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Information Aggregation and Innovation in Market Design*

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Abstract

The literature on information aggregation predicts that market growth unambiguously reduces uncertainty about the value of traded goods. The results were developed within the classical model, which assumes that traders' values for the exchanged good are determined by fundamental (common) shocks. At the same time, design innovation in contemporaneous markets seems to exploit demand interdependence among agents with similar tastes or common information sharing (e.g., Facebook ads, the practice of customer targeting). This paper demonstrates that with heterogeneous interdependence among agents' values or noise in signals about values, opportunities to innovate in smaller or less connected (in the network-theoretic sense) markets may dominate those in larger or better connected markets.

JEL Classification: D44, D82, L13, G14

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1 Introduction

This paper examines how innovation in demand creation — broadly understood as the introduction of new goods or innovations in targeting or advertising — interacts with information aggregation. An important idea in economic theory and practical market design is that innovation can reduce, and is often motivated by, agents' uncertainty about the quality of exchanged goods.[1] In particular, existing literature on information aggregation in markets suggests that

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[1] Other, often related motives for innovation include the creation of demand for new goods and services and the bundling of product characteristics.

increasing market size *weakens* incentives to innovate: Existing literature's goal has been to examine whether the market mechanism is capable of aggregating all payoff-relevant information in the economic system. The main result has been establishing that this happens under general conditions. The full aggregation result implies that the equilibrium uncertainty about product quality decreases as new market participants contribute new information.[2]

The classical predictions have been developed for a particular structure of interdependence in agents' preferences, assuming that a fundamental value exists by which demands for the exchanged good for all traders are determined (i.e., the *fundamental value* or *mixed values* assumption). Consequently, market participants' demands comove in the same way, with fundamental (common) shocks. The fundamental-value model precludes any heterogeneity in demand interdependence, whether due to inherent tastes (i.e., the primitive which determines agents' inherent *values* for the good) or information-sharing technology (i.e., *noise* in signals about values present as agents learn through common sources or communicate). More abstractly, such shocks impact groups of traders, but not the market as a whole. Instead, values are assumed to be common to all traders and signal noise is i.i.d. Indeed, design innovation in contemporaneous markets seems to exploit interdependence among agents with similar tastes or common information sharing (e.g., Facebook ads, customer targeting).[3,4]

This project shows that heterogeneously interdependent demands (through tastes or information sharing) change the predictions of the classical model: Opportunities to innovate in smaller or less connected (in the network-theoretic sense) markets may dominate those in large or better connected markets.

RESULTS: The paper examines trade-offs in information aggregation occurring between small and large as well as sparsely or locally connected and better connected markets. Specifically, we show that in markets in which demands are heterogeneously correlated, uncertainty about product quality (as measured by the lack of information aggregation) may not be increasing with market size or greater connectivity among agents' values or noise. In fact, for economically relevant information structures, equilibrium uncertainty may exhibit arbitrary, non-monotone behavior. The paper develops the necessary and sufficient conditions on the information structure under which uncertainty about product quality and, hence, incentives

[2] The literature on information aggregation has focused on large market environments – within the rational expectations or strategic, double-auction settings, and established limit results, as the number of traders grows large. Examining how market growth impacts information aggregation requires a model with an arbitrary, finite number of traders, which is a less explored setting. Still, all existing results demonstrate that markets are informationally efficient (Dubey, Geanakoplos, and Shubik (1987); Ostrovsky (2009); Vives (2011); Vives (2008) provides a review). Rostek and Weretka (2011) are an exception.

[3] For a review of the growing literature on interdependence in traders' demands for assets in financial markets, see Veldkamp (2011, Chapter 8).

[4] Targeting has recently been studied by Athey and Gans (2010), Bar Isaac, Caruana, and Cunat (2010), Levin and Milgrom (2010), Bergemann and Bonatti (2011).

to innovate are monotone in market size.

A key question that arises in the empirical literature is whether the demand comovements across markets or traders are driven by interdependence in values or noise. We establish conditions under which interdependence in demands due to preferences versus information can be separated. Unlike the fundamental-value model, with heterogeneously interdependent demands, separating the impact of values and noise is critical for determining whether and how efficiency and product innovation can be enhanced through appropriate policy response or market design.

The absence of monotonicity in learning about asset quality in general suggests that product design and targeting aimed at smaller, segmented, or less connected markets can improve agents' learning. Thus, the efficiency and profitability of innovation may improve when markets become segmented or decentralized.

Approach: The new predictions owe to the paper's modeling contribution: a tractable model of markets (one-sided or double auctions) with an arbitrary number of traders and a rich class of interdependencies in values and noise. All traders are Bayesian and strategic; in particular, there are no noise traders. Cast in a linear-normal setting, as a key innovation, the model permits heterogeneous correlations in values and noise in signals. The model accommodates aspects of heterogeneity such as interdependence in values and noise that varies with "distance" (e.g., geographical or social proximity), group-dependence in values or noise, and characteristics of network connectivity (degree distribution).

Other Related Literature: Rostek and Weretka (2011) study information aggregation in auctions with heterogeneous comovements exclusively in values. This paper instead considers the interaction between information aggregation and innovation and how it depends on market size and technology generating information among agents, and concerns markets in which both values and signal noise are heterogeneously interdependent.[5]

The paper also contributes to the literature on price formation in social networks, albeit in a very special class of networks. Research examining how details of the underlying network impact properties of prices is typically restricted to a monopolist selling to buyers embedded in a social network (a divisible good in Candogan et al. (2011) or an indivisible good in Hartline et al. (2008) and Akhlaghpour et al. (2010)). We further allow network models based on a double auction, thereby accommodating markets in which both sellers and buyers are part of the social network.

Furthermore, the existing economic network models concerning information aggregation typically do not study the question in a market context. For example, in Acemoglu, Dahleh,

[5] To the best of our knowledge, apart from spatial models, our model is the first to accommodate a large class of heterogeneity in the interdependence of signal noise.

Lobel, and Ozdaglar (2010), the agents' goal is to uncover the true value by sequentially updating their decisions based on direct observations of their peers' decisions. Our framework allows analysis of how the market price itself aggregates information.

2 Model and Equilibrium

2.1 A Market with Heterogeneously Interdependent Demands

Consider a market for a perfectly divisible good (e.g., an asset) with I traders. Each trader i has a quasilinear utility which is quadratic in the quantity of the asset.

$$U_i(q_i) = \theta_i q_i - \frac{\mu}{2} q_i^2. \tag{1}$$

Each trader is uncertain about how much the asset is worth. Trader uncertainty is captured by the randomness of the intercepts of marginal utility functions θ_i. That is, each trader i observes only a noisy signal of the true value, $s_i = \theta_i + \varepsilon_i$. Random vector $\{\theta, \varepsilon_i\}_{i \in I}$ is jointly normally distributed, noise ε_i has expectation $E(\varepsilon_i)$ and variance σ_ε^2, the same for all i, θ_i has expectation $E(\theta_i)$ and variance σ_θ^2. Values and noise are uncorrelated, for all i, j. The variance ratio $\sigma^2 \equiv \sigma_\theta^2 / \sigma_\varepsilon^2$ measures the relative importance of noise in the signal. The $I \times I$ variance-covariance matrix of the joint distribution of values $\{\theta_i\}_{i \in I}$, normalized by variance σ_θ^2, specifies the matrix of preference correlations,

$$\mathcal{C}^\theta \equiv \begin{pmatrix} 1 & \rho_{1,2}^\theta & \cdots & \rho_{1,I}^\theta \\ \rho_{2,1}^\theta & 1 & \cdots & \rho_{2,I}^\theta \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{I,1}^\theta & \rho_{I,2}^\theta & \cdots & 1 \end{pmatrix} = \left\{ cov\left(\theta_i, \theta_j\right) / \sigma_\theta^2 \right\}_{i,j}. \tag{2}$$

The $I \times I$ variance-covariance matrix of the joint distribution of noise $\{\varepsilon_i\}_{i \in I}$, normalized by variance σ_ε^2, specifies the matrix of noise correlations,

$$\mathcal{C}^\varepsilon \equiv \begin{pmatrix} 1 & \rho_{1,2}^\varepsilon & \cdots & \rho_{1,I}^\varepsilon \\ \rho_{2,1}^\varepsilon & 1 & \cdots & \rho_{2,I}^\varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{I,1}^\varepsilon & \rho_{I,2}^\varepsilon & \cdots & 1 \end{pmatrix} = \left\{ cov\left(\varepsilon_i, \varepsilon_j\right) / \sigma_\varepsilon^2 \right\}_{i,j}. \tag{3}$$

