NEW YORK UNIVERSITY
STERN SCHOOL OF BUSINESS
FINANCE DEPARTMENT

Working Paper Series, 1994

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FD-94-26

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1 September 1994

Abstract

Most valuation theory has ignored transaction costs, has either ignored taxes or treated them *ad hoc*, and has used NPV to value riskless cash flows. Recently, in a new approach, transaction costs were explicitly considered, taxes were treated in more detail, and the single term structure vector of NPV was replaced by a convex multifaceted packet of term structure vectors. Accordingly, simple vector multiplication of cash flows by that single term structure vector was replaced by linear programming over this entire packet. But is the resulting complication worth the trouble? This paper shows that NPV can create substantial errors due to the difference between long and short positions. The biggest errors stem from ignoring short-borrowing costs and can be exacerbated by ignoring taxes. Actual market prices are used along with the transaction cost and tax schedules of major investors, including their four tax payment dates each year.

The author gratefully acknowledges the assistance of three Treasury bond traders: Gregory D. Cellucci of Smith Barney, Harris Upman Co.; Corsan Malsey of ITT Hartford; and Mark Rzepcznski of First Chicago Corporation.

1. Introduction

Most theory on term structure has ignored transaction costs as too small or too bothersome to worry about, and has either neglected taxes or treated them *ad hoc*. Taxes have nearly always been treated as proportional to and paid with the payments of the cash flow being valued.[1] The traditional assumption of no transaction costs has meant that long and short trades could be established at the same price and has lead to a linear pricing operator, NPV, becoming a cornerstone of finance theory. This assumption has also lead to analogous valuation of risky cash flows in state space models. The traditional NPV approach, which is commonly used on Wall Street, will be referred to as Approach I. An enlightening summary of Approach I is found in Ruback (1986), which also extends the traditional approach by showing when to use the present value method and the adjusted (for tax shields) present value method.

In a model ignoring transaction costs, Approach I values any future default-free cash flow vector w for a given tax class of investors as the scalar product dw, where d is a single vector of discount factors associated with certain prices of traded bonds. This linear valuation has been an important mathematical convenience in the literature and in practice. However, Schaefer's (1982a) paradox shows that no set of bond prices could avoid arbitrage in a simple market with different tax classes and no transaction costs. In response to such problems, a new approach to valuation has been devised by Dermody and Rockafellar (1991). See that article for a sketch of the literature on a single term structure for each tax class and of "representative" or "marginal" investors. In particular, it discusses Schaefer's paradox and unsuccessful efforts to refute the paradox.

The new approach uses the duality theory of linear programming to develop a method of valuation for riskless cash flows in the face of transaction costs and taxes. These transaction costs are bid-ask spreads and, more importantly, short-borrowing costs. The taxes treated are the different tax payments or credits that occur on the taxed investor's four estimated tax payment dates per year as a result of trading each bond. These payments are different for buying, selling, short selling, and reversing a short sale.

The no-arbitrage valuation operator for each class of investors under this new approach is associated with a multifaceted convex set of term structures d, in contrast to the single term structure vector associated with Approach I. Each future cash flow vector $w = (w_1, \ldots, w_n)$, that pays w_i on future date i, is valued by more than one term structure d on the boundary of this set. One d provides the upper value bound dw, which is the minimum cost of obtaining w through trades at market prices that are net of the transaction costs of the class. A different d provides the lower value bound dw, which is the maximum cash that could be obtained in exchange for an obligation to pay w through such trades. These valuations are linear to scale but not additive. They apply to any investor in the given tax and transaction-cost class; any special prices particular to a given investor in the class are ignored. This first part of the new approach will be referred to as Approach II.

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[1] Some exceptions that treated transaction costs are Hodges and Schaefer (1977), Garman and Ohlson (1981), Prisman (1986) (1990a), Chen (1992), and Dermody and Rockafellar (1993) (1991). Articles that treated taxes include Schaefer (1986a) (1982b), Jordan (1984), Constantinides and Ingersoll (1984), Ruback (1986), Prisman (1986) (1990b), Bierwag and Suchanek (1990), Katz and Prisman (1991) (1992), and Dermody and Rockafellar (1993) (1991). See the latter for a lengthy discussion of transaction costs and taxes in the term structure literature.

Before the new approach was developed, several papers used linear programming in models of the term structure, and some of these, notably Hodges and Schaefer (1977), admitted transaction costs. In each case linear programming was employed in the customary task of finding a single term structure rather than the infinite multiplicity of no-arbitrage term structures (in the above mentioned multifaceted convex set) that stems from transaction costs. These papers assumed no short borrowing and, if they modeled taxes, did so *ad hoc*. See Dermody and Rockafellar (1991) for a discussion of these previous papers and their relation to the new approach.

Dermody and Rockafellar (1993) extended their theory to treat any special pricing opportunities that might face a particular investor because of previously acquired positions.[2] Since the investor's price worsens as successive best previously acquired positions are exhausted, such valuation is not even linear to scale. This second part of the new approach will be referred to as Approach III.

For a given set of no-arbitrage long and short bond prices an investor faces in a market and for any particular future cash stream w, the new approach provides the minimum cost $V(w)$ (i.e., upper value bound) of acquiring w through market trades. The new approach also provides the most current cash $v(w)$ (i.e., lower value bound) that can be obtained through any set of market trades that obligates the investor to pay at most w in the future. Since the investor can obtain w for $V(w)$ but not cheaper, it is the true long value of w. Since the investor can get $v(w)$ for w but not more, it is the true short value of w. Note that $V(w)$ may involve short-borrowing and selling a bond even though there is no negative component w_i in w. In general, Approach I provides a valuation of w different than both $V(w)$ and $v(w)$. In our nonnegative examples cash flows w^0 and w^1, Approach I values w above both $V(w)$ and $v(w)$. In our example w^2 with a negative cash flow component, Approach I values w above, between, and below $V(w)$ and $v(w)$, depending on whether the investor is taxed and whether w is valued long or short.

Thus, the new approach to valuation is accurate in markets with transaction costs while the traditional NPV approach (Approach I) is not in general. Recall that Schaefer's paradox shows that arbitrage must be available in many ordinary markets without transaction costs. But this new approach is also more complicated. However, this complication resides principally in theoretical modeling and in computer software for valuations of actual cash flows; investors need only have an appreciation for the distinctions involved, e.g., accounting for previously acquired position and taxes.

Each set of investors facing a particular tax schedule is a tax class. Untaxed investors constitute one class and investors paying the top US corporate rate of 34% constitute another. Each set of investors facing a particular transaction cost schedule is a transaction-cost class. Every combination of tax and transaction-cost class will have a different pricing operator. If one considers previously acquired positions as in Approach III, then a third dimension is added to the description of investor classes. Investors with the same transaction cost and tax schedules are in different classes if they have different previously acquired positions because

[2] A previously acquired long position allows an investor to make a short trade by selling a long position in a bond, instead of short-borrowing the bond and selling it. A previous short position allows an investor to make a long trade by unwinding a short position through buying the bond and selling the unused short-borrowing rights (to maturity), instead of just buying it.

they face more advantageous prices (until the previous positions are exhausted) in different bonds. Is the new approach worth the trouble, i.e., are there situations where substantial mistakes are made by Approach I, and just how much more complicated is the new approach?

The purpose of this paper is to answer both parts of this question empirically. It does so with simple examples of cash flows priced in a subset of the market for US Treasury debt that has maturities in five or eleven months. These examples include the precise prices, transaction costs, and tax situation (including the tax payment dates) facing a major investor at a particular instant in the US Treasury market. It shows step-by-step the valuation of each example under the traditional NPV approach (Approach I) and under the new approach. One can judge the importance of the errors stemming from the use of Approach I, which are summarized later in Table 7, in light of the difference in computations shown.

To bond trading firms, the errors for the nonnegative example cash flows are large (+.01% to +2.28%), and the errors for the example cash flow with a negative component are very large (−20% to +91%). From the viewpoint of an analyst engaged in long term capital budgeting, the former errors are less significant. A large part of some of the errors stem from the fact that Approach I ignores short borrowing costs. Bonds with payments further in the future have higher short borrowing costs *ceteris paribus*, and thus such errors in valuation are larger for cash flows further in the future. Firms that are not permitted to short borrow will have larger errors than otherwise identical firms.

2. Approaches and Data

Consider a market with three default-free bonds and two periods, which exhibits the actual market prices, transaction costs, and taxation faced by major investors for those bonds in the US Treasury market. Any such investor can long or short a given cash flow w by longing and shorting these three bonds. Note that for consistency, the cash flow being valued w is tax-free since there is no income associated with it. The bonds used to replicate w long or short do have tax consequences if held, if short-borrowed and sold, or if issued by a taxed investor.

Any of these major investors can create a long position in a bond by buying the bond at the ask price. But, if the investor has a previously acquired short position in that bond, then a long position can be established in a different way at a better (lower) price: unwind the short position, i.e., purchase the bond for the ask price and sell the unused short-borrowing rights until maturity for the bid price of short borrowing. Any investor can create a short position in a bond by short borrowing the bond at the ask price of short borrowing and then selling the bond for the bid price. But if the investor has a previously acquired long position in that bond, then a short position can be established in a different way at a better (higher) price: by unwinding the long position, i.e., sell the bond for the bid price. Thus long and short prices are more advantageous by the amount of the bid and ask price of short borrowing, respectively, for investors holding previously acquired opposite positions. Once the previously acquired opposite position is exhausted, the investor then faces the less advantageous price available to every investor in the transaction-cost and tax class.

We will treat three levels of sophistication. Approach I uses simple vector multiplication (scalar product) in the customary NPV approach common on Wall Street and described in Ruback (1986). Approaches II and III were developed in Dermody and Rockafellar (1991) and (1993), respectively, and use a particular no-arbitrage valuation of riskless cash flows.

Since intermediate trading would make the cash flows risky, only buy-and-hold positions are associated with this type of no-arbitrage valuation. Thus future price changes and tax-option effects (c.f., Ehrhardt, Jordan, and Prisman (1992)) as well as liquidity and reconstitution effects (c.f., Daves and Ehrhardt (1993)) are necessarily ignored. But any effects of incomplete markets (c.f. John (1984)) are captured in Approaches II and III.

Approach II uses linear programming with the term structure packet of no-arbitrage discount factors common to all investors facing the same transaction-cost and tax schedules. In contrast to the frictionless market assumed in Approach I, the transaction costs facing a class of investors imply a long price larger than the short price for each bond. The resulting discount factors are associated with no arbitrage in that they discount (or value) the (riskless) future cash flow of each bond to an amount between the long and short prices of that bond faced by those investors. See Dermody and Rockafellar (1991).

Extending this method of valuation, Approach III uses a far smaller term structure packet that reflects the more favorable prices of long and short trades peculiar to a given investor, if that investor is holding any previously acquired opposite positions. See Dermody and Rockafellar (1993). Note that as a consequence of Approaches II and III, the minimum cost set of trades that replicates a given w long may involve short borrowing and selling a bond even though there is no negative component in w.

We start with part of the actual US Treasury market facing a major bond trader at 10H30 EST Tuesday 26 January 1993. It is listed in Table 1. See the Appendix for the language and conventions of the bid and ask bond prices, as well as, the bid and ask reverse repo rates (which are used to compute the bid and ask short-borrowing cost). For ease of reference, note that inserted under the price and rate column headings of Table 1 are the symbols that will represent these quantities in later sections for Bonds $j = 1, 2, 3$.

