Assessing the Difference Between Shock Sharing and Demand Sharing in Supply Chains

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Abstract

We consider the problem of assessing value of demand sharing in a multi-stage supply chain in which the retailer observes stationary autoregressive moving average demand with Gaussian white noise (shocks). Similar to previous research, we assume each supply chain player constructs its best linear forecast of the leadtime demand and uses it to determine the order quantity via a periodic review myopic order-up-to policy. We demonstrate how a typical supply chain player can determine the extent of its available information under demand sharing by studying the properties of the moving average polynomials of adjacent supply chain players. Hence, we study how a player can determine its available information under demand sharing, and use this information to forecast leadtime demand. We characterize the value of demand sharing for a typical supply chain player. Furthermore, we show conditions under which (i) it is equivalent to no sharing, (ii) it is equivalent to full information shock sharing, and (iii) it is intermediate in value to the two previously described arrangements. We then show that demand propagates through a supply chain where any player may share nothing, its demand, or its full-information shocks with an adjacent upstream player as quasi-ARMA in - quasi-ARMA out. We also provide a convenient form for the propagation of demand in a supply chain that will lend itself to future research applications.

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1 Introduction

We consider the problem of assessing value of demand sharing in a multi-stage supply chain in which the retailer observes covariance-stationary autoregressive moving average demand with Gaussian white noise (shocks). We assume that all supply chain players use a myopic order-up- to inventory policy where negative order quantities are allowed, but the probability of negative demand or negative orders is negligible. It is assumed that the lead time guarantee holds, i.e., if an upstream player does not have enough stock to fill an order from the adjacent downstream player, then the upstream player will meet the shortfall from an alternative source, with additional cost representing the penalty cost to this shortfall. Excess demand at the retailer is backlogged. Similar to previous research, we assume each supply chain player constructs its best linear forecast of the leadtime demand and uses it to determine the order quantity via a periodic review myopic order-up-to policy.

With respect to the information structure, we assume, as others have (c.f. [Lee et al., 2000] (hereafter LST) that the form and parameters of the model generating a downstream player's demand are known to the adjacent upstream player. However the downstream player's demand realizations, and shocks that generate all of the player's information (the downstream player's *full information shocks*), may be private knowledge. When there is no information sharing, the upstream player receives only an order from the adjacent downstream player. When there is demand sharing, the downstream player provides its demand in addition to placing its order with the upstream player. Finally, when there is full information shock sharing, the downstream player provides its full information shocks in addition to placing its order with the upstream player.

The existing literature either does not distinguish between demand sharing and shock sharing [Gaur et al., 2005] (hereafter GGS) and [Zhang, 2004] (hereafter Zhang)) or focuses on the value of full information shock sharing in a supply chain without allowing for the possibility that a player may share its demand as opposed to its full information shocks [Giloni et al., 2012] (hereafter GHS). We demonstrate how a typical supply chain player can determine the extent of its available information under demand sharing by studying the properties of the moving average polynomials of adjacent supply chain players. We utilize the methods and results described in GHS (2012) where they demonstrate how a typical supply chain player can determine its available information under full information shock sharing or possibly under no sharing arrangement. We study how a player can determine its available information under demand sharing, and use this information to forecast leadtime demand. Furthermore, we show conditions under which (i) it is equivalent to no sharing, (ii) it is equivalent to full information shock sharing, and (iii) it is intermediate in value to the two previously described arrangements.

After characterizing a player's information set under demand sharing, we then study how demand propagates through a supply chain where any player may share nothing, its demand, or its full-information shocks with an adjacent upstream player. Specifically, we find that demand propagates as quasi-ARMA (QUARMA) in - quasi-ARMA out even with the possibility of demand sharing. We also introduce a convenient mathematical structure for the propagation of demand, notappearing in previous literature. This is done by studying QUARMA propagation as sums of polynomials rather than linear combinations of coefficients. This form provides more intuition behind how demand propagates upstream in the supply chain. Furthermore, it allows for the study of various supply chain dynamics, such as the bullwhip effect and the asymptotic behavior of supply chains with many stages.

We provide several important contributions to the literature. The first is in characterizing a player's information set when the adjacent downstream player shares demand. The second is in establishing the new result that demand sharing can be intermediate in value. We provide examples of this by demonstrating what a player's full information shocks and mean square forecast error (MSFE) would be under the three aforementioned sharing arrangements. The third is that we show that under the possibility of either no sharing, demand sharing, or full information shock sharing, demand propagates upstream the supply chain as quasi-ARMA in - quasi-ARMA out. The fourth is that we provide a convenient form for the propagation of demand in a supply chain that will lend itself to future research applications.

2 The Research Problem

2.1 Recovering Shocks from Historical Data

In this paper we represent a player's information in terms of a white noise series. It is therefore essential to understand if and when a series of shocks can be recovered from present and past observations. It is sometimes assumed (incorrectly) that this is always possible. The following example illustrates this problem for a simple moving average (MA) model.

Example 1. Part I

Consider the following MA(1) model:

$$D_t = c + \epsilon_t - \theta_1 \epsilon_{t-1} \tag{1}$$

Consider trying to solve for ϵ_t *in terms of present and past values of* $\{D_t\}$*. Note that (1) can be rewritten as*

$$\epsilon_t = D_t - c + \theta_1 \epsilon_{t-1}$$

or

$$\epsilon_t = D_t - c + \theta_1 (D_{t-1} - c + \theta_1 \epsilon_{t-2}).$$

Continuing in the same manner we have for any $N > 0$

$$\epsilon_t = c \sum_{n=0}^{N-1} \theta_1^n + \sum_{n=0}^{N-1} \theta_1^n D_{t-n} + \theta_1^N \epsilon_{t-N} \tag{2}$$

If $|\theta_1| < 1$ then the last term in (2) will approach 0 and we get the representation:

$$\epsilon_t = c\sum_{n=0}^{\infty}\theta_1^n + \sum_{n=0}^{\infty}\theta_1^n D_{t-n}$$

which shows that ϵ_t can be written as a convergent series of present and past observations $\{D_t\}$. Any $\{D_t\}$ that satisfies this property is said to be **invertible** *with respect to shocks $\{\epsilon_t\}$. Note that $|\theta_1| < 1$ if and only if the root of $1 - \theta_1 z$ is outside the unit circle. As we will discuss in Remark 1, the location of roots is central to a discussion of invertibility.*

However, if $|\theta_1| > 1$, we will show in Example 1 Part II that one cannot express the current shock as a convergent series of present and past observations. The case of $|\theta_1| > 1$ occurs if and only if the root of $1 - \theta_1 z$ is inside the unit circle. Here the demand series $\{D_t\}$ is said to be **non-invertible** *with respect to shocks $\{\epsilon_t\}$.*

If $|\theta_1| = 1$, then it is possible to recover ϵ_t from present and past values of $\{D_t\}$, however this is accomplished in a different way than described for the case when $|\theta_1| < 1$. We still say that $\{D_t\}$ is invertible with respect to shocks $\{\epsilon_t\}$ for this case. Refer to GHS for a discussion of invertibility.

The invertibility concepts described in this example extend naturally for an MA(q) model. Similarly, for an AR(p) model, we say that demand series $\{D_t\}$ is **causal** with respect to shocks $\{\epsilon_t\}$ if we can write D_t as a linear combination of present and past $\{\epsilon_t\}$.

Remark 1. *A series $\{D_t\}$ is causal and invertible ARMA(p,q) with respect to a series of independent Gaussian random variables $\{\epsilon_t\}$, called "shocks", having mean zero and variance σ_ϵ^2 if it can be written as*

$$D_t = c + \phi_1 D_{t-1} + \phi_2 D_{t-2} + ... + \phi_p D_{t-p} + \epsilon_t - \theta_1\epsilon_{t-1} - \theta_2\epsilon_{t-2} - ... - \theta_q\epsilon_{t-q}, \quad (3)$$

where c is a constant and the roots of the polynomials $1 - \phi_1 z - ... - \phi_p z^p$ and $1 - \theta_1 z - ... - \theta_q z^q$ are outside the unit circle for $z \in \mathbb{C}$.

It is often useful to express (3) in terms of the backshift operator, B, where $B^s\epsilon_t = \epsilon_{t-s}$ and $B^r D_t = D_{t-r}$. In order to do so, let $\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - ... - \phi_p B^p$ and $\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - ... - \theta_q B^q$. Then $\{D_t\}$ in (3) can be expressed as

$$\phi(B)D_t = c + \theta(B)\epsilon_t \quad (4)$$

For some more intuition behind invertibility and the use of the backshift operator, consider the following:

Example 1. Part II

We can rewrite the model (1) in terms of the backshift operator as

$$D_t = c + (1 - \theta_1 B)\epsilon_t \tag{5}$$

which can be rewritten as

$$\epsilon_t = -\frac{1}{1-\theta_1 B}c + \frac{1}{1-\theta_1 B}D_t \tag{6}$$

Suppose $|\theta_1| < 1$*. Through a formal Taylor series expansion of* $\frac{1}{1-\theta_1 B}$*, this can be rewritten as*

$$\epsilon_t = -\sum_{n=0}^{\infty}(\theta_1 B)^n c + \sum_{n=0}^{\infty}(\theta_1 B)^n D_t$$

or equivalently

$$\epsilon_t = -\sum_{n=0}^{\infty}(\theta_1 B)^n c + \sum_{n=0}^{\infty}\theta_1^n D_{t-n}$$

and hence we can write ϵ_t *as a linear combination of present and past values* $\{D_n\}_{n=-\infty}^{t}$*. Thus here the model in (5) is invertible.*

Suppose now that $|\theta_1| > 1$ *and consider the term* $\frac{1}{1-\theta_1 B}$ *in (6). Doing some manipulations we have that*

$$\frac{1}{1-\theta_1 B} = \frac{1/B}{B^{-1}-\theta_1} = \frac{1/B}{\theta_1(\theta_1^{-1}B^{-1}-1)} = (-\theta_1^{-1}/B)\frac{1}{1-\theta_1^{-1}B^{-1}}$$

Since $|\theta_1| > 1$*, it is obvious that* $|\frac{1}{\theta_1}| > 1$ *and through a formal Taylor series expansion of* $\frac{1}{1-\theta_1^{-1}B^{-1}}$*, we rewrite (6) as*

$$\epsilon_t = (\theta_1^{-1}/B)\sum_{n=0}^{\infty}(\theta_1^{-1}B^{-1})^n c + (-\theta_1^{-1}/B)\sum_{n=0}^{\infty}(\theta_1^{-1}B^{-1})^n D_t$$

which can be rewritten as

$$\epsilon_t = (\theta_1^{-1}/B)\sum_{n=-\infty}^{0}(\theta_1 B)^n c + (-\theta_1^{-1}/B)\sum_{n=-\infty}^{0}(\theta_1 B)^n D_t$$

or equivalently

$$\epsilon_t = (\theta_1^{-1}/B)\sum_{n=-\infty}^{0}(\theta_1 B)^n c + (-\theta_1^{-1}/B)\sum_{n=0}^{\infty}\theta_1^{-n} D_{t+n}$$

Here ϵ_t is expressed through values in the sequence $\{D_n\}_{n=t+1}^{\infty}$, which are in fact unknown in practice at time t. Thus the model in (5) is not invertible if $|\theta_1| < 1$.

From this example, we get the intuition behind Remark 1. We can see this by rewriting (4) as

$$\epsilon_t = -\theta^{-1}(B)c + \phi(B)\theta^{-1}(B)D_t$$

The polynomial $\theta(z) = 1 - \theta_1 z - ... - \theta_q z^q$, having roots $z_1, ..., z_q$, can be factorized as $\prod_{j=1}^{q}(1 - \frac{z}{z_j})$. Therefore the previous equation is equivalent to

$$\epsilon_t = -\theta^{-1}(B)c + \frac{\phi(B)}{\prod_{j=1}^{q}(1 - z_j^{-1}B)} D_t \tag{7}$$

We can treat the terms $\frac{1}{1 - z_j^{-1}B}$ in the same way we treated $\frac{1}{1 - \theta_1 B}$ in Example 1 Part II. In doing so, when we express ϵ_t through observations $\{D_t\}$, we will require some values in the sequence $\{D_n\}_{n=t+1}^{\infty}$ if and only if there is a root z_j such that $|z_j| < 1$.

In accordance with GHS, we say that $\{D_t\}$ is $QUARMA(p, q, J)$ with respect to shocks $\{\epsilon_t\}$ if it can be written as

$$D_t = c + \phi_1 D_{t-1} + \phi_2 D_{t-2} + ... + \phi_p D_{t-p} + \epsilon_{t-J} - \theta_1 \epsilon_{t-J-1} - \theta_2 \epsilon_{t-J-2} - ... - \theta_q \epsilon_{t-J-q} \tag{8}$$

or, in terms of the backshift operator,

$$\phi(B)D_t = c + B^J \theta(B)\epsilon_t \tag{9}$$

where $\phi(B)$ and $\theta(B)$ are as previously defined. We refer to $\phi(z)$ and $\theta(z)$ as the AR and MA polynomials in the QUARMA representation of $\{D_t\}$ with respect to $\{\epsilon_t\}$. We refer to J as the QUARMA degree. Note that $\{D_t\}$ which is QUARMA with respect to $\{\epsilon_t\}$ is ARMA with respect to $\{B^J \epsilon_t\}$. As in GHS, if $J > 0$ in (9), then $\{D_t\}$ is non-invertible with respect to $\{\epsilon_t\}$ since, at time t, there would be no way to recover ϵ_t from present and past values $\{D_n\}_{n=-\infty}^{t}$. The model in (9) will be central to our study of demand propagation as we will show that demand $\{D_{k,t}\}$ of player k may be QUARMA, with a QUARMA degree $J_k > 0$, even though the retailer observes ARMA demand. Henceforth $\{D_{k,t}\}$ and $\{\epsilon_{k,t}\}$ will refer to player k's demand series and full information shock (FIS) series, defined below in Definition 1. $D_{k,t}$ and $\epsilon_{k,t}$ will refer to player k's demand and shock at time t.

2.2 Assumptions

We consider a K-stage supply chain where at discrete equally-spaced time periods, the retailer (assumed to be at stage 1) faces external demand $\{D_{1,t}\}$, for a single item. Let $\{D_{1,t}\}$ follow a covariance stationary $ARMA\,(p, q_1)$ process with $p \geq 0$, $q_1 \geq 0$:

$$\phi(B)D_{1,t} = d + \theta_1(B)\epsilon_{1,t} \tag{10}$$

where $d > 0$ is a constant and the roots of $\phi(z)$ and $\theta_1(z)$ are outside the unit circle to insure that the retailer's demand is causal and invertible with respect to $\{\epsilon_{1,t}\}$. Following LST, Zhang and GHS, we assume that the shocks $\{\epsilon_{1,t}\}$ are Gaussian white noise. Let the replenishment leadtime from the retailer's supplier to the retailer be ℓ_1 periods. Excess demand at the retailer is backlogged. Let the replenishment leadtime from the player at stage $k+1$ to stage k be ℓ_k periods. We assume that all supply chain players use a myopic order-up-to inventory policy where negative order quantities are allowed, but d is sufficiently large so that the probability of negative demand or negative orders is negligible. Furthermore, h_k and p_k are player k's unit holding and shortage (or backorder) costs per time period. Player k's required service level is given by $c_k = \Phi^{-1}[\frac{p_k}{p_k+h_k}]$, where Φ is the standard Normal cdf. It is assumed that for $k \geq 1$ the ℓ_k period lead time guarantee holds, i.e., if the player at stage $k+1$ does not have enough stock to fill an order from the player at stage k, then the player at stage $k+1$ will meet the shortfall from an alternative source, with additional cost representing the penalty cost to this shortfall. [Gallego and Zipkin, 1999] show how this assumption allows one to decompose a multi-stage system with no alternative source into single-stage systems and to approximate the cost of the system.

Hence, at the end of time period t, after demand $D_{1,t}$ has been observed, the retailer observes the inventory position and places order $D_{2,t}$ with its supplier. The retailer receives the shipment of this order at the beginning of period $t + \ell_1 + 1$, where $\ell_1 \geq 0$. The sequence of events at all supply chain players is similar. However, it is further assumed that all upstream supply chain players observe their demand, observe their inventory positions and place their orders instantaneously at the end of time period t.