We use the word *attribute* when referring to either bidder value or noise, $a_i \in \{\theta_i, \varepsilon_i\}$, for each i. We allow comovements of values $\rho_{i,j}^\theta$ and noise $\rho_{i,j}^\varepsilon$ to be heterogeneous across all pairs of agents in the market. We impose one restriction: for each trader i, his attribute a_i is *on average* correlated with other traders' attributes a_j, $j \neq i$, in the same way; that is, for each

i,

$$\frac{1}{I-1}\sum_{j\neq i}\rho^{\theta}_{i,j}=\bar{\rho}^{\theta} \text{ and } \frac{1}{I-1}\sum_{j\neq i}\rho^{\varepsilon}_{i,j}=\bar{\rho}^{\varepsilon}. \tag{4}$$

Intuitively, the average correlation statistic for each attribute measures how each bidder attribute correlates on average with the other bidders' attributes. For some interpretations, it will be useful to express the model in terms of total average correlations, $r^a=\frac{1}{I}\sum_{j\in I}\rho^a_{i,j}=\frac{1}{I}\left(1+(I-1)\bar{\rho}^a\right)$, $a=\theta,\varepsilon$. Let $\gamma\equiv 1-1/(I-1)$ be an index of *auction size*; $\gamma\in[0.1]$. For a sequence of auctions indexed by market size γ, commonality function $\bar{\rho}^a(\gamma)$ specifies commonality $\bar{\rho}^a$ for any market size.

2.2 Interdependent Demands: Examples

The theoretical framework described thus far fits a range applications. Here, we present several examples of equicommonal markets, motivated by applications in industrial organization, social networks, and financial markets. We use the word *attribute* when referring to either values or noise. All models in this section are well-defined as long as they define appropriate correlation matrices (positive semidefiniteness, symmetry, correlations between -1 and 1). We begin with the classical model.[6]

Model 1 (Fundamental) *The attribute of each of I bidders is determined by a common shock plus an idiosyncratic (i.i.d.) shock. Therefore, pairwise correlations of attributes across bidders are the same; $\rho_{i,j}=\rho>0$ for all $j\neq i$. As a new bidder enters the auction, his correlation with each of the other bidders' attributes is equal to ρ. The commonality function, $\bar{\rho}^a(\gamma)=\rho$, is constant in γ, and the average correlation is decreasing in I,*

$$r^a(I)=\frac{1+(I-1)\rho}{I}.$$

Model 2 (Spatial) *I bidders are located on a circle. Bidders' attributes correlate proportionally to the distance between them. If the shortest distance between bidders i and j is $d_{i,j}$ (measured by the length of the shortest path between them, the length between immediate neighbors being 1), then $\rho_{i,j}=\beta^{d_{i,j}}$, where $\beta\in(0,1)$ is a decay parameter. As a new bidder enters the auction, the circle's circumference increases by one. The commonality function (assuming an odd I), $\bar{\rho}^a(\gamma)=(1-\gamma)2\beta\left(1-\beta^{\frac{1}{2}\frac{1}{1-\gamma}}\right)/(1-\beta)$, is decreasing in γ, and the average correlation is decreasing in I,*

$$r^a(I)=\frac{1+\beta}{1-\beta}\cdot\frac{1-\beta^{\frac{I}{2}}}{I}.$$

[6] For a more thorough interpretation of the first three models, see Rostek and Weretka (2011).

Model 3 (Group) *There are two groups of equal size $\frac{I}{2}$. Attributes are perfectly correlated within groups ($\rho_{i,j} = 1$ if i and j belong to the same group) and correlated $\alpha \in [-1, 1]$ across groups ($\rho_{i,j} = \alpha$ if i and j belong to different groups).*[7] *New bidders increase both group sizes; their attributes are perfectly correlated within their own group and correlated α with attributes of the other group. The commonality function $\bar{\rho}^a(\gamma) = ((2-\gamma)\alpha + \gamma)/2$ is decreasing in γ, and the average correlation is constant in I,*

$$r^a(I) = \frac{1+\alpha}{2}.$$

Adopting an explicit networks interpretation, we turn to applications with a fixed auction size, I, examining comparative statics with respect to statistics describing the network structure, indexed by $k = 1, 2, \ldots$. The exception is the Regular Network Model, in which both auction size and network connectivity change. The Condensing Links Model captures the idea of a social network becoming denser as new links form between bidders; for instance, gradual exploration of the market. The Strengthening Links Model captures that existing correlations between agents' attributes strengthen. The Regular Network Model describes a regular network in which each bidder is correlated with k other bidders, becoming larger and more connected. The Spreading Attention Model captures the idea of an increasing number of connections while the total correlation of each bidder's attribute is preserved as the number of agents grows.

Model 4 (Condensing Links) *There is a fixed circle of I bidders without links, but with independent attributes. At each step k, for each bidder, two new links are added to connect the bidder to the two closest unlinked bidders, each correlated by β^k. I is an odd, fixed number while $k \leq (I-1)/2$. The commonality function, $\bar{\rho}^a(k) = (1-\gamma)\, 2\beta\left(1-\beta^k\right)/(1-\beta)$, is increasing in k, and the average correlation, is increasing in k,*

$$r^a(k) = \frac{1 + 2\beta + 2\beta^2 + \cdots + 2\beta^k}{I} = \frac{1 + 2\beta\frac{1-\beta^k}{1-\beta}}{I}.$$

Model 5 (Strengthening Links) *In the Spatial Model, introduce parameter δ ($\delta \in (0,1)$) and fix the circle of I bidders. At each step k, the correlations of every bidder's attributes get each multiplied by δ^{-1}. The commonality function, $\bar{\rho}^a(k) = (1-\gamma)\, 2\delta^{-k}\beta\left(1-\left(\delta^{-k}\beta\right)^{\frac{1}{2}\frac{1}{1-\gamma}}\right)/\left(1-\delta^{-k}\beta\right)$, is decreasing in k and the average correlation is decreasing in k,*

$$r^a(k) = \frac{1+\delta^{-k}\beta}{1-\delta^{-k}\beta} \cdot \frac{1-\left(\delta^{-k}\beta\right)^{\frac{I}{2}}}{I}.$$

[7] For example, for $\alpha \in [0, 1]$, attributes are determined by a fundamental shock plus a shock that is the same for members within a group and idiosyncratic (i.i.d.) across groups.

Model 6 (Regular Network) *In a k-regular network,*[8] *let the attributes of bidders be correlated α to connected bidders and 0 to non-connected bidders. As new bidders enter the market, both the regularity of the market and the size of the market increase (as described by an non-decreasing function $I(k)$). The commonality function is $\bar{\rho}^a(k) = (1-\gamma(k))(\alpha k(2-\gamma(k))/(1-\gamma(k)))$, and the average correlation is*

$$r^a(k) = \frac{1+\alpha k}{I(k)}.$$

Model 7 (Spreading Attention) *In the Spatial Model with I bidders, the total correlation of each bidder with others' attributes is fixed and equal to 2β. For each step k, the correlation of each bidder's attribute with the $2k$ closest neighbors is β/k. The commonality function (for an odd I, and $k \leq (I-1)/2$) $\bar{\rho}^a(k) = (1-\gamma)2\beta$ is constant in k, and the average correlation is constant in I,*

$$r^a(k) = \frac{1+2\beta}{I}.$$

2.3 Equilibrium

In finite auctions, strategic bidders shade their bid: Relative to bid shading induced by the decreasing marginal utility itself, bids are further affected by price inference. Equilibrium existence in equicommonal auctions requires that interdependence, and hence bid shading, be bounded. Proposition 1 establishes a corresponding upper bound on commonality, $\bar{\rho}^+(\gamma, \sigma^2)$, which weakens with positively correlated information (the bound is characterized in the Appendix). Additionally, in auctions with negative correlations, value commonality has to be strictly above the lower bound of $\bar{\rho}^{\theta-}(\gamma) = -(1-\gamma) < 0$, which is independent of information correlations $\{\rho^{\varepsilon}_{i,j}\}_{i \neq j}$.[9] As a necessary and sufficient condition, Proposition 1 demonstrates that a symmetric linear Bayesian Nash equilibrium in equicommonal double auctions exists for any triple $(\gamma, \bar{\rho}^\theta, \bar{\rho}^\varepsilon)$ that strictly satisfies the two bounds restricting average but not pairwise correlations.