Table 1. Market Prices Faced By A Major Investor.*

Bonds name/type maturity	Prices (per $100 face value) Bid P_j^b	Ask P_j^a	Reverse Repo Rates Bid r_j^b	Ask r_j^a	Bond Payments 15 May 93 a_{2j}	15 Nov 93 a_{4j}
Bond 1 Strip 15 May 93	$ 99.082005	$ 99.084978	3.16%	3.14%	$100	$ 0
Bond 2 Strip 15 Nov 93	97.546780	97.554525	3.21	3.19	0	100
Bond 3 11.75% Coupon 15 Nov 93	108.883892	108.915142	3.21	3.19	5.875	105.875

* The same prices were reported by three major investors at 10H30 EST Tuesday 26 January 1993.

3. Approach I: Customary NPV Method

For an untaxed investor, the single term structure in this approach is the discount factor $\overline{d} = (\overline{d}_2, \overline{d}_4)$ for our two-period model that is derived from the ask prices P_j^a and cash flows a_{ij} of the strips, i.e., of Bonds 1 and 2. For an untaxed investor $\overline{d} = (\overline{d}_2, \overline{d}_4) = (P_1^a/a_{21}, P_2^a/a_{42})$ = (\$99.084978 ask price / \$100 Bond 1 payment, \$97.554525 ask price / \$100 Bond 2 payment) = (.990850, .975545).[3] For that investor, the Approach I value of any cash flow w is $\overline{V}(w) = \overline{d}w = (\overline{d}_2, \overline{d}_4)(w_2, w_4)$. The dates $i = 1, 3$ (for d_1 and d_3) are reserved for tax payments dates that do not fall on one of the two bond payment dates. Some bond traders use the average of bid and ask prices instead of the ask prices, while others extract discount factors econometrically from the prices of a series of coupon bonds. However, such alterations in method have a minuscule effect on the difference between valuation in Approach I and valuation in Approaches II and III.

Let us value under Approach I without taxes our three examples of future cash streams with payment dates $i = 2, 4$: $w^0 = (\$100, \$100)$, $w^1 = (\$0, \$100)$, and $w^2 = (\$100, -\$100)$. We employ the traditional NPV single after-tax term structure $\overline{d}$ computed above.[4]

Untaxed

$$\begin{aligned}\overline{V}(w^0) &= \overline{d}w^0 = (.990850, .975545)(\$100, \quad \$100) = \$196.639503\\ \overline{V}(w^1) &= \overline{d}w^1 = (.990850, .975545)(\$\ \ 0, \quad \$100) = \$\ 97.554525\\ \overline{V}(w^2) &= \overline{d}w^2 = (.990850, .975545)(\$100, -\$100) = \$\quad 1.530453\end{aligned}$$

Valuing the example cash flows under Approach I with taxes is similar. But, the bond payments (due bond holders) of the two strips (Bonds 1 and 2) are adjusted for tax payments, that are assumed to be due on the date of the bond payments instead of being due on the four estimated tax payment dates per year. Furthermore taxes are computed as if long and short positions are established at the same price. See the conventions in Ruback (1986). The after-tax term structure is computed from the strips. Let a_{ij} and $\widehat{a}_{ij}$ be the before and after tax bond payments, respectively, of Bond j on date i. Then $\widehat{a}_{12} = a_{12} - [a_{12} - P_1^a](\text{taxrate}) = \$100 - [\$100 - \$99.084978](.34) = \$99.688892$ and $\widehat{a}_{24} = a_{24} - [a_{24} - P_2^a](\text{taxrate}) = \$100 - [\$100 - 97.554525](.34) = \$99.168538$. Hence $\overline{d}_2 = P_1^a/\widehat{a}_{12} = \$99.084978/\$99.688892 = .993942$ and $\overline{d}_4 = P_2^a/\widehat{a}_{24} = \$97.554525/\$99.168538 = .983725$. Thus, the traditional NPV single after- tax term structure of Approach I $\overline{d} = (\overline{d}_2, \overline{d}_4)$ values our example cash flows as follows.

Taxed

$$\begin{aligned}\overline{V}(w^0) &= \overline{d}w^0 = (.993942, .983725)(\$100, \quad \$100) = \$197.766655\\ \overline{V}(w^1) &= \overline{d}w^1 = (.993942, .983725)(\$\ \ 0, \quad \$100) = \$\ 98.372454\\ \overline{V}(w^2) &= \overline{d}w^2 = (.993942, .983725)(\$100, -\$100) = \$\quad 1.021747\end{aligned}$$

In contrast to these tax computations of Approach I, the actual tax payments would be computed with P_j^b or $P_j^b - \sigma_j^a$ instead of P_j^a for the Bonds $j = 1, 2, 3$ used to create w^0, w^1,

[3] Computation is done with 11 digits, but display is rounded off to fewer digits.

[4] An obvious improvement to Approach I valuation of w^2 is to use the Bond 2 bid price, instead of the ask price, to determine the d_2 with which to multiply $w_2^2 = -\$100$. But using different d for different cash flows eliminates the main advantage of Approach I, its linearity, while reducing only a tiny portion of the error, which is shown in the next section.

and w^2 long or short. Furthermore, the tax payments and would be made in equal amounts on the four "estimated" tax payments dates. σ_j^a is the bid price of short borrowing Bond j. Approaches II and III compute tax payments in the way they are actually paid.

An investor "longing" or "shorting" a cash flow $w = (w_1, w_2)$, as opposed to a bond, trades (longs and/or shorts) bonds so as to add w to, or subtract w from, respectively, the investor's previously acquired cash flow. That previously acquired cash flow might be zero (e.g., the investor has no previously acquired position) and w might contain negative payments, i.e., $w_i < 0$. Many situations arise in finance that call for the valuation of cash flows like w^0, w^1, and w^2 long or short, e.g., an investment project (not traded in a market) may require or provide such a cash flow. These situations might be independent of the trading circumstances (including restrictions on short borrowing) of the investor in question.

4. Approach II Untaxed: Linear Programming

In this approach we consider the long and short prices our major untaxed investor faces, but ignore any more favorable pricing circumstances that might stem from previously acquired positions in the three bonds or other opportunities peculiar to the investor. That is, we consider the ask price of the bond as the bond's long price, on the one hand, and we consider the bid price of the bond minus the ask price of short-borrowing as the bond's short price, on the other hand. We write this as $P_j = P_j^a$ and $p_j = P_j^b - \sigma_j^a$. All untaxed investors having the same transaction costs as this investor face the same imputed long and short prices, i.e., the valuation operator we construct will apply to this class of investors.

Given these two sets of prices, long and short, any future cash stream vector w has an upper value bound and a lower value bound, rather than a single value as in Approach I. The upper value is, by mathematical specification, the minimum cost of acquiring (longing) w using any combination of long and short bond trades at those prices. Note that valuation methods were judged against the minimum cost of replication in Ruback (1986, p. 324), and that this pointed towards the new approach. In particular, Ruback compared the present value and adjusted present value methods to "... the minimum amount of money that has to be invested in riskless securities to replicate (long) the stream ..." in a market without transaction costs but with short borrowing. Similarly, the lower value is, the maximum current cash that can be obtained in exchange for an obligation to pay (short to maturity) w using any combination of long and short bond trades at those prices. We use the market prices in Table 1 to determine the actual long and short prices facing our investor. Finding those short prices required calculating the ask short-borrowing (to maturity) costs, which we did in the Appendix.

<u>Construction of the Investor's Term Structure Packet D</u>

Our investor faces: long prices $P_j = P_j^a$ and short prices $p_j = P_j^b - \sigma_j^a$, where P_j^a, P_j^b, and σ_j^a are the ask and bid bond prices and ask price of short-borrowing, respectively, for Bond $j = 1, 2, 3$. These prices are given in Table 2. See Table 1 for the long prices, and see Table 1A in the Appendix for the computation of the short prices.

For any particular investor or class of investors, a no-arbitrage pricing operator is defined as an operator that values every cash stream w available in the market so as to prevent

		Short Prices p_j	Long Prices P_j
Table 2. Approach II Prices	Bond 1	\$ 98.222439	\$ 99.084978
	Bond 2	95.377016	97.554525
	Bond 3	106.461450	108.915142

arbitrage by that investor or class. Thus, it must value every w at or below what the investor can obtain (long) w for in the market, i.e., by any combination of trades at the long and short prices in Table 2. Similarly, it must value every w at or above what the investor can get for (shorting) w in the market.

To accommodate additional dates for tax payments in Section 8 of this paper, let the bond payment dates of the three bonds in the model, 15 May and 15 November 1993, be denoted by $i = 2$ and $i = 4$, respectively. Hence, the packet D of no-arbitrage term structures for an untaxed investor is the set of all nonnegative vectors $d = (d_2, d_4)$ that value the future cash stream of each bond (i.e., discount its future payments $w = (w_2, w_4)$ to an amount) at or between the long and short prices the investor faces. We will say "untaxed D" when we want to emphasize that the investor with that D is untaxed. If an investor's valuation of Bond j long was above P_j, then the difference would be arbitrage on each Bond j longed. If an investor's valuation of any Bond j short was less than p_j, then the difference would be arbitrage on each Bond j shorted. This no-arbitrage property (two weak inequalities) holds for the future cash stream of each bond if, and only if, it holds for the cash stream created by any combination of those bonds.[5]

Hence, by definition, D is this entire set of nonnegative vectors $d = (d_2, d_4)$ such that

$$p_j \leq d_2 a_{2j} + d_4 a_{4j} \leq P_j \quad \text{for Bond } j \text{ and } j = 1, 2, 3 \tag{1}$$

where a_{ij} is the pretax bond payment of Bond $j = 1, 2, 3$ on payment date $i = 2, 4$. Thus, the term structure packet D of the untaxed investors in our three-bond two-period example is the set of all nonnegative (d_2, d_4) that satisfy these six weak inequality constraints that stem from the prices in Table 2. These inequalities are listed in Table 3.

Thus, each bond provides a pair of parallel faces , e.g., a long and a short face. Faces are lines in our two-period (i.e., two dimensional) world. For a given Bond j, these two faces are defined by (1) holding with equalities for that j. Geometrically, the set of no-arbitrage term structures for our investor's class is the set D of all d on or between each of the three pairs of parallel faces shown in Figure 1.[6] Note that treating the long and short price instead of a single price for each bond changes the description of the pricing operator from a yield

[5] Inequalities hold under addition and subtraction, and under multiplication by a positive number.

[6] Because, the markets always reflect a positive time value of money, i.e., $0 \leq d_2 \leq d_4 \leq 1$, we can define D in terms of nonnegative d. The four vertices of D are (.9822243900, .9742108703), (.9822243900, .9537701600), (.9537701600, .9908497800), and (.9908497800, .9737322474).

Table 3.

Constraint Names	Approach II Constraint Specifications that Define D for an Untaxed Investor			
Bond 1 Long	$d_2\$100$			$\leq \$\ 99.084978$
Bond 1 Short	$d_2\$100$			$\geq \$\ 98.222439$
Bond 2 Long			$d_4\$100$	$\leq \$\ 97.554525$
Bond 2 Short			$d_4\$100$	$\geq \$\ 95.377016$
Bond 3 Long	$d_2\$\ \ 5.875$	$+$	$d_4\$105.875$	$\leq \$108.915144$
Bond 3 Short	$d_2\$\ \ 5.875$	$+$	$d_4\$105.875$	$\geq \$106.461450$

curve, as shown in McCulloch (1975) and Carelton and Cooper (1976), to a packet of term structures in a discount factor space. This difference in treatment is illustrated by contrasting the $\bar{d}$ with the D in Figure 1. The yield curve is two-dimensional, i.e., it has a single interest rate (and thus a single discount factor) for each date, whereas D has a continuum of discount factors for each date. The dimension of D is the number of payment dates in the model, and each dimension conveys information about the market. Thus, in the two-dimensional framework of a yield curve, one can not completely describe our D if there is more than one period. There is not even a band of yield curves that can price each cash stream, because the particular continuum for one date depends on the cash flows at other dates. Put another way, the width (continuum) of d values in D is different in each direction, i.e., as the w being evaluated changes.

