Continuous time equilibrium pricing of nonredundant assets

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Abstract

The problem of fair pricing of contingent claims is well understood in the context of an arbitrage free, complete financial market, with perfect information. But in the more realistic context of an incomplete market or with imperfect information, the arbitrage approach does not enable us to obtain a unique fair price for all contingent claims but only a fair pricing interval, which is known to be too large to be of great interest.
We present here a new approach by exploiting partial conditions issued from equilibrium analysis. The explicit use of market clearing conditions enables us to obtain a unique preference-free admissible price.
On a practical point of view, this enables us to give a unique fair price to any contingent claim. Moreover, on a theoretical point of view, this unique price appears to be only dependent on the real economy, as opposed to the financial one.

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1. Introduction

The problem of fair pricing of contingent claims is well understood in the context of an arbitrage free, complete financial market. We know that in this context, there exists a unique equivalent martingale measure and that all contingent claims are attainable. The problem of fair pricing of contingent claims is then reduced to taking expected values with respect to the unique equivalent martingale measure (see Harrison and Kreps [1979]).
In the context of an incomplete market, there exist more than one equivalent martingale measure and we don't obtain a unique fair price for all contingent claims but only a fair pricing interval which consists of all expected values with respect to the equivalent martingale measures (Jouini-Kallal [1995]). This interval is known to be too large to be of great interest (Cvitanic-Pham-Touzi [1997]).

Our problem here is the following: we consider a market in which there are a certain number of productive assets, referred to as the primitive stocks, whose price processes are supposed to be driven by a Brownian motion and we assume that the number of sources of uncertainty are greater than the number of primitive stocks available so that the primitive market consisting only of these primitive productive assets is incomplete. The price processes are very general diffusion processes: in particular, we don't assume that they are of Markovian type. In this context, the set of state price densities (or of equivalent probability measures that make the productive assets martingales) is very large and so is the fair pricing interval for any contingent claim.
In addition to these productive assets, we consider purely financial assets; we assume that they complete the market and that the full market is in equilibrium. Our goal here is then to use equilibrium conditions to reduce the set of admissible state prices (or of admissible equivalent martingale measures).
Our approach is the following: we consider the complete full market and we assume that the agents of this economy can achieve optimal demands. For each agent, the optimal trading-consumption strategy as well as the optimal terminal wealth can be linked, through marginal utility, to the density of the unique equivalent martingale measure. Moreover, we assume that our full market is in equilibrium, i.e., that both the commodity and the financial markets clear: the aggregated optimal demand of agents must equal the total supply available in the economy. As the purely financial assets are in zero net supply, this brings out considerable restrictions on the possible set of equivalent martingale measures and enables us to greatly reduce the interval of admissible prices. For example, if we assume that the utility functions satisfy some regularity assumptions, then the interval of admissible prices is reduced to a single point. Moreover, if we assume that the dividend process only depends on the productive asset's price, which is not too restrictive an assumption, then our unique fair price is the price given by Föllmer and Schweizer [1991]. To be concise, we have productive assets in positive net supply and purely financial assets in zero net supply,

which complete the market and we show that the prices of these purely financial assets are completely determined as long as we know the productive assets price processes as well as their corresponding dividend processes.

On a practical point of view, these restrictions on the admissible state prices can first and foremost be used for pricing issues in different cases: firstly, if we consider a market consisting of the primitive and of the additional assets and if we want to find a fair price for one of the additional assets, using as little information as possible, i.e., not using all the additional assets price processes; secondly, if we want to do the same thing for any asset to be introduced on the market assuming that its introduction does not affect existing productive assets prices, i.e., in such a way that allowing the agents to trade in the new asset does not change the prices of the primitive assets; thirdly, if we consider an incomplete market consisting of the primitive assets (or of the primitive assets and of a few financial assets, not numerous enough to complete the market) and if we want to find a fair price for nonredundant contingent claims, assuming that we can introduce enough financial assets to complete the market without modifying the primitive assets prices and that this full completed market should be in equilibrium. Besides, it is common in the option pricing econometric literature to assume that assets prices follow a specific type of diffusion process; the relations obtained through our analysis between the underlying securities price processes and the contingent claims price processes have testable consequences and enable to check the validity of the models considered.

On a theoretical point of view, it is interesting to notice that the unique fair price we find only depends on the real economy, as opposed to the financial one.

Our model is not similar to the Black and Scholes model as their market is complete without the additional financial assets. In their model, pricing problems are solved without any equilibrium consideration.

Some authors have examined whether or not a certain type of diffusion price processes can be derived from an equilibrium. While we try to reduce the set of admissible state prices, they consider specific pricing models and want to find an economic justification for them, i.e., to find utility functions for which there is an equilibrium in the considered models. Such a problem has been raised by Bick [1987] for the Black and Scholes model and by Bick [1990] and He and Leland [1993] for a Markovian diffusion model in a complete market framework, where the unique risky asset is productive and available in one unit supply. They characterize the risk premium as the unique solution of a nonlinear partial differential equation, and they relate it to the shape of the representative agent utility function. In Pham-Touzi [1996], a specific stochastic volatility model, which is a particular case of a Markov setting, is considered; the same main "tools" as in this paper, to wit utility maximization (although it is considered in the framework of the representative agent's theory) and equilibrium characterization are used; they apply their results to provide an economic foundation to the Hull and White model and to the minimal martingale measure. Although it was not obvious from the beginning because the goal and the settings are quite different, our approach is very close to theirs. As far as our problem is concerned, they find that the minimal martingale measure of Föllmer and Schweizer is compatible with an equilibrium in

their model if and only if the dividend process is an affine function of the productive asset's price process. We show in this paper that there is a unique admissible price, which appears to be the minimal martingale measure if the dividend process is a regular function of the productive asset's price process.
In Bizid, Jouini and Koehl [1997], the same problem as ours is solved in discrete finite time and with a finite number of states of the world at each date. They show that, for a fixed node, the Arrow-Debreu prices associated to the possible successors of the considered node are decreasing with respect to the price of this asset.

This paper is organized as follows: in section 2, we introduce our market model; in section 3, we characterize the equivalent martingale measures; in section 4, we consider how economic agents actually trade in this market. In section 5, we are interested in utility maximization and optimal demand for a single agent. In section 6, we define what we call equilibrium and we use the preceding sections to obtain necessary conditions for equilibrium and provide the main results. Section 7 is made of extensions and remarks and studies more specifically sufficient conditions for equilibrium as well as the case of several productive assets.
All proofs are in the appendix.

We introduce a few notations; all vectors are column vectors and transposition is denoted by the superscript $*$. As usual, 1_d denotes the d-dimensional vector whose every component is one. If $Z = \left(Z^1, ..., Z^n\right)$ denotes a vector in R^n, then $diagZ$ denotes the $(n \times n)$ diagonal matrix whose diagonal entries are the components of Z. We denote by $\parallel Z \parallel^2$ the nonnegative real number $\sum_{i=1}^n \left(Z^i\right)^2$.
Two probability measures P and Q, defined on the same measurable space (Ω, F) are said to be equivalent if they agree on the null sets. Let (Ω, F, P) be a fixed probability space and $\mathbb{T}$ denote the interval $[0, T]$. Then $L_d^2(\mathbb{T})$ denotes the set of $(F_t)_{t\in\mathbb{T}}$-progressively measurable, R^d-valued processes $\{\Psi_t; t \in \mathbb{T}\}$ such that

$$\int_0^T \parallel \Psi_t \parallel^2 dt < \infty \quad a.s. \quad P.$$

For any R^d-valued process $\Psi = \{\Psi_t; t \in \mathbb{T}\}$ in $L_d^2(\mathbb{T})$, let the real-valued process $\mathcal{E}(\Psi) = \{\mathcal{E}_t(\Psi); t \in \mathbb{T}\}$ denote the exponential local martingale given for each t in $\mathbb{T}$ by

$$\mathcal{E}_t(\Psi) = \exp\left\{\int_0^t (\Psi_s)^* dW_s - 1/2 \int_0^t \parallel \Psi_s \parallel^2 ds\right\}.$$

For a real-valued process $u = \{u_t; t \in \mathbb{T}\}$ defined on a probability space (Ω, F, P), we denote by $u^- = \left\{u_t^-; t \in \mathbb{T}\right\}$ the process defined by $u_t^- = -\min(0, u_t)$ for all t in $\mathbb{T}$.
As usual, a function $F : \mathbb{T} \times R \to R$ is said to be of class $C^{m,n}$ if the m-th derivative of $F(\cdot, x) : \mathbb{T} \to R$ and the n-th derivative of $F(t, \cdot) : R \to R$ exist and are continuous.

2. The market model

We fix a finite-time horizon $\mathbb{T} \triangleq [0,T]$, on which we are going to treat our problem: T corresponds to the terminal date for all economic activity under consideration. All processes that we shall encounter in this paper are defined on $\mathbb{T}$.
We consider a "primitive" market consisting of one bond and one single productive asset. We shall refer to these two assets as the primitive assets. We will consider in section 7 the case of a market in which there are more than just one productive asset. As we have seen in the introduction, we assume that our full market consists not only of these primitive assets but also of additional "purely financial" assets "completing" the market. More precisely:

2.1. Conditions on the primitive market

The primitive market model is the same as in Karatzas [1989] taking $m = 1$, except that we consider here dividends paying assets.
We adopt a model for the primitive market consisting of one bond with price at time t denoted by S_t^0 such that

$$dS_t^0 = S_t^0 r_t dt, \quad S_0^0 = 1$$

and one stock (or one productive risky asset) with price per share at time t denoted by S_t satisfying the equation

$$dS_t = S_t \left[(b_t - \delta_t)\, dt + \sigma_t dW_t \right], \quad S_0 = 1. \tag{2.1}$$

Here, $W = \left\{ \left(W_t^1, ..., W_t^d\right)^* ; t \in \mathbb{T} \right\}$ is a d-dimensional Brownian motion on a probability space (Ω, F, P) and we let $(F_t)_{t\in\mathbb{T}}$ denote the P-augmentation of the natural filtration generated by W. We assume that the sample paths of W completely specify all the distinguishable events, which mathematically entails $F_T = F$. Since standard Brownian motions start from zero with probability one, F_0 is trivial.
It is assumed throughout that $d \geq 1$, i.e., the number of sources of uncertainty is larger than the number of stocks.

Assumption 1 *The real-valued interest rate process $\{r_t; t \in \mathbb{T}\}$, the real-valued process $\{b_t; t \in \mathbb{T}\}$, the real-valued dividend yield process paid by the stock $\{\delta_t; t \in \mathbb{T}\}$ as well as the volatility $(1 \times d)$-matrix-valued process $\left\{\sigma_t = \left(\sigma_t^1, ..., \sigma_t^d\right); t \in \mathbb{T}\right\}$ are taken to be progressively measurable with respect to $(F_t)_{t\in\mathbb{T}}$ and bounded uniformly in (t, ω) in $\mathbb{T} \times \Omega$.*

Under this assumption, we know that equation (2.1) admits a unique real-valued, $(F_t)_{t\in\mathbb{T}}$-adapted, continuous solution $\{S_t; t \in \mathbb{T}\}$, satisfying $E\left[\sup_{t\in\mathbb{T}} {S_t}^2\right] < \infty$ (see for instance Karatzas-Shreve [1988]).

Assumption 2 *For all t in $\mathbb{T}$, the volatility matrix σ_t has full rank* 1.

This amounts to assuming that, for all t in $\mathbb{T}$, $P\left[\sigma_t^j = 0 \text{ for all } j \text{ in } \{1, ..., d\}\right] = 0$. A d-dimensional process $\theta = \{\theta_t; t \in \mathbb{T}\}$ that we shall call in a natural way the relative risk process can then be defined by :

$$\theta_t \triangleq \left[(b_t - r_t) / \sum_{i=1}^{d} (\sigma_t^i)^2 \right] \sigma_t^* \quad a.s.\ P, \quad 0 \le t \le T.$$

With the above assumptions, θ is an $(F_t)_{t\in\mathbb{T}}$-progressively measurable process and we shall make in the remainder of the paper the following

Assumption 3 *The process θ is uniformly bounded.*

Let $\beta = \{\beta_t; t \in \mathbb{T}\}$ denote the process given by $\beta_t \triangleq 1/S_t^0 = \exp - \int_0^t r_s ds$. We shall also have the occasion to use the discounted price process $\tilde{S} = \left\{\tilde{S}_t; t \in \mathbb{T}\right\}$ defined by $\tilde{S}_t \triangleq S_t \exp \int_0^t (\delta_s - r_s)\, ds$ for all t in $\mathbb{T}$. Using Itô's lemma, we easily get that $\tilde{S}$ is the unique solution of the following stochastic differential equation:

$$d\tilde{S}_t = \tilde{S}_t \left[(b_t - r_t)\, dt + \sigma_t dW_t \right] = \tilde{S}_t \sigma_t \left[\theta_t dt + dW_t \right], \quad \tilde{S}_0 = 1.$$

2.2. Conditions on the purely financial assets

In this section, we consider a sufficiently large number of purely financial assets (or contingent claims), i.e., assets which are in zero net supply, in order to complete the market. It is easy to show that the minimum number of such assets is $(d-1)$. Moreover, if there are more assets and if the market is supposed to be complete, we can combine them in such a way as to obtain $(d-1)$ assets which complete the market. In the next, we will assume that there are exactly $(d-1)$ additional purely financial assets. Their prices C_t^i for i in $\{1, ..., d-1\}$ are driven by the d-dimensional Brownian motion W and we assume that they "complete" the market. More precisely, we assume that the prices C_t^i are governed by

$$dC_t^i = C_t^i \left[a_t^i dt + \mu_t^i dW_t \right] \qquad i = 1, ..., (d-1),$$

where the coefficients satisfy the following regularity conditions.

