T} : T$ satisfies $i.$ of Corollary 6.7.7$\}$. From now on, for a forbidden triangle $T \in \mathfrak{T}$ let $V_T = \{u_T, v_T, w_T\}$ with $u_T, v_T \in K$.

<u>**Case 1:**</u> $p, q \notin Z$

By Propositions 6.7.4 and 6.7.6, if $r_T \in Z$ for some triangle $T \in \mathfrak{T}$ then $T \in \mathfrak{T}_3$. Let $Z' = Z \setminus \{r_T : T \in \mathfrak{T}_3\}$, $H' = H \setminus \{pr_T : T \in \mathfrak{T}_3\} \cup \{u_T w_T, v_T w_T : T \in \mathfrak{T}_3\}$. It is easy to check that the tri-comb $(Z', H', \mathfrak{R})$ is odd, hence satisfy (iii) of (P_8) in the original graph. However, both sides of (iii) remains unchanged when considering $(Z, H, \mathfrak{R})$ instead in G°, hence the validity of (iii) follows from the same inequality for $(Z', H', \mathfrak{R}')$ in the original graph.

Case 2: $p, q \in Z$

Proposition 6.7.5 implies $\mathfrak{T} = \mathfrak{T}_4 \cup \mathfrak{T}_5 \cup \mathfrak{T}_6 \cup \mathfrak{T}_7$. However, $|\mathfrak{T}_7| \leq 1$. Indeed, $x^\circ(\dot{\delta}(p)) = |F| + |\mathfrak{T}|$, and the degree of p is $|F| + |\mathfrak{T}| + 1$, so any two edges incident to p must have x° value together at least 1. If $|\delta(Z)_p \setminus H_p| \geq 2$, then the addition of two edges from this set to H would not decrease the deficiency of the tri-comb, not increase $|Z|$ but increase $|H|$, a contradiction.

If $\mathfrak{T}_7 = \emptyset$ then let $S = K \cup (Z \cap \bar{K})$, $I = \{u_T w_T : r_T w_T^1 \in H\} \cup \{v_T w_T : r_T w_T^2 \in H\} \cup (H \cap E)$ and $\mathfrak{P} = \mathfrak{R} \cup \mathfrak{T}_6$. Then

$$\begin{aligned}
&x^\circ(E^\circ[Z]) + x^\circ(H) + \sum_{T \in \mathfrak{R}} x^\circ(E_T^\circ) \\
&\quad = x(E[S]) + x(I) + \sum_{T \in \mathfrak{P}} x(E_T) - x(E[K]) + x^\circ(pq) + \sum_{T \in \mathfrak{T}_1 \cup \mathfrak{T}_2 \cup \mathfrak{T}_3} x(e_T)) - 2|\mathfrak{T}_6| \\
&\quad = x(E[S]) + x(I) + \sum_{T \in \mathfrak{P}} x(E_T) - x(E[K]) + |F| + 3|\mathfrak{T}| \\
&\qquad - x(F) - \sum_{T \in \mathfrak{T}} x(E_T) - 2|\mathfrak{T}_6| \\
&\quad \leq \lfloor \tfrac{b(S) + |I| + 3|\mathfrak{P}|}{2} \rfloor - \tfrac{b(K) - |F| - 3|\mathfrak{T}| - 1}{2} - 2|\mathfrak{T}_6| \\
&\quad = \tfrac{b(K) + b^\circ(Z) - 1 - |F| - |\mathfrak{T}| - 2|\mathfrak{T}_4 \cup \mathfrak{T}_5| + |H| - |\mathfrak{T}_6| + 3|\mathfrak{R}| + 3|\mathfrak{T}_6| - 1}{2} - \tfrac{b(K) - |F| - 3|\mathfrak{T}| - 1}{2} - 2|\mathfrak{T}_6| \\
&\quad = \tfrac{b^\circ(Z) + |H| + 3|\mathfrak{R}| - 1}{2} - |\mathfrak{T}_4 \cup \mathfrak{T}_5 \cup \mathfrak{T}_6| + |\mathfrak{T}| \\
&\quad = \tfrac{b^\circ(Z) + |H| + 3|\mathfrak{R}| - 1}{2}.
\end{aligned}$$

This implies $\texttt{def}(Z, H, \mathfrak{R}) \leq 0$.

If $|\mathfrak{T}_7| = 1$ then take $Z' = Z \cap (\bar{K} \cup \{r_T : T \in \mathfrak{T}\})$, $F_2 = \{pr_T : T \in \mathfrak{T}_5\}$ and $H' = (H \setminus H_q) \cup F_2$. Thus

$$\begin{aligned}
&x^\circ(E^\circ[Z]) + x^\circ(H) + \sum_{T \in \mathfrak{R}} x^\circ(E_T^\circ) \\
&\quad = x^\circ(E^\circ[Z']) + x^\circ(H') + \sum_{T \in \mathfrak{R}} x^\circ(E_T^\circ) + x^\circ(pq) + x^\circ(E^\circ[q, Z']) + x^\circ(H_q) \\
&\quad \leq \lfloor \tfrac{b^\circ(Z') + |H'| + 3|\mathfrak{R}|}{2} \rfloor + x^\circ(pq) + x^\circ(E^\circ[q, Z']) + x^\circ(H_q) \\
&\quad \leq \lfloor \tfrac{b^\circ(Z) + |H| + 3|\mathfrak{R}| - |F^\circ| - 1 + |F_2|}{2} \rfloor + x^\circ(pq) + x^\circ(E^\circ[q, Z']) + x^\circ(H_q) \\
&\quad \leq \lfloor \tfrac{b^\circ(Z) + |H| + 3|\mathfrak{R}|}{2} \rfloor - 1 + x^\circ(pq) + x^\circ(E^\circ[q, Z']) + x^\circ(H_q) \\
&\quad \leq \lfloor \tfrac{b^\circ(Z) + |H| + 3|\mathfrak{R}|}{2} \rfloor.
\end{aligned}$$

This implies $\texttt{def}(Z, H, \mathfrak{R}) \leq 0$.

Case 3: $p \notin Z, q \in Z$ The proof of this case, by using the above propositions, goes exactly the same way as in case $(I)/3$.

Case 4: $p \in Z, q \notin Z$ The proof of this case, by using the above propositions, goes exactly the same way as in case $(I)/4$.

6.8 Proof of Lemma 6.6.8

Take a maximal independent set of tight equalities of form (ii), and extend this to a maximal independent set with bad equalities of type (IV) with $|K| = 1$, and then with equalities of type (V). Let $\mathcal{L}$ denote the set of equalities thus obtained.

Claim 6.8.1. *There is no bad pair (K, F) independent from $\mathcal{L}$.*

Proof. In the proof we will use Proposition 6.5.6 several times without mentioning it.

Assume that (K, F) is of type (I) independent from $\mathcal{L}$. First of all, $b(K) \geq |F| - 1$ as otherwise $x(E[K]) + x(F) = \lfloor \frac{1}{2}(b(K) + |F|) \rfloor \leq |F| - 2$, contradicting $x(F) \geq |F| - 1$. If $b(K) = |F| - 1$ then from $x(E[K]) + x(F) = |F| - 1$ we get $x(E[K]) = 0$ and $x(F) = b(K)$ which in turn implies $E[K] = \emptyset$ and $F = \delta(K)$, so $x(\delta(v)) = b(v)$ for each $v \in K$. But this is a contradiction as (K, F) is supposed to be independent from equalities of form (ii). Observe that $b(K) = |F|$ is not possible as (K, F) is an odd pair.

Assume that (K, F) is a bad pair of type (II), so $K = \{v\}, F \subseteq \delta(v), \ell(v) = \emptyset$ and $b(v) = |F| + 1$. Then the tightness of (v, F) means $x(F) = |F|$, which is only possible if $F = \emptyset$ by $x < 1$, contradicting independence.

Assume that (K, F) is a bad pair of type (III) independent from $\mathcal{L}$ and let $K = \{u, v\}$. Let C be the set of parallel edges between u and v. As $b(u) + b(v) = |F_u| + |F_v| + 1$, either $b(u) \leq |F_u|$ or $b(v) \leq |F_v|$, say the first one. In this case $x(C) + x(F_u) \leq b(u) \leq |F_u|$, so $x(C) + x(F_u) + x(F_v) \leq |F_u| + |F_v|$. Here $F_v = \emptyset$, otherwise even strict inequality holds by $x(F_v) < |F_v|$, contradicting the tightness of (K, F). By the tightness of the pair, $x(C) + x(F_u) = |F_u|$. We assumed that $b(u) \leq |F_u|$, so $b(u) = |F_u|$ and $b(v) = 1$ implying $\delta(u) \setminus (C \cup F_u) = \emptyset$. But then the tightness of the pair (K, F) is equivalent to $x(\dot{\delta}(u)) = b(u)$, contradicting linear independence.

Assume now that (K, F) is of type (IV) independent from $\mathcal{L}$ with $|K| \geq 2$. It can be seen similarly to the earlier cases that $b(\bar{K}) \geq |F| - 1$ must hold. If $b(\bar{K}) = |F| - 1$ then $x(E[\bar{K}]) + x(\delta(K) \setminus F) = 0$, hence $E[\bar{K}] = \emptyset$ and $\delta(K) = F$. So we have $x(E) = x(E[K]) + x(\delta(K)) = x(E[K]) + x(F) = \frac{1}{2}(b(K) + |F| - 1) = \frac{1}{2}b(V)$. That is, x is in fact a b-factor, a contradiction.

If $b(\bar{K}) = |F|$ then $x(E) \geq x(E[K]) + x(F) + x(E[\bar{K}]) = \frac{1}{2}(b(K) + |F| - 1) + x(E[\bar{K}]) = \lfloor \frac{1}{2}b(V) \rfloor + x(E[\bar{K}])$. But $x(E) \leq \lfloor \frac{1}{2}b(V) \rfloor$ so $E[\bar{K}] = \emptyset$ and also $\delta(K) = F$. That means that $\bar{K}$ consists of isolated nodes $v_1, ..., v_k$ and $\delta(K) = F = \delta(v_1) \cup ... \cup \delta(v_k)$. Let $F_i = \delta(v_i)$. We claim that $b(v_i) = |F_i|$ for each i. Indeed, otherwise there is an i with $b(v_i) \geq |F_i| + 1 > d(v_i)$, contradicting Proposition 6.6.6. So $b(v_i) = |F_i|$ for each i. Then $(K \cup \{v_1, ..., v_{k-1}\}, F_k)$ is also tight, and the tightness of (K, F) is identical to the tightness of this pair, a contradiction.

Now assume that (K, F) is a bad pair of type ((VI) independent from $\mathcal{L}$ and let $\bar{K} = \{u, v\}$. As $b(u) + b(v) = |F_u| + |F_v| + 1$, either $b(u) \leq |F_u|$ or $b(v) \leq |F_v|$, say the first one. By Proposition 6.6.7, $(K + v, F_u)$ is also tight and $\dot{\delta}(v) \setminus F = \emptyset$, hence the tightness of (K, F) is equivalent to the tightness of $(K + v, F_u)$, contradicting linear independence. □

Claim 6.8.1 implies that an upper bound for $|\mathcal{L}|$ is also an upper bound for the maximum number of independent bad constraints. Hence it suffices to bound $|\mathcal{L}|$. We say that a bad constraint in $\mathcal{L}$ **corresponds** to a node $v \in V$ if it is either of type $x(\dot{\delta}(v)) = b(v)$, or of type (IV) or (V) with $\bar{K} = \{v\}$. We give a bound on the number of bad constraints in $\mathcal{L}$ corresponding to a node $v \in V$.

Proposition 6.8.2. *If (K, F) is in $\mathcal{L}$ then $(K, F') \notin \mathcal{L}$ for $F' \subset F$.*

Proof. Assume indirectly that (K, F') is in $\mathcal{L}$ for some $F' \subset F$. Then $x(F \setminus F') = \frac{|F \setminus F'|}{2}$ from what $F' = \emptyset, |F| = 2, x(F) = 1$ follow by Propositions 6.5.6 and 6.6.2. But then each node is saturated in K and $(K, F') = (K, \emptyset)$ is not independent from equalities of form (ii). □

Claim 6.8.3. *If $x(\dot{\delta}(v)) = b(v)$ then there is no bad constraint of type (IV) or (V) in $\mathcal{L}$ corresponding to v.*

Proof. Let v be such that $x(\dot{\delta}(v)) = b(v)$ and $x(E[K])) + x(F) = \frac{b(K)+|F|-1}{2}$ for some $F \subseteq \delta(K)$ where $K = V - v$. Recall that $\ell(v) = \emptyset$.

Assume first that $b(v) \leq |F|$. By Proposition 6.6.7, $\dot{\delta}(v) \setminus F = \emptyset$. Hence $x(\dot{\delta}(v)) = b(v)$ is identical to $x(F) = |F|$, a contradiction.