5. Valuation Differences Between Approaches I and II

Under Approach I the same term structure vector $\bar{d}$ is multiplied by any cash flow $w = (w_2, w_4)$ in order to determine the value $\bar{d}w$ of w. This single term structure $\bar{d}$ is constructed with only the ask prices of the strip bonds and ignores both transaction costs and the possible advantages of using the other bonds in the market to construct cash flows.[7] If taxes are considered in Approach I, then there is such a single $\bar{d}$ for each tax class. In contrast, under Approach II each vector $d = (d_2, d_4)$ in D is a valid no-arbitrage term structure because it provides a value dw for every w that is consistent with the absence of arbitrage opportunity, i.e., satisfies the weak inequalities in Table 3. Furthermore, D is constructed with the long and short market prices (and thus all transaction costs) of every bond in the market. Note that the NPV term structure $\bar{d}$ in Figure 1 is not a no-arbitrage term structure, i.e., is not in D. Since $\bar{d}$ is above D, it exaggerates the magnitude of the value of any payment at time $i = 4$ in a cash stream. Because it does not touch the left side of D, it also exaggerates such

[7] Recall the minor exceptions to this given in Section 3.

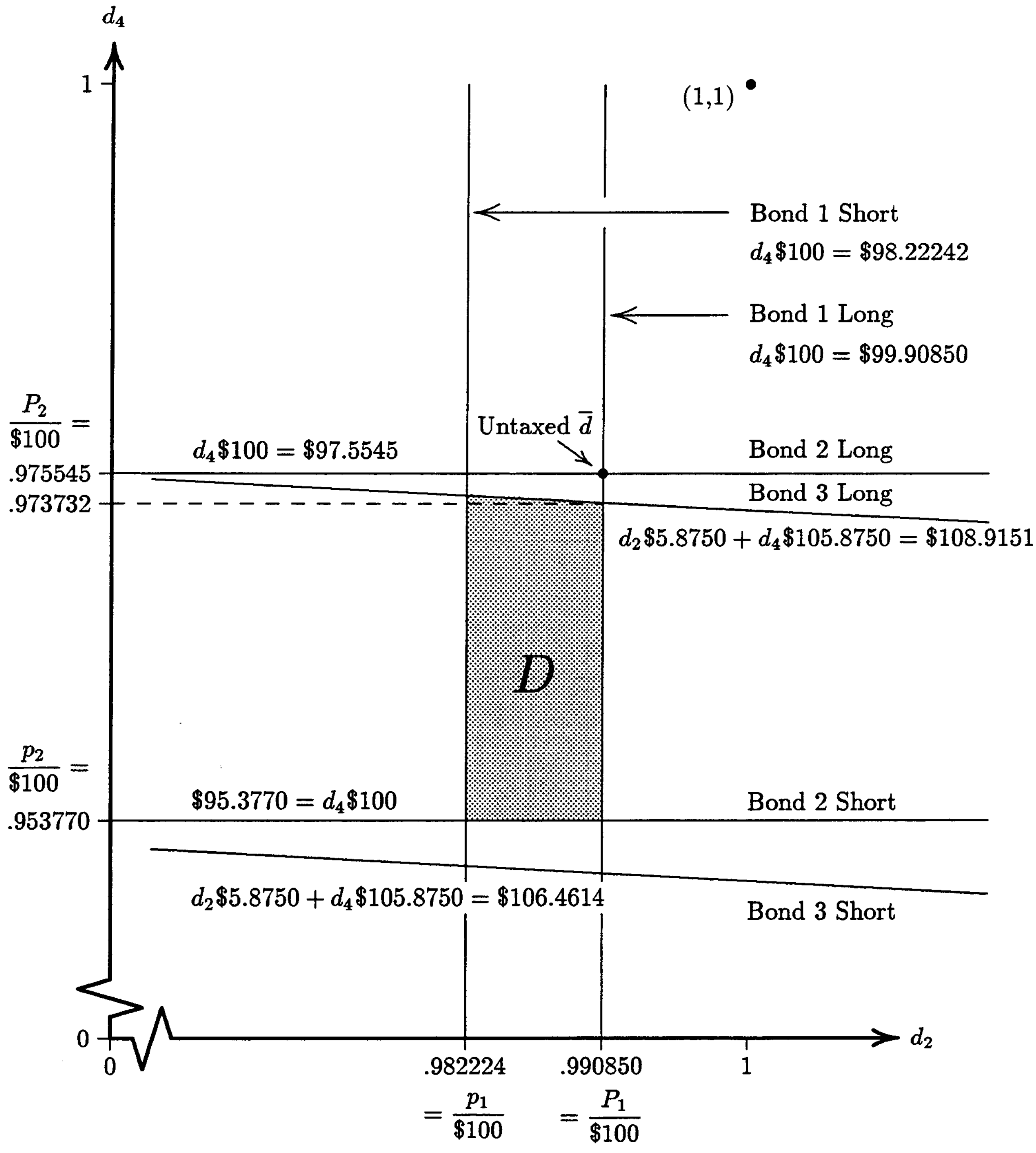


Figure 1. Constraints Determining Untaxed D from Table 3.

At 10H30 on 26 January 1993, a major untaxed investor's term structure packet D under Approach II is contrasted with the single-point term structure $\bar{d}$ under Approach I. Note that d_2 and d_4 are the discount factors for the two bond-payment dates, 15 May (109 days) and 15 November 1993 (293 days), respectively.

for any negative payment at time $i = 2$.

For any given w, the maximum value dw, across all d in the set D, is the upper value bound of w and is written $V(w)$. This $V(w)$ is, by mathematical specification, the minimum cost of obtaining at least the future cash flow w (i.e., at least w_2 at time $i = 2$ and at least w_4 at time $i = 4$) through any combination of long and/or short trades at the prices in Table 2 (i.e., with all transaction costs considered). Since w can be obtained for $V(w)$ and not cheaper, Approach II provides the accurate long value of w (when ignoring previously acquired positions), while Approach I does not.

Similarly, the minimum value dw across all d in D is the lower value bound of w and is written $v(w)$. This $v(w)$ is, by mathematical specification, the maximum proceeds (in current cash) that can be obtained in exchange for an obligation to pay at most w in the future through any combination of trades at the prices in Table 2. Since obligating to pay w can provide $v(w)$ and not more, Approach II provides the accurate short value of w (when ignoring previously acquired positions), while Approach I does not. The maximum value $V(w)$ equaling the minimum cost and the minimum value $v(w)$ equaling the maximum proceeds is the key feature of the duality theory of linear programming in the new approach to valuation, c.f., Dermody and Rockafellar (1991).

We now return our attention to the three example cash flows. First, consider $w^0 =$ (\$100, \$100), that in Section 3 was assigned a (common long and short) value of \$196.6395 by the single $\bar{d}$ of the customary NPV valuation under Approach I untaxed. In Approach II the greatest no-arbitrage value (upper value bound) of w^0 is obviously associated with the upper right corner $d = (.990850, .973732)$ of D in Figure 1, because that d is furthest in the w^0 direction. It is

$$V(w^0) = (.990850, .973732)(\$100, \$100) = \$196.4582\,.$$

The lowest is associated with the lower left corner $d = (.982224, .953770)$ of D. It is

$$v(w^0) = (.982224, .953770)(\$100, \$100) = \$193.5994\,.$$

Thus, Approach II values w^0 long as the \$196.4582 cost of buying .9445 units of Bond 1 and .94451 units of Bond 3. It values w^0 short as the \$193.5995 (negative) cost of short borrowing and selling one unit each of Bonds 1 and 2. This is in contrast to the Approach I valuation as the \$196.6395 cost associated (by $\bar{d}$ in Figure 1) with buying one unit of each of the strips, i.e., of Bonds 1 and 2. If one used the middle of the bid-ask spread, instead of the ask prices in Approach I, then these errors in the NPV valuation of w^0 long and short would reduce from \$.1813 and \$3.0400 to \$.1759 and \$3.0347, respectively.

Second, consider $w^1 =$ (\$0, \$100). Approach I values w^1 long and short as the \$97.5545 cost of buying Bond 2. In contrast, Approach II values w^1 long as the \$97.4211 net cost of short borrowing and selling .0555 units of Bond 1 and buying .9445 units of Bond 3, and it values w^1 short as the \$95.3770 (negative) cost of short borrowing and selling one unit of Bond 2.

Third, consider $w^2 =$ (\$100, −\$100). Approach I values it long and short as \$1.5305, which is the cost of buying one unit of Bond 1 minus the cost of buying one unit of Bond 2. In contrast, Approach II values w^2 long as the \$3.7080, which is the cost of buying one unit of Bond 1 minus the price of short borrowing and selling one unit of Bond 2, and values w^2

short as the \$.8014, which is the price of short borrowing and selling 1.0555 units of Bond 1 minus the cost of buying .9445 units of Bond 3.[8] These differences are the errors stemming from using the traditional NPV method of valuation instead of using Approach II without taxes. Such errors in the face of taxes are larger. The geometry of these errors for untaxed investors is analyzed in the next section.

6. The Geometry of NPV Errors: Simple Case

To focus on the fundamental insights, we will confine this section to untaxed investors with no previously acquired position. Let $A = \{a_{ij}\}$ be the matrix of bond payments, where each unit of Bond $j = 1, \ldots, n$ pays $a_{ij} \geq 0$ on bond-payment dates $i = 1, \ldots, m$. The investor faces a row vector of long prices $P = (P_1, \ldots, P_m)$ and a row vector of short prices $p = (p_1, \ldots, p_m)$, and can choose a column vector $X = (X_1, \ldots, X_m) \geq \vec{0}$ of long buys along with a column vector of the amounts of short borrowing and sales $x = (x_1, \ldots, x_m) \geq \vec{0}$. Prices are net of transaction costs, $p < P$, and $\vec{0}$ denotes a vector of zeros. The differences in valuations between Approaches I and II are related to the fundamental arbitrage- maximization problem (2), that is the basis of the valuation operator associated with D.

$$\text{maximize} \quad px - PX \qquad \text{subject to} \quad A(X - x) \geq \vec{0}\ , \ \ X \geq \vec{0}\ \ , \text{and} \ \ x \geq \vec{0}. \tag{2}$$

This primal problem is to choose the trades (X, x) that maximize present cash (net receipts px of shorting x minus net cost PX of longing X) subject to the trades not requiring a net provision of cash in the future (i.e., subject to future outflow Ax from shorting x not exceeding future inflow AX from longing X). If this present cash flow $px - PX$ is positive, then there is arbitrage to be had by any untaxed investor facing those prices P and p. Weak no-arbitrage is defined as the value of (2) being zero.[9] Note that it is certainly nonnegative since $(X, x) = (\vec{0}, \vec{0})$ is a feasible trade. If we perturb (2) by changing the first constraint constant from $\vec{0}$ to any cash flow $w = (w_1, \ldots, w_m)$ in question, then we obtain a valuation of w in problem (3).

$$\text{maximize} \quad px - PX \qquad \text{subject to} \quad A(X - x) \geq w\ , \ \ X \geq \vec{0}\ , \text{and} \ \ x \geq \vec{0}. \tag{3}$$

Since maximize $\{px - PX\} = -$ minimize $\{PX - px\}$, we see that the objective of (3) is also the minimum cost of obtaining the position (X, x), i.e., the cost of buying X minus the cost of short borrowing and selling x, that provides the investor a cashflow of at least w_i in each future period i. This minimum cost of creating w is the upper value bound $V(w)$ of w, i.e., no investor should ever pay more for w than the minimum cost of a set trades that provides w.