Proposition 1 (Equilibrium Existence) *There exist bounds $\bar{\rho}^{\theta-}(\gamma)$ and $\bar{\rho}^{\theta+}(\gamma, \sigma^2)$ such that, in an equicommonal double auction characterized by $(\gamma, \bar{\rho}^\theta)$, a symmetric linear Bayesian Nash equilibrium exists if, and only if, $\bar{\rho}^{\theta-}(\gamma) < \bar{\rho}^\theta < \bar{\rho}^{\theta+}(\gamma, \sigma^2)$. The symmetric equilibrium is unique.*

Proposition 1 contributes by arguing equilibrium existence for divisible goods, allowing for both heterogeneously interdependent values and information, and auctions with two bidders,

[8] A graph is *k-regular* if each of its nodes is of degree k.

[9] The lower bound is derived from the positive semidefiniteness of the variance-covariance matrix of the data-generating process underlying the joint distribution of values and is binding only for commonality equal to $\bar{\rho}^{\theta-}(\cdot)$.

provided $\bar{\rho}^{\theta} - \bar{\rho}^{\varepsilon} < 0$.[10] The bounds apply only to average correlations.

Proposition 2 derives the equilibrium bids for a class of markets characterized by $(\gamma, \bar{\rho}^{\theta}, \bar{\rho}^{\varepsilon})$. The following necessary and sufficient condition gives the best response of trader i with utility (1) to the linear bids of bidders $j \neq i$: for any p,

$$E(\theta|s_i, p) - \mu q_i = p + \lambda q_i \tag{5}$$

where $\lambda \equiv -(\partial q_i(p)/\partial p)^{-1}/(I-1)$ is the slope of the residual supply defined for i by bids of traders $j \neq i$, whose signals $\{s_j\}_{j\neq i}$ determine the supply's intercept. Using the first-order condition (5), market clearing gives the equilibrium price $p^* = \frac{1}{I}\sum_{i\in I} E(\theta|s_i, p^*)$. The conditional expectation of the affine information structure is linear $E(\theta|s_i,p) = c_{\theta+\varepsilon_i} E(\theta+\varepsilon_i) + c_s s_i + c_p p$. The inference coefficients c_s, c_p and $c_{\theta+\varepsilon_i}$ can be determined in terms of commonality and market size from two conditions and the projection theorem. The parameter λ is endogenized from individual bids. Define $\gamma = 1 - \frac{1}{I-1}$ as the index of market size, $\gamma \in [0,1]$.

Proposition 2 (Equilibrium Bids) *The equilibrium bid of trader i is*

$$q_i(p) = \frac{\gamma - c_p}{1-c_p}\frac{c_\theta}{\mu}E(\theta_i) + \frac{\gamma - c_p}{1-c_p}\frac{c_s}{\mu}s_i - \frac{\gamma - c_p}{\mu}p, \tag{6}$$

where inference coefficients in the conditional expectation $E(\theta_i|s_i,p)$ are given by

$$c_s = \frac{\left(1-\bar{\rho}^{\theta}\right)}{(1-\bar{\rho}^{\theta}) + \sigma^2\left(1-\bar{\rho}^{\theta\varepsilon}\right)}, \tag{7}$$

$$c_p = c_s\frac{I\sigma_\varepsilon^2\left(\bar{\rho}^{\theta}-\bar{\rho}^{\theta\varepsilon}\right)}{\sigma_\theta^2(1-\bar{\rho}^{\theta})\left(1+\bar{\rho}^{\theta}(I-1)\right)} = \frac{(2-\gamma)\sigma^2\left(\bar{\rho}^{\theta}-\bar{\rho}^{\theta\varepsilon}\right)}{(1-\gamma+\bar{\rho}^{\theta})\left((1-\bar{\rho}^{\theta})+\sigma^2\left(1-\bar{\rho}^{\theta\varepsilon}\right)\right)}, \tag{8}$$

$$c_\theta = 1 - c_s - c_p. \tag{9}$$

3 Price Inference and Demand Interdependence

3.1 Information Aggregation

Consider an agent who makes inference about the unknown θ_i based on the privately observed signal s_i and price p. Informational efficiency of price is typically evaluated relative to what a

[10] For auctions with $\bar{\rho}^{\varepsilon} = 0$, the bounds coincide with that of Rostek and Weretka (2011). For the Fundamental Value Model, the upper bound $\bar{\rho}^{\theta+}\left(\gamma,\sigma^2\right)$ coincides with that derived by Vives (2009), assuming an inelastic demand and downward-sloping bids.

Negative average correlation in demands weakens strategic interdependence among best responses, which in the absence of price inference leads to non-existence of equilibrium for $I = 2$ (cf. Wilson (1979) and Kyle (1989)), whereas positive average demand correlation amplifies via price informativeness the strategic interdependence of best responses.

bidder could learn about his value from the total information available in the market, measured by the profile of all bidders' signals, $s \equiv \{s_i\}_{i \in I}$. The equilibrium price is *privately revealing* if, for any bidder i, the conditional c.d.f.'s of the posterior of θ_i satisfy $F(\theta_i|s_i, p^*) = F(\theta_i|s)$ for every state s, given the corresponding equilibrium price $p^* = p^*(s)$. Proposition 3 establishes which auctions are efficient in this sense.

Proposition 3 (Aggregation of Private Information) *In a finite double auction, the equilibrium price is privately revealing if, and only if, $\rho_{i,j} = \bar{\rho}^\theta$ and $\rho^\varepsilon_{i,j} = \bar{\rho}^\varepsilon$ for all $j \neq i$.*

Informational efficiency is a non-generic property of price in the class of information structures considered and is lost whenever information structure features heterogeneity in interdependence among trader values or information. The argument behind the result can be related to the dimensionality argument by Jordan (1983), who demonstrated that, whether price aggregates information depends on the relative dimension of the payoff-relevant information (signals $j \neq i$) and price. Let us emphasize, however, that for each bidder, price does match the dimension of the sufficient statistic for the signals of other bidders and is, thus, capable of summarizing the information in the considered setting. A bidder's sufficient statistic is a weighted average signal, with the weights being functions of correlations $\mathcal{C}^\theta$ and $\mathcal{C}^\varepsilon$. Since the price in an equicommonal auction reveals the equally weighted average signal $\bar{s}$ (see (11)), the learning instrument matches, for each bidder, the sufficient statistic only in auctions with identical correlations.[11]

Given Proposition 3, beyond auctions with both identical correlations in values and information, the prediction that price aggregates more information in larger markets no longer follows.

3.2 Price Informativeness

Returning to our central question, in a market with I traders, will an additional market participant improve the informativeness of price for all traders? In the Gaussian setting, price informativeness can be quantified through variance reduction by defining an index of *price informativeness* as $\psi^+ \equiv \frac{Var(\theta_i|s_i) - Var(\theta_i|s_i,p^*)}{Var(\theta_i|s_i)}$ (Rostek and Weretka (2011)), $\psi^+ \in [0,1]$. A uninformative price brings no reduction in variance beyond a private signal ($\psi^+ = 0$); a price that, together with a private signal, reveals the value θ_i to trader i and accomplishes full reduction ($\psi^+ = 1$). Proposition 4 establishes the necessary and sufficient condition on an equicommonal information structure for a new bidder to increase price informativeness. For an auction of size γ in a sequence of equicommonal auctions, let $\Delta\bar{\rho}^\theta(\gamma)$ be the change in commonality that results from including an additional bidder.

[11] The generic inefficiency result is not driven by equilibrium symmetry.