Assume now that $b(v) = |F| + 1$. As $x(\delta(v)) = b(v) = |F| + 1$ and $x(F) \leq |F|$, $x(\delta(v) \setminus F) \geq 1$ must hold. Hence we have $x(E) = x(E[K]) + x(F) + x(\delta(v) \setminus F) \geq \frac{b(K)+|F|-1}{2} + 1 = \frac{b(V)}{2}$, which is only possible if x is a b-factor, a contradiction. □

Observe that if there is a bad constraint of type (IV) corresponding to v then this constraint is unique, namely $(V - v, \delta(v))$. Moreover, there is no bad constraint of type (V) corresponding to v by Proposition 6.8.2.

Claim 6.8.4. *For each $v \in V$, there is at most one bad constraint of type (V) in $\mathcal{L}$ corresponding to v.*

Proof. Assume that v is such that $x(E[K])) + x(F_1) = \frac{b(K)+|F_1|-1}{2}$ and $x(E[K])) + x(F_2) = \frac{b(K)+|F_2|-1}{2}$ for different $F_1, F_2 \subseteq \delta(K)$ where $K = V - v$.

Proposition 6.8.5. $|F_1| = |F_2|$.

Proof. Assume to the contrary that $|F_1| > |F_2|$. $(F_1 \setminus F_2) \subseteq F_1$ hence $x(F_1 \setminus F_2) \geq |F_1 \setminus F_2| - 1$. On the other hand, $(F_1 \setminus F_2) \subseteq (\delta(K) \setminus F_2)$, hence $x(F_1 \setminus F_2) \leq 1$. These imply $|F_1 \setminus F_2| \leq 2$. By parity arguments, $F_2 \subseteq F_1$, contradicting Proposition 6.8.2. □

Proposition 6.8.6. $|F_1 \cap F_2| = 0$.

Proof. Assume that $F_1 \cap F_2 = F \neq \emptyset$. From the tightness of (K, F_1) and (K, F_2) we get $2x(E[K]) + 2x(F) + x(F_1 \triangle F_2) = b(K) + |F| + \frac{|F_1 \triangle F_2|}{2} - 1 \geq b(K) + |F|$. On the other hand, we know that $2x(E[K]) + x(\delta(K)) \leq b(K)$ and $x(F) < |F|$ implying $2x(E[K]) + 2x(F) + x(\delta(K) \setminus F) < b(K) + |F|$, a contradiction. □

Proposition 6.8.7. $|F_1| = |F_2| = 1$

Proof. By Proposition 6.5.6, $x(F_1) \le 1$ as $F_1 \subseteq \delta(K) \setminus F_2$, hence $|F_1| \le 2$ by the same proposition.

Assume that $|F_1| = 2$. From the tightness of (K, F_1) and (K, F_2) we get

$$2x(E[K]) + x(F_1) + x(F_2) = b(K) + 1.$$

On the other hand, we know that $2x(E[K]) + x(\delta(K)) \le b(K)$, a contradiction. □

Let $F_1 = f_1, F_2 = f_2$. Clearly, $x(f_1) = x(f_2)$.

Proposition 6.8.8. $\delta(v) = \{f_1, f_2\}$

Proof. We have $x(E[K]) + x(f_1) = \frac{1}{2}b(K)$ and $x(E[K]) + x(f_2) = \frac{1}{2}b(K)$, so $2x(E[K]) + x(f_1) + x(f_2) = b(K)$. That means that each node is saturated in K by the x-values on $E[K]$ and $\{f_1, f_2\}$, hence there is no edge $f \in \delta(K) \setminus \{f_1, f_2\}$ by Proposition 6.6.2. □

Proposition 6.8.8 implies that there are at most two bad constraints of type (V) in $\mathcal{L}$ corresponding to a node. Assume that v is a node with two such constraints. The proof of Proposition 6.8.8 implies that all the other nodes are saturated by x, hence v is unique with this property by Claim 6.8.3.

We claim that $\mathcal{T} = \emptyset$. Indeed, assume first that there is a forbidden triangle $T \in \mathcal{T}$ containing v. Let $f_1 = vu$ and $f_2 = vw$ be the two edges incident to v. Both u and w have degree 3 as they are saturated and $x < 1$. Let $e_1 = \delta(u) \setminus E_T$ and $e_2 = \delta(w) \setminus E_T$. It is easy to see that $x(e_1) = x(e_2) > x(f_1) = x(f_2)$. Also, $x(e_i) > \frac{1}{2}$ by $x < 1$, the previous observation and $x(e_i) + x(f_i) + x(uw) = 2$.

Edges e_1, e_2, uw do not form the edge-set of a forbidden triangle T' as otherwise $x(E_T) + x(E_{T'}) = x(\delta(u)) + x(\delta(w)) = 4$, hence both T and T' are tight, a contradiction.

Delete the edges uv, uw from G, shrink u and w in a single node z with $b(z) = 2$ and add a new edge vz to the graph with $x(vz) = 2 - x(e_1) - x(e_2)$. Let $G', b', \mathcal{T}', x'$ denote the lexicographically smaller problem thus arising. An easy case-checking shows that x' satisfies (P_8) in G' with b' and $\mathcal{T}'$ hence it is a convex combination of $\mathcal{T}'$-free b'-matchings of G'. This convex combination can be extended to the original problem in a straightforward manner thus giving x, a contradiction.

Proposition 6.8.9. *There is no triangle $T \in \mathcal{T}$ whose nodes are all saturated.*

Proof. Assume that $x(\delta(v)) = 2$ for each $v \in V_T$ for some $T \in \mathcal{T}$. Recall that V_T does not span parallel edges by Proposition 6.6.1. Then $2x(E_T) + x(\delta(V_T)) = 6$, and so $x(E_T) + x(\delta(V_T)) \ge 5 - 2 = 4$. On the other hand, $(V_T, \delta(V_T))$ is an odd pair, so $x(E_T) + x(\delta(V_T)) \le \lfloor \frac{6+3}{2} \rfloor) = 4$. Hence we have equality everywhere, implying $x(E_T) = 2$, a contradiction. □

By Claim 6.8.9, there is no $T \in \mathcal{T}$ with $V_T \subseteq V - v$ either. Let $f_1 = vu$ and $f_2 = vw$ be the two edges incident to v. Delete v from G and add a new edge between u and w with x-value $x(f_1) = x(f_2) = C$. Let G', x' denote the graph and vector thus arising.

Proposition 6.8.10. *x' satisfies (P_8) in G'.*

Proof. It only suffices to verify (*iii*). Assume that there is an odd pair (Z, H) with $Z \subseteq V - v, H \subseteq \delta(Z) \setminus \{f_1, f_2\}$ violating (*iii*) in G'. It is easy to see that $u, w \in Z$ must hold otherwise there would be a violating pair in the original problem, too. That means that $x(E[Z]) + x(H) > \frac{b(Z)+|H|-1}{2} - C$. In other words, as each node different from v is saturated, $b(Z) - x(E[Z]) - x(\delta(Z) \setminus H) > \frac{b(Z)+|H|-1}{2} - C$, so $x(E[Z]) + x(\delta(Z) \setminus H) < \frac{b(Z)-|H|+1}{2} + C$. If (Z, H) is odd then $(V \setminus (Z + v), H)$ is also odd and $x(E[V \setminus (Z + v)]) + x(H) \leq \frac{(V\setminus(Z+v))+|H|-1}{2}$. Summing up these we get $x(E) < \frac{b(V-v)}{2} + C$.

As both $(V - v, f_1)$ and (V_v, f_2) are tight, $2x(E[V - v]) + x(\{f_1, f_2\}) = b(V - v)$, that is, $2x(E) = b(V - v) + 2C$, a contradiction. □

As G', x' provides a lexicographically smaller problem, x' is a convex combination of b-matchings (in fact factors) of G'. These b-matchings easily extends to G giving x, a contradiction. □

Claims 6.8.1, 6.8.3 and 6.8.4 imply that $|\mathcal{L}| \leq |V|$, and we are done.

6.9 Further remarks

The problem of giving a complete description of the triangle-free 2-matching polytope of arbitrary graphs is still open. As mentioned in Section 1.4, assumption (6.1) is essential: Theorem 6.1.2 is false if we remove the degree bound $d_G(v) \leq 3$ on nodes of forbidden triangles, as shown by the following example.

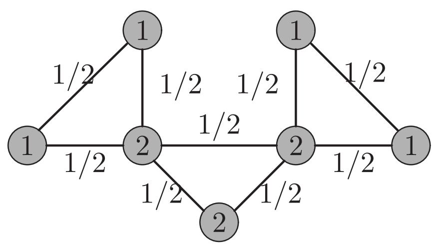

Figure 6.9: A counterexample for the non-subcubic case

The values on the nodes and on the edges represent b and x, respectively, and $\mathcal{T}$ contains the triangle in the center. One may check that x satisfies (P_8) with total value $\frac{9}{2}$, but the maximum size of a $\mathcal{T}$-free b-matchings is 4, hence x is definitely not contained in the $\mathcal{T}$-free b-matching polytope.

In [58], Grötschel and Pulleyblank introduced a new class of inequalities valid for the travelling salesman polytope. This new class, which is called clique tree inequalities, properly contains various classes of well known inequalities such as blossom inequalities, subtour elimination constraints, 2-matching constraints, Chvátal combs or comb inequalities.

An **articulation set** of a graph $G = (V, E)$ is minimal set of nodes whose deletion results in graph with more connected components that of G. A clique tree, according to [58], is defined as follows.

Definition 6.9.1. A **clique tree** is a connected graph C for which the maximal cliques satisfy the following properties:

1. The cliques are partitioned into the sets of **handles** and **teeth**.
2. No two teeth intersect.
3. No two handles intersect.
4. Each tooth contains at least two, at most $n - 2$ nodes, and at least one node belonging to no handle.

5. For each handel, the number of teeth intersecting it is odd and at least three.
6. If a tooth T and a handle H have nonempty intersection, then $H \cap T$ is an articulation set of the clique tree.

It follows from the definition that a clique tree indeed has a 'tree-like structure', see Figure 6.10.

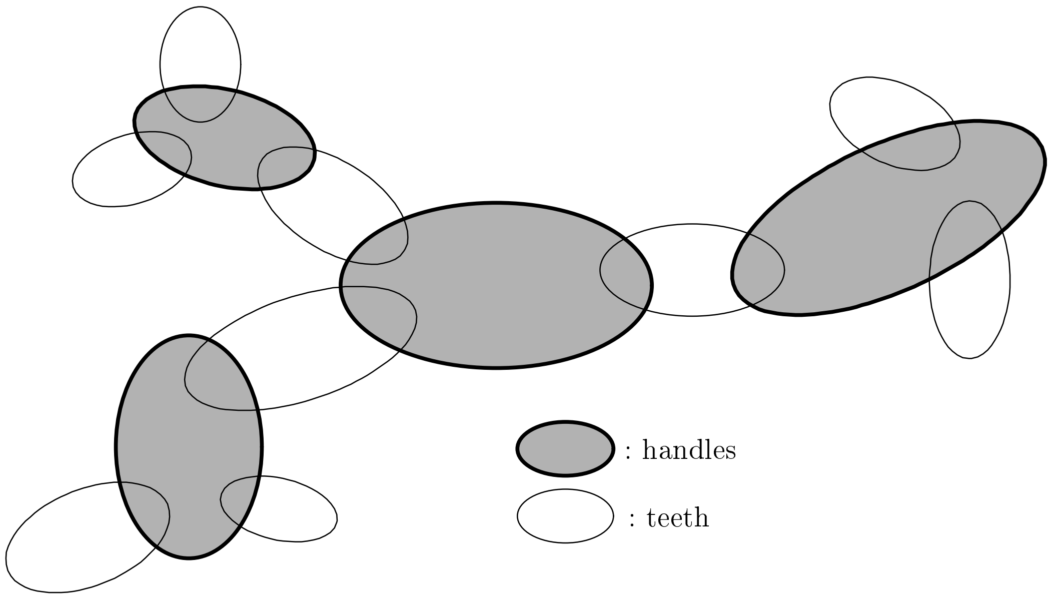

Figure 6.10: A clique tree

Grötschel and Pulleyblank showed the following.

Theorem 6.9.2 (Grötschel and Pulleyblank). *Let C be a clique tree in K_n with handles $H_1, \ldots, H_r$ and teeth $T_1, \ldots, T_s$. Then the* ***clique tree inequality***

$$\sum_{i=1}^{r} x(E[H_i]) + \sum_{j=1}^{s} x(E[T_j]) \leq \sum_{i=1}^{r} |H_i| + \sum_{j=1}^{s} (|T_j| - t_j) - \tfrac{s+1}{2} \tag{6.4}$$

is valid with respect to the travelling salesman polytope, where t_j denotes the number of handles intersecting tooth T_j.

In case of triangle-free 2-matchings, those clique trees are interesting in which the teeth are either triangles or single edges, see Figure 6.11.

