The Local Whittle Estimator of Long Memory Stochastic Volatility

Clifford M. Hurvich
Department of Statistics and Operations Research
New York University
44 W. 4^{th} Street
New York NY 10012 USA
churvich@stern.nyu.edu

Bonnie K. Ray*
Department of Mathematical Sciences
IBM T.J. Watson Research Center
P.O. Box 218
Yorktown Heights, NY 10598 USA
bonnier@us.ibm.com

Abstract

We propose a new semiparametric estimator of the degree of persistence in volatility for long memory stochastic volatility (LMSV) models. The estimator uses the periodogram of the log squared returns in a local Whittle criterion which explicitly accounts for the noise term in the LMSV model. Finite-sample and asymptotic standard errors for the estimator are provided. An extensive simulation study reveals that the local Whittle estimator is much less biased and yields more accurate confidence intervals than the widely-used GPH estimator. In an empirical analysis of the daily Deutschemark/Dollar exchange rate, the new estimator indicates stronger persistence in volatility than the GPH estimator, provided that a large number of frequencies is used.

Key Words: long-range dependence; nonlinearity; semiparametric estimation

1. INTRODUCTION

Long memory in volatility of financial returns has received considerable attention in recent years. See, *e.g.* Ding, Granger and Engle (1993), de Lima and Crato (1993), Baillie, Bollerslev and Mikkelsen (1996), Andersen and Bollerslev (1997a,b), Lobato and Savin (1998), Lobato and Robinson (1998), Ray and Tsay (2000), Lobato and Velasco (2000), Wright (2000), Andersen, Bollerslev, Diebold and Labys (2001), and Robinson (2001). A widely-used methodology for determining the degree of persistence in volatility, parameterized by d, is to estimate d semiparametrically using log periodogram regression based on squared or absolute returns. The log

*Corresponding author

periodogram regression estimator, $\widehat{d}_{GPH}$, was originally proposed by Geweke and Porter-Hudak (1983), in a non-volatility context. Properties of this estimator for stationary Gaussian processes, which are linear and hence free of volatility clustering, were derived by Robinson (1995a) and Hurvich, Deo and Brodsky (1998). In this case, $\widehat{d}_{GPH}$ is consistent and asymptotically normal under certain regularity conditions. The GPH method is practically appealing, as it is may be computed using simple linear regression.

To model observed persistence in volatility of financial returns, the long memory stochastic volatility (LMSV) model was introduced independently by Breidt, Crato and de Lima (1998) and Harvey (1998). The series of logarithms of squared values of an LMSV process is modeled as a long-range dependent process plus added noise (See Section 2). However, Deo and Hurvich (2001) show that $\widehat{d}_{GPH}$ based on log squared returns in the LMSV model suffers from a potentially severe negative bias which does not arise in the Gaussian case, and which depends on d, becoming worse as d goes to zero. Deo and Hurvich (2001) is, to the best of our knowledge, the first paper which derives theoretical properties for any semiparametric estimator of d in the context of volatility.

In this paper, we propose a new semiparametric estimator of d in the LMSV context, designed with a view towards bias reduction in comparison with $\widehat{d}_{GPH}$. The new estimator, $\widehat{d}_{LWN}$, is a local Whittle estimator which explicitly accounts for the noise term in the LMSV model. This noise term introduces a certain degree of roughness, which is determined by d, in the short memory component of the spectral density in a neighborhood of zero frequency. The estimator $\widehat{d}_{LWN}$ is implicitly defined, and may be computed using a two-dimensional nonlinear optimization algorithm.

1.1 Analysis of transformed returns

We focus in this paper on estimators of d for series of log squared returns. This choice of transformation seems to be justified empirically; Ding, Granger and Engle (1993) observed that autocorrelations of absolute returns raised to the power c were typically maximized by taking c close to 1. Deo and Hurvich (2001) have proposed an explanation in terms of leverage effects for the fact that absolute and squared returns typically have smaller sample autocorrelations than log squared returns. An analogous phenomenon presumably holds for the degree of persistence implied by periodograms. Indeed, Wright (2000) has shown using simulations under both LMSV and ARCH-type models that periodogram-based semiparametric estimators of d are less negatively biased if log squared returns are used, instead of absolute or squared returns.

1.2 Using GPH to assess persistence in volatility

Even using log squared returns for analysis, however, the GPH estimator of persistence in volatility in LMSV models still suffers from a potentially severe negative bias. This bias, which is given explicitly in Theorem 1 of Deo and Hurvich (2001), implies a slow rate of convergence for $\widehat{d}_{GPH}$. In general, in order to guarantee that $\sqrt{m}(\widehat{d}_{GPH} - d)$ will be asymptotically normal with zero mean, m must grow more slowly than $n^{4d/(4d+1)}$, where n is the sample size and m is the number of frequencies used in the regression. For example, if $d = .25$, then m must grow more slowly than $n^{1/2}$, while if $d = .1$, m must grow more slowly than $n^{2/7}$. In no situation with $0 < d < 1/2$ can m grow faster than $n^{2/3}$.

Now, when $d = 0$ in the LMSV model, Hurvich and Soulier (2000) have shown that $\sqrt{m}(\widehat{d}_{GPH} - d)$ is asymptotically normal with mean zero and variance $\pi^2/24$, as long as m grows more slowly than $n^{4/5}$. Thus, an asymptotically valid test for long memory in volatility is to reject the null hypothesis of $d = 0$ in favor of $d > 0$ if the test statistic $\widehat{d}_{GPH}/\sqrt{\pi^2/(24m)}$ is greater than the $1 - \alpha$ quantile of the standard normal distribution, where α is the desired significance level. This would seem to suggest that $\widehat{d}_{GPH}$ is satisfactory for assessing the existence of persistence in volatility.

Nevertheless, the fact that the bias in $\widehat{d}_{GPH}$ depends on d makes statistical inference based on $\widehat{d}_{GPH}$ difficult, if not impossible, in general. Indeed, even if we knew that $d > 0$, we could not construct an asymptotically valid confidence interval for d based on $\widehat{d}_{GPH}$ without an *a priori*, strictly positive lower bound for d. Such a bound, which would seldom if ever be available in practice, would be needed to prevent the practitioner from selecting too large a value of m, and thereby invalidating the confidence interval by introducing excessive bias in $\widehat{d}_{GPH}$. Thus, a better estimator of long memory in volatility is desirable.

1.3 Outline of paper

Here, we investigate the properties of the local Whittle estimator, $\widehat{d}_{LWN}$ compared to $\widehat{d}_{GPH}$ in practice. We also compare the proposed method to the local polynomial GPH estimator, $\widehat{d}_{LP-GPH}$ of Andrews and Guggenberger (2000), which reduces the bias of GPH for sufficiently regular linear processes. We present extensive simulation studies comparing the performance of $\widehat{d}_{GPH}$, $\widehat{d}_{LP-GPH}$ and $\widehat{d}_{LWN}$. The simulations reinforce the fact that $\widehat{d}_{GPH}$ can be extremely negatively biased. This is of considerable practical relevance, since it suggests, in conjunction with our data analysis, that many of the published data analyses may be understating the strength of the true persistence in volatility. The local polynomial GPH estimator is slightly less biased, but at the cost of increased variability. We find that $\widehat{d}_{LWN}$ has much less bias than $\widehat{d}_{GPH}$, and its variance inflation compared with $\widehat{d}_{GPH}$ is not unreasonably large. Thus, $\widehat{d}_{LWN}$ seems to hold great promise for estimating persistence in volatility. The theoretical properties of $\hat{d}_{LWN}$ have been studied by Hurvich, Moulines and Soulier (2002). We summarize here the most relevant aspects of that theory, including an expression for the asymptotic variance of $\hat{d}_{LWN}$, which depends on d. We also provide a feasible, finite-sample expression for the variance of $\hat{d}_{LWN}$. The accuracy of these approximations, as well as resulting confidence intervals, is assessed in our simulation study. Finally, we present an empirical analysis of the daily Dollar/Deutschemark exchange rate, and find a higher degree of persistence in volatility than suggested by the GPH estimator when a large number of frequencies is used.

2. ESTIMATION OF d IN THE LMSV MODEL

The LMSV model for returns $\{r_t\}$ takes form $r_t = \eta \exp(Y_t/2)e_t$ where $\eta > 0$ is a scale parameter, $\{e_t\}$ are independent identically distributed (*i.i.d.*) shocks with zero mean, and $\{Y_t\}$ is a zero-mean stationary Gaussian process, independent of $\{e_t\}$, with spectral density

$$f_Y(x) = x^{-2d} f_Y^*(x), \tag{1}$$

where f_Y^* is an even, positive, continuous function on $[-\pi, \pi]$ and d is the memory parameter, $0 \leq d < 1/2$. We assume hereafter that $d > 0$. Under the LMSV model, the logarithms of the squared returns, $X_t = \log(r_t^2)$, may be expressed as

$$X_t = Y_t + Z_t, \tag{2}$$

where $\{Z_t\} = \{\log e_t^2\}$ is $i.i.d.$ with variance $\sigma_Z^2 < \infty$.

The assumptions given above for the LMSV model imply that the spectral density of X_t may be written as

$$f_X(x) = f_Y(x) + \sigma_Z^2/(2\pi). \tag{3}$$

The LMSV model described above can be generalized in various ways. The $\{Y_t\}$ series can be non-Gaussian, subject to the regularity conditions described below. Additionally, the log squared returns can be nonstationary, with memory parameter $d \in (1/2, 1]$. In this nonstationary case, we define the model by $r_t = \eta \exp(U_t/2)e_t$ where $U_t = \sum_{s=1}^t Y_s$ and $f_Y(x) = x^{-2(d-1)} f_Y^*(x)$, so that here $\{Y_t\}$ has memory paremeter $d_Y \in (-1/2, 0]$. Since $\{U_t\}$ is nonstationary, it does not have a spectral density, but it does have a pseudo spectral density given by $|1 - e^{ix}|^{-2} f_Y(x)$. This pseudo spectral density plays a similar role to that of the ordinary spectral density in determining the properties of the periodogram when $d > 1/2$. See, e.g., Solo (1992), Hurvich and Ray (1995), Velasco (1999).

Overall, then, our generalized model is

$$r_t = \begin{cases} \eta \exp(Y_t/2)e_t \ , & d \in (0, 1/2) \\ \eta \exp(\sum_{s=1}^t Y_s/2)e_t \ , & d \in (1/2, 1) \end{cases}$$

such that $\{Y_t\}$ is independent of the $i.i.d.$ process $\{e_t\}$, where $\{Y_t\}$ is stationary and invertible with spectral density $f_Y(x) = x^{-2d_Y} f_Y^*(x)$, $d_Y \in (-1/2, 1/2)$, and

$$d_Y = \begin{cases} d \text{ if } d \in (0, 1/2) \\ d - 1 \text{ if } d \in (1/2, 1) \end{cases} .$$

The log squared return series $\{\log r_t^2\}$ is given by

$$X_t = \begin{cases} Y_t + Z_t \text{ if } d \in (0, 1/2) \\ \sum_{s=1}^t Y_s + Z_t \text{ if } d \in (1/2, 1) \end{cases} .$$

In both cases, $\{Z_t\} = \{\log e_t^2\}$ is an $i.i.d.$ process with finite variance, independent of $\{Y_t\}$.

2.1 The GPH Estimator

Define the periodogram of the observations $X_1, \cdots, X_n$ at the k^{th} Fourier frequency $x_k = 2\pi k/n$ by

$$I_{n,k}^X = \frac{1}{2\pi n} \left| \sum_{t=1}^n X_t e^{itx_k} \right|^2 .$$

The GPH estimator of d using the first m Fourier frequencies may be written as

$$\widehat{d}_{GPH} = -\frac{1}{2S_{ww}} \sum_{k=1}^{m} a_k \log I_{n,k}^X,$$

where $a_k = W_k - \overline{W}$, $W_k = \log|2\sin(x_k/2)|$, $\overline{W} = m^{-1} \sum_{k=1}^{m} W_k$ and $S_{ww} = \sum_{k=1}^{m} a_k^2$. Note that the intuition behind the GPH estimator in the standard Gaussian case is the linear relation at low frequencies between the logarithm of the spectral density of a long memory process and the logarithm of the corresponding frequencies, as can be seen from (1). The $\{Z_t\}$ process in (2) may be viewed as an additive noise term which corrupts this linear relationship and impairs our ability to estimate the memory parameter in the signal process $\{Y_t\}$.