We assume that all players place their orders based on the best linear forecast of their lead-time demand. This means that player k's order will be based on its best linear forecast of the demand it will observe through time period $t + \ell_k + 1$ (that is $\sum_{i=1}^{\ell_k+1} D_{k,t+i}$). It is assumed that all upstream supply chain players observe their demand, observe their inventory positions and place their orders instantaneously at the end of every time period t.

We assume that, at time t, along with placing its order, a player may choose to share nothing, its demand $D_{k,t}$, or its FIS $\epsilon_{k,t}$, with an adjacent upstream player. It is assumed that all players are aware of the retailer's model and all sharing arrangements that occur downstream. We will show that this assumption guarantees that all players know the model for their own demand $\{D_{k,t}\}$ with respect to their FIS $\{\epsilon_{k,t}\}$. The last assumption also guarantees the information structure assumed by GHS (2012), GGS (2005), LST (2000), Raghunathan (2001), and Zhang (2004), namely that, for $k \geq 2$ the form and parameters of the model generating player $k - 1$'s demand are known to player k. However player $k - 1$'s demand realizations and/or full information shocks may not be observable by player k.

2.3 Information Sets and Full Information Shocks

As mentioned above, each player will forecast lead-time demand based on their information set at time t. As in GHS, we denote the full information set available to player k as $\mathcal{M}_t^k$. Let $\mathcal{M}_t^{D_k} = \overline{sp}\{1, D_{k,t}, D_{k,t-1}, D_{k,t-2}, \ldots\}$, where "$\overline{sp}\{\}$" refers to the "closed linear span". Then $\mathcal{M}_t^{D_k}$ is the Hilbert space generated by $\{1, D_{k,t}, D_{k,t-1}, D_{k,t-2}, \ldots\}$ with inner product given by the covariance. We will at times refer to $\mathcal{M}_t^{D_k}$ as the "linear past" of $\{D_{k,t}\}$. Similarly let $\mathcal{M}_t^{\epsilon_k} = \overline{sp}\{1, \epsilon_{k,t}, \epsilon_{k,t-1}, \epsilon_{k,t-2}, \ldots\}$. For the linear past of two time series, for example, $\{D_{k-1,t}\}$ and $\{D_{k,t}\}$, we write $\mathcal{M}_t^{D_{k-1},D_k} = \overline{sp}\{1, D_{k-1,t}, D_{k,t}, D_{k-1,t-1}, D_{k,t-1}, \ldots\}$.

As an example on how to determine a player's information set, consider the retailer's information set $\mathcal{M}_t^1$. Since at any time period t the retailer knows the series $\{D_{1,t}\}$, the retailer can also compute any linear combination of $\{1, D_{1,t}, D_{1,t-1}, D_{1,t-2}, \ldots\}$. Since the retailer only observes $D_{1,t}$, we say that $\mathcal{M}_t^1 = \mathcal{M}_t^{D_1}$. However, if we recall our assumption that the retailer's demand is invertible and causal with respect to the shocks $\{\epsilon_{1,t}\}$, we find that the retailer can recover the series $\{\epsilon_{1,t}\}$ from the series $\{D_{1,t}\}$ and vice-versa (see [Brockwell and Davis, 1991], pp 83-88 for a complete discussion of invertibility and causality). Therefore we can say that $\mathcal{M}_t^{D_1} = \mathcal{M}_t^{\epsilon_1}$. Thus the retailer's information set is also $\mathcal{M}_t^1 = \mathcal{M}_t^{\epsilon_1}$. In the presence of information sharing, there are several possible forms for player k's information set $\mathcal{M}_t^k$.

Now that we have defined player k's information set, we can define player k's full information shocks as they appear in GHS (2012).

Definition 1. *Suppose for $k > 0$ we can represent player k's demand series $\{D_{k,t}\}$ as a QUARMA with respect to a series of shocks $\{\epsilon_{k,t}\}$. We say that $\{\epsilon_{k,t}\}$ are player k's Full Information Shocks (FIS) if $\mathcal{M}_t^k = \mathcal{M}_t^{\epsilon_k}$.*

This definition implies two key properties of full information shocks. Player k's information set can be used to characterize player k's full information shocks. Also, player k's information set can be characterized using player k's full information shocks.

We now introduce an example that we will study throughout the paper. This example will show how information is gained from various sharing arrangements. Furthermore this example demonstrates the importance of studying various sharing arrangements because the difference in value of the arrangements can be significant even for the very simple model provided below.

Example 2. Part I

Suppose the retailer observes ARMA(2,2) demand given by

$$(1 + \frac{1}{3}B + \frac{1}{2}B^2)D_{1,t} = d + (1 - \frac{83}{57}B + \frac{289}{456}B^2)\epsilon_{1,t} \tag{11}$$

We will assume that $\ell_1 = 1$ *and* $\ell_2 = 1$.

Note that $\phi(z) = 1 + \frac{1}{3}z + \frac{1}{2}z^2$ *has roots* $-0.333333 + 1.374369i$ *and* $-0.333333 - 1.374369i$ *which are outside the unit circle and* $\theta_1(z) = 1 - \frac{83}{57}z + \frac{289}{456}z^2$ *has roots* $1.148789 + 0.508074i$ *and* $1.148789 - 0.508074i$ *which are also outside the unit circle. Therefore the retailer's demand is causal and invertible with respect to* $\epsilon_{1,t}$.

Suppose the retailer shares its shocks with the supplier. Following the propagation described in GHS (under shock-sharing), with $\ell_1 = 1$, *we find that the supplier observes the following ARMA(2,2) demand:*

$$(1 + \frac{1}{3}B + \frac{1}{2}B^2)D_{2,t} = d + (1 - \frac{32}{3}B + \frac{20}{3}B^2)\epsilon_{2,t}$$

where $\epsilon_{2,t} = (-9/152)\epsilon_{1,t}$. *We denote the innovation variance of* $\{\epsilon_{2,t}\}$ *by* $\sigma^2_{\epsilon_2}$.

We stop the discussion of this example here for now and will continue it later in Section 3 once we derive the necessary tools to study it further.

2.4 Demand Propagation from Stage $k-1$ to k

GHS show that when players can either share nothing or their full information shocks, ARMA demand at the retailer given in equation (10) propagates up the supply chain such that player k (with $k > 1$) faces QUARMA(p, q_k, J_k) demand with respect to its full information shocks, $\{\epsilon_{k,t}\}$, i.e.,

$$D_{k,t} = d+\phi_1 D_{k,t-1}+\phi_2 D_{k,t-2}+\cdots+\phi_p D_{k,t-p}+\epsilon_{k,t-J_k}-\theta_{k,1}\epsilon_{k,t-J_k-1}-\theta_{k,2}\epsilon_{k,t-J_k-2}-\cdots-\theta_{k,q_k}\epsilon_{k,t-J_k-q_k}. \tag{12}$$

where $\theta_{k,q_k} \neq 0$. Note that in equation (12) the most recent J_k shocks do not appear. As long as $J_k < \infty$, the QUARMA(p, q_k, J_k) model for player k's demand with respect to shocks $\{\epsilon_{k,t}\}$ may be expressed using the backshift operator B as,

$$\phi(B)D_{k,t} = d + B^{J_k}\theta_k(B)\epsilon_{k,t}, \tag{13}$$

where $\theta_k(B) = 1 - \sum_{j=1}^{q_k} \theta_{k,j} B^j$.

A key contribution of this paper is showing that an equation of the form (13) holds when demand sharing is also allowed throughout the chain. We prove this by mathematical induction on k in Theorem 3 of Section 5. The inductive hypothesis in the proof is that for a particular $k > 1$ we can express player $k-1$'s demand $\{D_{k-1,t}\}$ in terms of $\{\epsilon_{k-1,t}\}$ as

$$\phi(B)D_{k-1,t} = d + B^{J_{k-1}}\theta_{k-1}(B)\epsilon_{k-1,t} \tag{14}$$

Sections 3-4 use (14) to obtain general formulas for player k's full information shocks $\{\epsilon_{k,t}\}$. Theorem 3 makes use of the results found in these sections to show that (13) indeed holds even when players can share their demand.

In Section 3 we will discuss how player $k-1$ will forecast its demand and place its order to player k according to a myopic order-up-to-policy sharing either nothing, $D_{k-1,t}$, or $\epsilon_{k-1,t}$. Section 3 will discuss how player k receives the order, which we show is QUARMA with respect to player $k-1$'s full information shocks. Note that when describing player $k-1$'s order it is unnecessary to consider the sharing arrangement between player $k-1$ and player k. Section 4 will discuss how player k will recover its full information shocks $\{\epsilon_{k,t}\}$ based on its information set, which may depend on its sharing arrangement with player $k-1$. These shocks determine the QUARMA representation of player k's demand $\{D_{k,t}\}$ with respect to $\{\epsilon_{k,t}\}$. At this stage we have come back to equation (14) with k replacing $k-1$ and demand propagation will continue from player k to player $k+1$.

Once we have described the concepts mentioned above, it will be possible to tackle the outstanding issue of showing that Equation (13) does in fact hold for all $k > 1$. As mentioned, this will be covered in detail in Section 5. In Section 6 we will compare the various sharing arrangments between players $k-1$ and k to see if there could be value gained in changing sharing arrangements. It will turn out that player $k-1$ sharing its demand can lead to the variance of player k's FIS being intermediate to the variance of player k's FIS when player $k-1$ shares nothing or its full information shocks. This would imply that player k's MSFE will also be intermediate when player k forecasts one step ahead. We will illustrate this with several examples. In Section 7, we will summarize the contributions of this paper.

3 Player $k-1$'s Order to Player k

As mentioned previously, the inductive hypothesis in Theorem 3 is that player $k-1>0$ observes QUARMA demand $\{D_{k-1,t}\}$ with respect to its full information shocks.

$$\phi(B)D_{k-1,t} = d + B^{J_{k-1}}\theta_{k-1}(B)\epsilon_{k-1,t}$$

We will call this player $k-1$'s "demand equation". Here we will discuss how player $k-1$ goes about creating its optimal order to player k. Since player $k-1$ has already recovered its FIS, it can forecast its lead-time demand using its demand equation. As in GHS, we call this forecast and its MSFE $m_{k-1,t}$ and $v_{k-1,t}$. Using a myopic-order-up-to-policy, player $k-1$ determines its order-up-to-level, $S_{k-1,t} = m_{k-1,t} + c_{k-1}\sqrt{v_{k-1,t}}$. Then player $k-1$ constructs its order to player k,

$$D_{k,t} = D_{k-1,t} + S_{k-1,t} - S_{k-1,t-1} = D_{k-1,t} + m_{k-1,t} - m_{k-1,t-1}$$

where the last equality holds because each player's MSFE is time invariant (ie. $v_{k-1,t} = v_{k-1}$). Note that both player $k-1$'s order as well as player k's demand is $D_{k,t}$. While it is indeed the case that numerically player $k-1$'s order is player k's demand, it is important to study $D_{k,t}$ with respect to the information that is available to player $k-1$ and player k separately.

Recall that $m_{k-1,t}$ and $m_{k-1,t-1}$ are player $k-1$'s best linear forecasts, at time t and $t-1$, of its lead-time demand and therefore will be a linear combination of present and past values of $\{\epsilon_{k-1,t}\}$ (see Lemma 1 of GHS). Player $k-1$'s demand, $D_{k-1,t}$, can also be written as a linear combination of present and past values of $\{\epsilon_{k-1,t}\}$ since these are player $k-1$'s FIS. Therefore it stands to reason that $D_{k,t}$ can be expressed as a linear combination of present and past values of $\{\epsilon_{k-1,t}\}$. As we will see by Proposition 1 and Theorem 1 below, $\{D_{k,t}\}$ will be QUARMA with respect to $\{C\epsilon_{k-1,t}\}$ where C is some constant when $J_{k-1} < \infty$. The case of constant demand ($J_{k-1} = \infty$) is trivial.

The following proposition gives a useful characterization of the QUARMA model.

Proposition 1. *We can represent a series $\{D_t\}$ in terms of a shock series $\{\epsilon_t\}$ as*

$$\phi(B)D_t = c + \lambda(B)\epsilon_t \tag{15}$$

where $\lambda(z)$ is some polynomial in $z \in \mathbb{C}$ such that we can write $\lambda(z) = \lambda_0+\lambda_1 z+\lambda_2 z^2+...+\lambda_{q+J}z^{q+J}$ with $\lambda_{q+J} \neq 0$ and $J = \inf\{j \geq 0|\lambda_j \neq 0\} \neq \infty$ **if and only if** *D_t is QUARMA with respect to $\{\lambda_J\epsilon_t\}$:*

$$\phi(B)D_t = c + B^J\theta(B)\lambda_J\epsilon_t$$

where $\theta(z) = z^{-J}\lambda(z)/\lambda_J$ has a leading coefficient of 1 and no roots at zero.

Proof. First assume that representation (15) holds and write the polynomial $\lambda(z)$ as the product of two polynomials and a constant term. The first polynomial will have roots only at 0 (if there are any roots at 0), and the other will have no roots at 0 and a leading coefficient of 1. To do this, we note that J represents the multiplicity of the 0-root of $\lambda(z)$ and that λ_J is the first non-zero coefficient of $\lambda(z)$. Let $\theta(z) = z^{-J}\lambda_J^{-1}\lambda(z)$. Therefore (15) can be rewritten as

$$\phi(B)D_t = c + B^J\theta(B)\lambda_J\epsilon_t, \tag{16}$$

Note the first polynomial is z^J and the second is $\theta(z)$ where either z^J has all its roots at 0 or $z^J \equiv 1$, while $\theta(z)$ has no roots at 0 and a leading coefficient of 1.

The necessity of (15) follows from the the definition of $\theta(z)$. By simple arithmetic we can express $\lambda(z) = z^J\theta(z)\lambda_J$ to get $\lambda(z)$ and equation (15) holds. □

At present we are interested in representing player $k-1$'s order $\{D_{k,t}\}$ in terms of player $k-1$'s FIS $\{\epsilon_{k-1,t}\}$. It follows from Proposition 1 that if we can find a polynomial $\lambda_k(z) = \lambda_0 + \lambda_1 z + \lambda_2 z^2 + ... + \lambda_{\tilde{q}_k+\tilde{J}_k} z^{\tilde{q}_k+\tilde{J}_k}$ with $\lambda_{\tilde{q}_k+\tilde{J}_k} \neq 0$ and $\tilde{J}_k = \inf\{j \geq 0 | \lambda_j \neq 0\} \neq \infty$ such that

$$\phi(B)D_{k,t} = d + \lambda_k(B)\epsilon_{k-1,t} \tag{17}$$

then $\{D_{k,t}\}$ will be QUARMA with respect to $\{\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}\}$. The parameters $\tilde{J}_k$ and $\tilde{q}_k$ are conceptually different from the J_k and q_k appearing in player k's demand equation since here we are expressing $\{D_{k,t}\}$ in terms of $\{\epsilon_{k-1,t}\}$. The following theorem shows how to find $\lambda_k(z)$ from the polynomials appearing in player $k-1$'s demand equation. The formula below is the backbone for many of the concepts discussed in this paper. It is crucial in finding an example of demand sharing being intermediate in value to no sharing and shock sharing, which we will see once Example 2 is completed in Section 6. Furthermore, it can be also used to study the asymptotic behavior of supply chains (including the bullwhip effect), which we leave to future research.

