Stabbing Pairwise Intersecting Disks by Five Points

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Abstract

Suppose we are given a set $\mathcal{D}$ of n pairwise intersecting disks in the plane. A planar point set P *stabs* $\mathcal{D}$ if and only if each disk in $\mathcal{D}$ contains at least one point from P. We present a deterministic algorithm that takes $O(n)$ time to find five points that stab $\mathcal{D}$. Furthermore, we give a simple example of 13 pairwise intersecting disks that cannot be stabbed by three points.

This provides a simple – albeit slightly weaker – algorithmic version of a classical result by Danzer that such a set $\mathcal{D}$ can always be stabbed by four points.

2012 ACM Subject Classification Mathematics of computing → Combinatorics

Keywords and phrases Disk graph, piercing set, LP-type problem

Digital Object Identifier 10.4230/LIPIcs.ISAAC.2018.50

[1] Partially supported by a NSF AF awards CCF-1421231, and CCF-1217462.
[2] Partially supported by DFG grant MU/3501/1 and ERC STG 757609.
[3] Partially supported by DFG grant MU/3501/1.
[4] Partially supported by ISF Grant 892/13, by the Israeli Centers of Research Excellence (I-CORE) program (Center No. 4/11), by the Blavatnik Research Fund in Computer Science at Tel Aviv University, and by the Hermann Minkowski-MINERVA Center for Geometry at Tel Aviv University.

29th International Symposium on Algorithms and Computation (ISAAC 2018).
Editors: Wen-Lian Hsu, Der-Tsai Lee, and Chung-Shou Liao; Article No. 50; pp. 50:1–50:12
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Related Version Also available on the arXiv as https://arxiv.org/abs/1801.03158.

Funding Work on this paper was supported in part by grant 1367/2016 from the German-Israeli Science Foundation (GIF).

Acknowledgements We would like to thank Timothy Chan for pointing us to the direct LP-type algorithm described in Lemma 4.7.

1 Introduction

Let $\mathcal{D}$ be a set of n disks in the plane. If every *three* disks in $\mathcal{D}$ intersect, then Helly's theorem shows that the whole intersection $\bigcap \mathcal{D}$ of $\mathcal{D}$ is nonempty [9, 10, 11]. In other words, there is a single point p that lies in all disks of $\mathcal{D}$, i.e., p *stabs* $\mathcal{D}$. More generally, when we know only that every *pair of* disks in $\mathcal{D}$ intersect, there must be a point set P of constant size such that each disk in $\mathcal{D}$ contains at least one point in P. It is fairly easy to give an upper bound on the size of P, but for some time, the exact bound remained elusive. Eventually, in July 1956 at an Oberwolfach seminar, Danzer presented the answer: four points are always sufficient and sometimes necessary to stab any finite set of pairwise intersecting disks in the plane (see [5]). Danzer was not satisfied with his original argument, so he never formally published it. In 1986, he presented a new proof [5]. Previously, in 1981, Stachó had already given an alternative proof [15], building on a previous construction of five stabbing points [14]. This line of work was motivated by a result of Hadwiger and Debrunner, who showed that three points suffice to stab any finite set of pairwise intersecting *unit* disks [8]. In later work, these results were significantly generalized and extended, culminating in the celebrated (p, q)-theorem that was proven by Alon and Kleitman in 1992 [1]. See also a recent paper by Dumitrescu and Jiang that studies generalizations of the stabbing problem for translates and homothets of a convex body [6].

Danzer's published proof [5] is fairly involved and uses a compactness argument, and part of it is based on an undetailed verification by computer. There seems to be no obvious way to turn it into an efficient algorithm for finding a stabbing set of size four. The two constructions of Stachó [15, 14] are simpler, but they start with three disks in $\mathcal{D}$ with empty intersection and maximum inscribed circle. It is not clear to us how to find such a triple quickly (in, say, near-linear time). Here, we present a new argument that yields five stabbing points. Our proof is constructive, and it lets us find the stabbing set in deterministic linear time.

As for lower bounds, Grünbaum gave an example of 21 pairwise intersecting disks that cannot be stabbed by three points [7]. Later, Danzer reduced the number of disks to ten [5]. This example is close to optimal, because every set of eight disks can be stabbed by three points [14]. It is hard to verify Danzer's lower bound example – even with dynamic geometry software, the positions of the disks cannot be visualized easily. Here, we present a simple construction that needs 13 disks and can be verified by inspection.

2 The Geometry of Pairwise Intersecting Disks

Let $\mathcal{D}$ be a set of n pairwise intersecting disks in the plane. A disk $D_i \in \mathcal{D}$ is given by its center c_i and its radius r_i. To simplify the analysis, we make the following assumptions: (i) the radii of the disks are pairwise distinct; (ii) the intersection of any two disks has a nonempty interior; and (iii) the intersection of any three disks is either empty or has a nonempty interior. A simple perturbation argument can then handle the degenerate cases.

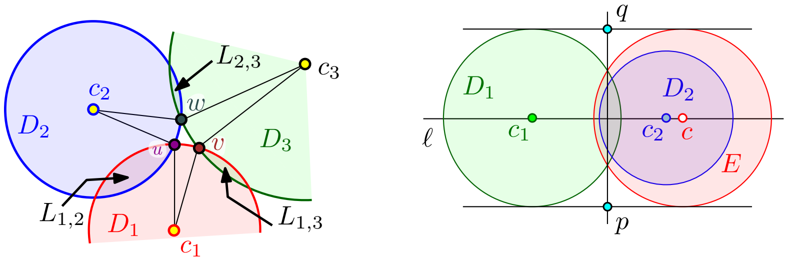

Figure 1 left: At least one lens angle is large. **right:** D_1 and E have the same radii and lens angle $2\pi/3$. By Lemma 2.2, D_2 is a subset of E. $\{c_1, c, p, q\}$ is the set P from Lemma 2.4.

The *lens* of two disks $D_i, D_j \in \mathcal{D}$ is the set $L_{i,j} = D_i \cap D_j$. Let u be any of the two intersection points of ∂D_i and ∂D_j. The angle $\angle c_i u c_j$ is called the *lens angle* of D_i and D_j. It is at most π. A finite set $\mathcal{C}$ of disks is *Helly* if their common intersection $\bigcap \mathcal{C}$ is nonempty. Otherwise, $\mathcal{C}$ is *non-Helly*. We present some useful geometric lemmas.