---

[8] If one modified Approach I to use the Bond 2 bid price, instead of the ask price, in determining the d_2 that values $w_2^0 = \$100$ short and $w_2^2 = -\$100$ long, then the errors would fall by \$.007745. But that uses more than one d and thus eliminates the simplicity of a linear pricing operator. In fact, such use of different d is mathematically similar to the new approach, but D would be much smaller because the still-ignored short-borrowing costs of all but the most immediate maturities dwarf the bid-ask spreads that would then be treated.

[9] See Sections 3 and 4 of Dermody and Rockafellar (1991) for the definitions of and discussion about three gradations of no-arbitrage assumption: weak, strong, and complete.

A dual problem to (3) is (5), where D is defined by (4), which is the n-period generalization of (1).

$$p_j \leq d_1 a_{1j} + \cdots + d_n a_{nj} \leq P_j \qquad \text{for securities } j = 1, \ldots, m. \tag{4}$$

The mathematical definition of the upper value bound is (5): the largest valuation dw for which d is in D.

$$V(w) = \max_{d \in D} dw \tag{5}$$

For our three-bond two-period model, (4) is given by Table 3, and for the first example w_1 = (\$100, \$100) in our model, (5) becomes (6) where D is the specific term structure packet defined by Table 3.

$$V(w^0) = \max_{d \in D} dw^0 = d_U w^0 = (.99085, .97373)(\$100, \$100) = \$196.458200 \tag{6}$$

That d in (6) is illustrated by d_U^0 in Figure 2. Formulas (5) and (3) provide the same valuation because the primal and dual problems in linear programming always have the same value. $V(-w)$ is the upper value bound of $-w$, which is the optimal value of problem (3) if w is replaced by $-w$. Call the new problem (3′) and observe that it is equivalent to both: (i) maximize $\{px - PX\}$ subject to $A(X - x) \geq -w$, $X \geq \vec{0}$, and $x \geq \vec{0}$ and (ii) minimize $\{PX - px\}$ subject to $-$ $A(X - x) \leq w$, $X \geq \vec{0}$, and $x \geq \vec{0}$. The latter has the straightforward interpretation of the minimum cost $(PX - px)$ of a set of trades that provides at least $-w$ future cash flow, i.e., requiring the investor to pay at most w. Note each payment w_i received by the investor on future payment date i can be positive, zero, or negative, and that $V(-w) = -v(w)$. A dual of (3′) is problem (7).

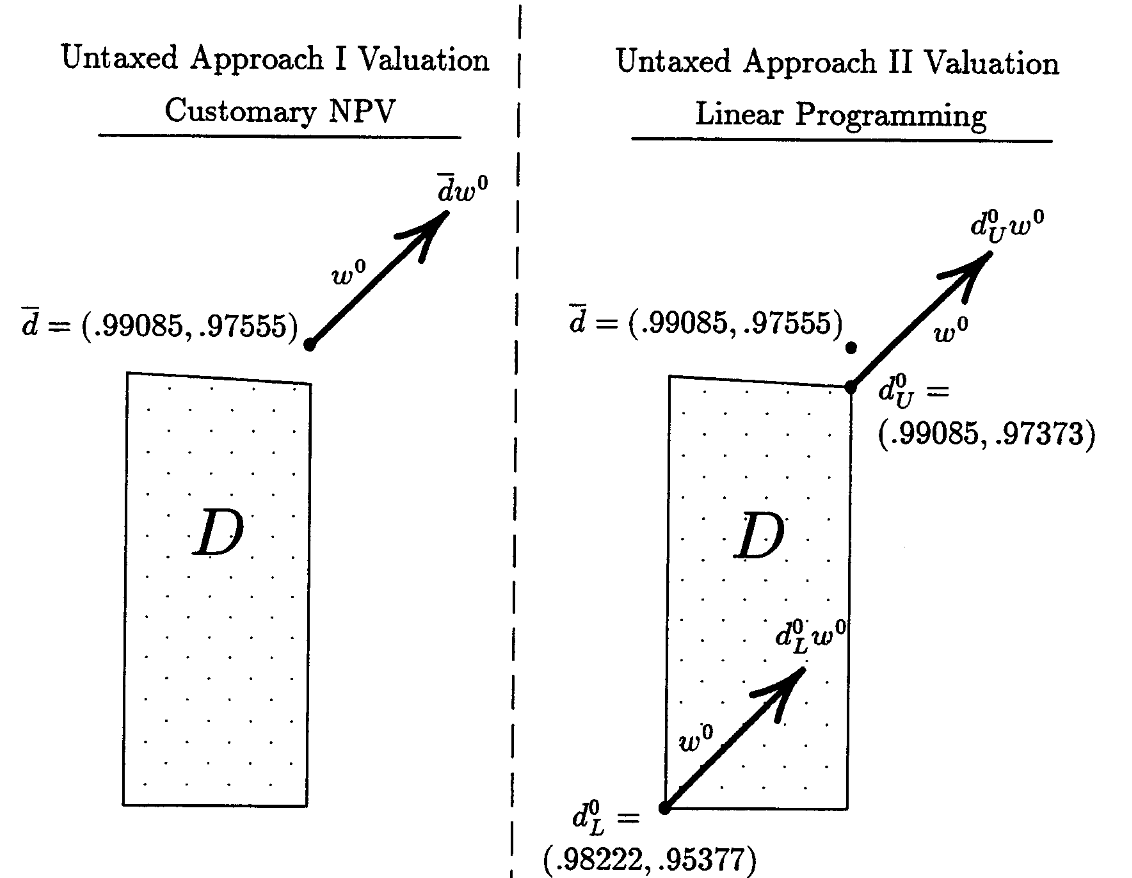
$$v(w) = \min_{d \in D} dw \tag{7}$$

In our first example $w = w^0$ and (7) becomes (8) where D is defined in Table 3.

$$v(w^0) = \min_{d \in D} dw^0 = d_L^0 w^0 = (.98222, .95377)(\$100, \$100) = \$193.599455 \tag{8}$$

In (7) we see the mathematical definition of lower value bound: the minimum dw for which d is in D. That d is illustrated for our first example by d_L^0 at a corner of the D in Figure 2. This takes us full circle back to the definition of D in general. Since all d in D are no-arbitrage pricing operators, the upper value bound of w_0 is the product of w^0 times the $d = (d_2, d_4)$ in D (defined by Table 3) that provides the greatest product $dw^0 = d_2 w_2^0 + d_4 w_4^0$, and the lower value bound is w^0 times the one that provides the lowest such product. These d occur at the upper right and lower left corners of D in Figure 2. Figures 3 and 4 show the appropriate term structure vectors d_U and d_L for the long and short values, respectively, of each of w^1 and w^2.

Figure 2. Comparison of Valuations of $w^0 = (\$100, \$100)$. The axes in each of these two diagrams are the same as those in Figure 1.



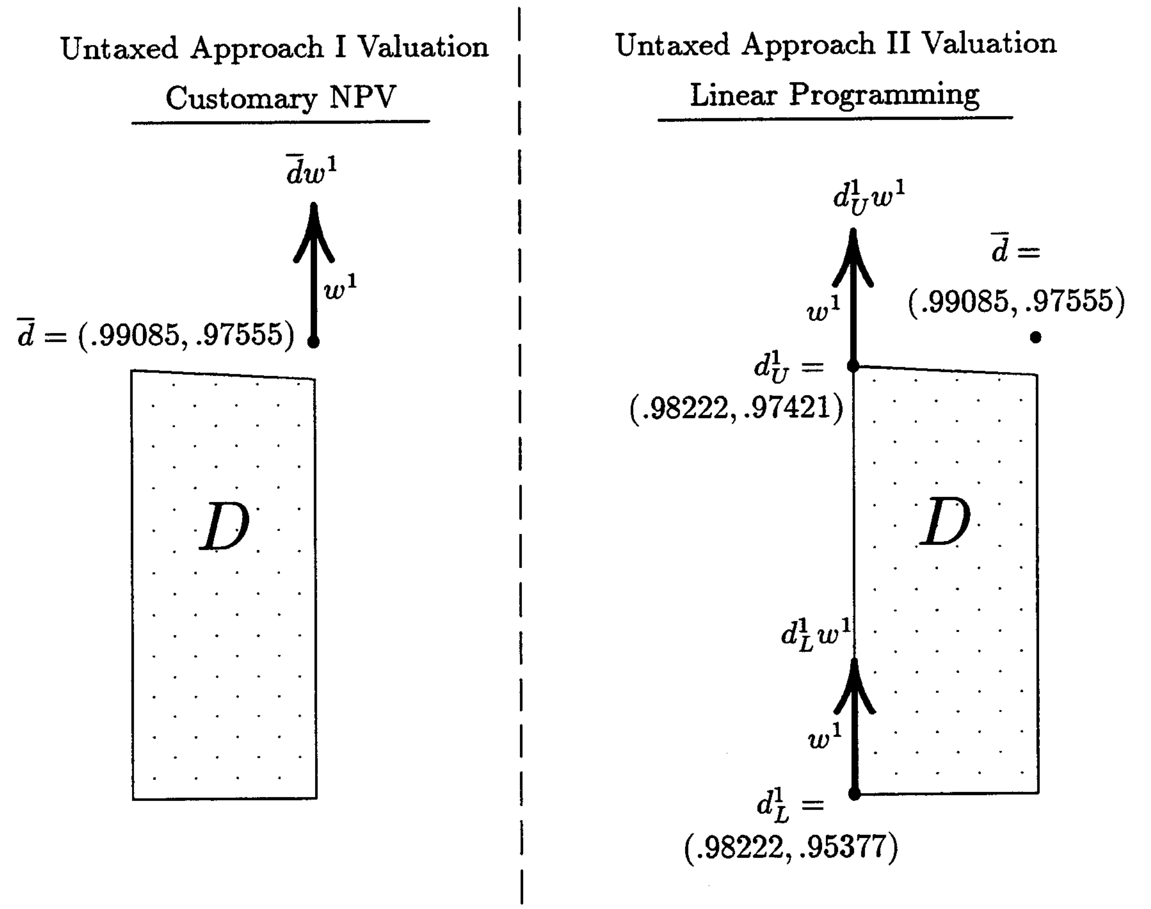
	Approach I	Approach II	NPV Errors
w^0 Long:	$\bar{d}w^0 = \$196.63950$	$d_U^0 w^0 = \$196.45820$	$+\$\ .18130 = +\ .092\%$
w^0 Short:	$\bar{d}w^0 = \$196.63950$	$d_L^0 w^0 = \$193.59945$	$+\$3.04005 = +1.570\%$

7. Approach III Untaxed: Linear Programming at the Margin

Investor's valuations of cash flows change with their previously acquired positions. Let Investor 1 be an untaxed investor with $\overline{X}_j$ units of a previously acquired long position in Bond j, while Investor 2 has none. Then Investor 1 can short the cash flow w^{bj} of Bond j by simply selling some of the long position for the current market bid price P_j^b. But to short w^{bj}, Investor 2 must first pay the current market ask short-borrowing cost σ_j^a to obtain Bond j in order to sell it. Thus, $p_j = P_j^b$ for Investor 1 and $p_j = P_j^b - \sigma_j^a$ for Investor 2, but this price advantage disappears after selling $\overline{X}_j$ units of Bond j.

On the other hand, suppose Investor 1 has $\overline{x}_j$ units of a previously acquired short position in Bond j while Investor 2 has none. Then Investor 1 can long the cash flow of w^{bj} by buying

Figure 3. Comparison of Valuations of $w^1 = (\$0, \$100)$. The axes in each of these two diagrams are the same as those in Figure 1.