Proposition 4 (Informational Impact) *Fix γ, $\bar{\rho}^{\varepsilon}$ and $\bar{\rho}^{\theta}-\bar{\rho}^{\varepsilon}>0$ ($\bar{\rho}^{\theta}-\bar{\rho}^{\varepsilon}<0$). A threshold $\tau<0$ (respectively, $\tau>0$) exists such that, in any auction that satisfies $\rho_{ij}=\bar{\rho}^{\theta}$ and $\rho_{ij}=\bar{\rho}^{\varepsilon}$, $j\neq i$, the contribution of an additional bidder to price informativeness is strictly positive if, and only if, $\Delta\left(\bar{\rho}^{\theta}-\bar{\rho}^{\varepsilon}\right)(\gamma)>\tau$ (respectively, $\Delta\left(\bar{\rho}^{\theta}-\bar{\rho}^{\varepsilon}\right)(\gamma)<\tau$).*

It is useful to express price informativeness in terms of the total average correlations,

$$\psi^{+}=\frac{\sigma^{2}\left(r^{\theta}-r^{\varepsilon}\right)^{2}}{\left(r^{\theta}+\sigma^{2}r^{\varepsilon}\right)\left[1-r^{\theta}+\sigma^{2}\left(1-r^{\varepsilon}\right)\right]}. \tag{10}$$

The average correlations in values and noise affect price informativeness ψ^{+} individually and not just as a difference $\bar{\rho}^{\theta}-\bar{\rho}^{\varepsilon}$. In particular, the relative importance of demand interdependence through values versus noise, as captured by σ^{2}, also matters. Similarly, $\bar{\rho}^{\theta}$ impacts the inference coefficients (8) and (7) in ways $\bar{\rho}^{\varepsilon}$ does not.

We report two sufficient conditions on the monotonicity of price informativeness. Let $r^{a\prime}\equiv\frac{\Delta r^{a+}}{\Delta I}$ be the change in variable r^{a} resulting from increasing I, $I\in\mathbb{N}_{++}$, by one.

Proposition 5 (Sufficient Condition for Monotone Price Informativeness 1) *Assume that r^{θ}, r^{ε} are increasing and $r^{\theta}>r^{\varepsilon}$ ($r^{\theta}<r^{\varepsilon}$). Price informativeness ψ^{+} increases in the number of bidders I if, $\frac{r^{\theta\prime}}{r^{\theta}}>\frac{r^{\varepsilon\prime}}{r^{\varepsilon}}$ $\left(\frac{r^{\theta\prime}}{r^{\theta}}<\frac{r^{\varepsilon\prime}}{r^{\varepsilon}}\right)$.*

Note that the sufficient condition does not depend on σ^{2}. Ranking growth rates ($\frac{r^{\theta\prime}}{r^{\theta}}>\frac{r^{\varepsilon\prime}}{r^{\varepsilon}}$) is equivalent to the condition $\frac{\left(r^{\theta}-r^{\varepsilon}\right)'}{r^{\theta}-r^{\varepsilon}}>\frac{\left(r^{\theta}+\sigma^{2}r^{\varepsilon}\right)'}{r^{\theta}+\sigma^{2}r^{\varepsilon}}$, which ranks the growth rate of the difference between the two average correlations versus that of the signal variance.

Proposition 6 (Sufficient Condition for Monotone Price Informativeness 2) *Assume that $r^{\theta}+\sigma^{2}r^{\varepsilon}<\frac{1+\sigma^{2}}{2}$ and $r^{\theta}>r^{\varepsilon}$ ($r^{\theta}<r^{\varepsilon}$). Price informativeness ψ^{+} increases in the number of bidders I if, $\frac{r^{\theta\prime}}{r^{\theta}}>\frac{r^{\varepsilon\prime}}{r^{\varepsilon}}$ $\left(\frac{r^{\theta\prime}}{r^{\theta}}<\frac{r^{\varepsilon\prime}}{r^{\varepsilon}}\right)$.*

3.3 Price Informativeness in Applications

The analysis thus far demonstrates that, in markets with heterogeneously interdependent values and/or noise, market growth or increasing connectivity of agents' attributes measured by the number or strength of links may have a non-monotone, essentially arbitrary, impact on price informativeness (Proposition 4 and (10)). In this section, we illustrate how price informativeness is affected by market growth or changes in network connectivity in the examples from Section 2.2.

Let us interpret Proposition 4 geometrically. For each value of $\psi^{+}\in[0,1]$, construct a ψ^{+}-curve consisting of all combinations of the average correlations $\left(r^{\theta},r^{\varepsilon}\right)$ for which price informativeness equals ψ^{+} (Figure 1). For $\psi^{+}=0$, price is uninformative; this happens if, and only if, $r^\theta = r^\varepsilon$, and the 0-curve corresponds to the 45-degree line. For an informative price, $\psi^+ > 0$, a ψ^+-curve consists of two segments, placed on the opposite sides of the 45 degree line. Price informativeness increases for curves located further away from the 0-curve and the 1-curve corresponding to points $(0,1)$ and $(1,0)$. The two schedules of each ψ^+-curve comprise the average correlations $\left(r^\theta, r^\varepsilon\right)$ and $\left(1-r^\theta, 1-r^\varepsilon\right)$. If $\sigma^2 = 1$, then price informativeness is exchangeable in r^θ and r^ε; hence, ψ^+-curves are symmetric with respect to the 45 degree line. A sequence of auctions is represented by a sequence of points $\left(r^\theta(I), r^\varepsilon(I)\right)$ in the case of increasing I or $\left(r^\theta(k), r^\varepsilon(k)\right)$ in the case of increasing k — tracing values of ψ^+.

In an application in which attributes' correlations are modeled as a Fundamental Value with Fundamental Noise, price informativeness is always increasing (Figure 2).[12] The convexity of the price informativeness, ψ^+, with respect to market size depends on the sign of $\bar{\rho}^\theta + \sigma^2\bar{\rho}^\varepsilon$,[13] which is always positive in the fundamental models. Thus, the "marginal" price informativeness, which is nonnegative, is always diminishing. Insofar as this translates into diminishing welfare, a welfare trade-off may exist, depending on participation costs.[14]

In markets with Fundamental Value and Group Noise, price informativeness can increase (when $\bar{\rho}^\theta \leq \alpha$), decrease (when $\bar{\rho}^\theta > \frac{1+\alpha}{2}$), or exhibit a U-shape, dropping to zero for an intermediate market size. In markets with Fundamental Value and Spatial Noise, depending on the relative correlation of value and noise, price informativeness may increase to reach a maximum value in a small market, then, as the market grows further, diminish to zero to increase again (e.g., when $\bar{\rho}^\theta = 0.2$ and $\beta^\varepsilon = 0.9$); when $\bar{\rho}^\theta = 0.6$ and $\beta^\varepsilon = 0.8$, the small market maximum is local while price informativeness is maximal in the large limit market.

Turning to network models, when the link formation corresponds to a Condensing Network for Values with a Condensing Network for Noise, price informativeness is always increasing in the number of links in values and noise. This is surprising given that, in general, the average correlations in value and in noise work against each other.[15] One can show that the

[12] To determine monotonicity, price informativeness of the Fundamental Value with Fundamental Noise can be written as

$$\psi^+ = \frac{\sigma^2\left(\bar{\rho}^\theta - \bar{\rho}^\varepsilon\right)^2}{\left(\frac{1+\sigma^2}{I-1} + \bar{\rho}^\theta + \sigma^2\bar{\rho}^\varepsilon\right)\left(1 - \bar{\rho}^\theta + \sigma^2\left(1-\bar{\rho}^\varepsilon\right)\right)}.$$

[13] The second derivative of price informativeness with respect to market size in the Fundamental Value and Fundamental Noise model is (treating I as a real number)

$$\psi^{+\prime\prime} = -\left(\bar{\rho}^\theta + \sigma^2\bar{\rho}^\varepsilon\right)\frac{2\sigma^2\left(1+\sigma^2\right)\left(\bar{\rho}^\theta - \bar{\rho}^\varepsilon\right)^2}{\left(1-\bar{\rho}^\theta + \sigma^2\left(1-\bar{\rho}^\varepsilon\right)\right)\left(1+\sigma^2+(I-1)\left(\bar{\rho}^\theta + \sigma^2\bar{\rho}^\varepsilon\right)\right)^3}.$$

[14] In markets with negative uniform correlations in values and noise, the marginal price informativeness is increasing in market size; thus, no such trade-off occurs.

[15] If $r^\theta > r^\varepsilon$, when r^θ is held fixed, an increase in r^ε decreases price informativeness and when r^ε is held fixed, an increase in r^θ increases informativeness. On the other hand, if $r^\varepsilon > r^\theta$, when r^ε is held fixed, an increase in r^θ decreases price informativeness and when r^θ is held fixed, an increase in r^ε increases informativeness. For a helpful visualization, see the price informativeness map in Figure 1.

attribute with the higher average correlation gives rise to a higher growth rate in the average correlation.[16] Similarly, in networks with Strengthening Links for Values and Strengthening Links for Noise, price informativeness is always increasing due to the fact that the attribute with the higher average correlation gives rise to the higher growth rate, while both average correlations are increasing,[17] which, along with increasing average correlations, by Proposition 5 implies the monotonicity.

