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The Optimal Dynamic Investment Policy for a Fund Manager Compensated with an Incentive Fee

Abstract

This paper solves the investment problem of a risk averse fund manager compensated with an incentive fee, a call option on the assets he controls. The optimal policy leads to all-or-nothing outcomes: the manager ends up either deep in or deep out of the money. The optimal trading strategy involves dynamically adjusting asset volatility as asset value changes. As assets grow large, the manager moderates portfolio risk. For example, if the manager has constant relative risk aversion, volatility converges to the Merton constant. On the other hand, as asset value goes to zero, portfolio volatility goes to infinity.

1 Introduction

Fund managers and other intermediaries paid with convex compensation schemes play an important role in financial markets. This paper provides a rigorous description of the effect of option-like compensation on these managers' investment policies. I present the optimal dynamic investment policy for a risk averse fund manager who receives an incentive fee that he cannot hedge in his personal account. This result represents the first step in determining equilibrium asset prices in an economy where option-compensated managers determine trading strategies for other investors. The solution technique, concavifying the objective function, applies to other problems in which option payoffs appear in the objective function.

An incentive fee is a share, for example, 15%, in the positive part of the returns on the client's portfolio net of some benchmark. Such a fee structure is typical for hedge fund and pension fund managers. Grinblatt and Titman (1989) study the fund manager's investment problem under the assumption that the manager can hedge the fee in his personal portfolio, so that his objective is to maximize the fee's market value. With this objective, the manager wants to maximize volatility and the problem has no solution. By contrast, this paper assumes that the manager cannot hedge the fee in his private account because shorting securities that he purchases on his client's behalf is a breach of fiduciary duty. This means the manager's objective is to maximize his expected utility of the incentive fee.

The paper begins with the investment problem of manager who earns an incentive fee only once. I cast the problem in a standard continuous-time financial market and use martingale methods to find the unique optimal investment policy for a manager with general concave utility and a general benchmark portfolio.[1] Under the optimal policy, the fund has an all-or-nothing payoff, either in the money or zero. The policy is

[1]Starks (1987) studies the portfolio manager's problem in a mean-variance framework and concludes that an asymmetric incentive fee will induce the manager to choose a higher beta than he would choose with a symmetric fee.

also a long-shot in the sense that the probability of bankruptcy is high, but the payoff, if in the money, is in the money by some strictly positive amount.

For the cases of constant relative and absolute risk averse utility functions with either the riskless asset or the market portfolio as benchmarks, I provide closed-form expressions for the optimal trading strategy. Rather than maximizing portfolio risk, the manager dynamically adjusts volatility in response to changes in the asset value over time. As the manager accumulates profits, so that he begins gambling with his own money, he moderates portfolio risk. For example, if the manager has constant relative risk averse utility and the benchmark is riskless, volatility converges to the Merton (1969, 1971) constant as fund value grows large. On the other hand, as bankruptcy approaches, portfolio volatility approaches infinity.

Consistent with these dynamics, Brown, Harlow, and Starks (1996) find evidence in the mutual fund industry that managers with relatively poor performance in the first half of their performance evaluation period increase fund volatility in the second half of the period more than managers who have done well. While, in a given year, mutual fund managers typically earn a fixed proportion of initial asset value, Sirri and Tufano (1992) show that new money tends to flow into winning funds faster than old money flows out of losers, making mutual fund managers' long run compensation convex in fund performance even though there is no explicit incentive fee. Similarly, Chevalier and Ellison (1995) estimate a nonlinear relationship between one year's performance and the next year's flow of new money for a large set of mutual funds and find that, for young funds, the function is relatively flat for moderately poor performance and then increasing for better performance. They conclude that this provides incentives for funds with moderately poor performance to gamble to recover losses. Then they study the relationship between performance from January to September and changes in portfolio riskiness from September to December and find that funds that are somewhat behind do tend to increase risk.

Finally, the paper solves the multi-period investment problem of a manager with constant relative risk aversion who earns a new incentive fee every year until retirement. In any given year prior to the last, the optimal investment policy limits the downside risk, and thus forgoes some upside potential, in order to protect the value of future incentive fees. Nevertheless, the policy is one of extreme outcomes: the manager's option finishes either deep in the money, or deep out of the money.

The paper proceeds as follows. Section 2 presents the single-period model. Section 2.1 describes the manager's preferences and opportunity set, section 2.2 uses martingale methods to transform the manager's dynamic trading problem strategy into a static problem of choosing an optimal random terminal portfolio value, section 2.3 solves the transformed problem, and sections 2.4 and 2.5 give examples of the optimal trading strategy. Section 2.6 explores implications of the results for contract theory. Section 3 presents the multi-period model, and section 4 concludes.

2 One-Period Model

At time zero, the client hires the manager for a fixed length of time T, and agrees to pay him an incentive fee. The manager's total terminal wealth, Y, is equal to his incentive fee plus a constant, K, that includes any fixed fees and personal wealth. Letting X_t represent fund value and B_t represent the value of a benchmark portfolio at time t,

$$Y = \alpha(X_T - B_T)^+ + K \ , \tag{1}$$

where $0 < \alpha < 1$. The manager chooses an investment policy to maximize his expected utility of terminal wealth.

2.1 Assumptions

The manager's utility function U is strictly increasing, strictly concave, at least twice continuously differentiable, and defined on a domain containing $(0, \infty)$. U'' is nondecreasing and $U'(W)$ approaches zero as W approaches infinity. Consequently, the function $I = U'^{-1}$ is a well-defined, strictly decreasing, convex, continuously differentiable function from $(0, \infty)$ onto a range containing $(0, \infty)$. For example, both the constant absolute and relative risk averse classes of utility functions satisfy these hypotheses.

The financial market consists of a riskless asset with interest rate r, and n risky assets. The risky asset prices, $P_i, i = 1, \ldots, n$ are diffusion processes governed by the equations

$$\frac{dP_{i,t}}{P_{i,t}} = (r + \mu_i)\, dt + \sigma_i'\, dWt \ ,$$

where $\mu_i \in \mathcal{R}$ and $\sigma_i \in \mathcal{R}^n$ are constants and W is standard n-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathcal{R}^n$, let σ be the matrix whose ith row is σ_i', and assume that σ is nondegenerate. Let $\{\mathcal{F}_t\}$ denote the $\mathcal{P}$-augmentation of the filtration generated by the Brownian motion; $\mathcal{F}_t$ represents the manager's information at time t.

A trading strategy for the manager is an n-dimensional process $\{\pi_t : 0 \leq t \leq T\}$ whose ith component, $\pi_{i,t}$, is the value of the holdings of risky asset i in the portfolio at time t. An admissible trading strategy, π, must be progressively measurable with respect to $\{\mathcal{F}_t\}$, must prevent fund value from falling below zero, and must satisfy $\int_0^T ||\pi_t||^2\, dt < \infty$, a.s. Under an admissible trading strategy π, portfolio value evolves according to

$$dX_t = (rX_t + \pi_t'\mu)\, dt + \pi_t'\sigma\, dW_t \ . \tag{2}$$

The benchmark portfolio value, B_t, is a geometric Brownian motion that can be replicated with a self-financing trading strategy involving the market securities:

$$\frac{dB_t}{B_t} = (r + \pi_B'\mu)\, dt + \pi_B'\sigma\, dWt \ , \tag{3}$$

where π_B is a constant known to the manager. For example, the most typical benchmark is cash: $\pi_B = 0$. Another benchmark of interest is the market portfolio, represented here as the mean-variance efficient portfolio: $\pi_B = (\sigma\sigma')^{-1}\mu$.