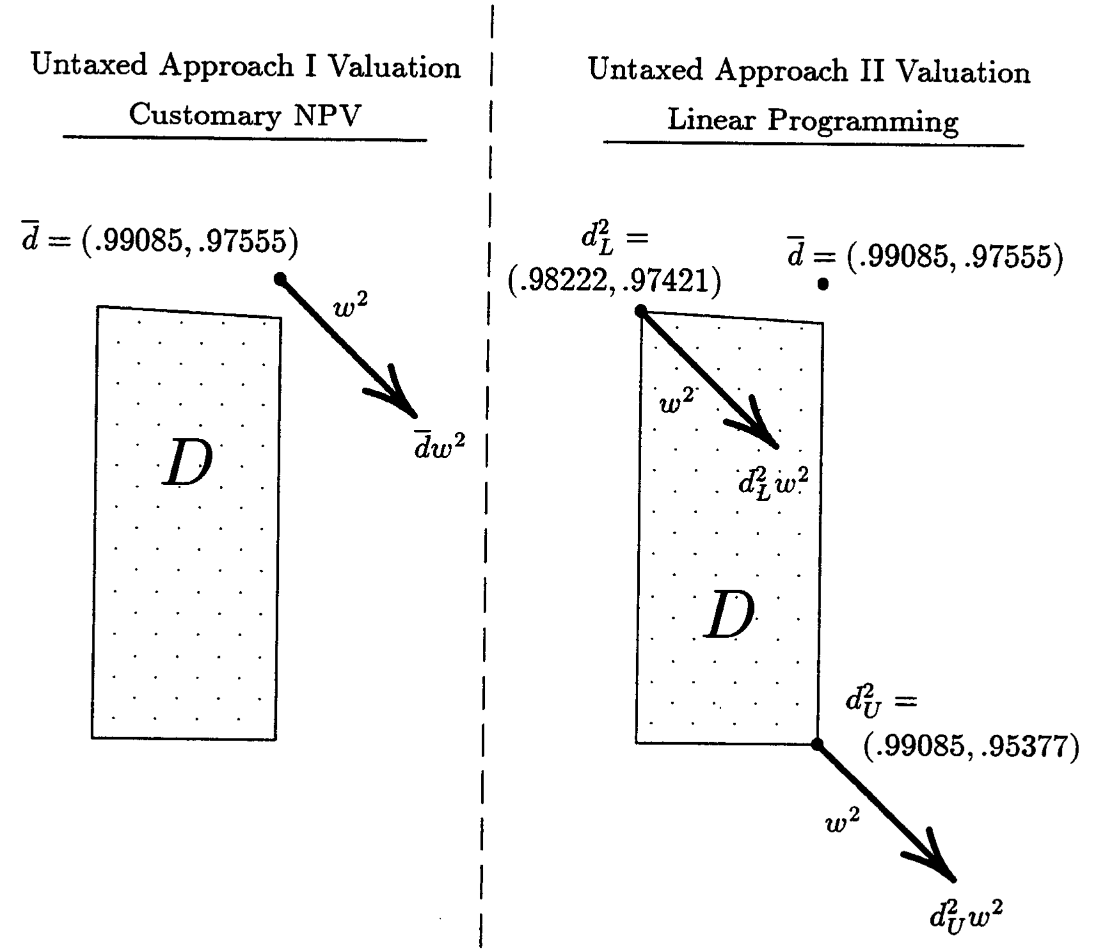


	Approach I	Approach II	NPV Errors
w^1 Long:	$\bar{d}w^0 = \$97.55452$	$d_U^0 w^0 = \$97.42108$	$+\$\ .13344 = +\ .137\%$
w^1 Short:	$\bar{d}w^0 = \$97.55452$	$d_L^0 w^0 = \$95.37702$	$+\$2.17750 = +2.283\%$

Bond j for its ask price P_j^a and then selling the unused short-borrowing (to maturity) rights to the previously acquired position for the current market bid short-borrowing price σ_j^b. In order to long w^{bj}, Investor 2 must pay P^a but can not offset part of this cost with the sale of short-borrowing rights. Thus, $P_j = P_j^a - \sigma_j^b$ for Investor 1 and $P_j = P_j^a$ for Investor 2, but this price advantage disappears after buying $\overline{x}_j$ units of Bond j. The Approach II prices are computed in the Appendix and listed in Table 4.

Investor 1 might have previously acquired long and/or short positions in many securities. This greatly complicates Investor 1's valuation of any given cash flow. One must compute its upper value and lower value bounds with each $\overline{X}_j$ and $\overline{x}_j$ in mind, because different portions of some long trades and some short trades will be at the advantageous price associated with closing out a different previously acquired opposite position. This is illustrated in

Figure 4. Comparison of Valuations of $w^2 = (\$100, -\$100)$. The axes in each of these two diagrams are the same as those in Figure 1.



			NPV Errors
w^2 Long:	$\bar{d}w^2 = \$1.53045$	$d_U^2 w^2 = \$3.70796$	$-\$2.17751 = -58.725\%$
w^2 Short:	$\bar{d}w^2 = \$1.53045$	$d_L^2 w^2 = \$\ .80135$	$+\$\ .72910 = +90.984\%$

Figure 5 by the untaxed Approach III D for each previously acquired position being a tiny subset of the entire shaded area representing the untaxed Approach II D. See Dermody and Rockafellar (1993) for the general treatment, which uses directional derivatives in place of linear programming.

In passing from Approach II, with no previously acquired position, to Approach III, with a previously acquired position only in Bond 1 long, D shrinks from the entire shaded area in Figure 5 to a sliver along and just inside the right side of the shaded area. The left solid-line short face moved across D to the dashed line near the right side. For such previous positions in Bond 2 or Bond 3, the corresponding bottom solid-line short bond face moves up to the

Table 4. Most Favorable Approach III Prices Possible

An investor faces short price p_j when holding a previously acquired long position in Bond j, and long price P_j when holding a previously acquired short position in Bond j.

	Short Prices $p_j = P_j^b$	Long Prices $P_j = P_j^a - \sigma_j^b$
Bond 1	\$ 99.082005	\$ 98.231423
Bond 2	97.546780	95.400204
Bond 3	108.883892	106.509941

respective dashed-line near the top. The new Approach III dashed-line constraint for each Bond j long or short, given a previously acquired opposite position in Bond j, is specified by Table 3 with the last column replaced by the respective prices from Table 4. When a previously acquired position is exhausted, the associated dashed-line bond face moves back across (left) or back down to the solid-line bond face on the other side. The symmetric movements hold for previously acquired short positions and for their exhaustion.

The dashed lines in Figure 5 show what D is for Investor 1 with any particular previously acquired position. D is the vertical sliver along the right side of the shaded area for an investor with only a previously acquired position in Bond 1 long. It is between the following vertical lines: Approach II Bond 1 long solid-line face and the Approach III Bond 1 short dashed-line face. If the investor's only previous position is in Bond 1 short, then D is the slightly thicker vertical sliver of shaded area on the left side of D between the solid- and dashed-line faces there. If the only such position is in Bond 2 long (Bond 3 short), then D would be the sliver along the top (bottom) of the shaded area between the respective dashed- and solid-line faces there.

In Approach III, the most interesting case is that of an investor holding a position including Bond 2 short and/or Bond 3 long. Then D is empty because: the Bond 2 dashed-line short face is above the Bond 3 solid-line long face, and/or the Bond 3 dashed-line long face is below the Bond 2 solid-line short face.[10]

Such a previously-acquired position offers a finite amount of "free" money, so at the margin future money might seem worthless.[11] See Dermody and Rockafellar (1993). Such an investor can cash in this accumulated advantage by a trade that provides current cash without any decrease in future cash flow, but only for a trade size that does not exceed the investor's previously acquired positions on which the "free money"depends. When those are

[10] Actually, these faces are above or below the other face for the segment that counts, i.e., between the Approach II Bond 1 long and short solid-line faces, rather than everywhere.

[11] The term "arbitrage" is reserved for infinite free lunches. Here it means that the set of d between each of the three pairs of solid lines in Figure 1 is empty. The finite "free" money between a dashed and a solid line is not really free, since it requires giving up some of the accumulated advantage of previously acquired positions. The Lagrange multiplier of (4) is zero, i.e., the shadow price of future cash is zero.

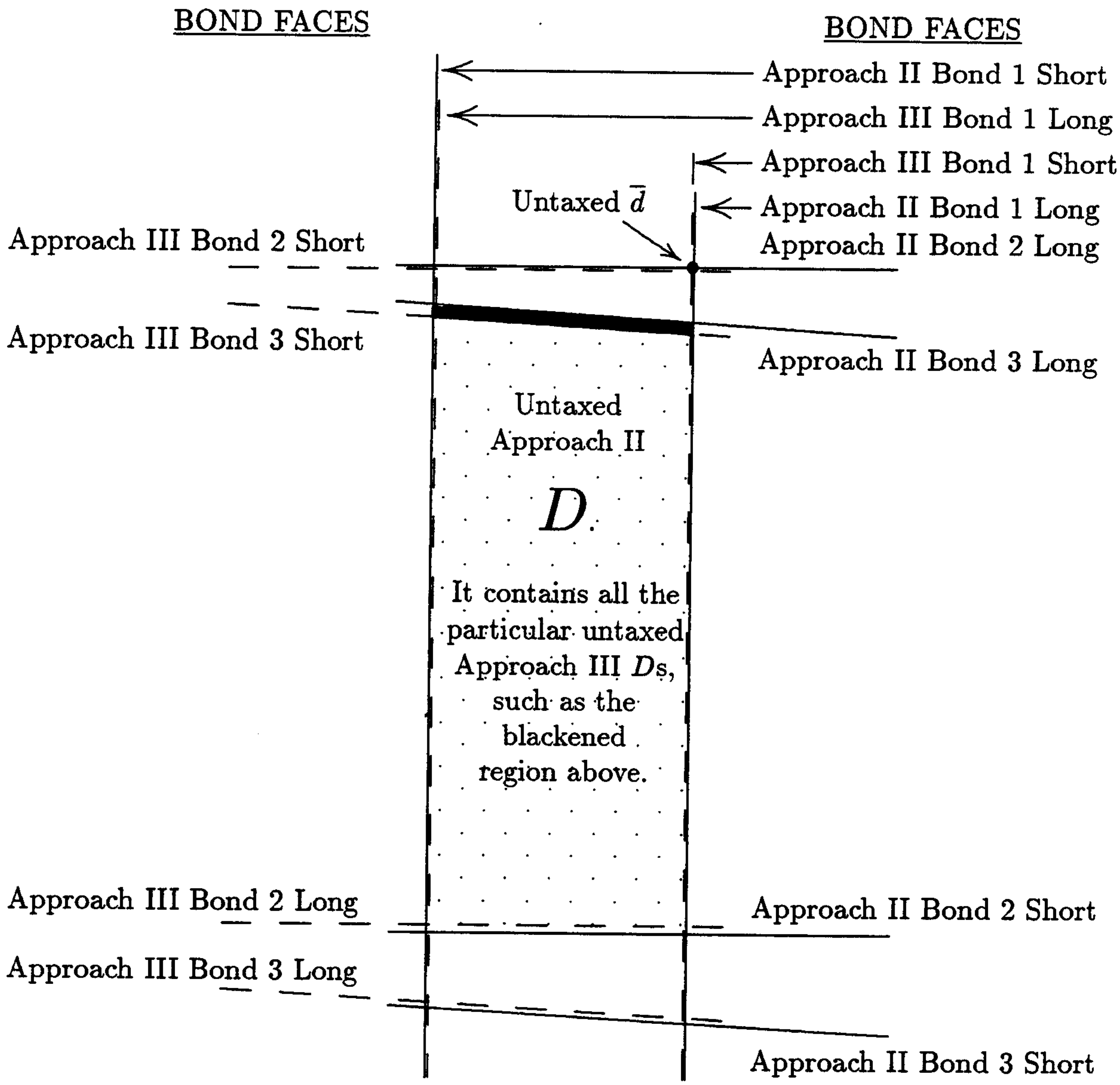


Figure 5. The D of an Investor with a Previously Acquired Position is a Sliver Along an Edge of the D for an Investor without such in Figure 1.