In networks described by Regular Network for Value with Regular Network for Noise, although connectedness of the network, k, increases, this does not necessarily increase price informativeness as the market grows. Price informativeness may diminish for a small number of individual links for small market sizes, achieve a minimum at a non-zero level, and increase with further increases in I and k (this is the case for small markets, $I \leq 10$, with $I = 2k$, $\alpha^{\theta} = 0.099$ and $\alpha^{\varepsilon} = 0.8$). For fixed market size, $I(k) = I$, price informativeness is monotone by Proposition 5 for positive α,[18] and by Proposition 6 for negative α.[19]

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[16] The growth rate of the average correlation in a Condensing Network Model is

$$\frac{r'(k)}{r(k)} = \frac{-2\ln\beta}{\beta^{-k-1} + \beta^{-k} - 2}.$$

Differentiating with respect to β, we have

$$\frac{\partial \frac{r'}{r}}{\partial \beta} = \frac{-\beta^{-k-2} - \beta^{-k-1} + 2\beta^{-1} - \ln\beta^{k+1}\beta^{-k-2} + \ln\beta^{k}\beta^{-k-1}}{2^{-1}\left(\beta^{-k-1} + \beta^{-k} - 2\right)^{2}} > 0,$$

where we used the inequality $\frac{\ln x}{x} < 1 - \frac{1}{x}$ ($x \in (0,1)$) in the terms $\ln\beta^{k+1}\beta^{-k-2} < \frac{1}{\beta} - \frac{1}{\beta^{k+2}}$ and $\ln\beta^{k}\beta^{-k-1} < \frac{1}{\beta} - \frac{1}{\beta^{k+1}}$. Hence, $\beta^{\theta} > \beta^{\varepsilon}$ is equivalent to $r^{\theta} > r^{\varepsilon}$ and to $\frac{r^{\theta\prime}}{r^{\theta}} > \frac{r^{\varepsilon\prime}}{r^{\varepsilon}}$, which, using the increasing average correlations proves that, for any choice of β^{θ} and β^{ε}, the conditions of Proposition 5 hold and. Thus, price informativeness is monotone increasing.

[17] The growth rate of the average correlation in the Condensing Links Model is

$$\frac{r'}{r} = \frac{\beta^{(I/2)}\delta^{(I/2k)}\frac{I\ln\delta}{2k^{2}}}{1 - \beta^{(I/2)}\delta^{(I/2k)}}.$$

If $\beta^{\theta} < \beta^{\varepsilon}$, then $r^{\theta} < r^{\varepsilon}$ and $\frac{r^{\theta\prime}}{r^{\theta}} < \frac{r^{\varepsilon\prime}}{r^{\varepsilon}}$.

[18] Assume fixed market size I and $\alpha > 0$. Then the average correlation $r(k) = \frac{1+\alpha k}{I}$ is increasing in k. The growth rate of the average correlation, $\frac{r'}{r} = \frac{\alpha}{1+\alpha k}$, is increasing in α, thus the model satisfies the conditions of Proposition 5.

[19] Assuming a fixed market size I and $\alpha < 0$, the average correlation $r(k) = \frac{1+\alpha k}{I}$ is decreasing in k, implying $r(k) < r(1) = \frac{1+\alpha}{I} < 1$. Thus, $r^{\theta} + \sigma^{2} r^{\varepsilon} < \frac{1+\alpha^{\theta}}{I} + \sigma^{2}\frac{1+\alpha^{\varepsilon}}{I} < \frac{1+\alpha^{\theta}}{2} + \sigma^{2}\frac{1+\alpha^{\varepsilon}}{2} < \frac{1+\sigma^{2}}{2}$. This condition, together with the increasing growth rate, ensures monotone price informativeness by Proposition 6.

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Appendix

Proof: Proposition 2 (Equilibrium Bids) Imposing market clearing on conditions(5), the equilibrium price satisfies $p^* = \frac{1}{I}\sum_{i\in I} E(\theta|s_i, p^*)$. Using the conditional expectations

$E(\theta|s_i, p) = c_{\theta+\varepsilon_i} E(\theta + \varepsilon_i) + c_s s_i + c_p p$, the equilibrium price is

$$p^* = \frac{c_{\theta+\varepsilon_i} E(\theta + \varepsilon_i)}{1 - c_p} + \frac{c_s}{1 - c_p} \bar{s}, \tag{11}$$

where $\bar{s} = \frac{1}{I} \sum_{i \in I} s_i$. By (11), random vector (θ_i, s_i, p^*) is jointly normally distributed,

$$\begin{pmatrix} \theta_i \\ s_i \\ p^* \end{pmatrix} = \mathcal{N} \left[\begin{pmatrix} E(\theta_i) \\ E(\theta_i) \\ E(\theta_i) \end{pmatrix}, \begin{pmatrix} \sigma_\theta^2 & \sigma_\theta^2 & cov(\theta_i, p^*) \\ \sigma_\theta^2 & \sigma_\theta^2 + \sigma_\varepsilon^2 & cov(s_i, p^*) \\ cov(p^*, \theta_i) & cov(p^*, s_i) & Var(p^*) \end{pmatrix} \right]. \tag{12}$$

Covariances of the joint distribution are given by

$$cov(\theta_i, p^*) = \frac{1}{I} \frac{c_s}{1 - c_p} \left(1 + (I - 1) \bar{\rho}^\theta\right) \sigma_\theta^2 \tag{13}$$

$$cov(s_i, p^*) == \frac{1}{I} \frac{c_s}{1 - c_p} \left(1 + \sigma^2 + (I - 1) \left(\bar{\rho}^\theta + \bar{\rho}^\varepsilon \sigma^2\right)\right) \sigma_\theta^2 \tag{14}$$

and

$$Var(p^*) = \frac{1}{I} \left(\frac{c_s}{1 - c_p}\right)^2 \left(1 + \sigma^2 + (I - 1) \left(\bar{\rho}^\theta + \bar{\rho}^\varepsilon \sigma^2\right)\right) \sigma_\theta^2. \tag{15}$$

Applying the projection theorem[20] and the method of undetermined coefficients yields determines the inference coefficients c_s and c_p, (7) and (8), respectively. By $E(\theta_i) = E(s_i) = E(p)$, coefficient c_θ, (**??**), is obtained. By (5), the equilibrium bid is

$$q_i(p) = \frac{1}{\left(\mu - (\partial q_i(p) / \partial p)^{-1} / (I - 1)\right)} \left[c_\theta E(\theta_i) + c_s s_i - (1 - c_p) p\right], \tag{16}$$

where $\partial q_i(p) / \partial p$ is constant in the linear equilibrium. Solving the price derivative of (16) for the bid slope gives $\partial q_i(p) / \partial p = -(\gamma - c_p) / \mu$, and the equilibrium bids (6). *Q.E.D.*

Proof: Proposition 1 (Existence of Equilibrium) (*Only if*) The bid profile (6), $i \in I$, from Proposition 2 constitutes an equilibrium with downward-sloping bids only if the slope of the residual supply λ satisfies $\infty > \lambda > 0$. This implies $\gamma > c_p > -\infty$, which, by (8), requires

$$\bar{\rho}^\theta \neq -(1 - \gamma) .ok \tag{17}$$

[20] Let θ and s be random vectors such that $(\theta, s) \sim N(\mu, \Sigma)$, where

$$\mu \equiv \begin{pmatrix} \mu_\theta \\ \mu_s \end{pmatrix} \text{ and } \Sigma \equiv \begin{pmatrix} \Sigma_{\theta,\theta} & \Sigma_{\theta,s} \\ \Sigma_{s,\theta} & \Sigma_{s,s} \end{pmatrix},$$

are partitional expectations and variance covariance matrix and $\Sigma_{s,s}$ is positive definite. The distribution of θ conditional on s is normal and given by $(\theta|s) \sim N(\mu_\theta + \Sigma_{\theta,s}\Sigma_{s,s}^{-1}(s - \mu_s), \Sigma_{\theta,\theta} - \Sigma_{\theta,s}\Sigma_{s,s}^{-1}\Sigma_{s,\theta})$.