2.2 The Manager's Investment Problem

The manager's dynamic problem is to choose an admissible trading strategy for the fund to maximize his expected utility of terminal wealth:

$$\begin{aligned} \max_{\pi} \quad & EU(\alpha(X_T - B_T)^+ + K) \\ \text{subject to} \quad & dX_t = (rX_t + \pi_t'\mu)\,dt + \pi_t'\sigma\,dW_t \\ \text{and} \quad & X_t \geq 0 \;\forall t \in [0,T]\,. \end{aligned} \tag{4}$$

Using martingale methods, I recast (4) as a static problem of choosing an optimal terminal fund value:[2]

$$\begin{aligned} \max_{X_T} \quad & EU(\alpha(X_T - B_T)^+ + K) \\ \text{subject to} \quad & E\zeta_T X_T \leq X_0 \\ \text{and} \quad & X_T \geq 0\,. \end{aligned} \tag{5}$$

where ζ_t is the "pricing kernal" or "state price density" defined by $\zeta_t \equiv e^{-rt-\theta' W_t - ||\theta||^2 t/2}$ and $\theta \equiv \sigma^{-1}\mu$.

The Market Value of the Incentive Fee

We know from first principles that the convexity of the incentive fee makes risky strategies relatively more attractive to the manager. This section develops an expression for the market value of the incentive fee that quantifies this effect simply and illuminates the key difference between the manager's problem and the standard investment problem. The market value expression also has implications for the efficiency of the contract explored in section 2.6.

Observe that under an optimal policy, $X_T \in \{0\} \cup (B_T, \infty)$, a.s.: whenever X_T takes on values in $(0,\ B_T]$, it uses resources without adding to utility, so an optimal

[2] See, for example, Harrison and Kreps (1979), Harrison and Pliska (1981), Pliska (1986), Karatzas, Lehoczky, and Shreve (1987), and Cox and Huang (1989) for the development of these methods and their application to optimal portfolio choice. See also the review article Karatzas (1989).

choice cannot do so with positive probability. In addition, an optimal policy satisfies the budget constraint in problem 5 with equality. These necessary conditions have implications for the market value of the incentive fee. By the market value of the incentive fee $\alpha(X_T - B_T)$, I mean the initial value of a self-financing trading strategy that replicates it. It is well known that this value can be expressed as $\tilde{E}\{e^{-rT}\alpha(X_T - B_T)^+\}$, where $\tilde{\mathcal{P}}$ is the usual equivalent martingale measure defined by $\frac{d\tilde{\mathcal{P}}}{d\mathcal{P}} = e^{rT}\zeta_T$. The proposition below simplifies this expression.

The proposition makes use of another equivalent probability measure, $\mathcal{P}^B$, defined by $\frac{d\tilde{\mathcal{P}}}{d\mathcal{P}} = \frac{B_T}{B_0}\zeta_T$. Whereas, under the usual martingale measure $\tilde{\mathcal{P}}$, asset prices are martingales when measured with a riskless money market account as numeraire, under the alternative measure $\mathcal{P}^B$, asset prices are martingales when measured with the benchmark portfolio as numeraire. For example, if the benchmark portfolio is the reciprocal of the pricing kernal, a representation of the market portfolio, then $\mathcal{P}^B = \mathcal{P}$, the true probability measure.

Proposition 1 *If $X_T \in \{0\} \cup (B_T, \infty)$, a.s., $\tilde{E}\{e^{-rT}X_T\} = X_0$, and $\tilde{E}\{e^{-rT}B_T = B_0\}$, then*

$$\tilde{E}\{e^{-rT}\alpha(X_T - B_T)^+\} = \alpha(X_0 - B_0 + B_0\mathcal{P}^B\{X_T = 0\}) \ . \tag{6}$$

Proof

$$\tilde{E}\{e^{-rT}\alpha(X_T - B_T)^+\} = \tilde{E}\{e^{-rT}\alpha(X_T - B_T)1_{\{X_T > B_T\}}\} \ , \tag{7}$$

where $1_A = 1$ whenever A occurs and $1_A = 0$ otherwise.

$$\tilde{E}\{e^{-rT}X_T 1_{\{X_T > B_T\}}\} = \tilde{E}\{e^{-rT}X_T\} = X_0 \ , \tag{8}$$

$$\tilde{E}\{e^{-rT}B_T 1_{\{X_T > B_T\}}\} = B_0\mathcal{P}^B\{X_T > B_T\} \ , \tag{9}$$

and the result follows.

In addition, when the assets have the all-or-nothing terminal distribution described above, the manager's terminal wealth Y from equation 1 is invertible for the terminal fund value X_T: $Y = K \Rightarrow X_T = 0$ and $Y > K \Rightarrow X_T = B_T + (Y - K)/\alpha$. Thus we can think of Y as the manager's choice variable and rewrite problem 5 as

$$\begin{aligned} \max_{Y} \quad & EU(Y) \\ \text{subject to} \quad & E\zeta_T Y \leq \alpha(X_0 - B_0 + B_0 \mathcal{P}^B\{Y = K\}) + K \\ \text{and} \quad & Y \geq K \; . \end{aligned} \tag{10}$$

So the manager's problem is like the standard terminal wealth problem except that his budget, $\alpha(X_0 - B_0 + B_0\mathcal{P}^B\{Y = K\}) + K$ is a function of his strategy–a "longer-shot" has more value.

2.3 The Optimal Terminal Portfolio Value

I solve problem 5 by concavifying the objective function. The concavification of a function u, if it exists, is the smallest concave function that dominates u.[3] The solution proceeds as follows. First I construct a concavification of the objective function in problem 5. Then I solve (5) with the concavified objective function using standard methods. Then I show that the policy that is optimal for the concavified objective function is also optimal for the true objective function because it never takes on values where the two functions disagree.

Define $u : \mathcal{R} \times (0, \infty) \rightarrow \mathcal{R}$ by

$$\begin{aligned} u(x, b) = \quad & U(\alpha(x - b)^+ + K) \quad && \text{for } x \geq 0 \; , \\ & -\infty && \text{otherwise.} \end{aligned} \tag{11}$$

In terms of u, the manager's problem is

$$\max_{X_T} Eu(X_T, B_T) \text{ subject to } E\zeta_T X_T \leq X_0 \; . \tag{12}$$

Because of the option-like incentive fee, the objective function u is not concave in the choice variable x. The dotted line in figure 1 plots u as a function of x. For each b,

[3]See Aumann and Perles (1965) for a formal definition.

however, $u(\cdot, b)$ has a concavification $\tilde{u}(\cdot, b)$, illustrated by the dashed line in figure 1. Roughly speaking, the function $\tilde{u}$ builds a bridge over the kink in u.