Assumption 4 *The process $\left\{ \mu_t = \left[\left(\mu_j^i \right)_t \right]_{\substack{1 \le i \le d-1 \\ 1 \le j \le d}}; t \in \mathbb{T} \right\}$ is an $(F_t)_{t\in\mathbb{T}}$-progressively measurable, uniformly bounded, $(d-1) \times d$ matrix-valued process such that for all t in $\mathbb{T}$, the $(d \times d)$-augmented volatility matrix $\bar{\sigma}_t \triangleq \left[\begin{smallmatrix} \sigma_t \\ \mu_t \end{smallmatrix}\right]$ admits an inverse[1]. The norms of $(\bar{\sigma}_t)^{-1}$ and of $(\bar{\sigma}_t^*)^{-1}$ are uniformly bounded[2]. The process $\left\{ a_t = \left(a_t^i \right)_{1 \le i \le d-1}; t \in \mathbb{T} \right\}$ is an $(F_t)_{t\in\mathbb{T}}$- progressively measurable, uniformly bounded $(d-1)$-dimensional vector process.*

Let $\bar{b} \triangleq \left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right]$ denote the d-dimensional augmented stock appreciation vector. A d-dimensional process $\bar{\theta} = \left\{ \bar{\theta}_t; t \in \mathbb{T} \right\}$ can then be defined by :

$$\bar{\theta}_t \triangleq (\bar{\sigma}_t)^{-1} \left[\left(\bar{b}_t - r_t 1_d \right) \right] \quad a.s. \quad P, \quad 0 \le t \le T.$$

[1] For example a matrix-valued process $\{\mu_t; t \in \mathcal{T}\}$ such that for all t, the rows of μ_t, thought of as vectors in R^d are orthonormal and in the kernel of σ_t, i.e., $\sigma_t \mu_t^* = 0$, like in Karatzas et al. [1991].

[2] We could for instance impose for all t in $\mathcal{T}$ a nondegeneracy condition on the matrix $\bar{\sigma}_t \bar{\sigma}_t^*$

With the above assumptions, $\bar{\theta}$ is $(F_t)_{t\in\mathbb{T}}$-progressively measurable and uniformly bounded.

So our full market consists of $(d+1)$ assets: the bond, the primitive stock and $(d-1)$ additional purely financial assets. We shall denote by Z the d-dimensional risky assets price process given by $Z \triangleq \left(S, C^1, ..., C^{d-1}\right)$ and by $\tilde{Z}$ the discounted price process $\tilde{Z} \triangleq \left(\tilde{S}, C^1/S^0, ..., C^{d-1}/S^0\right)$. Notice that assets prices can fluctuate in an almost arbitrary not necessarily Markovian fashion.

3. Equivalent martingale measures

Now that we have described both our primitive and our full markets, we can consider the problem of the existence and of the characterization of equivalent probability measures on (Ω, F, P) that make the discounted price processes $\tilde{S}$ in the primitive market and $\tilde{Z}$ in the full market martingales. We will see in section 6 that these probability measures are of great use for our problem. Besides, we already know that the fair price of any contingent claim, whose final payoff is in the form $h(S_T)$ for some function h, belongs to the following interval

$$\left[\inf_{Q\in\mathcal{M}_S} E^Q\left[\beta_T h(S_T)\right], \sup_{Q\in\mathcal{M}_S} E^Q\left[\beta_T h(S_T)\right]\right]$$

where $\mathcal{M}_S$ denotes the set of all equivalent probability measures that make the process $\tilde{S}$ a martingale (see Harrison-Kreps [1979], El Karoui-Quenez [1995] or Jouini-Kallal [1995]).

3.1. In the primitive market

With the notations of section 2, we still consider the filtered probability space $\left(\Omega, F, (F_t)_{t\in\mathbb{T}}, P\right)$.

Definition 3.1. *A probability measure Q defined on (Ω, F, P) is an S-equivalent martingale probability measure for $(F_t)_{t\in\mathbb{T}}$ if it satisfies :*

1. *The probability measures P and Q are equivalent.*
2. *The process $\tilde{S}$ is a Q-martingale for $(F_t)_{t\in\mathbb{T}}$.*

Using Itô's lemma and the fact that the dividend yield process is uniformly bounded, notice that an S-equivalent martingale probability measure is in fact an equivalent probability measure that makes the discounted "gain" process

$$G = \left\{\frac{S_t}{S_t^0} + \int_0^t \exp\left(\int_0^s -r_u du\right)\delta_s S_s ds; t\in\mathbb{T}\right\}$$

a martingale; and we know, since the fundamental theorem of asset pricing, that the existence of such a probability measure is essentially equivalent to the assumption of no arbitrage (see, among others, Harrison-Kreps [1979], Harrison-Pliska [1981],

Delbaen-Schachermayer [1984] and see also the end of section 4.1). Under such a probability measure, the expected return of the stock is equal to the (short term) interest rate minus the dividend yield.
Let $M_d^2(\mathbb{T})$ denote the set of R^d-valued processes $\Phi = \{\Phi_t; t \in \mathbb{T}\}$ in $L_d^2(\mathbb{T})$ such that

$$E\left[\mathcal{E}_T(\Phi)\right] \triangleq E\left[\exp\left\{\int_0^T (\Phi_s)^* \, dW_s - 1/2\int_0^T \parallel \Phi_s \parallel^2 ds\right\}\right] = 1.$$

Notice that $M_d^2(\mathbb{T})$ corresponds to the set of processes $\{\Phi_t; t \in \mathbb{T}\}$ in $L_d^2(\mathbb{T})$ such that the exponential local martingale

$$\mathcal{E}(\Phi) = \left\{\exp\left\{\int_0^t (\Phi_s)^* \, dW_s - 1/2\int_0^t \parallel \Phi_s \parallel^2 ds\right\}; t \in \mathbb{T}\right\}$$

is a true martingale: as a matter of fact, $\mathcal{E}(\Phi)$ being a nonnegative local martingale, we can use Fatou's lemma and get that this process is a supermartingale; therefore, it is a martingale if and only if its expected value is a constant for all t in $\mathbb{T}$.
We introduce the following set

$$K^\sigma \triangleq \left\{\nu \in L_d^2(\mathbb{T}) \text{ such that } \forall t,\ \sigma_t \nu_t = 0 \text{ and } -\theta^\nu \triangleq -(\theta + \nu) \in M_d^2(\mathbb{T})\right\}.$$

Notice that K^σ is never empty because the null process $n = \{n_t; t \in \mathbb{T}\}$ defined by $n_t = 0$ for all t always belongs to K^σ. Indeed, the first two conditions: n in $L_d^2(\mathbb{T})$ and $\sigma_t n_t = 0$ for all t are trivially satisfied and as θ is assumed to be uniformly bounded, the process $\mathcal{E}(-\theta)$ is a martingale -see, for instance, the Novikov condition in Karatzas and Shreve [1988] p. 199.

We now characterize all S-equivalent martingale probability measures:

Lemma 3.2. *Let Q be a probability measure defined on (Ω, F, P). The following are equivalent :*

1. *The probability measure Q is an S-equivalent martingale probability measure for $(F_t)_{t\in\mathbb{T}}$.*
2. *The probability measure Q is such that $dQ/dP = \mathcal{E}_T(-\theta^\nu)$ for some ν in K^σ.*

We shall denote by $\mathcal{M}_S$ the set of S-equivalent martingale probability measures for $(F_t)_{t\in\mathbb{T}}$. As we have seen, the null process n always belongs to K^σ so $\mathcal{M}_S$ is never reduced to the empty set and there always exists at least one $S-$equivalent martingale probability measure denoted by P^0 and given by $dP^0/dP = \mathcal{E}_T(-\theta)$; it is the so-called minimal martingale-measure of Föllmer-Schweizer [1991].
There exists a unique martingale probability measure if and only if we have $K^\sigma = \{0\}$ which is the case if and only if $d = 1$. In that case, the unique S-equivalent martingale probability measure is the one given by Föllmer and Schweizer [1991]. More generally,

$\mathcal{M}_S$ can be considered as indexed by K^σ: for each ν in K^σ, we shall denote by P^ν the corresponding martingale measure, i.e., such that

$$dP^\nu/dP = \exp\left\{\int_0^T -(\theta_s^\nu)^* \, dW_s - 1/2\int_0^T \| \theta_s \|^2 + \| \nu_s \|^2 \, ds\right\}$$

and by $M^\nu \triangleq \mathcal{E}\left[-\theta^\nu\right]$ the corresponding process and then we have

$$\mathcal{M}_S = \{P^\nu; \nu \in K^\sigma\} \neq \emptyset.$$

Notice that for each $S-$equivalent martingale probability measure P^ν, we have $d\tilde{S}_t = \tilde{S}_t\left[\sigma_t dW_t^{P^\nu}\right]$.

3.2. In the full market

We are interested in what we shall call $Z-$equivalent martingale probability measures, i.e., equivalent probability measures Q that make the full process

$$\tilde{Z} \triangleq \left(\tilde{S}, C^1/S^0, ..., C^{d-1}/S^0\right)$$

a Q-martingale for $(F_t)_{t\in\mathbb{T}}$. Notice that any $Z-$equivalent martingale probability measure is in an obvious way an S-equivalent martingale probability measure. Following exactly the same approach as in the preceding section for $d = 1$, we show the following result:

Lemma 3.3. *There exists a unique equivalent probability measure $\bar{P}$ defined on (Ω, F, P) that makes the full process $\tilde{Z}$ a martingale for $(F_t)_{t\in\mathbb{T}}$. It is given by*

$$d\bar{P}/dP = \mathcal{E}_T\left(-\bar{\theta}\right) = \exp\left\{-\int_0^T \left(\bar{\theta}_s\right)^* dW_s - 1/2\int_0^T \| \bar{\theta}_s \|^2 \, ds\right\}.$$

We then have $d\tilde{Z}_t = diag\tilde{Z}_t\left[\bar{\sigma}_t dW_t^{\bar{P}}\right]$ where $\left\{W_t^{\bar{P}}; t \in \mathbb{T}\right\}$ is the $\bar{P}-$Brownian motion for $(F_t)_{t\in\mathbb{T}}$ defined by $W_t^{\bar{P}} \triangleq W_t + \int_0^t \bar{\theta}_s ds$ for all t in $\mathbb{T}$. We shall in the remainder of the paper denote the martingale process $\left\{E\left[\frac{d\bar{P}}{dP} \mid F_t\right]; t \in \mathbb{T}\right\}$ by $\bar{M} = \left\{\bar{M}_t; t \in \mathbb{T}\right\}$. As $\bar{P}$ belongs to $\mathcal{M}_S$, it can be written in the form $P^{\bar{\nu}}$ for $\bar{\nu} = \bar{\theta} - \theta$ satisfying

$$\sigma_t \bar{\nu}_t = \sigma_t\left(\bar{\theta}_t - \theta_t\right) = (1\ 0\ ...0)\left(\bar{b}_t - r_t 1_d\right) - (b_t - r_t) = 0.$$

Notice also that $\bar{P}$ not only depends on the productive asset price process but also on the financial assets price processes.

4. Trading strategies

Let us now consider an economic agent, who invests in the full market.

4.1. Wealth process and admissible strategies

We shall denote respectively by π_t^S and $\pi_t^{C^i}$ the amounts that the agent invests at time t in the stock and in the ith contingent claim respectively, by c_t the rate at which he withdraws funds for consumption and by $X_t^{\pi^S,\left(\pi^{C^i}\right)_i,c}$ the corresponding wealth of this agent at time t. We allow here any π_t^S or $\pi_t^{C^i}$ to become negative, which amounts to allowing the agent to sell short any risky asset. Similarly, the amount of money $\pi_t^{S^0} = X_t^{\pi^S,\left(\pi^{C^1},\ldots,\pi^{C^{d-1}}\right),c} - \pi_t^S - \sum_{i=1}^{d-1} \pi_t^{C^i}$ invested in the bond at time t may also become negative, which is to be interpreted as borrowing at the interest rate r_t. More precisely:

Definition 4.1. *A trading strategy or a portfolio process*

$$\pi = \left\{ \left(\pi_t^S, \pi_t^{C^1}, \ldots, \pi_t^{C^{d-1}} \right)^* ; t \in \mathbb{T} \right\}$$

is an element of $L_d^2(\mathbb{T})$.

Definition 4.2. *A consumption strategy or a consumption rate process*

$$c = \{c_t; t \in \mathbb{T}\}$$

is a nonnegative, progressively measurable, real-valued process that satisfies $\int_0^T c_t dt < \infty \quad a.s.\ P$.