▶ **Lemma 2.1.** *Let $\{D_1, D_2, D_3\}$ be a set of three pairwise intersecting disks that is non-Helly. Then, the set contains two disks with lens angle larger than $2\pi/3$.*

Proof. Since $\{D_1, D_2, D_3\}$ is non-Helly, the lenses $L_{1,2}$, $L_{1,3}$ and $L_{2,3}$ are pairwise disjoint. Let u be the vertex of $L_{1,2}$ nearer to D_3, and let v, w be the analogous vertices of $L_{1,3}$ and $L_{2,3}$ (see Figure 1, left). Consider the simple hexagon $c_1 u c_2 w c_3 v$, and write $\angle u$, $\angle v$, and $\angle w$ for its interior angles at u, v, and w. The sum of all interior angles is 4π. Thus, $\angle u + \angle v + \angle w < 4\pi$, so at least one angle is less than $4\pi/3$. It follows that the corresponding exterior angle at u, v, or w must be larger than $2\pi/3$. ◀

▶ **Lemma 2.2.** *Let D_1 and D_2 be two intersecting disks with $r_1 \geq r_2$ and lens angle at least $2\pi/3$. Let E be the unique disk with radius r_1 and center c, such that (i) the centers c_1, c_2, and c are collinear and c lies on the same side of c_1 as c_2; and (ii) the lens angle of D_1 and E is exactly $2\pi/3$ (see Figure 1, right). Then, if c_2 lies between c_1 and c, we have $D_2 \subseteq E$.*

Proof. Let $x \in D_2$. Since c_2 lies between c_1 and c, the triangle inequality gives

$$|xc| \leq |xc_2| + |c_2 c| = |xc_2| + |c_1 c| - |c_1 c_2|. \tag{1}$$

Since $x \in D_2$, we get $|xc_2| \leq r_2$. Also, since D_1 and E have radius r_1 each and lens angle $2\pi/3$, it follows that $|c_1 c| = \sqrt{3}\, r_1$. Finally, $|c_1 c_2| = \sqrt{r_1^2 + r_2^2 - 2 r_1 r_2 \cos\alpha}$, by the law of cosines, where α is the lens angle of D_1 and D_2. As $\alpha \geq 2\pi/3$ and $r_1 \geq r_2$, we get $\cos\alpha \leq -1/2 = (\sqrt{3} - 3/2) - \sqrt{3} + 1 \leq (\sqrt{3} - 3/2) r_1/r_2 - \sqrt{3} + 1$, As such, we have

$$\begin{aligned}
|c_1 c_2|^2 &= r_1^2 + r_2^2 - 2 r_1 r_2 \cos\alpha \geq r_1^2 + r_2^2 - 2 r_1 r_2 \Big((\sqrt{3} - 3/2) \frac{r_1}{r_2} - \sqrt{3} + 1 \Big) \\
&= r_1^2 - 2(\sqrt{3} - 3/2) r_1^2 + 2(-\sqrt{3} + 1) r_1 r_2 + r_2^2 \\
&= (1 - 2\sqrt{3} + 3) r_1^2 + 2(-\sqrt{3} + 1) r_1 r_2 + r_2^2 = \big(r_1(\sqrt{3} - 1) + r_2 \big)^2.
\end{aligned}$$

Plugging this into Eq. (1) gives $|xc| \leq r_2 + \sqrt{3} r_1 - \big(r_1 (\sqrt{3} - 1) + r_2 \big) = r_1$, i.e., $x \in E$. ◀

▶ **Lemma 2.3.** *Let D_1 and D_2 be two intersecting disks with equal radius r and lens angle $2\pi/3$. There is a set P of four points so that any disk F of radius at least r that intersects both D_1 and D_2 contains a point of P.*

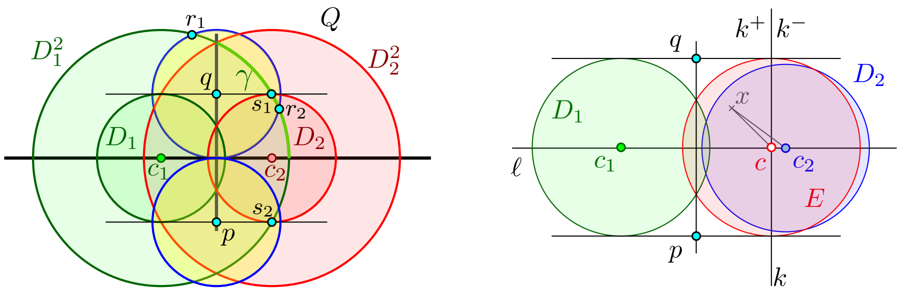

Figure 2 left: $P = \{c_1, c_2, p, q\}$ is the stabbing set. The green arc $\gamma = \partial(D_1^2 \cap D_2^2) \cap Q$ is covered by $D_1^2 \cap D_q$. **right:** Situation (ii) in the proof of Lemma 2.4: $D_2 \not\subseteq E$. x is an arbitrary point in $D_2 \cap F \cap k^+$. The angle at c in the triangle Δxcc_2 is $\geq \pi/2$.

Proof. Consider the two tangent lines of D_1 and D_2, and let p and q be the midpoints on these lines between the respective two tangency points. We set $P = \{c_1, c_2, p, q\}$ (see Figure 2, left).

Given the disk F that intersects both D_1 and D_2, we shrink its radius, keeping its center fixed, until either the radius becomes r or until F is tangent to D_1 or D_2. Suppose the latter case holds and F is tangent to D_1. We move the center of F continuously along the line spanned by the center of F and c_1 towards c_1, decreasing the radius of F to maintain the tangency. We stop when either the radius of F reaches r or F becomes tangent to D_2. We obtain a disk $G \subseteq F$ with center $c = (c_x, c_y)$ so that either: (i) radius$(G) = r$ and G intersects both D_1 and D_2; or (ii) radius$(G) \geq r$ and G is tangent to both D_1 and D_2. Since $G \subseteq F$, it suffices to show that $G \cap P \neq \emptyset$. We introduce a coordinate system, setting the origin o midway between c_1 and c_2, so that the y-axis passes through p and q. Then, as in Figure 2 (left), we have $c_1 = (-\sqrt{3}\,r/2, 0)$, $c_2 = (\sqrt{3}\,r/2, 0)$, $q = (0, r)$, and $p = (0, -r)$.

For case (i), let D_1^2 be the disk of radius $2r$ centered at c_1, and D_2^2 the disk of radius $2r$ centered at c_2. Since G has radius r and intersects both D_1 and D_2, its center c has distance at most $2r$ from both c_1 and c_2, i.e., $c \in D_1^2 \cap D_2^2$. Let D_p and D_q be the two disks of radius r centered at p and q. We will show that $D_1^2 \cap D_2^2 \subseteq D_1 \cup D_2 \cup D_p \cup D_q$. Then it is immediate that $G \cap P \neq \emptyset$. By symmetry, it is enough to focus on the upper-right quadrant $Q = \{(x, y) \mid x \geq 0, y \geq 0\}$. We show that all points in $D_1^2 \cap Q$ are covered by $D_2 \cup D_q$. Without loss of generality, we assume that $r = 1$. Then, the two intersection points of D_1^2 and D_q are $r_1 = (\frac{5\sqrt{3}-2\sqrt{87}}{28}, \frac{38+3\sqrt{29}}{28}) \approx (-0.36, 1.93)$ and $r_2 = (\frac{5\sqrt{3}+2\sqrt{87}}{28}, \frac{38-3\sqrt{29}}{28}) \approx (0.98, 0.78)$, and the two intersection points of D_1^2 and D_2 are $s_1 = (\frac{\sqrt{3}}{2}, 1) \approx (0.87, 1)$ and $s_2 = (\frac{\sqrt{3}}{2}, -1) \approx (0.87, -1)$. Let γ be the boundary curve of D_1^2 in Q. Since $r_1, s_2 \notin Q$ and since $r_2 \in D_2$ and $s_1 \in D_q$, it follows that γ does not intersect the boundary of $D_2 \cup D_q$ and hence $\gamma \subset D_2 \cup D_q$. Furthermore, the subsegment of the y-axis from o to the start point of γ is contained in D_q, and the subsegment of the x-axis from o to the endpoint of γ is contained in D_2. Hence, the boundary of $D_1^2 \cap Q$ lies completely in $D_2 \cup D_q$, and since $D_2 \cup D_q$ is simply connected, it follows that $D_1^2 \cap Q \subseteq D_2 \cup D_q$, as desired.