A previously acquired long position in Bond 2 or Bond 3 reduces untaxed D from the shaded area of Approach II by moving the respective bottom solid-line short bond face across the shaded area to the respective dashed line near the top. In the case of Bond 1 it does the same but from left to right. A previously acquired position in Bond 3 long shrinks the shaded Approach II D to just the blackened Approach III D. For previously acquired short positions and their exhaustion, the movements are symmetric.

exhausted, D changes from empty to the entire shaded area in Figure 5. Figure 6 enlarges the upper right corner of Figure 5 to show such a "free money" trade with a previously acquired position only in Bonds 1 and 2 long. D being empty does not prevent cash flow valuation. We simply take account of the 'free money" as (a negative) part of the minimum cost of obtaining the cash flow, or (a positive) part of the most cash that is available in exchange for obligating to pay the cash flow. This free money is the price obtained for selling an accumulated advantage.

Since some amount of Bond 1 and Bond 2 are held long, the investor can trade in the following proportions: sell .0588 units of Bond 1 for $99.0850/unit, sell 1.0588 units of Bond 2 for $97.5468/unit, and buy one unit of Bond 3 for $108.9151, all of which provides $.1836 of current cash without any future net obligations. If Investor 1 also had a previously acquired short position in Bond 3, then the cash stream of Bond 3 could have been longed for only $106.5099 (via buying Bond 3 and selling the unused short-borrowing rights until maturity) instead of paying $108.9151, thus raising the ("free money") current cash provided by these trades to $2.5888.

8. Taxed Approaches II and III

After-tax valuation must treat the tax cash flows as well as the bond payments. U.S. taxation changes the valuation operator by adding four estimated tax-payment dates per year to the bond payment dates specified in the bond. Hence, for a US corporation taxed at the top 34% rate, buying or selling a bond implies that the taxable income or loss stemming from that trade for each tax year will be reflected in 25% of 34% of that income or expense being paid or credited on each of the four dates.[12] Purchase or sale of a new thirty-year coupon bond or zero coupon bond would thus add between 120 and 124 tax-payment dates (depending on when during the tax year it was purchased) to the 60 or one bond-payment dates specified in the bonds, respectively. Note that some of these four tax-payment dates could fall on a bond-payment date.

The construction of the taxed investor's after-tax valuation operator associated with D follows the same steps described in Sections 4 or 7, except that the future cash flow is complicated by the additional cash flows on the tax-payment dates. Recall that D is the entire set of nonnegative vectors d that values every future cash flow available in the market at or below what the investor can long it for and at or above what the investor can short it for. Thus, Table 2 is unchanged but Table 3 must take on the additional terms for the d_i associated with the new tax-payment dates. Consider an investor, with no previously acquired position (i.e., Approach II), whose tax year starts on 1 December. Trading any bond in this model will change the estimated tax payments or credits due on the dates $i = 1, 2, 3, 4$, i.e., on the 15^{th} day of March, May, August, and November. This adds the March and August

[12] Corporations pay taxes on the 15^{th} day of the fourth, sixth, ninth, and twelfth month of their tax year. They may elect to start their tax year on the first day of any month. Note that only pretax cashflow can be traded, so that trades by investors of different tax classes or different previously acquired positions may have one investor selling an after-tax apple and the other buying an after-tax orange. Since we assume that the investor has enough income to use any credits resulting from these trades, the "estimated tax payments" are certain in our examples.

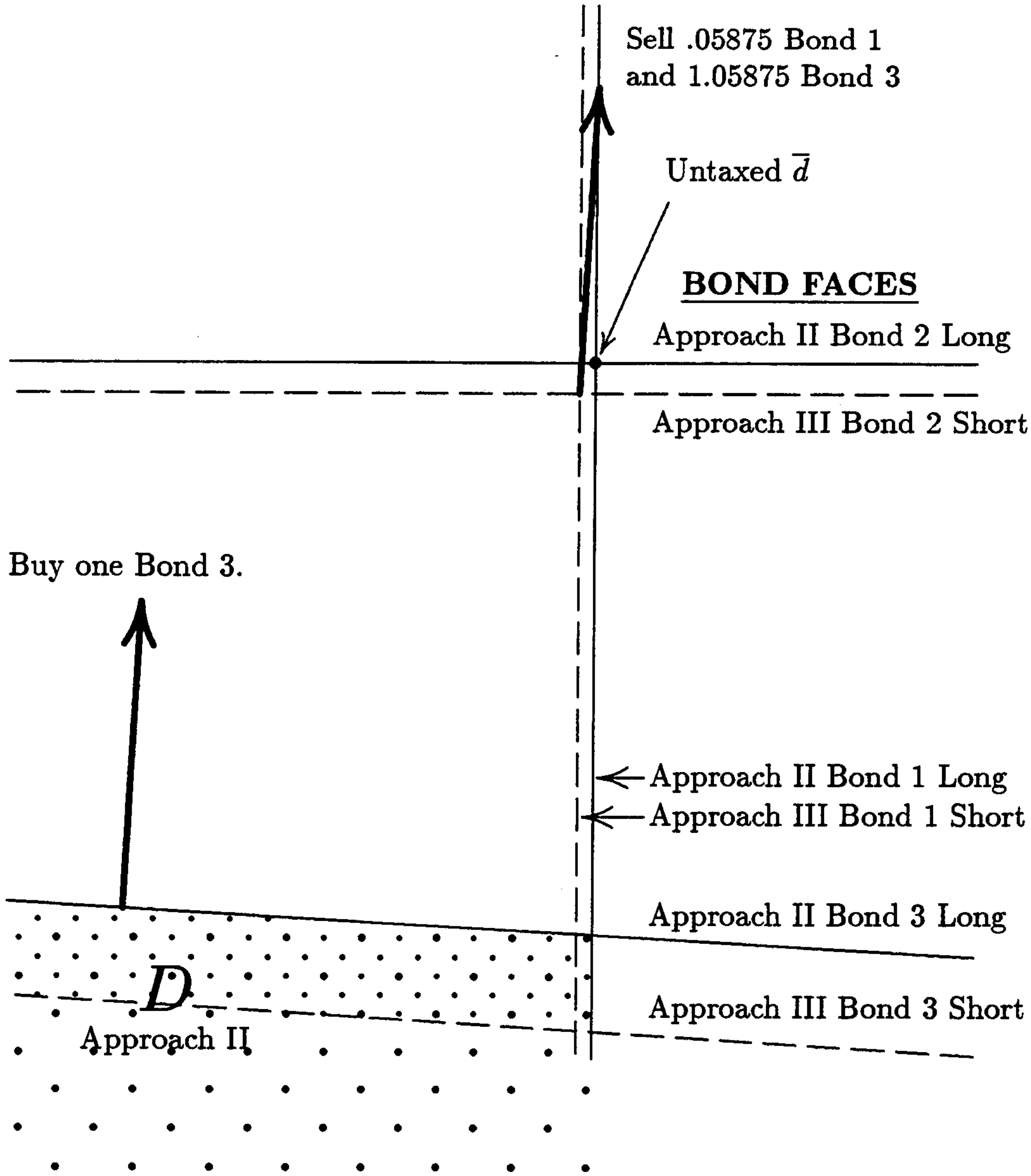


Figure 6. Upper Right Corner of Figure 5.

The Approach III D of an untaxed investor, with a previously acquired position only in Bonds 1 and 2 long, is the empty set and thus future cash flow is "free" at the margin. The investor could buy 1 unit of Bond 3 for $108.915142 and short the cash flow of Bond 3 ($5.875, $105.875), by selling .05875 units of Bond 1 and 1.05875 units of Bond 2 for $109.098721. This provides $.183579 current cash with no net change in future cash flow.

tax-payment dates to the May and November bond payment dates already considered in the untaxed case. We number the new dates as d_1 and d_3, respectively. For the taxed case, Table 3 becomes Table 5 based on the ordinary (discount) income on Bonds 1 and 2 and the ordinary (coupon) income and capital loss on Bond 3.

Table 5.

Constraint	Approach II Constraints that Define a Taxed D		
Bond 1 Long	$-\$.077777(d_1+d_3+d_4)+\$99.922223d_2$	$\leq$	\$ 99.084978
Bond 1 Short	$-\ .151093(d_1+d_3+d_4)+\ 99.848907d_2$	$\geq$	98.222439
Bond 2 Long	$-\ .207865(d_1+d_2+d_3)+\ 99.792135d_4$	$\leq$	97.554525
Bond 2 Short	$-\ .392954(d_1+d_2+d_3)+\ 99.607046d_4$	$\geq$	95.377016
Bond 3 Long	$-\ .041636(d_1+d_3)+\$5.833364d_2+\$105.833364d_4$	$\leq$	108.915142
Bond 3 Short	$-\ .250880(d_1+d_3)+\ 5.624120d_2+\ 105.624120d_4$	$\geq$	106.461450

Recall that untaxed valuation under Approach II is based on the untaxed investor's arbitrage-maximization problem (4). For the taxed investor, the analogous problem is

$$\text{maximize} \quad px - PX \quad \text{subject to} \quad A(X-x) - T(X) + S(x) \geq \vec{0},\ X \geq \vec{0},\ \text{and}\ x \geq \vec{0}, \tag{9}$$

where $T(\cdot)$ and $S(\cdot)$ are the tax payment matrices for the long and short trades, respectively, of all the bonds in the market considered. The matrix entries t_{ij} of T are the sums of the ordinary income and capital gain or loss of buying Bond j that is paid or credited to payments on estimated tax payment date i. Similarily, the entries s_{ij} of S are the ordinary expense and capital loss or gain of short borrowing and selling bonds on such dates. Each is 25% (one of four estimated tax payment dates) times 34% (top corporate rate on 26 January 1993) of the sum of the ordinary income and capital gains on Bond i. Note that the "estimated" tax payments are exact in this model as explained in Footnote 12. See Dermody and Rockafellar (1993a)(1993b) for more explanation of (9). For the three bonds in our example under Approach II, these two tax-payment matricies are

$$T = \begin{pmatrix} .077777 & .207865 & .041636 \\ .077777 & .207865 & .041636 \\ .077777 & .207865 & .041636 \\ .077777 & .207865 & .041636 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} .151093 & .392954 & .248224 \\ .151093 & .392954 & .248224 \\ .151093 & .392954 & .248224 \\ .151093 & .392954 & .248224 \end{pmatrix}.$$

If we project the four-dimensional D defined by Table 5 onto the d_2 and d_4 axes associated with the two bond payment dates, 15 May and 15 November 1993, then we have Figure 7.[13] The untaxed $\overline{d}$ and a dotted outline of the untaxed Approach II D from Figure 1 is drawn to facilitate comparison.

Figure 7 shows that the untaxed Approach II D in our example is compressed 33% horizontally and 70% vertically by the introduction of US corporate taxes. This introduction also moves the upper-right corner of the Approach II D up by .001388 (larger d_4) and right by .003082 (larger d_2). The error caused by the Approach I valuation of w^0 for a taxed investor is larger than that for an untaxed investor because the upper right corner of the taxed D is further from the taxed single NPV term structure $\overline{d}$ than is the respective corner from the $\overline{d}$ in the untaxed case.