Assuming that the trading-consumption strategy is self financing, i.e., that at each time t, sales and dividends must finance purchases and consumption, we obtain, with the above interpretations and definitions, for each t in $\mathbb{T}$, the following equation for the wealth of the agent

$$\begin{aligned} dX_t^{\pi,c} &= \frac{\pi_t^{S^0}}{S_t^0} dS_t^0 + \frac{\pi_t^S}{S_t} dS_t + \sum_{i=1}^{d-1} \frac{\pi_t^{C^i}}{C_t^i} dC_t^i \\ &\quad - c_t dt + \frac{\pi_t^S}{S_t} \delta_t S_t dt. \end{aligned} \tag{4.1}$$

where the terms on the right-hand side of the equation account respectively for capital gains or losses from the productive asset held, capital gains or losses from financial assets held, the decrease in wealth due to consumption and the increase in wealth due to dividends paid by the productive asset. It is easy to see that, with the assumptions made on the trading and consumption strategies, all quantities are well defined. Using what has been done in the preceding section, the dynamics of the wealth process can be rewritten

$$dX_t^{\pi,c} = \left[X_t^{\pi,c} - \pi_t^S - \sum_{i=1}^{d-1} \pi_t^{C^i} \right] r_t dt + \pi_t^S \left[(b_t - \delta_t)\, dt + \sigma_t dW_t \right]$$

$$
\begin{aligned}
&+\sum_{i=1}^{d-1} \pi_t^{C^i} \left[a_t^i dt + \mu_t^i dW_t\right] - c_t dt + \frac{\pi_t^S}{S_t} \delta_t S_t dt \\
=\ & \left[r_t X_t^{\pi,c} - c_t\right] dt + (\pi_t)^* \left[\bar{b}_t - r_t 1_d\right] dt + (\pi_t)^* \bar{\sigma}_t dW_t \\
=\ & \left[r_t X_t^{\pi,c} - c_t\right] dt + (\pi_t)^* \bar{\sigma}_t dW_t^{\bar{P}}.
\end{aligned}
$$

where, as above, $W_t^{\bar{P}} \triangleq W_t + \int_0^t \bar{\theta}_s ds$ is a $\bar{P}$-Brownian motion. The unique solution of this equation with initial wealth $X_0^{\pi,c} = x \geq 0$ is denoted by $\{X_t^{x;\pi,c}; t \in \mathbb{T}\}$ and is easily seen to be given for all t in $\mathbb{T}$ by

$$
\beta_t X_t^{x;\pi,c} = x - \int_0^t \beta_s c_s ds + \int_0^t \beta_s (\pi_s)^* \bar{\sigma}_s dW_s^{\bar{P}}. \tag{4.2}
$$

We shall now single out those pairs (π, c) for which the investor avoids negative wealth by defining admissible strategies:

Definition 4.3. *A pair (π, c) of portfolio and consumption rate processes is called admissible for the initial capital $x \geq 0$ if the unique corresponding wealth process $\{X_t^{x;\pi,c}; t \in \mathbb{T}\}$ given by equation (4.2) above satisfies*

$$
X_t^{x;\pi,c} \geq 0 \quad \text{for all } t \in \mathbb{T}. \tag{4.3}
$$

The class of such pairs is denoted by $\mathbf{A}(x)$. Notice that we don't need the assumption that π satisfies $E^{\bar{P}}\left[\int_0^T \|\pi_s\|^2 ds\right] < \infty$, which is often found in the literature and implies according to (4.2) that the process

$$
Y \triangleq \left\{\beta_t X_t^{x;\pi,c} + \int_0^t \beta_s c_s ds; t \in \mathbb{T}\right\}
$$

consisting of current discounted wealth plus total discounted consumption is a square integrable $\bar{P}$-martingale. One may note the requirement in (4.3) that wealth is always nonnegative which makes budget feasibility somewhat more restrictive than the usual notion. We impose a no-bankruptcy condition not only at terminal time but at each time t in $\mathbb{T}$, i.e., $X_t \geq 0$ for all t in $\mathbb{T}$, which amounts to saying that at each time t, the investor must be able to cover his debts -see e.g. Karatzas-Lehoczky-Shreve [1987] or Duffie [1994] where the same assumption is made. It is technically useful as it enables us to apply Fatou's lemma in equation (4.2) and get that the above mentioned process

$$
Y \triangleq \left\{\beta_t X_t^{x;\pi,c} + \int_0^t \beta_s c_s ds; t \in \mathbb{T}\right\} \tag{4.4}
$$

consisting of current discounted wealth plus total discounted consumption is a $\bar{P}$-supermartingale. We then get the inequality

$$
E^{\bar{P}}\left[\beta_t X_t^{x;\pi,c} + \int_0^t \beta_s c_s ds\right] \leq x \tag{4.5}
$$

which can be interpreted as a budget constraint: the expected total value of current wealth and consumption-to-date, both deflated to $t = 0$, does not exceed the initial capital.

If we consider an economic agent who only invests in the primitive market, all definitions and interpretations remain the same, provided we adapt them in a natural way to the primitive market: more precisely, a trading strategy is an element $\pi = \left\{ \left(\pi_t^S\right)^* ; t \in \mathbb{T} \right\}$ of $L_1^2(\mathbb{T})$; a consumption strategy is a nonnegative, progressively measurable, real-valued process $c = \{c_t; t \in \mathbb{T}\}$ that satisfies $\int_0^T c_t dt < \infty \quad a.s.\ P$. The wealth process corresponding to a trading-consumption strategy (π, c) is given for all t in $\mathbb{T}$ by

$$\beta_t X_t^{x;\pi,c} = x - \int_0^t \beta_s c_s ds + \int_0^t \beta_s (\pi_s)^* \sigma_s dW_s^{P^0}. \tag{4.6}$$

Finally, a pair (π, c) of trading and consumption strategies is called admissible for the initial capital $x \geq 0$ if the unique corresponding wealth process $\{X_t^{x;\pi,c}; t \in \mathbb{T}\}$ satisfies $X_t^{x;\pi,c} \geq 0 \quad$ for all $t \in \mathbb{T}$.

Notice that both our full market model and our primitive market model exclude arbitrage opportunities; an arbitrage opportunity is an admissible plan that yields through some combination of buying and selling a positive gain in some circumstances without a countervailing threat of loss in other circumstances or equivalently in our setting, a trading strategy that achieves with zero initial capital an amount of terminal wealth which is almost surely nonnegative and positive with positive probability; so here, an arbitrage opportunity consists of a pair (π, c) of portfolio and consumption rate processes such that (π, c) is in $A(0)$ and such that the corresponding wealth process with initial capital $x = 0$ is almost surely nonnull at terminal time. In both cases (full and primitive markets), the existence of at least one equivalent martingale probability measure (for $\tilde{Z}$ in the first case and for $\tilde{S}$ in the second case) rules out such opportunities: indeed, in both cases, as we have seen with equation (4.5), the wealth at initial time is greater than the expected value of the discounted wealth at terminal time: the discount process being positive and the wealth process being nonnegative, it is then impossible, starting from the initial capital $x = 0$ to reach a nonnull wealth at terminal time.

4.2. Achievable consumption and wealth processes

For every given real number $x \geq 0$, denote by $\mathbf{C}(x)$ the class of consumption rate processes c which satisfy $E^{\bar{P}}\left[\int_0^T \beta_s c_s ds\right] \leq x$ and by $\mathbf{L}(x)$ the class of nonnegative, F-measurable random variables B which satisfy $E^{\bar{P}}[\beta_T B] \leq x$.

We have seen with equation (4.5) that if (π, c) is in $A(x)$, then c is in $C(x)$ and X_T is in $L(x)$. We shall now study to which extent the "opposite implications" are true, i.e., for every c in $C(x)$, does there exist a trading strategy π such that (π, c) is in

$A(x)$; for every B in $L(x)$, does there exist (π, c) in $A(x)$ such that $X_T^{x;\pi,c} = B$ and for every pair (c, B) in $C(x) \times L(x)$ satisfying

$$E^{\bar{P}}\left[\beta_T B + \int_0^t \beta_s c_s ds\right] \leq x,$$

does there exist a trading strategy π such that (π, c) is in $A(x)$ and $X_T^{x;\pi,c} = B$.

Achievable consumption processes

Given an initial wealth $x > 0$, we want to know which consumption processes an investor can achieve and we give a positive answer to the first question just raised.

Proposition 4.4. *1. For every c in $C(x)$, there exists a portfolio process π such that (π, c) belongs to $A(x)$.*

2. For every c in $\mathbf{D}(x) \triangleq \left\{c \in C(x); E^{\bar{P}}\left[\int_0^T \beta_s c_s ds\right] = x\right\}$, the preceding π is unique, the corresponding wealth process satisfies $X_T^{x;\pi,c} = 0$ and the process M given in (4.4) is a $\bar{P}$-martingale.

Achievable terminal wealth and completeness issues

We shall see now that the primitive market in the case $d = 1$ as well as the full market enable agents to hedge against all risk.

Definition 4.5. *A contingent claim is a financial instrument consisting of a payment B at maturity, where B is a nonnegative, F_T-measurable random variable satisfying $E[B^\mu] < \infty$ for some $\mu > 1$.*

We shall denote any contingent claim by its payment B. Using the boundedness of the processes $\bar{\theta}$, θ and r as well as Hölder's inequality, it is not hard to see that any contingent claim B satisfies $E^{\bar{P}}[B\beta_T] < \infty$ as well as $E^{P^0}[B\beta_T] < \infty$.

Definition 4.6. *The market is complete if, for all contingent claim B, there exist a trading strategy π and an initial capital $x \geq 0$ such that $(\pi, 0)$ is in $A(x)$ and the terminal value of the corresponding wealth process is equal to B, i.e., $X_T^{x;\pi,0} = B$.*

We say that the full market -resp. the primitive market- is complete if the conditions of the definition are satisfied for a trading strategy in the form $\pi = \left(\pi^S, \pi^{C_1}, ..., \pi^{C_{d-1}}\right)$ -resp. in the form $\pi = \left(\pi^S\right)$- and for a wealth process satisfying equation (4.2) -resp. equation (4.6).

We can now prove the following

Theorem 4.7. *1. The primitive market is complete if and only if $d = 1$.*

2. The full market is complete.

Following exactly the same approach, we can characterize the levels of wealth attainable by an initial capital $x \geq 0$.

Proposition 4.8. *Given an initial wealth $x \geq 0$,*

1. *For every B in $L(x)$, there exists a pair (π, c) in $A(x)$ such that the corresponding wealth process $X^{x;\pi,c}$ satisfies $X_T^{x;\pi,c} = B$ almost surely.*
2. *For any B in $\mathbf{M}(x) \triangleq \left\{ B \in L(x); E^{\bar{P}}[\beta_T B] = x \right\}$, the pair (π, c) in $A(x)$ above is unique and $c \equiv 0$; moreover, the corresponding wealth process is given by $\beta_t X_t^{x;\pi,0} = E^{\bar{P}}[\beta_T B \mid F_t]$.*

Achievable pairs of terminal wealth and consumption processes

Let $\mathbf{A}$ denote the set of pairs (c, X) where c is an adapted nonnegative consumption rate process and X is a nonnegative F_T-measurable random variable describing terminal wealth; we want to know which pairs (c, X) in A an investor can achieve starting with an initial capital $x > 0$ and following an admissible strategy.

Proposition 4.9. *If a pair (c, X) in A is such that $E^{\bar{P}}\left[\int_0^T \beta_s c_s ds + \beta_T X\right] = x$, then there exists a trading strategy π such that (π, c) belongs to $A(x)$ and $X_T^{x;\pi,c} = X$.*

We are now in a position to answer the last question raised at the beginning of the section:

Corollary 4.10. *For any pair (c, X) in A such that $E^{\bar{P}}\left[\int_0^T \beta_s c_s ds + \beta_T X\right] < \infty$, there exist an initial wealth $x > 0$ and a trading strategy π such that (π, c) belongs to $A(x)$ and $X_T^{x;\pi,c} = X$.*

5. Optimal demand in the full market

We still consider an economic agent, who invests in the so-called full market. We assume that his preferences are represented by a utility function for consumption and terminal wealth. The problem is the following: how should this agent choose at every time his portfolio and his consumption rate processes from among admissible pairs in order to obtain a maximum expected utility from both consumption over the time-interval $\mathbb{T}$ and terminal wealth.
More precisely, the agent has a utility function $U : A \to R$ given by

$$U(c, X) = E\left[\int_0^T u(t, c_t)\, dt + V(X)\right]$$

where

- $V : R_+ \to R$ is strictly increasing and concave;
- $u : \mathbb{T} \times R_+ \to R$ is continuous and, for each t in $\mathbb{T}$, $u(t, \cdot) : R_+ \to R$ is strictly increasing and concave;
- V is strictly concave or, for each t in $\mathbb{T}$, $u(t, \cdot)$ is strictly concave.

We assume that the agent is endowed with an initial capital $x > 0$ and that there is no exogenous endowment during the trading period $\mathbb{T}$. We now have the problem for each initial wealth x,

$$\sup_{(\pi,c)\in A(x)} U\left(c, X_T^{x;\pi,c}\right).$$

Using proposition 4.9 and the strict monotonicity of either or both of V and $\{u(t,\cdot); t \in \mathbb{T}\}$, we get that the agent's optimization problem is equivalent to

$$\sup_{(c,X)\in A} U(c,X)$$

$$\text{subject to } E^{\bar{P}}\left[\int_0^T \beta_t c_t dt + \beta_T X\right] \leq x.$$

A first step in the characterization of optimal pairs in A is given by the following result.

Proposition 5.1. *A pair (c^*, X^*) in A is optimal for the agent if and only if*

$$E\left[\int_0^T \beta_t M_t c_t^* dt + \beta_T M_T X^*\right] = x$$

and there is a constant $\gamma^* > 0$ such that (c^*, X^*) solves

$$\sup_{(c,X)\in A} E\left[\int_0^T u(t,c_t) - \gamma^* \beta_t M_t c_t dt + V(X) - \gamma^* \beta_T M_T X\right]. \tag{5.1}$$

We can be a little more systematic about the properties of U and V in order to characterize optimal pairs.

Assumption A: *The function V is C^1 on $(0,\infty)$, strictly concave and satisfies Inada conditions*[3]*. For all $t \in \mathbb{T}, u(t,\cdot)$ is C^1 on $(0,\infty)$, strictly concave and satisfies Inada conditions.*

Under *Assumption A*, we shall denote by $u_c(t,\cdot)$ the derivative of $u(t,\cdot)$ and by $I_u(t,\cdot)$ the inverse function of $u_c(t,\cdot)$. Then

Proposition 5.2. *Under Assumption A, a pair (c^*, X^*) in A is optimal for the agent if and only if there exists a constant $\gamma^* > 0$ such that*

$$\beta_t \bar{M}_t = \gamma^* u_c(t, c_t^*) \qquad 0 \leq t \leq T \quad a.s.\ P \tag{5.2}$$

$$\beta_T \bar{M}_T = \gamma^* V'(X^*) \tag{5.3}$$

$$E\left[\int_0^T \beta_t \bar{M}_t c_t^* dt + \beta_T \bar{M}_T X^*\right] = x. \tag{5.4}$$

[3] A strictly concave increasing function $F : R^+ \to R$ that is C^1 on $(0,\infty)$ satisfies Inada conditions if $\inf_x F'(x) = 0$ and $\sup_x F'(x) = +\infty$. If F satisfies these Inada conditions, then the inverse I_F of F' is well defined as a strictly decreasing continuous function on $(0,\infty)$ whose image is $(0,\infty)$.

Now that we have characterized the optimal pairs (c^*, X^*) in A, we can turn to multi-agent equilibrium considerations.