For case (ii), since G is tangent to D_1 and D_2, the center c of G is on the perpendicular bisector of c_1 and c_2, so the points p, o, q and c are collinear. Suppose without loss of generality that $c_y \geq 0$. Then, it is easily checked that c lies above q, and radius$(G) + r = |c_1 c| \geq |oc| = r + |qc|$, so $q \in G$. ◀

▶ **Lemma 2.4.** *Consider two intersecting disks D_1 and D_2 with $r_1 \geq r_2$ and lens angle at least $2\pi/3$. Then, there is a set P of four points such that any disk F of radius at least r_1 that intersects both D_1 and D_2 contains a point of P.*

Proof. Let ℓ be the line through c_1 and c_2. Let E be the disk of radius r_1 and center $c \in \ell$ that satisfies the conditions (i) and (ii) of Lemma 2.2. Let $P = \{c_1, c, p, q\}$ as in the proof of Lemma 2.3, with respect to D_1 and E (see Figure 1, right). We claim that

$$D_1 \cap F \neq \emptyset \ \wedge\ D_2 \cap F \neq \emptyset \ \Rightarrow\ E \cap F \neq \emptyset. \tag{*}$$

Once (*) is established, we are done by Lemma 2.3. If $D_2 \subseteq E$, then (*) is immediate, so assume that $D_2 \nsubseteq E$. By Lemma 2.2, c lies between c_1 and c_2. Let k be the line through c perpendicular to ℓ, and let k^+ be the open halfplane bounded by k with $c_1 \in k^+$ and k^- the open halfplane bounded by k with $c_1 \notin k^-$. Since $|c_1c| = \sqrt{3}\,r_1 > r_1$, we have $D_1 \subset k^+$ (see Figure 2, right). Recall that F has radius at least r_1 and intersects D_1 and D_2. We distinguish two cases: (i) there is no intersection of F and D_2 in k^+, and (ii) there is an intersection of F and D_2 in k^+.

For case (i), let x be any point in $D_1 \cap F$. Since we know that $D_1 \subset k^+$, we have $x \in k^+$. Moreover, let y be any point in $D_2 \cap F$. By assumption (i), y is not in k^+, but it must be in the infinite strip defined by the two tangents of D_1 and E. Thus, the line segment $\overline{xy}$ intersects the diameter segment $k \cap E$. Since F is convex, the intersection of $\overline{xy}$ and $k \cap E$ is in F, so $E \cap F \neq \emptyset$.

For case (ii), fix $x \in D_2 \cap F \cap k^+$ arbitrarily. Consider the triangle Δxcc_2. Since $x \in k^+$, the angle at c is at least $\pi/2$ (Figure 2, right). Thus, $|xc| \leq |xc_2|$. Also, since $x \in D_2$, we know that $|xc_2| \leq r_2 \leq r_1$. Hence, $|xc| \leq r_1$, so $x \in E$ and (*) follows, as $x \in E \cap F$. ◀

3 Existence of Five Stabbing Points

With the tools from Section 2, we can now show that there is a stabbing set with five points.

▶ **Theorem 3.1.** *Let $\mathcal{D}$ be a set of n pairwise intersecting disks in the plane. There is a set P of five points such that each disk in $\mathcal{D}$ contains at least one point from P.*

Proof. If $\mathcal{D}$ is Helly, there is a single point that lies in all disks of $\mathcal{D}$. Thus, assume that $\mathcal{D}$ is non-Helly, and let $D_1, D_2, \ldots, D_n$ be the disks in $\mathcal{D}$ ordered by increasing radius. Let i^* be the smallest index with $\bigcap_{i \leq i^*} D_i = \emptyset$. By Helly's theorem [9, 10, 11], there are indices $j, k < i^*$ such that $\{D_{i^*}, D_j, D_k\}$ is non-Helly. By Lemma 2.1, two disks in $\{D_{i^*}, D_j, D_k\}$ have lens angle at least $2\pi/3$. Applying Lemma 2.4 to these two disks, we obtain a set P' of four points so that every disk D_i with $i \geq i^*$ contains at least one point from P'. Furthermore, by definition of i^*, we have $\bigcap_{i<i^*} D_i \neq \emptyset$, so there is a point q that stabs every disk D_i with $i < i^*$. Thus, $P = P' \cup \{q\}$ is a set of five points that stabs every disk in $\mathcal{D}$, as desired. ◀

4 Algorithmic Considerations

Theorem 3.1 leads to a simple $O(n \log n)$ time deterministic algorithm for finding a stabbing set of size 5: we sort the disks in $\mathcal{D}$ by radius, and we insert the disks one by one, while maintaining their intersection. Once the intersection becomes empty, we can use the method from Theorem 3.1 to find the stabbing set (otherwise, $\mathcal{D}$ is Helly, and we have a single stabbing point). As we will see next, there is also a deterministic linear time algorithm, using the *LP-type framework* by Sharir and Welzl [13, 3].

The LP-type framework. An *LP-type problem* $(\mathcal{H}, w, \leq)$ is an abstract generalization of a low-dimensional linear program. It consists of a finite set of *constraints* $\mathcal{H}$, a *weight function* $w : 2^{\mathcal{H}} \to \mathcal{W}$, and a *total order* $(\mathcal{W}, \leq)$ on the weights. The weight function w assigns a weight to each subset of constraints. It must fulfill the following three axioms:

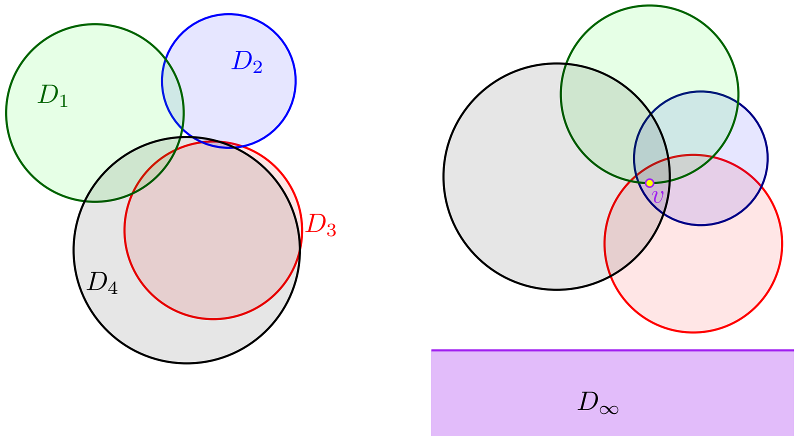

Figure 3 left: The disks D_3 and D_4 are destroyers of the set $\{D_1, D_2\}$. Moreover, D_3 is the smallest destroyer of the whole set $\{D_1, D_2, D_3, D_4\}$. **right:** The disks without D_∞ form a Helly set $\mathcal{C}$. Adding D_∞ leads to the non-Helly set $\overline{\mathcal{C}} = \mathcal{C} \cup \{D_\infty\}$ with smallest destroyer D_∞. The point v is the extreme point for $\mathcal{C}$ and D_∞, i.e., $\mathrm{dist}(\mathcal{C}) = d(v, D_\infty)$.