2.3.1 The Concavified Objective Function

The concavified objective function replaces part of the original function with a chord drawn between $x = 0$ and another point, $x = \hat{x} > b$, chosen to make the slope of the chord equal to the slope of u at $\hat{x}$, so that the resulting function is concave. For example, in figure 1, $b = 1$ and $\hat{x} = 1.4472$. To prove the existence of such an $\hat{x}$ in general, let $u'(x,b) = \frac{\partial u(x,b)}{\partial x}$, for $x > b$, and let

$$f(x,b) = u(x,b) - u(0,b) - xu'(x,b) \tag{13}$$

for all $b > 0$ and $x > b$.

Lemma 1 *For every b, there exists a unique $x > b$ such that $f(x,b) = 0$.*

Proof *Fix b and let $x > b$. $f(x,b) = U(\alpha(x-b)+K) - U(K) - \alpha x U'(\alpha(x-b)+K)$ is strictly increasing in x, for its derivative with respect to x is $-\alpha^2 x U''(\alpha(x-b)+K) > 0$. As $x \to b$, $f(x,b) \to -\alpha x U'(K) < 0$. As $x \to \infty$, $f(x,b)$ approaches a strictly positive limit, possibly infinity. To see this, rewrite f as*

$$f(x,b) = [U(\alpha(x-b)+K) - U(K) - \alpha(x-b)U'(\alpha(x-b)+K)] - \alpha b U'(\alpha(x-b)+K)\ .$$

The term in brackets above is strictly positive and increasing for all $x > b$, while the remaining term above approaches zero as x approaches ∞. Therefore, $f(\cdot, b)$ has a unique zero on (b, ∞).

Let $\hat{x}(b)$ be the unique $x > b$ such that $f(x,b) = 0$. Then $\tilde{u} : \mathcal{R} \times (0,\infty) \to \mathcal{R}$ defined by

$$\tilde{u}(x,b) = \begin{cases} -\infty & \text{for } x < 0 \\ u(0) + u'(\hat{x}(b), b)x & \text{for } 0 \le x \le \hat{x}(b) \\ u(x,b) & \text{for } x > \hat{x}(b) \end{cases} \tag{14}$$

is concave in x. Furthermore, $\tilde{u}(x,b) \geq u(x,b)$ for all $(x,b) \in \mathcal{R} \times (0,\infty)$ and $\tilde{u}(x,b) = u(x,b)$ for $x = 0$ and for all $x \geq \hat{x}(b)$.

Now I introduce what is essentially the derivative of $\tilde{u}$ with respect to x. The concavification $\tilde{u}$ is not differentiable at $x = 0$ but we can define a set-valued function $\tilde{u}'$ on $[0,\infty) \times (0,\infty)$ by

$$\begin{aligned} \tilde{u}'(x,b) = \quad & (\infty, u'(\hat{x}(b),b)] && \text{for } x = 0 \\ & \{u'(\hat{x}(b),b)\} && \text{for } 0 < x \leq \hat{x}(b) \\ & \{u'(x,b)\} && \text{for } x > \hat{x}(b) \; . \end{aligned} \tag{15}$$

Then, for every $x' \in \mathcal{R}$ and every $m \in \tilde{u}'(x,b)$, $\tilde{u}(x',b) - \tilde{u}(x,b) \leq m(x'-x)$. Strict inequality holds whenever $x > \hat{x}(b)$ and $x' \neq x$.[4]

2.3.2 The Optimal Policy

Karatzas, Lehoczky and Shreve (1987) and Cox and Huang (1989) show that in the standard investment problem, with no incentive fee, optimal random terminal wealth sets marginal utility equal to a multiple of the pricing kernal, where the multiple is determined by the budget constraint. The candidate policy we construct next does essentially the same for the concavified objective function. It turns out that this candidate is optimal for the true objective function.

To construct this candidate, I define an inverse function for $\tilde{u}'(\cdot,b)$, $i : (0,\infty) \times (0,\infty) \to [0,\infty)$, by

$$i(y,b) = [(I(y/\alpha) - K)/\alpha + b]1_{\{y < u'(\hat{x}(b),b)\}} \; . \tag{16}$$

The function i is the inverse of $\tilde{u}$ in the sense that $y \in \tilde{u}'(i(y,b),b)$ for all $b > 0$.

Next, let $\mathcal{X}(\lambda) = E\zeta_T i(\lambda\zeta_T, B_T)$ for $\lambda > 0$. $\mathcal{X}(\lambda)$ is just the market value of the policy $i(\lambda\zeta_T, B_T)$. Assume that

$$\mathcal{X}(\lambda) < \infty \text{ for all } \lambda \; . \tag{17}$$

[4]For each b, $\tilde{u}'(\cdot,b)$ is what is formally called the *subdifferential* of $\tilde{u}(\cdot,b)$. See Rockafellar (1970, p.214-215).

This holds for both constant absolute and relative risk aversion with either the riskless asset and market portfolio as benchmarks. Then $\mathcal{X}(\lambda)$ is continuous and strictly decreasing, $\mathcal{X}(\lambda) \to \infty$ as $\lambda \to 0$, and $\mathcal{X}(\lambda) \to 0$ as $\lambda \to \infty$. Therefore, there exists a unique $\lambda^* > 0$ such that $\mathcal{X}(\lambda^*) = X_0$.

Proposition 2 *Under assumption (17), $X_T^* \equiv i(\lambda^* \zeta_T, B_T)$ is the unique optimal solution to problem (12).*

Proof *If X' is any other feasible strategy that is not almost surely equal to X_T^*, then*

$$\begin{aligned} E\{u(X', B_T) - u(X_T^*, B_T)\} &= E\{u(X', B_T) - \tilde{u}(X_T^*, B_T)\} \\ &\leq E\{\tilde{u}(X', B_T) - \tilde{u}(X_T^*, B_T)\} \\ &< E\{\lambda^* \zeta_T (X' - X_T^*)\} \\ &\leq \lambda^*(E\zeta_T X' - X_0) \leq 0 \ . \end{aligned}$$

Notice that the optimal policy is one of extremes. The incentive fee is either as far out of the money as possible, or else it is in the money by at least $\alpha(\hat{x}(B_T) - B_T) > 0$. It does not pay for the manager to be just marginally in the money, since he must expend substantial resources to bring fund value into the money at all.

2.4 Typical Benchmarks

The most typical benchmark is a constant, or, in other words, a riskless portfolio. If the manager is measured against a riskless benchmark, $B_T = B_0 e^{rT}$, the manager's optimal terminal fund value is

$$X_T^1 = [(I(\lambda_1 \zeta_T / \alpha) - K)/\alpha + B_0 e^{rT}] 1_{\{\zeta_T < z_1\}} \ , \tag{18}$$

where λ_1 solves $E\zeta_T i(\lambda \zeta_T, B_0 e^{rT}) = X_0$ and $z_1 = \alpha U'(\alpha(\hat{x}(B_0 e^{rT}) - B_0 e^{rT})/\lambda_1$. A plot of the optimal terminal wealth X_T^1 as a function of the state price density ζ_T appears in figure 2. In the figure, the critical value of ζ_T is $z_1 = 1.2373$. Optimal terminal wealth X_T^1 is greater than $\hat{x} = 1.4472$ and decreasing in ζ_T until ζ_T hits z_1. Then X_T^1 jumps from $\hat{x}$ down to zero.