Since for any random vector $\{\theta_i\}_{i\in I}$, $\bar{\rho}^\theta \geq -(1-\gamma)$ holds, the lower bound on the commonality then follows from condition (17): $\bar{\rho}^\theta > -(1-\gamma) \equiv \bar{\rho}^{\theta-}(\gamma)$. The upper bound derives from condition $\gamma > c_p$, which by (8) is equivalent to

$$\frac{(2-\gamma)\sigma^2\left(\bar{\rho}^\theta - \bar{\rho}^\varepsilon\right)}{(1-\gamma+\bar{\rho}^\theta)\left((1-\bar{\rho}^\theta)+\sigma^2\left(1-\bar{\rho}^\varepsilon\right)\right)} < \gamma$$

and

$$\bar{\rho}^\theta < \frac{\gamma^2-(2-\gamma)\sigma^2+\sqrt{\left[(2-\gamma)\sigma^2-\gamma^2\right]^2+4\gamma\left[\gamma\left(1-\gamma+\sigma^2\right)+2\left(1-\gamma\right)\sigma^2\bar{\rho}^\varepsilon\right]}}{2\gamma} \equiv \bar{\rho}^{\theta+}\left(\gamma,\sigma^2\right).$$

For $\gamma = 0$, define the upper bound $\bar{\rho}^{\theta+}(0,\sigma^2)$ as the limit of $\bar{\rho}^{\theta+}(\gamma,\sigma^2)$ as $\gamma \to 0$, $\bar{\rho}^{\theta+}(0,\sigma^2) \equiv 0$. (*If*) Fix bids (6) for bidders $j \neq i$. For any commonality $\bar{\rho}^\theta$ such that $\bar{\rho}^{\theta-}(\gamma) < \bar{\rho}^\theta < \bar{\rho}^{\theta+}(\gamma,\sigma^2)$, the first-order condition (5) is then necessary and sufficient for optimality of the bid (6), for any price, for each i. Therefore, the bids from Proposition 2 constitute a unique symmetric linear Bayesian Nash equilibrium. *Q.E.D.*

Proof: Proposition 3 (Aggregation of Private Information) (*Only if*) Assume that the equilibrium price is privately revealing, that is, the posterior distributions coincide, $F(\theta_i|s_i,p^*) = F(\theta_i|s)$ for every i and s, given the equilibrium price $p^* = p^*(s)$. Fix i. Since the price is a deterministic function of the average signal (by (11)), $F(\theta_i|s_i,p^*) = F(\theta_i|s_i,\bar{s})$ holds.

Applying the projection theorem to (θ_i, s), $E(\theta_i|s_i,\bar{s}) = c_0 + c\cdot s$, where $c_{s_j} = c_{s_k}$ for all $j,k \neq i$ in the vector of constants $c^T = (c_{s_1}, c_{s_2}, ..., c_{s_I})$. Equality $E(\theta_i|s_i,\bar{s}) = E(\theta_i|s)$ for all s implies that the coefficients multiplying each s_k, $k \in I$, are the same in the two conditional expectations. Hence, in $E(\theta_i|s)$, the coefficients satisfy $c_{s_j} = c_{s_k}$ for all $j,k \neq i$.

Let $\Sigma_{s,s} \equiv \sigma_\theta^2\mathcal{C}^\theta + \sigma_\varepsilon^2\mathcal{C}^\varepsilon$ be the variance-covariance matrix of signals $\{s_i\}_{i\in I}$ and let $\Sigma_{\theta_i,s} = \{cov(\theta_i,s_k)\}_{k\in I}$ be the row vector of covariances. Using the positive semidefiniteness of $\mathcal{C}^\theta$ and $\mathcal{C}^\varepsilon$, and hence $\Sigma_{s,s}$, and applying the projection theorem, coefficients $c \in R^I$ in expectation $E(\theta_i|s)$ are characterized by $c^T = \Sigma_{\theta_i,s}\Sigma_{s,s}^{-1}$. For any $j \neq i$, write the j^{th} row of $(\Sigma_{\theta_i,s})^T = \Sigma_{s,s}c$ as, $cov(\theta_i,s_j) = \sum_{k\in I}(cov(\theta_j,\theta_k) + cov(\varepsilon_j,\varepsilon_k))\,c_{s_k}$. Using $cov(\theta_i,s_j) = cov(\theta_i,\theta_j)$ and $c_{s_j} = c_{s_k}$ for all $j,k \neq i$, the j^{th} row can be written as

$$cov(\theta_i,\theta_j) = c_{s_j}\left(\sum_{k\neq j} cov(\theta_j,\theta_k) + \sum_{k\neq j} cov(\varepsilon_j,\varepsilon_k)\right) + \left(c_{s_i} - c_{s_j}\right)\left(cov(\theta_i,\theta_j) + cov(\varepsilon_j,\varepsilon_k)\right). \tag{18}$$

(18) gives

$$cov(\theta_i,\theta_j)=\frac{c_{s_j}(I-1)\left(\sigma_\theta^2\bar{\rho}^\theta+\sigma_\varepsilon^2\bar{\rho}^\varepsilon\right)}{1-\left(c_{s_i}-c_{s_j}\right)}+\frac{\left(c_{s_i}-c_{s_j}\right)cov(\varepsilon_i,\varepsilon_j)}{1-\left(c_{s_i}-c_{s_j}\right)}. \tag{19}$$

Applying the projection theorem to $(\theta_i, s_i, \bar{s})$, coefficients $\bar{c}\in R^2$ in expectation $E(\theta_i|s_i,\bar{s})$ are characterized by $\bar{c}^T=\Sigma_{\theta_i,(s_i,\bar{s})}\Sigma^{-1}_{(s_i,\bar{s}),(s_i,\bar{s})}$, the first row of which is

$$cov(\theta_i,\theta_j)=\left(\sigma_\theta^2+\sigma_\varepsilon^2\right)c_{s_i}+cov(\theta_i,\bar{s})\,c_{\bar{s}}. \tag{20}$$

By (19) and (20), and since $c_{s_j}=c_{s_k}$ for all $j,k\neq i$, $cov(\varepsilon_i,\varepsilon_j)=cov(\varepsilon_i,\varepsilon_k)$ for all $j,k\neq i$ and hence, by equicommonality of $\mathcal{C}^\varepsilon$, $\rho^\varepsilon_{i,j}=\bar{\rho}^\varepsilon$ for all $j\neq i$. Since the argument holds for any i, it follows that $\rho^\varepsilon_{i,j}=\bar{\rho}^\varepsilon$ for all pairs i,j, $i\neq j$. Then, by (19) and equicommonality of $\mathcal{C}^\theta$, $\rho_{i,j}=\bar{\rho}^\theta$ for all $j\neq i$, and $\rho_{i,j}=\bar{\rho}^\theta$ for all pairs i,j, $i\neq j$. (*If*) The argument from Rostek and Weretka (2011) applies. *Q.E.D.*

Proof: Proposition 4 (Informational Impact) The projection theorem gives conditional variances $Var(\theta_i|s_i)$ and $Var(\theta_i|s_i,p^*)$, from which price informativeness ψ^+ is derived.

$$Var(\theta_i|s_i)=\sigma_\theta^2-\frac{\sigma_\theta^4}{\sigma_\theta^2+\sigma_\varepsilon^2}$$

$$Var(\theta_i|s_i,p^*)=\sigma_\theta^2-\frac{\sigma_\theta^4 Var(p^*)-2\sigma_\theta^2 cov(p^*,s_i)\,cov(\theta_i,p^*)+(\sigma_\theta^2+\sigma_\varepsilon^2)\left[cov(\theta_i,p^*)\right]^2}{(\sigma_\theta^2+\sigma_\varepsilon^2)\,Var(p^*)-\left[cov(p^*,s_i)\right]^2}.$$

$$\begin{aligned}\psi^+ &\equiv \frac{Var(\theta_i|s_i)-Var(\theta_i|s_i,p^*)}{Var(\theta_i|s_i)}\\ &= \frac{1}{\sigma^2}\left[\frac{(1+\sigma^2)^2\left[cov(\theta_i,p^*)\right]^2+\left[cov(s_i,p^*)\right]^2-2(1+\sigma^2)\,cov(p^*,s_i)\,cov(\theta_i,p^*)}{(\sigma_\theta^2+\sigma_\varepsilon^2)\,Var(p^*)-\left[cov(p^*,s_i)\right]^2}\right]\end{aligned}$$