- **Monotonicity:** for any $\mathcal{H}' \subseteq \mathcal{H}$ and $H \in \mathcal{H}$, we have $w\big(\mathcal{H}' \cup \{H\}\big) \leq w(\mathcal{H}')$;
- **Finite Basis:** there is a constant $d \in \mathbb{N}$ such that for any $\mathcal{H}' \subseteq \mathcal{H}$, there is a subset $\mathcal{B} \subseteq \mathcal{H}'$ with $|\mathcal{B}| \leq d$ and $w(\mathcal{B}) = w(\mathcal{H}')$; and
- **Locality:** for any $\mathcal{B} \subseteq \mathcal{H}' \subseteq \mathcal{H}$ with $w(\mathcal{B}) = w(\mathcal{H}')$ and for any $H \in \mathcal{H}$, we have that if $w\big(\mathcal{B} \cup \{H\}\big) = w(\mathcal{B})$, then also $w\big(\mathcal{H}' \cup \{H\}\big) = w(\mathcal{H}')$.

Given a subset $\mathcal{H}' \subseteq \mathcal{H}$, a *basis* for $\mathcal{H}'$ is an inclusion-minimal set $\mathcal{B} \subseteq \mathcal{H}'$ with $w(\mathcal{B}) = w(\mathcal{H}')$. The Finite-Basis-axiom states that any basis has at most d constraints. The goal in an LP-type problem is to determine $w(\mathcal{H})$ and a corresponding basis $\mathcal{B}$ for $\mathcal{H}$.

A generalization of Seidel's algorithm for low-dimensional linear programming [12] shows that we can solve an LP-type problem in expected time $O(|\mathcal{H}|)$, provided that an $O(1)$-time *violation test* is available: given a set $\mathcal{B} \subseteq \mathcal{H}$ and a constraint $H \in \mathcal{H}$, we say that H *violates* $\mathcal{B}$ if and only if $w\big(\mathcal{B} \cup \{H\}\big) < w(\mathcal{B})$. In a violation test, we are given $\mathcal{B}$ and H, and we must determine (i) whether $\mathcal{B}$ is a valid basis for some subset of constraints; and (ii) whether H violates $\mathcal{B}$.[5] Here and below, the constant factor in the O-notation may depend on d.

Chazelle and Matoušek [4] showed that an LP-type problem can be solved in $O(|\mathcal{H}|)$ *deterministic* time if (i) we have a constant-time violation test and (ii) the range space $(\mathcal{H}, \{\mathrm{vio}(\mathcal{B}) \mid \mathcal{B} \text{ is a basis for some } \mathcal{H}' \subseteq \mathcal{H}\})$ has bounded VC-dimension [3]. Here, for a basis $\mathcal{B}$, the set $\mathrm{vio}(\mathcal{B}) \subset \mathcal{H}$ consists of all constraints that violate $\mathcal{B}$. We will now show that the problem of finding a non-Helly triple as in Theorem 3.1 is LP-type and fulfills the requirements for the algorithm of Chazelle and Matoušek.

Geometric observations. The *distance* between two closed sets $A, B \subseteq \mathbb{R}^2$ is defined as $d(A, B) = \min\{d(a, b) \mid a \in A, b \in B\}$. From now on, we assume that all points in $\bigcup \mathcal{D}$ have positive y-coordinates. This can be ensured with linear overhead by an appropriate translation of the input. We denote by D_∞ the closed halfplane below the x-axis. It is

[5] Here, we follow the presentation of Chazelle [3]. Sharir and Welzl [13] do not require property (i) of a violation test. Instead, they need an additional *basis computation* primitive: given a basis $\mathcal{B}$ and a constraint $H \in \mathcal{H}$, find a basis for $\mathcal{B} \cup \{H\}$. Given a violation test with property (i), a basis computation primitive can easily be implemented by brute force enumeration.

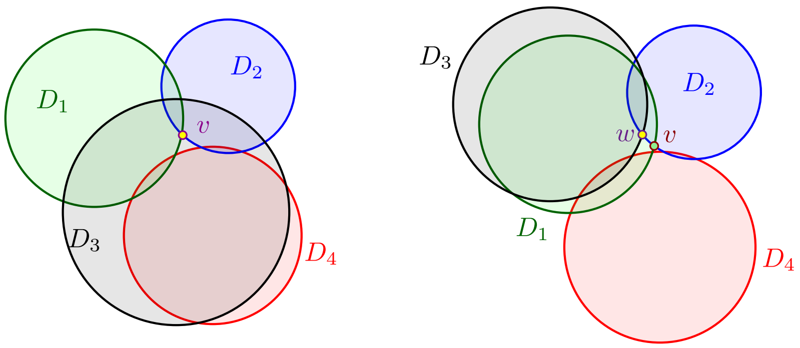

Figure 4 left: The disk D_4 is a destroyer for the Helly sets $\{D_1, D_2\}$ and $\{D_1, D_2, D_3\}$. The extreme point v for $\{D_1, D_2\}$ is also the extreme point for $\{D_1, D_2, D_3\}$. **right:** The disk D_4 is a destroyer for the Helly sets $\{D_1, D_2\}$ and $\{D_1, D_2, D_3\}$. The extreme point v for $\{D_1, D_2\}$ is not in $\mathcal{D}_3$. The distance to D_4 increases.

interpreted as a disk with radius ∞ and center at $(0, -\infty)$. For $\mathcal{C} \subseteq \mathcal{D}$ we set $\overline{\mathcal{C}} = \mathcal{C} \cup \{D_\infty\}$. Observe that for $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \overline{\mathcal{D}}$, if $\mathcal{C}_1$ is non-Helly, then $\mathcal{C}_2$ is non-Helly. Furthermore, for $r \in \mathbb{R}_{>0} \cup \{\infty\}$ and $\mathcal{C} \subseteq \overline{\mathcal{D}}$, we define $\mathcal{C}_{\le r}$ (resp., $\mathcal{C}_{<r}$) as the set of all disks in $\mathcal{C}$ with radius at most (resp., smaller than) r. Let $\mathcal{C} \subseteq \mathcal{D}$ be Helly. A disk $D \in \overline{\mathcal{D}}$ is a *destroyer* of $\mathcal{C}$ if $\mathcal{C} \cup \{D\}$ is non-Helly. Observe that D_∞ is a destroyer for every Helly subset of $\mathcal{D}$. Now, let $\mathcal{C} \subseteq \mathcal{D}$ be an arbitrary subset of $\mathcal{D}$ (either Helly or non-Helly). We say $D \in \overline{\mathcal{C}}$ is the *smallest destroyer* of $\mathcal{C}$ if $\mathcal{C}_{<r}$ is Helly and $\overline{\mathcal{C}}_{\le r}$ is non-Helly, where r is the radius of D. Note that D is the unique largest disk in $\overline{\mathcal{C}}_{\le r}$. Furthermore, D is the smallest disk in $\overline{\mathcal{C}}$ that causes a non-Helly triple. If $\mathcal{C}$ is Helly, then $D = D_\infty$. See Figure 3 for an example. We can make the following two observations.