6. Equilibrium and compatible state price densities

We have so far considered a single economic agent, trading in the full market. We shall now assume that our economy consists of a finite number n of agents, who all have utility functions $U_j : A \to R$ given by

$$U_j(c, X) = E\left[\int_0^T u_j(t, c_t)\, dt + V_j(X)\right] \quad \text{for } j = 1, ..., n$$

where u_j and V_j satisfy the same conditions as u and V at the beginning of section 5. Each agent j has an initial endowment x_j and tries to maximize his utility $U_j(c, X)$ from both consumption over the time-interval $\mathbb{T}$ and terminal wealth. So the optimal demand $(c_j)^*$ of each agent j in the consumption commodity as well as his optimal portfolio choice $(\pi_j)^*$ are determined by the optimization problem studied in the preceding section

$$\sup_{(\pi,c)\in A(x_j)} U_j\left(c, X_T^{x_j;\pi,c}\right).$$

Besides, the total supply in the economy at time t consists of one unit of the productive asset S_t and of the dividend paid by the stock D_t. In equilibrium, the aggregated optimal demands of the agents must equal the total supply available.

More precisely, an equilibrium consists in price processes $S^0, S, C^1, ..., C^{d-1}$ and trading-consumption choices $\left((\pi_j^*)^S, \left[(\pi_j^*)^{C^i}\right]_{i\leq d-1}; c_j^*\right)_{1\leq j\leq n}$ which are optimal for the agents and such that for all t in $\mathbb{T}$, the following market clearing conditions hold almost surely:

$$\sum_{j=1}^n \left(c_j^*\right)_t = D_t$$

$$\sum_{j=1}^n (\pi_j^*)_t^S = S_t$$

$$\sum_{j=1}^n (\pi_j^*)_t^{C^i} = 0 \qquad 1 \leq i \leq (d-1)$$

$$\sum_{j=1}^n X_t^{\pi_j^*, c_j^*} = S_t$$

where the last relation follows from the equilibrium condition on the amount invested in the bond: $\sum_j (\pi_j^*)_t^{S^0} = \sum_j X_t^{\pi_j^*, c_j^*} - \sum_j (\pi_j^*)_t^S - \sum_j \sum_{i=1}^{d-1} (\pi_j^*)_t^{C^i} = 0.$

As described in the introduction, our problem consists in finding a fair price for contingent claims using as little information as possible; more precisely, we want to know if it is possible to find a fair pricing interval, which is not too large, and that only uses information on the productive asset's price process.

As implied by the next lemma and mentioned in the introduction, by only using the assumption of no arbitrage, our problem is solved for any contingent claim in the case $d = 1$ and for contingent claims B, which are redundant with respect to the primitive market in the case $d > 1$.

Definition 6.1. *We say that a contingent claim B is redundant -with respect to the primitive market- if there exist a nonnegative initial capital x and an admissible trading-consumption strategy $(\pi, 0)$ in $A(x)$ such that the corresponding discounted wealth process is a P^0-martingale and has a terminal value equal to the discounted contingent claim, i.e., $X_T^{x;\pi,0} = B$.*

Notice that if the trading strategy π is in the form $\pi = \left(\pi^S, 0, ..., 0\right)$ then the corresponding wealth process is in the form

$$dX_t^{x;\pi,0} = [r_t X_t]\, dt + \left(\pi_t^S\right)^* \sigma_t dW_t^{P^0}$$

so that the condition $E^{P^0}\left[\int_0^T \left\|\pi_s^S\right\|^2 ds\right] < \infty$ ensures that $\beta X^{x;\pi,0}$ is a P^0-martingale.

Lemma 6.2. *Let B be a given contingent claim.*

1. *If $d = 1$, then the unique fair price for B is equal to $E^{P^0}[\beta_T B]$.*
2. *If $d > 1$ and if B is redundant then its unique fair price is also equal to $E^{P^0}[\beta_T B]$.*

Notice that P^0 only depends on the productive asset's price process so that in both cases the unique fair price for any contingent claim B is perfectly determined without any knowledge about the financial assets price processes.

Assume now that the contingent claim B is nonredundant (with respect to the primitive market); following the same approach as above in the case $d = 1$, its unique fair price is $E^{\bar{P}}[\beta_T B]$; the problem is that, as we have noticed at the end of section 3.2, we need to know all the additional purely financial assets price processes in order to compute this price, which is not supposed to be the case here. As the equivalent probability measure $\bar{P}$ belongs to the set $\mathcal{M}_S$ of all S-equivalent martingale measures, we know that this fair price lies in the interval consisting of the expected values of the discounted contingent claim with respect to all S-equivalent martingale measures. But this interval has been shown to be too large (Cvitanic-Pham-Touzi [1997]). Our purpose here is to find prices or equivalently S-equivalent martingale measures that are compatible with what we have called equilibrium and to restrict this way the fair pricing interval. In the remainder of the paper, a fair price will denote a price that is compatible with both equilibrium and the assumption of no arbitrage.

6.1. A necessary condition for equilibrium

We have seen in section 3 that as $\bar{P}$ belongs to $\mathcal{M}_S$, we can write it in the form $P^{\bar{\nu}}$ for some $\bar{\nu}$ in K^{σ}. We shall here suppose that there is an equilibrium and, in order to grab more information on $\bar{\nu}$, deduce necessary conditions that the process $\{M_t^{\bar{\nu}}; t \in \mathbb{T}\}$ must satisfy. We emphasize the fact that we are in this section only interested in necessary conditions for equilibrium; we shall consider sufficient conditions for equilibrium in section 7.
We assume that the agents' utility functions satisfy *Assumption A* of section 5.

As there is an equilibrium, each agent j must achieve an optimal consumption rate process c_j^* as well as an optimal terminal wealth X_j^*. According to section 5, this implies that for all $j = 1, ...n$, there exists a positive constant $\gamma_j^* > 0$ such that

$$\beta_t M_t^{\bar{\nu}} = \gamma_j^* \left(u_j\right)_c \left[t, \left(c_j^*\right)_t\right] \qquad 0 \leq t \leq T \quad a.s.\ P \tag{6.1}$$

or

$$\left(c_j^*\right)_t = I_{u_j}\left(t, \frac{1}{\gamma_j^*}\beta_t M_t^{\bar{\nu}}\right) \qquad 0 \leq t \leq T \quad a.s.\ P.$$

On the other hand, as there is an equilibrium, markets must clear. As we have seen at the beginning of this section, this implies that

$$\sum_{j=1}^{n} \left(c_j^*\right)_t = D_t.$$

So, with the notations introduced in the preceding sections, we get the following lemma, whose proof is immediate.

Lemma 6.3. *A necessary condition for an equilibrium to be reached in our model is that there exist positive constants γ_j^*, $j = 1, ...n$, such that*

$$\sum_{j=1}^{n} I_{u_j}\left(t, \frac{1}{\gamma_j^*}\beta_t M_t^{\bar{\nu}}\right) = D_t.$$

6.2. Assuming that the utility functions are "regular"

We shall now assume that for each j, the utility function u_j satisfies certain regularity conditions, to wit,

$$I_{u_j} : \mathbb{T} \times R_+^* \rightarrow R_+$$

is of class $C^{1,2}$. We also assume that the coefficients of the primitive risky asset are such that for all t in $\mathbb{T}$, $b_t \neq r_t$.

We show that the compatibility with equilibrium enables us to price the contingent claims in a unique way, by only using information on the primitive assets.

Let $\varphi(t,x) \triangleq \sum_{j=1}^{n} I_{u_j}\left(t, \frac{1}{\gamma_j^*}x\right)$. The regularity assumptions made on all utility functions imply that φ is of class $C^{1,2}$ on $R_+ \times R_+^*$ and enable us to apply Itô's lemma to the process $\Phi = \{\varphi(t, \beta_t M_t^{\bar{\nu}}); t \in \mathbb{T}\}$ and get that for all t in $\mathbb{T}$

$$d\Phi_t = a_t dt - \varphi_x\left(t, \beta_t M_t^{\bar{\nu}}\right) \beta_t M_t^{\bar{\nu}} \left(\theta_t^{\bar{\nu}}\right)^* dW_t$$

for some progressively measurable process $a = \{a_t; t \in \mathbb{T}\}$. Using lemma 6.3, if equilibrium is reached in our model, the dividend process $\{D_t; t \in \mathbb{T}\}$ must follow a diffusion process given by

$$dD_t = b_t^D dt + \sigma_t^D dW_t \qquad \text{for all } t \text{ in } \mathbb{T}$$

where

$$\sigma_t^D \triangleq -\varphi_x\left(t, \beta_t M_t^{\bar{\nu}}\right) \beta_t M_t^{\bar{\nu}} \left(\theta_t^{\bar{\nu}}\right)^*.$$

As for all j and for all t, $I_{u_j}(t, \cdot)$ is assumed to be strictly decreasing, $\varphi_x(t, \cdot)$ is negative; for all $t \in \mathbb{T}$, the random variable $\beta_t M_t^{\bar{\nu}}$ is positive; this implies that there exists a measurable positive process λ such that for all t in $\mathbb{T}$,

$$\theta_t^{\bar{\nu}} = \lambda_t \left(\sigma_t^D\right)^* \text{ or}$$

$$\theta_t + \bar{\nu}_t = \lambda_t (\sigma_t^D)^* \quad a.s.\ P.$$

Notice that this implies that, for all i in $\{1, ..., d\}$,

$$(\theta^{\bar{\nu}})_t^i = \lambda_t (\sigma^D)_t^i \quad a.s.\ P,$$

so that the coefficients $(\theta^{\bar{\nu}})_t^i$ and $(\sigma^D)_t^i$ are of the same sign.
The process $\bar{\nu}$ we are looking for satisfies

$$\sigma_t \bar{\nu}_t = 0 \text{ for all } t \text{ in } \mathbb{T}$$

in order to belong to K^σ. We must then have

$$\sigma_t \theta_t = \lambda_t \sigma_t (\sigma_t^D)^*. \tag{6.2}$$

For all t in $\mathbb{T}$, we have assumed that $b_t - r_t \neq 0$, so that $\sigma_t(\sigma_t^D)^* \neq 0$ and we get

$$\begin{aligned} \lambda_t &= \frac{(b_t - r_t)}{\sigma_t(\sigma_t^D)^*} \text{ and} \\ \bar{\nu}_t &= \hat{\nu}_t \triangleq \frac{(b_t - r_t)}{\sigma_t(\sigma_t^D)^*}(\sigma_t^D)^* - \theta_t \end{aligned} \tag{6.3}$$

The only martingale measure to be compatible with equilibrium is then $P^{\hat{\nu}}$. The problem of fair pricing of nonredundant contingent claims is then reduced to taking the expected value with respect to $P^{\hat{\nu}}$, which only involves the productive asset and its dividends price processes: our problem is solved.

Notice that, according to relation (6.2), the condition we have imposed on b_t and r_t is equivalent to the condition that for all t, $\sigma_t(\sigma_t^D)^* \neq 0$ which amounts to saying that the price process S and its associated dividend process are in a way correlated, which seems reasonable. We shall now consider the specific case where the dividends process can be expressed as a function of the corresponding asset price process.

6.3. Assuming that there exists a regular function $d : \mathbb{T} \times R_+^* \to R$ such that the dividend process can be written in the form $\{d(t, S_t); t \in \mathbb{T}\}$

We assume now that the dividend process can be written in the form

$$D_t = \{d(t, S_t); t \in \mathbb{T}\}$$

where $d : \mathbb{T} \times R_+^* \to R_+$ is of class $C^{1,2}$.
Notice that as the discounted gain process G given by

$$G_t = \frac{S_t}{S_t^0} + \int_0^t \exp\left(\int_0^s -r_u du\right) \delta_s S_s ds \qquad \text{for all } t \text{ in } \mathbb{T}$$

is a martingale under any S-equivalent martingale measure P^ν -see section 3-, we have

$$S_t = E^{P^\nu}\left[\exp\left(\int_t^T -r_u du\right) S_T + \int_t^T \exp\left(\int_t^s -r_u du\right) \delta_s S_s ds \mid F_t\right]$$

so that the assumption made seems reasonnable.
Using Itô's lemma, we get that the dividend process $\{D_t; t \in \mathbb{T}\}$ follows a diffusion process given by

$$dD_t = b_t^D dt + \sigma_t^D dW_t$$

where the volatility process σ^D satisfies

$$\sigma_t^D = d_s(t, S_t) S_t \sigma_t \qquad \text{for all } t \in \mathbb{T}.$$

Then, assuming the same regularity on the utility functions, the approach of the preceding section remains valid; we obtain that there exists a measurable process α such that for all t in $\mathbb{T}$

$$\theta_t^{\bar{\nu}} = \alpha_t (\sigma_t)^*.$$

As $\sigma_t \theta_t^{\bar{\nu}} = \sigma_t \theta_t = (b_t - r_t)$ and as for all t, $\sigma_t (\sigma_t)^* \neq 0$ because the matrix σ_t has full rank equal to one, this gives us

$$\alpha_t = \frac{(b_t - r_t)}{\sigma_t (\sigma_t)^*}$$

so that for all t,

$$\begin{aligned} \theta_t^{\bar{\nu}} &= \theta_t \text{ or} \\ \bar{\nu}_t &= 0. \end{aligned}$$

So if we assume that the dividends at time t can be written as a regular function of the productive asset's price, the unique equivalent martingale measure that can be compatible with equilibrium is the so-called minimal martingale measure of Föllmer-Schweizer [1991] denoted in this paper by P^0, which only involves the productive asset's price process. A fair pricing of contingent claims is then in this case a pricing that doesn't take into account all "orthogonal" risk. Notice that in the case where the rows of μ_t, thought of as vectors in R^d are orthonormal and in the kernel of σ_t, i.e., $\sigma_t \mu_t^* = 0$, like in Karatzas et al. [1991] (see footnote 1 p.11), the purely financial assets satisfy the same stochastic differential equation under P as under P^0. As pointed out in Hofmann-Platen-Schweizer [1992] (theorem 3.1), this is a characteristic of the minimal martingale measure.