Theorem 1. *For $k \geq 2$, assume that player $k-1$ observes demand series $\{D_{k-1,t}\}$ that is QUARMA(p, q_{k-1}, J_{k-1}) with respect to shocks $\{\epsilon_{k-1,t}\}$*

$$\phi(B)D_{k-1,t} = d + B^{J_{k-1}}\theta_{k-1}(B)\epsilon_{k-1,t} \tag{18}$$

Then, player $k-1$'s order to player k, $\{D_{k,t}\}$, will be

$$\phi(B)D_{k,t} = d + \lambda_k(B)\epsilon_{k-1,t} \tag{19}$$

where

$$
\begin{aligned}
\phi(B)D_{k,t} &= d + \Bigg\{ \Bigg[B^{J_{k-1}} + \mathbf{1}_{\{J_{k-1}>0\}}[B^{max(0,J_{k-1}-(\ell_{k-1}+1))} - B^{J_{k-1}}] \\
&+ \mathbf{1}_{\{\ell_{k-1}\geq J_{k-1}\}}[B^{J_{k-1}-(\ell_{k-1}+1)} - 1] \Bigg] \theta_{k-1}(B) \\
&+ \mathbf{1}_{\{\ell_{k-1}\geq J_{k-1}\}}\phi(B) \Bigg[\sum_{L=0}^{\ell_{k-1}-J_{k-1}} \psi_{k-1,L} - B^{J_{k-1}-(\ell_{k-1}+1)} \sum_{L=0}^{\ell_{k-1}-J_{k-1}} \psi_{k-1,L}B^{L} \Bigg] \Bigg\} \epsilon_{k-1,t}
\end{aligned}
$$

where $\psi_{k-1,L}$ is the L^{th} MA(∞) coefficient of $D_{k-1,t}$ with respect to $\{\epsilon_{k-1,n}\}_{-\infty}^{t-J_{k-1}}$.

A proof can be found in the Appendix. The constant term d in (19) is the same as the one appearing in (18). It can turn out that the sums in the above theorem have an upper limit that is smaller than its lower limit. If this is the case, the sum is 0 by convention. It is important that $\lambda_k(z)$ not have any negative powers of z. Indeed, this can be checked to be the case. The expression for $\lambda_k(z)$ is universal when player $k-1$ observes QUARMA demand and places its order according to the order-up-to policy. Combining this result with Proposition 1 we get that $\{D_{k,t}\}$ is QUARMA with respect to $\{\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}\}$.

We will write the QUARMA representation of $\{D_{k,t}\}$ with respect to $\{\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}\}$ as

$$\phi(B)D_{k,t} = d + B^{\tilde{J}_k}\tilde{\theta}_k(B)\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t} \tag{20}$$

where we use the "tilde" in $\tilde{\theta}_k(z)$ and $\tilde{J}_k$ to differentiate that we are expressing $\{D_{k,t}\}$ in terms of $\{\epsilon_{k-1,t}\}$ rather than $\{\epsilon_{k,t}\}$. We refer to (20) as player $k-1$'s order equation.

The expression for $\lambda_k(z)$ in Theorem 1 simplifies greatly when $J_{k-1} \geq \ell_{k-1}+1$ as demonstrated by the corollary below.

Corollary 1. *Consider the assumptions of Theorem 1 with $J_{k-1} \geq \ell_{k-1}+1$.*

Then, player $k-1$'s order to player k, $\{D_{k,t}\}$, will be

$$\phi(B)D_{k,t} = d + \lambda_k(B)\epsilon_{k-1,t} \tag{21}$$

where

$$\lambda_k(z) = z^{J_{k-1}-(\ell_{k-1}+1)}\theta_{k-1}(z)$$

Proof. If $J_{k-1} \geq \ell_{k-1} + 1$, then the expression in the conclusion of Theorem 1 simplifies to

$$\lambda_k(z) = z^{J_{k-1}}\theta_{k-1}(z) + z^{J_{k-1}-(\ell_{k-1}+1)}\theta_{k-1}(z) - z^{J_{k-1}}\theta_{k-1}(z)$$

which is simply

$$\lambda_k(z) = z^{J_{k-1}-(\ell_{k-1}+1)}\theta_{k-1}(z)$$

□

Corollary 1 shows an interesting relationship between player $k-1$'s demand equation and player $k-1$'s order equation. Specifically, when $J_{k-1} \geq \ell_{k-1} + 1$, we have that $\tilde{J}_k = J_{k-1} - (\ell_{k-1} + 1)$ and $\tilde{\theta}_k(z) = \theta_{k-1}(z)$.

Example 2. Part II

Recall that it was previously determined that the supplier observes demand equation:

$$(1 + \frac{1}{3}B + \frac{1}{2}B^2)D_{2,t} = d + (1 - \frac{32}{3}B + \frac{20}{3}B^2)\epsilon_{2,t} \tag{22}$$

Using Theorem 1 with $J_2 = 0$ and $\ell_2 = 1$ we have that

$$\lambda_3(z) = -\frac{1}{6} + \frac{13}{6}z - 5z^2$$

By Proposition 1 this means that the supplier's order equation is given by:

$$(1 + \frac{1}{3}B + \frac{1}{2}B^2)D_{3,t} = d + (1 - 13B + 30B^2)\frac{-1}{6}\epsilon_{2,t} \tag{23}$$

where $\tilde{\theta}_3(z) = 1 - 13z + 30z^2$.

4 The QUARMA representation of $\{D_{k,t}\}$ with respect to $\{\epsilon_{k,t}\}$

In order to establish the QUARMA representation of player k's demand, i.e., $\{D_{k,t}\}$ with respect to $\{\epsilon_{k,t}\}$, we must first establish player k's FIS $\{\epsilon_{k,t}\}$. As we will see, these will depend on the location of roots of $\tilde{\theta}_k$ in (20) and the sharing arrangement between player $k-1$ and k. Consider player $k-1$'s demand and order equations given in (14) and (20):

$$\phi(B)D_{k-1,t} = d + B^{J_{k-1}}\theta_{k-1}(B)\epsilon_{k-1,t}$$

$$\phi(B)D_{k,t} = d + B^{\tilde{J}_k}\tilde{\theta}_k(B)\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}$$

Recall that player k's FIS must satisfy two properties: $\{D_{k,t}\}$ is QUARMA with respect to $\{\epsilon_{k,t}\}$ and $\mathcal{M}_t^k = \mathcal{M}_t^{\epsilon_k}$. Therefore player k's FIS depend on player k's information set. Furthermore since the QUARMA representation of $\{D_{k,t}\}$ with respect to $\{\epsilon_{k,t}\}$ depends on player k's FIS, it inherently depends on player k's information set as well. Thus understanding player k's information set is crucial to the study of propagation.

Note that player k's information set consists of $D_{k,t}$ and anything shared by player $k-1$.

- If there is no sharing between player k and $k-1$, $\mathcal{M}_t^k = \mathcal{M}_t^{D_k}$
- If player $k-1$ shares its demand then $\mathcal{M}_t^k = \mathcal{M}_t^{D_k, D_{k-1}}$
- If player $k-1$ shares its shocks then $\mathcal{M}_t^k = \mathcal{M}_t^{D_k, \epsilon_{k-1}}$

This section will be divided into four subsections as we establish some notation and explore player k's FIS under the three possible sharing scenarios. The propositions found in Sections 4.2 and 4.4 are restatements of results found in GHS (2012), where the case of no sharing and shock sharing has previously been studied. They are presented here to keep this paper self-contained and because they yield insight into how to find and compare FIS under different sharing arrangements. Section 4.3 focuses on demand sharing and contains several key results of our paper.

4.1 Notation

Before we can show the form of player k's FIS under various sharing arrangments we must develop some notation. A lot of the theory from this point on will involve working with the roots of the polynomials $\theta_{k-1}(z)$ and $\tilde{\theta}_k(z)$. Furthermore, we will need to consider the multiplicity of the roots in the polynomials. To do this we introduce the following definition:

Definition 2. *For any $z \in \mathbb{C}$ and polynomial P, if z is a root of P we define $m(z, P)$ as the multiplicity of z in P. If z is not a root of polynomial P we define $m(z, P) = 0$.*

It will soon be useful to factorize $\tilde{\theta}_k(z)$ into factors having all roots on the the unit circle and all roots not on the unit circle, for this we utilize the following notation:

Consider player $k-1$'s demand and order equations. Suppose the polynomial $\tilde{\theta}_k(z)$ has r_k distinct roots $z_1, \ldots, z_{r_k}$ with respective multiplicities $m(z_1, \tilde{\theta}_k), \ldots, m(z_{r_k}, \tilde{\theta}_k)$. Note that $r_k \leq \tilde{q}_k$. Then $\tilde{\theta}_k(z)$ has the factorization:

$$\tilde{\theta}_k(z) = \prod_{j=1}^{r_k} (1 - \frac{z}{z_j})^{m(z_j, \tilde{\theta}_k)}$$

Define the following:

$$\tilde{\theta}_k^{IN} := \prod_{\{j:|z_j|<1\}} (1 - \frac{z}{z_j})^{m(z_j, \tilde{\theta}_k)} \tag{24}$$

$$\tilde{\theta}_k^{OUT} := \prod_{\{j:|z_j|>1\}} (1 - \frac{z}{z_j})^{m(z_j, \tilde{\theta}_k)} \tag{25}$$

$$\tilde{\theta}_k^{ON} := \prod_{\{j:|z_j|=1\}} (1 - \frac{z}{z_j})^{m(z_j, \tilde{\theta}_k)} \tag{26}$$

$$\tilde{\theta}_k^{OFF} := \prod_{\{j:|z_j|\neq 1\}} (1 - \frac{z}{z_j})^{m(z_j, \tilde{\theta}_k)} \tag{27}$$

If $\tilde{\theta}_k$ has no roots inside the unit circle, then $\tilde{\theta}_k^{IN} \equiv 1$. The same convention holds for the others. It should be clear that $\tilde{\theta}_k = \tilde{\theta}_k^{IN} \cdot \tilde{\theta}_k^{OUT} \cdot \tilde{\theta}_k^{ON}$ by construction.

We will also be interested in identifying any common roots between $\tilde{\theta}_k(z)$ and $\theta_{k-1}(z)$ inside the unit circle. To do this we define the following:

$$\tilde{\theta}_k^{I-C} := \prod_{\{j:|z_j|<1\}} (1 - \frac{z}{z_j})^{min(m(z_j, \tilde{\theta}_k), m(z_j, \theta_{k-1}))} \tag{28}$$

The roots of $\tilde{\theta}_k^{I-C}$ will all be inside the unit circle. Furthermore, since the term $m(z_j, \theta_{k-1})$ may be 0, $\tilde{\theta}_k^{I-C}$ will only consist of those roots that are common to both $\tilde{\theta}_k$ and θ_{k-1}. Also, the multiplicity of each root is the minimum of the multiplicities of the root in $\tilde{\theta}_k$ and θ_{k-1}. If $\theta_{k-1}(z)$ and $\tilde{\theta}_k(z)$ have no common roots inside the unit circle, then $\tilde{\theta}_k^{I-C} \equiv 1$.

Finally, define

$$\tilde{\theta}_k^{I-NC} := \prod_{\{j:|z_j|<1\}} (1 - \frac{z}{z_j})^{m(z_j,\tilde{\theta}_k)-min(m(z_j,\tilde{\theta}_k),m(z_j,\theta_{k-1}))} \tag{29}$$

The roots of $\tilde{\theta}_k^{I-NC}$ will all be inside the unit circle. Furthermore a root of $\tilde{\theta}_k$ is a root of $\tilde{\theta}_k^{I-NC}$ if $m(z_j, \tilde{\theta}_k) > m(z_j, \theta_{k-1})$ and the multiplicity of each root in $\tilde{\theta}_k^{I-NC}$ is $m(z_j, \tilde{\theta}_k) - min(m(z_j, \tilde{\theta}_k), m(z_j, \theta_{k-1}))$. If $m(z_j, \tilde{\theta}_k) \leq m(z_j, \theta_{k-1})$ for all j, then $\tilde{\theta}_k^{I-NC} \equiv 1$. Note that $\tilde{\theta}_k^{IN} = \tilde{\theta}_k^{I-C} \cdot \tilde{\theta}_k^{I-NC}$ by construction.

4.2 FIS Under No Sharing

If there is no sharing between player k and player $k-1$, player k's information set is

$$\mathcal{M}_t^k = \mathcal{M}_t^{D_k}.$$

Therefore player k's FIS $\{\epsilon_{k,t}\}$ must satisfy $\mathcal{M}_t^{\epsilon_k} = \mathcal{M}_t^{D_k}$.

Proposition 2. *If $z^{\tilde{J}_k}\tilde{\theta}_k(z)$ has no roots inside the unit circle then*

- $\{\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}\}$ *are player k's full information shocks.*
- $J_k = 0$ *and* $\theta_k(z) = \tilde{\theta}_k(z)$

Since $\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}$ are player k's full information shocks, we say $\epsilon_{k,t} = \lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}$ with

$$\phi(B)D_{k,t} = d + B^{J_k}\theta_k(B)\epsilon_{k,t}$$

where $J_k = 0$ and $\theta_k(z) = \tilde{\theta}_k(z)$.

To state player k's FIS when there is no sharing and $\theta_k(z)$ has roots inside the unit circle, some additional notation is required:

Definition 3. *Suppose a polynomial $P(z)$ factorizes as*

$$P(z) = \prod_{s=1}^{h}(1 - \frac{z}{a_s}) \prod_{s=h+1}^{q}(1 - \frac{z}{a_s})$$

such that $|a_s|< 1$ for $1 \leq s \leq h$ and $|a_s|\geq 1$ for $h+1 \leq s \leq q$.

Define $P^\dagger(z)$ as the polynomial

$$P^\dagger(z) = \prod_{s=1}^{h}(1 - \bar{a}_s z) \prod_{s=h+1}^{q}(1 - \frac{z}{a_s}) \tag{30}$$

where $\bar{a}_s$ is the complex conjugate of a_s

Proposition 3. *Suppose that $\tilde{\theta}_k(z)$ in player $k-1$'s order equation has $h > 0$ roots inside the unit circle. Then*

- $\{\frac{\tilde{\theta}_k(B)}{\tilde{\theta}_k^\dagger(B)} B^{\tilde{J}_k} \lambda_{k,\tilde{J}_k} \epsilon_{k-1,t}\}$ *are player k's full information shocks.*
- $J_k = 0$ *and* $\theta_k(z) = \tilde{\theta}_k^\dagger(z)$

The polynomial $\tilde{\theta}_k(z)$ can be factorized as

$$\tilde{\theta}_k(z) = \prod_{s=1}^{h}(1 - \frac{z}{z_s}) \prod_{s=h+1}^{\tilde{q}_k}(1 - \frac{z}{z_s})$$

where the roots $z_1, ..., z_{\tilde{q}_k}$ of $\tilde{\theta}_k(z)$ are such that $|z_s|< 1$ for $1 \leq s \leq h$ and $|z_s|\geq 1$ for $h+1 \leq s \leq \tilde{q}_k$

and

$$\tilde{\theta}_k^\dagger(z) = \prod_{s=1}^{h}(1 - \bar{z}_s z) \prod_{s=h+1}^{\tilde{q}_k}(1 - \frac{z}{z_s})$$

When $\{\frac{\tilde{\theta}_k(B)}{\tilde{\theta}_k^\dagger(B)} B^{\tilde{J}_k} \lambda_{k,\tilde{J}_k} \epsilon_{k-1,t}\}$ are player k's full information shocks, we say that

$$\epsilon_{k,t} = \frac{\tilde{\theta}_k(B)}{\tilde{\theta}_k^\dagger(B)} B^{\tilde{J}_k} \lambda_{k,\tilde{J}_k} \epsilon_{k-1,t}$$

which can be rewritten as

$$\frac{\tilde{\theta}_k^\dagger(B)}{\tilde{\theta}_k(B)} \epsilon_{k,t} = B^{\tilde{J}_k} \lambda_{k,\tilde{J}_k} \epsilon_{k-1,t}$$

Even if $\tilde{J}_k = 0$, there is no way to recover $\epsilon_{k-1,t}$ from present and past values of $\{\epsilon_{k,t}\}$ since any Laurent series representation of $\frac{1}{\tilde{\theta}_k(B)}$ will involve negative powers in B, as explained by the comments immediately following Example 1 Part II. Thus $\{\epsilon_{k-1,t}\}$ cannot be player k's full information shocks.

The only scenario not yet covered is that none of the roots of $\tilde{\theta}_k(z)$ are inside the unit circle, but $\tilde{J}_k > 0$. The following proposition gives the FIS in this case:

Proposition 4. *Suppose that $\tilde{\theta}_k(z)$ has no roots inside the unit circle and $\tilde{J}_k > 0$. Then*

- $\{B^{\tilde{J}_k}\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}\}$ *are player k's FIS*
- $J_k = 0$ *and* $\theta_k(z) = \tilde{\theta}_k(z)$

Proof. Let

$$\gamma_{k-1,t} = B^{\tilde{J}_k}\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t} \tag{31}$$

Substituting this into (20) we have

$$\phi(B)D_{k,t} = d + \tilde{\theta}_k(B)\gamma_{k-1,t}$$

Since $\tilde{\theta}_k(z)$ has no roots inside the unit circle Proposition 2 states that $\gamma_{k-1,t}$ are player k's full information shocks. We say that $\{\epsilon_{k,t}\} = \{B^{\tilde{J}_k}\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}\}$. Furthermore $\theta_k(z) = \tilde{\theta}_k(z)$ and $J_k = 0$. □

Thus we have found player k's FIS and how to express $\{D_{k,t}\}$ in terms of $\{\epsilon_{k,t}\}$ when there is no sharing. We now consider Example 2 for the case that the supplier shares nothing with player 3.