2.2 The Local Polynomial GPH Estimator, $\widehat{d}_{LP-GPH}$

Andrews and Guggenberger (2000) proposed a local polynomial GPH estimator of long memory. We will consider the simplest version here, in which the estimator $\hat{d}_{LP-GPH}$ is defined as the coefficient of $-2\log x_k$ in an ordinary least squares regression of $\log I_{n,k}^X$ on a constant, $-2\log x_k$ and x_k^2, for $k = 1, \ldots, m$. For a Gaussian (and therefore linear) process such that the spectral density of the short memory component is sufficiently smooth, specifically, smooth of order $s \geq 1$ at zero frequency, the optimal rate of convergence of mean squared error (MSE) of $\hat{d}_{LP-GPH}$ is proportional to $n^{-2\phi/(2\phi+1)}$ where $\phi = min\{s, 4\}$. Unfortunately, in the context of the LMSV model, we have $s = 2d$ (see Equation (4) below), presumably leading to an optimal mean squared error proportional to $n^{-4d/(4d+1)}$. This rate is identical to the rate attained by GPH in the LMSV context as given by Deo and Hurvich (2001), and is inferior to the optimal rate of $n^{-4/5+\epsilon}$ attained by the MSE of $\hat{d}_{LWN}$, as will be shown in Section 3 below. Nevertheless, for completeness we include $\hat{d}_{LP-GPH}$ in our comparative Monte Carlo study in Section 4.

2.3 The Local Whittle with Noise Estimator, $\widehat{d}_{LWN}$

We assume in this section that

$$f_Y^*(x) = f_Y^*(0) + Cx^2 + R(x),$$

where $R(x) = o(x^2)$ as $x \to 0$. This assumption holds for most short-memory models in current use, including all stationary invertible ARMA models, and exponential models (see Bloomfield, 1973). To avoid a conflict of notation, in this and the next section we denote the true value of the memory parameter by d_0. Then from Equations (1) and (3) we can write

$$f_X(x) = \frac{\sigma_Z^2}{2\pi} \left[1 + \frac{2\pi f_Y^*(0)}{\sigma_Z^2} x^{-2d_0}\right] + O(x^{2-2d_0}) \ . \qquad (4)$$

Stationarity is implicitly assumed in writing (4), but an argument based on pseudo-spectral densities shows that (4) holds even in the nonstationary case.

Since the final $O(x^{2-2d_0})$ term is negligible with respect to the other terms in (4) for x close to 0, it seems reasonable to try locally fitting a model of form

$$g_\theta(x) = b_0(1 + b_1 x^{-2d}) \tag{5}$$

in a neighborhood of zero frequency, where $\theta = (b_0, b_1, d)'$ is the vector of parameters. Model (5) explicitly accounts for the noise term in (2).

For local fitting of model (5), we propose to minimize the local Whittle criterion

$$L(\theta) = \sum_{j=1}^{m} \left[\log \; g_\theta(x_j) + \frac{I_{n,j}^X}{g_\theta(x_j)} \right] \;, \tag{6}$$

where the minimization is carried out in a compact set $\Theta \subset \mathbb{R}^+ \times \mathbb{R}^+ \times (0, 0.75)$, and m is a positive integer such that $1/m + m/n \to 0$ as $n \to \infty$. We assume that θ_0 is an interior point of Θ, where $\theta_0 = [\sigma_Z^2/(2\pi) \, , \, 2\pi f_Y^*(0)/\sigma_Z^2 \, , \, d_0]'$ is the vector of true parameters.

The parameter b_0 can be concentrated out of (6), so minimizing $L(\theta)$ is equivalent to finding (b_1, d) to minimize

$$\tilde{L}(b_1, d) = \sum_{j=1}^{m} \left[\log \; \tilde{g}_{\tilde{\theta}}(x_j) + \frac{I_{n,j}^X}{\tilde{g}_{\tilde{\theta}}(x_j)} \right] \;, \tag{7}$$

where $\tilde{\theta} = (b_1, d)'$,

$$\tilde{g}_{\tilde{\theta}}(x_j) = b_0^{\tilde{\theta}}(1 + b_1 x_j^{-2d}) \;, \tag{8}$$

and

$$b_0^{\tilde{\theta}} = \frac{1}{m} \sum_{j=1}^{m} \frac{I_{n,j}^X}{1 + b_1 x_j^{-2d}} \;. \tag{9}$$

The vector of estimated parameters is $\widehat{\theta} = (\widehat{b}_0 \, , \, \widehat{b}_1 \, , \, \widehat{d}_{LWN})'$, where $\widehat{b}_1, \widehat{d}_{LWN}$ minimize $\tilde{L}$, and $\widehat{b}_0 = b_0^{(\widehat{b}_1, \widehat{d}_{LWN})'}$. Here, the minimization is carried out in a compact set $\Theta \subset \mathbb{R}^+ \times (0, 0.75)$.

In the discussion above, it was implicitly assumed that the minimizer of $\tilde{L}$ occurs at an interior point of Θ. In this case, the estimators $\hat{b}_1$ and $\hat{d}_{LWN}$ satisfy the so-called first order conditions (FOC), that is, the partial derivatives of $\tilde{L}$ are zero at $(b_1, d) = (\hat{b}_1, \hat{d}_{LWN})$. In fact, we need to slightly modify the definition of $\widehat{d}_{LWN}$ to account for possible solutions to (7) on the boundary.

If the global minimizer of $\tilde{L}$ occurs at a boundary point of Θ, then, although there may be several interior points which satisfy the FOC, none of these local optima corresponds to a global optimum, and we define our estimator as follows. (1) If there are no solutions to the FOC, we use the global optimum (boundary point) as our estimator. (2) If there are any solutions to the FOC, then our estimator is defined to be that solution which is closest in the sense of ordinary Euclidean distance to the global optimum (boundary point).

It should be noted that the above algorithm implies that a local optimum will be chosen over the global optimum when the latter is a boundary point. The reason for this choice is to facilitate the development of theory, as suggested by Andrews and Sun (2001). The context for the suggestion of Andrews and Sun (2001) was a local polynomial Whittle estimator of long memory, in a non-volatity context. There, as here, the estimator involves minimization of a multidimensional criterion function, and the individual components of the estimator converge at different rates.

3 PROPERTIES OF $\widehat{d}_{LWN}$

The asymptotic properties of $\hat{d}_{LWN}$ and other related estimators are derived in Hurvich, Moulines and Soulier (2002). We present here the result for $\hat{d}_{LWN}$ under simplified assumptions.

We assume that $\{Y_t\}$ has an infinite order moving average representation

$$Y_t = \sum_{j=0}^{\infty} a_j \epsilon_{t-j} \quad , \tag{10}$$

where $\{\epsilon_t\}$ is a zero-mean white noise process with $Var[\epsilon_t] = \sigma_\epsilon^2$, and $\sum_{j=0}^{\infty} a_j^2 < \infty$. Note that $\{\epsilon_t\}$ is independent of $\{Z_t\}$. We lose no generality in assuming that $\{Y_t\}$ has zero mean, since the estimators considered in this paper are all functions of the periodogram at nonzero Fourier frequencies. In the nonstationary case, the assumption that $\{Y_t\}$ has mean zero ensures that $\{X_t\}$ is free of linear trends.

Define $a(x) = \sum_{j=0}^{\infty} a_j e^{\mathrm{i}jx}$. The spectral density of the process $\{Y_t\}$ is then $f_Y(x) = |a(x)|^2 \sigma_\epsilon^2/(2\pi)$, and we assume that it can be expressed as

$$f_Y(x) = x^{-2d_Y} f_Y^*(x), \tag{11}$$

with $d_Y \in (-1/2, 1/2)$.

To present our theoretical results, we require the following definition.

Definition 1. *For $\alpha \in (0, \pi]$, $\beta > 0$ and $0 < \mu < \infty$, $\mathcal{F}_0(\alpha, \beta, \mu)$ is the set of functions g defined on $[-\pi, \pi]$ satisfying $\int_{-\pi}^{\pi} |g(x)| dx \le \mu$ and for all $x \in [-\alpha, \alpha]$,*

$$|g(x)| \le \mu |x|^\beta. \tag{12}$$

We also require the following assumption, which was made in Robinson (1995b) as well.

(A1) $\{\epsilon_t\}$ is a martingale difference sequence such that for all t, $\mathbb{E}[\epsilon_t^4] := \mu_4 < \infty$ and $\mathbb{E}[\epsilon_t^2 \mid \epsilon_s, s < t] = 1$ almost surely.

Theorem 1. *Let $\{Y_t\}$ have a moving average representation representation (10) with respect to a white noise $\{\epsilon_t\}$ which satisfies (**A1**) and such that the function $a(x) = \sum_{j=0}^{\infty} a_j \mathrm{e}^{\mathrm{i}jx}$ can be expressed as $a(x) = x^{-d_Y} a^*(x)$, where $(a^*(0)^{-1} a^*(x) - 1) \in \mathcal{F}_0(\alpha, \beta, \mu)$ for some $\beta > 2d_0$, $\alpha > 0$ and $\mu > 0$. Assume that $d_0 \in (0, .75)$. If m is a non decreasing sequence of integers such that*

$$\lim_{n\to\infty} (m^{-4d_0-1} n^{4d_0} + n^{-2\beta} m^{2\beta+1} \log^2(m)) = 0, \tag{13}$$

then $m^{1/2}(\hat{d}_{LWN} - d_0)$ is asymptotically Gaussian with zero mean and variance $(1+2d_0)^2/(16{d_0}^2)$.

Thus, if $\beta = 2$ (as is most commonly assumed) and we use $m = n^{4/5-2\epsilon}$ for some small ϵ, then $\hat{d}_{LWN}$ is $n^{2/5-\epsilon}$-consistent, i.e., the same rate of convergence enjoyed by Robinson's (1995b) Gaussian semiparametric estimator in the linear case. The first term in (13) imposes a lower bound on the allowable value of m, requiring that m tend to ∞ faster than $n^{4d_0/(4d_0+1)}$. Thus, for example, if $d_0 = .4$ then m must tend to ∞ faster than $n^{8/13} \approx n^{.62}$ in order for Theorem 1 to be valid.

Note that the asymptotic variance of $\hat{d}_{LWN}$ in Theorem 1 depends only on d_0, and is a decreasing function of d_0. Unfortunately, unless the noise to signal ratio (nsr) is quite small, this asymptotic variance may not accurately reflect the actual variance, even in the relatively large sample sizes considered in this paper. An alternative approach is to construct a finite-sample approximation to the variance. Examination of the proofs in Hurvich, Moulines and Soulier (2002) suggests that we may approximate $Var(\hat{d}_{LWN})$ by M_{11}^{-1}, that is, the $(1,1)$ entry of the inverse of the matrix M, where M is the 2×2 matrix with entries given by

$$
\begin{aligned}
M_{11} &= 4\sum_{k=1}^{m}\left(\frac{\log x_k\, x_k^{-2d_0}}{x_k^{-2d_0}+b_{1,0}^{-1}}\right)^2 - \frac{4}{m}\left(\sum_{k=1}^{m}\frac{\log x_k\, x_k^{-2d_0}}{x_k^{-2d_0}+b_{1,0}^{-1}}\right)^2 \\
M_{12} &= -2\sum_{k=1}^{m}\frac{\log x_k\, x_k^{-2d_0}}{(x_k^{-2d_0}+b_{1,0}^{-1})^2} + \left(\frac{2}{m}\sum_{k=1}^{m}\frac{\log x_k\, x_k^{-2d_0}}{x_k^{-2d_0}+b_{1,0}^{-1}}\right)\left(\sum_{j=1}^{m}\frac{1}{x_j^{-2d_0}+b_{1,0}^{-1}}\right) \\
M_{21} &= M_{12} \\
M_{22} &= \sum_{k=1}^{m}\frac{1}{(x_k^{-2d_0}+b_{1,0}^{-1})^2} - \frac{1}{m}\left(\sum_{k=1}^{m}\frac{1}{x_k^{-2d_0}+b_{1,0}^{-1}}\right)^2 , \qquad (14)
\end{aligned}
$$

where $b_{1,0}$ is the signal to noise ratio, $b_{1,0} = 2\pi f_Y^*(0)/\sigma_Z^2$. The use of M_{11}^{-1} is not feasible in practice, since d_0 and $b_{1,0}$ are not known. We can, however, use the feasible version $\hat{M}_{11}^{-1}$ where d_0 and $b_{1,0}$ are replaced by $\hat{d}_{LWN}$ and $\hat{b}_1$ in the formulas above.