Another possible benchmark is a market index such as the S&P 500. A representation for a market index in this model is the portfolio $M_t \equiv 1/\zeta_t$, because risk averse investors solving standard investment/consumption problems in this framework always divide their portfolios between M and the riskless asset. With B_0/ζ_T as the benchmark, the manager's optimal terminal fund value is a simple function of the pricing kernal, similar to that with the riskless benchmark. Let λ_2 solve $E\zeta_T i(\lambda\zeta_T, B_0/\zeta_T) = X_0$ and let $g(\zeta_T) = u'(\hat{x}(B_0/\zeta_T), B_0/\zeta_T) - \lambda_2\zeta_T$.

Proposition 3 *Under assumption 17, the optimal policy for problem (12) with benchmark $B_T \equiv B_0/\zeta_T$ is*

$$X_T^2 = [(I(\lambda_2\zeta_T/\alpha) - K)/\alpha + B_0/\zeta_T]1_{\{\zeta_T < z_2\}} \ ,$$

where z_2 is the unique zero of g.

Proof *Given proposition (1), it remains only to show that $g(\zeta_T) > 0 \iff \zeta_T < z_2$, for some constant z_2. g cannot be nonpositive everywhere, by construction of λ_2, and $g(\zeta_T) \to -\infty$ as $\zeta_T \to \infty$, so, by continuity, g must have a zero.*

$$g'(\zeta_T) = g(\zeta_T)\frac{B_0/\zeta_t^2}{\hat{x}(B_0/\zeta_T)} - \lambda_2(1 - \frac{B_0/\zeta_T}{\hat{x}(B_0/\zeta_T)}) \ ,$$

so, whenever $g \leq 0$, g is decreasing. Therefore, g has a unique zero, z_2, and $g(\zeta_T) > 0 \iff \zeta_T < z_2$.

2.5 Examples of Optimal Trading Strategies

This section takes the benchmark to be either the riskless asset or the market portfolio and derives closed-form expressions for the manager's optimal trading strategy in the cases of constant absolute and relative risk aversion. At issue is whether or not we may use Ito's lemma to obtain a stochastic differential equation for the optimal portfolio value process, X_t^*, despite the fact that X_T^* is a discontinuous function of ζ_T.

With both benchmarks, final portfolio value $X_T^* = \psi(\zeta_T)$ for some function $\psi : (0,\infty) \to \mathcal{R}$. Therefore, intermediate portfolio value, $X_t^* = E((\zeta_T/\zeta_t)X_T^*|\mathcal{F}_t)$, is equal to $x^*(t,\zeta_t)$ for some function $x^* : [0,T) \times (0,\infty) \to \mathcal{R}$, because ζ is a Markov Process. Set $x^*(T,\zeta) \equiv \psi(\zeta)$. If the function $x^* : [0,T] \times (0,\infty) \to \mathcal{R}$ were continuous on $[0,T] \times (0,\infty)$ and $C^{1,2}$ on $[0,T) \times (0,\infty)$, then we could apply Ito's lemma to get an expression for dx^* and identify the resulting diffusion coefficient with the quantity $\pi_t'\sigma$ from (2). This would yield the following equation for the optimal trading strategy:

$$\pi_t^* = \rho(t,\zeta_t) \equiv -\zeta_t x_\zeta^*(t,\zeta_t)\Sigma^{-1}\mu \ , \tag{19}$$

where $\rho : [0,T) \times (0,\infty) \to \mathcal{R}^n$, x_ζ^* is the partial derivative of x^* with respect to its second argument, and the matrix $\Sigma = \sigma\sigma'$ is the covariance matrix of instantaneous stock returns.[5]

In the case of the manager's optimal policy, $x^*(T,\cdot)$ is not continuous. Nevertheless, in the cases of constant absolute and relative risk aversion, x^* is $C^{1,2}$ on $[0,T) \times (0,\infty)$, so (2) holds, with $X \equiv X^*$ and π^* defined by (19), for all $t < T$. In addition, for all values of $\zeta \neq z_i$, $i = 1$ for the riskless benchmark and $i = 2$ for the market benchmark, $x^*(\cdot,\zeta)$ is continuous on $[0,T]$. Furthermore, $\rho(\cdot,\zeta)$ defined by (19) has a continuous extension to $[0,T]$. Letting π^* be given by this extension, $X_T^* = X_0 + \int_0^T (rX_s^* + \pi_s^{*\prime}\mu)\,ds + \int_0^T \pi_s^{*\prime}\sigma\,dW_s$ a.s. because the equality holds for all $t < T$ and both the wealth process and the integrals are almost surely path-continuous on [0, T]. Therefore, equation 19 is valid for all $t \leq T$ and permits a closed-form expression for the optimal trading strategy.

2.5.1 Constant Relative Risk Aversion

Let $U(X) = \frac{X^{1-A}}{1-A}$ where $A > 0$ and $A \neq 1$.

[5]Similar arguments appear in Harrison and Pliska (1981, Subsection 5.3), Pliska (1986, Section 5), Karatzas, Lehoczky, and Shreve (1987, Section 7), and Karatzas (1989, Example 5.6).

The Riskless Portfolio as Benchmark

For this section, set $B_T \equiv B_0 e^{rT}$. Then portfolio value is the process

$$X_t^1 = e^{-r(T-t)}(B_T - \frac{K}{\alpha})N(d_{1,t}) + (\frac{e^{-r(T-t)}}{\alpha})^{1-1/A} e^{||\theta||^2(T-t)(1-A)/2A^2}(\lambda_1\zeta_t)^{-1/A}N(d_{2,t})$$

and the manager's optimal trading strategy is

$$\begin{aligned}\pi_t^1 = \quad & \{\frac{1}{A}[X_t^1 - e^{-r(T-t)}(B_T - \frac{K}{\alpha})N(d_{1,t})] \\ & + e^{-r(T-t)}(\frac{(\lambda_1 z_1/\alpha)^{-1/A} - K}{\alpha} + B_T)\frac{N'(d_{1,t})}{||\theta||\sqrt{T-t}}\}\Sigma^{-1}\mu \ ,\end{aligned}$$

where N is the standard cumulative normal distribution, $d_{1,t} = \frac{\ln(z_1/\zeta_t)+(r-||\theta||^2/2)(T-t)}{||\theta||\sqrt{T-t}}$, and $d_{2,t} = d_{1,t} + ||\theta||\sqrt{T-t}/A$.

It is easy to show that as $\zeta_t \to 0$, $X_t^1 \to +\infty$, $||\pi_t^1|| \to +\infty$, and $\frac{\pi_t^1}{X_t^1} \to \frac{\Sigma^{-1}\mu}{A}$. On the other hand, as $\zeta_t \to +\infty$, $X_t^1 \to 0$, $\pi_t^1 \to 0$, but $||\frac{\pi_t^1}{X_t^1}|| \to \infty$.