Example 2. Part III

Recall that we previously determined that the supplier's order equation is given by:

$$(1 + \frac{1}{3}B + \frac{1}{2}B^2)D_{3,t} = d + (1 - 13B + 30B^2)\frac{-1}{6}\epsilon_{2,t}$$

where $\tilde{\theta}_3(z) = 1 - 13z + 30z^2$ has roots .1 and 1/3. Since $\tilde{\theta}_3(z)$ has a root inside the unit circle and $\tilde{J}_3 = 0$, if the retailer shares nothing, we can use Proposition 3 to determine that player 3's *full information shocks are*

$$\epsilon_{3,t} = \frac{1 - 13z + 30z^2}{1 - 13/30z + 1/30z^2} \frac{-1}{6} \epsilon_{2,t}$$

The polynomial $\tilde{\theta}_3^{\dagger}(z) = 1 - 13/30z + 1/30z^2$ *is determined by (30). Furthermore player 3's demand equation is given by*

$$(1 + \frac{1}{3}B + \frac{1}{2}B^2)D_{3,t} = d + (1 - \frac{13}{30}B + \frac{1}{30}B^2)\epsilon_{3,t}$$

Also, from [Brockwell and Davis, 1991], pp. 125-127, the variance of the shocks $\{\epsilon_{3,t}\}$ *in this case (no sharing) is given by*

$$\sigma^2_{\epsilon_3,NS} = \frac{1}{(\frac{1}{10})^2} \cdot \frac{1}{(\frac{1}{3})^2} \cdot (\frac{1}{6})^2 \cdot \sigma^2_{\epsilon_2} = 900 \frac{1}{36} \sigma^2_{\epsilon_2} \tag{32}$$

4.3 FIS Under Demand Sharing

If player $k-1$ shares its demand with player k, player k's information set is

$$\mathcal{M}_t^k = \mathcal{M}_t^{D_k, D_{k-1}}$$

Therefore player k's FIS $\{\epsilon_{k,t}\}$ must satisfy $\mathcal{M}_t^{\epsilon_k} = \mathcal{M}^{D_k, D_{k-1}}$. Before we can state player k's full information shocks under demand sharing we need to develop the following crucial Lemma:

Lemma 1. *Suppose we can represent two sequences* $\{X_{1,t}\}$ *and* $\{X_{2,t}\}$ *in terms of a zero-mean stationary process* $\{\eta_t\}$ *as*

$$\phi(B)X_{1,t} = d + B^{J_1}\Theta_1(B)\eta_t \tag{33}$$

$$\phi(B)X_{2,t} = d + B^{J_2}\Theta_2(B)\lambda\eta_t \tag{34}$$

where $\phi(z)$ *has no roots inside the unit circle,* $\Theta_1(z)$ *and* $\Theta_2(z)$ *have a leading coefficient of 1 and no roots at zero, and* λ *is a non-zero constant.*

There exist functions $\vartheta(z)$ *and* $\omega(z)$ *with one sided Laurent series representations converging in a disk* $\mathcal{D}$ *that contains the unit circle such that* $\vartheta(B)\phi(B)X_{1,t} + \omega(B)\phi(B)X_{2,t} = \vartheta(1)d + \omega(1)d + \eta_t$ *if and only if the polynomials* $z^{J_1}\Theta_1(z)$ *and* $z^{J_2}\Theta_2(z)$ *have no common common roots inside or on the unit circle.*

Lemma 1 states that we can write η_t as a linear combination of present and past values of $X_{1,t}$ and $X_{2,t}$ if and only if $z^{J_1}\Theta_1(z)$ and $z^{J_2}\Theta_2(z)$ have no common common roots inside or on the unit circle. This concept will play a major role when searching for player k's FIS when there is knowledge of both $\{D_{k,t}\}$ and $\{D_{k-1,t}\}$. The importance of this lemma will be apparent when proving the following theorem, which establishes player k's FIS under demand sharing.

Theorem 2. *Suppose that player $k-1$ shares its demand series $\{D_{k-1,t}\}$ with player k*

(i) If $z^{\tilde{J}_k}\tilde{\theta}_k(z)$ and $z^{J_{k-1}}\theta_{k-1}(z)$ have no common roots inside the unit circle, then

- $\{\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}\}$ *are player k's FIS.*
- $J_k = \tilde{J}_k$ *and* $\theta_k(z) = \tilde{\theta}_k(z)$

(ii) If $z^{\tilde{J}_k}\tilde{\theta}_k(z)$ has at least one root inside the unit circle in common with $z^{J_{k-1}}\theta_{k-1}(z)$, then

- $\{\frac{\tilde{\theta}_k^{I-C}(B)}{\tilde{\theta}_k^{\dagger I-C}(B)}B^{min(\tilde{J}_k,J_{k-1})}\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}\}$ *are player k's FIS*
- $J_k = \tilde{J}_k - min(\tilde{J}_k, J_{k-1})$ *and* $\theta_k(z) = \tilde{\theta}_k^{OUT}(z)\tilde{\theta}_k^{ON}(z)\tilde{\theta}_k^{I-NC}(z)\tilde{\theta}_k^{\dagger I-C}(z)$

The polynomial $\tilde{\theta}_k^{\dagger I-C}(z)$ is defined using (28) and (30). Theorem 2 implies that if player $k-1$ shares its demand, player k can recover player $k-1$'s full information shocks if and only if $z^{\tilde{J}_k}\tilde{\theta}_k(z)$ and $z^{J_{k-1}}\theta_{k-1}(z)$ have no common roots inside the unit circle. One can see this by considering

$$\epsilon_{k,t} = \frac{\tilde{\theta}_k^{I-C}(B)}{\tilde{\theta}_k^{\dagger I-C}(B)}B^{min(\tilde{J}_k,J_{k-1})}\epsilon_{k-1,t}$$

which we can rewrite as

$$\epsilon_{k-1,t} = \frac{\tilde{\theta}_k^{\dagger I-C}(B)}{\tilde{\theta}_k^{I-C}(B)}B^{-min(\tilde{J}_k,J_{k-1})}\epsilon_{k,t}$$

If $z^{\tilde{J}_k}\tilde{\theta}_k(z)$ has at least one root inside the unit circle in common with $z^{J_{k-1}}\theta_{k-1}(z)$ then at least one of the following must be true:

- $min(\tilde{J}_k, J_{k-1}) > 0$

- $\dfrac{1}{\tilde{\theta}_k^{I-C}(z)}$ does not have a one-sided Laurent Series representation for $z \in \mathcal{D}$ where the disk $\mathcal{D}$ contains the unit circle

This means that it is impossible to write $\epsilon_{k-1,t}$ as a linear combination of present and past values of $\{\epsilon_{k,t}\}$ and $\mathcal{M}_t^{D_k,D_{k-1}} \neq \mathcal{M}_t^{\epsilon_{k-1}}$. We will compare the full information shocks we see here and those obtained when there is no sharing or full information shock sharing later in Section 6. For now we continue Example 2 for the case that the supplier shares its demand with player 3.

Example 2. Part IV

Recall that we previously determined that the supplier's order equation is given by:

$$(1 + \frac{1}{3}B + \frac{1}{2}B^2)D_{3,t} = d + (1 - 13B + 30B^2)\frac{-1}{6}\epsilon_{2,t}$$

where $\tilde{\theta}_3(z) = 1 - 13z + 30z^2$ *has roots 1/10 and 1/3, which are inside the unit circle. From before,* $\theta_2(z) = 1 - 32/3z + 20/3z^2$ *has roots 1/10 and 3/2. Therefore* $\tilde{\theta}_3(z)$ *has a root inside the unit circle in common with* $\theta_2(z)$*. Here we assume that the supplier shares its FIS with player 3. By Theorem 2(ii) player 3's full information shocks are*

$$\epsilon_{3,t} = \frac{1 - 10z}{1 - 1/10z}(-1/6)\epsilon_{2,t}$$

The polynomial $\tilde{\theta}_3^{I-C}(z) = 1 - 10z$ *is found from (28) and* $\tilde{\theta}_3^{\dagger I-C}(z) = 1 - 1/10z$ *is found using (30). Furthermore by Theorem 2(ii) we also have that* $J_3 = 0$ *and* $\theta_3(z) = 1 \cdot \tilde{\theta}_3^{I-NC}(z) \cdot \tilde{\theta}_3^{\dagger I-C} = (1 - 3z)(1 - 1/10z)$*. Therefore player 3's demand equation is given by*

$$(1 + \frac{1}{3}B + \frac{1}{2}B^2)D_{3,t} = d + (1 - \frac{31}{10}z + \frac{3}{10}z^2)\epsilon_{3,t}$$

Also, following [Brockwell and Davis, 1991], pp. 125-127, we have that the variance of the shocks $\{\epsilon_{3,t}\}$ *under demand sharing is*

$$\sigma^2_{\epsilon_3,DS} = \frac{1}{(\frac{1}{10})^2}(1/6)^2\sigma^2_{\epsilon_2} = 100\frac{1}{36}\sigma^2_{\epsilon_2} \qquad (35)$$

4.4 FIS Under Full Information Shock Sharing

We close this section by describing player k's full information shocks when player $k-1$ shares its shocks. In this case player k's information set is $\mathcal{M}_t^k = \mathcal{M}_t^{D_k,\epsilon_{k-1}}$. Recall that $\mathcal{M}_t^{D_k,\epsilon_{k-1}} =$

$\overline{sp}\{1, D_{k,t}, \epsilon_{k-1,t}, D_{k,t-1}, \epsilon_{k-1,t-1}, \ldots\}$. If we consider player $k-1$'s order equation

$$\phi(B)D_{k,t} = d + B^{\tilde{J}_k}\tilde{\theta}_k(B)\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}$$

and note that $\phi(z)$ has no roots inside the unit circle, we can use the same reasoning as in the proofs of the previous propositions to conclude that $D_{k,t}$ can be written as the linear combination of present and past $\{\epsilon_{k-1,t}\}$. This means that the space $\overline{sp}\{1, D_{k,t}, \epsilon_{k-1,t}, D_{k,t-1}, \epsilon_{k-1,t-1}, \ldots\}$ is the same as the space $\overline{sp}\{1, \epsilon_{k-1,t}, \epsilon_{k-1,t-1}, \ldots\}$. Therefore $\mathcal{M}_t^k = \mathcal{M}_t^{D_k,\epsilon_{k-1}} = \mathcal{M}_t^{\epsilon_{k-1}}$.

Consider $\gamma_{k-1} = \lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}$. It is readily seen that $\mathcal{M}_t^{\gamma_{k-1}} = \mathcal{M}_t^{\epsilon_{k-1}}$ and therefore $\mathcal{M}_t^k = \mathcal{M}_t^{\gamma_{k-1}}$. Since the order equation above shows that we can represent $\{D_{k,t}\}$ as QUARMA with repsect to $\{\gamma_{k-1,t}\}$ we conclude that $\{\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}\}$ are player k's FIS. Furthemore $J_k = \tilde{J}_k$ and $\theta_k(z) = \tilde{\theta}_k(z)$. The following proposition restates this result.

Proposition 5. *If player $k-1$ shares its shocks,*

- $\{\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}\}$ *are player k's FIS*
- $J_k = \tilde{J}_k$ *and* $\theta_k(z) = \tilde{\theta}_k(z)$

Example 2. Part V

Recall that we previously determined that the supplier's order equation is given by:

$$(1 + \frac{1}{3}B + \frac{1}{2}B^2)D_{3,t} = d + (1 - 13B + 30B^2)\frac{-1}{6}\epsilon_{2,t}$$

where $\tilde{\theta}_3(z) = 1 - 13z + 30z^2$ has roots 1/10 and 1/3, which are inside the unit circle. If the supplier shares its shocks, we can use Proposition 5 to conclude that player 3's full information shocks are

$$\epsilon_{3,t} = (-1/6)\epsilon_{2,t}$$

and that player 3's demand equation is

$$(1 + \frac{1}{3}B + \frac{1}{2}B^2)D_{3,t} = d + (1 - 13B + 30B^2)\epsilon_{3,t}$$

The variance of $\epsilon_{3,t}$ in this case is given by

$$\sigma^2_{\epsilon_3,SS} = \frac{1}{36}\sigma^2_{\epsilon_2} \tag{36}$$

We have now recovered player k's full information shocks and described player k's demand equation for every conceivable scenario when player $k-1$'s demand equation is given by (14). Having described player k's demand equation it is possible to continue the propagation forward in the same way to player $k+1$. The following section summarizes the results in the last three subsections and uses them to prove that demand does indeed propagate as QUARMA-in QUARMA-out.

5 QUARMA-in-QUARMA-out

In the previous section we found player k's FIS under no sharing, demand sharing, and full information shock sharing when player $k-1$'s demand follows demand equation (14). In this section we will use those results to show that indeed player $k-1$'s demand can be modeled as such. We summarize player k's FIS from the previous section in the following table:

	No Sharing	Demand Sharing	Full Information Shock Sharing
Scenario 1	$\{\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}\}$	$\{\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}\}$	$\{\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}\}$
Scenario 2	$\{\frac{\tilde{\theta}_k(B)}{\tilde{\theta}_k^{\dagger}(B)}B^{\tilde{J}_k}\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}\}$	$\{\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}\}$	$\{\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}\}$
Scenario 3	$\{\frac{\tilde{\theta}_k(B)}{\tilde{\theta}_k^{\dagger}(B)}B^{\tilde{J}_k}\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}\}$	$\{\frac{\tilde{\theta}_k^{I-C}(B)}{\tilde{\theta}_k^{\dagger I-C}(B)}B^{min(\tilde{J}_k,J_{k-1})}\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}\}$	$\{\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}\}$

Table 1: FIS under various conditions and sharing arrangements.

The scenarios described in Table 1 are as follows:

- Scenario 1:= $z^{J_{k-1}}\theta_{k-1}(z)$ has no roots inside the unit circle
- Scenario 2:= $z^{\tilde{J}_k}\tilde{\theta}_k(z)$ has no roots in common with $z^{J_{k-1}}\theta_{k-1}(z)$ inside the unit circle
- Scenario 3:= $z^{\tilde{J}_k}\tilde{\theta}_k(z)$ has a root in common with $z^{J_{k-1}}\theta_{k-1}(z)$ inside the unit circle

Note that there are three unique forms for the full information shocks in the above table. We will refer to this table when proving the central theorem of this section, stated here.

Theorem 3. *Suppose the retailer observes causal and invertible ARMA demand*

$$\phi(B)D_{1,t} = d + \theta_1(B)\epsilon_{1,t}$$

and that any player can share nothing, its demand, or its full information shocks with an adjacent upstream player. Then for any $k \geq 1$ we can express player k's demand as QUARMA with respect to player k's full information shocks:

$$\phi(B)D_{k,t} = d + B^{J_k}\theta_k(B)\epsilon_{k,t} \tag{37}$$

where $\theta_k(z)$ has a leading coefficient 1 and no roots at zero.

The only assumption of the above theorem is that the retailer observes causal and invertible ARMA demand. As we have done throughout this paper, we assume that players can share nothing, demand, or full information shocks with adjacent upstream players. The conclusion of the theorem states that demand will propagate as QUARMA throughout the supply chain.