In the next section, we compare the performance of $\widehat{d}_{LWN}$ relative to that of $\widehat{d}_{GPH}$ and $\widehat{d}_{LP-GPH}$ and assess the accuracy of the asymptotic and finite-sample expressions for $Var(\widehat{d}_{LWN})$ using simulation.

4 SIMULATION RESULTS

4.1 Assessment of Empirical Bias and Variance for $\widehat{d}_{LWN}$

We simulated logarithms of squared LMSV processes by first simulating Gaussian ARFIMA(p,d,q) data. The PACF method of Hosking (1984) was used to generate data from a Gaussian ARFIMA$(0,d,0)$ process. An ARMA(p,q) filter was then applied to give ARFIMA(p,d,q) data. An independent sequence of logarithms of squared standard normal random variates was added to the ARFIMA data to produce a series of logarithms of a squared LMSV-ARFIMA(p,d,q) process. One thousand realizations were generated for each value of $n = (1000, 5000, 10000)$, and for each of two values of the noise to signal ratio, $nsr = b_{1,0}^{-1}$. Since we take the $\{e_t\}$ to be standard normal, we have $\sigma_Z^2 = \pi^2/2$. The values $nsr = 5$ and $nsr = 10$ were chosen to correspond to the large nsr values observed in other empirical studies of LMSV models in finance (e.g., Breidt, Crato, and de Lima, 1998) and to see how the estimates of d are influenced by nsr in practice. For each realization, the $\widehat{d}_{GPH}$, $\widehat{d}_{LP-GPH}$ and $\widehat{d}_{LWN}$ estimators were evaluated for $m = ([n^{.4}], [n^{.5}], [n^{.6}], [n^{.7}], [n^{.8}])$. We investigated the LMSV-ARFIMA$(0, d, 0)$ model for values of $d = 0.3, 0.4, 0.45, 0.49$. These values were chosen based on previous findings of relatively strong persistence in financial time series (e.g. Lobato and Savin, 1999; Ray and Tsay, 2000). We also investigated the influence of ARMA components on the estimates by considering three LMSV-ARFIMA models having nonzero ARMA terms, that of an LMSV-ARFIMA$(1, d, 0)$ model with $d = 0.4$ and $\phi = 0.5, 0.8$ where ϕ is the autoregressive parameter in the ARFIMA$(1, d, 0)$ model, that is, $(1-B)^d(1-\phi B)y_t = \eta_t$ with $\{\eta_t\}$ $i.i.d$ normal random variates having standard deviation such that the specified nsr is obtained, and that of an LMSV-ARFIMA$(0, d, 1)$ model with $d = 0.4$ and $\theta = -0.8$, where θ is the moving-average parameter in the ARFIMA$(0, d, 1)$ model, that is, $(1-B)^d y_t = (1-\theta B)\eta_t$. The $\widehat{d}_{LWN}$ estimator was obtained by numerical optimization of (7) as a function of d and b_1. The value of d was constrained to lie in the range $[0.01, .75]$, while $\log(b_1)$ was constrained to the region $[-8, 20]$. The IMSL function DBCONF with default control parameters was used for optimization. The initial value used in computing $\widehat{d}_{LWN}$ for a given m was the $\widehat{d}_{GPH}$ estimator based on the same value of m. To find solutions to the FOC when the global optimum was obtained at a boundary point, we divided Θ into 16 equal-sized, non-overlapping rectangular regions. For each of these regions, (7) was optimized using DBCONF with starting value given by the midpoint of the region. Any interior solutions obtained by DBCONF were assumed to be solutions to the FOC.

Tables 1 and 2 provide representative results for the LMSV-ARFIMA$(0, d, 0)$ model for the cases $d = 0.3$ and $d = 0.4$, while Table 3 presents results for the LMSV-ARFIMA$(1, 0.4, 0)$ model with $\phi = 0.8$. Figures 1-3 present these results graphically in the form of box-plots, for the $nsr = 5$ case. Complete simulation results are available from the authors upon request.

We start by discussing the results for the LMSV-ARFIMA$(0, d, 0)$ processes. Overall, in most situations studied, $\widehat{d}_{LWN}$ has a smaller root mean squared error (RMSE) than either $\widehat{d}_{GPH}$ or $\widehat{d}_{LP-GPH}$. As m increases for given values of n, nsr and d, the RMSE for $\widehat{d}_{LWN}$ typically decreases, while the RMSE for $\widehat{d}_{GPH}$ and $\widehat{d}_{LP-GPH}$ is typically a convex function of m. The minimum RMSE with respect to m for a given situation is typically smaller for $\widehat{d}_{LWN}$ than for $\widehat{d}_{GPH}$ or $\widehat{d}_{LP-GPH}$.

The bias of $\widehat{d}_{LWN}$ is uniformly small, while the biases of $\widehat{d}_{GPH}$ and $\widehat{d}_{LP-GPH}$ become increasingly negative as either m or nsr is increased. This is in agreement with the theoretical results of Deo and Hurvich (2001). Even for samples of size $n = 10000$, the bias of $\widehat{d}_{GPH}$ may be quite severe. For example, for the LMSV-ARFIMA$(0, 0.49, 0)$ process with $n = 10000$, $m = [n^{.8}]$, $nsr = 10$, the bias in $\widehat{d}_{GPH}$ is -0.287, rendering the estimate nearly useless. The bias in $\widehat{d}_{LP-GPH}$, although smaller, is still -0.169.

The standard errors of both $\widehat{d}_{GPH}$ and $\widehat{d}_{LWN}$ decrease as m or n is increased, holding everything else fixed. Consistent with theory, the standard error of $\widehat{d}_{GPH}$ is often smaller than that of the corresponding $\widehat{d}_{LWN}$. For a given n, m, d, the standard error for $\widehat{d}_{GPH}$ is insensitive to nsr while the standard error for $\widehat{d}_{LWN}$ increases as nsr increases. Thus, for large nsr, the standard error for $\widehat{d}_{LWN}$ can become dramatically larger than the standard error for $\widehat{d}_{GPH}$ (except when m is small). However, this inflation in standard error for $\widehat{d}_{LWN}$ is usually not enough to offset the inflation in bias for $\widehat{d}_{GPH}$, so that $\widehat{d}_{LWN}$ typically has the smaller RMSE. The box-plots illustrate very nicely the trade-off between bias and variance, clearly showing the superiority of $\widehat{d}_{LWN}$ when m is large.

As d is increased, holding everything else fixed, the standard error for $\widehat{d}_{LWN}$ goes down, while that for $\widehat{d}_{GPH}$ remains stable. Furthermore, as d is increased, the bias for $\widehat{d}_{LWN}$ remains stable, while negative bias for $\widehat{d}_{GPH}$ becomes more severe. These findings are consistent with the theoretical results of Theorem 1 for $\widehat{d}_{LWN}$ and those of Deo and Hurvich (2001) for $\widehat{d}_{GPH}$, showing strong superiority of $\widehat{d}_{LWN}$ to $\widehat{d}_{GPH}$ in terms of RMSE when d is large.

For the LMSV-ARFIMA$(1, d, 0)$ model (Table 3), $\widehat{d}_{GPH}$ appears less biased than it was when the autoregressive parameter was absent. This can be explained by noting that the presence of the autoregressive parameter tends to increase the expected value of $\widehat{d}_{GPH}$, and thereby results in a less negatively biased estimator. Nevertheless, in almost all situations considered in Table 3, $\widehat{d}_{LWN}$ has a smaller RMSE than $\widehat{d}_{GPH}$. This is true despite the strong positive short-range correlation induced by the autoregressive parameter $\phi = 0.8$. Similar results were found for the other ARMA component models considered.

Overall, our simulation results suggest that $\widehat{d}_{LWN}$ is preferable to $\widehat{d}_{GPH}$ since the latter estimator may suffer from a very strong negative bias due to the noise term in the LMSV model, while the former estimator suffers from no such bias.

4.2 Assessment of Approximate Variance Expression for $\widehat{d}_{LWN}$

According to the asymptotic theory given in Theorem 1, the variance of $\widehat{d}_{LWN}$ does not depend on nsr. Our simulations appear to be at least somewhat at odds with that theory, as seen from the above discussion. The first two rows of each table in Tables 4-6 give the average and median standard errors across replications obtained using the asymptotic expression $(1 + 2d)/(4dm^{1/2})$ evaluated using $\hat{d}_{LWN}$, while the third row gives the value computed using the true value of d. The mean values are much larger than the values obtained using the asymptotic expression with the true value of d, especially when $n = 1000$ and m is small. We attribute this to a few outlying values of $\widehat{d}_{LWN}$, as can be seen from the box-plots. Although the median value for the standard errors based on estimated d values is close to that based on the true value of d, the values typically do not match closely the standard errors observed in the simulations, which increase as nsr increases (see row seven of each table in Tables 4-6). Thus, for the sample sizes typically encountered in practice, the asymptotic expression does not seem to provide a reliable approximation to the actual standard error of $\widehat{d}_{LWN}$.

We also explored whether $M_{1,1}^{-1}$ provides a better approximation, where the entries of M are given by (14). Note that $M_{1,1}^{-1}$ depends not only on d, but also on b_1. A feasible version can be computed by substituting estimates of the unknown parameters in the expression for $M_{1,1}^{-1}$. The fourth and fifth rows of each of the tables shown in Tables 4-6 give the mean and median values of the standard errors computing using (14) with estimated parameter values, while the sixth row gives the value obtained when the true parameter values are used. Again we see that the mean value of the standard errors computed using estimated parameter values can be extremely large, in particular when $n = 1000$ and also when n is larger but m is small. This is due to large variations in the estimated nsr values used in the computation of (14). Large sample sizes and large values of m, i.e. $m = [n^{.7}], [n^{.8}]$ are needed to accurately estimate nsr. When this is the case, both the mean and median values are very close to the values observed in the simulations (shown in row seven of each table).

We also compared the empirical 90% and 95% coverage obtained for Gaussian-based confidence intervals on d constructed using the estimated standard errors based on the asymptotic formula, the formula of (14) with estimated parameters, and the formula of (14) with known parameters. For completeness, these coverages were compared to those obtained from the GPH estimator with variance $\pi^2/(24S_{ww})$. Tables 7-9 show the results of these comparisons. The values in parentheses denote the median lengths of the constructed intervals. The LWN-based confidence intervals provide close to nominal coverage when d is estimated using a large number of Fourier frequencies and the interval is constructed using the finite-sample variance approximation based on (14) with estimated parameters. The GPH-based confidence intervals, in contrast, provide very poor coverage. These results indicate that reliable determination of the degree of persistence in an LMSV-ARFIMA model can be made using the Local Whittle method.

5 ANALYSIS OF CURRENCY EXCHANGE RATES

We consider a data set previously analyzed in Li, Deo and Hurvich (2001) consisting of daily returns on the Deutsche Mark / US Dollar exchange rate, from Jan 2 1985 to May 12 1998, $n = 3485$. Several of the returns r_t were zero. Adjusted log squared returns were constructed, using the method of Fuller (1996), computing

$$X_t = log(r_t^2 + \kappa) - \frac{\kappa}{r_t^2 + \kappa} \ ,$$

where $\kappa = \tau(n^{-1} \sum r_t^2)$ and $\tau = 0.02$. Time series plots of the returns series and volatility series are shown in Figure 4, while Figure 5 shows the sample autocorrelation function for the volatility series. The volatilities of DM/$ exchange rates exhibit the apparently changing mean levels characteristic of long-range dependent processes. The sample ACF values, although small, are positive even at large lags.