Let us now compare our result with the one concerning our problem obtained in Pham-Touzi [1996]. They consider a model in which there is a bond S^0, one risky productive asset, whose price process S is described by a stochastic volatility model, and one contingent claim completing the market. They fix the productive asset price process S and the contingent claim's one and this determines a unique equivalent martingale measure. Then they study the consistency of such a martingale measure with an equilibrium model; they answer the question: do there exist utility functions such that the price process S is an equilibrium price process? If the answer is yes, they say that the martingale measure is viable, and they show that this induces strong constraints on the coefficients of the price diffusion process. More specifically related to our problem, it is shown -see proposition 5.1- that in the positive dividend case, as long as the coefficients of the model respect the above mentioned constraints, the minimal martingale measure of Föllmer and Schweizer is viable if and only if the dividend process is in the form $D_t = a(t) S_t + b(t)$ for some continuous functions $a(t) > 0$ and $b(t) \geq 0$.
We have shown in section 6.2 that in a model which is not necessarily markovian anymore, if the dividend process is a diffusion process then there exists a unique admissible martingale measure. If we use the result of Pham-Touzi [1996], then we obtain that in a stochastic volatility setting, if the dividend process is in the form mentioned above, the minimal martingale measure being viable, it is necessarily the unique viable martingale measure. This is what we find in section 6.3 if we adapt our result to their specific setting and if we let the function d be an affine function.
Besides, we have shown that there is also a unique admissible price in any other case.

7. Extensions and remarks

We have so far assumed that there is an equilibrium and deduced properties that the state price densities must then satisfy; we have in fact found a unique state price density that could be compatible. We have only used necessary conditions. Conversely, we shall now study under which conditions equilibrium can be reached in our set-up (necessarily with this particular state price density).

7.1. Sufficient conditions for equilibrium

We shall use representative agent's theory to characterize equilibrium in our economy. We have seen in theorem 4.7 that our full market is complete, so we can apply the results of Huang [1987]. Our economy can be supported by a representative agent endowed with the aggregate individual endowments which consist of one unit of stock S and whose preferences for consumption and terminal wealth are characterized by

$$U(c, X) = E\left[\int_0^T u(t, c_t)\, dt + V(X)\right].$$

where the aggregated utility functions $u(t, .)$ and $V(.)$ inherit regularity from that of agents utility functions in the economy. From there, the problem of the existence and of the characterization of an optimal strategy for the representative agent is reduced to the study made in section 5, where a single agent is considered.

Following exactly the same approach as in section 6, and using the fact that at equilibrium the optimal consumption process of the representative agent must equal the dividend process, we obtain that a necessary and sufficient condition for an equilibrium to be reached in our set-up is that there exists a positive real number γ^* such that

$$\beta_t \bar{M}_t = \gamma^* u_c(t, D_t) \qquad a.s.\ P \quad 0 \leq t \leq T \tag{7.1}$$

$$\beta_T \bar{M}_T = \gamma^* V'(S_T) \tag{7.2}$$

with no additional condition because in this case, the budget constraint is automatically binding -see the end of the appendix for a precise proof.

We start by proving that we have not "worked on the empty set". Let d denote, like in section 6.3, a function of class $C^{1,2}$ from $\mathbb{T} \times R_+^*$ to R_+. We introduce the following definition:

Definition 7.1. *A d-equilibrium is a price system $\left[S^0, S, \left(C^i\right)_{i=1,\ldots,d-1}\right]$ satisfying the assumptions A_1, A_2, A_3 and A_4 made in section 2 and such that the economy defined by the dividend process $\{d(t, S_t); t \in \mathbb{T}\}$ is at the equilibrium.*

We want to know under which conditions there exists a d-equilibrium.

Lemma 7.2. *If there exist coefficients b, r ,σ, a and μ and a dividend function d satisfying the assumptions A_1, A_2, A_3 and A_4 of section 2 as well as the following equalities*

$$\begin{cases} b_t\left[1 + \frac{f_x(t,S_t)}{f(t,S_t)} S_t\right] = \beta_t \\ b_t - r_t + \frac{f_x(t,S_t)}{f(t,S_t)} S_t \left\|\sigma_t\right\|^2 = 0 \\ a_t = r_t 1_{(d-1)} - \frac{f_x(t,S_t)}{f(t,S_t)} S_t \mu_t \sigma_t^* \\ d(T, S_T) \triangleq I_u\left[T, V'(S_T)\right] \end{cases}$$

for $\beta_t \triangleq -\frac{f_t(t,S_t)}{f(t,S_t)} + \frac{f_x(t,S_t)}{f(t,S_t)} d(t, S_t) - \frac{f_x(t,S_t)}{f(t,S_t)}\left[\left\|\sigma_t\right\|^2 S_t\right] - \frac{1}{2}\frac{f_{xx}(t,S_t)}{f(t,S_t)}\left\|\sigma_t\right\|^2 S_t^2$ and $f(t, x) \triangleq u_c(t, d(t, x))$, then there exists a d-equilibrium.

Using this lemma, we consider the main types of utility functions and we show that equilibrium is reached.
Notice that the utility functions we consider for the logarithmic and exponential cases do not satisfy *Assumption A*; nevertheless it is easy to check that the approach that we have adopted remains valid: as the utility functions given in examples 7.3 and 7.5 satisfy $\inf_x u_c(t,x) = 0$, we may define $I_u(t,\cdot)$ to be the inverse of $u_c(t,\cdot)$ from $(0, u_c(t,0))$ onto $(0,+\infty)$ and we extend the domain of $I_u(t,\cdot)$ by setting

$$I_u(t,y) = 0 \qquad \text{for all } y \text{ in } [u_c(t,0), +\infty).$$

Then the equality

$$\min_{c\geq 0}[cy - u(t,c)] = yI_u(t,y) - u(t, I_u(t,y)) \qquad \text{for all } y \text{ in } (0,\infty) \text{ and } t \text{ in } \mathbb{T}$$

-where the minimum is uniquely attained at $I_u(t,y)$- remains true, so that the characterization of optimal pairs of terminal wealth and consumption rate processes given in proposition 5.2 and therefore the preceding lemma still hold.

Example 7.3. *Logarithmic utility function:*

We show here -like in Pham-Touzi [1996]- that equilibrium can be reached for a constant interest rate process. Suppose $u(t,c) = \exp(-\alpha t)\alpha\log(c+\alpha)$, $V(x) = \exp(-\alpha T)\log(x+1)$, $D_t = \alpha S_t$ for some positive constant α. As far as the terminal condition is concerned, we have $u_c[T, \alpha S_T] = \frac{\exp(-\alpha T)}{S_T+1} = V'(S_T)$. With the notations of the lemma, $f(t,x) = \frac{\exp(-\alpha T)}{x+1}$. Then $\frac{f_x(t,S_t)}{f(t,S_t)}S_t = -\frac{S_t}{S_t+1} \neq -1$. We find $\beta_t = \alpha - \frac{\alpha S_t}{S_t+1} + \frac{S_t}{S_t+1}\|\sigma_t\|^2 - \frac{(S_t)^2\|\sigma_t\|^2}{(S_t+1)^2}$ so that if

$$\begin{cases} b_t = \alpha + \frac{S_t}{S_t+1}\|\sigma_t\|^2 \\ r_t = \alpha \\ a_t = \alpha 1_{(d-1)} + \frac{S_t}{S_t+1}\mu_t\sigma_t^* \end{cases}$$

with σ and μ progressively measurable, uniformly bounded, and such that σ_t has full rank equal to one and $\bar{\sigma}_t$ is invertible, then the coefficients satisfy assumptions A_1, A_2, A_3 and A_4 and equilibrium is reached in our model.
Notice that the process σ is not submitted to any restriction, apart from being progressively measurable and uniformly bounded.

Example 7.4. *Power utility function*

Suppose $u(t,c) = \frac{c^\delta}{\delta}$ for $\delta \in]0,1[$, $V(x) = \frac{a^{(\delta-1)}}{\delta}x^\delta$, $D_t = aS_t$ for some positive constant a. We have $I_u[T, V'(S_T)] = [V'(S_T)]^{\frac{1}{(\delta-1)}} = aS_T$. With the notations of the lemma $f(t,x) = (ax)^{\delta-1}$. Then $\frac{f_x(t,S_t)}{f(t,S_t)}S_t = (\delta-1) \neq -1$, $\beta_t = -\frac{1}{2}\delta(\delta-1)\|\sigma_t\|^2 + a(\delta-1)$, so that if

$$\begin{cases} b_t = -\frac{1}{2}(\delta-1)\|\sigma_t\|^2 + \frac{a(\delta-1)}{\delta} \\ r_t = \frac{1}{2}(\delta-1)\|\sigma_t\|^2 + \frac{a(\delta-1)}{\delta} \\ a_t = r_t 1_{(d-1)} + (1-\delta)\mu_t\sigma_t^* \end{cases}$$

with σ and μ satisfying the same (regularity) conditions as in the preceding example, then equilibrium is reached for our model of section 2.

Example 7.5. *Exponential utility function*

Suppose $u(t,c) = 1 - \exp(-c)$, $V(x) = \frac{(1+x)^\delta}{\delta}$, $D_t = (1-\delta)\log(1+S_t)$ for some $\delta \in]0,1[$. Then $u_c[T, D_T] = (1+S_T)^{\delta-1} = V'(S_T)$. With the notations of the lemma, we have $f(t,x) = (1+x)^{\delta-1}$. Then $\frac{f_x(t,S_t)}{f(t,S_t)} S_t = \frac{(\delta-1)S_t}{1+S_t} \neq -1$ and $\beta_t = -(\delta-1)^2 \frac{\log(1+S_t)}{1+S_t} - (\delta-1)\frac{S_t}{1+S_t}\|\sigma_t\|^2 - \frac{1}{2}(\delta-1)(\delta-2)\left(\frac{S_t}{1+S_t}\right)^2 \|\sigma_t\|^2$ so that if

$$\begin{cases} b_t = -(\delta-1)^2 \frac{\log(1+S_t)}{1+\delta S_t} - (\delta-1)\frac{S_t}{1+\delta S_t}\|\sigma_t\|^2 \left[1 + \frac{1}{2}(\delta-2)\frac{1}{(1+S_t)}\right] \\ r_t = b_t + \frac{(\delta-1)S_t}{1+S_t}\|\sigma_t\|^2 \\ a_t = r_t 1_{(d-1)} - \frac{(\delta-1)S_t}{1+S_t}\mu_t \sigma_t^* \end{cases}$$

with σ and μ satisfying the usual regularity assumptions, then equilibrium is reached.

More generally, we obtain the following corollary which proves the existence of a d-equilibrium under some restrictions on the utility function for terminal wealth. We assume that V is of class C^2.

Corollary 7.6. *If there exist coefficients b, r, σ, a, μ satifying the assumptions A_1, A_2, A_3 and A_4 in section 2 as well as the following equalities*

$$\begin{cases} b_t \left[1 + \frac{V''(S_t)}{V'(S_t)} S_t\right] = \beta_t^V \\ r_t = b_t + \frac{V''(S_t)}{V'(S_t)} S_t \|\sigma_t\|^2 \\ a_t = r_t 1_{(d-1)} - \frac{V''(S_t)}{V'(S_t)} S_t \mu_t \sigma_t^* \end{cases}$$

where $\beta_t^V \triangleq -\frac{V''(S_t)}{V'(S_t)}\left[\|\sigma_t\|^2 S_t\right] + \frac{V''(S_t)}{V'(S_t)} I_u[t, V'(S_t)] - \frac{1}{2}\frac{V'''(S_t)}{V'(S_t)}\|\sigma_t\|^2 S_t^2$, *then there exists a d-equilibrium -for d given by* $d(t,x) = I_u[t, V'(x)]$.

Suppose for instance that $V(x) = \frac{x^\delta}{\delta}$ for $\delta \in]0,1[$, then $\frac{V''(S_t)}{V'(S_t)} S_t = (\delta - 1) \neq -1$, $\beta_t = -\frac{1}{2}\delta(\delta-1)\|\sigma_t\|^2 + (\delta-1)\frac{d(t,S_t)}{S_t}$ which ensures the existence of a solution if

$$\begin{cases} b_t = -\frac{1}{2}(\delta-1)\|\sigma_t\|^2 + \frac{(\delta-1)}{\delta}\frac{I_u\left[t,(S_t)^{(\delta-1)}\right]}{S_t} \\ r_t = b_t + (\delta-1)\|\sigma_t\|^2 \\ a_t = r_t 1_{(d-1)} - (\delta-1)\mu_t\sigma_t^* \end{cases}$$

and if $\frac{d(t,S_t)}{S_t} = \frac{I_u\left[t,(S_t)^{(\delta-1)}\right]}{S_t}$ is uniformly bounded, which corresponds to the fact that the dividend yield process is uniformly bounded (and which is the case for instance if $u(t,c) = \frac{c^\delta}{\delta}$ for $\delta \in]0,1[$).

We have seen that we have not worked on the empty set: as a matter of fact, we have seen that there exists a d-equilibrium for a large class of utility functions.

Notice that the dividend functions that we have obtained ($d(t,x) = ax$ for some positive constant a or $d(t,x) = (1-\delta)\log(1+x)$ for some $\delta \in]0,1[$) are increasing in x.

We shall now consider the problem in a different way: for a "given" dividend process, we study if there exists an equilibrium.

Proposition 7.7. *For all dividend process $D = \{D_t; t \in \mathbb{T}\}$ in the form*

$$dD_t = b_0 D_t dt + \sigma_0 D_t dW_t$$

for some constants b_0 in R and $\sigma_0 = (\sigma_0^1, ..., \sigma_0^d)$ in R^d, for all utility functions $u(t,\cdot)$ and V such that

$$V(\cdot) = u_c(T,\cdot) \text{ and } \tilde{u} : x \mapsto x u_c(t,x) \text{ is increasing,}$$

there exist price processes $S^0, (C^i)_{i=1,\ldots,d-1}$ and S in the form

$$S_t = s(t, D_t) \qquad \text{for all } t \text{ in } \mathbb{T}$$

where $s(t,\cdot)$ is increasing such that equations (7.1) and (7.2) are satisfied.

If assumptions A_1, A_2, A_3 and A_4 are satisfied, then the full market is in equilibrium.