The Market Portfolio as Benchmark

Now set $B_T \equiv B_0/\zeta_T$. Then portfolio value is the process

$$X_t^2 = (B_0/\zeta_t)N(d_{5,t}) - e^{-r(T-t)}\frac{K}{\alpha}N(d_{3,t}) + (\frac{e^{-r(T-t)}}{\alpha})^{1-1/A} e^{||\theta||^2(T-t)(1-A)/2A^2}(\lambda_2\zeta_t)^{-1/A}N(d_{4,t})$$

and the manager's optimal trading strategy is

$$\begin{aligned}\pi_t^2 = \quad & \{\frac{1}{A}[X_t^2 - (B_0/\zeta_t)N(d_{5,t}) + e^{-r(T-t)}\frac{K}{\alpha}N(d_{3,t})] \\ & + e^{-r(T-t)}(\frac{(\lambda_2 z_2/\alpha)^{-1/A} - K}{\alpha} + B_0/z_2)\frac{N'(d_{3,t})}{||\theta||\sqrt{T-t}} + (B_0/\zeta_t)N(d_{5,t})\}\Sigma^{-1}\mu \ ,\end{aligned}$$

where $d_{3,t} = \frac{\ln(z_2/\zeta_t)+(r-||\theta||^2/2)(T-t)}{||\theta||\sqrt{T-t}}$, $d_{4,t} = d_{3,t} + ||\theta||\sqrt{T-t}/A$, and $d_{5,t} = d_{3,t} + ||\theta||\sqrt{T-t}$.

As $\zeta_t \to 0$, $X_t^2 \to +\infty$, $||\pi_t^2|| \to +\infty$, and $\frac{\pi_t^2}{X_t^2}$ goes either to $\frac{\Sigma^{-1}\mu}{A}$ if $A < 1$ or $\Sigma^{-1}\mu$ if $A > 1$. Yet as $\zeta_t \to +\infty$, $X_t^2 \to 0$, $\pi_t^2 \to 0$, but $||\frac{\pi_t^2}{X_t^2}|| \to \infty$.

2.5.2 Constant Absolute Risk Aversion

Now let $U(W) = -e^{-AW}$ where $A > 0$.

The Riskless Portfolio as Benchmark

For this section, set $B_T \equiv B_0 e^{rT}$. Then portfolio value is the process

$$X_t^{1'} = e^{-r(T-t)}\{[\frac{1}{\alpha A}(\ln \frac{\alpha A}{\lambda_1' \zeta_t} + (r - ||\theta||^2/2)(T-t) + B_T - K/\alpha]N(d_{1,t}') + \frac{||\theta||\sqrt{T-t}}{\alpha A}N'(d_{1,t}')\} ,$$

and the manager's optimal trading strategy is

$$\pi_t^{1'} = e^{-r(T-t)}\{\frac{N(d_{1,t}')}{\alpha A} + \frac{N'(d_{1,t}')}{||\theta||\sqrt{(T-t)}}[\frac{1}{\alpha A}\ln \frac{\alpha A}{\lambda_1' z_1'} + B_T - K/\alpha]\}\Sigma^{-1}\mu ,$$

where $d_{1,t}' = \frac{\ln(z_1'/\zeta_t) + (r - ||\theta||^2/2)(T-t)}{||\theta||\sqrt{T-t}}$.

In this case, as $\zeta_t \to 0$, $X_t^{1'} \to +\infty$, $\pi_t^{1'} \to \frac{\Sigma^{-1}\mu}{\alpha A}$, and $\frac{\pi_t^{1'}}{X_t^{1'}} \to 0$, while as $\zeta_t \to +\infty$, $X_t^{1'} \to 0$, $\pi_t^{1'} \to 0$, but $||\frac{\pi_t^{1'}}{X_t^{1'}}|| \to \infty$.

The Market Portfolio as Benchmark

Now set $B_T \equiv B_0/\zeta_T$. Portfolio value is the process

$$\begin{aligned} X_t^{2'} = \quad & e^{-r(T-t)}[\frac{1}{\alpha A}(\ln \frac{\alpha A}{\lambda_2' \zeta_t} + (r - ||\theta||^2/2)(T-t) + -K/\alpha]N(d_{3,t}') \\ & +(B_0/\zeta_t)N(d_{5,t}') + \frac{e^{-r(T-t)}||\theta||\sqrt{T-t}}{\alpha A}N'(d_{3,t}'), \end{aligned}$$

and the manager's optimal trading strategy is

$$\pi_t^{2'} = \{\frac{e^{-r(T-t)}N(d_{1,t}')}{\alpha A} + (B_0/\zeta_t)N(d_{5,t}') + \frac{e^{-r(T-t)}N'(d_{3,t}')}{||\theta||\sqrt{(T-t)}}[\frac{1}{\alpha A}\ln \frac{\alpha A}{\lambda_2' z_2'} - K/\alpha + \zeta/z_2']\}\Sigma^{-1}\mu ,$$

where $d_{3,t}' = \frac{\ln(z_2'/\zeta_t) + (r - ||\theta||^2/2)(T-t)}{||\theta||\sqrt{T-t}}$ and $d_{5,t}' = d_{3,t}' + ||\theta||\sqrt{T-t}$..

Here, as $\zeta_t \to 0$, $X_t^{2'} \to +\infty$, $||\pi_t^{2'}|| \to \infty$, and $\frac{\pi_t^{2'}}{X_t^{2'}} \to 1$, while as $\zeta_t \to +\infty$, $X_t^{2'} \to 0$, $\pi_t^{2'} \to 0$, but $||\frac{\pi_t^{2'}}{X_t^{2'}}|| \to \infty$.

In all cases, when the portfolio value is very high so that the manager is deep in the money, his portfolio choice looks like the choice he would make if the performance fee were linear, that is, if he were maximizing $EU(\alpha(X_T - B_T) + K)$. For instance, with the riskless asset as benchmark and constant relative risk aversion, the proportional portfolio holdings of risky assets approach the Merton constant, $\frac{\Sigma^{-1}\mu}{A}$.

The effect of the convexity of the incentive fee becomes dramatic as wealth level falls to zero. As the manager gets farther out of the money, he takes on as much risk as possible, subject to the constraint that wealth must be nonnegative (in all cases, $||\frac{\pi_t^*}{x_t^*}|| \to \infty$, even though $\pi_t^* \to 0$). To illustrate, figure 3 plots the proportional portfolio holdings of risky assets as a function of portfolio value for the case of constant relative risk aversion and the riskless benchmark.

The manager's trading position becomes very unstable if he is near the money as the evaluation date draws near. As ζ_t vibrates around the critical point z_i, the manager's portfolio π^m oscillates between zero and a strictly positive value. Thus, small changes in the value of the market portfolio precipitate large trades as the manager alternates between the desire to gamble and the need to remain solvent.

2.6 The Cost of the Incentive Fee Contract

Although the paper focuses on the manager's investment problem, the results have some implications for contract theory. The corollary below shows that a linear performance fee in this model would give the manager greater utility at lower cost to the client. If the client were a profit-maximizer, then the incentive fee would be a costly form of risk-sharing. For example, to the extent that the problem of a corporate manager choosing an optimal leverage policy bears any resemblance to the problem considered here, and to the extent that firms are profit-maximizers, the use of options as compensation instead of restricted shares of stock incurs this cost. This does not mean that the option-like compensation scheme is not optimal for a profit-maximizing principal, however, because the model does not take the manager's choice of effort into consideration. The presence of this cost merely suggests that to be optimal, convex compensation schemes must compensate by better motivating managers to exert effort on behalf of their clients. Although the principal-agent literature does not find that convex sharing rules are necessarily best,[6] the widespread use of option compensation for corporate

[6] The optimal shape of the sharing rule can be arbitrary. See Holmstrom and Hart (1987).

managers suggests that options have better incentive effects than stock, which outweigh their risk-sharing costs.