Table 10 presents the $\widehat{d}_{GPH}$ and $\widehat{d}_{LWN}$ estimators for various values of m. The $\widehat{d}_{GPH}$ values decrease as m increases, a pattern which is consistent with the theoretical fact that the bias in $\widehat{d}_{GPH}$ becomes strongly negative for large values of m. On the other hand, the $\widehat{d}_{LWN}$ values increase with m, reaching 0.556 for $m = [n^{0.8}]$. For each given value of m, except for $m = [n^{0.5}]$, $\widehat{d}_{LWN}$ exceeds the corresponding value of $\widehat{d}_{GPH}$.

To gain some insight on the proper choice of m for $\widehat{d}_{LWN}$ in this exchange rate dataset, we carried out some additional simulations, using a fully parametric LMSV-ARFIMA$(1, d, 0)$ model fitted to the periodogram of $\{X_t\}$ at all Fourier frequencies using the Whittle likelihood. This model was found to fit well according to diagnostic tests performed in Li, Deo and Hurvich (2000). The fitted model has spectral density

$$f_X(x) = f_Y(x) + f_Z(x) = \frac{|2\ \sin(x/2)|^{-2\widehat{d}}\, \widehat{\sigma}_\eta^2}{2\pi|1 - \widehat{\phi}\ \exp(-ix)|^2} + \widehat{\sigma}_Z^2/(2\pi) \ ,$$

with $\widehat{d} = 0.4086$, $\widehat{\phi} = -0.1556$, $\widehat{\sigma}_\eta = 0.8452$, and $\widehat{\sigma}_Z = 2.4652$. The simulations were done by generating data from this model, using a Gaussian $\{Y_t\}$ process and a noise process given by

$Z_t = \log e_t^2$ where e_t are *i.i.d.* with a $t(3)$ distribution. The value of the degrees of freedom for e_t was chosen so that the standard error for Z_t nearly matches the estimated value, $\widehat{\sigma}_Z = 2.4652$. Note that the asymptotic results of Theorem 1 are not dependent on a Gaussian assumption for the multiplicative noise in the LMSV-ARFIMA model.

Table 11 gives the bias and RMSE of $\widehat{d}_{LWN}$ based on one hundred simulated realizations. It is seen that the bias is stable with respect to m, and is quite small, while the RMSE decreases uniformly in m. Overall, $m = [n^{0.8}]$ would appear to be the best choice for this data set, leading to $\widehat{d}_{LWN} = 0.556$. It is notable that this value is so large that it lies outside the range of d values corresponding to a weakly stationary process. The estimated nsr for this series is 23.89. Using (14) with $\hat{d} = 0.556$ and $\hat{b}_1 = 1/23.89$, we obtain an estimated standard error of 0.095. A corresponding confidence interval for d includes values in both the stationary and non-stationary range.

6 SUMMARY

We have investigated the efficacy of a modified Local Whittle method for semiparametrically estimating the degree of long memory in an LMSV process. Our simulation study has focused on the weakly stationary case, $d < 0.5$. The LWN estimator clearly dominates existing methods, such as GPH and the local polynomial GPH method of Andrews and Guggenberger (2000), in the presence of noisy observations. Reliable estimates of standard errors can be obtained using a finite-sample approximation to the asymptotic variance of the modified Local Whittle estimator.

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Table 1: Bias, standard error (SE), and root-mean-squared error (RMSE) for semi-parametric estimators of d in the LMSV-ARFIMA(0,0.30,0) model

$n = 1000$

		$m = [n^{.4}]$			$m = [n^{.5}]$			$m = [n^{.6}]$			$m = [n$	
		GPH	$LP-GPH$	LWN	GPH	$LP-GPH$	LWN	GPH	$LP-GPH$	LWN	GPH	$LP-G$
$nsr = 5$	Bias	-0.112	-0.069	0.028	-0.131	-0.094	0.033	-0.156	-0.109	0.042	-0.177	-0.134
	SE	0.213	0.598	0.285	0.137	0.320	0.265	0.088	0.194	0.246	0.063	0.126
	RMSE	0.241	0.602	0.286	0.190	0.334	0.267	0.179	0.222	0.249	0.188	0.184
$nsr = 10$	Bias	-0.165	-0.122	-0.018	-0.183	-0.145	-0.016	-0.205	-0.163	0.018	-0.221	-0.187
	SE	0.214	0.620	0.283	0.137	0.323	0.277	0.088	0.194	0.273	0.064	0.126
	RMSE	0.270	0.632	0.284	0.229	0.354	0.278	0.223	0.254	0.274	0.230	0.225

$n = 5000$

		$m = [n^{.4}]$			$m = [n^{.5}]$			$m = [n^{.6}]$			$m = [n$	
		GPH	$LP-GPH$	LWN	GPH	$LP-GPH$	LWN	GPH	$LP-GPH$	LWN	GPH	$LP-G$
$nsr = 5$	Bias	-0.067	-0.030	0.050	-0.095	-0.054	0.054	-0.122	-0.080	0.019	-0.154	-0.107
	SE	0.139	0.344	0.235	0.084	0.181	0.199	0.053	0.105	0.151	0.033	0.065
	RMSE	0.154	0.345	0.240	0.127	0.189	0.206	0.133	0.132	0.152	0.158	0.125
$nsr = 10$	Bias	-0.109	-0.066	0.030	-0.142	-0.093	0.042	-0.170	-0.126	0.018	-0.199	-0.156
	SE	0.140	0.338	0.255	0.084	0.183	0.232	0.052	0.106	0.196	0.033	0.065
	RMSE	0.177	0.344	0.257	0.165	0.205	0.236	0.178	0.165	0.197	0.202	0.169

$n = 10000$

		$m = [n^{.4}]$			$m = [n^{.5}]$			$m = [n^{.6}]$			$m = [n$	
		GPH	$LP-GPH$	LWN	GPH	$LP-GPH$	LWN	GPH	$LP-GPH$	LWN	GPH	$LP-G$
$nsr = 5$	Bias	-0.050	-0.029	0.066	-0.078	-0.042	0.044	-0.110	-0.066	0.012	-0.145	-0.097
	SE	0.120	0.278	0.204	0.070	0.147	0.163	0.042	0.085	0.117	0.027	0.050
	RMSE	0.130	0.279	0.215	0.105	0.153	0.169	0.118	0.108	0.118	0.147	0.109
$nsr = 10$	Bias	-0.091	-0.060	0.041	-0.123	-0.081	0.035	-0.158	-0.110	0.013	-0.191	-0.144
	SE	0.121	0.274	0.222	0.071	0.147	0.190	0.043	0.086	0.149	0.027	0.051
	RMSE	0.151	0.281	0.226	0.142	0.168	0.193	0.164	0.139	0.150	0.193	0.153

Table 2: Bias, standard error (SE), and root-mean-squared error (RMSE) for semi-parametric estimators of d in the LMSV-ARFIMA(0,0.40,0) model

$n = 1000$

		$m=[n^{.4}]$			$m=[n^{.5}]$			$m=[n^{.6}]$			$m=[n$	
		GPH	$LP-GPH$	LWN	GPH	$LP-GPH$	LWN	GPH	$LP-GPH$	LWN	GPH	$LP-G$
$nsr=5$	Bias	-0.091	-0.040	0.039	-0.120	-0.068	0.032	-0.160	-0.088	0.023	-0.200	-0.123
	SE	0.218	0.616	0.257	0.140	0.329	0.227	0.090	0.199	0.204	0.064	0.128
	RMSE	0.236	0.617	0.260	0.184	0.336	0.229	0.184	0.218	0.205	0.210	0.177
$nsr=10$	Bias	-0.146	-0.078	-0.006	-0.183	-0.114	0.004	-0.225	-0.146	0.006	-0.260	-0.189
	SE	0.220	0.615	0.274	0.140	0.329	0.254	0.089	0.198	0.241	0.064	0.128
	RMSE	0.264	0.620	0.274	0.230	0.348	0.254	0.242	0.247	0.241	0.268	0.228

$n = 5000$

		$m=[n^{.4}]$			$m=[n^{.5}]$			$m=[n^{.6}]$			$m=[n$	
		GPH	$LP-GPH$	LWN	GPH	$LP-GPH$	LWN	GPH	$LP-GPH$	LWN	GPH	$LP-G$
$nsr=5$	Bias	-0.041	-0.009	0.051	-0.071	-0.029	0.038	-0.111	-0.052	0.006	-0.163	-0.088
	SE	0.142	0.354	0.187	0.084	0.187	0.149	0.053	0.106	0.111	0.033	0.065
	RMSE	0.148	0.354	0.193	0.110	0.190	0.154	0.123	0.118	0.112	0.167	0.109
$nsr=10$	Bias	-0.076	-0.029	0.036	-0.120	-0.058	0.030	-0.169	-0.096	0.004	-0.223	-0.143
	SE	0.141	0.354	0.204	0.083	0.186	0.175	0.053	0.105	0.142	0.033	0.066
	RMSE	0.160	0.355	0.207	0.146	0.194	0.177	0.177	0.142	0.142	0.225	0.157

$n = 10000$

		$m=[n^{.4}]$			$m=[n^{.5}]$			$m=[n^{.6}]$			$m=[n$	
		GPH	$LP-GPH$	LWN	GPH	$LP-GPH$	LWN	GPH	$LP-GPH$	LWN	GPH	$LP-G$
$nsr=5$	Bias	-0.021	-0.001	0.057	-0.052	-0.012	0.035	-0.093	-0.037	0.009	-0.148	-0.074
	SE	0.117	0.283	0.155	0.069	0.145	0.123	0.042	0.083	0.088	0.027	0.049
	RMSE	0.119	0.283	0.165	0.086	0.145	0.127	0.102	0.091	0.088	0.150	0.089
$nsr=10$	Bias	-0.049	-0.018	0.049	-0.093	-0.037	0.028	-0.147	-0.073	0.008	-0.207	-0.124
	SE	0.119	0.282	0.168	0.070	0.146	0.141	0.042	0.086	0.107	0.027	0.050
	RMSE	0.128	0.283	0.175	0.117	0.151	0.144	0.153	0.113	0.107	0.209	0.134

Table 3: Bias, standard error (SE), and root-mean-squared error (RMSE) for semi-parametric estimators of d in the LMSV-ARFIMA(1,0.40,0) model with $\phi = 0.8$

$n = 1000$

		$m = [n^{.4}]$			$m = [n^{.5}]$			$m = [n^{.6}]$			$m = [n$	
		GPH	$LP-GPH$	LWN	GPH	$LP-GPH$	LWN	GPH	$LP-GPH$	LWN	GPH	$LP-G$
$nsr = 5$	Bias	-0.068	-0.049	0.043	-0.085	-0.057	0.036	-0.114	-0.071	0.023	-0.173	-0.07
	SE	0.228	0.640	0.251	0.143	0.334	0.208	0.097	0.210	0.177	0.066	0.135
	RMSE	0.238	0.642	0.255	0.166	0.339	0.211	0.149	0.222	0.179	0.185	0.155
$nsr = 10$	Bias	-0.132	-0.087	0.004	-0.159	-0.111	0.009	-0.195	-0.135	-0.003	-0.241	-0.15
	SE	0.227	0.623	0.267	0.143	0.345	0.240	0.093	0.200	0.221	0.065	0.132
	RMSE	0.263	0.629	0.267	0.213	0.362	0.240	0.216	0.241	0.221	0.249	0.206