Definition 6.9.3. A **tri-clique tree** is a connected graph C satisfying the following properties:

1. C is the union of subgraphs partitioned into two sets, **handles** and **teeth**.
2. No two teeth intersect.
3. No two handles intersect.
4. Each tooth is an edge or a triangle and contains at least one node belonging to no handle.
5. For each handel, the number of teeth intersecting it is odd and at least three.
6. If a tooth T and a handle H have nonempty intersection, then $H \cap T$ is an articulation set of the clique tree.

Using the same idea as in [58] the following can be proved.

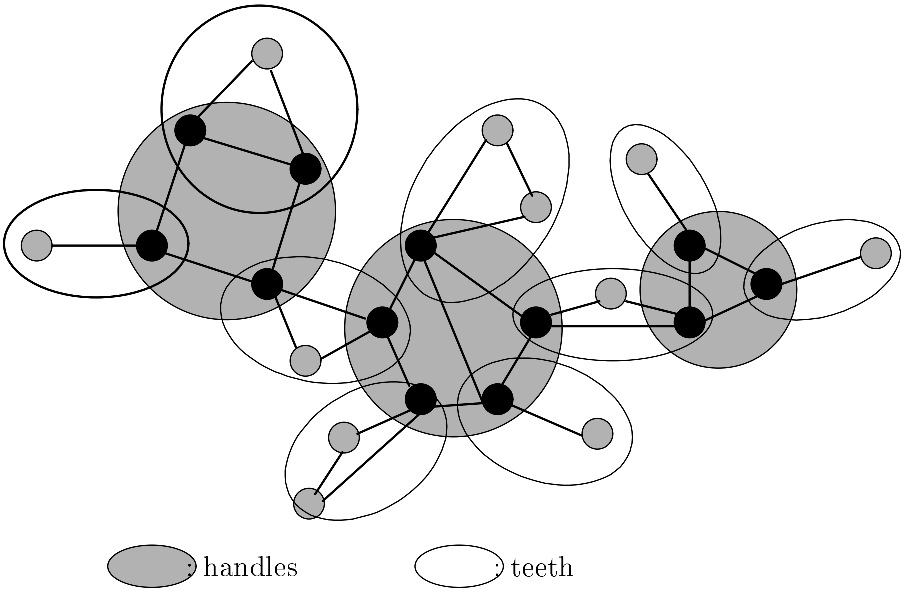

Figure 6.11: A clique tree for the C_3-free 2-matching case

Theorem 6.9.4. *Let C be a tri-clique tree in a simple graph G with handles $H_1, \ldots, H_r$ and teeth $T_1, \ldots, T_s$. Then the* ***tri-clique tree inequality***

$$\sum_{i=1}^{r} x(E[H_i]) + \sum_{j=1}^{s} x(E[T_j]) \leq \sum_{i=1}^{r} |H_i| + \sum_{j=1}^{s} (|T_j| - t_j) - \tfrac{s+1}{2} \qquad (6.5)$$

is valid with respect to the triangle-free 2-matching polytope, where t_j denotes the number of handles intersecting tooth T_j.

It was also showed in [58] that the clique tree inequalities are facet-inducing for the travelling salesman polytope and almost always induce distinct facets. Moreover, these inequalities -in some sense- can not be further generalized in a facet-inducing manner. Hence it would be interesting to see whether the addition of these inequalities to the description of the triangle-free 2-matchings in subcubic graphs would give a complete description of the polytope in question for arbitrary graphs.

Chapter 7

Splitting property via shadow systems

7.1 Shadow systems

The main result of the chapter is the following theorem.

Theorem 7.1.1. *In the poset* $(M_k, \prec)$*, the maximal antichain* M_k^k *has the splitting property, that is,* M_k^k *can be partitioned into disjoint sets* A_1 *and* A_2 *such that* $\mathcal{U}(A_1) \cup \mathcal{L}(A_2) = M_k$.

In Theorem 7.1.1, the required property of $A_1 \subset M_k^k$ is that for every vector $c \in M_k^{k+1}$, A_1 must contain at least one shadow of A_1. Generalizing this notion, for $r < t$ we call $A \subseteq M_k^r$ a $(t, r; k)$**-shadow system**, if for every colour vector $c \in M_k^t$, A contains at least one shadow of c. With this terminology, A_1 in Theorem 7.1.1 is a $(k+1, k; k)$-shadow system.

Consider a vector $s \in \mathbb{Z}_k^r$. The **colour profile** $a = M(s) \in M_k^r$ can be naturally defined so that a_i equals the number of i's in s for $1 \leq i \leq k$. First of all we give a proof of Theorem 1.5.4 by using the following.

Theorem 7.1.2. *For integers* $t > r$*, there exists a* $(t, r; t-1)$*-shadow system* $\mathcal{A}_r^t \subseteq M_{t-1}^r$ *so that if we pick a vector* $s \in \mathbb{Z}_{t-1}^r$ *uniformly at random, then the probability of* $M(s) \in \mathcal{A}_r^t$ *equals* $\left(\frac{r-1}{t-1}\right)^{r-1}$.

Proof of Theorem 1.5.4. Let us take a uniform random colouring with $t-1$ colours of a ground set V with $|V| = n$ nodes. Consider a $(t, r; t-1)$-shadow system $\mathcal{A}_r^t \subseteq M_{t-1}^r$ as in Theorem 7.1.2, and let the r-uniform hypergraph $(V, \mathcal{E})$ contain those r-element subsets X whose colour profile is contained in $\mathcal{A}_r^t$. (An r-element set coloured by $t-1$ colours naturally corresponds to a vector in $\mathbb{Z}_{t-1}^r$.) The $(t, r; t-1)$-shadow system property implies that every vector $c \in M_{t-1}^t$ has a shadow in $\mathcal{A}_r^t$. Consequently, every t-element subset of V has a subset in $\mathcal{E}$, that is, $\mathcal{E}$ is a Turán (n, t, r)-system. Theorem 1.5.4 follows since the expected size of $\mathcal{E}$ is $\left(\frac{r-1}{t-1}\right)^{r-1}\binom{n}{r}$ by Theorem 7.1.2. □

In what follows, we give a proof of Theorem 7.1.2.

Let $x = (x_1, \ldots, x_k) \in M_k$ be a k-colour vector. If $x_j = 0$ and $x_{j+1} \neq 0$ then $x' = (x_1, \ldots, x_{j-1}, x_{j+1} - 1, x_{j+2}, \ldots, x_k) \in M_{k-1}$ is called **the reduction of** x **at the** j**th position** and is denoted by $\texttt{red}[j](x)$ (indices are in a cyclic order, i.e. x_{k+1} refers to x_1). A vector with no zero entries is called **irreducible**. Assume that a series of reduction steps at positions $j_1, \ldots, j_t$ is applied on vector $x \in M_k$ which results in another vector $x' \in M_m$ where $t = k - m$. We define the **ancestor** $\operatorname{anc}(i)$ of a position $1 \leq i \leq m$ as the original position of that entry in the starting vector. Formally, these can be obtained by Procedure 2.

The following proposition unravels an important property of the reduction operation.

Procedure 2 Computing anc(i)

```
Set anc(i) := i.
Set q := t.
while q > 0 do
    if j_q > anc(i) then
        anc(i) := anc(i)
    else
        anc(i) := anc(i) + 1
    end if
    q := q − 1
end while
return anc(i)
```

Proposition 7.1.3. *Let $x \in M_k$ be a k-colour vector. Assume that after some reduction steps we obtain an irreducible vector x'. Then x' and the ancestors of its positions are independent from the choice of the reduction steps.*

Proof. For a contradiction, assume there exists a k-colour vector $x \in M_k$ that can be reduced to two vectors x' and x'' that are either different or are identical but one of the positions has different ancestors in them. Choose k as the minimum value where this may occur; clearly $k > 2$. By this minimal choice, the two reduction sequences must differ in the very first step. Assume the first sequence reduces at position j' and the second at position j'', resulting in $y' = \texttt{red}[j'](x)$ and $y'' = \texttt{red}[j''](x)$. W.l.o.g. assume $j' < j''$; then $j'' > j' + 1$ follows as we cannot reduce at position j' if $x_{j'+1} = 0$. Consider now the reductions $\texttt{red}[j'](y'')$ and $\texttt{red}[j'' - 1](y')$. These must be identical. Moreover, the ancestors of the positions in $\texttt{red}[j'](y'')$ and $\texttt{red}[j'' - 1](y')$ also coincide. However, by the minimal choice of k, any reduction sequence of y' and y'' must result in the same vector z with the same ancestors, a contradiction. ☐

As an alternative proof, we can define the following quantity. Let $\text{sum}(j, k) = \sum_{i=j}^{k-1}(x_i - 1)$ where indices are in cyclic order and $\text{sum}(k, k)$ is defined as 0. Let $x_i^{red} = \max\{0, x_i + \min_j \text{sum}(j, i)\}$. Observe that the reduction stops with an x' which is obtained from x^{red} by deleting its zero entries. Moreover, the ancestor of position i is just the position of the corresponding nonzero entry in x^{red}.

The irreducible vector arising by applying a sequence of reductions on x is hence uniquely defined; it is called the **complete reduction** of x and is denoted by $\texttt{red}(x)$. The ancestor of position i in a complete reduction is denoted by anc(i). Let us define the **rank** of x, denoted by $rk(x)$, as the length of the vector $\texttt{red}(x)$, and let

$$\mathcal{A}_k := \{x \in M_k^k : rk(x) = 1\}. \tag{7.1}$$

Note that reducing a vector in M_k^k gives a vector in M_{k-1}^{k-1} and the only irreducible vector in M_k^k is an all-one vector (that is, all its entries are 1). Consequently, the complete reduction of any vector in M_k^k is an all-one vector of dimension $m \leq k$, and $x \in \mathcal{A}_k$ if and only if $m = 1$. Theorem 7.1.1 follows by the next lemma, showing that partitioning M_k^k to $\mathcal{A}_k$ and $M_k^k \setminus \mathcal{A}_k$ satisfies the splitting property.

Lemma 7.1.4. *Let $\mathcal{B}_k = M_k^k \setminus \mathcal{A}_k$. Then $M_k = \mathcal{U}(\mathcal{A}_k) \cup \mathcal{L}(\mathcal{B}_k)$.*

The proof needs one more operation. For $x = (x_1, \dots, x_k) \in M_k$ we call $x' = (x_1, x_2, \dots, x_{j-1}, 0, x_j + 1, x_{j+1}, \dots, x_k) \in M_{k+1}$ **the extension of x at the jth position** and denote it by $\texttt{ext}[j](x)$. The extension can be considered as a reverse counterpart of the reduction. However, there are no restrictions on the elements of x in this case and applying `ext` does not modify the result of `red`, namely $\texttt{red}(x) = \texttt{red}(\texttt{ext}[j](x))$.

Proof of Lemma 7.1.4. We have to show that *(a)* for every $c \in M_k^{k+1}$, $\mathcal{A}_k$ contains a shadow of c, that is, $\mathcal{A}_k$ is a $(k+1, k; k)$-shadow system; and *(b)* for every $d \in M_k^{k-1}$, there exists a $b \in \mathcal{B}_k$ such that d is a shadow of b.

Both statements are proved by induction on k. For $k = 2$, $\mathcal{A}_2 = \{(2,0),(0,2)\}$ and $\mathcal{B}_2 = \{(1,1)\}$, and both statements clearly hold. Assume both (a) and (b) hold for all values strictly less than k.

For *(a)*, consider an arbitrary vector $c \in M_k^{k+1}$. We distinguish two cases.

Case 1. *c is irreducible, that is, every entry is strictly positive.*

Since the sum of the elements of c is $k+1$, this is only possible if for some $1 \le p \le k$, $c_p = 2$ and $c_i = 1$ for $1 \le i \le k$, $i \ne p$. Consider the vector $a \in M_k^k$ with $a_p = 2$, $a_{p+1} = 0$, $a_i = 1$ for every other index i. Then a is a shadow of c and it is easy to verify that $rk(a) = 1$, that is, $a \in \mathcal{A}_k$ as required.

Case 2. *There exists an index i with $c_i = 0$, $c_{i+1} \ne 0$.*

Let $c' = \texttt{red}[i](c) \in M_{k-1}^k$. By induction, there exists an $a' \in \mathcal{A}_{k-1}^{k-1}$ that is a shadow of c'. Let $a = \text{ext}[i](a') \in M_k^k$. Then $rk(a) = rk(a') = 1$, and therefore $a \in \mathcal{A}_k$. Now a is a shadow of c, completing the proof.