Now consider some previously acquired positions of a taxed investor. Under Approach III the long price in Table 4 applies to Bond j if the previously acquired position has Bond j short, and the short prices there apply to Bond j if it has Bond j long. Otherwise the less favorable Table 2 prices apply. The investor's taxed D is the intersection of all nonnegative d that lie between the long and short taxed faces of the three bonds, where these faces are computed with the respective Table 2 or 4 prices. The taxed faces associated with the unfavorable and favorable prices are specified in Tables 5 and 6, respectively. Hence, an Approach III taxed D is defined by the inequalities from Table 5 associated with bonds for which the investor does not have a previously acquired opposite position, and with those inequalities from Table 6 associated with bonds for which the investor does have such.

Introducing an investor's previously acquired position, shrinks D from the shaded area of Figure 7 to a sliver or intersection of slivers on the edges of the shaded area or to the empty set. If the investor has a previously acquired position only in Bonds 1 and 2 short, then D shrinks from the entire shaded area in Figure 7 to an area 4.6% as wide and 1.9% as tall inside the shaded area and touching its bottom left corner.[14] See Figure 8.

There is a different Approach III D for each possible combination of previously acquired long and short positions in the bonds. The relation between the taxed Approach II D and each of the many possible taxed Approach III D are similar to those for the respective untaxed approaches. However, if we were to introduce the dashed-line taxed Approach III faces in Figure 8, they would not be parallel to the respective solid-line taxed Approach II faces, in constrast to the relationships of the untaxed faces in Figure 5. Later in Table 7, we list the values of w^0, w^1, and w^2 for a taxed and an untaxed investor with a previously acquired position only in Bonds 1 and 2 short.

9. Summary

The values of the three example cash streams (w^0, w^1, and w^2) under the (customary NPV) Approach I are compared in Table 7 with such under the new approach (Approaches II and

[13] The six vertices in the projection of the taxed Approach II D onto the (d_2, d_4) plane are (.9881953104, .9754335842), (.9881580568, .9754259553), (.9881390423, .9691526548), (.9939034575, .9691983167), (.9939275463, .9693200206), and (.9939320641, .9751196405).

[14] The vertices of this tiny area are: (.9882460964, .9694299103), (.9881862111, .9694294401), (.9881395150, .9693086045), (.9881390423, .9691526548), (.9881991674, .9691531311), and (.9882458631, .9692751917).

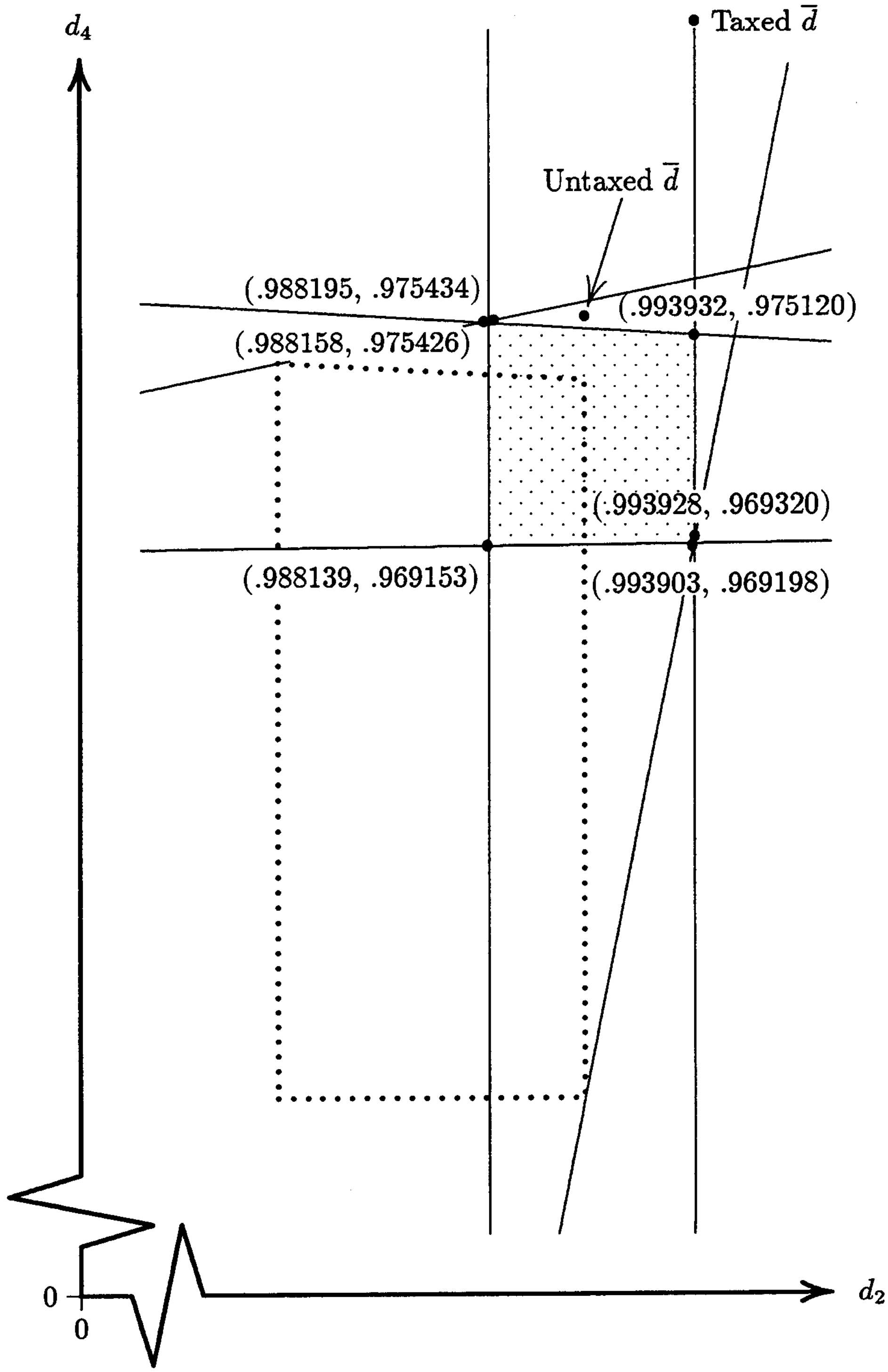


Figure 7. A major investor's term-structure packet D in three circumstances: **(i)** Approach II taxed D is the shaded area, **(ii)** Approach II untaxed D is dot-outlined area, and **(iii)** The (customary NPV) Approach I taxed and untaxed D is the single term structures labeled Untaxed $\bar{d}$ and Taxed $\bar{d}$. None of the six faces (lines) defining (i) is parallel to another or to an axis.

III). Many situations arise in finance that call for the valuation of such cash flows, e.g., an investment project (not traded in a market) may require or provide such cash flows. These situations are often independent of the trading circumstances of the investor involved. As explained in Section 3, Approach I is not accurate, while by mathematical specification, Approach II is accurate when there are no previously acquired positions and Approach III is accurate when there are such positions.

Table 6. Constraints Under Most Favorable Approach III Prices

Constraint for Bond j long or short used in defining D under the most favorable price (given in Table 4), which stems from holding a previously acquired opposite position in Bond j.

Constraint	For An Untaxed Investor
Bond 1 Long	$\$100 \; d_2 \leq \$\,98.231423$
Bond 1 Short	$100 \; d_2 \geq 99.082005$
Bond 2 Long	$\$100 \; d_4 \leq 95.400204$
Bond 2 Short	$100 \; d_4 \geq 97.546780$
Bond 3 Long	$5.875 d_2 + 105.875 d_4 \leq 106.509941$
Bond 3 Short	$5.875 d_2 + 105.875 d_4 \geq 108.883892$
	For A Taxed Investor
Bond 1 Long	$-\$.150329(d_1 + d_3 + d_4) + \$99.849671 d_2 \leq \$\,98.231423$
Bond 1 Short	$-\ 078030(d_1 + d_3 + d_4) + 99.921970 d_2 \geq 99.082005$
Bond 2 Long	$-\ .390983(d_1 + d_2 + d_3) + 99.609017 d_4 \leq 95.400204$
Bond 2 Short	$-\ .208524(d_1 + d_2 + d_3) + 99.791476 d_4 \geq 97.546780$
Bond 3 Long	$-\$.248429(d_1 + d_3) + \$5.626571 d_2 + \$105.626571 d_4 \leq 106.509941$
Bond 3 Short	$-\ .046644(d_1 + d_3) + 5.921644 d_2 + 105.921644 d_4 \geq 108.883892$

The errors made by an untaxed investor using Approach I are listed in the top half of the two right-most columns of Table 7. These errors stem from the differences in location between the single untaxed NPV term structure $\overline{d}$ and the corners d_L^k and d_U^k of D (of Approaches II or III) associated with the value of the cash flow $w^k = w^0, w^1, w^2$ long and short, respectively.

These differences for w^0, w^1, and w^2 were illustrated in Figures 2, 3, and 4, respectively. A taxed investor's errors from using Approach I are listed in the bottom half of those two columns. They stem from the analogous differences of location in Figures 7 or 8. Some of these distances in Figure 7 are larger and some smaller than the analogous ones in Figures 2, 3, and 4. All the errors in Table 7 are significant from the perspective of a bond trading desk. From the perspective of some capital budgeting analyst, the errors for w^2 are large and those for w^0 and w^1 are less significant. There are many perspectives in finance for which the significance of these errors lies between that of a trading desk and that of capital budgeting.

Table 7. Comparison of valuation approaches for three example cash flows w^k with payments on 15 May and/or 15 November 1993: $w^0 = (\$100, \$100), w^1 = (\$0, \$100)$, and $w^2 = (\$100, -\$100)$.

	Valuations for an Untaxed Investor				
	Customary			NPV Errors, i.e.,	
Cash	NPV Approach	New Approaches		Approach I errors vis-a-vis	
Flows	Approach I	Approach II*	Approach III**	Approach II*	Approach III**
w^0 Long	\$196.639503	\$196.458200	\$193.631627	+ .092%	+ 1.553%
w^0 Short	196.639503	193.599455	193.599455	+ 1.570	+ 1.570
w^1 Long	97.554525	97.421085	95.400204	+ .137	+ 2.258
w^1 Short	97.554525	95.377016	95.377016	+ 2.283	+ 2.283
w^2 Long	1.530453	3.707962	2.831219	−58.725	−45.944
w^2 Short	1.530453	.801355	2.822236	+90.983	−45.772
	Valuations for a Taxed Investor				
w^0 Long	\$197.766655	\$196.905170	\$195.767601	+ .438%	+ 1.021%
w^0 Short	197.766655	195.729170	195.729170	+ .990	+ .990
w^1 Long	98.372455	97.542596	96.942944	+ .851	+ 1.475
w^1 Short	98.372455	96.915265	96.915265	+ 1.504	+ 1.504
w^2 Long	1.021746	2.460753	1.904604	−58.478	−46.354
w^2 Short	1.021746	1.276173	1.876926	+19.937	−67.011

* For an investor with no previously acquired position.