$n = 5000$

		$m = [n^{.4}]$			$m = [n^{.5}]$			$m = [n^{.6}]$			$m = [n$	
		GPH	$LP-GPH$	LWN	GPH	$LP-GPH$	LWN	GPH	$LP-GPH$	LWN	GPH	$LP-G$
$nsr = 5$	Bias	-0.040	0.001	0.059	-0.060	-0.025	0.039	-0.079	-0.049	-0.007	-0.122	-0.06
	SE	0.141	0.351	0.192	0.086	0.188	0.146	0.053	0.107	0.098	0.037	0.068
	RMSE	0.147	0.351	0.201	0.105	0.190	0.151	0.095	0.118	0.098	0.127	0.093
$nsr = 10$	Bias	-0.066	-0.025	0.040	-0.107	-0.042	0.029	-0.148	-0.085	-0.008	-0.200	-0.12
	SE	0.141	0.322	0.202	0.085	0.186	0.168	0.054	0.108	0.128	0.036	0.067
	RMSE	0.156	0.323	0.206	0.137	0.191	0.171	0.158	0.138	0.128	0.203	0.141

$n = 10000$

		$m = [n^{.4}]$			$m = [n^{.5}]$			$m = [n^{.6}]$			$m = [n$	
		GPH	$LP-GPH$	LWN	GPH	$LP-GPH$	LWN	GPH	$LP-GPH$	LWN	GPH	$LP-G$
$nsr = 5$	Bias	-0.027	0.001	0.056	-0.052	-0.020	0.030	-0.071	-0.044	-0.009	-0.106	-0.06
	SE	0.119	0.279	0.159	0.069	0.149	0.118	0.042	0.082	0.076	0.027	0.052
	RMSE	0.122	0.279	0.169	0.086	0.150	0.122	0.083	0.093	0.076	0.109	0.080
$nsr = 10$	Bias	-0.049	-0.005	0.042	-0.088	-0.034	0.023	-0.133	-0.073	-0.007	-0.184	-0.11
	SE	0.122	0.271	0.172	0.071	0.147	0.134	0.043	0.086	0.102	0.029	0.051
	RMSE	0.131	0.271	0.177	0.113	0.150	0.136	0.140	0.113	0.103	0.186	0.126

Table 4: Comparison of simulation standard errors for LWN estimator of LMSV-ARFIMA(0,0.30,0) to standard errors obtained using the simple asymptotic formula and the Hessian formula with both estimated and true parameter values*

$n = 1000$

			$m = [n^{.4}]$	$m = [n^{.5}]$	$m = [n^{.6}]$	$m = [n^{.7}]$	$m = [n^{.8}]$
	Asymptotic	mean	1.897	0.828	0.341	0.200	0.125
		med	0.368	0.264	0.188	0.128	0.089
		true	0.344	0.239	0.168	0.119	0.084
$nsr = 5$	Hessian	mean	614.705	87.342	99.937	31.131	10.413
		med	0.963	0.563	0.389	0.296	0.227
		true	1.169	0.666	0.427	0.298	0.219
	Simulation		0.285	0.265	0.246	0.219	0.191
	Asymptotic	mean	2.363	1.289	0.595	0.338	0.218
		med	0.469	0.340	0.220	0.142	0.094
		true	0.344	0.239	0.168	0.119	0.084
$nsr = 10$	Hessian	mean	1003.785	251.093	318.519	140.068	76.771
		med	1.223	0.744	0.484	0.377	0.305
		true	1.553	0.905	0.593	0.423	0.316
	Simulation		0.283	0.277	0.273	0.259	0.238

$n = 5000$

			$m = [n^{.4}]$	$m = [n^{.5}]$	$m = [n^{.6}]$	$m = [n^{.7}]$	$m = [n^{.8}]$
	Asymptotic	mean	0.966	0.353	0.196	0.134	0.091
		med	0.337	0.238	0.171	0.119	0.084
		true	0.344	0.239	0.168	0.119	0.084
$nsr = 5$	Hessian	mean	22.634	0.534	0.190	0.131	0.093
		med	0.459	0.274	0.184	0.128	0.091
		true	0.521	0.295	0.185	0.124	0.089
	Simulation		0.235	0.199	0.151	0.116	0.091
	Asymptotic	mean	1.259	0.490	0.249	0.152	0.101
		med	0.372	0.257	0.175	0.121	0.085
		true	0.344	0.239	0.168	0.119	0.084
$nsr = 10$	Hessian	mean	143.806	10.770	6.108	0.168	0.128
		med	0.519	0.312	0.219	0.165	0.124
		true	0.621	0.362	0.233	0.162	0.119
	Simulation		0.255	0.232	0.196	0.161	0.125

$n = 10000$

			$m = [n^{.4}]$	$m = [n^{.5}]$	$m = [n^{.6}]$	$m = [n^{.7}]$	$m = [n^{.8}]$
	Asymptotic	mean	0.621	0.267	0.182	0.127	0.087
		med	0.324	0.239	0.169	0.119	0.084
		true	0.344	0.239	0.168	0.119	0.084
$nsr = 5$	Hessian	mean	9.447	0.207	0.137	0.092	0.064
		med	0.345	0.210	0.135	0.090	0.064
		true	0.400	0.219	0.135	0.089	0.063
	Simulation		0.204	0.163	0.117	0.085	0.062
	Asymptotic	mean	0.856	0.332	0.200	0.135	0.090
		med	0.345	0.248	0.168	0.120	0.084
		true	0.344	0.239	0.168	0.119	0.084
$nsr = 10$	Hessian	mean	17.546	0.405	0.161	0.118	0.086
		med	0.378	0.239	0.163	0.115	0.084
		true	0.460	0.260	0.165	0.113	0.083
	Simulation		0.222	0.190	0.149	0.113	0.083

* **Asymptotic**: standard errors computed using the asymptotic formula of Theorem 1.
Hessian: Standard errors computed using the finite-sample approximation to the variance-covariance matrix of $\hat{d}$ and $\hat{b}_1$.
Simulation: Standard errors obtained from simulations.

Table 5: Comparison of simulation standard errors for LWN estimator of LMSV-ARFIMA(0,0.40,0) to standard errors obtained using the simple asymptotic formula and the Hessian formula with both estimated and true parameter values*

$n = 1000$

			$m = [n^{.4}]$	$m = [n^{.5}]$	$m = [n^{.6}]$	$m = [n^{.7}]$	$m = [n^{.8}]$
	Asymptotic	mean	0.920	0.378	0.178	0.117	0.080
		med	0.275	0.201	0.146	0.102	0.072
		true	0.290	0.202	0.142	0.101	0.071
$nsr = 5$	Hessian	mean	151.019	5.493	0.433	0.223	0.169
		med	0.619	0.414	0.288	0.212	0.162
		true	0.757	0.439	0.288	0.207	0.157
	Simulation		0.257	0.227	0.204	0.179	0.153
	Asymptotic	mean	1.312	0.529	0.248	0.142	0.098
		med	0.302	0.215	0.151	0.103	0.072
		true	0.290	0.202	0.142	0.101	0.071
$nsr = 10$	Hessian	mean	65.478	11.211	1.912	0.367	0.253
		med	0.721	0.459	0.334	0.267	0.218
		true	0.905	0.541	0.366	0.271	0.210
	Simulation		0.274	0.254	0.241	0.216	0.197

$n = 5000$

			$m = [n^{.4}]$	$m = [n^{.5}]$	$m = [n^{.6}]$	$m = [n^{.7}]$	$m = [n^{.8}]$
	Asymptotic	mean	0.406	0.210	0.147	0.104	0.072
		med	0.276	0.199	0.143	0.100	0.070
		true	0.290	0.202	0.142	0.101	0.071
$nsr = 5$	Hessian	mean	1.073	0.197	0.134	0.092	0.067
		med	0.333	0.204	0.132	0.091	0.066
		true	0.365	0.208	0.131	0.090	0.066
	Simulation		0.187	0.149	0.111	0.091	0.068
	Asymptotic	mean	0.489	0.229	0.153	0.107	0.073
		med	0.283	0.203	0.143	0.101	0.070
		true	0.290	0.202	0.142	0.101	0.071
$nsr = 10$	Hessian	mean	1.498	0.213	0.151	0.113	0.085
		med	0.350	0.220	0.153	0.112	0.084
		true	0.398	0.234	0.153	0.109	0.083
	Simulation		0.204	0.175	0.142	0.113	0.087

$n = 10000$

			$m = [n^{.4}]$	$m = [n^{.5}]$	$m = [n^{.6}]$	$m = [n^{.7}]$	$m = [n^{.8}]$
	Asymptotic	mean	0.321	0.201	0.144	0.102	0.071
		med	0.276	0.197	0.141	0.100	0.070
		true	0.290	0.202	0.142	0.101	0.071
$nsr = 5$	Hessian	mean	0.258	0.148	0.099	0.066	0.047
		med	0.266	0.156	0.098	0.066	0.047
		true	0.289	0.158	0.098	0.065	0.047
	Simulation		0.155	0.123	0.088	0.065	0.047
	Asymptotic	mean	0.327	0.207	0.146	0.103	0.071
		med	0.280	0.202	0.141	0.100	0.070
		true	0.290	0.202	0.142	0.101	0.071
$nsr = 10$	Hessian	mean	0.273	0.156	0.111	0.079	0.059
		med	0.274	0.169	0.112	0.078	0.059
		true	0.307	0.174	0.111	0.078	0.058
	Simulation		0.168	0.141	0.107	0.079	0.059

* **Asymptotic**: standard errors computed using the asymptotic formula of Theorem 1.
Hessian: Standard errors computed using the finite-sample approximation to the variance-covariance matrix of $\hat{d}$ and $\hat{b}_1$.
Simulation: Standard errors obtained from simulations.

Table 6: Comparison of simulation standard errors for LWN estimator of LMSV-ARFIMA(1,0.40,0) with $\phi = 0.8$ to standard errors obtained using the asymptotic formula and the Hessian formula with both estimated and true parameter values*

$n = 1000$

			$m = [n^{.4}]$	$m = [n^{.5}]$	$m = [n^{.6}]$	$m = [n^{.7}]$	$m = [n^{.8}]$
	Asymptotic	mean	0.906	0.322	0.162	0.107	0.069
		med	0.275	0.199	0.142	0.097	0.064
		true	0.290	0.202	0.142	0.101	0.071
$nsr = 5$	Hessian	mean	141.677	19.484	10.622	0.202	0.165
		med	0.605	0.384	0.257	0.192	0.156
		true	0.757	0.439	0.288	0.207	0.157
	Simulation		0.251	0.208	0.177	0.166	0.148
	Asymptotic	mean	1.171	0.483	0.213	0.131	0.084
		med	0.295	0.211	0.153	0.102	0.067
		true	0.290	0.202	0.142	0.101	0.071
$nsr = 10$	Hessian	mean	358.211	29.378	11.557	14.687	0.231
		med	0.684	0.440	0.316	0.248	0.212
		true	0.905	0.541	0.366	0.271	0.210
	Simulation		0.267	0.240	0.221	0.206	0.190

$n = 5000$

			$m = [n^{.4}]$	$m = [n^{.5}]$	$m = [n^{.6}]$	$m = [n^{.7}]$	$m = [n^{.8}]$
	Asymptotic	mean	0.451	0.205	0.148	0.104	0.067
		med	0.274	0.198	0.147	0.101	0.067
		true	0.290	0.202	0.142	0.101	0.071
$nsr = 5$	Hessian	mean	0.601	0.193	0.123	0.081	0.063
		med	0.323	0.200	0.122	0.081	0.062
		true	0.365	0.208	0.131	0.090	0.066
	Simulation		0.192	0.146	0.098	0.078	0.062
	Asymptotic	mean	0.509	0.227	0.153	0.107	0.070
		med	0.282	0.202	0.146	0.102	0.068
		true	0.290	0.202	0.142	0.101	0.071
$nsr = 10$	Hessian	mean	0.949	0.209	0.147	0.103	0.082
		med	0.342	0.216	0.145	0.102	0.080
		true	0.398	0.234	0.153	0.109	0.083
	Simulation		0.202	0.168	0.128	0.103	0.079

$n = 10000$

			$m = [n^{.4}]$	$m = [n^{.5}]$	$m = [n^{.6}]$	$m = [n^{.7}]$	$m = [n^{.8}]$
	Asymptotic	mean	0.339	0.202	0.147	0.105	0.068
		med	0.279	0.200	0.145	0.103	0.068
		true	0.290	0.202	0.142	0.101	0.071
$nsr = 5$	Hessian	mean	2.026	0.149	0.094	0.059	0.044
		med	0.265	0.155	0.093	0.059	0.044
		true	0.289	0.158	0.098	0.065	0.047
	Simulation		0.159	0.118	0.076	0.059	0.044
	Asymptotic	mean	0.419	0.212	0.149	0.105	0.070
		med	0.279	0.201	0.144	0.103	0.069
		true	0.290	0.202	0.142	0.101	0.071
$nsr = 10$	Hessian	mean	0.557	0.158	0.108	0.073	0.056
		med	0.272	0.166	0.107	0.072	0.056
		true	0.307	0.174	0.111	0.078	0.058
	Simulation		0.172	0.134	0.102	0.074	0.056

* **Asymptotic**: standard errors computed using the asymptotic formula of Theorem 1.
Hessian: Standard errors computed using the finite-sample approximation to the variance-covariance matrix of $\hat{d}$ and $\hat{b}_1$.
Simulation: Standard errors obtained from simulations.