▶ **Lemma 4.1.** *Let $\mathcal{C} \subseteq \mathcal{D}$ be Helly and $D \in \overline{\mathcal{D}}$ a destroyer of $\mathcal{C}$. Then, the point $v \in \bigcap \mathcal{C}$ with minimum distance to D is unique.*

Proof. Suppose there are two distinct points $v \neq w \in \bigcap \mathcal{C}$ with $d(v, D) = d\big(\bigcap \mathcal{C}, D\big) = d(w, D)$. Since $\bigcap \mathcal{C}$ is convex, the segment $\overline{vw}$ lies in $\bigcap \mathcal{C}$. Now, if $D \neq D_\infty$, then every point in the relative interior of $\overline{vw}$ is strictly closer to D than v and w. If $D = D_\infty$, then all points in $\overline{vw}$ have the same distance to D, but since $\bigcap \mathcal{C}$ is strictly convex, the relative interior of $\overline{vw}$ lies in the interior of $\bigcap \mathcal{C}$, so there must be a point in $\bigcap \mathcal{C}$ that is closer to D than v and w. In either case, we obtain a contradiction to the assumption $v \neq w$ and $d(v, D) = d\big(\bigcap \mathcal{C}, D\big) = d(w, D)$. The claim follows. ◀

The unique point $v \in \bigcap \mathcal{C}$ with minimum distance to a destroyer D is called the *extreme point* for $\mathcal{C}$ and D (see Figure 3).

▶ **Lemma 4.2.** *Let $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{D}$ be two Helly sets and $D \in \overline{\mathcal{D}}$ a destroyer of $\mathcal{C}_1$ (and thus of $\mathcal{C}_2$). Let $v \in \bigcap \mathcal{C}_1$ be the extreme point for $\mathcal{C}_1$ and D. We have $d\big(\bigcap \mathcal{C}_1, D\big) \le d\big(\bigcap \mathcal{C}_2, D\big)$. In particular, if $v \in \bigcap \mathcal{C}_2$, then $d\big(\bigcap \mathcal{C}_1, D\big) = d\big(\bigcap \mathcal{C}_2, D\big)$ and v is also the extreme point for $\mathcal{C}_2$. If $v \notin \bigcap \mathcal{C}_2$, then $d\big(\bigcap \mathcal{C}_1, D\big) < d\big(\bigcap \mathcal{C}_2, D\big)$.*

Proof. The first claim holds trivially: let $w \in \bigcap \mathcal{C}_2$ be the extreme point for $\mathcal{C}_2$ and D. Since $\mathcal{C}_1 \subseteq \mathcal{C}_2$, it follows that $w \in \bigcap \mathcal{C}_1$, so $d\big(\bigcap \mathcal{C}_1, D\big) \le d(w, D) = d\big(\bigcap \mathcal{C}_2, D\big)$. If $v \in \bigcap \mathcal{C}_2$, then $d\big(\bigcap \mathcal{C}_1, D\big) \le d\big(\bigcap \mathcal{C}_2, D\big) \le d(v, D) = d\big(\bigcap \mathcal{C}_1, D\big)$, so $v = w$, by Lemma 4.1. If $v \notin \bigcap \mathcal{C}_2$, then $d\big(\bigcap \mathcal{C}_1, D\big) < d\big(\bigcap \mathcal{C}_2, D\big)$, by Lemma 4.1 and the fact that $\mathcal{C}_1 \subseteq \mathcal{C}_2$. See Figure 4. ◀

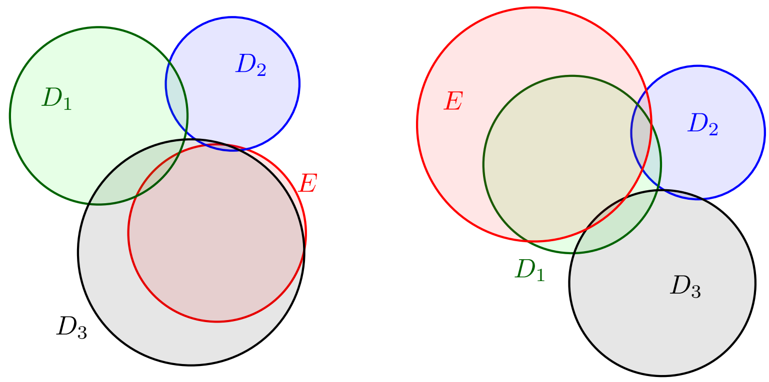

Figure 5 Monotonicity: In both cases, $\{D_1, D_2, D_3\}$ is non-Helly with smallest destroyer D_3. Adding a disk E either decreases the radius of the smallest destroyer (**left**) or increases the distance to the smallest destroyer (**right**).

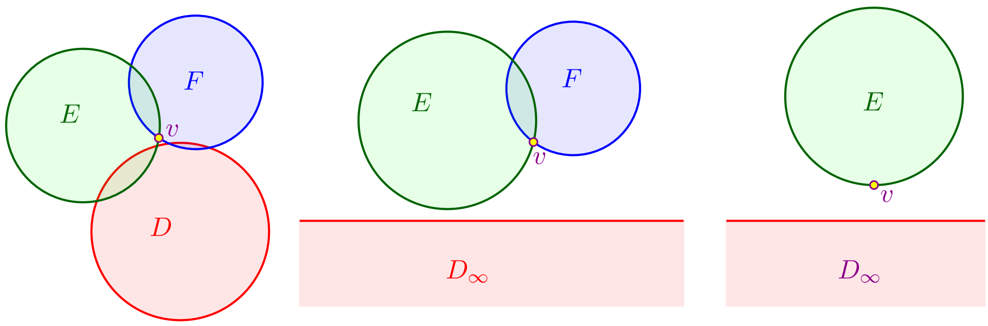

Figure 6 A basis can either be a non-Helly triple (left), a pair of intersecting disks E and F where the point of minimum y-coordinate in $E \cap F$ is a vertex (middle), or a single disk.