** For an investor with a previously acquired position only in Bonds 1 and 2 short.

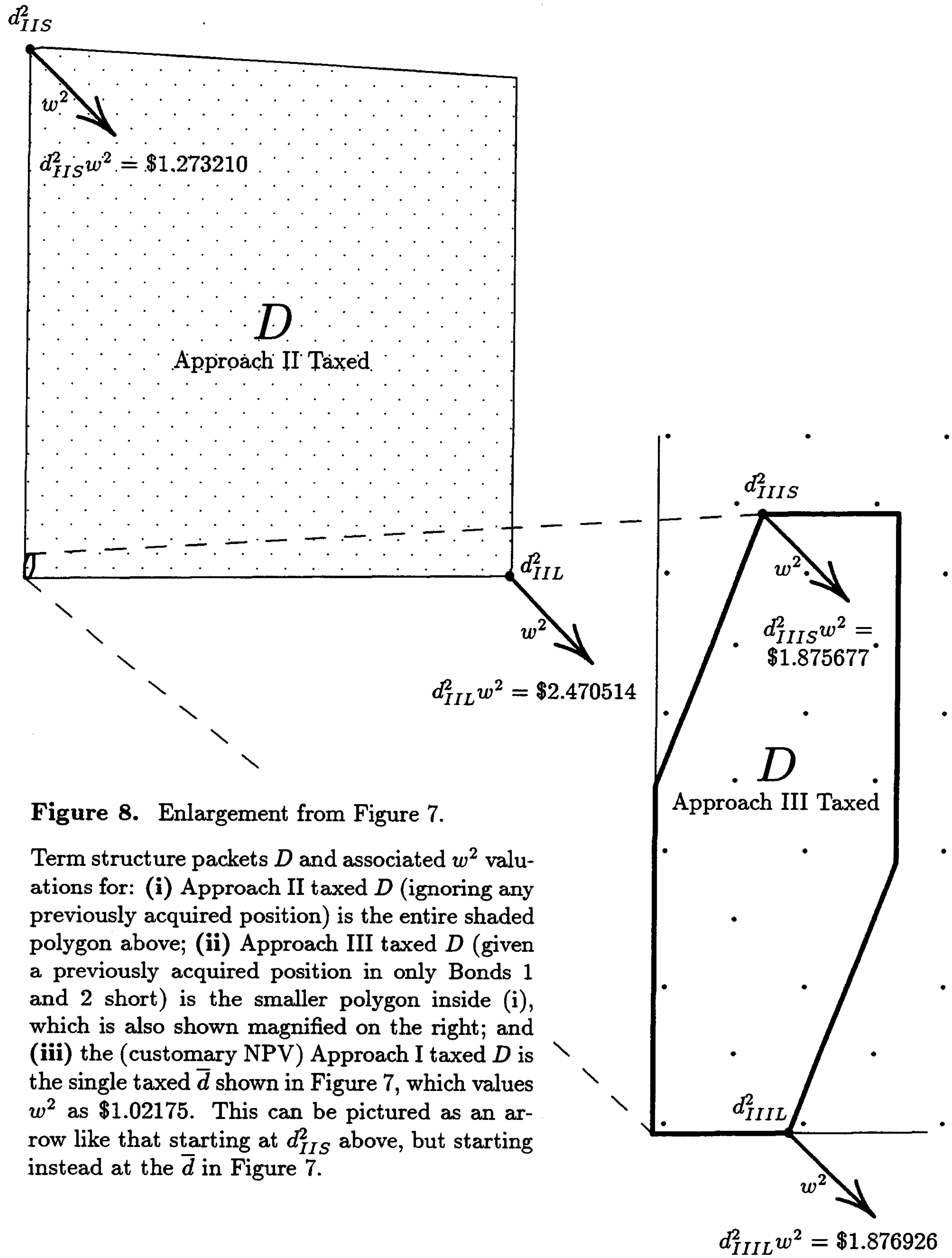


Figure 8. Enlargement from Figure 7.

Term structure packets D and associated w^2 valuations for: **(i)** Approach II taxed D (ignoring any previously acquired position) is the entire shaded polygon above; **(ii)** Approach III taxed D (given a previously acquired position in only Bonds 1 and 2 short) is the smaller polygon inside (i), which is also shown magnified on the right; and **(iii)** the (customary NPV) Approach I taxed D is the single taxed $\overline{d}$ shown in Figure 7, which values w^2 as \$1.02175. This can be pictured as an arrow like that starting at d^2_{IIS} above, but starting instead at the $\overline{d}$ in Figure 7.

In many cases most of the error results from ignoring the short-borrowing costs. Ignoring such costs can be thought of as ignoring the distance between $\overline{d}$ and a corner of D that is on the other side of D from $\overline{d}$ in Figures 2, 3, 4, and 7. An investor's most advantageous trade may involve short borrowing even though there is no negative component to the cash flow being evaluated, e.g., the minimum cost trades that provide $w^1 = (\$0, \$100)$ long in Section 6. In the examples of this paper short-borrowing costs range from 76 to 289 times as large as the bid-ask spreads. Since short-borrowing (to maturity) costs are higher for bonds with cash flows further in the future, *certerus paribus*, the errors from ignoring short borrowing costs are also higher for such cash flows. Our example cash flows have payments in 109 and/or 293 days. The reverse repo contracts that facilate short-borrowing are now written for up to two years. Many traders believe that as the Treasury market continues to enlarge, the introduction of progressively longer contracts will also continue.

Suppose reverse repo contracts are not available to an investor who lacks a previously acquired long position in Bond j (i.e., the investor can neither sell, nor short borrow and sell, some Bond j). Then the short price p_j to that investor is \$0, i.e., the investor receives nothing for an obligation to pay the future cash flow of Bond j because the ask short-borrowing cost equals the bond's bid price. We write this condition as $p_j = 0$. If this applies to Bond 2, then in Figure 1 such an untaxed investor's D extends down to touch the d_2 axis between $d_2 = .982224$ and $d_2 = .990850$. (Recall that d_2 is the discount factor for the date when Bond 1, not Bond 2, makes its only payment.) Thus, investors who are not permitted to short borrow bonds, will have larger D and therefore larger errors in valuing many cashflows under the traditonal NPV approach.

A different source of error results from ignoring bonds other than strips, e.g., ignoring Bond 3 when valuing: w^0 long in Figure 2, w^1 long in Figure 3, and w^2 short in Figure 4. Still another source is incomplete markets, but we have none in our model.

Appendix

Price Quotations and Conventions

Strip prices are quoted in terms of basis points of yields, where the actual number of days in each month and year (called Actual/Actual) are used to determine the number of days in a compounding period (between dates of purchase, anniversary of issue, and maturity), but the quoted yields are adjusted to a 360-day year. That is, interest for a period accrues as the quoted yield times the fraction of a 360-day period equal to the period length. In the market of our examples, the bid and ask prices of Bond 1 and Bond 2 were stated as 3.06% 3.05% and 3.09% 3.08%, respectively. Coupon bonds prices are the quoted price, in increments of 1/64 of a percent of face value (use \$100 face value to ease explication), plus the accrued coupon interest since the previous coupon payment date. That interest accrues Actual/Actual and is quoted relative to the actual number of days in a year. In the market of our examples, the bid and ask prices were stated as 106 17/32+ and 106 18/32+, respectively, where "+" denotes "plus 1/64".[15]

[15] To facilate comparison with strips, Treasury coupon bonds Actual/Actual yields are sometimes quoted relative to 360-day years (called money market yields), but we do not use this convention.

Accrual timing for taxation is computed on an actual/actual basis, relative to the actual number of days in a year. If accrual is quoted in terms of yield with 360-day years, then the accrual amount is computed as the 360-day yield times the number of 360-days periods of accrual timing. Interest accrues as of the settlement date, which is the day after the trade. Thus for Bond 3, 72 of the 181 days in the coupon period had accrued after the previous coupon date of 15 November 1992 and before the settlement date of 27 January 1993. Reverse repo rates for both strips and coupon bonds are quoted Actual/Actual adjusted to 360-day years.

Calculation of Short-Borrowing (to maturity) Ask Cost

If an investor has no previously acquired long position position in Bond j, then the short price is $p_j = P_j^b - \sigma_j^a$ where P_j^b is the market bid price of Bond j in Table 1 and σ_j^a is the ask price of short-borrowing computed below for Bonds $j = 1, 2, 3$.

To short borrow a bond (i.e., obtain a bond to sell that the investor does not already own) to maturity, the investor borrows cash (say, from the investor's corporate treasurer) at a rate that reflects the cost of that capital, and then posts it as collateral with the bond owner, who pays interest on the collateral at a lower rate. This spread in rates provides the incentive for the bond holder to lend the bond and creates the investor's cost of short-borrowing. The investor is funding the owner's purchase (carry) of the bond at a favorable rate in exchange for the owner losing the flexibility to sell the bond before maturity.

The details are as follows for each of the three bonds. A bond owner lends the bond to a repo desk in exchange for the desk posting cash collateral equal to 102% of the ask price of the bond, on which the owner pays the reverse repo bid rate of interest r_j^b (e.g., $r_2^b = 3.21\%$) until the bond matures. The repo desk lends, in turn, the bond to the investor in exchange for the investor posting the same amount of collateral, on which the desk pays only the ask reverse repo rate r_j^a (e.g., $r_2^a = 3.19\%$). The investor borrows (until the bond's maturity), at 6% interest (cost of capital), the amount of cash for which the repayment equals the sum of the return of the collateral (from the repo desk) plus interest at the reverse repo ask rate. The investor pays, as the short-borrowing cost, the difference between the proceeds of the 6% loan λ_j and the amount of collateral posted. This difference is due to the spread between the cost of capital and the reverse repo ask rate, and is paid at the time the bond is shorted.[16]

Thus, the short-borrowing cost σ_j of the bonds $j = 1, 2, 3$ is calculated as follows, where P_j^a is the ask price in Table 1, λ_j is the amount of cash the investor borrows, and θ_j is the proportion of a 360-day period until maturity, i.e., in which interest accrues.

$$\sigma_j^a = 1.02P_j^a - \lambda_j \qquad \text{where} \qquad [1 + (.06\theta_j)]\lambda_j = [1 + (r_j^a\theta_j)](1.02P_j^a) \qquad (A1)$$

Hence,

$$\sigma_j^a = \left[1 - \frac{1 + (r_j^a\theta_j)}{1 + (.06\theta_j)}\right]1.02P_j^a \qquad (A2)$$

[16] In the future, the short borrower pays the desk the future bond payments on the payment dates and receives the collateral plus (reverse repo ask) interest at maturity from the desk. The desk passes the bond payments to the bond holder and receives from the holder the collateral plus (reverse repo bid) interest at maturity.

The values of σ_j^a and the other variables in (A2) for the three bonds in our model are given in Table 1A. Reverse Repo rates are computed Actual/Actual, but quotes are adjusted relative to a 360-day year.

Table 1A. Ask Short-Borrowing Costs for a particular major investor at 10H30 EST on Tuesday 26 January 1993.

	Ask Price of Bond P_j^a	Ask Reverse Repo Rate r_j^a	Proportion of Period θ_j	Cash Borrowed λ_j	Ask Short Borrowing Cost σ_j^a
Bond 1	\$ 99.084978	.0314	109/360 = .302778	\$100.206711	\$ .859566
Bond 2	97.554525	.0319	293/360 = .813889	97.335851	2.169764
Bond 3	108.915142	.0319	293/360 = .813889	108.671002	2.422442

Calculation of Short-Borrowing (to maturity) Bid Price

If an investor has a short position in Bond j, then that investor can long Bond j at a more favorable (lower) price $P_j^a - \sigma_j^b$ by unwinding that short position instead of just buying the bond. The investor buys the bond at the market ask price P_j^a in Table 1 and sells the unused short-borrowing rights at the bid price of short-borrowing σ_j^b. σ_j^b is computed just as was σ_j^a in (A.2) except that r_j^a is replaced by r_j^b and P_j^a is replaced by P_j^b from Table 1. This provides $\sigma_1^b = \$.853555$, $\sigma_2^b = \$2.154321$, and $\sigma_3^b = \$2.405200$.

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