Substituting from (13), (14), and (**??**), and using $\sigma^2=\sigma_\varepsilon^2/\sigma_\theta^2$, we have

$$\psi^+=\frac{\sigma^2\left(\bar{\rho}^\theta-\bar{\rho}^\varepsilon\right)^2}{(1+\sigma^2)^2(1-\gamma)+\gamma\left(\bar{\rho}^\theta+\bar{\rho}^\varepsilon\sigma^2\right)(1+\sigma^2)-\left(\bar{\rho}^\theta+\bar{\rho}^\varepsilon\sigma^2\right)^2}=\frac{\sigma^2\left(r^\theta-r^\varepsilon\right)^2}{\left(r^\theta+\sigma^2 r^\varepsilon\right)\left[1-r^\theta+\sigma^2\left(1-r^\varepsilon\right)\right]}. \tag{21}$$

Let $d\equiv\bar{\rho}^\theta-\bar{\rho}^\varepsilon$. Given $\bar{\rho}^\varepsilon$, equation (21) is quadratic in d and has roots

$$d=\frac{\psi^+(1+\sigma^2)(\gamma-2\bar{\rho}^\varepsilon)\pm(1+\sigma^2)\sqrt{\psi^{+2}\gamma^2+4\psi^+\left[(1-\gamma)\left(\sigma^2+\psi^+\right)+\sigma^2\bar{\rho}^\varepsilon(\gamma-\bar{\rho}^\varepsilon)\right]}}{2\left(\sigma^2+\psi^+\right)}. \tag{22}$$

For any $\psi^+\in[0,1]$ and $\gamma\in[0,1)$, (22) gives the values of d that induce price informativeness equal to ψ^+. For $\psi^+ > 0$, equation (21) has a positive and a negative root. For a fixed pair $\left(\gamma, \bar{\rho}^\theta\right)$ and the corresponding ψ^+, the threshold τ is determined as the change in d that maintains constant the value of price informativeness ψ^+ with an additional trader, whose inclusion increases γ by $\Delta\gamma \equiv 1/\left[I\left(I-1\right)\right] = \left(1-\gamma\right)^2/\left(2-\gamma\right)$. Using (22) for $d>0$, threshold τ can be found,

$$\begin{aligned}\tau &= \frac{\left(1+\sigma^2\right)}{2\left(\sigma^2+\psi^+\right)}\left[\psi^+\Delta\gamma + \sqrt{\psi^{+2}\left(\gamma+\Delta\gamma\right)^2 + 4\psi^+\left[\left(1-\gamma-\Delta\gamma\right)\left(\sigma^2+\psi^+\right)+\sigma^2\bar{\rho}^\varepsilon\left(\gamma-\Delta\gamma-\bar{\rho}^\varepsilon\right)\right]}\right. \\ &\quad \left.\sqrt{\psi^{+2}\gamma^2 + 4\psi^+\left[\left(1-\gamma\right)\left(\sigma^2+\psi^+\right)+\sigma^2\bar{\rho}^\varepsilon\left(\gamma-\bar{\rho}^\varepsilon\right)\right]}\right].\end{aligned}$$

The positive root in (22) is decreasing in γ and increasing in ψ^+, hence $\tau < 0$. The threshold for $d < 0$ is derived analogously. *Q.E.D.*

Proof: Proposition 5 (Sufficient Condition for Monotone Price Informativeness)

$$\psi^+\left(I\right) - \psi^+\left(I-1\right) > 0$$

is equivalent to

$$\frac{\left(r_I^\theta - r_I^\varepsilon\right)^2\left(r_{I-1}^\theta + \sigma^2 r_{I-1}^\varepsilon\right)\left[1 - r_{I-1}^\theta + \sigma^2\left(1-r_{I-1}^\varepsilon\right)\right] - \left(r_{I-1}^\theta - r_{I-1}^\varepsilon\right)^2\left(r_I^\theta + \sigma^2 r_I^\varepsilon\right)\left[1 - r_I^\theta + \sigma^2\left(1-r_I^\varepsilon\right)\right]}{\left(r_I^\theta + \sigma^2 r_I^\varepsilon\right)\left(r_{I-1}^\theta + \sigma^2 r_{I-1}^\varepsilon\right)} > 0$$

since $\left(r_I^\theta + \sigma^2 r_I^\varepsilon\right)\left(r_{I-1}^\theta + \sigma^2 r_{I-1}^\varepsilon\right) > 0$, we have

$$\begin{aligned} &\left(r_I^\theta - r_I^\varepsilon\right)^2\left[\left(r_{I-1}^\theta + \sigma^2 r_{I-1}^\varepsilon\right)\left(1+\sigma^2\right) - \left(r_{I-1}^\theta + \sigma^2 r_{I-1}^\varepsilon\right)^2\right] \\ > \; &\left(r_{I-1}^\theta - r_{I-1}^\varepsilon\right)^2\left[\left(r_I^\theta + \sigma^2 r_I^\varepsilon\right)\left(1+\sigma^2\right) - \left(r_I^\theta + \sigma^2 r_I^\varepsilon\right)^2\right] \end{aligned} \tag{23}$$

$$\begin{aligned} &\left(r_I^\theta - r_I^\varepsilon\right)^2\left(r_{I-1}^\theta + \sigma^2 r_{I-1}^\varepsilon\right)\left[\left(1+\sigma^2\right) - \left(r_{I-1}^\theta + \sigma^2 r_{I-1}^\varepsilon\right)\right] \\ > \; &\left(r_{I-1}^\theta - r_{I-1}^\varepsilon\right)^2\left(r_I^\theta + \sigma^2 r_I^\varepsilon\right)\left[\left(1+\sigma^2\right) - \left(r_I^\theta + \sigma^2 r_I^\varepsilon\right)\right]. \end{aligned}$$

For $\psi^+\left(I\right) - \psi^+\left(I-1\right) > 0$, it is sufficient that

$$r_I^\theta + \sigma^2 r_I^\varepsilon > r_{I-1}^\theta + \sigma^2 r_{I-1}^\varepsilon$$

$$r_I^\theta - r_{I-1}^\theta + \sigma^2\left(r_I^\varepsilon - r_{I-1}^\varepsilon\right) > 0 \tag{24}$$

Q.E.D.

Special cases:

If $\bar{\rho}^{\varepsilon}=0$, then

$$\begin{aligned} c_s &= \frac{1-\bar{\rho}^{\theta}}{1-\bar{\rho}^{\theta}+\sigma^2}, \\ c_p &= \frac{(2-\gamma)\,\sigma^2\bar{\rho}^{\theta}}{(1-\gamma+\bar{\rho}^{\theta})(1-\bar{\rho}^{\theta}+\sigma^2)} = c_s\frac{(2-\gamma)\,\sigma^2\bar{\rho}^{\theta}}{(1-\bar{\rho}^{\theta})(1-\gamma+\bar{\rho}^{\theta})}. \end{aligned}$$

If $\bar{\rho}^{\theta}=0$, then

$$\begin{aligned} c_s &= \frac{1}{1+\sigma^2(1-\bar{\rho}^{\varepsilon})}, \\ c_p &= \frac{-(2-\gamma)\,\sigma^2\bar{\rho}^{\varepsilon}}{(1-\gamma)(1+\sigma^2(1-\bar{\rho}^{\varepsilon}))} = -\frac{(2-\gamma)}{(1-\gamma)}\sigma^2 c_s\bar{\rho}^{\varepsilon}. \end{aligned}$$

Alternative conditions for monotonicity of price informativeness:[21] (1) Assume that $\bar{\rho}^{\theta}$ and $\bar{\rho}^{\varepsilon}$ do not change with I. $\psi^+(I)-\psi^+(I-1)>0$ if, and only if,

$$\frac{\sigma^2\left(\bar{\rho}^{\theta}-\bar{\rho}^{\varepsilon}\right)^2(1+\sigma^2)}{[(1+\sigma^2)-(\bar{\rho}^{\theta}+\sigma^2\bar{\rho}^{\varepsilon})]}\frac{1}{[(1+\sigma^2)+(I-1)(\bar{\rho}^{\theta}+\sigma^2\bar{\rho}^{\varepsilon})][(1+\sigma^2)+(I-2)(\bar{\rho}^{\theta}+\sigma^2\bar{\rho}^{\varepsilon})]}>0$$

which is equivalent to

$$\left[\left(1+\sigma^2\right)+(I-1)\left(\bar{\rho}^{\theta}+\sigma^2\bar{\rho}^{\varepsilon}\right)\right]\left[\left(1+\sigma^2\right)+(I-2)\left(\bar{\rho}^{\theta}+\sigma^2\bar{\rho}^{\varepsilon}\right)\right]>0$$

If $\bar{\rho}^{\theta},\bar{\rho}^{\varepsilon}>0$, then $\psi^+(I)-\psi^+(I-1)>0$. If $\bar{\rho}^{\theta},\bar{\rho}^{\varepsilon}<0$, then for $I=2$, $\psi^+(I)-\psi^+(I-1)>0$, for $I=3$, parameter values matter, then for I large enough, $\psi^+(I)-\psi^+(I-1)>0$.