Corollary *There exists a linear performance fee that has lower cost to the client than the incentive fee and gives the manager greater expected utility.*

Proof *Let* $p = \mathcal{P}^B\{X_T^* > 0\}$ *and let* $\alpha' = \alpha - pB_0/X_0$. *Under any investment policy, the linear fee* $\alpha' X_T$, *has the same market value as the optimized incentive fee. Under the optimal policy, the linear fee gives the manager strictly greater expected utility. Indeed, letting* $y = \alpha' X_T + K$, *the manager's problem is*

$$\begin{aligned} \max_y \quad & EU(y) \\ \textit{subject to} \quad & E\zeta_T y \leq \alpha(X_0 - pB_0) + K \\ \textit{and} \quad & y \geq K \ . \end{aligned} \tag{20}$$

This is just like the incentive fee problem, (10), without the constraint on the probability of bankruptcy. Relaxing that constraint allows the manager to achieve a better policy. Reducing α' *just slightly will still leave the manager better off than he is under the incentive fee and will cost the client less.*

3 Multi-Period Model

With the one-time incentive fee in the last section, the manager chooses an all-or-nothing policy: he either substantially outperforms the benchmark, or else bankrupts the client. This solution seems unrealistic given that real managers play a repeated game and probably try to avoid bankrupting their clients. This section addresses this limitation by presenting a simple model of the optimal investment policy for a manager who gets a new incentive fee every year. The result is that the manager limits downside risk, and thus forgoes some upside potential, in order to preserve the value of future incentive fees. Nevertheless, his optimal policy is still one of extremes.

The model abstracts from some of the features of the multi-period problem real managers face. Most importantly, the model does not incorporate flows of new money that respond to past performance, and therefore, does not incorporate a reputation effect. To do so would involve building an equilibrium model that is beyond the scope of the paper.

In addition, the model assumes that each year the incentive fee is reset at the money. Generally, managers more must make up losses from previous periods before earning an incentive fee in the current period. This would make the benchmark or strike price of any year's incentive fee the all-time high value of the fund. However, some funds periodically renew long term contracts, resetting the strike price of the incentive fee to the current fund level. For example, the prospectus of E.F. Hutton Commodity Limited Partnership II (1980) states, "In each fiscal quarter, each advisor will be paid, as an incentive fee, 12.5% of ... the excess of net asset value ... as of the last day of such fiscal quarter ... over the highest net asset value as of the last day of any preceding fiscal quarter." However, the prospectus states further that "upon termination of the current advisors' contracts (12 months following commencement of trading operations), the partnership may employ other advisory services whose compensation may be calculated without regard to losses incurred by the current advisors ... the partnership may renew its relationship with any of the advisors on the same or different terms ..." Elton, Gruber, and Rentzler (1987) note that with "the basing of incentive fees on shorter-term performance, incentive fees can be high even with poor long-term performance."

3.1 Assumptions

The manager controls a fund with time t value X_t. At the end of each year, $t = 1, 2, \ldots, T$, he receives and consumes a total fee equal to a fixed proportion of the fund value at the beginning of the year, $\alpha_0 X_{t-1}$, plus an incentive fee, $\alpha_1(X_t - R_0 X_{t-1})$ where R_0 is a constant. I assume for simplicity that the client pays the manager fees out of a separate account. The manager trades in the continuous-time financial market described in section 2.1. He has constant relative risk aversion with coefficient A, $A > 0, A \neq 1$. He chooses an investment policy to maximize his expected discounted lifetime utility. The value function for his problem at any year $t = 0, 1, \ldots, T-1$ is

$$\begin{aligned} V_t(x) = \max_{\{\pi_s, t \le s \le T\}} \quad & \sum_{j=0}^{T-t-1} \beta^j E\{U(\alpha_0 X_{t+j} + \alpha_1 (X_{t+j+1} - R_0 X_{t+j})^+) | \mathcal{F}_t\} \\ \text{subject to} \quad & dX_s = (rX_s + \pi_s' \mu)\, ds + \pi_s' \sigma\, dW_s \,; X_t = x \\ \text{and} \quad & X_s \ge 0 \ \forall s \in [t, T] \,, \end{aligned} \tag{21}$$

where $0 < \beta \le 1$. Martingale methods transform this problem to

$$\begin{aligned} V_t(x) = \max_{\{R_s, s = t+1, t+2, \ldots, T\}} \quad & \sum_{j=0}^{T-t-1} \beta^j E\{U(\alpha_0 X_{t+j} + \alpha_1 X_{t+j} (R_{t+j+1} - R_0)^+) | \mathcal{F}_t\} \\ \text{subject to} \quad & X_{t+j+1} = R_{t+j+1} X_{t+j}; X_t = x \,, \\ & E\{\frac{\zeta_{t+j+1}}{\zeta_{t+j}} R_{t+j+1} | \mathcal{F}_{t+j}\} \le 1 \,, \\ \text{and} \quad & R_{t+j} \ge 0 \ \forall j = 1, 2, \ldots, T-t \,. \end{aligned} \tag{22}$$

3.2 Reduction to a One-Period Problem

With constant relative risk averse utility, the multi-period problem 22 reduces to a single-period problem.

Lemma 2

$$V_t(x) = c_t \frac{x^{1-A}}{1-A} \tag{23}$$

for a positive constant c_t, defined recursively by

$$\begin{aligned} c_t = (1-A) \max_R \quad & E\{\frac{(\alpha_0 + \alpha_1 (R - R_0)^+)^{1-A} + \beta c_{t+1} R^{1-A}}{1-A}\} \\ \textit{subject to} \quad & E\{\zeta R\} \le 1 \textit{ and } R \ge 0 \,; \end{aligned} \tag{24}$$

$$c_T = \quad 0 \tag{25}$$

where $\zeta = \zeta_1$.