Proof of Theorem 3. The proof follows by induction. It is true for $k = 1$ by assumption since we can take $J_1 = 0$. Assume that (37) holds for $k \geq 1$. We need to show that we can find J_{k+1} and $\theta_{k+1}(z)$ such that we can express player $k+1$'s demand as

$$\phi(B)D_{k+1,t} = d + B^{J_{k+1}}\theta_{k+1}(B)\epsilon_{k+1,t}$$

As given by Equation (20) in Section 3, with $k+1$ and k replacing k and $k-1$, we can write $D_{k+1,t}$ as

$$\phi(B)D_{k+1,t} = d + B^{\tilde{J}_{k+1}}\tilde{\theta}_{k+1}(B)\lambda_{k+1,\tilde{J}_{k+1}}\epsilon_{k,t} \tag{38}$$

In accordance with Table 1, player $k+1$'s full information shocks are one of the following:

$$\begin{aligned} (i)\ \epsilon_{k+1,t} &= \lambda_{k+1,\tilde{J}_{k+1}}\epsilon_{k,t} \\ (ii)\ \epsilon_{k+1,t} &= \frac{\tilde{\theta}_{k+1}(B)}{\tilde{\theta}^{\dagger}_{k+1}(B)}B^{\tilde{J}_{k+1}}\lambda_{k+1,\tilde{J}_{k+1}}\epsilon_{k,t} \\ (iii)\ \epsilon_{k+1,t} &= \frac{\tilde{\theta}^{I-C}_{k+1}(B)}{\tilde{\theta}^{\dagger I-C}_{k+1}(B)}B^{min(\tilde{J}_{k+1},J_k)}\lambda_{k+1,\tilde{J}_{k+1}}\epsilon_{k,t} \end{aligned}$$

For (i), we would write (38) as

$$\phi(B)D_{k+1,t} = d + B^{\tilde{J}_{k+1}}\tilde{\theta}_{k+1}(B)\lambda_{k+1,\tilde{J}_{k+1}}\epsilon_{k+1,t}$$

Here we can take $J_{k+1} = \tilde{J}_{k+1}$ and $\theta_{k+1}(z) = \tilde{\theta}_{k+1}(z)$ to get the required equation

$$\phi(B)D_{k+1,t} = d + B^{J_{k+1}}\theta_{k+1}(B)\epsilon_{k+1,t}$$

For (ii), we would write (38) as

$$\phi(B)D_{k+1,t} = d + B^{\tilde{J}_{k+1}}\tilde{\theta}_{k+1}(B)\frac{\tilde{\theta}^{\dagger}_{k+1}(B)}{\tilde{\theta}_{k+1}(B)}B^{-\tilde{J}_{k+1}}\epsilon_{k+1,t}$$

which simplifies as

$$\phi(B)D_{k+1,t} = d + \tilde{\theta}^{\dagger}_{k+1}(B)\epsilon_{k+1,t}$$

We would take $J_{k+1} = 0$ and $\theta_{k+1}(z) = \tilde{\theta}^{\dagger}_{k+1}(z)$ to get the required equation.

Finally, for (iii), we would write (38) as

$$\phi(B)D_{k+1,t} = d + B^{\tilde{J}_{k+1}}\tilde{\theta}_{k+1}(B)\frac{\tilde{\theta}^{\dagger I-C}_{k+1}(B)}{\tilde{\theta}^{I-C}_{k+1}(B)}B^{-min(\tilde{J}_{k+1},J_k)}\epsilon_{k+1,t}$$

which simplifies as

$$\phi(B)D_{k+1,t} = d + B^{\tilde{J}_{k+1}-min(\tilde{J}_{k+1},J_k)}\tilde{\theta}^{OUT}_{k+1}(B)\tilde{\theta}^{I-NC}_{k+1}(B)\tilde{\theta}^{\dagger I-C}_{k+1}(B)\epsilon_{k+1,t} \tag{39}$$

Here we can take $J_{k+1} = \tilde{J}_{k+1} - min(\tilde{J}_{k+1}, J_k)$ and $\theta_{k+1}(z) = \tilde{\theta}^{OUT}_{k+1}(z)\tilde{\theta}^{I-NC}_{k+1}(z)\tilde{\theta}^{\dagger I-C}_{k+1}(z)$

Thus we have found a suitable J_{k+1} and θ_{k+1} in every case and induction is proved. $\square$

6 Comparison of Various Sharing Arrangements

We present here a discussion of the value of demand sharing within a supply chain in contrast to full information shock sharing and no sharing. We do this by studying the best linear forecast of lead-time demand for the three sharing arrangements.

Given player k's demand equation and Lemma 1 of GHS, the best linear forecast of player $k's$ leadtime demand is given by $m_{k,t} = \sum_{i=\ell_k+1}^{\infty}\omega_{k,i}\epsilon_{k,t+\ell_k+1-i} + (\ell_k+1)\mu_d = \sum_{i=0}^{\infty}\omega_{k,i+\ell_k+1}\epsilon_{k,t-i} +$

$(\ell_k + 1)\mu_d$ with an associated Mean Squared Forecast Error $MSFE_k = \sigma^2_{\epsilon_k} \sum_{i=0}^{\ell_k} \omega^2_{k,i}$, where $\sigma^2_{\epsilon_k} = Var(\epsilon_{k,t})$ and $\omega_{k,i}$ are given by

$$\omega_{k,i} = \begin{cases} 0 & i < 0 \\ \psi_{k,i} & i = 0 \\ \omega_{k,i-1} + \psi_{k,i} & 0 < i < \ell_k + 1 \\ \omega_{k,i-1} + \psi_{k,i} - \psi_{k,i-\ell_k-1} & i \geq \ell_k + 1 \end{cases} \tag{40}$$

where $\psi_{k,j}$ is the j^{th} coefficient in the $MA(\infty)$ representation of player k's demand with respect to its FIS. From this it is clear that player k's $MSFE_k$ is related to the variance of its full information shocks, $\sigma^2_{\epsilon_k}$. The following proposition states the variance of player k's full information shocks under the three arrangements of no sharing ($\sigma^2_{\epsilon_k,NS}$), demand sharing ($\sigma^2_{\epsilon_k,DS}$) and shock sharing ($\sigma^2_{\epsilon_k,SS}$).

Proposition 6. *Below are the variances of player k's full information shocks when player $k-1$'s shares its shocks, its demand, or nothing with player k.*

- *(i)* $\sigma^2_{\epsilon_k,SS} = \lambda^2_{k,\tilde{J}_k} \sigma^2_{\epsilon_{k-1}}$
- *(ii)* $\sigma^2_{\epsilon_k,DS} = \prod_{j:|z_j|<1} |z_j|^{-2\cdot min(m(z_j,\tilde{\theta}_k),m(z_j,\theta_{k-1}))} \lambda^2_{k,\tilde{J}_k} \sigma^2_{\epsilon_{k-1}}$
- *(iii)* $\sigma^2_{\epsilon_k,NS} = \prod_{j:|z_j|<1} |z_j|^{-2\cdot m(z_j,\tilde{\theta}_k)} \lambda^2_{k,\tilde{J}_k} \sigma^2_{\epsilon_{k-1}}$

This Proposition follows immediately from the form of player k's FIS under the three sharing arrangements and [Brockwell and Davis, 1991], pp. 125-127. The expressions for $\sigma^2_{\epsilon_k,DS}$ and $\sigma 2_{\epsilon_k,NS}$ are not necessarily in simplest form. For example if $min(m(z_j,\tilde{\theta}_k), m(z_j,\theta_{k-1})) = 0$ for all roots z_j of $\tilde{\theta}_k$ with $|z_j| < 1$ then (ii) would become $\sigma^2_{\epsilon_k,DS} = \lambda^2_{k,\tilde{J}_k} \sigma^2_{\epsilon_{k-1}}$.

The following theorem illustrates the relationship of the variances given in Proposition 6. Note that we are still considering all the assumptions in Section 2.2.

Theorem 4. *Suppose the retailer observes causal and invertible ARMA demand. For any* $k > 1$,
(i) Suppose $\tilde{\theta}_k(z)$ *has at least one root in common with* $\theta_{k-1}(z)$ *inside the unit circle. Suppose further that there is a root* z_j *of* $\tilde{\theta}_k(z)$ *such that* $|z_j| < 1$ *and* $m(z_j, \tilde{\theta}_k) > m(z_j, \theta_{k-1})$*. Then* $\sigma^2_{\epsilon_k,SS} < \sigma^2_{\epsilon_k,DS} < \sigma^2_{\epsilon_k,NS}$.
(ii) Suppose $\tilde{\theta}_k(z)$ *has at least one root in common with* $\theta_{k-1}(z)$ *inside the unit circle. Suppose further that any root* z_j *of* $\tilde{\theta}_k(z)$ *where* $|z_j| < 1$ *is such that* $m(z_j, \tilde{\theta}_k) \leq m(z_j, \theta_{k-1})$*. Then* $\sigma^2_{\epsilon_k,SS} < \sigma^2_{\epsilon_k,DS} = \sigma^2_{\epsilon_k,NS}$.
(iii) Suppose $\tilde{\theta}_k(z)$ *has no roots in common with* $\theta_{k-1}(z)$ *inside the unit circle.*

(a) If $\tilde{\theta}_k(z)$ *has a root inside the unit circle, then* $\sigma^2_{\epsilon_k,SS} = \sigma^2_{\epsilon_k,DS} < \sigma^2_{\epsilon_k,NS}$

(b) If $\tilde{\theta}_k(z)$ *has no roots inside the unit circle, then* $\sigma^2_{\epsilon_k,SS} = \sigma^2_{\epsilon_k,DS} = \sigma^2_{\epsilon_k,NS}$

Cases (i) and (ii) exhaust the event that $\tilde{\theta}_k(z)$ and $\theta_{k-1}(z)$ have at least one common root inside the unit circle. Case (iii) considers what happens when $\tilde{\theta}_k(z)$ and $\theta_{k-1}(z)$ have no common roots inside the unit circle. It should be further noted that any roots of $\tilde{\theta}_k(z)$ outside or on the unit circle have no impact on the variance of the full information shocks.

Example 2. Part VI

Recall the variance of player 3's full information shocks under the three different sharing arrangements given by (32) (35) and (36):

$$\begin{aligned}
\sigma^2_{\epsilon_3,NS} &= 900\frac{1}{36}\sigma^2_{\epsilon_2} \\
\sigma^2_{\epsilon_3,DS} &= 100\frac{1}{36}\sigma^2_{\epsilon_2} \\
\sigma^2_{\epsilon_3,SS} &= \frac{1}{36}\sigma^2_{\epsilon_2}
\end{aligned}$$

We see that indeed

$$\sigma^2_{\epsilon_3,SS} < \sigma^2_{\epsilon_3,DS} < \sigma^2_{\epsilon_3,NS}. \tag{41}$$

Furthermore the differences are stark. The polynomial $\tilde{\theta}_3(z)$ *has a root, 1/10, in common with* $\theta_2(z)$ *inside the unit circle. Furthermore,* $\tilde{\theta}_3(z)$ *has a root, 1/3, inside the unit circle, such that* $1 = m(1/3, \tilde{\theta}_3) > m(1/3, \theta_2) = 0$*. Therefore the conditions of (i) hold and thus (41) is to be expected.*

Theorem 4 also leads to the following useful Corollaries:

Corollary 2. *For any player k, the full information shocks $\{\epsilon_k\}$ will be such that $\sigma^2_{\epsilon_k,SS} \leq \sigma^2_{\epsilon_k,DS} \leq \sigma^2_{\epsilon_k,NS}$.*

The proof follows immediately from Theorem 4.

Corollary 3. *For any $k > 1$ where $\ell_k = 0$ and $\tilde{J}_k = 0$, $\tilde{\theta}_k(z)$ has at least one root in common with $\theta_{k-1}(z)$ inside the unit circle and there is a root z_j of $\tilde{\theta}_k(z)$ such that $|z_j| < 1$ and $m(z_j, \tilde{\theta}_k) > m(z_j, \theta_{k-1})$* **if and only if**

$$MSFE_{k,SS} < MSFE_{k,DS} < MSFE_{k,NS}$$

This corollary only applies when player k has to forecast one step ahead. In this case $MSFE_k = \sigma^2_{\epsilon_k}$ and the corollary follows immediately from Theorem 4.

Next we consider Example 2 and explore player 3's MSFE for the three sharing arrangements given various lead-times.

Example 2. Part VII

Recall that player 3's MSFE is given by $MSFE_3 = \sigma^2_{\epsilon_3} \sum_{i=0}^{\ell_3} \omega^2_{3,i}$ where $\omega_{3,i}$'s are given by (40). From Example 2 we have the variance of player 3's full information shocks under the three different sharing arrangements given by (32), (35), and (36):

$$\begin{aligned}
\sigma^2_{\epsilon_3,NS} &= 900\frac{1}{36}\sigma^2_{\epsilon_2} \\
\sigma^2_{\epsilon_3,DS} &= 100\frac{1}{36}\sigma^2_{\epsilon_2} \\
\sigma^2_{\epsilon_3,SS} &= \frac{1}{36}\sigma^2_{\epsilon_2}
\end{aligned}$$

We can use (40) and player 3's demand equations under the three sharing scenarios to find $\omega_{3,i}$ for $i = 1, ..., \ell_k$ for any lead-time ℓ_k. We can hence compute the ratios of the MSFEs that arise given the different sharing arrangements to the MSFE that arises when nothing is shared, which is displayed in Figure 1 below.

We see that the MSFE that arises under demand sharing is strictly between the other two. We continue with an example of intermediate value to demand for $k > 1$ in which $\tilde{J}_k > J_{k-1} > 0$

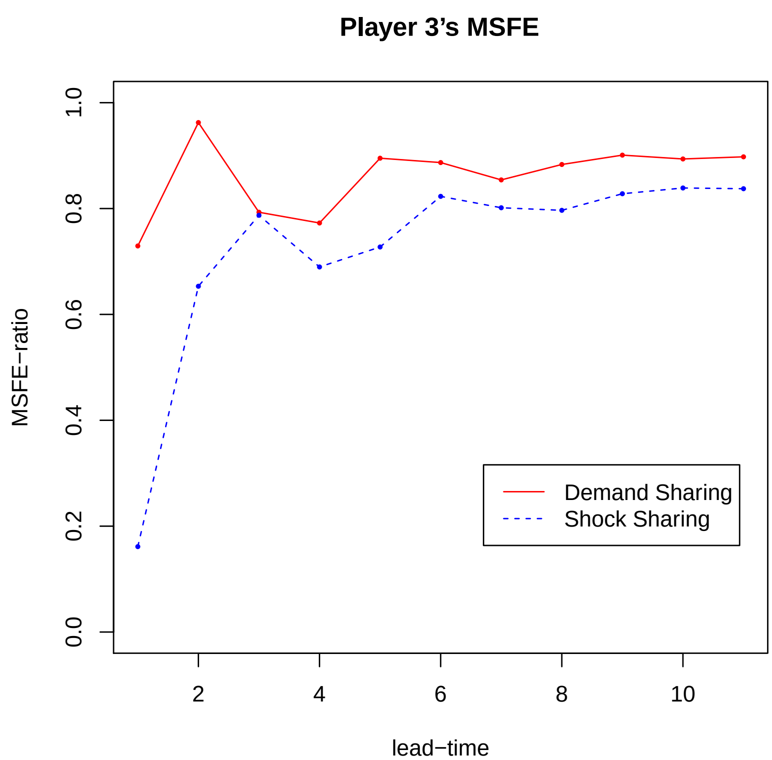

Figure 1: The ratios of player 3's MSFE under demand sharing and shock sharing to no sharing for the model given in Example 2

Example 3. *Suppose the retailer has lead time $\ell_1 = 0$, shares its full information shocks with the supplier and observes the following ARMA(4,4) model of its demand:*

$$(1 + .5B - .2B^2 - .4B^3 + .4B^4)D_{1,t} = d + (1 - .5B + .3B^2 - .7B^3 + .1B^4)\epsilon_{1,t} \tag{42}$$

Using Theorem 1 we can compute

$$\begin{aligned}
\lambda_2(z) &= (1 - .5z + .3z^2 - .7z^3 + .1z^4) + (z^{-1} - 1)((1 - .5z + .3z^2 - .7z^3 + .1z^4) \\
&\quad - (1 + .5z - .2z^2 - .4z^3 + .4z^4)) \\
&= (1 - .5z + .3z^2 - .7z^3 + .1z^4) + (z^{-1} - 1)(-z + .5z^2 - .3z^3 - .3z^4) \\
&= 1 - .5z + .3z^2 - .7z^3 + .1z^4 + -1 + .5z - .3z^2 - .3z^3 + z - .5z^2 + .3z^3 + .3z^4 \\
&= z - .5z^2 - .7z^3 + .4z^4
\end{aligned}$$