Notice that the existence of price processes $S^0, (C^i)_{i=1,\ldots,d-1}$ and S in the form $S_t = s(t, D_t)$, where $s(t,\cdot)$ is increasing and such that the full market is in equilibrium is equivalent to the existence of a d-equilibrium for $d(t,\cdot) = [s(t,\cdot)]^{-1}$.

We prove now that equilibrium can be reached for a whole class of dividends processes.

Lemma 7.8. *For all dividend process $D = \{D_t; t \in \mathbb{T}\}$ in the form*

$$dD_t = b_t^D dt + \sigma_t^D dW_t$$

where the coefficients satisfy

$$b_t^D [\alpha(t, D_t)] = \beta(t, D_t) + \gamma(t, D_t) \left\|\sigma_t^D\right\|^2$$

for $\alpha = \left[I'_V \circ u_c + \frac{I_V \circ u_c}{u_c}\right] u_{cc}$, $\beta = -(\alpha)\frac{u_{ct}}{u_{cc}} - x$, and

$$\gamma = -\frac{1}{2} I_V \circ u_c \frac{u_{ccc}}{u_c} - I'_V \circ u_c \left[\frac{1}{2} u_{ccc} + \frac{(u_{cc})^2}{u_c}\right] - \frac{1}{2} I''_V \circ u_c (u_{cc})^2,$$

there exist price processes $S^0, (C^i)_{i=1,\ldots,d-1}$ and S in the form $S_t = I_V[u_c(t, D_t)]$ for all t in $\mathbb{T}$ such that equations (7.1) and (7.2) are satisfied.

If assumptions A_1, A_2, A_3 and A_4 are satisfied, then the full market is in equilibrium.

Notice that if $\alpha(t,x) \neq 0$, then the coefficient σ_t^D is not subject to any constraint. Suppose for instance that $V(x) = \frac{x^\delta}{\delta}$ for $\delta \in]0,1[$; if

$$b_t^D\left[\alpha^V(t,D_t)\right] = \beta^V(t,D_t) + \gamma^V(t,D_t)\left\|\sigma_t^D\right\|^2$$

where $\alpha^V = \frac{\delta}{\delta-1}[u_c]^{-\frac{\delta-2}{\delta-1}} u_{cc} \neq 0$, $\beta^V = -\frac{\delta}{\delta-1}[u_c]^{-\frac{\delta-2}{\delta-1}} u_{ct} - x$ and

$$\gamma = -\frac{\delta}{2(\delta-1)}(u_c)^{\frac{2-\delta}{\delta-1}} u_{ccc} - \frac{\delta}{2(\delta-1)^2}(u_c)^{\frac{-2\delta+3}{\delta-1}} u_{cc}^2$$

then equilibrium is reached.

7.2. Considering more than one productive asset

As we have mentioned in the introduction, our main result remains valid if there is more than one productive asset.

We now consider a primitive market consisting of one bond with price at time t denoted by S_t^0 such that

$$dS_t^0 = S_t^0 r_t dt$$

and m stocks with price per share at time t denoted for each $i = 1,...,m$ by S_t^i such that the m-dimensional process $S = \left\{\left(S^1,...,S^m\right)_t;\ t \in \mathbb{T}\right\}$ satisfies the equation

$$dS_t = diag S_t\left[(b_t - \delta_t)\,dt + \sigma_t dW_t\right]$$

where $W = \left\{\left(W_t^1,...,W_t^d\right)^*; t \in \mathbb{T}\right\}$ is still a d-dimensional Brownian motion on a probability space (Ω,F,P), the interest rate r_t, the m-dimensional dividend rate process paid by the stocks $\delta = \left\{\left(\delta^1,...,\delta^m\right)_t^*; t \in \mathbb{T}\right\}$, the volatility $(m \times d)$-matrix $\sigma = \left\{(\sigma_{ij})_t; t \in \mathbb{T}; 1 \leq i \leq m,\ 1 \leq j \leq d\right\}$ as well as the m-dimensional process $b = \left\{\left(b^1,...,b^m\right)_t^*; t \in \mathbb{T}\right\}$ are the coefficients of the model and are taken to be progressively measurable with respect to $(F_t)_{t\in\mathbb{T}}$ and bounded uniformly in (t,ω) in $[0,T]\times\Omega$. The number of sources of uncertainty is larger than the number of stocks, i.e., $d \geq m$. We assume that for all t, the $m \times d$ volatility matrix σ_t has full rank equal to m so that for all t, the matrix $\sigma_t\sigma_t^*$ is invertible.

Our full market consists then of the primitive market and of at least $(d-m)$ purely financial additional assets C^i such that the full market is complete. More precisely, we assume that the prices C_t^i are governed by

$$dC_t = diag C_t\left[a_t dt + \mu_t dW_t\right]$$

where the coefficients are as follows: the process $\left\{\mu_t = \left(\mu_{ij}\right)_t; t \in \mathbb{T}\right\}$ is an $(F_t)_{t\in\mathbb{T}}$-progressively measurable, uniformly bounded, $(d-m)\times d$ matrix-valued process such that the $d \times d$-augmented volatility matrix $\bar{\sigma}_t \triangleq \begin{bmatrix}\sigma_t \\ \mu_t\end{bmatrix}$ admits an inverse and the process $\left\{a_t = \left(a_t^1, ..., a_t^{d-m}\right)^* ; t \in \mathbb{T}\right\}$ is an $(F_t)_{t\in\mathbb{T}}$- progressively measurable, uniformly bounded $(d-m)$-dimensional vector process. Let $\bar{b} \triangleq \left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right]$ denote the d-dimensional augmented stock appreciation vector.

Adapting notations and definitions, all that we have done in sections 2-4 remains valid: letting the d-dimensional vector processes θ and $\bar{\theta}$ be such that for all t in $\mathbb{T}$

$$\theta_t \triangleq \sigma_t^* \left(\sigma_t \sigma_t^*\right)^{-1} \left(b_t - r_t 1_m\right) \quad a.s.\ P, \text{ and}$$
$$\bar{\theta}_t \triangleq \left(\bar{\sigma}_t\right)^{-1} \left[\left(\bar{b}_t - r_t 1_d\right)\right] \quad a.s.\ P,$$

we obtain the following results:

- All S-equivalent martingale measures, i.e., all equivalent probability measures under which the m-dimensional process $\tilde{S} = \left\{\tilde{S}_t ; t \in \mathbb{T}\right\}$ defined by

$$\tilde{S}_t^i \triangleq S_t^i \exp \int_0^t \left(\delta_s^i - r_s\right) ds \qquad \text{for all } t \in \mathbb{T} \text{ and } i \in \{1, ...m\}$$

  is a martingale, are given by the set $\mathcal{M}_S = \{P^\nu ; \nu \in K^\sigma\}$ with $dP^\nu/dP = \mathcal{E}_T\left(-\theta^\nu\right)$.

- The unique equivalent probability measure under which the d-dimensional process $\tilde{Z} \triangleq \left(\tilde{S}, C^1/S^0, ..., C^{d-m}/S^0\right)$ is a martingale, is given by $\bar{P}$ such that $d\bar{P}/dP = \bar{M}_T = \mathcal{E}_T\left(-\bar{\theta}\right)$.

- The primitive market is complete if and only if $m = d$ and the full market is complete.

Our economy consists of a finite number n of agents, who all have utility functions for consumption and terminal wealth that satisfy the regularity assumptions made in section 5 and in particular *Assumption A*. If there is an equilibrium, then each agent j, for $j = 1, ...n$, must be able to lead an optimal trading-consumption strategy $\left(\pi_j^*, c_j^*\right)$. This optimal strategy must be such that there exist positive real numbers $\gamma_j^* > 0$, for $j = 1, ...n$, for which

$$\beta_t \bar{M}_t = \gamma_j^* \left(u_j\right)_c \left[t, \left(c_j^*\right)_t\right] \qquad 0 \le t \le T \quad a.s.\ P \tag{7.3}$$

or

$$\left(c_j^*\right)_t = I_{u_j}\left(t, \frac{1}{\gamma_j^*} \beta_t \bar{M}_t\right) \qquad 0 \le t \le T \quad a.s.\ P.$$

On the other hand, if there is an equilibrium, markets must clear: the aggregated optimal demands of the agents must equal the total supply available. More precisely, an equilibrium consists in price processes $S^0, S = \left(S^1, ..., S^m\right), C = \left(C^1, ..., C^{d-m}\right)$ and trading-consumption choices

$$\left(\left(\pi_j^*\right)^{S^1}, ..., \left(\pi_j^*\right)^{S^m}, \left(\pi_j^*\right)^{C^1}, ..., \left(\pi_j^*\right)^{C^{d-m}} ; c_j^*\right)_{1 \le j \le n}$$

which must be optimal for the agents and such that for all t in $\mathbb{T}$, the following market clearing conditions hold almost surely:

$$\sum_{j=1}^{n} \left(c_j^*\right)_t = \sum_{i=1}^{m} D_t^i$$

$$\sum_{j=1}^{n} (\pi_j^*)_t^{S^i} = S_t^i \qquad 1 \leq i \leq m$$

$$\sum_{j=1}^{n} (\pi_j^*)_t^{C^i} = 0 \qquad 1 \leq i \leq (d-m)$$

$$\sum_{j=1}^{n} X_t^{\pi_j^*, c_j^*} = \sum_{i=1}^{m} S_t^i.$$

Then if there is equilibrium in this set-up, there must exist positive constants $\gamma_j^* > 0$, $j = 1, ...n$ such that

$$\sum_{j=1}^{n} I_{u_j}\left(t, \frac{1}{\gamma_j^*}\beta_t \bar{M}_t\right) = \sum_{i=1}^{m} D_t^i. \tag{7.4}$$

Then all results obtained in sections 6.2 and 6.3 remain true, replacing the dividend process $\{D_t; t \in \mathbb{T}\}$ by the so-called aggregate dividend process

$$\tilde{D} = \left\{\sum_{i=1}^{m} D_t^i; t \in \mathbb{T}\right\}.$$

We assume first that for all $t \in \mathbb{T}$, there exists $i \in \{1, ..., m\}$ such that $b_t^i \neq r_t$. According to (7.4), the aggregate dividend process $\tilde{D}$ must follow a diffusion process. More precisely, if we denote by $\sigma^{\tilde{D}}$ the volatility vector of the aggregate dividend process, we obtain that there exists a progressively measurable positive process $\tilde{a}$ such that for all t in $\mathbb{T}$

$$\bar{\theta}_t = \tilde{a}_t(\sigma_t^{\tilde{D}})^*.$$

As $\bar{P}$ belongs to the set $\mathcal{M}_S$, it can be written in the form $P^{\bar{\nu}}$ for some $\bar{\nu}$ in K^σ. Then $\sigma_t\theta_t^{\bar{\nu}} = \sigma_t\theta_t = (b_t - r_t 1_m)$ and

$$\tilde{a}_t = \frac{\left(b_t^i - r_t\right)}{(e_i)^* \, \sigma_t(\sigma_t^{\tilde{D}})^*}$$

for any $i \leq m$, where e_i denotes as usual the vector whose components are all zero except the ith, which is equal to one. Notice that the condition that for all $t \in \mathbb{T}$, there exists $i \in \{1, ..., m\}$ such that $b_t^i \neq r_t$ is equivalent to the assumption that for all $t \in \mathbb{T}$, the matrix $\sigma_t \left(\sigma_t\right)^*$ is not equal to the null matrix.

If an agent knows the aggregate dividend process $\tilde{D}$, then he can fairly price, in a unique way, all contingent claims in this market without any information on the additional purely financial assets.

If we now assume that the aggregate dividend process can be written in the form

$$D_t = d\left(t, S_t^1, ..., S_t^m\right)$$

for some regular enough function d -for instance if, like in the case of a single productive asset, for all i, the dividend process D^i can be written in the form $D_t^i = d_i\left(t, S_t^i\right)$ for all t for some regular enough function d_i- then we can apply Itô's lemma and obtain that there exists a progressively measurable, m-dimensional process γ such that for all t in $\mathbb{T}$

$$\theta_t^{\bar{\nu}} = (\sigma_t)^* \gamma_t.$$

This implies that for all t,

$$\gamma_t = \left(\sigma_t \sigma_t^*\right)^{-1}\left(b_t - r_t 1_m\right)$$

and

$$\theta_t^{\bar{\nu}} = (\sigma_t)^*\left(\sigma_t \sigma_t^*\right)^{-1}\left(b_t - r_t 1_m\right) \triangleq \theta_t.$$

Here again, we find that the unique equivalent martingale measure to be compatible with equilibrium is the minimal martingale measure of Föllmer-Schweizer.

On a practical point of view, if the aggregate dividend process is somewhat difficult to compute, the aggregate consumption can be observed and gives us a link between the unique equivalent martingale measure and the productive assets price processes. Moreover, if we assume that the dividends processes only depend on the productive assets price processes, the pricing of contingent claims is reduced to taking expected values with respect to the minimal martingale measure which only involves the productive assets prices.

Besides, on a theoretical point of view, the economic interpretation is interesting as it tells us that the unique martingale measure that can be compatible with an economic equilibrium -and therefore the unique "fair" price- appears to be only dependent on the real economy, as opposed to the financial one.

APPENDIX

Proof of lemma 3.2 1) Let us first show that Q is an equivalent probability measure if and only if it is such that $dQ/dP = \mathcal{E}_T(\rho)$ for some process $\rho = \{\rho_t; t \in \mathbb{T}\}$ in $M_d^2(\mathbb{T})$.

One implication is immediate: as ρ belongs to $L_d^2(\mathbb{T})$, the random variable $\mathcal{E}_T(\rho)$ is well defined. As it is nonnegative, it can be expressed as a measure density respectively to P, i.e., we can define on (Ω, F) a measure denoted by P^ρ given by $dP^\rho/dP = \mathcal{E}_T(\rho)$ and as $E[\mathcal{E}_T(\rho)] = 1$ -because ρ belongs to $M_d^2(\mathbb{T})$- this measure P^ρ is a probability measure. As M^ρ is positive, the probability measure P^ρ is equivalent to P.