Let $\mathcal{C}$ be a subset of $\mathcal{D}$. The radius of the smallest destroyer D of $\overline{\mathcal{C}}$ is denoted by $\mathrm{rad}(\mathcal{C})$. Note that $\mathrm{rad}(\mathcal{C}) \in \mathbb{R}_{>0} \cup \{\infty\}$. Moreover, let $\mathrm{dist}(\mathcal{C})$ be the distance between D and the set $\bigcap \mathcal{C}_{<\mathrm{rad}(\mathcal{C})}$, i.e., $\mathrm{dist}(\mathcal{C}) = d\big(\bigcap \mathcal{C}_{<\mathrm{rad}(\mathcal{C})}, D\big)$. Then, $\mathcal{C}$ is Helly if and only if $\mathrm{rad}(\mathcal{C}) = \infty$. In this case, $\mathrm{dist}(\mathcal{C})$ is the distance between $\bigcap \mathcal{C}$ and the x-axis. We define the *weight* $w(\mathcal{C})$ of $\mathcal{C}$ as $w(\mathcal{C}) = (\mathrm{rad}(\mathcal{C}), -\mathrm{dist}(\mathcal{C}))$, and we denote by $\leq$ the lexicographic order on $\mathbb{R}^2$. Chan observed, in a slightly different context, that $(\mathcal{D}, w, \leq)$ is LP-type [2]. However, Chan's paper does not contain a detailed proof for this fact. Thus, in the following lemmas, we show that the three LP-type axioms hold.

▶ **Lemma 4.3.** *For any $\mathcal{C} \subseteq \mathcal{D}$ and $E \in \mathcal{D}$, we have $w\big(\mathcal{C} \cup \{E\}\big) \leq w(\mathcal{C})$.*

Proof. Set $\mathcal{C}^* = \mathcal{C} \cup \{E\}$. Let D be the smallest destroyer of $\overline{\mathcal{C}}$, and let $r = \mathrm{rad}(\mathcal{C})$ be the radius of D. Then, D is the largest disk in $\overline{\mathcal{C}}_{\leq r}$. The set $\overline{\mathcal{C}}_{\leq r}$ is non-Helly. Adding E does not change this, i.e., $\overline{\mathcal{C}}^*_{\leq r}$ is also non-Helly. Thus, the smallest destroyer of $\overline{\mathcal{C}}^*_{\leq r}$ is either D or some smaller disk in $\mathcal{C}^*_{<r}$. In the latter case, we have $\mathrm{rad}(\mathcal{C}^*) < \mathrm{rad}(\mathcal{C})$. In the former case, we have $\mathrm{rad}(\mathcal{C}^*) = \mathrm{rad}(\mathcal{C})$, and Lemma 4.2 gives $-\mathrm{dist}(\mathcal{C}^*) = -d\big(\bigcap \mathcal{C}^*_{<r}, D\big) \leq -d(\bigcap \mathcal{C}_{<r}, D) = -\mathrm{dist}(\mathcal{C})$. In either case, $w\big(\mathcal{C}^*\big) \leq w(\mathcal{C})$. See Figure 5 for an illustration. ◀

▶ **Lemma 4.4.** *For any $\mathcal{C} \subseteq \mathcal{D}$, there is a set $\mathcal{B} \subseteq \mathcal{C}$ with $|\mathcal{B}| \leq 3$ and $w(\mathcal{B}) = w(\mathcal{C})$.*

Proof. Let D be the smallest destroyer of $\overline{\mathcal{C}}$. Let $r = \mathrm{rad}(\mathcal{C})$ be the radius of D, and let $v \in \bigcap \mathcal{C}_{<r}$ be the extreme point for $\mathcal{C}_{<r}$ and D. By general position, there are at most two disks $E, F \in \mathcal{C}_{<r}$ with $v \in \partial(E \cap F)$. Note that E and F may be the same disk.

Set $\mathcal{B} = \{D, E, F\} \setminus \{D_\infty\}$. There are three possibilities. If $\mathcal{C}$ is non-Helly, then $D \neq D_\infty$ and $\mathcal{B}$ is a non-Helly triple (indeed, as the disks in $\mathcal{D}$ are pairwise intersecting, the extreme point v must lie at the intersection of two disk boundaries). If $\mathcal{C}$ is Helly, then $D = D_\infty$ and $|\mathcal{B}| \leq 2$. If $|\mathcal{B}| = 2$, then v is the vertex of $\partial(E \cap F)$ with minimum y-coordinate. If $|\mathcal{B}| = 1$, then v is the point on ∂E with minimum y-coordinate. In either case, $\mathrm{dist}(\mathcal{B})$ is the value of the smallest y-coordinate of a point in $\bigcap \mathcal{B}$. See Figure 6 for an illustration.

We claim that $w(\mathcal{B}) = w(\mathcal{C})$. Firstly, $\mathrm{rad}(\mathcal{B}) = \mathrm{rad}(\mathcal{C})$, because $\mathcal{B}$ and $\mathcal{C}$ have the same smallest destroyer. Secondly, we show $\mathrm{dist}(\mathcal{B}) = \mathrm{dist}(\mathcal{C})$: since $\mathcal{B}_{<r} \subseteq \mathcal{C}_{<r}$, by Lemma 4.2, we get $\mathrm{dist}(\mathcal{B}) = d\big(\bigcap \mathcal{B}_{<r}, D\big) \leq d\big(\bigcap \mathcal{C}_{<r}, D\big) = \mathrm{dist}(\mathcal{C})$. Suppose that $\mathrm{dist}(\mathcal{B}) < \mathrm{dist}(\mathcal{C})$. Then, there is a point $w \in E \cap F$ with $d(w, D) < d(v, D)$. Furthermore, by general position and since v is the intersection of two disk boundaries, there is a relatively open neighborhood N around v in $\bigcap \mathcal{C}_{<r}$ such that N is also relatively open in $E \cap F$. Since $E \cap F$ is convex, there is a point $x \in N$ that also lies in the relative interior of the line segment $\overline{wv}$. Then, $d(x, D) < d(v, D)$ and $x \in \bigcap \mathcal{C}_{<r}$, which is impossible, as v is the extreme point.

The set $\mathcal{B}$ is actually a basis for $\mathcal{C}$: if $\mathcal{B}$ is a non-Helly triple, then removing any disk from $\mathcal{B}$ creates a Helly set and increases the radius of the smallest destroyer to ∞. If $|\mathcal{B}| \leq 2$, then D_∞ is the smallest destroyer of $\mathcal{B}$ and the minimality follows directly from the definition. ◀

▶ **Lemma 4.5.** *Let $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{D}$ with $w(\mathcal{B}) = w(\mathcal{C})$ and let $E \in \mathcal{D}$. Then, if $w\big(\mathcal{B} \cup \{E\}\big) = w(\mathcal{B})$ we also have $w\big(\mathcal{C} \cup \{E\}\big) = w(\mathcal{C})$.*