Let us now turn to statement *(b)*. Consider an arbitrary colour vector $d \in M_k^{k-1}$. Since the sum of the elements of d is $k-1$, there is an index $1 \le i \le k$ such that $d_i = 0$ and $d_{i+1} \ne 0$. Let $d' = \texttt{red}[i](d)$ which is in M_{k-1}^{k-2}. By induction, there exists a $b' \in \mathcal{B}_{k-1}$ such that d' is a shadow of b'. Let $b = \texttt{ext}[i](b') \in M_k^k$. Since $\texttt{red}(b) = \texttt{red}(b')$, it follows that $b \in \mathcal{B}_k$, as required. $\square$

The construction of the $(t, r; t-1)$-shadow system in Theorem 7.1.2 is also based on $\mathcal{A}_k$. We first need to define some further operations. For a vector $x \in \mathbb{Z}_k^r$, we obtain the vector $x' = \delta x \in \mathbb{Z}_k^r$ by increasing every coordinate by 1: $x'_i = x_i + 1$. We call δ the **k-shifting operator**; the j'th power is denoted by δ^j. Clearly δ^k is the identity but $\delta^j x \ne x$ for $0 < j < k$. The set $\{x, \delta x, \delta^2 x, \dots, \delta^{k-1} x\}$ is called the **k-orbit** of x. Being in the same k-orbit defines an equivalence relation on $\mathbb{Z}_k^r$.

The k-shifting operation induces a natural operation on the colour vectors in M_k^r. For $a \in M_k^r$, let $a' = \Delta a \in M_k^r$ be the vector with $a'_i = a_{i-1}$ (with indices modulo k, i.e. $a'_1 = a_k$). We call Δ the **cyclic shifting operator**. Clearly, $M(\delta x) = \Delta M(x)$ for every $x \in \mathbb{Z}_k^r$ (recall that $M(x)$ denotes the colour profile of x). Again, $\{a, \Delta a, \Delta^2 a, \dots, \Delta^{k-1} a\}$ defines the **cyclic orbits** of M_k^r, and being in the same orbit is again an equivalence relation. However, note that $\Delta^j a = a$ may occur even for $j < k$. (For example, let $k = 4$, $r = 4$, $j = 2$, $a = (2020)$.) If a and b are on the same cyclic orbits, then so are $\texttt{red}(a)$ and $\texttt{red}(b)$. We denote the cyclic orbit of an $a \in M_k^r$ by $CO(a)$. The above notions are illustrated on Figure 7.1.

Remark 7.1.5. It is worth mentioning that in Lemma 7.1.4, both sets $\mathcal{A}_k$ and $\mathcal{B}_k$ are closed under the operation Δ.

$\mathbb{Z}_3^2$	M_3^2	3-orbits of $\mathbb{Z}_3^2$	cyclic orbits of M_3^2
$(1,1)$	$(2,0,0)$	$\{(1,1),(2,2),(3,3)\}$	$\{(2,0,0),(0,2,0),(0,0,2)\}$
$(1,2)$	$(0,2,0)$	$\{(1,2),(2,3),(3,1)\}$	$\{(1,1,0),(0,1,1),(1,0,1)\}$
$(1,3)$	$(0,0,2)$	$\{(1,3),(2,1),(3,2)\}$	
$(2,1)$	$(1,1,0)$		
$(2,2)$	$(1,0,1)$		
$(2,3)$	$(0,1,1)$		
$(3,1)$			
$(3,2)$			
$(3,3)$			

Figure 7.1: The members and orbits of $\mathbb{Z}_3^2$ and M_3^2.

We are ready to define $\mathcal{A}_r^t$ as in Theorem 7.1.2. Consider $\mathcal{A}_r$ as in (7.1), and let $a \in \mathcal{A}_r$. By definition, $\texttt{red}(a) = (1)$. Let us call the ancestor of this single element the **tip** of the vector a. Let $\texttt{blow}(a) \in M_{t-1}^r$ denote the vector arising from a by inserting $t-1-r$ zeros just after the tip of a. Define

$$\mathcal{A}_r^t := \bigcup_{a \in \mathcal{A}_r} CO(\texttt{blow}(a)). \qquad \textbf{(SHA)}$$

For example, let $r = 3$, $t = 5$, and $a = (2,0,1) \in \mathcal{A}_3$. The tip of a is the first element, and $\texttt{blow}(a) = (2,0,0,0,1)$. Finally, $CO(\texttt{blow}(a)) = \{(2,0,0,0,1),(1,2,0,0,0),(0,1,2,0,0),(0,0,1,2,0),(0,0,0,1,2)\}$. Also, note that if $a' \in CO(a)$, then $CO(\texttt{blow}(a)) = CO(\texttt{blow}(a'))$. Further, $\cup_{a' \in CO(a)} \texttt{blow}(a') \subsetneq CO(\texttt{blow}(a))$: in the above example, $(0,0,0,1,2)$ is contained in the latter set but not in the first.

We show that $\mathcal{A}_r^t$ is a $(t,r;t-1)$-shadow system satisfying the requirement of Theorem 7.1.2. The shadow system property can be verified using an argument almost identical to that in the proof of Lemma 7.1.4.

Lemma 7.1.6. *For integers $t > r$, $\mathcal{A}_r^t \subseteq M_{t-1}^r$ defined by* (**SHA**) *is a $(t,r;t-1)$-shadow system.*

Proof. The proof is by induction on r. For $r = 2$, $\mathcal{A}_2 = \{(2,0),(0,2)\}$, and for any $t > r$, $\mathcal{A}_2^t$ contains the vectors with one entry being 2 and all other entries 0. Every $c \in M_{t-1}^t$ must contain at least one entry ≥ 2, and therefore it has a shadow in $\mathcal{A}_2^t$. Assume we have proved the statement for all values strictly less than r and consider an arbitrary colour vector $c \in M_{t-1}^t$.

Case 1. *c is irreducible, that is, every entry is strictly positive.*

Since the sum of the elements of c is t, this is only possible if for some $1 \leq p \leq t-1$, $c_p = 2$ and $c_i = 1$ for $1 \leq i \leq t-1$, $i \neq p$. Consider the vector $a \in M_{t-1}^r$ with

$$a_i = \begin{cases} 2 \text{ if } i = p, \\ 0 \text{ if } i = p+1, \ldots, p+t-r, \\ 1 \text{ otherwise,} \end{cases}$$

where we use the indexing cyclically, i.e. t means 1. Clearly, a is a shadow of c, and $a \in \mathcal{A}_r^t$ since removing $t-1-r$ 0's after the 2, we obtain $a' = (1,\ldots,1,2,0,1,\ldots,1) \in M_r^r$, and it is easy to verify $a' \in \mathcal{A}_r$.

Case 2. *There exists an index i with $c_i = 0$, $c_{i+1} \neq 0$.*

Let $c' = \texttt{red}[i](c) \in M^{t-1}_{t-2}$. By induction, there exists an $a' \in \mathcal{A}^{r-1}_{t-2}$ that is a shadow of c'. Let $a = \texttt{ext}[i](a') \in M^r_{t-1}$. It is easy to verify $a \in \mathcal{A}^t_r$. Now a is a shadow of c, completing the proof. □

The following lemma considers elements of $\mathbb{Z}^r_{t-1}$ instead of colour vectors, and gives the exact number of those having their colour profile in $\mathcal{A}^t_r$.

Lemma 7.1.7. *Let $\mathcal{S} \subseteq \mathbb{Z}^r_{t-1}$ denote the set of vectors whose colour profile is in $\mathcal{A}^t_r$. Then $|\mathcal{S}| = (r-1)^{r-1}(t-1)$.*

Before proving the lemma, let us derive Theorem 7.1.2 as a consequence.

Proof of Theorem 7.1.2. We show that $\mathcal{A}^t_r$ as defined by (**SHA**) satisfies the conditions. Lemma 7.1.6 shows that it is a $(t, r; t-1)$-shadow system. The total number of vectors in $\mathbb{Z}^r_{t-1}$ is $(t-1)^r$. The probability that a randomly picked $s \in \mathbb{Z}^r_{t-1}$ has its colour profile in $\mathcal{A}^t_r$ is $|\mathcal{S}|/(t-1)^r = \left(\frac{r-1}{t-1}\right)^{r-1}$ by Lemma 7.1.7 as required. □

By definition, $\mathcal{A}^t_r$ is closed under the operation Δ. While certain cyclic orbits may be shorter than $t-1$, the next claim shows this cannot be the case for orbits contained in $\mathcal{A}^t_r$.

Claim 7.1.8. *If $a \in \mathcal{A}^t_r$, then $\Delta^j a \neq a$ for $0 < j < t-1$. Consequently, all cyclic orbits contained in $\mathcal{A}^t_r$ have size exactly $t-1$.*

Proof. Every cyclic orbit in $\mathcal{A}^t_r$ can be obtained as $CO(\texttt{blow}(a))$ for some $a \in \mathcal{A}_r$. It suffices to show that for any $0 < j < t-1$, $\Delta^j\texttt{blow}(a) \neq \texttt{blow}(a)$. For a contradiction, assume there exists such a j and a for which $\Delta^j\texttt{blow}(a) = \texttt{blow}(a)$; let $b = \texttt{blow}(a)$ and $b' = \Delta^j\texttt{blow}(a)$. Without loss of generality, assume the tip of a is its first element.

As $a \in \mathcal{A}_r$, it can be reduced to (1), which means that b can be reduced to $(0, \ldots, 0)$ consisting of $t-r-1$ zeros and the ancestor of the ith zero is i. Recall that the complete reduction of b and the ancestors of the elements of $\texttt{red}(b)$ are uniquely defined by Proposition 7.1.3. By $b' = b$, b' also has complete reduction $(0, \ldots, 0)$ consisting of $t-r-1$ zeros where the ancestor of the ith zero is i. On the other hand, by $b' = \Delta^j b$, the ancestors of the elements of $\texttt{red}(b')$ are just the ancestors of the elements of $\texttt{red}(b)$ shifted by j, a contradiction as $0 < j < t-1$. □

Proof of Lemma 7.1.7. The cardinality of $\mathbb{Z}^r_{r-1}$ is $(r-1)^r$ and the number of $(r-1)$-orbits is $(r-1)^{r-1}$. Since $\mathcal{A}^t_r$ is closed under Δ, it follows that $\mathcal{S}$ is closed under δ and is hence a union of $(t-1)$-orbits. In what follows, we define a bijection φ between the $(r-1)$-orbits of $\mathbb{Z}^r_{r-1}$ and the $(t-1)$-orbits of $\mathcal{S}$. Since every $(t-1)$-orbit has cardinality $t-1$ by Lemma 7.1.7, this proves the lemma.

Consider a colour vector $a \in M^r_{r-1}$. It is easy to verify that its complete reduction has one entry that is 2 and all other entries are 1, that is $\texttt{red}(a) = (1, \ldots, 1, 2, 1, \ldots, 1)$. Analogously as for elements of $\mathcal{A}_r$, we call the ancestor of the entry 2 the **tip** of a. Clearly, the tip of Δa is the tip of a plus one (in a cyclic sense).

Take an arbitrary $(r-1)$-orbit X in $\mathbb{Z}^r_{r-1}$. The colour profiles of the vectors in X map to a cyclic-orbit T of M^r_{r-1}. T must have an element a whose tip is the last ($(r-1)$'st) coordinate; pick an $s \in X$ such that $M(s) = a$. Let us inject $\mathbb{Z}_{r-1}$ into $\mathbb{Z}_{t-1}$ by mapping $i \in \mathbb{Z}_{r-1}$ to $i \in \mathbb{Z}_{t-1}$ for $1 \leq i \leq r-1$, and let $\bar{s} \in \mathbb{Z}^r_{t-1}$ be the image of s under this mapping. Let us define $\varphi(X)$ as the $(t-1)$-orbit of $\bar{s}$ in

$\mathbb{Z}_{t-1}^r$. In what follows, we verify that φ is a good bijection.

Well-defined. We first have to show that $\bar{s} \in \mathcal{S}$, that is, $M(\bar{s}) \in \mathcal{A}_r^t$. Observe that $\bar{a} = M(\bar{s}) \in M_{t-1}^r$ can be obtained from $a = M(s) \in M_{r-1}^r$ by adding $t-r$ zero coordinates at the $(r-1)$'st position. The vector a can be reduced to $(1,1,\ldots,1,2)$; apply the same reduction steps to $\bar{s}$. This gives a vector $b = (1,1,\ldots,1,2,0,\ldots,0)$ (with $t-r$ zeros at the end), which can be further reduced to (1) after deleting the last $t-r-1$ zeros.

Injective. Assume indirectly that X_1 and X_2 are different $(r-1)$-orbits of $\mathbb{Z}_{r-1}^r$, such that $\varphi(X_1) = \varphi(X_2)$. For $i = 1,2$, let T_i be the corresponding cyclic orbit, $a^i \in T_i$ the element with tip $(r-1)$ and $s^i \in X_i$ with $M(s^i) = a^i$. Define $\overline{s}^i \in \mathcal{S}$ by mapping $\mathbb{Z}_{r-1}$ to $\mathbb{Z}_{t-1}$ and $\overline{a}^i \in M_{t-1}^r$ as the colour profile of $\overline{s}^i$. Now $s^1 \neq s^2$ are on different $(r-1)$-orbits but $\overline{s}^1 \neq \overline{s}^2$ are on the same $(t-1)$-orbit. That means that there is a j such that $\overline{s}^2 = \delta^j \overline{s}^1$, and so $\overline{a}^2 = \Delta^j \overline{a}^1$.