For the converse implication, we shall denote dQ/dP by M_T and consider the process $M = \{M_t; t \in \mathbb{T}\}$ given by $M_t \triangleq E[M_T \mid F_t]$ for all t in $\mathbb{T}$. The process $\{M_t; t \in \mathbb{T}\}$ is well defined because $M_T \in L^1(\Omega, F, P)$; it is in an obvious way a continuous $(F_t)_{t\in\mathbb{T}}$-martingale and $(F_t)_{t\in\mathbb{T}}$ is the P-augmentation of the filtration generated by W so, by the fundamental martingales representation theorem (see Karatzas-Shreve [1988], p.170), M can be written as a stochastic integral with respect to W: there exists a process $\{\gamma_t; t \in \mathbb{T}\}$ in $L_d^2(\mathbb{T})$ such that

$$M_t = E[M_T] + \int_0^t (\gamma_s)^* \, dW_s \quad 0 \le t \le T$$

so $M_t = 1 + \int_0^t (\gamma_s)^* \, dW_s$. As $M_T > 0$, the process M satisfies $M_t > 0$ for all t in $\mathbb{T}$, so we can apply Itô's lemma and obtain for all t in $\mathbb{T}$

$$\ln M_t = \ln M_0 + \int_0^t (\gamma_s/M_s)^* \, dW_s - 1/2 \int_0^t \| \gamma_s/M_s \|^2 \, ds$$

or $M_t = \exp\left\{\int_0^t (\rho_s)^* \, dW_s - 1/2 \int_0^t \| \rho_s \|^2 \, ds\right\} = \mathcal{E}_t(\rho)$ for the d-dimensional process $\rho = \{\rho_t; t \in \mathbb{T}\}$ in $L_d^2(\mathbb{T})$, defined by $\rho_t \triangleq \frac{\gamma_t}{M_t}$ for all t in $\mathbb{T}$. As $E[M_T] = 1 = E[\mathcal{E}_T(\rho)]$, the process ρ belongs to $M_d^2(\mathbb{T})$ and this completes the proof.

So we can index the set of equivalent probability measures by $M_d^2(\mathbb{T})$ and for each process $\rho = \{\rho_t; t \in \mathbb{T}\}$ in $M_d^2(\mathbb{T})$ denote by P^ρ the equivalent probability measure such that $dP^\rho/dP = \mathcal{E}_T(\rho)$.

2) Let us now show the lemma: by Girsanov's theorem (see e.g. Karatzas-Shreve [1988], p.191), for all process ρ in $M_d^2(\mathbb{T})$, the d-dimensional process $W^{P^\rho} = \left\{W_t^{P^\rho}; t \in \mathbb{T}\right\}$ defined for all t in $\mathbb{T}$ by

$$W_t^{P^\rho} \triangleq W_t - \int_0^t (\rho_s) \, ds$$

is a P^ρ- Brownian motion for $(F_t)_{t\in\mathbb{T}}$. As $d\tilde{S}_t = \tilde{S}_t[(b_t - r_t)\,dt + \sigma_t dW_t]$, we have for all process ρ in $M_d^2(\mathbb{T})$, $d\tilde{S}_t = \tilde{S}_t\left[(b_t - r_t + \sigma_t\rho_t)\,dt + \sigma_t dW_t^{P^\rho}\right]$; defining the process $\{\nu_t; t \in \mathbb{T}\}$ by $\nu_t \triangleq -(\rho_t + \theta_t)$ for all t in $\mathbb{T}$, we have $d\tilde{S}_t = \tilde{S}_t\left[-\sigma_t\nu_t dt + \sigma_t dW_t^{P^\rho}\right]$ and $\tilde{S}$ is a P^ρ-martingale for $(F_t)_{t\in\mathbb{T}}$ if and only if $\sigma_t\nu_t = 0$ for all t. This ends the proof of the lemma.□

Proof of lemma 3.3 Analogous to lemma 3.2 in the primitive market for $d = 1$. As a matter of fact, replacing σ with $\bar{\sigma}$ and θ with $\bar{\theta}$, if we let

$$K^{\bar{\sigma}} \triangleq \left\{\nu \in L_d^2(\mathbb{T}) \text{ such that } \bar{\sigma}_t \nu_t = 0 \text{ for all } t \text{ and } -\left(\bar{\theta} + \nu\right) \in M_d^2(\mathbb{T})\right\},$$

then $K^{\bar{\sigma}} = \{0\}$, because $\bar{\sigma}$ admits an inverse. So there exists a unique equivalent martingale measure which is in the form given above.□

Proof of proposition 4.4 See Karatzas [1989]: the proof uses a representation result and is analogous to the proof of the completeness of the full market, given in theorem 4.8 below.□

Proof of theorem 4.7 We first state a representation result, which is an easy corollary of the fundamental martingales representation theorem:

- *Lemma* *Let $Y = \{Y_t; t \in \mathbb{T}\}$ be a $\bar{P}-$martingale for $(F_t)_{t\in\mathbb{T}}$. Then there exists a d-dimensional process Φ in $L_d^2(\mathbb{T})$ such that*

$$Y_t = Y_0 + \int_0^t (\Phi_s)^* \, dW_s^{\bar{P}} \qquad 0 \leq t \leq \mathbb{T}.$$

Proof of the lemma Apply the martingales representation theorem (see Karatzas and Shreve [1988]) to the process $\bar{M}_t Y_t$, which is a continuous P-martingale for $(F_t)_{t\in\mathbb{T}}$, where $(F_t)_{t\in\mathbb{T}}$ is the P-augmentation of the filtration generated by W and the lemma is obtained through the use of Itô's lemma (see lemma 8.4 in Karatzas-Lehoczy-Shreve [1990] for a detailed proof).□

We now prove the theorem

2. For each contingent claim B, we consider the process X given for all t in $\mathbb{T}$ by $X_t = \frac{1}{\beta_t} E^{\bar{P}}[\beta_T B \mid F_t]$. The process $\beta X = \{\beta_t X_t; t \in \mathbb{T}\}$ is in a trivial way a $\bar{P}-$martingale for $(F_t)_{t\in\mathbb{T}}$. Using the lemma, we can write βX in the form

$$\beta_t X_t = E^{\bar{P}}[\beta_T B] + \int_0^t (\Phi_s)^* \, dW_s^{\bar{P}}, \quad 0 \leq t \leq T$$

for some d-dimensional process Φ in $L_d^2(\mathbb{T})$. Defining the process π by

$$\pi_t = (1/\beta_t)\left(\bar{\sigma}_t^{-1}\right)^* \Phi_t, \quad 0 \leq t \leq T$$

we get that π is a portfolio process and that

$$\beta_t X_t = E^{\bar{P}}[\beta_T B] + \int_0^t \beta_s (\pi_s)^* \, \bar{\sigma}_s dW_s^{\bar{P}},$$

which shows that X is the wealth process corresponding to the trading strategy $(\pi, 0)$ with initial value $E^{\bar{P}}[\beta_T B]$, i.e., $X = X^{E^{\bar{P}}[\beta_T B];\pi,0}$ -see equation (4.2). The terminal value satisfies $X_T = B$ and as X is nonnegative, the trading strategy $(\pi, 0)$ is admissible.

1. The proof is analogous to the proof of 2. and can be found for instance in Musiela-Rutkowski [1997], p.250. □

Proof of proposition 4.8 See Karatzas [1989].

Proof of proposition 4.9 We consider the following quantities

$$x_1 \triangleq E^{\bar{P}}\left[\int_0^T \beta_t c_t dt\right] \text{ and } x_2 \triangleq x - x_1 = E^{\bar{P}}\left[\beta_T X\right],$$

for which it is easy to see that we have $c \in D(x_1)$ and $X \in M(x_2)$.
As c belongs to $D(x_1)$, according to proposition 4.4, there exists a unique trading strategy π_1 such that (π_1, c) is in $A(x_1)$ and the corresponding wealth process satisfies $X_T^{x_1;\pi_1,c} = 0$.
As X belongs to $M(x_2)$, according to proposition 4.9, there exists a unique pair (π_2, c_2) in $A(x_2)$ such that $X_T^{x_2;\pi_2,c_2} = X$ and it satisfies $c_2 \equiv 0$.
We then consider the strategy π given by $\pi \triangleq \pi_1 + \pi_2$ and it is easy to check that (π, c) belongs to $A(x)$ and that

$$X_T^{x;\pi,c} = X_T^{x_1;\pi_1,c} + X_T^{x_2;\pi_2,c_2} = X$$

which completes the proof.
We can sketch the proof of a direct approach, that leads to the same result using the martingales representation theorem: consider the martingale process

$$\left\{M_t \triangleq E^{\bar{P}}\left[\int_0^T \beta_s c_s ds + \beta_T X \mid F_t\right]; t \in \mathbb{T}\right\}.$$

Using the martingale representation theorem, M_t can be written in the form

$$M_t = E^{\bar{P}}\left[\int_0^T \beta_s c_s ds + \beta_T X\right] + \int_0^t \beta_s \pi_s^* \bar{\sigma}_s dW_s^{\bar{P}}$$

for some portfolio process π. Then, according to equation (4.2), $X_t \triangleq M_t - \int_0^t \beta_s c_s ds = \beta_t X_t^{x;\pi,c}$ and $X_T^{x;\pi,c} = X$.□

Proof of corollary 4.10 Immediate using the proof of the preceding theorem and considering $x \triangleq E^{\bar{P}}\left[\int_0^T \beta_s c_s ds + \beta_T X\right]$.□

Proof of proposition 5.1 By the Saddle Point Theorem (see Duffie [1994], p.231) and the strict monotonicity of U, $(c^*, X^*) \in A$ solves our problem if and only if there is a Lagrange multiplier $\gamma > 0$ such that (c^*, X^*) solves the unconstrained problem

$$\sup_{(c,X)\in A} U(c, X) - \gamma E^{\bar{P}}\left[\int_0^T \beta_t c_t dt + \beta_T X - x\right]$$

with the complementary slackness condition

$$E^{\bar{P}}\left[\int_0^T \beta_t c_t dt + \beta_T X\right] = x.$$

Then, by Fubini's theorem, the fact that c is an adapted process, the law of iterated expectations and the fact that the process $\left\{\bar{M}_t; t \in \mathbb{T}\right\}$ is a martingale, we get that

$$\begin{aligned}
E^{\bar{P}}\left[\int_0^T \beta_t c_t dt + \beta_T X\right] &= E\left[M_T\left(\int_0^T \beta_t c_t dt + \beta_T X\right)\right] \\
&= E\left[\int_0^T M_T \beta_t c_t dt + M_T \beta_T X\right] \\
&= E\left[\int_0^T E_t\left[M_T\right] \beta_t c_t dt + M_T \beta_T X\right] \\
&= E\left[\int_0^T M_t \beta_t c_t dt + M_T \beta_T X\right]
\end{aligned}$$

which completes the proof.□

Proof of proposition 5.2 As conditions (5.2) and (5.3) are equivalent to

$$c_t^* = I_u\left(t, \gamma^* \beta_t \bar{M}_t\right) \;\; \forall t \in \mathbb{T} \text{ and } X^* = I_V\left(\gamma^* \beta_T \bar{M}_T\right), \tag{7.5}$$

we only need to check that the solution of the optimization problem (5.1) is given by (c^*, X^*) like in (7.5). We easily get from elementary calculus that for all t in $\mathbb{T}$,

$$\min_{c \geq 0}\left[cy - u(t,c)\right] = yI_u(t,y) - u\left(t, I_u(t,y)\right) \qquad \text{for all } y \text{ in } (0,\infty)$$

and that for all y in $(0,\infty)$, the minimum is uniquely attained at $I_u(t,y)$, so that

$$u\left(t, I_u(t,y)\right) \geq u(t,c) + y\left[I_u(t,y) - c\right] \qquad \text{for all } c \geq 0 \text{ and all } y \text{ in } (0,\infty),$$

the inequality being strict for $c \neq I_u(t,y)$. Then

$$u\left(t, c_t^*\right) \geq u\left(t, c_t\right) + \gamma^* \beta_t \bar{M}_t\left[I_u\left(t, \gamma^* \beta_t \bar{M}_t\right) - c_t\right] \text{ and}$$

$$E\left[\int_0^T u\left(t, c_t^*\right) - \gamma^* \beta_t \bar{M}_t c_t^* dt\right] \geq E\left[\int_0^T u\left(t, c_t\right) - \gamma^* \beta_t \bar{M}_t c_t dt\right]$$

the inequality being strict for $c \neq c^*$, which proves our proposition.□

Proof of lemma 6.2 1. As any contingent claim B belongs to $M\left(E^{P^0}\left[\beta_T B\right]\right)$, proposition 4.9 tells us that in the case $d = 1$, all contingent claims are redundant: for any contingent claim B, there exists a trading strategy π such that $(\pi, 0)$ is in $A\left(E^{P^0}\left[\beta_T B\right]\right)$, $\beta X^{E^{P^0}[\beta_T B];\pi,0}$ is a martingale and $X_T^{E^{P^0}[\beta_T B];\pi,0} = B$. Then, by absence of arbitrage opportunity, the price for B at time 0 is necessarily equal to $E^{P^0}[\beta_T B]$.

2. As $B = X_T^{x;\pi,0}$ for some x in $(0,\infty)$ and some pair $(\pi, 0)$ in $A(x)$, by absence of arbitrage opportunity, the contingent claim B must have a price equal to x, so we only need to compute the value of x. As the corresponding discounted wealth process $\beta X^{x;\pi,0}$ is a P^0-martingale, we have

$$x = X_0^{x;\pi,0} = E^{P^0}\left[\beta_T X_T^{x;\pi,0}\right] = E^{P^0}[\beta_T B],$$

which is the result announced.□

Proof of lemma 7.2 Our approach is similar to the one adopted in Karatzas-Lehoczky-Shreve [1990]. We know that a necessary and sufficient condition for an equilibrium to be reached in our set-up is that there exists a positive real number γ^* such that

$$\begin{aligned} \beta_t \bar{M}_t &= \gamma^* u_c(t, D_t) \qquad a.s.\ P \quad 0 \le t \le T \\ \beta_T \bar{M}_T &= \gamma^* V'(S_T). \end{aligned} \tag{7.6}$$

We can take without loss of generality $u_c(0, D_0) = 1$ so that $\gamma^* = 1$. We have $d(T, S_T) \triangleq I_u[T, V'(S_T)]$, so that we only need to prove the first equality for $D_t = d(t, S_t)$.