Table 7: Coverage results for GPH and LWN estimators of LMSV-ARFIMA(0,0.30,0) model based on different standard error calculation methods. Values in parentheses denote median lengths of the computed confidence intervals.*

$n = 1000$

			$m = [n^{.4}]$	$m = [n^{.5}]$	$m = [n^{.6}]$	$m = [n^{.7}]$	$m = [n^{.8}]$
	GPH	90%	52.700 (0.721)	58.100 (0.452)	54.100 (0.297)	45.200 (0.203)	36.200 (0.142)
		95%	58.200 (0.863)	65.000 (0.540)	63.600 (0.355)	54.800 (0.243)	45.200 (0.170)
	LWN-A	90%	76.100 (1.211)	75.100 (0.867)	71.500 (0.618)	73.300 (0.421)	73.600 (0.292)
		95%	77.300 (1.443)	76.600 (1.034)	74.000 (0.737)	76.400 (0.502)	77.000 (0.348)
$nsr = 5$	LWN-H	90%	78.100 (3.169)	78.100 (1.853)	82.300 (1.279)	89.900 (0.974)	94.400 (0.747)
		95%	79.000 (3.776)	79.500 (2.207)	82.500 (1.524)	90.100 (1.161)	95.000 (0.891)
	LWN-IH	90%	00.000 (3.845)	100.000 (2.190)	100.000 (1.404)	100.000 (0.981)	93.300 (0.719)
		95%	00.000 (4.581)	100.000 (2.609)	100.000 (1.673)	100.000 (1.169)	94.300 (0.857)
	GPH	90%	46.200 (0.297)	48.200 (0.297)	38.400 (0.297)	32.900 (0.297)	27.800 (0.297)
		95%	51.300 (0.354)	55.100 (0.354)	47.000 (0.354)	41.400 (0.354)	36.200 (0.354)
	LWN-A	90%	79.500 (1.542)	77.100 (1.118)	70.600 (0.725)	69.900 (0.466)	69.500 (0.311)
		95%	80.800 (1.838)	77.900 (1.332)	72.900 (0.864)	71.900 (0.555)	71.800 (0.370)
$nsr = 10$	LWN-H	90%	80.800 (4.022)	78.500 (2.448)	77.400 (1.591)	82.800 (1.241)	86.700 (1.003)
		95%	82.300 (4.792)	80.300 (2.917)	78.200 (1.895)	83.300 (1.478)	87.100 (1.195)
	LWN-IH	90%	00.000 (5.111)	100.000 (2.978)	100.000 (1.951)	100.000 (1.390)	100.000 (1.038)
		95%	00.000 (6.089)	100.000 (3.548)	100.000 (2.325)	100.000 (1.657)	100.000 (1.237)

$n = 5000$

			$m = [n^{.4}]$	$m = [n^{.5}]$	$m = [n^{.6}]$	$m = [n^{.7}]$	$m = [n^{.8}]$
	GPH	90%	68.900 (0.721)	73.000 (0.452)	63.800 (0.297)	49.500 (0.203)	41.500 (0.142)
		95%	74.400 (0.859)	78.400 (0.538)	71.700 (0.354)	58.500 (0.242)	47.700 (0.170)
	LWN-A	90%	83.300 (1.109)	81.900 (0.784)	87.800 (0.563)	91.000 (0.391)	91.600 (0.275)
		95%	86.600 (1.322)	85.600 (0.935)	90.100 (0.671)	93.900 (0.466)	94.300 (0.328)
$nsr = 5$	LWN-H	90%	73.400 (1.509)	77.900 (0.900)	92.100 (0.607)	96.300 (0.421)	91.200 (0.299)
		95%	73.800 (1.798)	78.000 (1.072)	92.400 (0.723)	98.300 (0.501)	98.700 (0.357)
	LWN-IH	90%	100.000 (1.714)	100.000 (0.972)	94.900 (0.607)	94.800 (0.409)	88.400 (0.293)
		95%	100.000 (2.043)	100.000 (1.158)	97.000 (0.723)	97.100 (0.488)	93.700 (0.349)
	GPH	90%	62.300 (0.297)	58.900 (0.297)	46.700 (0.297)	37.200 (0.297)	30.400 (0.297)
		95%	68.500 (0.354)	65.800 (0.354)	54.900 (0.354)	45.300 (0.354)	39.800 (0.354)
	LWN-A	90%	80.900 (1.223)	78.900 (0.845)	81.300 (0.575)	83.300 (0.397)	83.400 (0.278)
		95%	82.700 (1.457)	80.800 (1.006)	84.800 (0.685)	85.700 (0.473)	86.800 (0.331)
$nsr = 10$	LWN-H	90%	72.500 (1.709)	72.700 (1.026)	81.900 (0.721)	91.100 (0.542)	93.900 (0.409)
		95%	72.900 (2.036)	73.500 (1.222)	82.300 (0.859)	91.500 (0.646)	97.100 (0.488)
	LWN-IH	90%	100.000 (2.042)	100.000 (1.192)	92.700 (0.768)	93.300 (0.534)	86.600 (0.393)
		95%	100.000 (2.433)	100.000 (1.421)	100.000 (0.915)	95.800 (0.636)	95.000 (0.468)

$n = 10000$

			$m = [n^{.4}]$	$m = [n^{.5}]$	$m = [n^{.6}]$	$m = [n^{.7}]$	$m = [n^{.8}]$
	GPH	90%	74.800 (0.721)	79.000 (0.452)	68.400 (0.297)	53.500 (0.203)	40.400 (0.142)
		95%	80.000 (0.863)	84.000 (0.540)	77.200 (0.355)	62.200 (0.243)	49.000 (0.170)
	LWN-A	90%	87.100 (1.065)	88.700 (0.788)	94.400 (0.557)	96.400 (0.391)	97.300 (0.275)
		95%	89.800 (1.268)	91.800 (0.939)	95.900 (0.664)	98.200 (0.465)	98.900 (0.328)
$nsr = 5$	LWN-H	90%	73.600 (1.134)	82.400 (0.691)	95.200 (0.445)	96.200 (0.297)	91.400 (0.209)
		95%	73.900 (1.351)	82.500 (0.824)	96.300 (0.530)	98.700 (0.354)	95.800 (0.249)
	LWN-IH	90%	100.000 (1.317)	93.700 (0.720)	95.100 (0.443)	90.500 (0.294)	90.400 (0.207)
		95%	100.000 (1.570)	96.500 (0.858)	96.900 (0.528)	96.600 (0.350)	94.800 (0.246)
	GPH	90%	67.200 (0.297)	64.500 (0.297)	49.900 (0.297)	39.200 (0.297)	32.900 (0.297)
		95%	73.300 (0.354)	71.700 (0.354)	59.800 (0.354)	46.000 (0.354)	40.000 (0.354)
	LWN-A	90%	86.400 (1.136)	85.300 (0.817)	89.300 (0.553)	91.600 (0.395)	91.400 (0.277)
		95%	88.800 (1.354)	88.500 (0.974)	92.500 (0.659)	93.800 (0.471)	94.800 (0.330)
$nsr = 10$	LWN-H	90%	71.800 (1.244)	76.600 (0.787)	87.300 (0.535)	95.400 (0.379)	92.800 (0.278)
		95%	72.500 (1.482)	76.800 (0.938)	87.500 (0.638)	96.900 (0.452)	96.800 (0.331)
	LWN-IH	90%	100.000 (1.513)	93.800 (0.854)	94.100 (0.543)	88.900 (0.373)	89.900 (0.271)
		95%	100.000 (1.802)	100.000 (1.017)	96.500 (0.647)	96.200 (0.445)	94.300 (0.323)

* GPH denotes empirical coverage percentages based on GPH estimates of d with standard errors computed using the finite-sample approximation to the theoretical GPH standard error. LWN-H denotes empirical coverage percentages based on LWN estimates of d with standard errors computed using the finite-sample Hessian-based approximation to the theoretical standard errors with estimated parameters. LWN-IH denotes empirical coverage percentages based on LWN estimates of d with standard errors computed using the finite-sample Hessian-based approximation to the theoretical standard errors with known values of the parameters.

Table 8: Coverage results for GPH and LWN estimators of LMSV-ARFIMA(0,0.40,0) model based on different standard error calculation methods. Values in parentheses denote median lengths of the computed confidence intervals.*

$n = 1000$

			$m=[n^{.4}]$	$m=[n^{.5}]$	$m=[n^{.6}]$	$m=[n^{.7}]$	$m=[n^{.8}]$
	GPH	90%	53.600(0.721)	58.800 (0.452)	53.300 (0.297)	37.700 (0.203)	27.300 (0.142)
		95%	59.700(0.863)	66.400 (0.540)	61.200 (0.355)	46.500 (0.243)	35.000 (0.170)
	LWN-A	90%	100.000(0.906)	74.600 (0.662)	74.800 (0.480)	75.800 (0.336)	60.500 (0.236)
		95%	100.000(1.080)	77.500 (0.789)	77.600 (0.571)	79.200 (0.401)	79.600 (0.281)
$nsr = 5$	LWN-H	90%	77.500(2.037)	83.900 (1.360)	91.600 (0.946)	96.500 (0.697)	97.800 (0.532)
		95%	78.500(2.427)	84.400 (1.621)	91.600 (1.127)	97.300 (0.831)	99.000 (0.634)
	LWN-IH	90%	100.000(2.490)	100.000 (1.444)	100.000 (0.948)	93.200 (0.681)	90.000 (0.516)
		95%	100.000(2.966)	100.000 (1.721)	100.000 (1.129)	100.000 (0.811)	95.400 (0.615)
	GPH	90%	50.900(0.297)	48.000 (0.297)	34.100 (0.297)	24.600 (0.297)	19.800 (0.297)
		95%	55.700(0.354)	55.500 (0.354)	41.400 (0.354)	32.100 (0.354)	25.400 (0.354)
	LWN-A	90%	100.000(0.994)	72.900 (0.706)	70.300 (0.496)	72.000 (0.340)	54.700 (0.237)
		95%	100.000(1.184)	75.500 (0.841)	72.800 (0.591)	74.000 (0.405)	73.700 (0.283)
$nsr = 10$	LWN-H	90%	76.800(2.373)	78.100 (1.509)	80.400 (1.099)	89.600 (0.878)	94.400 (0.718)
		95%	78.500(2.828)	79.100 (1.797)	80.600 (1.309)	90.000 (1.046)	94.600 (0.856)
	LWN-IH	90%	100.000(2.977)	100.000 (1.781)	100.000 (1.205)	100.000 (0.890)	89.500 (0.692)
		95%	100.000(3.547)	100.000 (2.123)	100.000 (1.436)	100.000 (1.061)	100.000 (0.825)