We know that both $\overline{a}^1$ and $\overline{a}^2$ can be reduced to $(1,\ldots,1,2,0,\ldots,0)$ (with $t-r$ zeros at the end) by applying the same reductions steps as for a^1 and a^2, and this vector can be further reduced to the all-zero $(0,\ldots,0)$ vector consisting of $t-r-1$ zeros where the ancestor of the ith element is $t-r$. Again, the complete reduction of a vector and the ancestors of the elements of the reduction are uniquely defined by Proposition 7.1.3. We have seen that $\overline{a}^1$ and $\overline{a}^2$ has the same complete reduction. On the other hand, by $\overline{a}^2 = \Delta^j \overline{a}^1$, the ancestors of the elements of $\texttt{red}(\overline{a}^2)$ are just the ancestors of the elements of $\texttt{red}(\overline{a}^1)$ shifted by j, a contradiction as $0 < j < t-1$.

Surjective. Consider any orbit Y of $\mathcal{S}$, and let $a \in \mathcal{A}_r^t$ be the colour profile of an element $s \in Y$. We may choose s such that $a_r = \ldots = a_{t-1} = 0$. This is since a is a vector in $CO(\texttt{blow}(a_0))$ for some $a_0 \in \mathcal{A}_r$, that is, we insert $t-1-r$ zeros after the tip of a_0 and apply Δ^j for some j. It is easy to verify that the element of a_0 following the tip must be 0 because of $rk(a_0) = 1$.

Let us apply reduction steps on a avoiding the last $t-r$ zeros but reducing all others. It is easy to verify that this reduces a to $(1,\ldots,1,2,0,\ldots,0)$ (with $t-r$ zeros at the end). Now let us map $s \in \mathbb{Z}_{t-1}^r$ to $s^* \in \mathbb{Z}_{r-1}^r$ by mapping $i \in \mathbb{Z}_{t-1}$ to $i \in \mathbb{Z}_{r-1}$ for $1 \leq i \leq r-1$ (this is well-defined as s does not contain colors $r,\ldots,t-1$ by $a_r = \ldots = a_{t-1} = 0$). Observe that φ maps the orbit of s^* to Y, proving the claim. □

7.1.1 Relation to Sidorenko's construction

Sidorenko's construction is based on the following observation.

Lemma 7.1.9. *Let $b_1,\ldots,b_k$ be cyclically ordered reals, and $b = \frac{b_1+\ldots+b_k}{k}$. Then there exists an index m such that*

$$b_m + \ldots + b_{m-s+1} \geq sb \quad \forall s = 1,\ldots,k.$$

The construction is as follows: Divide the n elements into $t-1$ groups $A_1, A_1, \ldots, A_{t-1}$. Let B be an r-element subset and $b_i = |B \cap A_i|$. Then set B is included into the set system $\mathcal{T}$ if and only if there is an index m such that

$$\sum_{i=1}^{s} b_{m-i+1} \geq s+1 \qquad \forall s = 1, \ldots, r-1, \tag{7.2}$$

where indices are meant in cyclic order, that is, $b_t = b_1$. It follows from Lemma 7.1.9 that $\mathcal{T}$ thus obtained is a Turán (n, t, r)-system.

The following lemma shows the connection between Sidorenko's construction and that of $\mathcal{A}_r^t$.

Lemma 7.1.10. *Assume that the n elements are divided into $t-1$ groups $A_1, A_1, \ldots, A_{t-1}$. An r-element subset B is included into $\mathcal{T}$ if and only if $(b_1, \ldots, b_{t-1}) \in \mathcal{A}_r^t$.*

Proof. Consider a set B with $b = (b_1, \ldots, b_{t-1}) \in \mathcal{A}_r^t$. Then $b \in CO(\texttt{blow}(a))$ for some $a \in \mathcal{A}_r$ where $\mathcal{A}_r$ is defined by (7.1), say $b = \Delta^j \texttt{blow}(a)$. Let p be the tip of a and define $m = p+j$. We claim that m and b satisfies (7.2). Indirectly, assume that there is an $1 \leq s \leq r-1$ violating (7.2), that is, $\sum_{i=1}^{s} b_{m-i+1} \leq s$. From $s \leq r-1$ and the definitions of b and m, $\sum_{i=1}^{s} b_{m-i+1} = \sum_{i=1}^{s} a_{p-i+1}$. Choose s to be maximal. Then $s < r-1$ as $\sum_{i=1}^{r-1} a_{p-i+1} = r$. Indeed, $a \in \mathcal{A}_r$ so $\sum_{i=1}^{r} a_{p-i+1} = r$, and $a \neq (1, \ldots, 1)$ as it can be reduced to (1).

Recall that $a' = \texttt{red}(a)$ is obtained from a^{red} by deleting its zero entries, where $a_i^{red} = \max\{0, a_i + \min_j \operatorname{sum}(j, i)\}$ and $\operatorname{sum}(j, k) = \sum_{i=j}^{k-1}(a_i - 1)$ (we defined $\operatorname{sum}(k, k)$ as 0). However, $\sum_{i=1}^{s} a_{p-i+1} \leq s$ means that in fact $\sum_{i=1}^{s} a_{p-i+1} = s$, otherwise $a_p^{red} = 0$ contradicting p being a tip. The maximal choice of s implies $\sum_{i=1}^{q} a_{p-s-i+1} \geq q$ for $1 \leq q \leq r$ and $\sum_{i=1}^{r-s} a_{p-s-i+1} = r - s > 0$. Hence $a_{r-s}^{red} > 0$, contradicting $a \in \mathcal{A}_r$.

Now take a $B \in \mathcal{T}$ and an index m satisfying (7.2). W.l.o.g. assume that $m = r$. That is, $\sum_{i=1}^{s} b_{r-i+1} \geq s+1$ for $1 \leq s \leq r-1$. As $\sum_{i=1}^{t-1} b_{r-i+1} = r$, we immediately have $b_{r+1} = \ldots = b_{t-1} = b_1 = 0$. Let $a = (a_1, \ldots, a_r) = (b_1, \ldots, b_r)$. Then $\sum_{i=1}^{r} a_{r-i+1} = r$ and $\sum_{i=1}^{s} a_{r-i+1} \geq s+1$ for $1 \leq s \leq r-1$. We claim that $a \in \mathcal{A}_r$. To see this, it suffices to show that $a_p^{red} = 0$ for $p = 1, \ldots, r-1$. Assume indirectly that $a_p^{red} > 0$ for some p. This implies $\sum_{i=1}^{q} a_{p-i+1} \geq q$ for $1 \leq q \leq r$. We have $r = \sum_{i=1}^{r} a_i = \sum_{i=1}^{p} a_{p-i+1} + \sum_{i=1}^{r-p} a_{r-i+1} \geq p + r - p + 1 = r+1$, a contradiction. □

In the proof of Theorem 1.5.4, we took a uniform random colouring of the ground set with $t-1$ colours and showed that the expected number of r-element subsets whose colour profile is contained in $\mathcal{A}_r^t$ is 'small enough'. Sidorenko's construction takes a deterministic colouring instead with almost equal groups, that is, $\big||A_i| - |A_j|\big| \leq 1$ for $1 \leq i < j \leq t-1$, and shows that for such a colouring the number of r-element subsets with colour profile in $\mathcal{A}_r^t$ does not exceeds the bound, thus proving (1.11).

7.2 Weighted Turán number

Recall the definition of the weighted Turán number $tw(t, r)$ from the Introduction. The following easy observation shows that the presence of weights does not affect the upper bound for $tw(t, r)$.

Theorem 7.2.1. *For any integers $t > r$, we have $tw(t, r) = t(t, r)$, and therefore $tw(t, r) \leq \left(\frac{r-1}{t-1}\right)^{r-1}$.*

Proof. Clearly, $tw(t, r) \geq t(t, r)$ as the unweighted Turán number corresponds to the special case $w \equiv 1$. To see the other direction, take an arbitrary Turán (n, t, r)-system (without taking weights into account). If we consider the weight of this system in a random permutation of the elements, then the expected value of its weight is exactly $\frac{T(n,t,r)}{\binom{n}{r}} \cdot w^*$, which means that there exists a Turán (n, t, r)-system with weight at most that, completing the proof. The second half follows by Theorem 1.5.4. □

Theorem 7.2.1 ensures the existence of a Turán (n, t, r)-system with 'small' weight. However, it is still not clear how to find and represent such a system. For $t = 3$ and $k = 2$, Theorems 1.5.4 and 7.2.1 imply that in a weighted graph, we can choose a set of edges whose weight is at most the half of the total weight w^* covering every triangle. Indeed, the most simple maximum cut algorithm delivers such an edge set. Let us colour the nodes of the graph by two colours uniformly at random, and choose the set of edges whose two endpoints receive the same colour. Clearly, these edges must cover every triangle. Since every individual edge gets chosen by probability $\frac{1}{2}$, the expected cost of the chosen edge set will be $\frac{w^*}{2}$.

The proof of Theorem 1.5.4 using Theorem 7.1.2 presented in the Introduction also yields a simple randomized algorithm for finding an (n, t, r)-Turán system in question. We colour the nodes uniformly at random by $(t-1)$-colours, and choose r-element subsets according to their colour profiles. Note that we must obtain a Turán system of cost at most $\left(\frac{r-1}{t-1}\right)^{r-1} w^*$ with probability at least $\left(\frac{r-1}{t-1}\right)^{r-1}$. The construction of the $(t, r; t-1)$-shadow system $\mathcal{A}_r^t$ in Theorem 7.1.2 will give a simple and efficient way to decide whether a colour vector is contained in $\mathcal{A}_r^t$. Consequently, although the size of the construction is $O(n^r)$, the colouring provides a simple linear representation.

7.3 Tuza's conjecture

As outlined earlier, the minimum number of edges covering all of the triangles in an arbitrary graph is the weighted Turán number $T_w(n, 3, 2)$ for $w_e = 1$ on the edges of the graph and $w_e = 0$ otherwise. Given an undirected graph $G = (V, E)$, a set of pairwise edge-disjoint triangles is called a **triangle packing**, while a set of edges sharing an edge with all triangles is called a **triangle cover**. Let

$$\nu(G) = \text{maximum cardinality of a triangle packing in } G,$$
$$\tau(G) = \text{minimum cardinality of a triangle cover in } G.$$

Hence the unweighted Turán number $T(n, 3, 2)$ is the same as $\tau(K_n)$. The problem of determining the exact values of $\nu(G)$ and $\tau(G)$ is showed to be NP-complete by Holyer [68] and Yannakakis [136], respectively. Still, it would be interesting to give a connection between these parameters. Clearly, $\nu(G) \leq \tau(G)$ holds so a natural approach would be to give an upper bound for $\tau(G)$ as a function of $\nu(G)$. In [127], Tuza proposed the following conjecture.

Conjecture 7.3.1 (Tuza)**.** *$\tau(G) \leq 2\nu(G)$ for any simple undirected graph G.*

It is worth mentioning that equality holds for infinitely many graphs. Indeed, take any graph with all maximal two-connected subgraphs isomorphic to either K_2, K_4 or K_5. That is, if Conjecture 7.3.1 is true then it is sharp.

The conjecture has been proved for various classes of graphs (see [24, 56, 65, 66, 67, 99, 128]). The first nontrivial bound for general graphs was given by Haxell by proving that for any graph G, we have $\tau(G) \leq (3 - \varepsilon)\nu(G)$, where $\varepsilon > \frac{3}{23}$ [64]. A fractional weakening of the conjecture was given by

Krivelevich [99] who showed that $\tau(G) \leq 2\tau^*(G)$ and $\nu^*(G) \leq 2\nu(G)$ where $\tau^*(G)$ and $\nu^*(G)$ stand for the optimal fractional solutions of the corresponding covering and packing problems, respectively.

The problem of determining $\nu(G)$ and $\tau(G)$ can be generalized in two ways. In [37], Erdős and Tuza proposed a 'clique version' of the original problem by considering the covering of complete subgraphs with complete subgraphs, while in [17] Chapuy et al. studied an edge-weighted version of the conjecture, and weighted analogues of results of Tuza, Krivelevich and Haxell were proved. Putting together these two ideas, we formalize a more general version of the problem.