Proof *By backward induction:*

$$\begin{aligned} V_{T-1}(x) = \max_R \quad & E\{\frac{(\alpha_0 x + \alpha_1 x(R-R_0)^+)^{1-A}}{1-A}\} \\ \textit{subject to} \quad & E\{\zeta R\} \le 1 \textit{ and } R \ge 0 \end{aligned} \tag{26}$$

$$\begin{aligned} = \frac{x^{1-A}}{1-A}(1-A)\max_R \quad & E\{\frac{(\alpha_0 + \alpha_1 (R-R_0)^+)^{1-A}}{1-A}\} \\ \textit{subject to} \quad & E\{\zeta R\} \le 1 \textit{ and } R \ge 0 \end{aligned} \tag{27}$$

$$= c_{T-1}\frac{x^{1-A}}{1-A}\,. \tag{28}$$

Now suppose $V_{t+1}(x) = c_{t+1}\frac{x^{1-A}}{1-A}$. *Then*

$$\begin{aligned} V_t(x) = \max_R \quad & E\{\frac{(\alpha_0 x + \alpha_1 x(R-R_0)^+)^{1-A} + \beta c_{t+1}(Rx)^{1-A}}{1-A}\} \\ \textit{subject to} \quad & E\{\zeta R\} \le 1 \textit{ and } R \ge 0 \end{aligned} \tag{29}$$

$$\begin{aligned} = \frac{x^{1-A}}{1-A}(1-A)\max_R \quad & E\{\frac{(\alpha_0 + \alpha_1 (R-R_0)^+)^{1-A} + \beta c_{t+1}R^{1-A}}{1-A}\} \\ \textit{subject to} \quad & E\{\zeta R\} \le 1 \textit{ and } R \ge 0 \end{aligned} \tag{30}$$

$$= c_t\frac{x^{1-A}}{1-A}\,. \tag{31}$$

Thus, each year, the manager's problem is of the form

$$\begin{aligned} \max_R \quad & Ev(R) \\ \text{subject to} \quad & E\{\zeta R\} \le 1 \text{ and } R \ge 0 \end{aligned} \tag{32}$$

where $v : (0,\infty) \to \mathcal{R}$ is defined by

$$v(R) = \frac{(\alpha_0 + \alpha_1 (R-R_0)^+)^{1-A} + \beta c(R)^{1-A}}{1-A} \tag{33}$$

for some positive constant c which varies from year to year.

3.3 Optimal Random Fund Return

As before, the option-like payoff of the incentive fee makes the objective function v nonconcave. The dotted line in figure 4 plots v as a function of the choice variable R. Again, v has a concavification $\tilde{v}$ represented by the dashed line in figure 4. Concavifying

v amounts to replacing part of v by a chord between points R_1 and R_2, $0 < R_1 < R_0 < R_2$, chosen to make the slope of the chord equal to the slope of v at the points R_1 and R_2, so that the resulting function is concave. For example, in figure 4, $R_1 = 0.2126$ and $R_2 = 1.3557$ serve as the endpoints of a "bridge" over the kink point $R_0 = 1$ that renders the resulting function concave. In general, the points R_1 and R_2 are defined by the equations

$$\begin{aligned} \beta c U'(R_1) &= \alpha_1 U'(\alpha_0 + \alpha_1(R_2 - R_0)) + \beta c U'(R_2) \\ U(\alpha_0) + \beta c U(R_1) + \beta c U'(R_1)(R_2 - R_1) &= U(\alpha_0 + \alpha_1(R_2 - R_0)) + \beta c U(R_2)\,. \end{aligned} \quad (34)$$

Using an argument similar to that in the proof of lemma 1, it is not hard to show that such an R_1 and R_2 exist uniquely for any utility function U satisfying the conditions of section 2.1, not just constant relative risk averse utility. Now define $\tilde{v} : (0, \infty) \to \mathcal{R}$ by

$$\tilde{v}(R) = \begin{cases} v(R_1) + v'(R_1)(R - R_1) & \text{whenever } R_1 < R < R_2 \text{ and} \quad (35) \\ v(R) & \text{otherwise.} \quad (36) \end{cases}$$

Note that

$$\tilde{v}'(\hat{R}) = \begin{cases} v'(R_1) = v'(R_2) & \text{whenever } R_1 < R < R_2 \text{ and} \quad (37) \\ v'(R) & \text{otherwise.} \quad (38) \end{cases}$$

To construct the optimal policy for problem 32, define an inverse function $j : (0, \infty) \to \mathcal{R}$ for $\tilde{v}'$ by

$$j(y) = \begin{cases} R_2 & \text{for } y = \tilde{v}'(R_2) \text{ and} \quad (39) \\ \tilde{v}'^{-1}(y) & \text{otherwise.} \quad (40) \end{cases}$$

Then $\tilde{v}'(j(y)) = y$. Note that j is monotonically decreasing and has a jump discontinuity at $y = \tilde{v}'(R_2)$.

The optimal policy for problem 32 is $R^* = j(\lambda^* \zeta)$ where λ^* solves $E\{\zeta j(\lambda^* \zeta)\}$. The proof is virtually identical to that of propostion 2. Figure 5 plots the optimal return policy R^* as a function of the state price density ζ. As figure 5 shows, R^* decreases monotonically to R_2, then jumps down to R_1 at the critical value of ζ and then continues to decrease monotonically, approaching zero as ζ goes to infinity.

It is not hard to show that as the value of c increases to infinity, the points R_1 and R_2 monotonically approach R_0. Thus, the greater the value of future incentive fees, the smoother the current policy will be. The change in the optimal policy as retirement approaches depends on the behavior of c_t over time. In the case of log utility ($A = 1$), the manager's time t value function is $V_t(x) = c_t \log(x) + a_t$ where a_t is a constant and $c_t = \sum_{j=0}^{T-t-1} \beta^j$. Therefore, c_t decreases over time, and the optimal policy becomes increasingly risky as retirement draws near.

4 Conclusion

This paper presents the optimal investment policy for a risk averse manager who is paid with an option on the assets he controls. With a one-time fee, the assets optimally have an all-or-nothing distribution of outcomes: either deep in the money or zero. The policy is also a long-shot in the sense that the probability of bankruptcy is relatively high, but the payoff, if nonzero, is quite large. In a multi-period setting, the manager limits downside risk to protect the value of future compensation, but the essence of the solution, that the manager does not want to end up too near the money, remains the same.

The martingale approach sheds light on the manager's preference for a long-shot by revealing that the market value of the option is an increasing function of the probability of bankruptcy under a martingale measure. This relationship implies that the contract is inefficient in the sense that lower cost linear contracts exist that give the manager greater expected utility.

Explicit expressions for the optimal trading strategy for constant relative and absolute risk averse utility functions with either the riskless asset or the market portfolio as benchmarks show how the manager dynamically adjusts volatility as asset value changes. When the manager is near the money, small changes in the value of the mean-variance efficient portfolio lead to large trades as the manager alternates between the desire to gamble and the need to remain solvent. As asset value grows large, the manager moderates portfolio risk. On the other hand, as bankruptcy approaches, portfolio volatility approaches infinity.

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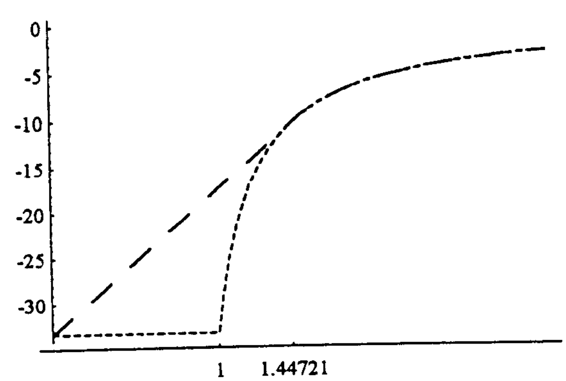
Figure 1

Fund Manager's Single-Period Incentive Fee Problem: Original and Concavified Objective Functions

Plots of the manager's objective function $u(x) = U(\alpha(x-b)^{+} + K)$, the dotted line, and its concavification $\tilde{u}(x)$, the dashed line. The manager's utility function U is constant relative risk averse with coefficient 2, $\alpha = 0.15$, $b = 1$, and $K = 0.03$. The concavification replaces part of the original function with a chord drawn from the point $x = 0$ to the point $x = \hat{x} = 1.4472$ chosen to make the slope of the chord match the slope of the original function at $\hat{x}$ so that the resulting function is concave.