Thus we have $\lambda_2(z) = z(1 - .5z - .7z^2 + .4z^3)$. *From Proposition 1 we have that* $\tilde{\theta}_2(z) = (1 - .5z - .7z^2 + .4z^3)$ *and* $\tilde{J}_2 = 1$. *Furthermore since the retailer shared its shocks, Proposition 5 tells us that* $\theta_2(z) = \tilde{\theta}_2(z)$ *and* $J_2 = 1$ *so the supplier observes the following QUARMA(4,1,3) model:*

$$(1 + .5B - .2B^2 - .4B^3 + .4B^4)D_{2,t} = d + B(1 - .5B - .7B^2 + .4B^3)\epsilon_{2,t} \tag{43}$$

The roots of $\theta_2(z)$ *are* $1.4575 + 0.147i, 1.4575 - 0.147i, and - 1.165$, *which are all outside the unit circle. Assuming that the Supplier has a leadtime* $\ell_2 = 1$ *and continuing the propagation using Theorem 1 we get,*

$$\begin{aligned}
\lambda_3(z) &= (1 + .5z - .2z^2 - .4z^3 + .4z^4) + z^{-1}(1 - .5z - .7z^2 + .4z^3 \\
&\quad - (1 + .5z - .2z^2 - .4z^3 + .4z^4)) \\
&= (1 + .5z - .2z^2 - .4z^3 + .4z^4) + z^{-1}(-z - .5z^2 + .8z^3 - .4z^4) \\
&= 1 + .5z - .2z^2 - .4z^3 + .4z^4 - 1 - .5z + .8z^2 - .4z^3 \\
&= .6z^2 - .8z^3 + .4z^4
\end{aligned}$$

Thus we have that $\lambda_3(z) = z^2(.6 - .8z + .4z^2)$. *Again by Proposition 1 we get* $\tilde{\theta}_3(z) = 1 - \frac{4}{3}z + \frac{2}{3}z^2$ *and* $\tilde{J}_3 = 2$. *Player 3's demand model with respect to player 2's full information shocks would be*

$$(1 + .5B - .2B^2 - .4B^3 + .4B^4)D_{3,t} = d + B^2(1 - \frac{4}{3}B + \frac{2}{3}B^2)(\frac{3}{5})\epsilon_{2,t} \tag{44}$$

Note that $\tilde{\theta}_3(z)$ *has roots* $1 + 0.707i$ *and* $1 - 0.707i$ *which are outside the unit circle, but* $\tilde{J}_3 > J_2$. *This will be central to the intermediate value of demand sharing in this case. For the three sharing arrangements we have player 3's FIS and demand equation given by*

NS:

$$\epsilon_{3,t} = \frac{3}{5}B^2\epsilon_{2,t}$$

and

$$(1 + .5B - .2B^2 - .4B^3 + .4B^4)D_{3,t} = d + (1 - \frac{4}{3}B + \frac{2}{3}B^2)\epsilon_{3,t}$$

DS:

$$\epsilon_{3,t} = \frac{3}{5}B\epsilon_{2,t}$$

and

$$(1 + .5B - .2B^2 - .4B^3 + .4B^4)D_{3,t} = d + B(1 - \frac{4}{3}B + \frac{2}{3}B^2)\epsilon_{3,t}$$

SS:

$$\epsilon_{3,t} = \frac{3}{5}\epsilon_{2,t}$$

and

$$(1 + .5B - .2B^2 - .4B^3 + .4B^4)D_{3,t} = d + B^2(1 - \frac{4}{3}B + \frac{2}{3}B^2)\epsilon_{3,t}$$

Note that the variance of player 3's full information shocks ($\sigma^2_{\epsilon_3} = 9/25\sigma^2_{\epsilon_2}$) is the same for all three sharing arrangements. Furthermore $\theta_3(z)$ is also the same. The only difference is in J_3. We can compute $\omega^2_{3,i}$ using (40) for $i = 0, ..., \ell_k$ for any $\ell_k \geq 0$. Thereby we obtain the ratio of the MSFEs resulting from the three different sharing arrangements to the MSFE that arises when nothing is shared given in Figure 2 below for lead-times 1,..., 11.

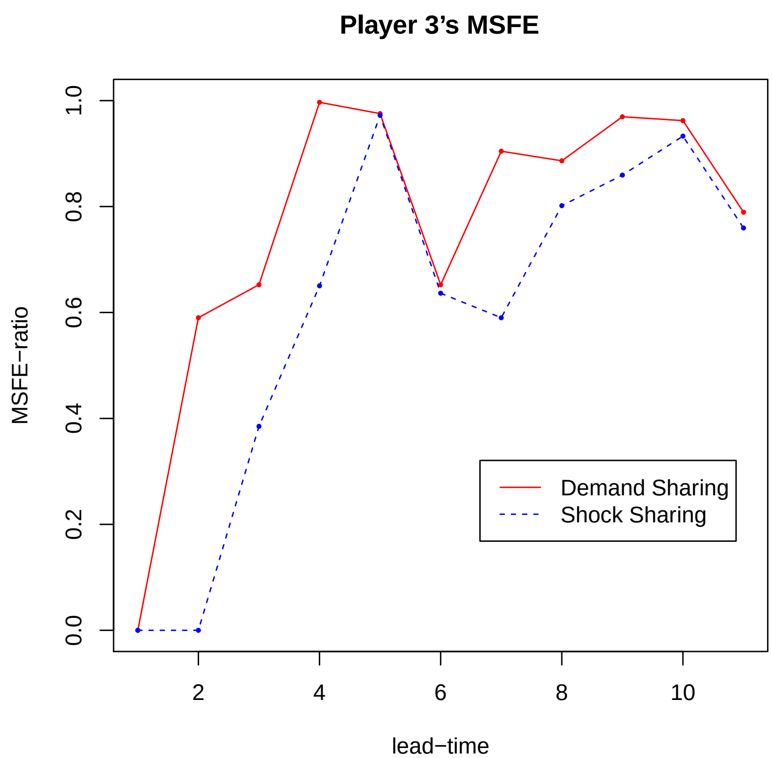

Figure 2: The ratios of player 3's MSFE under demand sharing and shock sharing to no sharing for the model given in Example 3

If the supplier shares its demand, player 3 would have a perfect forecast when forecasting 1-step ahead. If the supplier shares its FIS, player 3 would have a perfect forecast when forecasting 1 *or 2-steps ahead. Furthermore the resulting MSFE when demand is shared is strictly between the MSFE when nothing is shared and when shocks are shared for the lead-times considered.*

At time t, with no knowledge of the supplier's demand, player 3 has no way to recover $\epsilon_{2,t}$ or $\epsilon_{2,t-1}$. However if the supplier shares its demand then player 3 can recover $\epsilon_{2,t-1}$. Furthermore, if the supplier shares its full information shock series then player 3 would know both $\epsilon_{2,t}$ and $\epsilon_{2,t-1}$.

Having completed Examples 2 and 3, we see that demand sharing between player $k-1$ and k can be intermediate in value in the case when $\tilde{J}_k = 0$ as well as in the case of strict-QUARMA ($\tilde{J}_k > 0$). A discussion on how to find such examples is provided in the Appendix.

7 Conclusions and Direction for future research

The major contribution of this paper is that we extended the existing literature by assuming that there may be one of three possible sharing arrangements between adjacent players:

- no information sharing,
- demand sharing, or
- full information shock sharing.

We demonstrated that the value provided by a demand sharing arrangement can be equivalent to no sharing, equivalent to full information shock sharing, or intermediate to no sharing and full information shock sharing. We further characterized when each of these three cases will occur under demand sharing. We also derive a player's full information set, its full information shocks, as well as its best linear forecast under demand sharing and show how demand propagates upstream.

We proved that demand propagates according to QUARMA-in-QUARMA-out in the presence of either no sharing, demand sharing, or full information shock sharing. We further showed that demand sharing provides intermediate value to player k when (i) the MA polynomials for player k-1's demand and order have at least one common root inside the unit circle, and (ii) the MA polynomial for player k-1's order has at least one additional root inside the unit circle.

Finally, we have provided a simpler methodology for the way in which demand propagates in a supply chain in the presence of no sharing or shock sharing. Based upon this approach, we have a convenient way of exploring other features of possibly large supply chains which we leave to future research.

8 Appendix

Proof of Theorem 1. Let $\psi_{k-1}(B) = \frac{\theta_{k-1}(B)}{\phi(B)} = \sum_{j=0}^{\infty} \psi_{k-1,j} B^j$ with $\psi_{k-1,0} = 1$. Such a representation exists because $\phi(z)$ is assumed to have all its roots outside the unit circle. We can then write

$$D_{k-1,t} = \frac{d}{\phi(1)} + B^{J_{k-1}} \psi_{k-1}(B) \epsilon_{k-1,t} \tag{45}$$

Note that $\frac{d}{\phi(1)}$ is indeed the correct constant term since we know that $E[\phi(B)D_{k-1,t}] = \phi(1)E[D_{k-1,t}]$. According to the order-up-to policy, $D_{k,t} = D_{k-1,t} + m_{k-1,t} - m_{k-1,t-1}$ where $m_{k-1,t}$ and $m_{k-1,t-1}$ are the best linear forecasts of leadtime demand at time t and $t-1$ respectively.

Because player $k-1$'s leadtime is ℓ_{k-1}, the player would have to forecast $D_{k-1,t+1}, ..., D_{k-1,t+\ell_{k-1}+1}$. Using (45) we have, for any nonnegative integer n,

$$D_{k-1,t+n} = \frac{d}{\phi(1)} + B^{J_{k-1}-n} \psi_{k-1}(B) \epsilon_{k-1,t}$$

And consequently, since $m_{k-1,t}$ is the best linear forecast of $\sum_{i=1}^{\ell_{k-1}+1} D_{k-1,t+i}$

$$m_{k-1,t} = \frac{d}{\phi(1)} + \sum_{n=1}^{\ell_{k-1}+1} B^{J_{k-1}-n} \sum_{j=max(0,n-J_{k-1})}^{\infty} B^j \psi_{k-1,j} \epsilon_{k-1,t} \tag{46}$$

If $\ell_{k-1} + 1 > J_{k-1}$ we can write (46) as

$$m_{k-1,t} = \frac{d}{\phi(1)} + \left\{ \sum_{n=1}^{J_{k-1}} B^{J_{k-1}-n} \psi_{k-1}(B) + \sum_{n=J_{k-1}+1}^{\ell_{k-1}+1} B^{J_{k-1}-n} \left[\psi_{k-1}(B) - \sum_{L=0}^{n-J_{k-1}-1} \psi_{k-1,L} B^L \right] \right\} \epsilon_{k-1,t} \tag{47}$$

If $\ell_{k-1} + 1 \leq J_{k-1}$ we can write (46) as

$$m_{k-1,t} = \frac{d}{\phi(1)} + \sum_{n=1}^{\ell_{k-1}+1} B^{J_{k-1}-n} \psi_{k-1}(B) \epsilon_{k-1,t} \tag{48}$$

Combining (47) and (48), and using the convention that if the upper limit is smaller than the lower limit of a sum, that sum is 0, we have that

$$m_{k-1,t} = \frac{d}{\phi(1)}+\left\{\sum_{n=1}^{min(J_{k-1},\ell_{k-1}+1)} B^{J_{k-1}-n}\psi_{k-1}(B)+\sum_{n=J_{k-1}+1}^{\ell_{k-1}+1} B^{J_{k-1}-n}\left[\psi_{k-1}(B)-\sum_{L=0}^{n-J_{k-1}-1}\psi_{k-1,L}B^L\right]\right\}\epsilon_{k-1,t} \tag{49}$$

or equivalently,

$$m_{k-1,t} = \frac{d}{\phi(1)}+\left\{\sum_{n=max(0,J_{k-1}-(\ell_{k-1}+1))}^{J_{k-1}-1} B^{n}\psi_{k-1}(B)+\sum_{n=1}^{\ell_{k-1}+1-J_{k-1}} B^{-n}\left[\psi_{k-1}(B)-\sum_{L=0}^{n-1}\psi_{k-1,L}B^L\right]\right\}\epsilon_{k-1,t}. \tag{50}$$

Using the backshift operator, the order-up-to policy dictates that $D_{k,t} = D_{k-1,t}+(1-B)m_{k-1,t}$. By (50) and (18) we have that

$$\begin{aligned} \phi(B)D_{k,t} = d \quad &+ \quad \Bigg\{B^{J_{k-1}}\theta_{k-1}(B) \quad + \quad (1-B)\sum_{j=max(0,J_{k-1}-(\ell_{k-1}+1))}^{J_{k-1}-1} B^j\theta_{k-1}(B) \\ &+ \quad (1-B)\sum_{j=1}^{\ell_{k-1}+1-J_{k-1}} B^{-j}\left[\theta_{k-1}(B)-\phi(B)\sum_{L=0}^{j-1}\psi_{k-1,L}B^L\right]\Bigg\}\epsilon_{k-1,t} \end{aligned} \tag{51}$$

Note that $(1-B)\sum_{j=max(0,J_{k-1}-(\ell_{k-1}+1))}^{J_{k-1}-1} B^j\theta_{k-1}(B)$

$$\begin{aligned} &= \sum_{j=max(0,J_{k-1}-(\ell_{k-1}+1))}^{J_{k-1}-1} B^j\theta_{k-1}(B)-\sum_{j=max(1,J_{k-1}-(\ell_{k-1}))}^{J_{k-1}} B^j\theta_{k-1}(B) \\ &= \mathbf{1}_{\{J_{k-1}>0\}}[B^{max(0,J_{k-1}-(\ell_{k-1}+1))}-B^{J_{k-1}}] \end{aligned} \tag{52}$$

Furthermore

$$\begin{aligned} (1-B)\sum_{j=1}^{\ell_{k-1}+1-J_{k-1}} B^{-j}\theta_{k-1}(B) &= \sum_{j=1}^{\ell_{k-1}+1-J_{k-1}} B^{-j}\theta_{k-1}(B)-\sum_{j=0}^{\ell_{k-1}-J_{k-1}} B^{-j}\theta_{k-1}(B) \\ &= \mathbf{1}_{\{\ell_{k-1}\geq J_{k-1}\}}[B^{J_{k-1}-(\ell_{k-1}+1)}-1]\theta_{k-1}(B) \end{aligned} \tag{53}$$

and likewise,

$$
\begin{aligned}
&(1-B)\sum_{j=1}^{\ell_{k-1}+1-J_{k-1}} B^{-j}\sum_{L=0}^{j-1}\psi_{k-1,L}B^L \\
&= \sum_{j=1}^{\ell_{k-1}+1-J_{k-1}} B^{-j}\sum_{L=0}^{j-1}\psi_{k-1,L}B^L - \sum_{j=0}^{\ell_{k-1}-J_{k-1}} B^{-j}\sum_{L=0}^{j}\psi_{k-1,L}B^L \\
&= B^{J_{k-1}-(\ell_{k-1}+1)}\sum_{L=0}^{\ell_{k-1}-J_{k-1}}\psi_{k-1,L}B^L + \sum_{j=1}^{\ell_{k-1}-J_{k-1}} B^{-j}\sum_{L=0}^{j-1}\psi_{k-1,L}B^L \\
&- \sum_{j=1}^{\ell_{k-1}-J_{k-1}} B^{-j}\sum_{L=0}^{j}\psi_{k-1,L}B^L - B^0\psi_{k-1,0}B^0 \\
&= \mathbf{1}_{\{\ell_{k-1}\geq J_{k-1}\}}\left[B^{J_{k-1}-(\ell_{k-1}+1)}\sum_{L=0}^{\ell_{k-1}-J_{k-1}}\psi_{k-1,L}B^L - \sum_{j=1}^{\ell_{k-1}-J_{k-1}} B^{-j}\psi_{k-1,j}B^j - \psi_{k-1,0}\right] \\
&= \mathbf{1}_{\{\ell_{k-1}\geq J_{k-1}\}}\left[B^{J_{k-1}-(\ell_{k-1}+1)}\sum_{L=0}^{\ell_{k-1}-J_{k-1}}\psi_{k-1,L}B^L - \sum_{j=0}^{\ell_{k-1}-J_{k-1}}\psi_{k-1,j}\right] \qquad (54)
\end{aligned}
$$