Proof. Set $\mathcal{C}^* = \mathcal{C} \cup \{E\}, \mathcal{B}^* = \mathcal{B} \cup \{E\}$. Let $r = \mathrm{rad}(\mathcal{C})$ and D the smallest destroyer of $\overline{\mathcal{C}}$. Since $w(\mathcal{C}) = w(\mathcal{B}) = w(\mathcal{B}^*)$, we have that D is also the smallest destroyer of $\overline{\mathcal{B}}$ and of $\overline{\mathcal{B}}^*$. If E has radius $> r$, then E cannot be the smallest destroyer of $\overline{\mathcal{C}}^*$, so $w\big(\mathcal{C}^*\big) = w(\mathcal{C})$. Assume that E has radius $< r$. Let v be the extreme point of $\mathcal{C}_{<r}$ and D. Since $w(\mathcal{B}^*) = w(\mathcal{B})$, we know that $d\big(\bigcap \mathcal{B}_{<r}, D\big) = d\big(\bigcap \mathcal{B}^*_{<r}, D\big) = d(v, D)$. Now, Lemma 4.2 implies $v \in E$, since $E \in \mathcal{B}^*_{<r}$. Thus, the set $\mathcal{C}^*_{<r} = \mathcal{C}_{<r} \cup \{E\}$ is Helly. Furthermore, $\overline{\mathcal{C}}^*_{\leq r}$ is non-Helly, because the subset $\overline{\mathcal{C}}_{\leq r}$ is non-Helly. Therefore, D is also the smallest destroyer of $\overline{\mathcal{C}}^*$, so $\mathrm{rad}(\mathcal{C}^*) = r = \mathrm{rad}(\mathcal{C})$. Finally, since $\mathcal{B}^*_{<r} \subseteq \mathcal{C}^*_{<r}$ we can use Lemma 4.2 to derive

$$d\Big(\bigcap \mathcal{C}_{<r}, D\Big) = d\Big(\bigcap \mathcal{B}^*_{<r}, D\Big) \leq d\Big(\bigcap \mathcal{C}^*_{<r}, D\Big) \leq d(v, D) = d\Big(\bigcap \mathcal{C}_{<r}, D\Big). \qquad ◀$$

Next, we describe a violation test for $(\mathcal{D}, w, \leq)$: given a set $\mathcal{B} \subseteq \mathcal{D}$ and a disk $E \in \mathcal{D}$ with radius r, determine (i) whether $\mathcal{B}$ is a basis for some subset of $\mathcal{D}$, and (ii) whether E violates $\mathcal{B}$, i.e., whether $w\big(\mathcal{B} \cup \{E\}\big) < w(\mathcal{B})$. This is done as follows:

- If (i) $|\mathcal{B}| > 3$; or (ii) $|\mathcal{B}| = 3$ and $\mathcal{B}$ is Helly; or (iii) $|\mathcal{B}| = 2$ and the y-minimum of $\bigcap \mathcal{B}$ is also the y-minimum of a single disk of $\mathcal{B}$, return **"$\mathcal{B}$ is not a basis"**.
- If $|\mathcal{B}| = 1$, then, if the y-minimum in $E \cap \bigcap \mathcal{B}$ differs from the y-minimum in $\bigcap \mathcal{B}$, return **"E violates $\mathcal{B}$"** ; otherwise, return **"E does not violate $\mathcal{B}$"**.
- If $\mathcal{B} = \{D_1, D_2\}$, find the y-minimum v of $D_1 \cap D_2$ and return **"E violates $\mathcal{B}$"** if $v \notin E$, and **"E does not violate $\mathcal{B}$"**, otherwise.
- Finally, if $\mathcal{B} = \{D, D_1, D_2\}$ is non-Helly with smallest destroyer D.[6] Let $r = \mathrm{rad}(\mathcal{B})$ be the radius of D and r' be the radius of E:

[6] Note that since $\mathcal{B}$ is a subset of $\mathcal{D}$ and since $\mathcal{B}$ is non-Helly, the smallest destroyer D of $\mathcal{B}$ cannot be the disk D_∞.

- If $r' > r$, return "E **does not violate** $\mathcal{B}$".
- If $r' < r$, find the vertex v of $D_1 \cap D_2$ that minimizes the distance to E and return "E **violates** $\mathcal{B}$" if $v \notin E$, and "E **does not violate** $\mathcal{B}$", otherwise.

The violation test obviously needs constant time. Finally, to apply the algorithm of Chazelle and Matoušek, we still need to check that the range space $(\mathcal{D}, \mathcal{R})$ with $\mathcal{R} = \{\text{vio}(\mathcal{B}) \mid \mathcal{B} \text{ is a basis of a subset in } \mathcal{D}\}$ has bounded VC dimension.

▶ **Lemma 4.6.** *The range space $(\mathcal{D}, \mathcal{R})$ has VC-dimension at most 3.*

Proof. The discussion above shows that for any basis $\mathcal{B}$, there is a point $v_{\mathcal{B}} \in \mathbb{R}^2$ such that $E \in \mathcal{D}$ violates $\mathcal{B}$ if and only if $v_{\mathcal{B}} \notin E$. Thus, for any $v \in \mathbb{R}^2$, let $\mathcal{R}'_v = \{D \in \mathcal{D} \mid v \notin D\}$ and let $\mathcal{R}' = \{\mathcal{R}'_v \mid v \in \mathbb{R}^2\}$. Since $\mathcal{R} \subseteq \mathcal{R}'$, it suffices to show that $(\mathcal{D}, \mathcal{R}')$ has bounded VC-dimension. For this, consider the complement range space $(\mathcal{D}, \mathcal{R}'')$ with $\mathcal{R}'' = \{\mathcal{R}''_v \mid v \in \mathbb{R}^2\}$ and $\mathcal{R}''_v = \{D \in \mathcal{D} \mid v \in D\}$, for $v \in \mathbb{R}^2$. It is well known that $(\mathcal{D}, \mathcal{R}')$ and $(\mathcal{D}, \mathcal{R}'')$ have the same VC-dimension [3], and that $(\mathcal{D}, \mathcal{R}'')$ has VC-dimension 3 (e.g., this follows from the classic homework exercise that there is no planar Venn-diagram for four sets). ◀

Finally, the following lemma summarizes discussion so far.

▶ **Lemma 4.7.** *Given a set $\mathcal{D}$ of n pairwise intersecting disks in the plane, we can decide in $O(n)$ deterministic time whether $\mathcal{D}$ is Helly. If so, we can compute a point in $\bigcap \mathcal{D}$ in $O(n)$ deterministic time. If not, we can compute the smallest destroyer D of $\mathcal{D}$ and two disks $E, F \in \mathcal{D}_{<r}$ that form a non-Helly triple with D. Here, r is the radius of D.*

Proof. Since $(\mathcal{D}, w, \leq)$ is LP-type, the violation test needs $O(1)$ time, and the VC-dimension of $(\mathcal{D}, \mathcal{R})$ is bounded, we can apply the deterministic algorithm of Chazelle and Matoušek [4] to compute $w(\mathcal{D}) = (\text{rad}(\mathcal{D}), -\text{dist}(\mathcal{D}))$ and a corresponding basis $\mathcal{B}$ in $O(n)$ time. Then, $\mathcal{D}$ is Helly if and only if $\text{rad}(\mathcal{D}) = \infty$. If $\mathcal{D}$ is Helly, then $|\mathcal{B}| \leq 2$. We compute the unique point $v \in \bigcap \mathcal{B}$ with $d(v, D_\infty) = d\big(\bigcap \mathcal{B}, D_\infty\big)$. Since $\mathcal{B} \subseteq \mathcal{D}$ and $d\big(\bigcap \mathcal{B}, D_\infty\big) = d\big(\bigcap \mathcal{D}, D_\infty\big)$, we have $v \in \bigcap \mathcal{D}$ by Lemma 4.2. We output v. If $\mathcal{D}$ is non-Helly, we simply output $\mathcal{B}$, because $\mathcal{B}$ is a non-Helly triple with the smallest destroyer D of $\mathcal{D}$ and two disks $E, F \in \mathcal{D}_{<r}$, where r is the radius of D. ◀