For an $(r-1)$-uniform simple hypergraph $H = (V, \mathcal{E})$, an r-block is a subset of r nodes spanning a complete subhypergraph. The set of r-blocks is denoted by $\mathcal{B}_r$. A r-packing is a set of disjoint r-blocks, while an r-cover is a set of hyperedges such that each r-block spans at least one of them. Assume now that a weight function $w : \mathcal{E} \to \mathbb{R}_+$ is also given. A **weighted r-packing** is a family of - not necessarily disjoint - r-blocks such that each hyperedge e is contained in at most $w(e)$ of them. For the weighted case, let

$$\nu_w(H) = \text{maximum cardinality of a weighted } r\text{-packing in } H,$$
$$\tau_w(H) = \text{minimum weight of a } r\text{-cover in } H.$$

Here $\nu_w(H)$ and $\tau_w(H)$ are called **weighted r-packing** and **weighted r-covering** numbers, respectively. These parameters can be interpreted as optimal solutions to the following integer programs. Let A be the hyperedge - r-block incidence matrix of H, that is, $A_{e,R} = 1$ if $e \in \mathcal{E}$ is spanned by r-block R, and 0 otherwise. Then

$$\nu_w(H) = \max\{\mathbf{1} \cdot x |\ Ax \leq w,\ x \in \mathbb{Z}_+^{\mathcal{B}_r}\},$$
$$\tau_w(H) = \min\{w \cdot y |\ A^T y \geq 1,\ y \in \mathbb{Z}_+^{\mathcal{E}}\}.$$

By relaxing the integrality constraints we get the following primal-dual pair of linear programs.

$$\nu_w^*(H) = \max\{\mathbf{1} \cdot x |\ Ax \leq w,\ x \in \mathbb{R}_+^{\mathcal{B}_r}\},$$
$$\tau_w^*(H) = \min\{w \cdot y |\ A^T y \geq 1,\ y \in \mathbb{R}_+^{\mathcal{E}}\},$$

where $\nu_w^*(H)$ and $\tau_w^*(H)$ are called the **weighted fractional r-packing** and **weighted fractional r-covering** numbers, respectively. The linear programming duality theorem gives

$$\nu_w(H) \leq \nu_w^*(H) = \tau_w^*(H) \leq \tau_w(H).$$

As a generalization of Tuza's, we propose the following conjecture.

Conjecture 7.3.2. *Let $H = (V, \mathcal{E})$ be a simple $(r-1)$-uniform hypergraph and $w : \mathcal{E} \to \mathbb{R}_+$ a weight function. Then $\tau_w(H) \leq \lceil \frac{r+1}{2} \rceil \nu_w(H)$.*

Tuza's conjecture corresponds to the case when $r = 3$, $w \equiv 1$ and H is a simple graph. Similarly to the original conjecture, if Conjecture 7.3.2 is true then it is sharp. Indeed, let $w \equiv 1$ and take an $(r-1)$-uniform complete hypergraph $H = (V, \mathcal{E})$ on $r+1$ nodes. We claim that $\nu_w(H) = 1$ and $\tau_w(H) = \lceil \frac{r+1}{2} \rceil$.

It is easy to see that $\nu_w(H) = 1$ as the graph has only $r+1$ nodes, so any two r-blocks share $r-1$ nodes in common. As the graph is complete, there is a hyperedge spanned by these nodes, so $w \equiv 1$ implies that at most one r-block is contained in any weighted r-packing.

To see $\tau_w(H) \geq \lceil \frac{r+1}{2} \rceil$ it suffices to show that for any set $\mathcal{C}$ of r-blocks with cardinality at most $\lceil \frac{r+1}{2} \rceil - 1$ there exists a node v which is contained in all members of $\mathcal{C}$. That would clearly prove the lower bound as $\mathcal{C}$ does not cover the r-block $H - v$. Assume indirectly that there is no such node, that is, each node is contained in at most $|\mathcal{C}| - 1$ of them. We have

$$\sum_{v \in V} |\{e \in \mathcal{C} : \ v \in e\}| \leq (r+1)(|\mathcal{C}| - 1).$$

On the other hand,

$$\sum_{v \in V} |\{e \in \mathcal{C} : \ v \in e\}| = \sum_{e \in \mathcal{C}} |e| = (r-1)|\mathcal{C}|.$$

These together gives $(r+1)(|\mathcal{C}| - 1) \geq (r-1)|\mathcal{C}|$, hence $|C| \geq \lceil \frac{r+1}{2} \rceil$, a contradiction.

It remains to show an r-cover with cardinality $\lceil \frac{r+1}{2} \rceil$. Let $V = \{v_1, \ldots, v_{r+1}\}$ and $\mathcal{C} = \{V \setminus \{v_{2i-1}, v_{2i}\} | \ i = 1, \ldots, \lceil \frac{r+1}{2} \rceil\}$ where indices are meant in cyclic order, so $v_{r+2} = v_1$. Then for any $v \in V$ there is at least one $e \in \mathcal{C}$ not containing v. Hence $\mathcal{C}$ is an r-cover as for any r-block B there is an $e \in \mathcal{C}$ not containing $V \setminus B$, thus $e \subseteq B$.

Conjecture 7.3.2 is widely open. With the help of the shadow system appearing in Theorem 7.1.2, we prove a fractional weakening of the conjecture which can be considered as a weighted counterpart of Krivelevich's result.

Theorem 7.3.3. *Let $H = (V, \mathcal{E})$ be a simple $(r-1)$-uniform hypergraph and $w : \mathcal{E} \to \mathbb{R}_+$ a weight function. Then $\tau_w(H) \leq (r-1)\tau^*_w(H)$.*

Proof. Suppose that the theorem does not hold and let H be a minimal counterexample, that is, $\tau_w(H) > (r-1)\tau^*_w(H)$ but $\tau_w(H') \leq (r-1)\tau^*_w(H')$ for every proper subhypergraph H' of H. This implies that each hyperedge $e \in E$ is contained in an r-block as otherwise it could be left out from H thus giving a smaller counterexample. Take a pair of optimal solutions of the weighted fractional r-packing and r-cover problems denoted by x^* and y^*, respectively.

Case 1. $y^*_e \geq \frac{1}{r-1}$ *for some* $e \in \mathcal{E}$.

Let H' be the graph obtained by deleting the hyperedge e from H. Clearly, $\tau_w(H') \geq \tau_w(H) - w(e)$. On the other hand, z^* is a fractional r-cover in H' where $z^*(e') = y^*(e')$ for $e' \neq e$. Hence $\tau^*_w(H') \leq \tau^*_w(H) - \frac{w(e)}{r-1}$. By the minimal choice of H we get

$$\tau_w(H) \leq \tau_w(H') + w(e) \leq (r-1)\tau^*_w(H') + w(e) \leq (r-1)\tau^*_w(H),$$

a contradiction.

Case 2. $y^*_e < \frac{1}{r-1}$ *for each* $e \in \mathcal{E}$.

We claim that $y^*_e > 0$ for each $e \in \mathcal{E}$. Indeed, an r-block spans r different hyperedges. If one of these hyperedges had y^* value 0 then the total y^* sum on them would be strictly smaller than 1, contradicting the assumption that y^* is a fractional r-cover. As mentioned earlier, each hyperedge is spanned by one of the r-blocks, hence the statement follows. By complementary slackness, we have

$$\sum_{\substack{B \in \mathcal{B}_r \\ B \text{spans } e}} x^*(B) = w(e) \text{ for each } e \in \mathcal{E}.$$

That also implies that the exact value of the optimum for the fractional problem can be computed as

$$\tau_w^*(H) = \nu_w^*(H) = \sum_{B\in\mathcal{B}_r} x^*(B) = \frac{1}{r}\sum_{e\in\mathcal{E}} \sum_{\substack{B\in\mathcal{B}_r \\ B \text{ spans } e}} x^*(B) = \frac{1}{r}\sum_{e\in\mathcal{E}} w(e) = \frac{1}{r}w^*.$$

So it suffices to show that $\tau_w(H) \leq \frac{r-1}{r}w^*$. We do the same as in the proof of Theorem 7.2.1: colour the nodes uniformly at random with the colours $1, \dots, r-1$ and define the r-cover as the set of hyperedges e with colour profile in $\mathcal{A}_{r-1}^r$ defined in (**SHA**). We have already seen that there exist a colouring of the nodes such that the total weight of the covering is at most $\left(\frac{r-1}{r}\right)^{r-1}w^* \leq \frac{r-1}{r}w^*$, and we are done. $\square$

Bibliography

[1] R. Ahlswede, P. Erdős, and N. Graham. A splitting property of maximal antichains. *Combinatorica*, 15(4):475–480, 1995. 17

[2] N. Apollonio and A. Sebő. Minsquare factors and maxfix covers of graphs. *Integer Programming and Combinatorial Optimization*, pages 1–2, 2004. 13, 56

[3] J. Aráoz, W. H. Cunningham, J. Edmonds, and J. Green-Krótki. Reductions to 1-matching polyhedra. *Networks*, 13(4):455–473, 1983. 16

[4] V. Auletta, Y. Dinitz, Z. Nutov, and D. Parente. A 2-approximation algorithm for finding an optimum 3-vertex-connected spanning subgraph. *Journal of Algorithms*, 32(1):21–30, 1999. 12

[5] J. Bang-Jensen. Edge-disjoint in-and out-branchings in tournaments and related path problems. *Journal of Combinatorial Theory, Series B*, 51(1):1–23, 1991. 8

[6] K. Bérczi. Packings and coverings in directed graphs. Master's thesis, Eötvös Loránd University, Budapest, 2008. 1, 3

[7] K. Bérczi. The triangle-free 2-matching polytope of subcubic graphs. Technical Report TR-2012-02, Egerváry Research Group, Budapest, 2012. `www.cs.elte.hu/egres`. 3, 131, 133

[8] K. Bérczi, P. Csikvári, E. R. Kovács, and L. A. Végh. Splitting property via shadow systems. Technical Report TR-2013-02, Egerváry Research Group, Budapest, 2013. `www.cs.elte.hu/egres`. 3, 131, 133

[9] K. Bérczi and A. Frank. Variations for Lovász' submodular ideas. *Building Bridges*, pages 137–164, 2008. 5, 8, 9, 36

[10] K. Bérczi and A. Frank. Packing arborescences. *Lecture Notes, in: RIMS Kokyuroku Bessatsu B*, 23:1–31, 2009. 1, 3, 31, 131, 133

[11] K. Bérczi, S. Fujishige, and N. Kamiyama. A linear-time algorithm to find a pair of arc-disjoint spanning in-arborescence and out-arborescence in a directed acyclic graph. *Information Processing Letters*, 109(23):1227–1231, 2009. 1, 3, 131, 133

[12] K. Bérczi and Y. Kobayashi. An algorithm for $(n-3)$-connectivity augmentation problem: Jump system approach. *Journal of Combinatorial Theory, Series B*, 102(3):565–587, 2012. 2, 3, 13, 81, 131, 133

[13] K. Bérczi and E. R. Kovács. A note on strongly edge-disjoint arborescences. In *Proceedings of 7th Japanese-Hungarian Symposium on Discrete Mathematics and its Applications*, pages 10–18, Kyoto, 2011. 1, 3, 131, 133

[14] K. Bérczi and L. A. Végh. Restricted b-matchings in degree-bounded graphs. In F. Eisenbrand and F. Shepherd, editors, *Integer Programming and Combinatorial Optimization*, volume 6080 of *Lecture Notes in Computer Science*, pages 43–56. Springer, 2010. 2, 3, 13, 131, 133

[15] A. Bhalgat, R. Hariharan, T. Kavitha, and D. Panigrahi. Fast edge splitting and Edmond's arborescence construction for unweighted graphs. In *Proceedings of the Nineteenth annual ACM-SIAM Symposium on Discrete Algorithms, San Francisco*, pages 455–464, 2008. 8

[16] A. Bouchet and W. H. Cunningham. Delta-matroids, jump systems, and bisubmodular polyhedra. *SIAM Journal on Discrete Mathematics*, 8(1):17–32, 1995. 44

[17] G. Chapuy, M. DeVos, J. McDonald, B. Mohar, and D. Scheide. Packing triangles in weighted graphs. *Arxiv preprint arXiv:1012.0372*, 2010. 117

[18] L. Colussi, M. Conforti, and G. Zambelli. Disjoint paths in arborescences. *Discrete mathematics*, 292(1):187–191, 2005. 1, 7

[19] M. Conforti, D. G. Corneil, and A. R. Mahjoub. K_i-covers I: Complexity and polytopes. *Discrete mathematics*, 58(2):121–142, 1986. 3

[20] M. Conforti, D. G. Corneil, and A. R. Mahjoub. K_i-covers II: K_i-perfect graphs. *Journal of graph theory*, 11(4):569–584, 1987. 3

[21] G. Cornuéjols, D. Naddef, and W. Pulleyblank. The traveling salesman problem in graphs with 3-edge cutsets. *Journal of the ACM*, 32:383–410, 1985. 15

[22] G. Cornuéjols and W. Pulleyblank. A matching problem with side conditions. *Discrete Mathematics*, 29(2):135–159, 1980. 2, 10