Therefore using (52), (53) and (54), (51) becomes

$$
\begin{aligned}
\phi(B)D_{k,t} &= d + \Bigg\{\Bigg[B^{J_{k-1}} + \mathbf{1}_{\{J_{k-1}>0\}}[B^{max(0,J_{k-1}-(\ell_{k-1}+1))} - B^{J_{k-1}}] \\
&+ \mathbf{1}_{\{\ell_{k-1}\geq J_{k-1}\}}[B^{J_{k-1}-(\ell_{k-1}+1)} - 1]\Bigg]\theta_{k-1}(B) \\
&+ \mathbf{1}_{\{\ell_{k-1}\geq J_{k-1}\}}\phi(B)\left[\sum_{L=0}^{\ell_{k-1}-J_{k-1}}\psi_{k-1,L} - B^{J_{k-1}-(\ell_{k-1}+1)}\sum_{L=0}^{\ell_{k-1}-J_{k-1}}\psi_{k-1,L}B^L\right]\Bigg\}\epsilon_{k-1,t}
\end{aligned}
$$

□

Proof of Proposition 2. Let $\gamma_{k-1,t} = \lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}$

We can rewrite player $k-1$'s order equation as

$$D_{k,t} = \frac{d}{\phi(1)} + B^{\tilde{J}_k}\frac{\tilde{\theta}_k(B)}{\phi(B)}\gamma_{k-1,t}$$

The term $\frac{d}{\phi(1)}$ is indeed the correct constant here since we know that $E[\phi(B)D_{k-1,t}] = \phi(1)E[D_{k-1,t}]$. Since $\phi(B)$ has no roots inside the unit circle, there exists a one-sided Laurent series expansion $L^{\phi}(z)$ of $\frac{1}{\phi(z)}$ for $z \in \mathcal{D}$ such that disk $\mathcal{D}$ contains the unit circle. Inserting this into the previous expression,

$$D_{k,t} = \frac{d}{\phi(1)} + B^{\tilde{J}_k}\tilde{\theta}_k(B)L^{\phi}(B)\gamma_{k-1,t}$$

The Laurent series expansion $L^{\phi}(z)$ has the form

$$L^{\phi}(z) = \sum_{n=0}^{\infty} \Psi_n z^n \text{for} z \in \mathcal{D}$$

This shows that, for any t, we can write $D_{k,t}$ as a linear combination of present and past $\gamma_{k-1,t}$ ie. $D_{k,t} \in \overline{sp}\{1, \gamma_{k-1,t}, \gamma_{k-1,t-1}, \gamma_{k-1,t-2}, ...\}$ and therefore $\mathcal{M}_t^{D_k} \subset \mathcal{M}_t^{\gamma_k - 1}$.

Furthermore, we can also rewrite player $k-1$'s order equation as

$$\frac{\phi(B)}{\tilde{\theta}_k^{OFF}(B)} D_{k,t} - \frac{d}{\tilde{\theta}_k^{OFF}(1)} = B^{\tilde{J}_k} \tilde{\theta}_k^{ON}(B) \gamma_{k-1,t}$$

where $\tilde{\theta}_k^{OFF}(z)$ and $\tilde{\theta}_k^{ON}(z)$ are defined in 27 and 26. Let $\{\nu_{k-1,t}\} = \{\tilde{\theta}^{ON}(B)\gamma_{k-1,t}\}$ and rewrite this as

$$\frac{\phi(B)}{\tilde{\theta}_k^{OFF}(B)} D_{k,t} - \frac{d}{\tilde{\theta}_k^{OFF}(1)} = B^{\tilde{J}_k} \nu_{k-1,t}$$

Note that $z^{\tilde{J}_k} \tilde{\theta}_k(z)$ has no roots inside or on the unit circle, $\tilde{J}_k = 0$ and there exists a one-sided Laurent Series Expansion $L^{\tilde{\theta}_k}(z)$ of $\dfrac{1}{\tilde{\theta}_k^{OFF}(z)}$ for $z \in \mathcal{D}$ such that disk $\mathcal{D}$ contains the unit circle. Therefore, for any t, we can write $\nu_{k-1,t}$ as a linear combination of present and past $D_{k,t}$.

Thus $\mathcal{M}_t^{\nu_{k-1}} \subset \mathcal{M}_t^{D_k}$. Finally, by [Brockwell and Davis, 1991] Proposition 4.4.1 $\mathcal{M}_t^{\gamma_{k-1}} \subset \mathcal{M}_t^{\nu_{k-1}}$. Therefore $\mathcal{M}_t^{\gamma_{k-1}} \subset \mathcal{M}_t^{D_k}$.

Thus we have shown that $\mathcal{M}_t^{D_k} = \mathcal{M}_t^{\gamma_{k-1}}$. Since $\mathcal{M}_t^k = \mathcal{M}_t^{D_k}$ we have that $\mathcal{M}_t^{\gamma_{k-1}} = \mathcal{M}_t^k$. If we can show that player k's demand $\{D_{k,t}\}$ can be written as QUARMA with respect to $\{\gamma_{k-1,t}\}$ then these will be player k's FIS. To do this recall that $\epsilon_{k-1,t} = \gamma_{k-1,t}/\lambda_{k,\tilde{J}_k}$. Substituting this into player $k-1$'s order equation we get

$$\phi(B) D_{k,t} = d + B^{\tilde{J}_k} \tilde{\theta}_k(B) \gamma_{k-1,t}$$

Thus if we take $J_k = \tilde{J}_k = 0$ and $\theta_k(z) = \tilde{\theta}_k(z)$ we will get the QUARMA representation of $D_{k,t}$ with respect to $\gamma_{k-1,t}$. This completes the proof. □

Proof of Proposition 3. Let

$$\gamma_{k-1,t} = \frac{\tilde{\theta}_k(B)}{\tilde{\theta}_k^{\dagger}(B)} B^{\tilde{J}_k} \lambda_{k,\tilde{J}_k} \epsilon_{k-1,t} \tag{55}$$

We can rewrite $\epsilon_{k-1,t}$ in terms of $\gamma_{k-1,t}$ as

$$\lambda_{k,\tilde{J}_k} \epsilon_{k-1,t} = \frac{\tilde{\theta}_k^{\dagger}(B)}{\tilde{\theta}_k(B)} B^{-\tilde{J}_k} \gamma_{k-1,t}$$

Substituting this into (20) we have

$$\phi(B)D_{k,t} = d + B^{\tilde{J}_k}\tilde{\theta}_k(B)\frac{\tilde{\theta}^{\dagger}_k(B)}{\tilde{\theta}_k(B)}B^{-\tilde{J}_k}\gamma_{k-1,t}$$

and simplifying

$$\phi(B)D_{k,t} = d + \tilde{\theta}^{\dagger}_k(B)\gamma_{k-1,t}$$

The polynomial $\tilde{\theta}^{\dagger}_k(B)$ has no roots inside the unit circle. Therefore by Proposition 2 we have that $\gamma_{k-1,t}$ are player k's full information shocks. Thus we say that $\{\epsilon_{k,t}\} = \frac{\tilde{\theta}_k(B)}{\tilde{\theta}^{\dagger}_k(B)}B^{\tilde{J}_k}\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}$. Furthermore $\theta_k(z) = \tilde{\theta}^{\dagger}_k(z)$ and $J_k = 0$. □

Proof of Lemma 1. We can rewrite (33) and (34) as

$$\frac{\phi(B)}{\Theta_1^{OUT}(B)}X_{1,t} = \frac{d}{\Theta_1^{OUT}(1)} + B^{J_1}\Theta_1^{IN}(B)\Theta_1^{ON}(B)\eta_t \tag{56}$$

$$\frac{\phi(B)}{\Theta_2^{OUT}(B)}X_{2,t} = \frac{d}{\Theta_2^{OUT}(1)} + B^{J_2}\Theta_2^{IN}(B)\Theta_2^{ON}(B)\lambda\eta_t \tag{57}$$

Where Θ_1^{IN}, Θ_2^{IN}, Θ_1^{OUT}, Θ_2^{OUT}, Θ_1^{ON} and Θ_2^{ON} are polynomials defined in the same way as $\tilde{\theta}_k^{IN}(z)$, $\tilde{\theta}_k^{OUT}(z)$ and $\tilde{\theta}_k^{ON}$ are defined in (24), (25) and (26).

Consider the polynomials $P_1(z) = z^{J_1}\Theta_1^{IN}(z)\Theta_1^{ON}(z)$ and $P_2(z) = \lambda z^{J_2}\Theta_2^{IN}(z)\Theta_2^{ON}(z)$. Suppose $P_2(z)$ has r_2 distinct non-zero roots $b_1, ..., b_{r_2}$.

Define

$$GCD(P_1, P_2) := z^{min(J_1,J_2)}\prod_{j=1}^{r_2}(1 - \frac{z}{b_j})^{min\{m(b_j,P_1),m(b_j,P_2)\}}$$

The roots of $GCD(P_1, P_2)$ are those roots that are common to both P_1 and P_2. Furthermore the multiplicity of each root is the minimum of the multiplicities of the root in P_1 and P_2. By construction, the coefficient in front of the lowest power of z of $GCD(P_1, P_2)$ is 1.

By the Euclidean Algorithm for polynomials (cf. [Koblitz, 1998] pg 63) we know that there exist polynomials $Q_1(z)$ and $Q_2(z)$ such that

$$Q_1P_1 + Q_2P_2 = GCD(P_1, P_2) \tag{58}$$

Suppose $\Theta_1^{ON}(z)$ has $r_{1,on}$ distinct roots $b_1, ..., b_{r_{1,on}}$. Define Θ_1^{ON-C} as

$$\Theta_1^{ON-C} := \prod_{j=1}^{r_{1,on}}(1 - \frac{z}{b_j})^{min(m(b_j,\Theta_1^{ON}),m(b_j,\Theta_2^{ON}))}$$

Note that if $\Theta_1^{ON}(z)$ and $\Theta_2^{ON}(z)$ have no common roots, then $\Theta_1^{ON-C} \equiv 1$.

Noting that $GCD(z^{J_1}\Theta_1^{IN}(z)\Theta_1^{ON}(z), \lambda z^{J_2}\Theta_2^{IN}(z)\Theta_2^{ON}(z)) = z^{min(J_1,J_2)}\Theta_1^{I-C}(z)\Theta_1^{ON-C}(z)$ the Euclidean Algorithm tells us how to find polynomials $Q_1(z)$ and $Q_2(z)$ such that

$$z^{J_1}Q_1(z)\Theta_1^{IN}(z) + \lambda z^{J_2}Q_2(z)\Theta_2^{IN}(z) = z^{min(J_1,J_2)}\Theta_1^{I-C}(z)\Theta_1^{ON-C}(z) \tag{59}$$

Therefore multiplying (56) and (57) by $Q_1(B)$ and $Q_2(B)$ and summing we get

$$\frac{1}{\Theta_1^{OUT}(B)}\phi(B)Q_1(B)X_{1,t} + \frac{1}{\Theta_2^{OUT}(B)}\phi(B)Q_2(B)X_{2,t} = C + B^{min(J_1,J_2)}\Theta_1^{I-C}(B)\Theta_1^{ON-C}\eta_t \tag{60}$$

where $C = \frac{Q_1(1)d}{\Theta_1^{OUT}(1)} + \frac{Q_2(1)d}{\Theta_2^{OUT}(1)}$ is a constant.

If $B^{J_1}\Theta_1^{IN}(z)\Theta_1^{ON}(z)$ and $B^{J_2}\Theta_2^{IN}(z)\Theta_2^{ON}(z)$ have no common roots then $\Theta_1^{I-C}(z) \equiv 1$, $\Theta_1^{ON-C}(z) \equiv 1$ (and $min(J_1, J_2) = 0$) in (60) and therefore we can take $\vartheta(z) = \frac{Q_1(z)}{\Theta_1^{OUT}(z)}$ and $\omega(z) = \frac{Q_2(z)}{\Theta_2^{OUT}(z)}$ to get

$$\vartheta(B)\phi(B)X_{1,t} + \omega(z)\phi(B)X_{2,t} = C + \eta_t$$

Furthermore since $\Theta_1^{OUT}(z)$ and $\Theta_2^{OUT}(z)$ have no roots inside or on the unit circle by construction, their reciprocals have one-sided Laurent series representations that converge in a disk $\mathcal{D}$ that contains the unit circle. Therefore the constructed $\vartheta(z)$ and $\omega(z)$ have one-sided Laurent Series Representations that converge for all $z \in \mathcal{D}$. Note that $C = \vartheta(1)d + \omega(1)d$.

Now suppose that there exist functions $\vartheta(z)$ and $\omega(z)$ with one sided Laurent Series Representations that converge in $\mathcal{D}$ such that $\vartheta(B)\phi(B)X_{1,t}+\omega(B)\phi(B)X_{2,t} = C+\eta_t$ where $C = \vartheta(1)d+\omega(1)d$.

From (33) and (34) we can rewrite this as

$$\vartheta(1)d + \omega(1)d + B^{J_1}\vartheta(B)\Theta_1(B)\eta_t + B^{J_2}\omega(B)\Theta_2(B)\lambda\eta_t = C + \eta_t$$

which simplifies to

$$B^{J_1}\vartheta(B)\Theta_1(B)\eta_t + B^{J_2}\omega(B)\Theta_2(B)\lambda\eta_t = \eta_t \tag{61}$$

Define $L(z) := z^{J_1}\vartheta(z)\Theta_1(z) + z^{J_2}\omega(z)\Theta_2(z)\lambda - 1$. Note that (61) implies that for all $\mu \in [-\pi, \pi]$, $L(e^{-i\mu}) \equiv 0$. Consider the Laurent series expansion of $L(z)$ for $z \in \mathcal{D}$,

$$L(z) = \sum_{k=-\infty}^{\infty} g_k z^k = z^{J_1}\vartheta(z)\Theta_1(z) + z^{J_2}\omega(z)\Theta_2(z)\lambda - 1 \tag{62}$$

The Laurent Series (62) must have the same coefficients g_k as the Fourier series expansion

$$L(e^{-i\mu}) = \sum_{k=-\infty}^{\infty} g_k e^{-i\mu k} = e^{-i\mu J_1}\vartheta(e^{-i\mu})\Theta_1(e^{-i\mu}) + e^{-i\mu J_2}\omega(e^{-i\mu})\Theta_2(e^{-i\mu})\lambda - 1 \tag{63}$$

Since $L(e^{-i\mu}) \equiv 0, g_k \equiv 0$ for all k in (63) and therefore in (62). This shows that $L(z) \equiv 0$ for all $z \in \mathcal{D}$.

If $\Theta_1(z)$ and $\Theta_2(z)$ had a common root z_0 inside or on the unit circle then we would have $L(z_0) = -1$ in (62) which is a contradiction. □

Proof of Theorem 2. To show (i) suppose $z^{\tilde{J}_k}\tilde{\theta}_k(z)$ and $z^{J_{k-1}}\theta_{k-1}(z)$ have no common roots inside the unit circle. We would like to show that $\mathcal{M}_t^{D_k,D_{k-1}} = \mathcal{M}_t^{\epsilon_{k-1}}$. Since $\phi(z)$ has no roots inside the unit circle we can rewrite (14) and (20) as:

$$D_{k-1,t} = \frac{d}{\phi(1)} + \sum_{j=0}^{\infty} \Psi_{k-1,j}\epsilon_{k-1,t-j-J_{k-1}}$$

$$D_{k,t} = \frac{d}{\phi(1)} + \sum_{j=0}^{\infty} \tilde{\Psi}_{k,j}\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t-j-\tilde{J}_k}$$

where $\{\Psi_{k-1,j}\}$ and $\{\tilde{\Psi}_{k,j}\}$ converge exponentially fast to zero. This shows that $\mathcal{M}_t^{D_k,D_{k-1}} \subset \mathcal{M}_t^{\epsilon_{k-1}}$.