$n = 5000$

			$m=[n^{.4}]$	$m=[n^{.5}]$	$m=[n^{.6}]$	$m=[n^{.7}]$	$m=[n^{.8}]$
	GPH	90%	71.000 (0.721)	79.200 (0.452)	67.500 (0.297)	46.400 (0.203)	32.000 (0.142)
		95%	75.600 (0.859)	84.800 (0.538)	76.400 (0.354)	53.900 (0.242)	41.300 (0.170)
	LWN-A	90%	100.000 (0.908)	88.800 (0.656)	94.400 (0.470)	94.700 (0.330)	93.700 (0.232)
		95%	100.000 (1.082)	91.500 (0.781)	96.700 (0.560)	96.500 (0.394)	96.200 (0.276)
$nsr = 5$	LWN-H	90%	76.800 (1.096)	86.700 (0.672)	96.900 (0.434)	90.300 (0.299)	89.000 (0.218)
		95%	77.200 (1.306)	86.800 (0.800)	98.200 (0.517)	97.400 (0.356)	94.600 (0.260)
	LWN-IH	90%	100.000 (1.201)	93.900 (0.685)	95.900 (0.431)	88.600 (0.296)	88.900 (0.217)
		95%	100.000 (1.431)	100.000 (0.816)	97.800 (0.514)	94.000 (0.352)	93.300 (0.258)
	GPH	90%	67.100 (0.297)	65.300 (0.297)	46.700 (0.297)	30.000 (0.297)	20.300 (0.297)
		95%	73.000 (0.354)	72.500 (0.354)	55.000 (0.354)	41.200 (0.354)	28.900 (0.354)
	LWN-A	90%	100.000 (0.930)	85.300 (0.669)	88.600 (0.470)	90.700 (0.331)	85.500 (0.232)
		95%	100.000 (1.109)	87.800 (0.797)	92.400 (0.560)	93.500 (0.394)	92.500 (0.276)
$nsr = 10$	LWN-H	90%	72.900 (1.150)	78.500 (0.725)	92.700 (0.504)	90.600 (0.368)	90.400 (0.278)
		95%	73.600 (1.370)	78.600 (0.864)	92.800 (0.601)	96.300 (0.438)	95.300 (0.331)
	LWN-IH	90%	100.000 (1.310)	99.700 (0.771)	93.900 (0.504)	88.300 (0.359)	88.800 (0.273)
		95%	100.000 (1.561)	100.000 (0.919)	96.900 (0.601)	93.100 (0.428)	93.100 (0.325)

$n = 10000$

			$m=[n^{.4}]$	$m=[n^{.5}]$	$m=[n^{.6}]$	$m=[n^{.7}]$	$m=[n^{.8}]$
	GPH	90%	76.700 (0.721)	85.500 (0.452)	76.600 (0.297)	52.000 (0.203)	36.900 (0.142)
		95%	81.300 (0.863)	89.600 (0.540)	83.500 (0.355)	60.500 (0.243)	40.600 (0.170)
	LWN-A	90%	100.000 (0.909)	94.100 (0.649)	97.300 (0.464)	98.600 (0.328)	98.900 (0.232)
		95%	100.000 (1.083)	96.400 (0.774)	98.800 (0.553)	99.100 (0.391)	99.400 (0.276)
$nsr = 5$	LWN-H	90%	75.900 (0.877)	88.800 (0.514)	95.900 (0.322)	90.000 (0.216)	89.500 (0.155)
		95%	76.200 (1.044)	89.000 (0.613)	97.600 (0.384)	95.500 (0.257)	95.800 (0.184)
	LWN-IH	90%	100.000 (0.950)	94.200 (0.521)	94.700 (0.321)	89.600 (0.215)	89.700 (0.154)
		95%	100.000 (1.132)	96.600 (0.620)	97.300 (0.382)	94.900 (0.256)	94.700 (0.184)
	GPH	90%	74.700 (0.297)	75.600 (0.297)	53.000 (0.297)	35.400 (0.297)	20.200 (0.297)
		95%	80.400 (0.354)	81.300 (0.354)	63.100 (0.354)	42.000 (0.354)	33.300 (0.354)
	LWN-A	90%	100.000 (0.922)	91.600 (0.664)	95.100 (0.463)	96.000 (0.329)	96.000 (0.232)
		95%	100.000 (1.099)	94.200 (0.791)	96.800 (0.552)	98.100 (0.393)	98.100 (0.276)
$nsr = 10$	LWN-H	90%	73.600 (0.903)	83.900 (0.555)	94.800 (0.367)	90.400 (0.257)	91.400 (0.193)
		95%	73.800 (1.076)	83.900 (0.661)	95.900 (0.437)	95.800 (0.306)	94.700 (0.230)
	LWN-IH	90%	100.000 (1.012)	93.500 (0.571)	92.300 (0.366)	89.200 (0.256)	90.200 (0.191)
		95%	100.000 (1.205)	95.900 (0.681)	95.800 (0.436)	94.700 (0.305)	94.600 (0.228)

* GPH denotes empirical coverage percentages based on GPH estimates of d with standard errors computed using the finite-sample approximation to the theoretical GPH standard error. LWN-H denotes empirical coverage percentages based on LWN estimates of d with standard errors computed using the finite-sample Hessian-based approximation to the theoretical standard errors with estimated parameters. LWN-IH denotes empirical coverage percentages based on LWN estimates of d with standard errors computed using the finite-sample Hessian-based approximation to the theoretical standard errors with known values of the parameters.

Table 9: Coverage results for GPH and LWN estimators of LMSV-ARFIMA(1,0.40,0) model with $\phi = 0.8$ based on different standard error calculation methods. Values in parentheses denote median lengths of the computed confidence intervals.*

$n = 1000$

			$m = [n^{.4}]$	$m = [n^{.5}]$	$m = [n^{.6}]$	$m = [n^{.7}]$	$m = [n^{.8}]$
	GPH	90%	55.200 (0.721)	66.200 (0.452)	66.100 (0.297)	46.700 (0.203)	28.800 (0.142)
		95%	60.900 (0.863)	73.000 (0.540)	73.000 (0.355)	55.100 (0.243)	37.800 (0.170)
	LWN-A	90%	100.000 (0.904)	78.200 (0.656)	79.600 (0.466)	72.700 (0.318)	50.700 (0.212)
		95%	100.000 (1.077)	80.900 (0.781)	82.800 (0.556)	76.500 (0.379)	62.000 (0.252)
$nsr = 5$	LWN-H	90%	79.400 (1.989)	89.600 (1.264)	97.500 (0.845)	98.300 (0.632)	93.400 (0.515)
		95%	79.900 (2.370)	90.000 (1.506)	97.500 (1.007)	99.300 (0.753)	97.800 (0.613)
	LWN-IH	90%	100.000 (2.490)	100.000 (1.444)	100.000 (0.948)	94.100 (0.681)	86.900 (0.516)
		95%	100.000 (2.966)	100.000 (1.721)	100.000 (1.129)	100.000 (0.811)	91.600 (0.615)
	GPH	90%	50.700 (0.297)	54.200 (0.297)	41.800 (0.297)	29.100 (0.297)	21.100 (0.297)
		95%	57.000 (0.354)	60.300 (0.354)	50.400 (0.354)	37.100 (0.354)	25.400 (0.354)
	LWN-A	90%	100.000 (0.969)	74.900 (0.695)	75.700 (0.503)	71.600 (0.334)	51.400 (0.220)
		95%	100.000 (1.155)	77.300 (0.828)	78.000 (0.600)	73.800 (0.398)	64.000 (0.263)
$nsr = 10$	LWN-H	90%	77.200 (2.252)	80.800 (1.449)	87.500 (1.040)	93.300 (0.815)	95.100 (0.698)
		95%	78.000 (2.683)	81.600 (1.726)	87.800 (1.239)	93.400 (0.971)	95.700 (0.832)
	LWN-IH	90%	100.000 (2.977)	100.000 (1.781)	100.000 (1.205)	100.000 (0.890)	89.100 (0.692)
		95%	100.000 (3.547)	100.000 (2.123)	100.000 (1.436)	100.000 (1.061)	100.000 (0.825)

$n = 5000$

			$m = [n^{.4}]$	$m = [n^{.5}]$	$m = [n^{.6}]$	$m = [n^{.7}]$	$m = [n^{.8}]$
	GPH	90%	68.600 (0.721)	79.600 (0.452)	80.500 (0.297)	62.700 (0.203)	37.900 (0.142)
		95%	76.800 (0.859)	85.100 (0.538)	86.400 (0.354)	72.000 (0.242)	43.500 (0.170)
	LWN-A	90%	100.000 (0.903)	88.800 (0.651)	95.700 (0.484)	96.900 (0.333)	82.200 (0.219)
		95%	100.000 (1.076)	92.300 (0.775)	97.500 (0.577)	98.400 (0.397)	89.700 (0.261)
$nsr = 5$	LWN-H	90%	75.200 (1.061)	87.400 (0.658)	97.000 (0.402)	92.400 (0.265)	81.500 (0.205)
		95%	75.600 (1.264)	87.700 (0.784)	98.600 (0.478)	98.500 (0.316)	89.700 (0.245)
	LWN-IH	90%	100.000 (1.201)	94.500 (0.685)	96.800 (0.431)	94.100 (0.296)	83.500 (0.217)
		95%	100.000 (1.431)	100.000 (0.816)	98.700 (0.514)	97.900 (0.352)	90.700 (0.258)
	GPH	90%	68.000 (0.297)	67.500 (0.297)	53.700 (0.297)	37.400 (0.297)	20.900 (0.297)
		95%	73.100 (0.354)	73.800 (0.354)	61.200 (0.354)	43.800 (0.354)	31.500 (0.354)
	LWN-A	90%	100.000 (0.928)	87.800 (0.665)	92.200 (0.479)	92.700 (0.335)	80.300 (0.224)
		95%	100.000 (1.106)	90.600 (0.793)	95.200 (0.571)	95.300 (0.399)	87.100 (0.267)
$nsr = 10$	LWN-H	90%	74.000 (1.125)	81.000 (0.710)	96.100 (0.478)	89.600 (0.335)	90.300 (0.265)
		95%	74.200 (1.340)	81.000 (0.846)	96.900 (0.570)	96.600 (0.399)	96.500 (0.316)
	LWN-IH	90%	100.000 (1.310)	99.700 (0.771)	96.900 (0.504)	89.800 (0.359)	89.200 (0.273)
		95%	100.000 (1.561)	100.000 (0.919)	98.700 (0.601)	96.400 (0.428)	95.700 (0.325)

$n = 10000$

			$m = [n^{.4}]$	$m = [n^{.5}]$	$m = [n^{.6}]$	$m = [n^{.7}]$	$m = [n^{.8}]$
	GPH	90%	74.600 (0.721)	85.600 (0.452)	85.600 (0.297)	70.200 (0.203)	40.100 (0.142)
		95%	80.000 (0.863)	89.600 (0.540)	92.000 (0.355)	79.300 (0.243)	47.900 (0.170)
	LWN-A	90%	100.000 (0.918)	95.900 (0.657)	99.000 (0.478)	99.700 (0.339)	94.700 (0.222)
		95%	100.000 (1.093)	97.100 (0.783)	99.500 (0.570)	99.900 (0.404)	97.900 (0.265)
$nsr = 5$	LWN-H	90%	74.900 (0.872)	89.800 (0.509)	97.500 (0.305)	87.500 (0.194)	79.700 (0.144)
		95%	75.000 (1.038)	90.400 (0.606)	98.900 (0.364)	95.100 (0.231)	87.300 (0.172)
	LWN-IH	90%	100.000 (0.950)	95.900 (0.521)	97.400 (0.321)	91.600 (0.215)	82.700 (0.154)
		95%	100.000 (1.132)	97.400 (0.620)	99.000 (0.382)	97.200 (0.256)	89.500 (0.184)
	GPH	90%	74.100 (0.297)	74.700 (0.297)	57.300 (0.297)	40.600 (0.297)	21.300 (0.297)
		95%	79.900 (0.354)	80.400 (0.354)	67.300 (0.354)	48.000 (0.354)	36.600 (0.354)
	LWN-A	90%	100.000 (0.919)	93.300 (0.662)	96.700 (0.475)	98.900 (0.339)	93.500 (0.226)
		95%	100.000 (1.095)	96.300 (0.789)	98.500 (0.566)	99.600 (0.403)	96.700 (0.270)
$nsr = 10$	LWN-H	90%	73.900 (0.896)	86.200 (0.545)	96.200 (0.352)	89.400 (0.237)	89.000 (0.183)
		95%	74.800 (1.068)	86.300 (0.649)	98.100 (0.419)	94.700 (0.283)	94.800 (0.218)
	LWN-IH	90%	100.000 (1.012)	95.000 (0.571)	94.200 (0.366)	91.200 (0.256)	89.700 (0.191)
		95%	100.000 (1.205)	97.400 (0.681)	98.100 (0.436)	95.400 (0.305)	95.200 (0.228)

* GPH denotes empirical coverage percentages based on GPH estimates of d with standard errors computed using the finite-sample approximation to the theoretical GPH standard error. LWN-H denotes empirical coverage percentages based on LWN estimates of d with standard errors computed using the finite-sample Hessian-based approximation to the theoretical standard errors with estimated parameters. LWN-IH denotes empirical coverage percentages based on LWN estimates of d with standard errors computed using the finite-sample Hessian-based approximation to the theoretical standard errors with known values of the parameters.