As the process $\{\beta_t \bar{M}_t, t \in \mathbb{T}\}$ is the unique solution of the stochastic differential equation

$$dX_t = X_t\left[-r_t dt - (\bar{\theta}_t)^* dW_t\right]$$

satisfying $X_0 = 1$, there is equilibrium if

$$df(t, S_t) = f(t, S_t)\left[-r_t dt - (\bar{\theta}_t)^* dW_t\right]$$

where $f(t,x) \triangleq u_c(t, d(t,x))$. With the regularity assumptions made on the utility functions as well as on the function d, the function f is of class $C^{1,2}$ and we can apply Itô's lemma:

$$\begin{aligned} df(t, S_t) &= \left[f_t(t, S_t) + f_x(t, S_t)(b_t - \delta_t) S_t + \frac{1}{2} f_{xx}(t, S_t) \parallel \sigma_t \parallel^2 S_t^2\right] dt \\ &\quad + f_x(t, S_t) S_t \sigma_t dW_t \end{aligned}$$

which we can write in the form

$$df(t, S_t) = f(t, S_t)[-A_t dt - B_t dW_t]$$

with

$$\begin{aligned} A_t &\triangleq -\frac{1}{f(t, S_t)}\left[f_t(t, S_t) + f_x(t, S_t)(b_t - \delta_t) S_t + \frac{1}{2} f_{xx}(t, S_t) \parallel \sigma_t \parallel^2 S_t^2\right] \\ B_t &\triangleq -\frac{1}{f(t, S_t)}[f_x(t, S_t) S_t \sigma_t]. \end{aligned}$$

So, if we can let the interest rate be given by

$$r_t = A_t \qquad \text{for all } t \text{ in } \mathbb{T} \tag{7.7}$$

and the coefficients of the assets price processes be such that

$$\left(\bar{\theta}_t\right)^* = B_t \qquad \text{for all } t \text{ in } \mathbb{T} \tag{7.8}$$

then $f(t, S_t) = u_c(t, D_t) = \beta_t \bar{M}_t$ and equilibrium is reached. We shall now try to make these conditions clearer.

Conditions (7.7) and (7.8) are satisfied if

$$\left\{ \begin{array}{c} r_t = A_t \\ \bar{\nu}_t \triangleq \bar{\theta}_t - \theta_t = 0 \\ (\theta_t)^* = B_t \end{array} \right.$$

Notice that we find the equivalent martingale measure $\bar{P}$ to be equal to the martingale measure P^0, which is coherent with what we have seen in section 6.3.
These conditions are satisfied if the coefficients of the price processes are such that

$$\left\{ \begin{array}{c} b_t \left[1 + \frac{f_x(t,S_t)}{f(t,S_t)} S_t\right] = \beta_t \\ b_t - r_t = -\frac{f_x(t,S_t)}{f(t,S_t)} S_t \left\|\sigma_t\right\|^2 \\ a_t = r_t 1_{(d-1)} - \frac{f_x(t,S_t)}{f(t,S_t)} S_t \mu_t \sigma_t^* \end{array} \right.$$

where a_t denotes like in section 2.2 the purely financial assets appreciation rates and $\beta_t \triangleq -\frac{f_t(t,S_t)}{f(t,S_t)} + \frac{f_x(t,S_t)}{f(t,S_t)} d(t, S_t) - \frac{f_x(t,S_t)}{f(t,S_t)} \left[\left\|\sigma_t\right\|^2 S_t\right] - \frac{1}{2} \frac{f_{xx}(t,S_t)}{f(t,S_t)} \left\|\sigma_t\right\|^2 S_t^2$. □

Proof of corollary 7.6 Immediate using the lemma and noticing that in this case $f(t, x) = V'(x)$.□

Proof of proposition 7.7 We must have $s(T, x) = I_V[u_c(T, x)]$. Adopting the same approach as in the proof of lemma 7.2, it is easy to obtain that there is equilibrium if the coefficients satisfy

$$r_t = \bar{A}_t \triangleq -\frac{1}{u_c(t, D_t)} \left[u_{ct}(t, D_t) + u_{cc}(t, D_t) b_t^D + \frac{1}{2} u_{ccc}(t, D_t) \parallel \sigma_t^D \parallel^2\right]$$

$$\frac{b_t - r_t}{\sigma_t \sigma_t^*} \sigma_t = -\frac{u_{cc}(t, D_t)}{u_c(t, D_t)} \sigma_t^D$$

$$a_t = r_t 1_{(d-1)} - \frac{u_{cc}(t, D_t)}{u_c(t, D_t)} \mu_t \left(\sigma_t^D\right)^*$$

We can let r (and the corresponding price process S^0) be given by the first equation; the coefficient a and μ can be chosen as to satisfy the third equation; we shall prove that there exist increasing functions $s(t, \cdot)$ such that $S_t = s(t, D_t)$ and such that the second equation is satisfied.

As $S_t = s(t, D_t)$ for all t, using Itô's lemma, we must have

$$\begin{aligned} dS_t &= \left[s_t(t, D_t) + s_x(t, D_t) b_t^D + \frac{1}{2} s_{xx}(t, D_t) \left\| \sigma_t^D \right\|^2 \right] dt + s_x(t, D_t) \sigma_t^D dW_t \\ &= S_t \left[(b_t - \delta_t) dt + \sigma_t dW_t \right]. \end{aligned}$$

We are then reduced to finding an increasing solution to the stochastic differential equation

$$\begin{aligned} & s_t(t, D_t) + s_x(t, D_t) b_t^D + \frac{1}{2} s_{xx}(t, D_t) \left\| \sigma_t^D \right\|^2 + D_t - \bar{A}_t s(t, D_t) \\ = & -\frac{u_{cc}(t, D_t)}{u_c(t, D_t)} s_x(t, D_t) \left\| \sigma_t^D \right\|^2, \end{aligned} \tag{7.9}$$

with terminal condition $s(T, x) = I_V[u_c(T, x)]$.
This is in turn equivalent to

$$(su_c)_t(t, D_t) + b^D (su_c)_x(t, D_t) + \frac{1}{2} \left\| \sigma_t^D \right\|^2 (su_c)_{xx}(t, D_t) = -D_t u_c(t, D_t) \tag{7.10}$$

with terminal condition $(su_c)(T, D_T) = I_V[u_c(T, D_T)] u_c(T, D_T)$.
Writing Y for su_c, and using the specific form of the considered dividend process, we want to solve the partial differential equation

$$xu_c(t, x) + Y_t(t, x) + b_0 x Y_x(t, x) + \frac{1}{2} \|\sigma_0\|^2 x^2 Y_{xx}(t, x) = 0 \tag{7.11}$$

with terminal condition $Y(T, x) = H(x) \triangleq I_V[u_c(T, x)] u_c(T, x)$.
Let $Z(t, x) \triangleq Y[T - t, \exp(\alpha x + \beta t)]$ with $\alpha = \frac{1}{\sqrt{2}} \|\sigma_0\|$ and $\beta = \frac{1}{2} \|\sigma_0\|^2 - b_0$. Then, equation (7.11) can be written in the following form

$$-Z_t(t, x) = Z_{xx}(t, x) + F(t, x)$$

with initial condition $Z(0, x) = H\left(\exp \frac{1}{\sqrt{2}} \|\sigma_0\| x\right) \triangleq \tilde{H}(x)$ where

$$F(t, x) \triangleq \exp(\alpha x + \beta t) u_c(T - t, \exp(\alpha x + \beta t)).$$

The solution of this partial differential equation (see Cannon [1984]) is given by

$$\begin{aligned} Z(t, x) &= \int_{-\infty}^{+\infty} K(t, x - y) \tilde{H}(y) dy + \\ & \int_0^t \int_{-\infty}^{+\infty} K(t - \tau, x - y) F(\tau, y) dy d\tau \end{aligned} \tag{7.12}$$

where

$$K(x, t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right).$$

We now need to check that the solution Z found can be such that the associated functions $s(t,\cdot)$ are increasing in x. As $Y \triangleq s u_c$, we want $\frac{Y}{u_c}$ to be increasing in x or $Y_x u_c - Y u_{cc} \geq 0$. We know that Y is positive and that u_{cc} is negative so that we only need to prove that $Y_x u_c \geq 0$ or $Y_x \geq 0$ or equivalently $Z_x \geq 0$. According to (7.12), the solution Z obtained satisfies

$$\begin{aligned} Z_x(t,x) &= \int_{-\infty}^{+\infty} K_x(t,x-y)\,\tilde{H}(y)\,dy + \\ &\quad \int_0^t \int_{-\infty}^{+\infty} K_x(t-\tau,x-y)\,f(\tau,y)\,dy\,d\tau \\ &= K(t,\cdot) * \tilde{H}'(x) + \int_0^t K(t-\tau,\cdot) * f_x(\tau,\cdot)(x)\,d\tau, \end{aligned}$$

so that if $\tilde{H}'(x) \geq 0$ and $F_x(t,x) \geq 0$, then $Z_x(t,x) \geq 0$. Now, $\tilde{H}'(x) \geq 0$ if and only if $H'(x) \geq 0$ or equivalently if $I_V[u_c(T,x)]\,u_c(T,x)$ is increasing in x. Besides, the relation $F_x \geq 0$ holds if

$$\alpha \exp(\alpha x + \beta t)\, u_c(T-t, \exp(\alpha x + \beta t)) +$$

$$[\exp(\alpha x + \beta t)]^2\, u_{cc}(T-t, \exp(\alpha x + \beta t)) \geq 0$$

or if $u_c + x u_{cc} \geq 0$ or if $\tilde{u} : x \mapsto x u_c$ is increasing. Taking $V : x \mapsto u_c(T,x)$, we obtain that if for all t, the function $x u_c(t,x)$ is increasing, then the function s that we have found is increasing in x, which completes the proof of the proposition.□

Proof of lemma 7.8 Using the proof of the preceding proposition, and especially equation (7.9), we know that the full market is in equilibrium for a dividend process D in the form $dD_t = b_t^D dt + \sigma_t^D dW_t$ and a price system $\left[S^0, S, \left(C^i\right)_{i=1,\ldots,d-1}\right]$ with $S_t = s(t,D_t) = I_V[u_c(t,D_t)]$ if and only if, with the same notations, the equality

$$\begin{aligned} & s_t(t,D_t) + s_x(t,D_t)\,b_t^D + \frac{1}{2} s_{xx}(t,D_t)\left\|\sigma_t^D\right\|^2 + D_t - \bar{A}_t s(t,D_t) \\ = & -\frac{u_{cc}(t,D_t)}{u_c(t,D_t)} s_x(t,D_t)\left\|\sigma_t^D\right\|^2 \end{aligned}$$

holds for all t. Replacing $s(t,x)$ with its value $I_V[u_c(t,x)]$ gives us the relation wanted.□

Proof of the characterization of equilibrium with representative agent's theory We have to check that in this case, the budget constraint is automatically binding: as a matter of fact, as $d\tilde{S}_t = \tilde{S}_t \sigma_t dW_t^{\bar{P}}$,we have, using Itô's lemma and the fact that the process $\tilde{S}$ is a martingale under $\bar{P}$

$$\begin{aligned} d\left[\tilde{S}_t \exp\left\{-\int_0^t \delta_s ds\right\}\right] &= -\delta_t \tilde{S}_t \exp\left\{-\int_0^t \delta_s ds\right\} dt + \\ &\quad \exp\left\{-\int_0^t \delta_s ds\right\} \tilde{S}_t \sigma_t dW_t^{\bar{P}}. \end{aligned}$$

As δ and σ are uniformly bounded and $E\left[\sup_{t\in\mathbb{T}} {S_t}^2\right] < \infty$, we get

$$E\left[\int_0^t \| \exp\left\{-\int_0^t \delta_s ds\right\} \tilde{S}_t\sigma_t \|^2\right] < \infty$$

so $E^{\bar{P}}\left[\int_0^T \exp\left\{-\int_0^t \delta_s ds\right\} \tilde{S}_t\sigma_t dW_t^{\bar{P}}\right] = 0$ and

$$\begin{aligned} E^{\bar{P}}\left[\tilde{S}_T \exp\left\{-\int_0^T \delta_s ds\right\}\right] &= E^{\bar{P}}\left[\tilde{S}_0\right] - E^{\bar{P}}\left[\int_0^T \delta_t\tilde{S}_t \exp\left\{-\int_0^t \delta_s ds\right\} dt\right] \\ &= S_0 - E^{\bar{P}}\left[\int_0^T \delta_t\tilde{S}_t \exp\left\{-\int_0^t \delta_s ds\right\} dt\right] \quad (7.13) \end{aligned}$$

Then, using Fubini's theorem and the fact that the processes D and β are $(F_t)_{t\in\mathbb{T}}$-adapted

$$\begin{aligned} E\left[\int_0^T \bar{M}_t\beta_t D_t dt\right] &= E\left[\int_0^T E_t\left[\bar{M}_T\beta_t D_t\right] dt\right] \\ &= E\left[\int_0^T \bar{M}_T\beta_t D_t dt\right] \\ &= E^{\bar{P}}\left[\int_0^T \beta_t\delta_t S_t dt\right] \end{aligned}$$

so that

$$\begin{aligned} & E\left[\int_0^T \bar{M}_t\beta_t D_t dt + \bar{M}_T\beta_T S_T\right] \\ = \; & E^{\bar{P}}\left[\int_0^T \delta_t\tilde{S}_t \exp\left\{-\int_0^t \delta_s ds\right\} dt\right] + E^{\bar{P}}\left[\tilde{S}_T \exp\left\{-\int_0^T \delta_s ds\right\}\right] \end{aligned}$$

which, according to equation (7.13), is equal to S_0 and this completes the proof because the initial endowment of the representative agent is equal to S_0.□

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