▶ **Theorem 4.8.** *Given a set $\mathcal{D}$ of n pairwise intersecting disks in the plane, we can find in $O(n)$ time a set P of five points such that every disk of $\mathcal{D}$ contains at least one point of P.*

Proof. Using the algorithm from Lemma 4.7, we decide whether $\mathcal{D}$ is Helly. If so, we return the point computed by the algorithm. Otherwise, the algorithm gives us a non-Helly triple $\{D, E, F\}$, where D is the smallest destroyer of $\mathcal{D}$ and $E, F \in \mathcal{D}_{<r}$, with r being the radius of D. Since $\mathcal{D}_{<r}$ is Helly, we can obtain in $O(n)$ time a stabbing point $q \in \bigcap \mathcal{D}_{<r}$ by using the algorithm from Lemma 4.7 again. Next, by Lemma 2.1, there are two disks in $\{D, E, F\}$ whose lens angle is at least $2\pi/3$. Let P' be the set of four points from the proof of Lemma 2.4. Then, $P = P' \cup \{q\}$ is a set of five points that stabs every disks in $\mathcal{D}$. ◀

5 A Simple Lower Bound

We now exhibit a set of 13 pairwise intersecting disks in the plane such that no point set of size three can pierce all of them. The construction begins with an inner disk A of radius 1 and three larger disks D_1, D_2, D_3 of equal radius, so that A is tangent to all three disks and so that each two disks are tangent to each other. For $i = 1, 2, 3$, we denote the contact point of A and D_i by ξ_i.

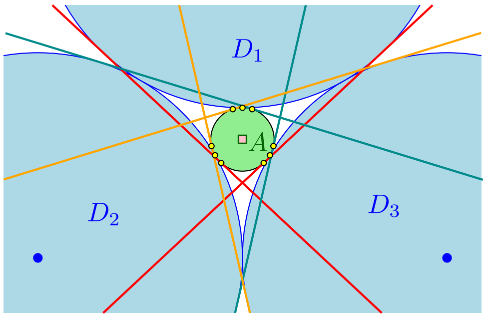

Figure 7 Each common tangent ℓ between A and D_i represents a very large disk, whose interior is disjoint from A. The nine points of tangency are pairwise distinct.

We add six more disks as follows. For $i = 1, 2, 3$, we draw the two common outer tangents to A and D_i, and denote by T_i^- and T_i^+ the halfplanes that are bounded by these tangents and are openly disjoint from A. The labels T_i^- and T_i^+ are chosen such that the points of tangency between A and T_i^-, D_i, and T_i^+, appear along ∂A in this counterclockwise order. One can show that the nine points of tangency between A and the other disks and tangents are pairwise distinct (see Figure 7). We regard the six halfplanes T_i^-, T_i^+, for $i = 1, 2, 3$, as (very large) disks; in the end, we can apply a suitable inversion to turn the disks and halfplanes into actual disks, if so desired.

Finally, we construct three additional disks A_1, A_2, A_3. To construct A_i, we slightly expand A into a disk A_i' of radius $1 + \varepsilon_1$, while keeping the tangency with D_i at ξ_i. We then roll A_i' clockwise along D_i, by a tiny angle $\varepsilon_2 \ll \varepsilon_1$, to obtain A_i.

This gives a set of 13 disks. For sufficiently small ε_1 and ε_2, we can ensure the following properties for each A_i: (i) A_i intersects all other 12 disks; (ii) the nine intersection regions $A_i \cap D_j$, $A_i \cap T_j^-$, $A_i \cap T_j^+$, for $j = 1, 2, 3$, are pairwise disjoint; and (iii) $\xi_i \notin A_i$.

▶ **Theorem 5.1.** *The construction yields a set of* 13 *disks that cannot be stabbed by* 3 *points.*

Proof. Consider any set P of three points. Set $A^* = A \cup A_1 \cup A_2 \cup A_3$. If $P \cap A^* = \emptyset$, we have unstabbed disks, so suppose that $P \cap A^* \neq \emptyset$. For $p \in P \cap A^*$, property (ii) implies that p stabs at most one of the nine remaining disks D_j, T_j^+ and T_j^-, for $j = 1, 2, 3$. Thus, if $P \subset A^*$, we would have unstabbed disks, so we may assume that $|P \cap A^*| \in \{1, 2\}$.

Suppose first that $|P \cap A^*| = 2$. As just argued, at most two of the remaining disks are stabbed by $P \cap A^*$. The following cases can then arise.

(a) None of D_1, D_2, D_3 is stabbed by $P \cap A^*$. Since $\{D_1, D_2, D_3\}$ is non-Helly and a non-Helly set must be stabbed by at least two points, at least one disk remains unstabbed.

(b) Two disks among D_1, D_2, D_3 are stabbed by $P \cap A^*$. Then the six unstabbed halfplanes form many non-Helly triples, e.g., T_1^-, T_2^-, and T_3^-, and again, a disk remains unstabbed.

(c) The set $P \cap A^*$ stabs one disk in $\{D_1, D_2, D_3\}$ and one halfplane. Then, there is (at least) one disk D_i such that D_i and its two tangent halfplanes T_i^-, T_i^+ are all unstabbed by $P \cap A^*$. Then, $\{D_i, T_i^-, T_i^+\}$ is non-Helly, and at least two more points are needed to stab it.

Suppose now that $|P \cap A^*| = 1$, and let $P \cap A^* = \{p\}$. We may assume that p stabs all three disks A_1, A_2, A_3, since otherwise a disk would stay unstabbed. At most one of the nine remaining disks is stabbed by p. Thus, $p \notin \{\xi_1, \xi_2, \xi_3\}$, so the other disk that it stabs (if any) must be a halfplane. That is, p does not stab any of D_1, D_2, D_3. Since $\{D_1, D_2, D_3\}$ is non-Helly, it requires two stabbing points. Moreover, since $|P \setminus \{p\}| = 2$, we may assume that one point q of $P \setminus A^*$ is the point of tangency of two of these disks, say $q = D_2 \cap D_3$. Then, q stabs only two of the six halfplanes, say, T_1^- and T_1^+. But then, $\{D_1, T_2^+, T_3^-\}$ is non-Helly and does not contain any point from $\{p, q\}$. At least one disk remains unstabbed. ◀

6 Conclusion

We gave a simple linear-time algorithm to find five stabbing points for a set of pairwise intersecting disks in the plane. It remains open how to use the proofs of Danzer or Stachó [15, 5] (or any other technique) for an efficient construction of four stabbing points. It is also not known whether nine disks can be stabbed by three points or not (for eight disks, this is the case [14]). Furthermore, it would be interesting to find a simpler construction, than the one by Danzer, of ten pairwise intersecting disks that cannot be stabbed by three points.

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