[23] G. Cornuéjols and W. R. Pulleyblank. Perfect triangle-free 2-matchings. In *Combinatorial Optimization II*, Mathematical Programming Studies. 15

[24] Q. Cui, P. Haxell, and W. Ma. Packing and covering triangles in planar graphs. *Graphs and Combinatorics*, 25(6):817–824, 2009. 116

[25] W. H. Cunningham. Matching, matroids, and extensions. *Mathematical programming*, 91(3):515–542, 2002. 44

[26] D. de Caen. Extension of a theorem of Moon and Moser on complete subgraphs. *Ars Combinatoria*, 16:5–10, 1983. 18

[27] Y. Dinitz and Z. Nutov. A 3-approximation algorithm for finding optimum 4,5-vertex-connected spanning subgraphs. *Journal of Algorithms*, 32(1):31–40, 1999. 12

[28] A. W. M. Dress and W. Wenzel. Valuated matroids: A new look at the greedy algorithm. *Applied Mathematics Letters*, 3(2):33–35, 1990. 56

[29] A. W. M. Dress and W. Wenzel. A greedy-algorithm characterization of valuated Δ-matroids. *Applied mathematics letters*, 4(6):55–58, 1991. 13, 56

[30] A. W. M. Dress and W. Wenzel. Valuated matroids. *Advances in Mathematics*, 93(2):214–250, 1992. 56

[31] D. Duffus and B. Sands. Finite distributive lattices and the splitting property. *Algebra Universalis*, 49(1):13–33, 2003. 17

[32] O. Durand de Gevigney, V. H. Nguyen, and Z. Szigeti. Basic packing of arborescences. *SIAM Journal of Discrete mathematics*, 27(1):567–574, 2013. 5, 6

[33] J. Edmonds. Maximum matching and a polyhedron with $0, 1$ vertices. *Journal of Research of the National Bureau of Standards*, 69:125–130, 1965. 13, 14

[34] J. Edmonds. Edge-disjoint branchings. *Combinatorial Algorithms*, 9:91–96, 1973. 1, 4

[35] J. Edmonds and R. Giles. A min-max relation for submodular functions on graphs. *Annals of Discrete Mathematics*, 1(1977):185–204, 1977. 41

[36] P. Erdős. On the combinatorial problems which I would most like to see solved. *Combinatorica*, 1(1):25–42, 1981. 18

[37] P. Erdős, T. Gallai, and Zs. Tuza. Covering and independence in triangle structures. *Discrete Mathematics*, 150(1):89–101, 1996. 117

[38] K. P. Eswaran and R. E. Tarjan. Augmentation problems. *SIAM Journal on Computing*, 5(4):653–665, 1976. 11, 12

[39] A. Frank. Covering branchings. *Acta Scientiarum Mathematicarum (Szeged)*, 41(1-2):77–82, 1979. 20

[40] A. Frank. Kernel systems of directed graphs. *Acta Scientiarum Mathematicarum (Szeged)*, 41(1-2):63–76, 1979. 9

[41] A. Frank. Restricted t-matchings in bipartite graphs. *Discrete Applied Mathematics*, 131(2):337–346, 2003. 2, 12

[42] A. Frank. Rooted k-connections in digraphs. *Discrete Applied Mathematics*, 157(6):1242–1254, 2009. 41

[43] A. Frank and T. Jordán. Minimal edge-coverings of pairs of sets. *Journal of Combinatorial Theory, Series B*, 65(1):73–110, 1995. 11, 12

[44] A. Frank and T. Jordán. Directed vertex-connectivity augmentation. *Mathematical programming*, 84(3):537–553, 1999. 11

[45] A. Frank and É. Tardos. Generalized polymatroids and submodular flows. *Mathematical Programming*, 42(1):489–563, 1988. 19

[46] A. Frank and É. Tardos. An application of submodular flows. *Linear algebra and its applications*, 114:329–348, 1989. 5

[47] S. Fujishige. *Submodular functions and optimization*, volume 58. Elsevier, 2005. 44

[48] S. Fujishige. A note on disjoint arborescences. *Combinatorica*, 30(2):247–252, 2010. 5, 8

[49] Z. Füredi. Turán type problems. *Surveys in combinatorics, London Mathematical Society Lecture Note Series*, 166:253–300, 1991. 18

[50] H. N. Gabow. An efficient reduction technique for degree-constrained subgraph and bidirected network flow problems. In *STOC '83: Proceedings of the fifteenth annual ACM symposium on Theory of computing*, pages 448–456, New York, 1983. ACM. 73

[51] H. N. Gabow. Data structures for weighted matching and nearest common ancestors with linking. In *Proceedings of the first annual ACM-SIAM symposium on Discrete algorithms*, pages 434–443. Society for Industrial and Applied Mathematics, 1990. 56

[52] H. N. Gabow. A matroid approach to finding edge connectivity and packing arborescences. In *Proceedings of the twenty-third annual ACM symposium on Theory of computing*, pages 112–122. ACM, 1991. 8

[53] H. N. Gabow and R. E. Tarjan. Faster scaling algorithms for general graph matching problems. *Journal of the ACM*, 38(4):815–853, 1991. 56

[54] M. R. Garey, D. S. Johnson, and L. Stockmeyer. Some simplified NP-complete graph problems. *Theoretical computer science*, 1(3):237–267, 1976. 54

[55] L. Georgiadis and R. Tarjan. Dominator tree verification and vertex-disjoint paths. *16th ACM-SIAM Symposium on Discrete Algorithms*, pages 433–442, 2005. 7, 22, 23

[56] S. Ghosh and P. Haxell. Packing and covering tetrahedra. *Discrete Applied Mathematics*, 161(9):1209 – 1215, 2013. <ce:title>Jubilee Conference on Discrete Mathematics</ce:title>. 116

[57] A. V. Goldberg and A. V. Karzanov. Maximum skew-symmetric flows and matchings. *Mathematical programming*, 100(3):537–568, 2004. 45

[58] M. Grötschel and W. R. Pulleyblank. Clique tree inequalities and the symmetric travelling salesman problem. *Mathematics of operations research*, 11(4):537–569, 1986. 106, 107, 108

[59] D. Hartvigsen. *Extensions of matching theory*. PhD thesis, Carnegie-Mellon University, 1984. 2, 10

[60] D. Hartvigsen. The square-free 2-factor problem in bipartite graphs. In *Proceedings of the 7th International IPCO Conference on Integer Programming and Combinatorial Optimization*, pages 234–241, Berlin, 1999. Springer-Verlag. 12

[61] D. Hartvigsen. Finding maximum square-free 2-matchings in bipartite graphs. *Journal of Combinatorial Theory, Series B*, 96(5):693–705, 2006. 12

[62] D. Hartvigsen and Y. Li. Triangle-free simple 2-matchings in subcubic graphs (extended abstract). In M. Fischetti and D. Williamson, editors, *Integer Programming and Combinatorial Optimization*, volume 4513 of *Lecture Notes in Computer Science*, pages 43–52. Springer, 2007. 2, 3, 12, 15, 16

[63] D. Hartvigsen and Y. Li. Polyhedron of triangle-free simple 2-matchings in subcubic graphs. *Mathematical Programming*, pages 1–40, 2012. 3, 15, 16, 75

[64] P. Haxell. Packing and covering triangles in graphs. *Discrete mathematics*, 195(1-3):251–254, 1999. 3, 116

[65] P. Haxell and Y. Kohayakawa. Packing and covering triangles in tripartite graphs. *Graphs and Combinatorics*, 14(1):1–10, 1998. 116

[66] P. Haxell, A. Kostochka, and S. Thomassé. Packing and covering triangles in K_4-free planar graphs. *Graphs and Combinatorics*, pages 1–10, 2011. 116

[67] P. Haxell, A. Kostochka, and S. Thomassé. A stability theorem on fractional covering of triangles by edges. *European Journal of Combinatorics*, 2011. 116

[68] I. Holyer. The NP-completeness of some edge-partition problems. *SIAM Journal on Computing*, 10(4):713–717, 1981. 116

[69] J. E. Hopcroft and R. M. Karp. An $n^{\frac{5}{2}}$ algorithm for maximum matchings in bipartite graphs. *SIAM Journal on Computing*, 2(4):225–231, 1973. 26

[70] T. S. Hsu. Undirected vertex-connectivity structure and smallest four-vertex-connectivity augmentation. *Algorithms and Computations*, pages 274–283, 1995. 11

[71] T. S. Hsu and V. Ramachandran. A linear time algorithm for triconnectivity augmentation. In *Foundations of Computer Science, 1991. Proceedings., 32nd Annual Symposium on*, pages 548–559. IEEE, 1991. 11

[72] T. S. Hsu and V. Ramachandran. Finding a smallest augmentation to biconnect a graph. *SIAM Journal on Computing*, 22(5):889–912, 1993. 11

[73] A. Huck. Disproof of a conjecture about independent branchings in k-connected directed graphs. *Journal of Graph Theory*, 20:235–239, 1995. 7, 24

[74] A. Huck. Independent branchings in acyclic digraphs. *Discrete Mathematics*, 199:245–249, 1999. 7

[75] A. Huck. Independent trees and branchings in planar multigraphs. *Graphs and Combinatorics*, 15:211–220, 1999. 7

[76] T. Ishii and H. Nagamochi. On the minimum augmentation of an l-connected graph to a k-connected graph. *Algorithm Theory-SWAT 2000*, pages 603–611, 2000. 11

[77] B. Jackson and T. Jordán. A near optimal algorithm for vertex connectivity augmentation. *Algorithms and Computation*, pages 189–224, 2000. 11

[78] B. Jackson and T. Jordán. Independence free graphs and vertex connectivity augmentation. *Journal of Combinatorial Theory, Series B*, 94(1):31–77, 2005. 11

[79] T. Jordán. Conjecture on node-connectivity augmentation by one. 62

[80] T. Jordán. On the optimal vertex-connectivity augmentation. *Journal of Combinatorial Theory, Series B*, 63(1):8–20, 1995. 11

[81] T. Jordán. A note on the vertex-connectivity augmentation problem. *Journal of Combinatorial Theory, Series B*, 71(2):294–301, 1997. 11

[82] N. Kamiyama, N. Katoh, and A. Takizawa. Arc-disjoint in-trees in directed graphs. In *Proceedings of the nineteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 518–526. Society for Industrial and Applied Mathematics, 2008. 1, 5, 8

[83] P. Keevash. Hypergraph Turán problems, Surveys in Combinatorics, 2011. 18

[84] Cs. Király. On maximal independent arborescence-packing. Technical Report TR-2013-03, Egerváry Research Group, March 2013. 5, 6

[85] Z. Király. Personal communication. 22

[86] Z. Király. C_4-free 2-factors in bipartite graphs. Technical Report TR-2001-13, Egerváry Research Group, Budapest, 2001. `www.cs.elte.hu/egres`. 2, 12

[87] Z. Király. Restricted t-matchings in bipartite graphs. Technical Report QP-2009-04, Egerváry Research Group, Budapest, 2009. `www.cs.elte.hu/egres`. 2, 12, 13, 15

[88] Y. Kobayashi. A simple algorithm for finding a maximum triangle-free 2-matching in subcubic graphs. *Discrete Optimization*, 7(4):197–202, 2010. 2, 12, 13, 15, 90

[89] Y. Kobayashi, K. Murota, and K. Tanaka. Operations on M-convex functions on jump systems. *SIAM Journal on Discrete Mathematics*, 21(1):107–129, 2007. 13, 56

[90] Y. Kobayashi, J. Szabó, and K. Takazawa. A proof of Cunningham's conjecture on restricted subgraphs and jump systems. *Journal of Combinatorial Theory, Series B*, 102(4):948–966, 2012. 13, 45, 46

[91] Y. Kobayashi and K. Takazawa. Even factors, jump systems, and discrete convexity. Technical Report METR 2007-36, Department of Mathematical Engineering, University of Tokyo, June 2007. 13

[92] Y. Kobayashi and K. Takazawa. Square-free 2-matchings in bipartite graphs and jump systems. Technical Report METR 2008-40, Department of Mathematical Engineering, University of Tokyo, October 2008. 13

[93] Y. Kobayashi and K. Takazawa. Square-free 2-matchings in bipartite graphs and jump systems. Technical report, Technical report, Department of Mathematical Engineering, University of Tokyo, METR 2008-40, 2008. 13, 49

[94] Y. Kobayashi and K. Takazawa. Even factors, jump systems, and discrete convexity. *Journal of Combinatorial Theory, Series B*, 99(1):139–161, 2009. 13, 56

[95] Y. Kobayashi and X. Yin. An algorithm for finding a maximum t-matching excluding complete partite subgraphs. *Discrete Optimization*, 9(2):98–108, 2012. 13

[96] B. Korte and J. Vygen. *Combinatorial optimization: theory and algorithms*, volume 21. Springer, 2012. 27

[97] G. Kortsarz and Z. Nutov. Approximating minimum cost connectivity problems. *T. F. Gonzalez, editor*, 2007. 12