Table 10: GPH and LWN estimators for Deutschemark/Dollar exchange rate, $n = 3485$.

	$m = [n^{0.5}]$	$m = [n^{0.6}]$	$m = [n^{0.7}]$	$m = [n^{0.8}]$
$\widehat{d}_{LWN}$	0.365	0.378	0.387	0.556
$\widehat{d}_{GPH}$	0.370	0.355	0.274	0.135

Table 11: Bias and RMSE of $\hat{d}_{LWN}$ in 100 simulated replications of LMSV-ARFIMA$(1, d, 0)$ process fitted to Deutschemark/Dollar exchange rate

	$m = [n^{0.5}]$	$m = [n^{0.6}]$	$m = [n^{0.7}]$	$m = [n^{0.8}]$
Bias	0.012	0.019	0.011	0.011
RMSE	0.144	0.145	0.134	0.100

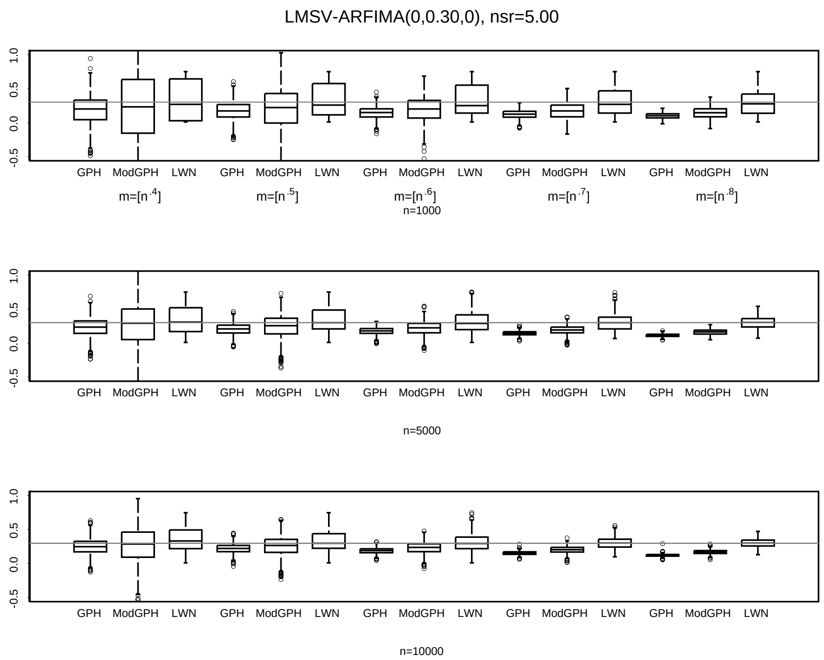

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Figure 1: Box-plots of $\hat{d}_{GPH}$, $\hat{d}_{LP-GPH}$, and $\hat{d}_{LWN}$ for the LMSV-ARFIMA$(0, 0.3, 0)$ model with $nsr = 5$. Estimates were obtained using $m = [n^x]$ Fourier frequencies, where $x = 0.4, 0.5, 0.6, 0.7, 0.8$. The solid line indicates the true value of $d = 0.3$.

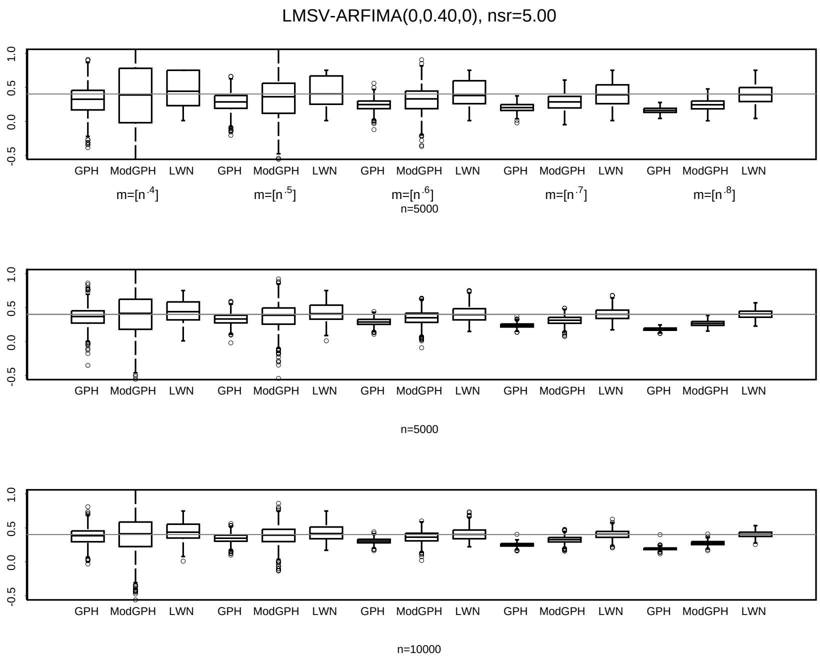

Figure 2: Box-plots of $\hat{d}_{GPH}$, $\hat{d}_{LP-GPH}$, and $\hat{d}_{LWN}$ for the LMSV-ARFIMA$(0, 0.4, 0)$ model with $nsr = 5$. Estimates were obtained using $m = [n^x]$ Fourier frequencies, where $x = 0.4, 0.5, 0.6, 0.7, 0.8$. The solid line indicates the true value of $d = 0.4$.

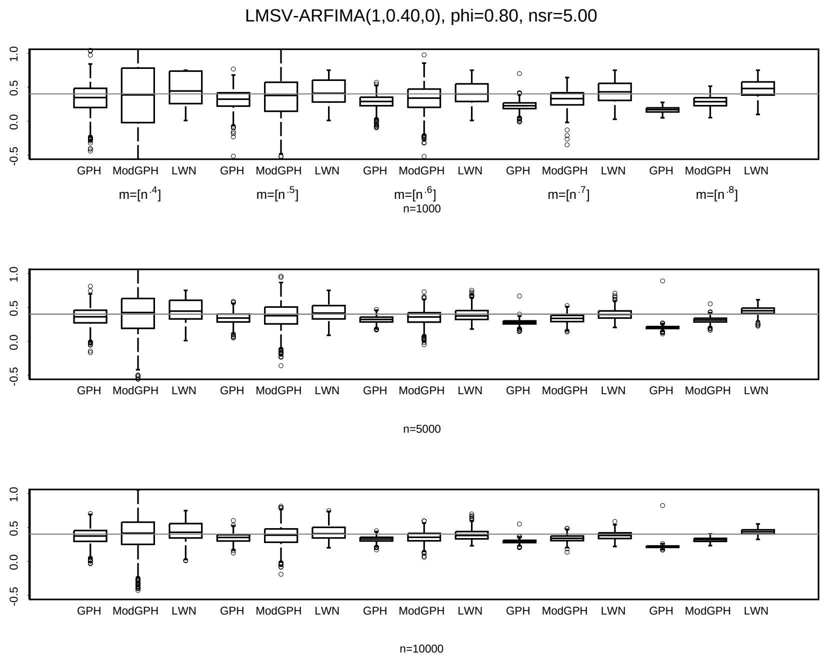

Figure 3: Box-plots of $\hat{d}_{GPH}$, $\hat{d}_{LP-GPH}$, and $\hat{d}_{LWN}$ for the LMSV-ARFIMA$(1, 0.4, 0)$ model with $\phi = 0.8$ and $nsr = 5$. Estimates were obtained using $m = [n^x]$ Fourier frequencies, where $x = 0.4, 0.5, 0.6, 0.7, 0.8$. The solid line indicates the true value of $d = 0.4$.

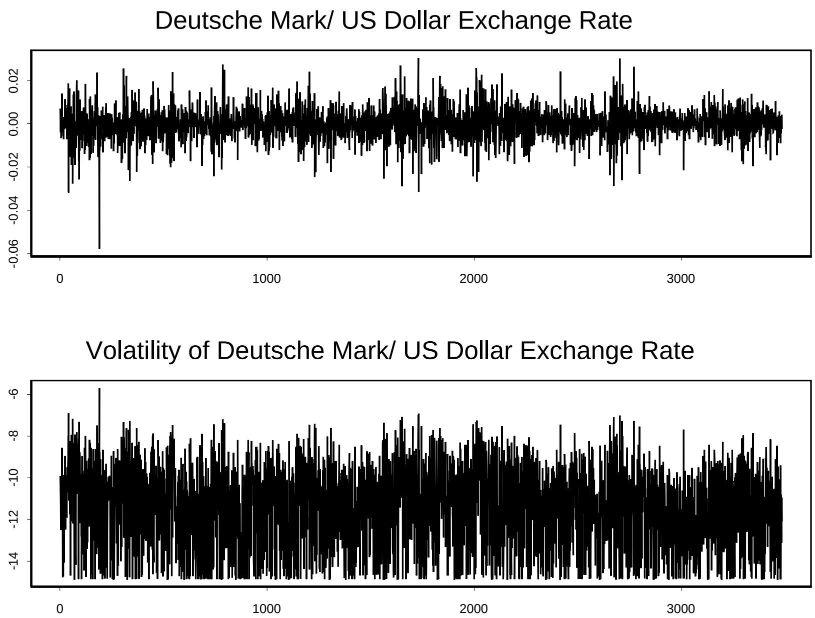

Figure 4: Top plot: Deutsche Mark/ US Dollar exchange rate from Jan 2, 1985 to May 12, 1998. Bottom plot: Volatility series for Deutsche Mark/ US Dollar exchange rate constructed using adjusted log squared returns.

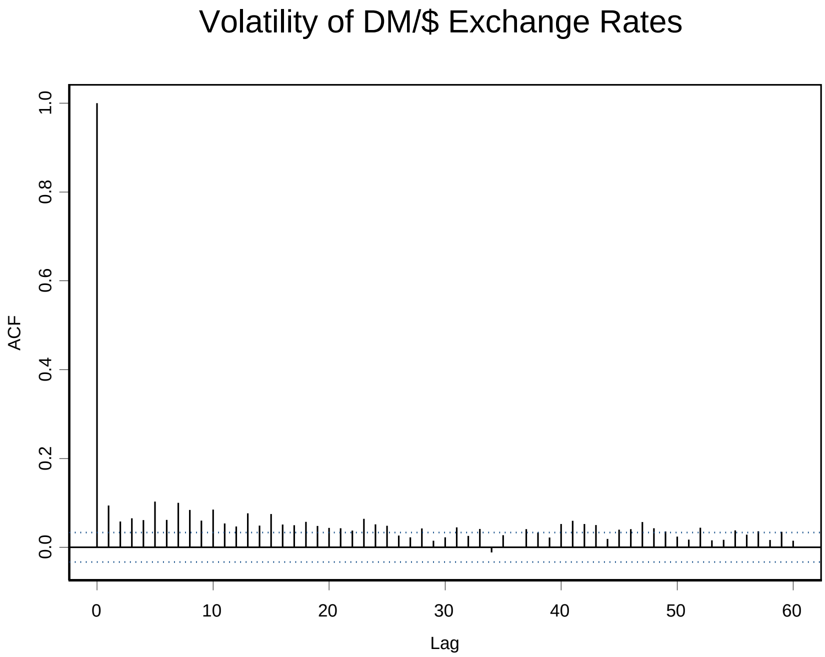

Figure 5: Sample ACF for volatility series of Deutsche Mark/ US Dollar exchange rates