To show that $\mathcal{M}_t^{\epsilon_{k-1}} \subset \mathcal{M}_t^{D_k,D_{k-1}}$ first suppose that $\theta_{k-1}(z)$ and $\tilde{\theta}_k$ have no common roots on the unit circle. Since $z^{\tilde{J}_k}\tilde{\theta}_k(z)$ and $z^{J_{k-1}}\theta_{k-1}(z)$ have no common roots inside the unit circle, from Lemma 1, there exist functions $\vartheta(z)$ and $\omega(z)$ with one-sided Laurent series representations such that

$$\vartheta(B)D_{k-1,t} + \omega(B)D_{k,t} = \vartheta(1)d + \omega(1)d + \epsilon_{k-1,t}$$

Thus $\mathcal{M}_t^{\epsilon_{k-1}} \subset \mathcal{M}_t^{D_k,D_{k-1}}$.

Now suppose that $\theta_{k-1}(z)$ and $\tilde{\theta}_k(z)$ have $h > 0$ distinct common roots on the unit circle and $\tilde{\theta}_k(z)$ has $r_{k,on}$ distinct roots $b_1, ..., b_{r_{k,on}}$ on the unit circle. Define $\tilde{\theta}_k^{ON-C}$ as

$$\tilde{\theta}_k^{ON-C} := \prod_{j=1}^{r_{k,on}} (1 - \frac{z}{b_j})^{min(m(b_j,\theta_{k-1}),m(b_j,\tilde{\theta}_k))}$$

Thus we can rewrite player $k-1$'s demand and order equations as

$$\phi(B)D_{k-1,t} = d + B^{J_{k-1}}\theta_{k-1}^*(B)\tilde{\theta}_k^{ON-C}(B)\epsilon_{k-1,t} \tag{64}$$

$$\phi(B)D_{k,t} = d + B^{\tilde{J}_k}\tilde{\theta}_k^*(B)\tilde{\theta}_k^{ON-C}(B)\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t} \tag{65}$$

where $\theta_{k-1}^* = \frac{\theta_{k-1}}{\tilde{\theta}_k^{ON-C}}$ and $\tilde{\theta}_k^* = \frac{\tilde{\theta}_k}{\tilde{\theta}_k^{ON-C}}$. Let $\nu_{k-1,t} = \theta_{k-1}^{ON-C}(B)\epsilon_{k-1,t}$. Note that we can rewrite (64) and (65) as

$$\phi(B)D_{k-1,t} = d + B^{J_{k-1}}\theta_{k-1}^*(B)\nu_{k-1,t}$$

$$\phi(B)D_{k,t} = d + B^{\tilde{J}_k}\tilde{\theta}^*_k(B)\lambda_{k,\tilde{J}_k}\nu_{k-1,t}$$

where the last equality comes from the fact that $\theta^{ON-C}_{k-1}(z) = \tilde{\theta}^{ON-C}_k(z)$. Noting that $\theta^*_{k-1}(z)$ and $\tilde{\theta}^*_k(z)$ have no common roots on or inside the unit circle we can use Lemma 1 to get functions $\vartheta(z)$ and $\omega(z)$ with one sided Laurent series representations converging in a disk $\mathcal{D}$ that contains the unit circle such that

$$\vartheta(B)\phi(B)D_{k-1,t} + \omega(B)\phi(B)D_{k,t} = \vartheta(1)d + \omega(1)d + \nu_{k-1,t}$$

Therefore $\mathcal{M}^{\nu_{k-1}}_t \subset \mathcal{M}^{D_k,D_{k-1}}_t$. Furthermore by [Brockwell and Davis, 1991] Proposition 4.4.1 $\mathcal{M}^{\epsilon_{k-1}}_t \subset \mathcal{M}^{\nu_{k-1}}_t$ and thus $\mathcal{M}^{\epsilon_{k-1}}_t \subset \mathcal{M}^{D_k,D_{k-1}}_t$ in this case as well.

Finally let $\gamma_{k-1,t} = \lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}$. Noting that $\mathcal{M}^{\gamma_{k-1}}_t = \mathcal{M}^{\epsilon_{k-1}}_t = \mathcal{M}^k_t$ and plugging $\gamma_{k-1,t}$ into player k's order equation we get

$$\phi(B)D_{k,t} = d + B^{\tilde{J}_k}\tilde{\theta}_k(B)\gamma_{k-1,t}$$

Thus we can write $\{D_{k,t}\}$ as QUARMA with respect to $\{\gamma_{k-1,t}\}$ and therefore these are player k's full information shocks. Furthemore $J_k = \tilde{J}_k$ and $\theta_k(z) = \tilde{\theta}_k(z)$ and the proof of (i) is complete.

To show part (ii) suppose that $z^{\tilde{J}_k}\tilde{\theta}_k(z)$ and $z^{J_{k-1}}\theta_{k-1}(z)$ have a common root inside the unit circle. Let $\xi_{k-1,t} = \frac{\tilde{\theta}^{I-C}_k(B)}{\tilde{\theta}^{\dagger I-C}_k(B)}B^{min(\tilde{J}_k,J_{k-1})}\epsilon_{k-1,t}$ where $\tilde{\theta}^{I-C}_k$ is defined in (28) and $\tilde{\theta}^{\dagger I-C}_k$ is defined by (30). Since $\tilde{\theta}^{\dagger I-C}_k(B)$ has all its roots outside the unit circle, $\xi_{k-1,t} \in \mathcal{M}^{\epsilon_{k-1}}_t$. We will show that $\mathcal{M}^{D_k,D_{k-1}}_t = \mathcal{M}^{\xi_{k-1}}_t$.

Define $\theta^{I-C^c}_{k-1} := \frac{\theta_{k-1}}{\tilde{\theta}^{I-C}_k}$ and $\tilde{\theta}^{I-C^c}_k := \frac{\tilde{\theta}_k}{\tilde{\theta}^{I-C}_k}$. Then we can rewrite player $k-1$'s demand and order equations as

$$\phi(B)D_{k-1,t} = d + B^{J_{k-1}}\theta^{I-C^c}_{k-1}(B)\tilde{\theta}^{I-C}_k(B)\epsilon_{k-1,t}$$

$$\phi(B)D_{k,t} = d + B^{\tilde{J}_k}\tilde{\theta}^{I-C^c}_k(B)\tilde{\theta}^{I-C}_k(B)\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}$$

Replacing $\epsilon_{k-1,t}$ with $\frac{\tilde{\theta}^{\dagger I-C}_k(B)}{\tilde{\theta}^{I-C}_k(B)}B^{-min(\tilde{J}_k,J_{k-1})}\xi_{k-1,t}$ we get

$$\phi(B)D_{k-1,t} = d + B^{J_{k-1}-min(\tilde{J}_k,J_{k-1})}\theta^{I-C^c}_{k-1}(B)\tilde{\theta}^{\dagger I-C}_k(B)\xi_{k-1,t} \quad (66)$$

$$\phi(B)D_{k,t} = d + B^{\tilde{J}_k-min(\tilde{J}_k,J_{k-1})}\tilde{\theta}^{I-C^c}_k(B)\tilde{\theta}^{\dagger I-C}_k(B)\lambda_{k,\tilde{J}_k}\xi_{k-1,t} \quad (67)$$

The polynomials $\theta^{I-C^c}_{k-1}(z)$ and $\tilde{\theta}^{I-C^c}_k(z)$ have no common roots inside the unit circle by definition. Therefore the polynomials $z^{J_{k-1}-min(\tilde{J}_k,J_{k-1})}\theta^{I-C^c}_{k-1}(B)\tilde{\theta}^{\dagger I-C}_k(B)$ and $z^{\tilde{J}_k-min(\tilde{J}_k,J_{k-1})}\tilde{\theta}^{I-C^c}_k(z)\tilde{\theta}^{\dagger I-C}_k(z)$

have no common roots inside the unit circle. Thus by part (i) we have that $\lambda_{k,\tilde{J}_k}\xi_{k-1,t}$ are player k's full information shocks. Thus we have the result that $\epsilon_{k,t} = \lambda_{k,\tilde{J}_k}\frac{\tilde{\theta}_k^{I-C}(B)}{\tilde{\theta}_k^{\dagger I-C}(B)}B^{min(\tilde{J}_k,J_{k-1})}\epsilon_{k-1,t}$. Furthermore $\theta_k(z) = \tilde{\theta}_k^{I-C^c}(B)\tilde{\theta}_k^{\dagger I-C}(B)$ and $J_k = \tilde{J}_k - min(\tilde{J}_k, J_{k-1})$. Noting that $\tilde{\theta}_k^{I-C^c}(B) = \tilde{\theta}_k^{OUT}(z)\tilde{\theta}_k^{ON}\tilde{\theta}_k^{I-NC}$ we get the intended result. □

Proof of Theorem 4. Recall Proposition 6 which states that

$$\sigma^2_{\epsilon_k,SS} = \lambda^2_{k,\tilde{J}_k}\sigma^2_{\epsilon_{k-1}} \tag{68}$$

$$\sigma^2_{\epsilon_k,DS} = \prod_{j:|z_j|<1}|z_j|^{-2\cdot min(m(z_j,\tilde{\theta}_k),m(z_j,\theta_{k-1}))}\lambda^2_{k,\tilde{J}_k}\sigma^2_{\epsilon_{k-1}} \tag{69}$$

$$\sigma^2_{\epsilon_k,NS} = \prod_{j:|z_j|<1}|z_j|^{-2\cdot m(z_j,\tilde{\theta}_k)}\lambda^2_{k,\tilde{J}_k}\sigma^2_{\epsilon_{k-1}} \tag{70}$$

To prove (i) consider (69). Since $\tilde{\theta}_k(z)$ and $\theta_{k-1}(z)$ share a root inside the unit circle, there is a z_j in the product such that $m(z_j,\tilde{\theta}_k) > 0$ and $m(z_j,\theta_{k-1}) > 0$. Since $|z_j| < 1$ we have that $\sigma^2_{\epsilon_k,SS} < \sigma^2_{\epsilon_k,DS}$.

Now consider (70). There exists z_j with $|z_j| < 1$ such that $m(z_j,\tilde{\theta}_k) > m(z_j,\theta_{k-1})$ by assumption and therefore $\sigma^2_{\epsilon_k,DS} < \sigma^2_{\epsilon_k,NS}$. Combining this with the previous result and we have that $\sigma^2_{\epsilon_k,SS} < \sigma^2_{\epsilon_k,DS} < \sigma^2_{\epsilon_k,NS}$.

To prove (ii) consider (69) again. Since $\tilde{\theta}_k(z)$ and $\theta_{k-1}(z)$ share a root inside the unit circle, there is a z_j in the product such that $m(z_j,\tilde{\theta}_k) > 0$ and $m(z_j,\theta_{k-1}) > 0$. Since $|z_j| < 1$ we have that $\sigma^2_{\epsilon_k,SS} < \sigma^2_{\epsilon_k,DS}$.

Furthermore, by assumption, all roots z_j of $\tilde{\theta}_k$ where $|z_j| < 1$ are such that $m(z_j,\tilde{\theta}_k) \leq m(z_j,\theta_{k-1})$. Therefore for all j, $min(m(z_j,\tilde{\theta}_k), m(z_j,\theta_{k-1})) = m(z_j,\tilde{\theta}_k)$ and (70) is equivalent to (69). Therefore $\sigma^2_{\epsilon_k,DS} = \sigma^2_{\epsilon_k,NS}$ and the result is proved.

To prove (iii) consider (69) again. If $\tilde{\theta}_k(z)$ has no roots in common with $\theta_{k-1}(z)$ inside the unit circle then $min(m(z_j,\tilde{\theta}_k), m(z_j,\theta_{k-1})) = 0$ for all j. Thus (69) is equivalent to (68) and we have that $\sigma^2_{\epsilon_k,SS} = \sigma^2_{\epsilon_k,DS}$.

For part (a), since $\tilde{\theta}_k$ has a root inside the unit circle, there exists a root z_j such that $|z_j| < 1$ and $m(z_j,\tilde{\theta}_k) > 0$. Therefore $\sigma^2_{\epsilon_k,NS}$ given by (70) is such that $\sigma^2_{\epsilon_k,NS} > \sigma^2_{\epsilon_k,SS}$ and we have the result that $\sigma^2_{\epsilon_k,SS} = \sigma^2_{\epsilon_k,DS} < \sigma^2_{\epsilon_k,NS}$.

For part (b), assuming that $\tilde{\theta}_k(z)$ has no roots inside the unit circle, we note from (70) that $\sigma^2_{\epsilon_k,NS} = \lambda^2_{k,\tilde{J}_k}\sigma^2_{\epsilon_{k-1}}$ and therefore by (68) we get that $\sigma^2_{\epsilon_k,NS} = \sigma^2_{\epsilon_k,SS}$. Therefore $\sigma^2_{\epsilon_k,SS} = \sigma^2_{\epsilon_k,DS} =$

$\sigma^2_{\epsilon_k,NS}$. □

Finding Examples of Intermediate Value of Demand Sharing

There are several examples in this paper that illustrate how intermediate value to demand sharing can arise. Here we present a discussion on how such examples can be found. The main focus here is finding some $k > 0$ such that player $k-1$ sharing its demand will be intermediate in value to the other two possible sharing arrangements and $J_{k-1} = \tilde{J}_k = 0$ (the non-strict-QUARMA case) where the retailer observes casual and invertible ARMA demand. In particular we show a set of conditions for the coefficients of the retailer's model such that all the requirements hold and there is intermediate value to player 2 sharing its demand with player 3 where $J_2 = \tilde{J}_3 = 0$ and $\ell_3 = 0$.

Since we need for $J_2 = \tilde{J}_3 = 0$, it could be shown that player 2 and player 3 must observe ARMA(2,2) demand with respect to player 2's full information shocks in this case. According to Corollary 3, there will be intermediate value to demand sharing if $\theta_2(z)$ and $\tilde{\theta}_3(z)$ have a root inside the unit circle in common and $\tilde{\theta}_3(z)$ has a root r inside the unit circle such that $|r| < 1$ $m(r, \tilde{\theta}_3) > m(r, \theta_2)$. We can therefore express the roots of $\theta_2(z)$ as z_0 and $z_{2,1}$ and the roots of $\tilde{\theta}_3(z)$ as z_0 and $\tilde{z}_{3,1}$ where z_0 is the common root and $\tilde{z}_{3,1} \neq z_{2,1}$.

The following Remark lists conditions under which the retailer observes causal and invertible ARMA and there is intermediate value to player 2 sharing its demand with player 3 where $J_2 = \tilde{J}_3 = 0$.

Remark 2. *Suppose the retailer observes ARMA(2,2) demand such that the following conditions* *hold for $\theta_1(z)$ and $\phi(z)$:*

$$\phi_1 + \phi_2 < 1,\ \phi_2 - \phi_1 < 1,\ -1 < \phi_2 < 1 \tag{71}$$

$$\theta_{1,1} + \theta_{1,2} < 1,\ \theta_{1,2} - \theta_{1,1} < 1,\ -1 < \theta_{1,2} < 1 \tag{72}$$

$$1 + \phi_1 = 1/z_{2,1} \text{ where } z_{2,1} \text{ is a root of } \theta_2(z) \tag{73}$$

$$|1 + \phi_1 - \theta_{2,1}| > 1 \tag{74}$$

$$|\phi_2 - \phi_1| < |\phi_2| \tag{75}$$

$$\frac{1}{1+\phi_1} \neq \frac{\phi_2 - \phi_1}{\phi_1} \tag{76}$$

Suppose further that the retailer shares the equivalent of its full information shocks with player 2 and that $J_2 = \tilde{J}_3 = 0$, $\ell_1 = 1$, $\ell_2 = 1$ and $\ell_3 = 0$. Then the retailer's demand is causal and invertible with respect to its full information shocks and player 2 sharing its demand will be intermediate to no sharing or full information shock sharing.

Note that $z_{2,1}$ and $\theta_{2,1}$ in (73) and (74) are not free parameters. They will depend on choices of $\phi(z)$ and $\theta_1(z)$. Constraints (71) and (72) are triangle conditions that guarantee that the retailer observes a causal and invertible ARMA(2,2) model. Constraints (73)-(76) guarantee that we have intermediate value to demand sharing between player 2 and player 3. The proof of this latter fact is done by analyzing the relationship of the parameters of the retailers ARMA model on the roots of $\theta_2(z)$ and $\tilde{\theta}_3(z)$.

The constraints in Remark 2 form the backbone for finding the Examples of intermediate value of demand sharing. The space defined by these constraints is certainly non-empty as demonstrated by the Examples in this paper.

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