[98] E. R. Kovács. Kódolás hálózatokban (Network coding, in hungarian). Master's thesis, Eötvös Loránd University, Budapest, 2007. 22, 23

[99] M. Krivelevich. On a conjecture of Tuza about packing and covering of triangles. *Discrete Mathematics*, 142(1):281–286, 1995. 18, 116, 117

[100] G. Liberman and Z. Nutov. On shredders and vertex connectivity augmentation. *Journal of Discrete Algorithms*, 5(1):91–101, 2007. 11

[101] L. Lovász. On two minimax theorems in graph. *Journal of Combinatorial Theory, Series B*, 21(2):96–103, 1976. 8

[102] L. Lovász. The membership problem in jump systems. *Journal of Combinatorial Theory, Series B*, 70(1):45–66, 1997. 13, 44

[103] M. Makai. On maximum cost $K_{t,t}$-free t-matchings of bipartite graphs. *SIAM J. Discret. Math.*, 21(2):349–360, 2007. 2, 12

[104] C. Mao-cheng. Arc-disjoint arborescences of digraphs. *Journal of Graph Theory*, 7(2):235–240, 1983. 20

[105] K. Murota. *Discrete convex analysis*, volume 10. Society for Industrial Mathematics, 1987. 56

[106] K. Murota. Convexity and Steinitz's exchange property. *Integer Programming and Combinatorial Optimization*, pages 260–274, 1996. 56

[107] K. Murota. M-convex functions on jump systems: a general framework for minsquare graph factor problem. *SIAM Journal on Discrete Mathematics*, 20(1):213–226, 2006. 12, 13, 44, 56

[108] K. Murota and K. Tanaka. A steepest descent algorithm for m-convex functions on jump systems. *IEICE transactions on fundamentals of electronics, communications and computer sciences*, 89(5):1160–1165, 2006. 45, 56

[109] Z. Nutov. *Improved Approximation Algorithms for Finding Optimum k-Vertex Connected Spanning Subgraphs*. Citeseer, 1998. 12

[110] Gy. Pap. Alternating paths revisited II: Restricted b-matchings in bipartite graphs. Technical Report TR-2005-13, Egerváry Research Group, Budapest, 2005. `www.cs.elte.hu/egres`. 12, 13, 73

[111] R. Ravi and D. P. Williamson. An approximation algorithm for minimum-cost vertex-connectivity problems. *Algorithmica*, 18(1):21–43, 1997. 12

[112] A. Schrijver. Fractional packing and covering. *Packing and Covering in Combinatorics*, 1979. 7

[113] A. Schrijver. Short proofs on the matching polyhedron. *Journal of Combinatorial Theory, Series B*, 34(1):104–108, 1983. 16

[114] A. Schrijver. *Combinatorial Optimization - Polyhedra and Efficiency*. Springer, 2003. 1, 14, 37, 65

[115] T. sheng Hsu. On four-connecting a triconnected graph. *Journal of Algorithms*, 35(2):202–234, 2000. 11

[116] A. Shioura and K. Tanaka. Polynomial-time algorithms for linear and convex optimization on jump systems. *SIAM journal on discrete mathematics*, 21(2):504–522, 2007. 13, 45, 56, 57

[117] A. Sidorenko. Systems of sets that have the T-property. *Vestnik Moskovskogo Universiteta. Seriya I. Matematika, Mekhanika*, pages 19–22, 1981. 18

[118] A. Sidorenko. Upper bounds for Turán numbers. *Journal of Combinatorial Theory, Series A*, 77(1):134–147, 1997. 18

[119] J. Szabó. Ugrácsok és matroid parity (Jump systems and matroid parity, in hungarian). Master's thesis, Eötvös Loránd University, Budapest, 2002. 13, 45, 46

[120] L. Szegő. On covering intersecting set-systems by digraphs. *Discrete Mathematics*, 234(1):187–189, 2001. 9

[121] K. Takazawa. A weighted $K_{t,t}$-free t-factor algorithm for bipartite graphs. *Mathematics of Operations Research*, 34(2):351–362, 2009. 12

[122] R. E. Tarjan. A good algorithm for edge-disjoint branching. *Information Processing Letters*, 3(2):51–53, 1974. 8

[123] C. Thomassen. Configurations in graphs of large minimum degree, connectivity, or chromatic number. *Annals of the New York Academy of Sciences*, 555(1):402–412, 1989. 30

[124] P. Tong and E. L. Lawler. A faster algorithm for finding edge-disjoint branchings. *Information Processing Letters*, 17(2):73–76, 1983. 8

[125] P. Turán. Research problems. *Magyar Tudományos Akadémia Matematikai Kutató Intézet Közleménye*, 6:417–423, 1961. 17

[126] W. Tutte. A short proof of the factor theorem for finite graphs. *Canadian Journal of Mathematics*, 6:347–352, 1954. 59

[127] Zs. Tuza. Conjecture. In *Finite and Infinite Sets, Proc. Colloq. Math. Soc. János Bolyai*, page 888. North-Holland, 1981. 3, 18, 116

[128] Zs. Tuza. A conjecture on triangles of graphs. *Graphs and Combinatorics*, 6(4):373–380, 1990. 116

[129] L. A. Végh. Augmenting undirected node-connectivity by one. In *Proceedings of the 42nd ACM symposium on Theory of computing*, pages 563–572. ACM, 2010. 2, 11, 12, 13

[130] L. A. Végh and A. A. Benczúr. Primal-dual approach for directed vertex connectivity augmentation and generalizations. *ACM Transactions on Algorithms*, 4(2):20, 2008. 11

[131] K. Vidyasankar. Covering the edge set of a directed graph with trees. *Discrete Mathematics*, 24(1):79–85, 1978. 31

[132] T. Watanabe and A. Nakamura. A smallest augmentation to 3-connect a graph. *Discrete applied mathematics*, 28(2):183–186, 1990. 11

[133] W. Wenzel. Δ-matroids with the strong exchange conditions. *Applied Mathematics Letters*, 6:67–70, 1993. 44

[134] W. Wenzel. Pfaffian forms and Δ-matroids. *Discrete Mathematics*, 115(1):253–266, 1993. 44

[135] R. Whitty. Vertex-disjoint paths and edge-disjoint branchings in directed graphs. *Journal of Graph Theory*, 11(3):349–358, 1987. 7

[136] M. Yannakakis. Edge-deletion problems. *SIAM Journal on Computing*, 10(2):297–309, 1981. 3, 116

Abstract

The thesis has two main topics, the first of them is arborescence packing. We consider extensions of Edmonds' fundamental result on packing disjoint spanning arborescences. The problem can be naturally generalized in two directions: the edge-disjointness condition may be strengthened, and the set of nodes spanned by the arborescences may be decreased.

- *We give a disproof of the conjecture of Colussi, Conforti and Zambelli on strongly edge-disjoint arborescences.* For $k = 2$ the conjecture is true; we give its generalization for dicycle-disjoint Steiner arborescences.
- *We present a linear time algorithm for finding a pair of disjoint in- and out-arborescences in an acyclic digraph.* Deciding the existence of such arborescences is NP-complete in general. Our algorithm is based on a reduction to bipartite matching in an associated bipartite graph.
- *We present a strongly polynomial time algorithm for finding disjoint arborescences spanning convex sets under capacity constraints.* Our solution is based on the deep understanding of the connection between packing arborescences and covering intersecting bi-set families.
- *We give a polyhedral description of arborescence packable subgraphs and prove that the system is TDI.* The proof strongly relies on the special intersecting bi-set families appearing in the proof of Fujishige's theorem.

The second part of the thesis deals with restricted b-matchings, mainly with C_k-free k-matchings. It has been known that the C_k-free 2-matching problem is NP-complete for $k \geq 5$. We consider the C_3-free and the C_4-free 2-matching, and the $K_{t,t}$- and K_{t+1}-free t-matching problems in graphs that satisfy certain degree bounds.

- *We give a min-max theorem and an algorithm for the square-free 2-matching problem in subcubic graphs.* We show that the weighted version of the problem is NP-hard even in planar bipartite cubic graphs, but is polynomially solvable when the weight function is node-induced on each square.
- *We give a min-max theorem and an algorithm for the $K_{t,t}$- and K_{t+1}-free t-matching problem in degree bounded graphs.* Note that this problem is a generalization of the C_3-free, C_4-free and $C_{\leq 4}$-free 2-matching problems.
- *We give a description of the triangle-free 2-matching polytope of subcubic graphs.* The description was conjectured by Hartvigsen and Li; the complete proof appeared recently. We give an independent proof of the result which relies on a shrinking method.

The last chapter examines arbitrary triangle-free subgraphs, that is, when the degree bound on the nodes in the subgraph is omitted. The problem is approached through shadow systems and Turán numbers.

- *We prove that the set of multisets with size k over a ground set with size also k has the so-called splitting property.* From this, we show that a weighted extension of the Turán number admits the same upper bounds as the unweighted one. We also prove a combinatorial colouring theorem and a fractional version of an extension of Tuza's conjecture to hypergraphs.

The results are based on the papers [7], [8], [10], [11], [12], [13] and [14].

Összefoglalás

Az értekezés két fő témával foglalkozik, melyek közül az első a fenyők pakolásának kérdésköre. A probléma két irányban is általánosítható: egyrészt szigorítható a fenyőkre vonatkozó éldiszjunktsági megkötés, másrészt a fenyők által feszített pontok halmaza is szűkíthető.

- *Megcáfoljuk Colussi, Conforti és Zambelli erősen éldiszjunkt fenyőkre vonatkozó sejtését.* A sejtés a $k = 2$ esetben igaz; ezt a zeredményt általánosítjuk irányított kördiszjunkt Steiner fenyőkre.
- *Lineáris idejű algoritmust adunk egy pár éldiszjunkt ki- és be-fenyő megtalálására aciklikus gráfokban.* A kérdéses fenyők létezésének eldöntése általában NP-teljes probléma. Az általunk adott algoritmus visszavezeti a problémát egy páros gráfban való maximális párosítás megkeresésére.
- *Erősen polinomiális algoritmust adunk adott konvex halmazokat feszítő éldiszjunkt fenyők megkeresésére egy élkapacitásokkal rendelkező gráfban.* Megoldásunk a fenyő-pakolások és a metsző párhalmazrendszerek fedése közti szoros kapcsolaton alapul.
- *Megadjuk a fenyő-pakolható részgráfok poliéderes leírását„ és igazoljuk, hogy a kapott rendszer TDI.* A bizonyítás a Fujishige tételének bizonyításában megjelenő speciális párhalmaz családok szerkezetére épül.

A dolgozat második része tiltott részgráfokat nem tartalmazó b-matchingekkel foglalkoznak, különös tekintettel a C_k-mentes 2-matchingekre. Ismert volt korábban, hogy a C_k-mentes 2-matching probléma NP-teljes $k \geq 5$ esetén. Mi a C_3-mentes és C_4-mentes 2-matchingek, illetve a $K_{t,t}$- és K_{t+1}-mentes t-matchingek problémáját vizsgáljuk fokszámkorlátozott gráfokban.

- *Min-max tételt és algoritmust adunk a négyszög-mentes 2-matching feladatra szubkubikus gráfokban.* Megmutatjuk, hogy a probléma súlyozott változata már síkbarajzolható páros kubikus gráfokban is NP-nehéz, ugyanakkor pont-indukált költségfüggvény esetén polinomiális algoritmus adható.
- *Min-max tételt és algoritmust adunk a $K_{t,t}$- és K_{t+1}-mentes t-matching feladatra fokszámkorlátozott gráfokban.* Ez a probléma könnyen láthatóan általánosítja a C_3-mentes, a C_4-mentes, illetve a $C_{\leq 4}$-mentes 2-matching problémákat.
- *Megadjuk a szubkubikus gráfok háromszög-mentes 2-matching poliéderének leírását.* A leíró rendszert Hartvigsen és Li sejtette meg; teljes bizonyítása nemrégiben jelent meg. Egy független bizonyítást adunk az említett leírás helyességére, mely egy új összehúzási műveleten alapul.

Az utolsó fejezetben tetszőleges háromszög-mentes részgráfokkal foglalkozik, azaz mikor a vizsgált részgráfokban a pontokra vonatkozó fokszámkorlátot elhagyjuk. A problémát más ismert területeket érintve közelítjük meg, mint például az árnyék-rendszerek, avagy a Turán-szám.

- *Igazoljuk, hogy eg k méretű alaphalmazon értelmezett k elemű multihalmazok rendszere rendelkezik az úgynevezett splitting tulajdonsággal.* Ennek segítségével bizonyítunk egy kombinatorikus színezési tételt, melyből aztán a Tuza-sejtés egy hipergráfokra való általánosításának törtirányú gyengítése következik.

A bemutatott eredmények a [7], [8], [10], [11], [12], [13] és [14] cikkekben jelentek meg.