Manager's objective functions u and $\tilde{u}$



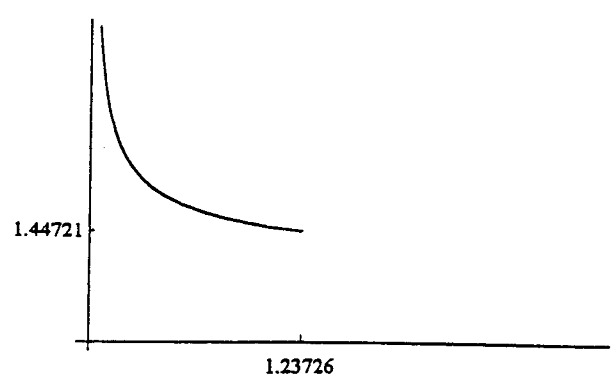
Terminal fund value x

Figure 2

Fund Manager's Single-Period Incentive Fee Problem: Optimal Random Terminal Fund Value

Plot of the manager's optimal random terminal fund value X_T^1 as a function of the state price density ζ_T. The random variable X_T^1 maximizes $EU(\alpha(X_T - B_T)^+ + K)$ subject to $E\zeta_T X_T \leq X_0$ and $X_T \geq 0$. The manager's utility function U is constant relative risk averse with coefficient 2, $\alpha = 0.15$, $B_T = 1$, $K = 0.03$, $X_0 = 1$. The state price density is $\zeta_T = e^{-rT-\theta W_T-\theta^2 T/2}$ where $r = 0$, $\theta = 0.4$, $T = 1$, and W_T is Brownian motion at time T. The nonconcavity of the objective function makes the optimal terminal fund value a discontinuous function of ζ_T. At the critical level $\zeta_T = 1.2373$, X_T^1 jumps from $\hat{x} = 1.4472$ to zero.

Optimal terminal fund value X_T^1



State price density ζ_T

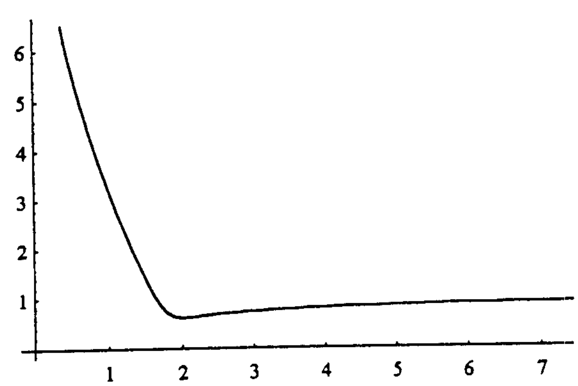
Figure 3

Fund Manager's Single-Period Incentive Fee Problem:

Optimal Trading Strategy

Plot of the manager's optimal proportion of fund value invested in the risky asset, $\frac{\pi_t^1}{X_T^1}$, as a function of fund value, X_t^1, one year prior to the evaluation date T. Terminal fund value X_T^1 maximizes $EU(\alpha(X_T - B_T)^+ + K)$ subject to $E\zeta_T X_T \leq X_0$ and $X_T \geq 0$. The manager's utility function U is constant relative risk averse with coefficient 2, $\alpha = 0.15$, $B_T = 1$, $K = 0.03$, $X_0 = 1$. Intermediate fund value is $X_t^1 = E_t\{\frac{\zeta_T}{\zeta_t} X_T^1\}$ where the state price density process is $\zeta_t = e^{-rt-\theta W_t - \theta^2 t/2}$ with $r = 0$, $\theta = 0.4$, and W_t Brownian motion at time t. The Sharpe ratio θ on the risky asset is $\frac{\mu}{\sigma}$ where the risky asset's excess expected return $\mu = 0.08$ and its volatility $\sigma = 0.2$. As fund value X_t^1 grows large, the proportion of the fund invested in the risky asset approaches the Merton constant $\frac{\mu}{A\sigma^2} = 1$ that would be optimal in a standard investment problem with no incentive fee. As fund value X_t^1 approaches zero, the proportion of the fund invested in the risky asset, and thus fund volatility, go to infinity.

Optimal proportion of fund value in risky asset $\frac{\pi_t^1}{X_t^1}$



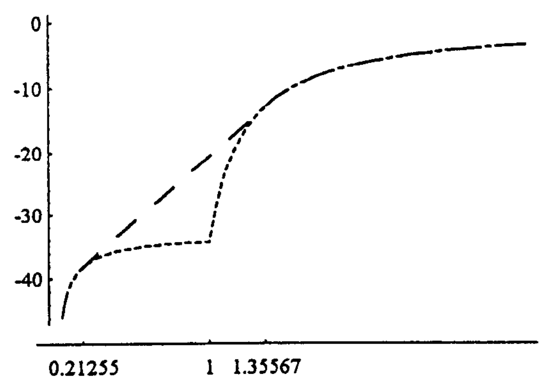
Intermediate fund value X_t^1

Figure 4

Fund Manager's Multi-Period Incentive Fee Problem: Original and Concavified Objective Functions

Plots of the manager's objective function $v(R) = U(\alpha_0 + \alpha_1(R - R_0)^+) + \beta c U(R)$, the dotted line, and its concavification $\tilde{v}(R)$, the dashed line. The manager's utility function U is constant relative risk averse with coefficient 2, $\alpha_0 = 0.03$, $\alpha_1 = 0.15$, $R_0 = 1$, $\beta = 1$, and $c = 1$. The concavification replaces part of the original function with a chord drawn from the point $R_1 = 0.2126$ to the point $R_2 = 1.3557$ chosen to make the slope of the chord match the slope of the original function at R_1 and R_2 so that the resulting function is concave.

Manager's objective functions v and $\tilde{v}$



Terminal fund return R

Figure 5

Fund Manager's Multi-Period Incentive Fee Problem:

Optimal Random Fund Return

Plot of the manager's optimal random terminal fund value R^* as a function of the state price density ζ. The random variable R^* maximizes $E\{U(\alpha_0+\alpha_1(R-R_0)^+)+\beta cU(R)\}$, subject to $E\zeta R \leq 1$ and $R \geq 0$. The manager's utility function U is constant relative risk averse with coefficient 2, $\alpha_0 = 0.03$, $\alpha_1 = 0.15$, $R_0 = 1$, $\beta = 1$, and $c = 1$. The state price density is $\zeta = e^{-r-\theta W_1-\theta^2/2}$ where $r = 0$, $\theta = 0.4$, and W_1 is Brownian motion at time 1. The nonconcavity of the objective function makes the optimal terminal fund value a discontinuous function of ζ_T. At the critical level $\zeta = 1.2371$, R^* jumps from $R_2 = 1.3557$ to $R_1 = 0.2126$.