Next, allow $\bar{\rho}^{\theta}$ and $\bar{\rho}^{\varepsilon}$ to change with I.

[21] Another once but improper:

$$\begin{aligned} \psi^+ &= \frac{\sigma^2 d^2}{\left[(1+\sigma^2)\left(1-\gamma+\bar{\phi}\right)+d\right]\left[(1+\sigma^2)\left(1-\bar{\phi}\right)-d\right]} \\ &= \frac{(I-1)\,\sigma^2 d^2}{\left[(1+\sigma^2)\left(1+(I-1)\,\bar{\phi}\right)+d\,(I-1)\right]\left[(1+\sigma^2)\left(1-\bar{\phi}\right)-d\right]} \end{aligned}$$

Differentiating with respect to d, we obtain

$$\psi^{+\prime}=\frac{\sigma^2 d^2}{(1+\sigma^2)\left[\left(1+(I-1)\,\bar{\phi}\right)+d\,(I-1)\right]^2\left[(1+\sigma^2)\left(1-\bar{\phi}\right)-d\right]}$$

where we used that $\left(1+\sigma^2\right)\left(1-\bar{\phi}\right)-d>0$.

$$\psi^+(I) = \frac{(I-1)\,\sigma^2\left(\bar{\rho}^\theta(I) - \bar{\rho}^\varepsilon(I)\right)^2}{\left[(1+\sigma^2) + (I-1)\left(\bar{\rho}^\theta(I) + \bar{\rho}^\varepsilon(I)\,\sigma^2\right)\right]\left[(1+\sigma^2) - \left(\bar{\rho}^\theta(I) + \bar{\rho}^\varepsilon(I)\,\sigma^2\right)\right]}$$

(2) If $r_I^\theta - r_{I-1}^\theta > -\sigma^2\left(r_I^\varepsilon - r_{I-1}^\varepsilon\right)$, then $\psi^+(I) - \psi^+(I-1) > 0$ if

$$\left(r_I^\theta + \sigma^2 r_I^\varepsilon\right) + \left(r_{I-1}^\theta + \sigma^2 r_{I-1}^\varepsilon\right) > 1 + \sigma^2. \tag{25}$$

Using $r_I^\theta - r_{I-1}^\theta > -\sigma^2\left(r_I^\varepsilon - r_{I-1}^\varepsilon\right)$, (25) gives $r_I^\theta + \sigma^2 r_I^\varepsilon > \frac{1+\sigma^2}{2}$.

If $r_I^\theta - r_{I-1}^\theta < -\sigma^2\left(r_I^\varepsilon - r_{I-1}^\varepsilon\right)$, then $\psi^+(I) - \psi^+(I-1) > 0$ if

$$\left(r_I^\theta + \sigma^2 r_I^\varepsilon\right) + \left(r_{I-1}^\theta + \sigma^2 r_{I-1}^\varepsilon\right) < 1 + \sigma^2. \tag{26}$$

Using $r_I^\theta - r_{I-1}^\theta < -\sigma^2\left(r_I^\varepsilon - r_{I-1}^\varepsilon\right)$, (26) gives $r_I^\theta + \sigma^2 r_I^\varepsilon < \frac{1+\sigma^2}{2}$.

Proof: Back to (23), for $\psi^+(I) - \psi^+(I-1) > 0$, it is sufficient that

$$\begin{aligned}
\left[\left(r_{I-1}^\theta + \sigma^2 r_{I-1}^\varepsilon\right)\left(1+\sigma^2\right) - \left(r_{I-1}^\theta + \sigma^2 r_{I-1}^\varepsilon\right)^2\right] &> \left[\left(r_I^\theta + \sigma^2 r_I^\varepsilon\right)\left(1+\sigma^2\right) - \left(r_I^\theta + \sigma^2 r_I^\varepsilon\right)^2\right] \\
\left[\left(r_I^\theta + \sigma^2 r_I^\varepsilon\right) - \left(r_{I-1}^\theta + \sigma^2 r_{I-1}^\varepsilon\right)\right]\left[\left(r_I^\theta + \sigma^2 r_I^\varepsilon\right) + \left(r_{I-1}^\theta + \sigma^2 r_{I-1}^\varepsilon\right)\right] &> \left[\left(r_I^\theta + \sigma^2 r_I^\varepsilon\right) - \left(r_{I-1}^\theta + \sigma^2 r_{I-1}^\varepsilon\right)\right]\left(1 +
\end{aligned}$$

If $\left(r_I^\theta + \sigma^2 r_I^\varepsilon\right) > \left(r_{I-1}^\theta + \sigma^2 r_{I-1}^\varepsilon\right)$, then for $\psi^+(I) - \psi^+(I-1) > 0$, it is sufficient that

$$\begin{aligned}
\left(r_I^\theta + \sigma^2 r_I^\varepsilon\right) + \left(r_{I-1}^\theta + \sigma^2 r_{I-1}^\varepsilon\right) &> 1 + \sigma^2 \\
r_I^\theta + \sigma^2 r_I^\varepsilon &> \frac{1+\sigma^2}{2}.
\end{aligned}$$

If $\left(r_I^\theta + \sigma^2 r_I^\varepsilon\right) < \left(r_{I-1}^\theta + \sigma^2 r_{I-1}^\varepsilon\right)$, then for $\psi^+(I) - \psi^+(I-1) > 0$, it is sufficient that

$$\begin{aligned}
\left(r_I^\theta + \sigma^2 r_I^\varepsilon\right) + \left(r_{I-1}^\theta + \sigma^2 r_{I-1}^\varepsilon\right) &< 1 + \sigma^2 \\
r_I^\theta + \sigma^2 r_I^\varepsilon &< \frac{1+\sigma^2}{2}.
\end{aligned}$$

Q.E.D.

Figure 1

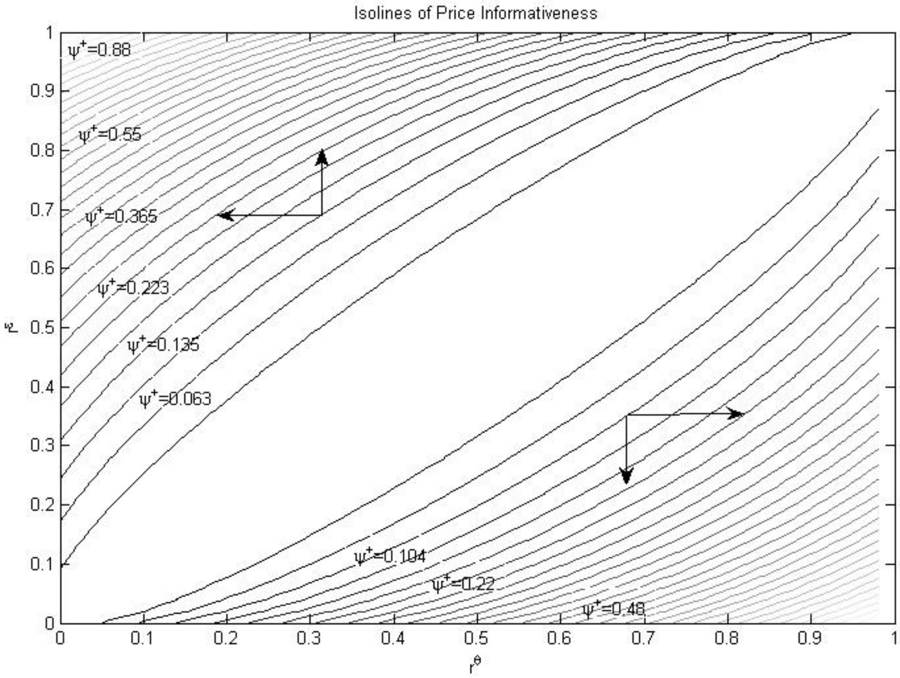

Figure 2